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Research paper

On the decaying property of quintic NLS on 3D hyperbolic space

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ABSTRACT

In this paper, we study the (pointwise) decaying property of quintic NLS on the three-dimensional hyperbolic space $\mathbb{H}^3$. We show the nonlinear solution enjoys the same decay rate as the linear solution does. This result is based on the associated global well-posedness and scattering result in Ionescu et al. (2012). This extends (Fan and Zhao, 2021)' Euclidean works to the hyperbolic space with additional improvements in regularity requirement (lower and almost critical regularity assumed). Realizing such improvements also work for the Euclidean case, we obtain a result for the fourth-order NLS analogue studied in Yu et al. (2023) recently with better, i.e. almost critical regularity assumption.

1. Introduction

1.1. Background and motivations

In this paper, we consider the quintic nonlinear Schrödinger equations (NLS) on hyperbolic space as follows,

$$\begin{cases} (i\partial_t + \Delta_{\mathbb{H}^3})u = |u|^4 u, \\ u(0,x) = u_0 \in H^s(\mathbb{H}^3), \end{cases} \tag{1.1}$$

and the center of this work is the decay property of nonlinear solutions in curved spaces.

If one considers the linear solution to (1.1) (i.e. letting the right-hand-side be 0), the following decay estimate holds, (see [1] for more details)

$$\|u(t)\|_{L_x^\infty(\mathbb{H}^3)} \lesssim t^{-\frac{3}{2}} \|u_0\|_{L_x^1(\mathbb{H}^3)}. \tag{1.2}$$

The goal of this paper is to show the nonlinear solution to (1.1) also satisfies the decay property

$$\|u(t)\|_{L_x^\infty(\mathbb{H}^3)} \lesssim_{\text{data}} t^{-\frac{3}{2}},$$

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where the constant depends on the initial data. Our results for (1.1) are based on Ionescu–Pausader–Staffilani [2] where the global well-posedness and scattering for model (1.1) are proved.

The well-posedness problem of NLS has drawn a lot of attention in the last half-century. In Euclidean spaces, where the sectional curvature is constant zero, the global well-posedness and scattering problem of NLS (at least in the subcritical regime and much of the critical setting) is well understood now (see for instance [3–12]). However, our universe is not flat. In hyperbolic spaces ($\mathbb{H}^d$) which are the simplest symmetric spaces of non-compact type with a constant negative sectional curvature, due to different geometric properties given by the negative curvature metric, there are only very few results studying the global well-posedness and scattering effect of solutions to NLS, see [2,13–21]. Given that NLS is classified as a dispersive equation, understanding its dispersive estimates and decay properties is of great importance. In this paper, we aim to establish a decay property of solutions to NLS on the curved spaces $\mathbb{H}^d$.

1.2. Statement of the main results

The main result of this paper is as follows.

Theorem 1.1. *Let u solves (1.1) with initial data u_0, which is in $H^{1+} \cap L^1$. Then, there exists a constant C_{u_0} depending on u_0, such that for $t \geq 0$,*

$$\|u(t,x)\|_{L^\infty_x} \leq C_{u_0} t^{-\frac{3}{2}}.$$

Remark 1.2. H^{1+}-space indicates the nonhomogeneous Sobolev space $H^{1+\epsilon}$ for arbitrarily small $\epsilon > 0$ (i.e. a Sobolev space that is a little bit more regular than the energy space H^1); L^1-space assumption is natural in viewing of the standard dispersive estimate (1.2).

Remark 1.3. We note that the constant C_{u_0} is dependent on the size of the initial data u_0, not the profile of the initial data u_0. See [22] for more explanations. Since the arguments via profile decompositions are now standard, we omit this part.

Remark 1.4. We note that Theorem 1.1 also covers for the Euclidean analogue, i.e. consider 3D, quintic NLS (1.1) on $\mathbb{R}^3$. Based on the seminal global result Colliander–Keel–Staffilani–Takaoka–Tao [3], the scheme works with little modifications.

The result above can be regarded as the 'nonlinear decaying property', i.e. the nonlinear solution of a dispersive equation enjoys the same (pointwise) decay property as its linear solution does. Heuristically, scattering means the nonlinear solution behaves like a linear solution asymptotically. Thus for a dispersive model with scattering property, it is natural to study if the nonlinear solution decays pointwise like its linear solution.

We note that the 'nonlinear decaying properties' are results that have their own interests. There are also some further applications of such decaying properties. We mention two examples: 1. *Scattering for energy-subcritical models:* To show scattering for subcritical models (such as NLS), nonlinear decaying properties of the solutions are first investigated. (See [23]); 2. *The applications for mathematical physics (such as many body problems):* (pointwise) decay estimates for the Hartree equations are used to study the many-body problems (see [24]).

Now we briefly discuss the main strategy for the proof of Theorem 1.1. Compared to the 3D quintic NLS model studied in [25], firstly, we need the corresponding global result [2] and estimates/tools for the hyperbolic case, together with a persistence of regularity argument (see Sections 2 and 3 for more details); secondly, we apply some new tricks to lower the regularity assumption in [25] (from H^3 to H^{1+}) based on the Gagliardo–Nirenberg inequality and a Sobolev inequality trick respectively (see Section 4 for more details). We also include an improvement for the fourth-order NLS case studied in [26]. See 5 for more details.

To the best knowledge of the authors, this paper is the first result on the nonlinear decaying property for NLS on hyperbolic spaces. (We also improve the Euclidean analogues in the meantime.)

Moreover, we also include improvements for recent results: Fan–Zhao [25] and Yu–Yue–Zhao [26]. These are the three main points of this paper.[1]

1.3. Organization of the rest of this paper

In Section 2, we discuss the preliminaries; in Section 3, we overview the global result in Ionescu–Pausader–Staffilani [2] for the main model (1.1); in Section 4, we give the proof for the main theorem (we will discuss it from two aspects); in Section 5, we discuss an improvement (in the sense of lowering the regularity assumption for the initial data) for the fourth-order NLS case (studied in [26]); in Appendix, we make a few comments on this research line.

[1] Here we mention a very recent development in this research line: (Fan C, Killip R, Visan M, et al. Dispersive decay for the mass-critical nonlinear Schrödinger equation[J]. arXiv preprint arXiv:2403.09989, 2024.), which deals with the nonlinear decay problem for mass-critical NLS. This result is new and it involves new ideas and ingredients.

1.4. Notations

Throughout this note, we use C to denote the universal constant and C may change line by line. We say $A \lesssim B$, if $A \leq CB$. We say $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We also use the notation C_B to denote a constant depends on B. We use the standard notation for L^p spaces and L^2-based Sobolev spaces H^s.

2. Preliminaries

In this section, we give some preliminaries on 'NLS on hyperbolic spaces'. We refer to [2] and the references therein. We start with hyperbolic spaces.

2.1. Geometry of $\mathbb{H}^3$

We review the model for 3-dimensional hyperbolic space $\mathbb{H}^3$.

Let $\mathbb{R}^{1+3}$ be the standard Minkowski space endowed with the metric

$$-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$$

and the bilinear form

$$[x, y] = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3.$$

We define $\mathbb{H}^3$ as the sub-manifold

$$\{x \in \mathbb{R}^{1+3} : [x, x] = 1\},$$

whose metric is induced from the Minkowski metric. We will also use the following polar coordinates for our computation. Pick $(1, 0, 0, 0)$ as the origin. For any point $x \in \mathbb{H}^3 \setminus \{0\}$, let r denote the distance from x to origin. We can express $\mathbb{H}^3$ by

$$\{x = (r \cosh \omega, r \sinh \omega) : r > 0, \omega \in \mathbb{S}^2\}.$$

The induced metric is

$$g_{\mathbb{H}^3} = dr^2 + \sinh^2 r \, d\omega^2,$$

where $d\omega^2$ is the metric on the sphere $\mathbb{S}^2$

The integral can be expressed by

$$\int_{\mathbb{H}^3} f \, dvol_{\mathbb{H}^3} = \int_0^\infty \int_{\mathbb{S}^2} f(r, \omega) \sinh^2 r \, dr d\omega$$

2.2. Analysis on $\mathbb{H}^d$

Now, we define the L^p and Sobolev spaces on $\mathbb{H}^3$. Given a smooth function, we define the L^p norm for $1 \leq p < \infty$ in the usual sense:

$$\|f\|_{L_x^p(\mathbb{H}^3)} = \left(\int_{\mathbb{H}^3} |f|^p \, dvol_{\mathbb{H}^3}\right)^{\frac{1}{p}}$$

and

$$\|f\|_{L_x^\infty(\mathbb{H}^3)} = \sup_{x \in \mathbb{H}^3} |f(x)|.$$

The Sobolev spaces $W^{k,p}(\mathbb{H}^3)$ are defined by the following norms as usual

$$\|f\|_{W^{k,p}(\mathbb{H}^3)} = \sum_{0 \leq l \leq k} \|\nabla^l f\|_{L^p(\mathbb{H}^3)}$$

Alternatively, we may define Sobolev spaces using the spectrum of Laplacian. In polar coordinates, we can write $\Delta_{\mathbb{H}^3}$ as

$$\Delta_{\mathbb{H}^3} = \partial_r^2 + \frac{2\cosh r}{\sinh r} \partial_r + \frac{1}{\sinh^2 r} \Delta_{\mathbb{S}^2}$$

For any $s \in \mathbb{R}$, we can define the fractional Sobolev norm

$$\|f\|_{\tilde{W}^{s,p}(\mathbb{H}^3)} = \|(-\Delta_{\mathbb{H}^3})^{\frac{s}{2}} f\|_{L^p(\mathbb{H}^3)}$$

and define the corresponding space as the completion of $C_0^\infty(\mathbb{H}^3)$ under the norm.

Lemma 2.1 (*Boundedness of Riesz Transform, See [27]*). *Suppose $k \in \mathbb{N}$, the two Sobolev norm defined above is equivalent. That is, for $f \in C_0^\infty(\mathbb{H}^3)$,*

$$\|f\|_{W^{k,p}(\mathbb{H}^3)} \sim \|f\|_{\tilde{W}^{k,p}(\mathbb{H}^3)}$$

Remark 2.2. The spectrum of the Laplacian on $\mathbb{H}^d$ has a spectral gap of $\rho = \frac{(d-1)^2}{4}$, rather than 0 which is the case on $\mathbb{R}^d$. See for example [28].

Thus, we have the following Poincaré inequalities:

Lemma 2.3 (*Poincaré Inequalities*). *For $f \in C_0^\infty(\mathbb{H}^3)$ and $-\infty < s_1 < s_2 < \infty$,*

$$\|f\|_{L^2} \le \frac{1}{\rho}\|\nabla f\|_{L^2},$$
$$\|(-\Delta)^{\frac{s_1}{2}} f\|_{L^2} \le \frac{1}{\rho^{s_2-s_1}}\|(-\Delta)^{\frac{s_2}{2}} f\|_{L^2}.$$

Next we include more useful inequalities in the hyperbolic setting. The proofs of the following inequalities from [29] on $\mathbb{H}^2$ can be adapted to $\mathbb{H}^3$ with minor modifications.

Lemma 2.4 (*Product Rule*). *For $f,g \in C_0^\infty(\mathbb{H}^3)$, given $s>0$, $1 \le r,p,q,\tilde{p},\tilde{q} < \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} = \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}}$,*

$$\|fg\|_{W^{s,r}(\mathbb{H}^3)} \lesssim \|f\|_{W^{s,p}(\mathbb{H}^3)}\|g\|_{L^q(\mathbb{H}^3)} + \|f\|_{L^{\tilde{p}}(\mathbb{H}^3)}\|g\|_{W^{s,\tilde{q}}(\mathbb{H}^3)}.$$

Lemma 2.5 (*Sobolev Embedding Theorem*). *For any $1 \le p < q < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{s}{3}$, we have*

$$W^{s,p}(\mathbb{H}^3) \hookrightarrow L^q(\mathbb{H}^3).$$

Remark 2.6. For a proof, please refer to [30]

Lemma 2.7 (*Embedding into L^∞*). *For any $\epsilon > 0$, we also have*

$$W^{1+\epsilon,3}(\mathbb{H}^3) \hookrightarrow L^\infty(\mathbb{H}^3),$$

Remark 2.8. This lemma is a special case of Lemma 2.12 in [29] adjusted to three dimensions. Following the same procedure of Lemma 2.12 in [29], we have for any $\epsilon > 0$ and $\delta(\epsilon) > 0$ depending on ϵ,

$$\|f\|_{L_x^\infty(\mathbb{H}^3)} \lesssim \|\nabla f\|_{L_x^{3+\delta(\epsilon)}(\mathbb{H}^3)} \lesssim \|\nabla^{1+\epsilon} f\|_{L^3(\mathbb{H}^3)}.$$

Lemma 2.9 (*Gagliardo–Nirenberg Inequality on $\mathbb{H}^3$*). *For $f \in C_0^\infty(\mathbb{H}^3)$, given any $1 < p < \infty$, $p \le q \le \infty$ and $0 \le \theta \le 1$ such that $\frac{1}{q} = (1-\theta)(\frac{1}{p} - \frac{1}{3}) + \frac{\theta}{p}$, we have*

$$\|f\|_{L^q} \lesssim \|\nabla f\|_{L^p}^{1-\theta}\|f\|_{L^p}^{\theta}.$$

2.3. Strichartz estimates for Schrödinger equations on $\mathbb{H}^d$

In this subsection, we recall the Strichartz estimate for Schrödinger equations on $\mathbb{H}^d$ as in [13].

Let $2 \le q,r \le \infty$. We say a couple (q,r) is admissible if $(\frac{1}{q},\frac{1}{r})$ belongs to the area

$$T_d = \{(\frac{1}{q},\frac{1}{r}) \mid 0 < \frac{1}{q},\frac{1}{r} < \frac{1}{2}, \frac{2}{q} + \frac{d}{r} = \frac{d}{2}\} \cup \{(0,\frac{1}{2})\}.$$

Proposition 2.10 (*See [13,31]*). *Suppose u is the solution on time interval I to the Schrödinger equations*

$$iu_t + \Delta_{\mathbb{H}^d} u = F, \qquad u(0) = u_0 \tag{2.1}$$

then for admissible couples $(q,r),(\tilde{q}',\tilde{r}')$, we have

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{H}^d)} \lesssim_{q,r,\tilde{q},\tilde{r}} \|u_0\|_{L_x^2(\mathbb{H}^d)} + \|F\|_{L_x^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{H}^d)}.$$

Proposition 2.10 follows from the following dispersive estimates with a standard T-T* argument. In particular, we need the dispersive estimates below in our proof.

Proposition 2.11 (*Dispersive Estimates in [13]*). *Suppose u is the solution to the free Schrödinger equation on I, i.e. (2.1) with $F = 0$. Assume $2 < q \le \frac{2d+4}{d}$, $r = \frac{2dq}{dq-4}$, $p_1 \in \{q', r'\}$, $p_2 \in \{q, r\}$. Then*

$$\|u(t)\|_{L_x^{p_2}(\mathbb{H}^d)} \le C_q B(t)\|u_0\|_{L^{p_1}(\mathbb{H}^d)}$$

where $B(t)$ is

$$B(t) = \begin{cases} t^{-2/q} & \text{if } |t| \le 1, \\ t^{-1} & \text{if } |t| > 1. \end{cases}$$

After introducing the basic setting for NLS on hyperbolic spaces, we will discuss the known global results for (1.1) in the next section. (See [2] for more details.)

3. A quick overview for the global result: Ionescu–Pausader–Staffilani [2]

In this section, we overview the global result in Ionescu–Pausader–Staffilani [2] for the main equation. In Euclidean spaces, critical well-posedness problems are proved via the concentration–compactness/rigidity method, also known as the Kenig and Merle road map. On non-Euclidean manifolds, Ionescu–Pausader–Staffilani [2] were able to for the first time obtain a critical global result on the hyperbolic space $\mathbb{H}^3$, where the authors managed to transfer the already available energy-critical global existence results in Euclidean spaces into the corresponding $\mathbb{H}^3$ settings using the Kenig–Merle road map and an ad hoc profile decomposition technique. More precisely, they showed that (1.1) is globally well-posed and scatters in $H^1(\mathbb{H}^3)$ for initial data in the energy space. As a quick consequence, one also has the persistence of regularity.

Proposition 3.1 (*Theorem 1.1 in [2]*). *If $\phi \in H^1(\mathbb{H}^3)$, then there exists a unique global solution $u \in C(\mathbb{R} : H^1(\mathbb{H}^3))$ of the initial-value problem (1.1). In addition, the mapping $\phi \to u$ is a continuous mapping from $H^1(\mathbb{H}^3)$ to $C(\mathbb{R} : H^1(\mathbb{H}^3))$, and both of the mass and energy are conserved. Moreover, we have the bound*

$$\|u\|_{L^{10}_{t,x}(\mathbb{R}\times\mathbb{H}^3)} \leq C(\|\phi\|_{H^1(\mathbb{H}^3)}).$$

As a consequence, there exist unique $u_\pm \in H^1(\mathbb{H}^3)$ such that

$$\|u(t) - e^{it\Delta_{\mathbb{H}^3}} u_\pm\|_{H^1(\mathbb{H}^3)} = 0 \text{ as } t \to \pm\infty.$$

Furthermore, one also has the following persistence of regularity for (1.1) as follows. To better understand the following proposition, we will first introduce some relevant notations.

Let $I \times \mathbb{H}^3$ be a spacetime slab. We define the L^2 Strichartz norm $S^0(I \times \mathbb{H}^3)$ by

$$\|u\|_{S^0(I\times\mathbb{H}^3)} := \sup_{(q,r)\text{ admissible}} \|u\|_{L^q_t L^r_x(I\times\mathbb{H}^3)}$$

and for $k = 1, 2$ we also can define the H^k Strichartz norm $S^k(I \times \mathbb{H}^3)$ by

$$\|u\|_{S^k(I\times\mathbb{H}^3)} := \| \langle\nabla\rangle^k u\|_{S^0(I\times\mathbb{H}^3)}.$$

Then, one has

Proposition 3.2 (*Persistence of Regularity*). *Let $k > 1$, I be a compact time interval, and let u be a finite energy solution to (1.1) on $I \times \mathbb{H}^3$ obeying the bounds*

$$\|u\|_{L^{10}_{t,x}(I\times\mathbb{H}^3)} \leq M.$$

Then, if $t_0 \in I$ and $u(t_0) \in H^k(\mathbb{H}^3)$,

$$\|u\|_{S^k(I\times\mathbb{H}^3)} \leq C(M)\|u(t_0)\|_{H^k(\mathbb{H}^3)}, \tag{3.1}$$

Proof of Proposition 3.2. We first divide the time interval I into N subintervals $I_j := [T_j, T_{j+1}]$ such that $I = \cup_{j=1}^N I_j$ and on each I_j

$$\|u\|_{L^{10}_{t,x}(I_j\times\mathbb{H}^3)} \leq \delta, \tag{3.2}$$

where δ will be chosen later. We have on each I_j by the Strichartz estimates, Hölder inequality and (3.2),

$$\begin{aligned}\|u\|_{S^k(I_j\times\mathbb{H}^3)} &\leq \|u(T_j)\|_{H^k(\mathbb{H}^3)} + C\| \langle\nabla\rangle^k (|u|^4 u)\|_{L^{\frac{10}{7}}_{t,x}(I_j\times\mathbb{H}^3)} \\ &\leq \|u(T_j)\|_{H^k(\mathbb{H}^3)} + C\| \langle\nabla\rangle^k u\|_{S^0(I_j\times\mathbb{H}^3)}\|u\|^4_{L^{10}_{t,x}(I_j\times\mathbb{H}^3)} \\ &\leq \|u(T_j)\|_{H^k(\mathbb{H}^3)} + C\delta^4\|u\|_{S^k(I_j\times\mathbb{H}^3)},\end{aligned}$$

where the constant C might vary from line to line.

Choosing δ small enough (for example $C\delta^4 < 1/2$), we obtain the bound for every j

$$\|u\|_{S^k(I_j\times\mathbb{H}^3)} \leq 2\|u(T_j)\|_{H^k(\mathbb{H}^3)}. \tag{3.3}$$

Then the bound (3.1) follows by adding up the bounds (3.3) we have on each subinterval. We note that the constant $C(M)$ depends on the number of subintervals, which is M.[2] □

In particular, based on this proposition, we can assume in our article that there exists M_1 such that

$$\|u\|_{L^\infty_t H^1_x(I\times\mathbb{H}^3)} \leq M_1.$$

[2] In fact, more terms need to be considered when one applies fractional Leibniz rule in this proof and in the following sections. The new estimates can be done via the Sobolev and the Hölder which can be handled similarly, so we leave it for interested readers.

4. The proof of the main theorem

In this section, we give the proof for the main theorem (Theorem 1.1). We first give the proof for a weaker version of Theorem 1.1 (compared with Fan–Zhao [25], it is already an improvement; more precisely, we reduce the regularity requirement from H^3 to H^{2+}). Then we give the proof for the main theorem (i.e. the second improvement).

4.1. The first improvement

Proving Theorem 1.1 with H^{2+}-regularity. We note that the key idea of this improvement is the application of the Gagliardo–Nirenberg inequality.

We define

$$A(\tau) := \sup_{0\leq s\leq \tau} s^{\frac{3}{2}} \|u(s)\|_{L_x^\infty(\mathbb{H}^3)}.$$

Note that $A(\tau)$ is monotone increasing. We will prove that there exists some constants, depending on u_0, so that

$$A(\tau) \leq C_{u_0}, \quad \text{for any } \tau \geq 0.$$

Recall we have persistence of regularity, thus for any given l, one can find C_l so that

$$A(\tau) \leq C_l, \quad \text{for any } 0 \leq \tau \leq l,$$

and the solution is continuous in time in L^∞ since we are working on high regularity data.

Thus, as shown in [25], Theorem 1.1 follows from the following bootstrap lemma.

Lemma 4.1. *There exists a constant C_{u_0}, such that if one has $A(\tau) \leq C_{u_0}$, then for $\tau \geq 0$, one has $A(\tau) \leq \frac{C_{u_0}}{2}$.*

Proof of Lemma 4.1. It is worth noting that the selection of C_{u_0} is critical in this lemma, and we will see the reason for choosing C_{u_0} during the proof. For fixed τ, we only need to prove that for any $t \leq \tau$, one has

$$\|u(t)\|_{L_x^\infty(\mathbb{H}^3)} \leq \frac{C_{u_0}}{2} t^{-\frac{3}{2}}.$$

We recall here, by bootstrap assumption, we apply the following estimates in the proof

$$\|u(t)\|_{L_x^\infty(\mathbb{H}^3)} \leq C_{u_0} t^{-\frac{3}{2}}. \tag{4.1}$$

Observe, for any δ, we can choose L, so that for one has

$$\left(\int_{L/2}^{\infty} \|u(t)\|_{L_x^{10}(\mathbb{H}^3)}^{10} \, dt\right)^{\frac{1}{10}} \leq \delta.$$

We will fix two special δ, L in the proof, though the exact way choice of those two parameters will only be made clear later.

We will only study $t \geq T$, and estimate all $t \leq L$ directly via

$$\|u(t)\|_{L_x^\infty} \leq A(L) t^{-\frac{3}{2}}, \quad t \leq L.$$

Next, by Duhamel's Formula, one can write the nonlinear solution $u(t,x)$ as follows,

$$u(t,x) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} (|u|^4 u)(s) \, ds = u_l + u_{nl}.$$

It is clear that the dispersive estimate gives for some constant C_0,

$$\|u_l(t)\|_{L_x^\infty(\mathbb{H}^3)} \leq C_0 t^{-\frac{3}{2}} \|u_0\|_{L_x^1(\mathbb{H}^3)}.$$

Then, we split u_{nl} into

$$u_{nl} = F_1 + F_2 + F_3,$$

where

$$F_1(t) := -i \int_0^M e^{i(t-s)\Delta} (|u|^4 u)(s) \, ds,$$

$$F_2(t) := -i \int_M^{t-M} e^{i(t-s)\Delta} (|u|^4 u)(s) \, ds,$$

$$F_3(t) := -i \int_{t-M}^{t} e^{i(t-s)\Delta} (|u|^4 u)(s) \, ds.$$

We can estimate F_1 as

$$\begin{aligned}\|F_1(t)\|_{L_x^\infty(\mathbb{H}^3)} &\le \int_0^M \|e^{i(t-s)\Delta}|u|^4u(s)\|_{L_x^\infty(\mathbb{H}^3)}\,ds\\ &\lesssim M(t-M)^{-\frac{3}{2}}\sup_s\|u^5(s)\|_{L_x^1(\mathbb{H}^3)}\\ &\lesssim Mt^{-\frac{3}{2}}\sup_s\|u(s)\|^5_{H_x^1(\mathbb{H}^3)}\\ &\lesssim MM_1^5t^{-\frac{3}{2}}.\end{aligned}$$

We note that we will choose M satisfying $M<\frac{t}{2}$, then we can bound $(t-M)^{-\frac{3}{2}}$ by $t^{-\frac{3}{2}}$ (multiplying a constant), which has been used in the above estimates.

For F_2, using the pointwise estimate (1.2), the Hölder inequality, Sobolev inequality and (4.1), we obtain

$$\begin{aligned}\|F_2(t)\|_{L_x^\infty} &\le \int_M^{t-M}\|e^{i(t-s)\Delta}|u(s)|^4u(s)\|_{L_x^\infty(\mathbb{H}^3)}\,ds\\ &\le \int_M^{t-M}(t-s)^{-\frac{3}{2}}\|u(s)\|_{L_x^\infty(\mathbb{H}^3)}\|u(s)\|^4_{L_x^4(\mathbb{H}^3)}\,ds\\ &\le \int_M^{t-M}(t-s)^{-\frac{3}{2}}\|u(s)\|_{L_x^\infty(\mathbb{H}^3)}\|u(s)\|^2_{L_x^3(\mathbb{H}^3)}\|u(s)\|^2_{L_x^6(\mathbb{H}^3)}\,ds\\ &\le \int_M^{t-M}(t-s)^{-\frac{3}{2}}\|u(s)\|_{L_x^\infty(\mathbb{H}^3)}\|u(s)\|^4_{H_x^1(\mathbb{H}^3)}\,ds\\ &\le CC_{u_0}M_1^4\int_M^{t-M}(t-s)^{-\frac{3}{2}}s^{-\frac{3}{2}}\,ds\\ &\le CC_{u_0}M_1^4\int_M^{\frac{t}{2}}(t-s)^{-\frac{3}{2}}s^{-\frac{3}{2}}\,ds\\ &\quad +CC_{u_0}M_1^4\int_{\frac{t}{2}}^{t-M}(t-s)^{-\frac{3}{2}}s^{-\frac{3}{2}}\,ds\\ &\le CC_{u_0}M_1^4M^{-\frac{1}{2}}t^{-\frac{3}{2}}.\end{aligned}$$

Now, choosing M, so that

$$CC_{u_0}M_1^4M^{-\frac{1}{2}}t^{-\frac{3}{2}} \le \frac{1}{10}C_{u_0}t^{-\frac{3}{2}}.$$

Therefore, we can estimate F_2 as

$$\|F_2(t)\|_{L_x^\infty} \le \frac{1}{10}C_{u_0}t^{-\frac{3}{2}}.$$

In order to estimate F_3, we will first state the following Lemma,

Lemma 4.2. *Let f be an $H_x^{2+\epsilon}$ function in $\mathbb{H}^3$, with*

$$\|f\|_{L_x^2(\mathbb{H}^3)} \le a, \quad \|f\|_{H_x^{2+\epsilon}(\mathbb{H}^3)} \le b,$$

then one has

$$\|f\|_{L_x^\infty(\mathbb{H}^3)} \lesssim a^{\frac{1+2\epsilon}{4+2\epsilon}}b^{\frac{3}{4+2\epsilon}}.$$

Proof of Lemma 4.2. With the help of the Gagliardo–Nirenberg inequality, we have

$$\|f\|_{L_x^\infty(\mathbb{H}^3)} \lesssim \|f\|^{\frac{1+2\epsilon}{4+2\epsilon}}_{L_x^2(\mathbb{H}^3)}\cdot\||\nabla|^{2+\epsilon}f\|^{\frac{3}{4+2\epsilon}}_{L_x^2(\mathbb{H}^3)}. \quad \square$$

Following Lemma 4.2, we estimate the L_x^2-norm and $H_x^{2+\epsilon}$-norm of F_3. Note that $H_x^{2+\epsilon}$ is a Banach algebra under pointwise multiplication, and $e^{i(t-s)\Delta}$ is unitary in $H_x^{2+\epsilon}$, we directly estimate $\|F_3(t)\|_{H_x^{2+\epsilon}}$ as

$$\|F_3(t)\|_{H_x^{2+\epsilon}(\mathbb{H}^3)} \le MM_1^5.$$

For $\|F_3(t)\|_{L_x^2}$, we will use the fact $t-M \ge L/2$ and rely on the scattering decay assumption. Also note $t-M\sim t$ since $t\ge L\ge 100M$. Then, applying the Gagliardo–Nirenberg inequality, Hölder inequality and (4.1), We obtain

$$\begin{aligned}\Big\|\int_{t-M}^t e^{i(t-s)\Delta}|u|^4u\,ds\Big\|_{L_x^2(\mathbb{H}^3)} &\le \int_{t-M}^t \||u|^4u\|_{L_x^2(\mathbb{H}^3)}\,ds\\ &\le \int_{t-M}^t \|u\|^{\frac{4+2\epsilon}{1+2\epsilon}}_{L_x^\infty(\mathbb{H}^3)}\cdot\|u\|^{\frac{2+\epsilon}{2+4\epsilon}}_{L_x^2(\mathbb{H}^3)}\cdot\|u\|^{\frac{15\epsilon}{2+4\epsilon}}_{L_x^{10}(\mathbb{H}^3)}\,ds\end{aligned}$$

$$
\begin{aligned}
&\leq CM_1^{\frac{2+\epsilon}{2+4\epsilon}}(C_{u_0}t^{-\frac{3}{2}})^{\frac{4+2\epsilon}{1+2\epsilon}}\int_{t-M}^{t}\|u\|_{L_x^{10}(\mathbb{H}^3)}^{\frac{15\epsilon}{2+4\epsilon}}\,ds\\
&\leq CM_1^{\frac{2+\epsilon}{2+4\epsilon}}(C_{u_0}t^{-\frac{3}{2}})^{\frac{4+2\epsilon}{1+2\epsilon}}\cdot\|u\|_{L_t^{10}L_x^{10}([t-M,t]\times\mathbb{H}^3)}^{\frac{15\epsilon}{2+4\epsilon}}\cdot M^{\frac{4+5\epsilon}{4+8\epsilon}}\\
&\leq CM_1^{\frac{2+\epsilon}{2+4\epsilon}}M^{\frac{4+5\epsilon}{4+8\epsilon}}\delta^{\frac{15\epsilon}{2+4\epsilon}}(C_{u_0}t^{-\frac{3}{2}})^{\frac{4+2\epsilon}{1+2\epsilon}}.
\end{aligned}
$$

Thus, via Lemma 4.2, we derive

$$
\begin{aligned}
\Big\|\int_{t-M}^{t}e^{i(t-s)\Delta}|u|^4u\,ds\Big\|_{L_x^\infty(\mathbb{H}^3)} &\leq \Big(CM_1^{\frac{2+\epsilon}{2+4\epsilon}}M^{\frac{4+5\epsilon}{4+8\epsilon}}\delta^{\frac{15\epsilon}{2+4\epsilon}}(C_{u_0}t^{-\frac{3}{2}})^{\frac{4+2\epsilon}{1+2\epsilon}}\Big)^{\frac{1+2\epsilon}{4+2\epsilon}}(MM_1^5)^{\frac{3}{4+2\epsilon}}\\
&\leq C^{\frac{1+2\epsilon}{4+2\epsilon}}M_1^{\frac{32+\epsilon}{8+4\epsilon}}M^{\frac{16+5\epsilon}{16+8\epsilon}}\delta^{\frac{15\epsilon}{8+4\epsilon}}C_{u_0}t^{-\frac{3}{2}}.
\end{aligned}
$$

Thus, by choosing δ small enough, according to M, M_1, we can ensure

$$
\|F_3(t)\|_{L_x^\infty} \leq \frac{1}{10}C_{u_0}t^{-\frac{3}{2}}.
$$

We note that we choose L, depending on δ, so that the above estimate holds.

It should be mentioned that the choice of M, L does not depend on C_{u_0}. Indeed, we will choose C_{u_0} depending on M, L.

To summarize, for all $t \leq \tau$, assuming $A(\tau) \leq C_{u_0}$, we derive

- For $t \leq L$, one has

$$
u(t) \leq A(L)t^{-\frac{3}{2}}.
$$

- For $L \leq t \leq \tau$, one has,

$$
u(t) \leq \{C(\|u_0\|_{L_x^1} + MM_1^5) + \frac{1}{10}C_{u_0} + \frac{1}{10}C_{u_0}\}t^{-\frac{3}{2}}.
$$

Thus, if one chooses

$$
C_{u_0} := 10A(L) + C(\|u_0\|_{L_x^1} + MM_1^5),
$$

then the desired estimates follow. This ends the proof of the main theorem. □

4.2. The second improvement

(With H^{1+}-regularity.) We note that the key idea of this improvement is the application of the Sobolev inequality trick inspired from [32].

We will now focus our attention on proving the case of H^{1+}-regularity in this section. More precisely, the proof of Theorem 1.1 follows from the same strategy as we deal with the case of $H^{2+\epsilon}$-regularity (the previous subsection), but we will adopt a different trick to treat 'F_3-term' with the help of the Sobolev inequality.[3] Let us discuss the treatment of F_3 term explicitly and skip other same treatments.

Recall, using the standard persistence of regularity argument (see Section 3 in [3]), given M, M_1, there exists $\delta > 0$ (small enough and to be decided), such that if L is chosen, so that,

$$
\Big(\int_{L/2}^{\infty}\|u\|_{L_x^{18}(\mathbb{H}^3)}^{6+\epsilon}\,dt\Big)^{\frac{1}{6+\epsilon}} \leq \delta.
$$

Next, by the Sobolev inequality and Hölder inequality, we can estimate F_3 term as follows

$$
\begin{aligned}
\|F_3(t)\|_{L_x^\infty(\mathbb{H}^3)} &\lesssim \Big\|\int_{t-M}^{t}e^{i(t-s)\Delta}|\nabla|^{1+\epsilon}(|u|^4u)\,ds\Big\|_{L_x^3(\mathbb{H}^3)}\\
&\lesssim \int_{t-M}^{t}(t-s)^{-\frac{1}{2}}\||\nabla|^{1+\epsilon}(|u|^4u)\|_{L_x^{\frac{3}{2}}(\mathbb{H}^3)}\,ds\\
&\lesssim \|u\|_{L_x^\infty(\mathbb{H}^3)}\int_{t-M}^{t}(t-s)^{-\frac{1}{2}}\||\nabla|^{1+\epsilon}u\|_{L_x^2(\mathbb{H}^3)}\|u\|_{L_x^{18}(\mathbb{H}^3)}^3\,ds\\
&\lesssim \|u\|_{L_x^\infty(\mathbb{H}^3)}\|u\|_{H_x^{1+\epsilon}(\mathbb{H}^3)}\int_{t-M}^{t}(t-s)^{-\frac{1}{2}}\|u\|_{L_x^{18}(\mathbb{H}^3)}^3\,ds\\
&\lesssim \|u\|_{L_x^\infty(\mathbb{H}^3)}\|u\|_{H_x^{1+\epsilon}(\mathbb{H}^3)}\Big(\int_{t-M}^{t}(t-s)^{-\frac{1}{2}\times\frac{6+\epsilon}{3+\epsilon}}\,ds\Big)^{\frac{3+\epsilon}{6+\epsilon}}\Big(\int_{t-M}^{t}\|u\|_{L_x^{18}(\mathbb{H}^3)}^{3\times(2+\frac{\epsilon}{3})}\,ds\Big)^{\frac{1}{2+\frac{\epsilon}{3}}}\\
&\lesssim C_{u_0}t^{-\frac{3}{2}}M_1\Big(\int_{t-M}^{t}(t-s)^{-\frac{6+\epsilon}{6+2\epsilon}}\,ds\Big)^{\frac{3+\epsilon}{6+\epsilon}}\Big(\int_{t-M}^{t}\|u\|_{L_x^{18}(\mathbb{H}^3)}^{6+\epsilon}\,ds\Big)^{\frac{3}{6+\epsilon}}
\end{aligned}
$$

[3] Another small difference for this case is the treatment for very small time since the rgularity is now not enough to allow a Sobolev inequality to control $A(t)$ for small t. We refer to [32] for such a treatment (some modifications are required).

$$\lesssim C_{u_0} t^{-\frac{3}{2}} M_1 \|u\|^3_{L_t^{6+\epsilon}[t-M,t] L_x^{18}(\mathbb{H}^3)}$$
$$\lesssim M_1 \delta^3 C_{u_0} t^{-\frac{3}{2}}.$$

Thus, by choosing δ small enough, according to M, M_1, we can ensure

$$\|F_3(t)\|_{L_x^\infty} \le \frac{1}{10} C_{u_0} t^{-\frac{3}{2}},$$

as desired.

The other steps are standard as before, so we omit them. (See [25] for more details.)

5. An improved decay result for the fourth-order NLS case (Yu–Yue–Zhao [26])

In this section, we consider the cubic, defocusing fourth-order nonlinear Schrödinger equations (4NLS) initial value problem as follows,

$$\begin{cases} (i\partial_t + \Delta^2_{\mathbb{R}^d})u = \mu |u|^2 u, \\ u(0,x) = u_0 \in H^{2+\epsilon}(\mathbb{R}^d), \end{cases} \tag{5.1}$$

where $\mu = -1, 5 \le d \le 8$.

It is noteworthy that the technique we utilized in Section 4.2 is also applicable to the Euclidean case. More specifically, our technique has effectively improved the results of the 4NLS equation obtained recently in [26] with better almost critical regularity assumption.

We first recall known results on the cubic 4NLS model (5.1). For the linear solution to (5.1), the following decay estimates holds (see [1] for more details)

$$\|u(t,x)\|_{L_x^\infty(\mathbb{R}^d)} \lesssim t^{-\frac{d}{4}} \|u(t,x)\|_{L_x^1(\mathbb{R}^d)}.$$

The global well-posedness and scattering theory for (5.1) has been established in [33].

Proposition 5.1 (*Theorem 1.1 in [33]*). *Let $1 \le d \le 8$. For any $u_0 \in H^2(\mathbb{R}^d)$, there exists a global solution $u \in C(\mathbb{R}, H^2(\mathbb{R}^d))$ of (5.1) with initial datum $u(0) = u_0$. If $5 \le d \le d$, the global solution also scatters in $H^2(\mathbb{R}^d)$. That is, there exist $f^\pm \in H^2(\mathbb{R}^d)$ such that*

$$\|u(t,x) - e^{it\Delta^2} f^\pm\|_{H^2(\mathbb{R}^d)} = 0 \text{ as } t \to \pm\infty.$$

We say that a pair (q,r) is Schrödinger admissible, for short S-admissible, if $2 \le q, r \le \infty$, $(q,r,d) \ne (2,\infty,2)$, and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$. We define a pair (q,r) is biharmonic admissible, for short B-admissible, if $2 \le q, r \le \infty$, $(q,r,d) \ne (2,\infty,4)$, and $\frac{4}{q} + \frac{d}{r} = \frac{d}{2}$. One has the following Strichartz type estimates:

Proposition 5.2 (*Proposition 3.1 in [34]*). *Let $u \in C(I, H^{-4})$ be a solution of*

$$(i\partial_t + \Delta^2_{\mathbb{R}^d})u + h = 0,$$

and $u(0) = u_0$. Then, for any B-admissible pairs (q,r) and $(\tilde{q},\tilde{r})$,

$$\|u\|_{L^q(I,L^r)} \le C(\|u_0\|_{L^2} + \|h\|_{L^{\tilde{q}'}(I,L^{\tilde{r}'})}),$$

where C depends only on d, and $\tilde{q}'$ and $\tilde{r}'$ are the conjugate exponents of $\tilde{q}$ and $\tilde{r}$.

Our main result in this section can be stated as follows

Theorem 5.3. *Let u solves (5.1) with initial data u_0, which is in $L_x^1 \cap H_x^{2+\epsilon}(\mathbb{R}^d)$ for any $\epsilon > 0$. Then, there exists a constant C_{u_0} depending on u_0, such that for $t \ge 0$,*

$$\|u(t,x)\|_{L_x^\infty(\mathbb{R}^d)} \le C_{u_0} t^{-\frac{d}{4}}.$$

Remark 5.4. The global well-posedness and scattering theory for (5.1) has been established in [33]. (See also [34,35] and the references therein for related results.) As mentioned in the introduction, the purpose of this section is devoted to improving the nonlinear decay results obtained in [26] in the sense of lowering the regularity assumption for the initial data, i.e. showing Theorem 5.3. See [26] for more background and related discussions.

We note that the key idea of this improvement is the application of the Sobolev inequality trick inspired from [32].

We now give the sketch for the proof of Theorem 5.3 as follows. The proof follows from the same scheme used in [26] and we will only emphasize that the 'F_3-term' (where the high regularity is required) can be now treated in a different way.

We define

$$A(\tau) := \sup_{0 \le s \le \tau} s^{\frac{d}{4}} \|u(s)\|_{L_x^\infty(\mathbb{R}^d)}.$$

Note that $A(\tau)$ is monotone increasing. We will prove that there exists some constants, depending on u_0, so that

$$A(\tau) \le C_{u_0}, \quad \text{for any } \tau \ge 0.$$

Recall we have persistence of regularity, thus for any given l, one can fine c_l so that

$$A(\tau) \le C_l, \quad \text{for any } 0 \le \tau \le l,$$

and the solution is continuous in time in L^∞ since we are working on high regularity data.

Thus, Theorem 1.1 follows from the following bootstrap lemma.

Lemma 5.5. *There exists a constant C_{u_0}, such that if one has $A(\tau) \le C_{u_0}$, then for $\tau \ge 0$, one has $A(\tau) \le \frac{C_{u_0}}{2}$.*

Proof of Lemma 5.5. It is worth noting that the selection of C_{u_0} is critical in this lemma, and we will see the reasons for choosing C_{u_0} during the proof. For fixed τ, we only need to prove that for any $t \le \tau$, one has

$$\|u(t)\|_{L_x^\infty(\mathbb{R}^d)} \le \frac{C_{u_0}}{2} t^{-\frac{d}{4}}.$$

We recall here, by bootstrap assumption, we apply the following estimates in the proof

$$\|u(t)\|_{L_x^\infty(\mathbb{R}^d)} \le C_{u_0} t^{-\frac{d}{4}}. \tag{5.2}$$

Similar to the NLS case, using the standard persistence of regularity argument (see Section 2 in [26]), we have, for any δ, we can choose L, so that for one has

$$\left(\int_{L/2}^{\infty} \|u\|_{L_x^{\frac{2d}{d-4-\epsilon}}(\mathbb{R}^d)}^{\frac{8(d-4)-2\epsilon d+8\epsilon+2\epsilon^2}{(d-4)(8-d)+\epsilon^2}} ds\right)^{\frac{(d-4)(8-d)+\epsilon^2}{8(d-4)-2\epsilon d+8\epsilon+2\epsilon^2}} \le \delta.$$

We will fix two special δ, L in the proof, though the exact way choice of those two parameters will only be made clear later. We will only study $t \ge T$, and estimate all $t \le L$ directly via

$$\|u(t)\|_{L_x^\infty(\mathbb{R}^d)} \le A(L) t^{-\frac{d}{4}}, \quad t \le T.$$

Next, by Duhamel's Formula, one can write the nonlinear solution $u(t,x)$ as follows,

$$u(t,x) = e^{it\Delta^2} u_0 + i \int_0^t e^{i(t-s)\Delta^2} (|u|^2 u)(s)\, ds = u_l + u_{nl}.$$

It is clear that the dispersive estimate gives for some constant C_0,

$$\|u_l(t)\|_{L_x^\infty(\mathbb{R}^d)} \le C_0 t^{-\frac{d}{4}} \|u_0\|_{L_x^1(\mathbb{R}^d)}.$$

Then, we split u_{nl} into

$$u_{nl} = F_1 + F_2 + F_3$$

where

$$F_1(t) = i \int_0^M e^{i(t-s)\Delta^2} (|u|^2 u)(s)\, ds,$$
$$F_2(t) = i \int_M^{t-M} e^{i(t-s)\Delta^2} (|u|^2 u)(s)\, ds,$$
$$F_3(t) = i \int_{t-M}^t e^{i(t-s)\Delta^2} (|u|^2 u)(s)\, ds.$$

We will estimate F_1 as

$$\begin{aligned} \|F_1(t)\|_{L_x^\infty(\mathbb{R}^d)} &\le \int_0^M \|e^{i(t-s)\Delta^2} |u|^2 u(s)\|_{L_x^\infty(\mathbb{R}^d)}\, ds \\ &\lesssim M(t-M)^{-\frac{d}{4}} \sup_s \|u^3(s)\|_{L_x^1(\mathbb{R}^d)} \\ &\lesssim M t^{-\frac{d}{4}} \sup_s \|u(s)\|_{H_x^1(\mathbb{R}^d)}^3 \\ &\lesssim M M_1^3 t^{-\frac{d}{4}}. \end{aligned}$$

We note that we will choose M satisfying $M < \frac{t}{2}$, then we can bound $(t-M)^{-\frac{d}{4}}$ by $t^{-\frac{d}{4}}$ (multiplying a constant), which has been used in the above estimates.

For F_2, by (5.2), and pointwise estimate, we obtain

$$\|e^{i(t-s)\Delta^2} |u(s)|^2 u(s)\|_{L_x^\infty(\mathbb{R}^d)} \lesssim (t-s)^{-\frac{d}{4}} \|u(s)\|_{H_x^1(\mathbb{R}^d)}^2 \|u(s)\|_{L_x^\infty(\mathbb{R}^d)} \lesssim C_{u_0} M_1^2 (t-s)^{-\frac{d}{4}} s^{-\frac{d}{4}}.$$

And one estimate F_2 as follows

$$
\begin{aligned}
\|F_2(t)\|_{L_x^\infty} &\le CC_{u_0}M_1^2\int_M^{t-M}(t-s)^{-\frac{d}{4}}s^{-\frac{d}{4}}\,ds\\
&\le CC_{u_0}M_1^2\int_M^{\frac{t}{2}}(t-s)^{-\frac{d}{4}}s^{-\frac{d}{4}}\,ds\\
&\quad + CC_{u_0}M_1^2\int_{\frac{t}{2}}^{t-M}(t-s)^{-\frac{d}{4}}s^{-\frac{d}{4}}\,ds\\
&\le 2CC_{u_0}M_1^2t^{-\frac{d}{4}}\int_M^{\frac{t}{2}}s^{-\frac{d}{4}}\,ds\\
&\quad + 2CC_{u_0}M_1^2t^{-\frac{d}{4}}\int_{\frac{t}{2}}^{t-M}(t-s)^{-\frac{d}{4}}\,ds\\
&\le \frac{1}{10}C_{u_0}t^{-\frac{d}{4}}.
\end{aligned}
$$

In order to estimate F_3, we will use the Sobolev inequality and Hölder inequality to obtain

$$
\begin{aligned}
\|F_3(t)\|_{L_x^\infty(\mathbb{R}^d)} &\lesssim \Big\|\int_{t-M}^{t} e^{i(t-s)\Delta^2}|\nabla|^{2+\epsilon}(|u|^2u)\,ds\Big\|_{L_x^{\frac{d}{2+\frac{\epsilon}{2}}}(\mathbb{R}^d)}\\
&\lesssim \int_{t-M}^{t}(t-s)^{-\frac{d}{4}(1-\frac{4+\epsilon}{d})}\||\nabla|^{2+\epsilon}(|u|^2u)\|_{L_x^{\frac{d}{d-2-\frac{\epsilon}{2}}}(\mathbb{R}^d)}\,ds\\
&\lesssim \|u\|_{L_x^\infty(\mathbb{R}^d)}\int_{t-M}^{t}(t-s)^{-\frac{d-4-\epsilon}{4}}\||\nabla|^{2+\epsilon}u\|_{L_x^2(\mathbb{R}^d)}\|u\|_{L_x^{\frac{2d}{d-4-\epsilon}}(\mathbb{R}^d)}\,ds\\
&\lesssim \|u\|_{L_x^\infty(\mathbb{R}^d)}\|u\|_{H_x^{2+\epsilon}(\mathbb{R}^d)}\int_{t-M}^{t}(t-s)^{-\frac{d-4-\epsilon}{4}}\|u\|_{L_x^{\frac{2d}{d-4-\epsilon}}(\mathbb{R}^d)}\,ds\\
&\lesssim \|u\|_{L_x^\infty(\mathbb{R}^d)}\|u\|_{H_x^{2+\epsilon}(\mathbb{R}^d)}\Big(\int_{t-M}^{t}(t-s)^{-\frac{d-4-\epsilon}{4}\times(\frac{4}{d-4-\epsilon}-\frac{\epsilon}{d-4})}\,ds\Big)^{\frac{(d-4)^2-d\epsilon+4\epsilon}{4(d-4)-\epsilon d+4\epsilon+\epsilon^2}}\\
&\quad\times\Big(\int_{t-M}^{t}\|u\|_{L_x^{\frac{2d}{d-4-\epsilon}}(\mathbb{R}^d)}^{\frac{8(d-4)-2\epsilon d+8\epsilon+2\epsilon^2}{(d-4)(8-d)+\epsilon^2}}\,ds\Big)^{\frac{(d-4)(8-d)+\epsilon^2}{8(d-4)-2\epsilon d+8\epsilon+2\epsilon^2}}M^{\frac{(d-4)(8-d)+\epsilon^2}{8(d-4)-2\epsilon d+8\epsilon+2\epsilon^2}}\\
&\lesssim M_1M^{\frac{(d-4)(8-d)+\epsilon^2}{8(d-4)-2\epsilon d+8\epsilon+2\epsilon^2}}\delta C_{u_0}t^{-\frac{d}{4}}.
\end{aligned}
$$

Thus, by choosing δ small enough, according to M_1, we can ensure

$$\|F_3(t)\|_{L_x^\infty}\le\frac{1}{10}C_{u_0}t^{-\frac{d}{4}},$$

as desired.

It should be mentioned that the choices of M, L do not depend on C_{u_0}. Indeed, we will choose C_{u_0} depending on M, L.

To summarize, for all $t\le\tau$, assuming $A(\tau)\le C_{u_0}$, we derive

- For $t\le L$, one has

$$u(t)\le A(L)t^{-\frac{d}{4}}.$$

- For $L\le t\le\tau$, one has,

$$u(t)\le\{C(\|u_0\|_{L_x^1}+MM_1^3)+\frac{1}{10}C_{u_0}+\frac{1}{10}C_{u_0}\}t^{-\frac{d}{4}}.$$

Thus, if one choose

$$C_{u_0}:=10A(L)+C(\|u_0\|_{L_x^1}+MM_1^3),$$

then the desired estimates follow. This ends the proof of Lemma 5.5. Therefore, we finish the proof of Theorem 5.3. □

Data availability

No data was used for the research described in the article.

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Appendix

In the appendix, we make a few quick comments on the research line of ‘the nonlinear decaying property for nonlinear dispersive equations’.

1. *On the sharp decaying result.* We note that for the proof of the main theorem, initial data that is a little more regular than the energy level is required, say $H^{1+\epsilon}$. A natural question is: is it possible to obtain the ‘sharp decaying result’ in the sense of lying data in the energy space? (i.e. removing the ϵ) That would be an interesting and challenging problem. It seems that other ingredients are required. The same question can be also considered for the 4NLS case, i.e. proving the decaying result in H^2 space (energy space). See Section 5.

2. *Other dispersive models.* We note that one may study the nonlinear decaying problems for other dispersive models rather than NLS. (or study NLS on other spaces rather than Euclidean spaces, like in this paper.) We refer to the appendix of [32] for more discussions.

3. *On the pointwise decay estimate.* We note that, compared with (1.2) (the L_x^∞-type in-time-decay), the pointwise decay estimate in the hyperbolic space setting can be stated for both space and time. See [31] (Proposition 3.1 and Remark 3.2) for more details. One may consider how to recover Proposition 3.1 and Remark 3.2 in [31] for nonlinear solutions.

4. *Other models on hyperbolic spaces.* In this paper, we mainly study the quintic NLS on 3D hyperbolic space, i.e. (1.1). One may also consider other NLS models (or even nonlinear dispersive models) on hyperbolic spaces. Analogous nonlinear decaying results are expected via the same method with suitable modifications.

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