How students reason with derivatives of vector field diagrams

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Introduction

Physics students are introduced to vector fields in introductory courses, typically in the contexts of electric and magnetic fields. Vector calculus provides several ways to describe how vector fields vary in space including the gradient, divergence, and curl. Physics majors use vector calculus extensively in a junior-level electricity and magnetism (E&M) course. Our focus here is exploring student reasoning with the partial derivatives that constitute divergence and curl in vector field representations, adding to the current understanding of how students reason with derivatives.

Background

Several previous studies in physics education research (PER) have examined student understanding of divergence and curl in post-introductory and graduate courses (Baily & Astolfi, 2014; Bollen et al., 2015; Gire & Price, 2012; Singh & Maries, 2013). These studies involved two-dimensional representations of a field as an array of vectors and asked students to determine the divergence and/or curl from these representations. Baily & Astolfi (2014) and Bollen et al. (2015) performed similar studies with different diagrams and reported that around 50% of their students could correctly determine whether the divergence and curl of the vector field diagrams are zero or not. Klein et al. (2019) conducted a teaching experiment in which they split students into two groups. Both groups received textual instruction on determining the sign of the partial derivatives of a vector field; one was given visual cues to accompany the text. No difference in improvement between the groups was seen on a transfer task. In these previous studies, students determined the sign or value of the divergence and/or curl for a given field diagram. There has not been as much focus on the partial derivatives that constitute these operations, e.g., $\frac{\partial V_x}{\partial x}$ and $\frac{\partial V_y}{\partial y}$ for divergence or $\frac{\partial V_x}{\partial y}$ and $\frac{\partial V_y}{\partial x}$ for curl in Cartesian coordinates. This study explores student understanding of constituent derivatives of divergence and curl with vector field representations.

Zandieh (2000) developed a theoretical framework for student understanding of derivatives, which Roundy et al. (2015) extended to include physical representations for derivatives of physics contexts. Wangberg and Gire (2019) investigated student understanding of partial derivatives of scalar fields using Zandieh's framework. These works are restricted to derivatives of scalar functions. Gire & Price (2012) reported confusion among students in an upper-division E&M course when sketching vector fields given expressions that "cross" components and coordinates.

Study and setting

The study was conducted in the Mathematical Methods for Physics course, an intermediate course intended to prepare students for the advanced mathematics they will encounter in upper-level physics courses. All students had completed introductory sequences in both physics and calculus. Written data were collected in the course after instruction on vector calculus. In the tasks, students were shown a 2-d field representation (see Figure 1) and asked to determine the signs first of the divergence and curl, then of the constituent derivatives. Field 1 has only V_x components; students were asked to determine $\frac{\partial V_x}{\partial x}$ and $\frac{\partial V_x}{\partial y}$. For Field 2 all constituent derivatives of divergence and curl were asked. Field 1 (N=14) and Field 2 (N=32) were asked in different semesters at two public universities; due to small N, data are combined. Only the results for constituent derivatives will be discussed here.

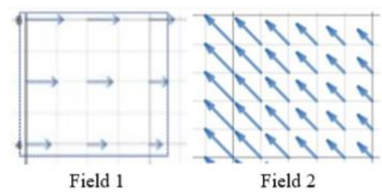

Figure 1: Vector field diagrams used in the tasks

Results and discussion

To determine the partial derivatives of the vector fields, students were first expected to determine which component to consider and then identify how that component changes with respect to the corresponding direction in the denominator (see Figure 2).

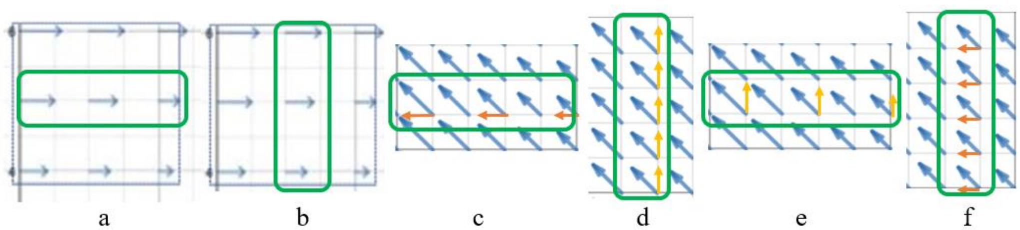

Figure 2: The vector field components for each field that students were expected to examine for each task. The selected vectors are those needed to examine the derivative in question, and the lighter colored arrows are the components in the direction of interest: $\frac{\partial V_x}{\partial x}$ and $\frac{\partial V_x}{\partial y}$ for Field 1 in a and b, respectively, and $\frac{\partial V_x}{\partial x}$, $\frac{\partial V_y}{\partial y}$, $\frac{\partial V_y}{\partial x}$ and $\frac{\partial V_x}{\partial y}$ for Field 2 in c, d, e, and f, respectively

Constituent derivatives for divergence

For Field 1, 9 of 14 students were able to identify the sign of $\frac{\partial V_x}{\partial x}$. Determining $\frac{\partial V_x}{\partial x}$ for Field 2 was more challenging: 34% of the students (N=32) answered correctly. An example of the most common incorrect reasoning was "arrows get smaller in the *x*-component as you move towards positive *dx* direction". We have suggested that some students recognize that the vector magnitude is decreasing, but do not account for the negative direction of the vector and thus find the sign of $\frac{\partial V_x}{\partial x}$ to be negative (Topdemir et al., 2023). More students correctly determined $\frac{\partial V_y}{\partial y}$ to be zero for Field 2 (78%).

Constituent derivatives for curl

The constituent derivatives for curl are 'mixed' in that the component that is differentiated does not match the coordinate with respect to which one differentiates (i.e., $\frac{\partial V_x}{\partial y}$ as opposed to $\frac{\partial V_x}{\partial x}$). For Field 1, only 2 of 14 students were able to identify the change in V_x with respect to the y-direction. For Field 2, 72% of the students (N=32) answered each of $\frac{\partial V_x}{\partial y}$ and $\frac{\partial V_y}{\partial x}$ correctly, but only 50% answered both derivatives correctly. When finding $\frac{\partial V_x}{\partial y}$ for Field 1, a few student responses suggested incorrect notation mapping (see Figure 3a), with explanations consistent with reversed components and variables. For example, the explanation for the sign of $\frac{\partial V_x}{\partial y}$ is consistent with the reasoning for the sign of $\frac{\partial V_y}{\partial x}$. Other responses for Field 1 (Figure 3b) stated that there is no change in the y-direction, which could be interpreted as no change in the y-component (correct but not relevant) or as no change with change in the y-coordinate (incorrect). This response might thus be explaining $\frac{\partial V_y}{\partial x}$ or $\frac{\partial V_y}{\partial y}$ rather than $\frac{\partial V_x}{\partial y}$. We suspect that determining $\frac{\partial V_x}{\partial x}$ is more similar to finding traditional scalar derivatives of a function than finding $\frac{\partial V_x}{\partial y}$, resulting in higher performance in determining $\frac{\partial V_x}{\partial x}$ for Field 1.

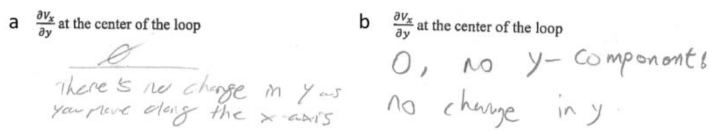

Figure 3: Student responses showing incorrect notation mapping to derivatives for Field 1

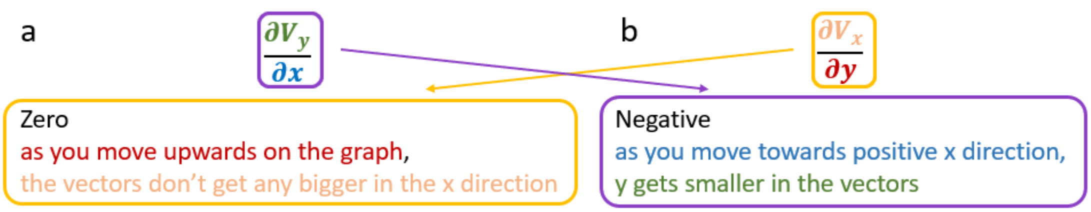

Figure 4: Student responses showing incorrect notation mapping to constituent derivatives of the curl for Field 2 (a, b). Colored text in response corresponds to similarly colored elements of derivative

For Field 2, 78% students answered $\frac{\partial V_x}{\partial y}$ correctly. Some students incorrectly mapped notations to derivatives. Figures 4a and 4b show responses from a student for $\frac{\partial V_y}{\partial x}$ and $\frac{\partial V_x}{\partial y}$, respectively. In Figure 4a, the response explains how V_x changes along the y-axis even though the question asked about $\frac{\partial V_y}{\partial x}$. Similarly, the response in Figure 4b explains how V_y changes along the x-axis instead of $\frac{\partial V_x}{\partial y}$. This reasoning may inflate student performance on the partial derivative tasks for which the component and the direction were the same (i.e., $\frac{\partial V_x}{\partial x}$ and $\frac{\partial V_y}{\partial y}$).

Conclusions and reflections

Determining the signs of the constituent derivatives of divergence and curl was a challenging task for students: only 5 of 32 students (16%) correctly determined all four partial derivative signs. Some challenges were dependent on the properties of the specific vector fields, e.g., when the vector field had a single component or when a vector field component was negative. Incorrect student responses suggested confusion between the change in a component and the change in a coordinate, confirming the informal observation of Gire & Price (2012) and a report in these proceedings by Walker & Dray (2023). While the vector field representation is complex, our data suggest that mapping the notation to the quantities in the ratio is the more challenging aspect of the task for most students. We note that, in general, students both recognized covariation in the constituent partial derivatives and treated these partial derivatives as ratios of small changes (along one coordinate). While consistent with the framework of Zandieh (2000) and Roundy et al. (2015), this suggests the need for extending the framework to account for specifics of vector partial derivatives.

Instructors may wish to integrate tasks including partial derivatives using vector field diagrams into instruction to provide an avenue for students to link more procedural understanding with graphical representations and explicitly attend to features of vector partial derivatives.

Acknowledgment

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