Physics Student Understanding of Divergence and Curl and Their Constituent Partial Derivatives

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This work is part of a broader project to investigate student understanding of mathematical ideas used in upper-division physics. This study in particular probes students' understanding of the divergence and curl operators as applied to vector field diagrams. We examined how students reason with partial derivatives that constitute divergence and curl of the vector field diagrams. Students' written responses to a task on derivatives, divergence, and curl of a 2D vector field were collected and coded. Students were generally successful in determining the sign of some of the constituent derivatives of div and curl, but struggled in one case in which components were negative. Analysis of written explanations showed confusion between the sign, direction, and change in the magnitude of vector field components.

Keywords: Partial Derivatives, Divergence, Curl, Vector Field Diagrams

Introduction

Many physical quantities, such as force and momentum, are represented with vectors. For several topics in physics, e.g., interactions in gravitation and electricity and magnetism, it is useful to define a vector field: a vector quantity is assigned to every point of a subset of space. Vector fields can be represented in different ways, with field lines, an array of arrows, or a symbolic expression like $\hat{V} = ay\hat{\imath} + bx\hat{\jmath}$. Students are introduced to vector fields in introductory courses, typically in the contexts of electric and magnetic fields. These vector fields vary in space, and vector calculus provides several ways to describe this variation, including the gradient, divergence, and curl of the vector field. Several significant physical quantities are associated with vector derivatives: Maxwell's equations for electromagnetism describe relationships between the divergence or curl of electric or magnetic fields and other physical quantities, and the fields themselves can be expressed in terms of derivatives of scalar and/or vector potentials. While most students have not encountered vector calculus the first time they study vector fields, those who go on to major in physics and electrical engineering will use these ideas extensively in a junior-level course in electricity and magnetism. Students encounter vector field representations for electric and magnetic fields in an electromagnetism course, and research involves electric and magnetic fields and even gravitational fields. Students are expected to reason with symbolic equations but also with vector field representations of divergence ($\nabla . \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}$) and curl ($\nabla \times \vec{V} = det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$). We expect students to look at the vector field and find the appropriate components and see how the proper components are changing as we move to certain directions.

Several previous studies in physics education research (PER) have examined student understanding of divergence and curl in post-introductory courses (Baily & Astolfi, 2014; Bollen et al., 2015; Gire & Price, 2012; Singh & Maries, 2013). Generally these studies have involved a two-dimensional representation of a field as an array of vectors. Singh and Maries (2013) reported that even though graduate students successfully calculated divergence and curl of the vector fields, they were unable to interpret the div and curl of vector field plots. Baily and Astolfi (2014) and Bollen et al. (2015) repeated the previous study with different diagrams, reporting that around 50% of their students could correctly determine divergence and curl given the vector field diagrams. Bollen et al. (2015) also qualitatively studied students' responses, categorizing them into three approaches: description based, concept based, and formula based. For example, determining div or curl based on a description of its meaning was categorized as description based. If students used concepts like flux or the "paddle wheel" to determine div or curl, respectively, it was categorized as concept based.

While previous studies have focused on the divergence and curl, the classroom experience of one author of this study suggested that reasoning with the partial derivatives that constitute these operations, e.g., $\frac{\partial V_x}{\partial x}$ and $\frac{\partial V_y}{\partial y}$ for divergence or $\frac{\partial V_x}{\partial y}$ and $\frac{\partial V_y}{\partial x}$ for curl in Cartesian coordinates, might be one element of the challenges faced by students. Previous studies have asked students to determine the sign or value of the divergence and/or curl for a given field diagram, but there has not been as much focus on their constituent derivatives.

Prior PER studies have documented student difficulties with partial derivatives, often in thermal physics contexts (Bajracharya & Thompson, 2016; Thompson et al., 2006). Student understanding of derivatives is studied widely in RUME. Zandieh (2000) developed a theoretical framework for student understanding of derivatives, which was extended by Roundy et al. (2015) to include partial derivatives. Wangberg and Gire (2019) investigated student understanding of partial derivatives of scalar fields represented as surfaces using Zandieh's framework.

The derivatives in the expressions for divergence and curl have the additional complication that they are derivatives of vector components. In these partial derivatives, V_x refers to the x-component of the vector field, so $\frac{\partial V_x}{\partial x}$ is the partial derivative with respect to x of the x-component of the vector V. Extracting information about the derivatives from a vector field diagram involves multiple steps. Existing frameworks are restricted to derivatives of scalar functions, and need to be extended to deal with the derivatives of vector quantities. While prior frameworks have offered insights into student reasoning, they do not account for functions of multiple variables, nor for the vector nature of the derivatives.

We set out to develop and implement tasks that used similar questions to prior studies investigating div and curl for vector field diagrams, but with added explicit questions about the constituent partial derivatives. The goal is to begin to answer the following research questions:

- To what extent can students determine the sign of the constituent derivatives of divergence and curl given a vector field diagram?
- To what extent are student responses to tasks focused on the signs of constituent derivatives related to success in determining the sign of divergence and curl?

Methods

Written data were collected at two universities in sections of Mathematical Methods for Physics, post-introductory courses for physics and engineering physics majors and minors that are intended to cover the advanced mathematics students will encounter in upper-level theory core courses like Electricity and Magnetism or Quantum Mechanics. All students (N=32) had completed introductory sequences in both physics and calculus, and the data were collected in the course after instruction on vector calculus. One campus serves a diverse student population in the southwest, the other is a predominantly white institution in the northeast. Responses from the two universities were similar and are thus combined and reported together in this paper.

In each course, the students had considered similar representations of vector fields in class and answered questions relating the features of a field to its divergence and curl. In one university, students had completed a research-based instructional tutorial; in the other, these questions were presented as a whole-class discussion. After instruction, the questions shown in Fig. 1a were posed on a course midterm exam. Students are shown a 2-d field representation and asked to determine the signs first of the divergence and curl, then of the constituent derivatives.

In the broader study, different versions of this task were used. In this report, students responded to a task asking first about the derivatives and then about div and curl. With this sequence, we hoped to see whether the reasoning for div/curl included derivatives or if other reasoning would emerge.

The coding process began with general codes for both correctness of the sign and the correctness of reasoning provided by students. After the lead author generated the initial codes, other members of the team independently coded several student responses to refine the coding scheme. Answers without explanations (i.e., "$\frac{\partial V_x}{\partial x}$ is zero") or explanations that were not clear enough to be understood were coded as unclear reasoning.

Figure 1b (not given to students) shows which components students are expected to use to determine the sign of $\frac{\partial V_x}{\partial x}$.

A slice of **a vector field V** (for z=0) is shown. Assume that the field has no components in the z-direction (into and out of the page) and that other slices for other values of z would look the same (i.e., ignore z components or direction).

A. For each of the quantities below, state whether the quantity is *positive*, *negative*, or *zero*. Show or explain briefly (stating any assumptions you are making).

The (z component of) the curl of the field V

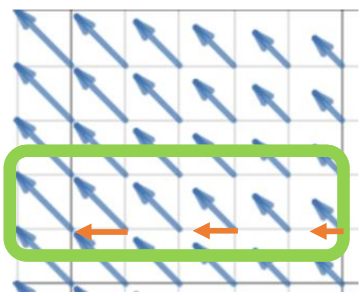
the divergence of the field V

B. Indicate whether the following derivatives are *positive, negative,* or *zero.* Show or explain briefly (stating any assumptions you are making).

$\frac{\partial V_x}{\partial x}$ in the region $\qquad$ $\frac{\partial V_y}{\partial x}$ in the region

$\frac{\partial V_x}{\partial y}$ in the region $\qquad$ $\frac{\partial V_y}{\partial y}$ in the region

a



b

Figure 1. (a) The research task asked to probe students' reasoning about div and curl. (b) The figure for the task, with components of the vector field students are expected to examine to determine $\frac{\partial V_x}{\partial x}$.

Results and Discussion

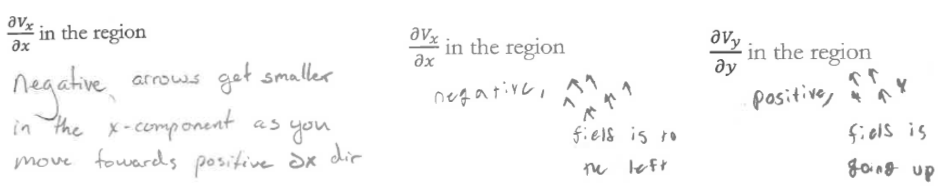
The numbers of students that correctly determined the constituent derivatives and the distribution of correctness of student reasoning are shown in Table 1.

Only 4 of 32 (13%) students answered all parts of the question (all derivatives, divergence, and curl) correctly, suggesting that the set of questions was especially challenging. Most students successfully identified the constituent derivatives in three out of four cases: for $\frac{\partial V_y}{\partial y}$, $\frac{\partial V_x}{\partial y}$, and $\frac{\partial V_y}{\partial x}$, success rates were over 75%. For two of these derivatives the vector field component was not changing in the indicated direction, and for the third it was positive and decreasing.

In contrast, only 21% of the students correctly determined that $\frac{\partial V_x}{\partial x}$ was positive with correct reasoning. Because this derivative was the most challenging, we examined the reasoning required in some detail. The first step was identifying the appropriate components; Figure 1b shows the components that students were expected to examine to determine $\frac{\partial V_x}{\partial x}$. Analysis of student responses showed that most (<90%) used the correct components to determine $\frac{\partial V_x}{\partial x}$. For the given vector field diagram, the absolute value of V_x is getting smaller with respect to the x-axis, but due to the direction of V, the change in V_x (∂V_x) is positive. Written responses did not show evidence of explicitly attending to subtracting the two components. Instead, students wrote about the trends in the magnitude of the components, whether that component was increasing, decreasing, or staying constant when you move in a certain direction. In one incorrect student responses, shown in Figure 2a, the student wrote "arrows getting smaller" and seemed to associate this with the resulting negative value for $\frac{\partial V_x}{\partial x}$. This response, while incorrect, included some correct reasoning. Responses that associated the component with the appropriate direction but reversed the sign were coded as correct reasoning with a sign mistake. However, this student wrote about change in magnitude rather than change in component.

Table 1. Performances and distribution of student reasoning for $\frac{\partial V_x}{\partial x}$, $\frac{\partial V_y}{\partial y}$, $\frac{\partial V_x}{\partial y}$, and $\frac{\partial V_y}{\partial x}$ of the diagram. Correct answers for the derivatives are given in parentheses. (C: Correct, I: Incorrect).

N=32	*Constituent derivatives for divergence*				*Constituent derivatives for curl*			
	$\frac{\partial V_x}{\partial x}$ (+)		$\frac{\partial V_y}{\partial y}$ (0)		$\frac{\partial V_x}{\partial y}$ (0)		$\frac{\partial V_y}{\partial x}$ (−)	
Reasoning	*C*	*I*	*C*	*I*	*C*	*I*	*C*	*I*
Correct	7	17	21	2	18	2	19	1
Incorrect	1	4	0	1	1	3	3	2
Unclear or none	3	0	4	4	6	2	1	6



a b c

Figure 2. (a) Incorrect response to $\frac{\partial V_x}{\partial x}$, reasoning about the decreasing arrow length to justify the response. (b) and (c) Example responses of students relating derivatives with the direction of the vector field component.

Previous studies conducted in introductory physics courses reported student difficulties in subtracting negative vectors when given as a graphical representation, even though students were very successful enacting the procedure of subtracting when vectors are provided in an equation with coordinates (Barniol & Zavala, 2014; Susac et al., 2018). Our results suggest similar difficulty among students in this more advanced context.

While students had good success with many of the constituent derivative tasks, only 38% of students determined the correct signs for divergence and curl with correct reasoning, as shown in Table 2. For the divergence, most of the responses coded as incorrect signs with correct reasoning stem from finding $\frac{\partial V_x}{\partial x}$ as negative, as described above. Student responses included several alternative forms of reasoning for divergence, such as inferring flux from the vector field diagram, or identifying changes relative to a perceived "source" of the arrows.

There was also no clear relationship between success on derivative task and on the curl task. Many of the students coded with incorrect reasoning for curl answered that the z component of curl was zero and referred to the problem statement that the vector field had no z component. This may reflect a misinterpretation of the question or a misunderstanding of the relationship between components of the vector field and those of the curl. A previous version of this question did not include this text and more students did find the curl correctly for the diagram.

Table 2. Student performances for divergence and curl of the diagram (C: Correct, I: Incorrect).

N=32	*Divergence*		*Curl*	
Reasoning	*C*	*I*	*C*	*I*
Correct	12	10	12	3
Incorrect	1	6	1	13
Unclear or none	2	1	1	2

Another small set of student responses show confusion between the direction of the vector field component and the change of the vector field component, as shown in Figure 2b and 2c. Similar confusion between a quantity and its change or rate of change has been widely reported in both mathematics and physics contexts at the introductory level (Meltzer, 2004; Trowbridge & McDermott, 1981). Our data show examples in a more advanced population, suggesting the persistence of this confusion.

Conclusions and Future Work

Our results were consistent with previous studies that reported that divergence and curl are challenging for students. For this task and population, students were largely successful in determining the sign of the constituent derivatives of div and curl. Incorrect responses showed confusion between the sign, direction, and change in the magnitude of vector field components; this confusion seems reminiscent of previous findings in both introductory physics and mathematics classes.

Determining the constituent derivative incorrectly did result in an incorrect divergence sign, but correct responses on constituent derivatives were not sufficient for success on divergence and curl. Students seemed to confuse components of curl with components of the field itself.

Explicit attention to the derivatives in expressions for divergence and curl seems to be a fruitful direction for future research and curriculum development. We plan to collect additional data, including student interviews, to further investigate the understanding of these derivatives as well as of divergence and curl, including relationships between quantities. We also intend to re-examine existing data through the lens of covariation (Carlson et al., 2002) and to examine whether it is possible to extend existing frameworks to vector derivatives.

Acknowledgments

This work is supported in part by the National Science Foundation under Grant Nos. PHY-1912087 and PHY-1912660.

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