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Max cut and semidefinite rank

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ABSTRACT

This paper considers the relationship between semidefinite programs (SDPs), matrix rank, and maximum cuts of graphs. Utilizing complementary slackness conditions for SDPs, we investigate when the rank 1 feasible solution corresponding to a max cut is the unique optimal solution to the Goemans-Williamson max cut SDP by showing the existence of an optimal dual solution with rank $n-1$. Our results consider connected bipartite graphs and graphs with multiple max cuts. We conclude with a conjecture for general graphs.

1. Introduction

Given an undirected graph $G=(V,E)$ and positive weights w_{ij} for every $(i,j)\in E$, a *cut* of G is a partitioning of V into A and B, denoted (A,B). If $M\subset E$ is the set of edges with exactly one endpoint each in A and B, then the *value* of the cut (A,B) is $\sum_{(i,j)\in M} w_{ij}$; the *max cut* is the cut with maximum value. The problem of finding a max cut has been extensively studied. It appeared on Karp's initial list of NP-complete problems [6], and there is a wealth of literature researching approximation algorithms and heuristics to find good cuts [8,5,3,4,9].

This paper considers the use of semidefinite programming for max cut. Goemans and Williamson [5] initially demonstrated the connection by introducing a semidefinite programming relaxation of the max cut problem producing a .878-approximation algorithm. The relaxation is given by the following vector program, which can be solved via semidefinite programming:

$$\begin{array}{lll} \text{maximize} & \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1-v_i\cdot v_j) & \\ \text{subject to} & v_i\cdot v_i=1, & \forall i\in V, \\ & v_i\in\mathbb{R}^n, & \forall i\in V. \end{array} \qquad \text{(MC-P)}$$

For any cut (A,B), there is a feasible solution to the vector program with objective value equal to the value of (A,B): let $u_1\in\mathbb{R}^n$ be any unit vector and $u_2=-u_1$ so that $u_1\cdot u_2=-1$. Then let $v_i=u_1$ if $i\in A$ and $v_i=u_2$ if $i\in B$ for all $i\in V$. Note that this solution is 1-dimensional, and the best 1-dimensional solution to the vector program corresponds to the max cut. We will call the feasible solution where there are two unit vectors $u_1,u_2\in\mathbb{R}^n$ with $u_1=-u_2$ and each v_i equal to u_1 or u_2 the *reference solution.*

In this paper, we consider situations when the reference solution is the unique optimal primal solution to the (MC-P). If the reference solution is the unique optimal primal solution, then we can obtain an optimal cut from the SDP instead of an approximation. To identify these situations, we utilize complementary slackness conditions for semidefinite programs. Consider the following general primal and dual SDPs (in what follows we assume all matrices are symmetric):

$$(P)\quad \begin{array}{lll} \text{maximize} & C\bullet X & \\ \text{subject to} & A_i\bullet X=b_i & \text{for } i=1,\dots,m, \\ & X\succeq 0, & \\ & X\in\mathbb{R}^{\ell\times\ell}, & \end{array}$$

$$(D)\quad \begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & S=\sum_{i=1}^m y_i A_i - C, \\ & S\succeq 0, \\ & S\in\mathbb{R}^{\ell\times\ell}. \end{array}$$

Here, $X\succeq 0$ represents the constraint that X must be a positive semidefinite matrix and $C\bullet X$ denotes the outer product of matrices given by $\sum_{i=1}^{\ell}\sum_{j=1}^{\ell} c_{ij}x_{ij}$. For any feasible primal solution X and feasible dual solution y, duality theory of SDPs shows that $C\bullet X\geq b^T y$. Furthermore, if $C\bullet X=b^T y$, then X and y are optimal primal and dual solutions, respectively, and duality theory shows that $XS=0$ and $\text{rank}(X)+\text{rank}(S)\leq\ell$. Therefore, if we want to show that any optimal primal solution has rank at most 1, it suffices to show the existence of an optimal dual solution with rank at least $\ell-1$. Given an instance of max cut on a graph $G=(V,E)$, let $W\in\mathbb{R}^{n\times n}$ be the matrix of edge weights, $W_{tot}=\sum_{(i,j)\in E} w_{ij}$, and $C\in\mathbb{R}^{n\times n}$ such that $C_{ij}=\sum_k w_{ik}$ if $i=j$ and $-w_{ij}$ otherwise.

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Then we can consider the semidefinite program in standard form that is equivalent to (MC-P):

$$\begin{array}{ll} \text{maximize} & \frac{1}{4} C \bullet X \\ \text{subject to} & E_i \bullet X = 1, \quad i = 1, \dots, n, \\ & X \succeq 0, \\ & X \in \mathbb{R}^{n \times n}, \end{array} \qquad \text{(MC-P-SDP)}$$

where $E_i \in \mathbb{R}^{n\times n}$ has a 1 in the ith spot on the diagonal and 0 everywhere else.

Then the corresponding dual program to (MC-P-SDP) in standard form is given by

$$\begin{array}{ll} \text{minimize} & \vec{1}^T y \\ \text{subject to} & S = \sum_{i=1}^n y_i E_i - \frac{1}{4} C, \\ & S \succeq 0, \\ & S \in \mathbb{R}^{n \times n}, \end{array} \qquad \text{(MC-D-SDP)}$$

which is equivalent to the following program:

$$\begin{array}{ll} \text{minimize} & \frac{1}{2} W_{tot} + \frac{1}{4} \sum_{i=1}^n \gamma_i \\ \text{subject to} & W + diag(\gamma) \succeq 0. \end{array} \qquad \text{(MC-D)}$$

Observe that (MC-D-SDP) and (MC-D) are equivalent since for any solution γ to (MC-D) or y to (MC-D-SDP) we can define an equivalent solution given by the relationship $y_i = \frac{1}{4}(C_{ii} - \gamma_i)$ where the objective values of (MC-D) and (MC-D-SDP) are equal at γ and y, respectively, and the slack matrices are the same except for a scaling of $\frac{1}{4}$.

By the previous discussion, if we can find a feasible dual solution for our max cut instance that has objective value equal to the value of the max cut of G and with rank $n-1$, then the complementary slackness conditions for SDPs tell us any optimal primal solution has rank at most 1; in this case, we will show that the solution must be the reference solution. We will refer to a dual solution as a slack matrix S and optimal dual solutions with rank $n-1$ as having *sufficiently high rank*.

Similar analysis and techniques were previously introduced for graph coloring by Mirka, Smedira, and Williamson [7]. In the setting of graph colorings, it was known that the corresponding reference solution is always an optimal primal solution for a k-colorable graph with an induced k-clique, but it was unknown whether it was the *unique* optimal solution and thus would always be the returned solution. In the max cut setting, the reference solution is always a feasible primal solution, but it may not be optimal. In this work, we explore both when the reference solution is an optimal primal solution and when it is uniquely so.

The rest of the paper is organized as follows. Section 2 presents some preliminary facts about semidefinite matrices and semidefinite programs that will be used in subsequent sections. Section 3 describes the max cut semidefinite program and its dual, as well as a partial characterization of graphs for which the reference solution corresponding to a max cut is the unique optimal solution. In particular, we show that dual solutions of rank $n-1$ exist for connected bipartite graphs but not for graphs which do not have unique max cuts. In Section 4, we consider more general graphs and present a conjecture on when the reference solution is the unique optimal primal solution. Finally, Section 5 concludes with a few further thoughts and open questions.

2. Preliminaries

In this section, we recall some basic facts about semidefinite matrices and semidefinite programs that we will use in subsequent sections. Recall the general primal and dual semidefinite programs (P) and (D) given in the introduction.

We always have weak duality for semidefinite programs, so that the following holds.

Fact 1. *Given any feasible X for (P) and y for (D), $C \bullet X \geq b^T y$.*

Thus if we can produce a feasible X for (P) and a feasible y for (D) such that $C \bullet X = b^T y$, then X must be optimal for (P) and y optimal for (D).

The following is also known, and is the semidefinite programming version of complementary slackness conditions for linear programming.

Fact 2. *[1, Theorem 2.10, Corollary 2.11] For optimal X for (P) and y for (D), $XS = 0$ and $rank(X) + rank(S) \leq \ell$.*

Semidefinite programs and vector programs are equivalent because a symmetric $X \in \mathbb{R}^{n\times n}$ is positive semidefinite (PSD) if and only if $X = QDQ^T$ for a real matrix $Q \in \mathbb{R}^{n\times n}$ and diagonal matrix D in which the entries of D are the eigenvalues of X, and the eigenvalues are all nonnegative. We can then consider $D^{1/2}$, the diagonal matrix in which each diagonal entry is the square root of the corresponding entry of D. Then $X = (QD^{1/2})(QD^{1/2})^T$. If we let $v_i \in \mathbb{R}^n$ be the ith row of $QD^{1/2}$, then $x_{ij} = v_i \cdot v_j$, and similarly, given the vectors v_i, we can construct a semidefinite matrix X with $x_{ij} = v_i \cdot v_j$. We also make the following observation based on this decomposition.

Observation 3. *Given a semidefinite matrix $X = QDQ^T \in \mathbb{R}^{n\times n}$, $rank(X) = d$ if and only if the vectors $v_i \in \mathbb{R}^n$ with v_i the ith row of $QD^{1/2}$ are supported on just d coordinates.*

Throughout the rest of the paper, we may refer to semidefinite programs and vector programs interchangeably due to this equivalence.

In the introduction, we defined the reference solution to be the rank 1 feasible solution to (MC-P) with two unit vectors $u_1, u_2 \in \mathbb{R}^n$ such that $u_1 = -u_2$ and v_i is equal to u_1 or u_2 for each $i \in V$. By Observation 3, the positive semidefinite matrix $X = WW^T$ (with v_i the ith row of W) must also have rank 1 and each entry of X is either 1 or -1.

We will also say that a positive semidefinite matrix X is the reference solution if it has rank 1 and there is some W such that $X = WW^T$ and W has exactly two distinct rows $u_1, u_2 \in \mathbb{R}^n$ with $u_1 = -u_2$.

We also observe that, in fact, any rank 1 feasible solution to (MC-P) must be the reference solution. Let u_1 be a unit vector from a rank 1 feasible solution to (MC-P). For any vertex i with $v_i \neq u_1$, it must be the case that $v_i = -u_1$ since $-u_1$ is the only other unit vector in the span of u_1. If $u_2 = -u_1$, then $u_1 \cdot u_2 = u_1 \cdot (-u_1) = -1$, and this is a reference solution as claimed.

3. Max cut semidefinite program

Recall the max cut SDP (MC-P) given in the introduction. In this section, we give a partial characterization of graphs for which the reference solution corresponding to a max cut is the unique optimal primal solution to (MC-P). We do so by investigating the rank of the optimal solutions to (MC-D).

We will refer to the primal and dual solutions as $X = (v_i \cdot v_j)_{i,j\in V}$ and $S = W + diag(\gamma)$, respectively. Our first result shows the reference solution is the unique optimal primal solution for connected bipartite graphs. Note that the value of a max cut in a bipartite graph is always W_{tot}: if $V = A \cup B$ with $A \cap B = \emptyset$ and all edges have exactly one endpoint in each A and B, then all edges are contained in the cut given by (A, B). Thus, we can prove the claim by constructing a feasible dual matrix with objective value W_{tot} and rank $n-1$.

Theorem 4. *If G is a connected bipartite graph with n vertices, there exists an optimal dual solution S to (MC-D) with rank $n-1$.*

Proof. Let $V = A \cup B$ with $A \cap B = \emptyset$ so that A and B are the two parts of G. For $v \in V$, let $\gamma_v = \sum_{i:(i,v)\in E} w_{iv}$. Then the matrix $S = W + diag(\gamma)$ is PSD: for any $x \in \mathbb{R}^n$,

$$\begin{aligned} x^T Sx &= \sum_{1\le i\le n} \gamma_i x_i^2 + \sum_{(i,j)\in E} 2w_{ij}x_ix_j \\ &= \sum_{1\le i\le n} \left(\sum_{j:(i,j)\in E} w_{ij} \right) x_i^2 + \sum_{(i,j)\in E} 2w_{ij}x_ix_j \\ &= \sum_{(i,j)\in E} w_{ij}(x_i + x_j)^2 \ge 0. \end{aligned}$$

Since S is PSD, it is a feasible dual slack matrix. Dual optimality of S is given by the objective value:

$$\begin{aligned} \frac{1}{2}W_{tot} + \frac{1}{4}\sum_{v\in V}\gamma_v &= \frac{1}{2}W_{tot} + \frac{1}{4}\sum_{v\in V}\sum_{i:(i,v)\in E} w_{iv} \\ &= \frac{1}{2}W_{tot} + \frac{1}{4}\sum_{(i,j)\in E} 2w_{ij} \\ &= \frac{1}{2}W_{tot} + \frac{1}{2}W_{tot} \\ &= W_{tot}, \end{aligned}$$

which is also the value of the max cut since G is bipartite.

It remains to show $rank(S) = n-1$. For a vector $x \in ker(S)$, we have $x^T Sx = \sum_{(i,j)\in E} w_{ij}(x_i + x_j)^2 = 0$. Then since all edge-weights are positive, it must be the case that $x_i + x_j = 0$ for every edge $(i, j) \in E$. Let $y \in \mathbb{R}^n$ such that $y_v = 1$ if $v \in A$ and $y_v = -1$ if $v \in B$. Since G is bipartite, every edge is between some vertex in A and some vertex in B, so $(y_i + y_j)^2 = 0$ for every edge. Therefore, $y \in ker(S)$. The connectivity of G guarantees that scalings of y are the only vectors in the $ker(S)$. Assume this is not the case, and there is a vector $y' \in ker(S)$ such that there exist vertices $a, a' \in A$ where $y'_a \ne y'_{a'}$. Because the graph is connected, there must be a path P from a to a' where vertices in the path alternate being in A and B. Let $a* \in A$ be the first vertex in A on the path such that $y'_a \ne y'_{a*}$ and $a^\dagger \in A$ the last vertex in A on the path before a^*. Then $a^\dagger$ and $a*$ must share a neighbor $b \in B$ on P, and therefore it must be the case that $(y'_{a^\dagger} + y'_b)^2 = 0$ and $(y'_{a*} + y'_b)^2 = 0$. This means $y'_{a^\dagger} = y'_{a*}$ which contradicts our selection of $a^\dagger$ and a^*. Since scalings of y are the only vectors in $ker(S)$, $rank(S) = n-1$. □

The connectivity of a bipartite graph G plays a subtle but important role in Theorem 4; without connectivity, the max cut of G would not have been unique, and the kernel of the constructed matrix S would be higher-dimensional. As the next result illustrates, it turns out the uniqueness of a max cut is a necessary condition for the existence of an optimal solution to (MC-D) with sufficiently high rank. We show this by finding a feasible solution to (MC-P) with rank greater than 1 but with objective value equal to the objective value attained by the reference solution. The key insight behind the construction is that the objective function of (MC-P) is affine, so any convex combination of optimal solutions gives an optimal solution. However, taking convex combinations does not necessarily preserve rank, so some of these optimal solutions will have rank greater than 1. Because there must be an optimal solution to (MC-P) with rank greater than 1, there cannot be an optimal dual solution with rank $n-1$.

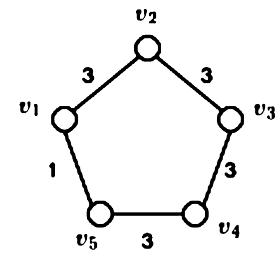

Fig. 1. A graph with a unique max cut and optimal solution to (MC-P) with rank greater than 1.

Theorem 5. *Let G be a graph with two distinct max cuts. Then there is an optimal solution to (MC-P) with rank greater than 1.*

Proof. Let (A_1, B_1) and (A_2, B_2) be two distinct max cuts. Consider the rank 1 solutions f and g associated with each cut given by $f(v) = 1$ if $v \in A_1$ and $f(v) = -1$ if $v \in B_1$ and $g(v) = 1$ if $v \in A_2$ and $g(v) = -1$ if $v \in B_2$. Let F and G be the corresponding PSD matrices where $F_{ij} = f(i)f(j)$ and $G_{ij} = g(i)g(j)$. Observe for any $\alpha \in (0, 1)$, the convex combination of F and G given by $H = \alpha F + (1-\alpha)G$ is PSD and a feasible solution to (MC-P) since each diagonal entry of H is equal to 1.

Now we consider the objective values. Let

$$C = \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - F_{ij}) = \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - G_{ij})$$

where the second equality follows since F and G both correspond to optimal cuts and thus produce the same objective value. Since the objective function is affine, the objective value for H is also C:

$$\begin{aligned} \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - H_{ij}) &= \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(1 - \alpha F_{ij} - (1-\alpha)G_{ij}) \\ &= \alpha C + (1-\alpha)C = C. \end{aligned}$$

Now that we have shown H attains the same objective value as the max cut, we must determine the rank of H. Let $y \in (A_1 \cap A_2) \cup (B_1 \cap B_2)$ so that $f(y) = g(y)$. Note that because the cuts are distinct, such a y must exist. If not, then it must be the case that $A_1 = B_2$ and $B_1 = A_2$, contradicting the distinctness of the cuts. Similarly, there must exist a $v \in (A_1 \cap B_2) \cup (A_2 \cap B_1)$ which means $f(v) = -g(v)$. Together, this implies $f(y)f(v) = -g(y)g(v)$. Consider $\alpha = \frac{1}{2}$, so that $H_{yv} = 0$. Since H is PSD, $H = V^T V$ for some V, and $\text{rank}(H) = \text{rank}(V)$. If $\text{rank}(V) = 1$, then $H_{ij} = \pm 1$ for $1 \le i, j \le n$. Therefore $H_{yv} = 0$ implies $1 < \text{rank}(V) = \text{rank}(H)$. Since H is a feasible solution to (MC-P) with rank greater than and obtaining the same objective value as F and G (the optimal rank 1 solutions), the claim is complete. □

Theorem 5 tells us that only graphs with unique max cuts can have the corresponding reference solution as the unique optimal solution. Unfortunately, uniqueness of a max cut is not sufficient to guarantee uniqueness of the reference solution as the optimal primal solution (or existence of a rank $(n-1)$ optimal dual solution) in general as Fig. 1 illustrates: the graph has a unique max cut with value 12 given by $(\{v_1, v_3, v_5\}, \{v_2, v_4\})$ and a rank 2 primal solution with objective value ~ 12.05.

4. General graphs

In Section 3, we showed that the uniqueness of a max cut for a graph is a necessary but insufficient condition to guarantee the reference solution is the unique optimal solution to (MC-P). We also proved that bipartiteness is sufficient in connected graphs, though it turns out it is not necessary. The following theorem provides an equivalent condition for the reference solution to be an optimal primal solution.

Theorem 6. *Let $G = (V, E)$ be a graph with a unique max cut, W its weight matrix, and $M \subset E$ the set of edges in its max cut. For $v \in V$, let $\gamma_v = \sum_{(v,j)\in M} w_{vj} - \sum_{(v,j)\in E\setminus M} w_{vj}$. Then the rank 1 reference solution corresponding to the max cut is an optimal primal solution if and only if $S = W + diag(\gamma)$ is PSD.*

Proof. We begin by showing optimality of the reference solution implies S is PSD. Let X be the primal solution corresponding to the reference solution; an off-diagonal entry $X_{ij} = 1$ if i and j are in the same side of the cut and -1 otherwise. Furthermore, since X is assumed to be optimal, by complementary slackness conditions we know that $XT = 0$ for any optimal dual solution T. In particular, $X_i \cdot T_i = 0$ for $i = 1, \dots n$, where X_i and T_i are the ith row of X and T, respectively. This means

$$0 = X_i \cdot T_i = \gamma_i + \sum_{(i,j)\in E\setminus M} w_{ij} - \sum_{(i,j)\in M} w_{ij}.$$

Rearranging shows that $\gamma_i = \sum_{(i,j)\in M} w_{ij} - \sum_{(i,j)\in E\setminus M} w_{ij}$ and T must be equal to S. Since T was assumed to be optimal for (MC-D), this implies S is optimal for (MC-D) and therefore is PSD as claimed.

On the other hand, if S is PSD, it is a feasible dual solution. Note the value of the max cut for G, and thus the reference solution, is $\sum_{e\in M} w_e$. The corresponding objective value for S is

$$\begin{aligned}
&\frac{1}{2}W_{tot} + \frac{1}{4}\sum_{v\in V}\gamma_v \\
&= \frac{1}{2}W_{tot} + \frac{1}{4}\sum_{v\in V}\left(\sum_{(v,j)\in M} w_{vj} - \sum_{(v,j)\in E\setminus M} w_{vj}\right) \\
&= \frac{1}{2}W_{tot} + \frac{1}{4}\left(2\sum_{e\in M} w_e - 2\sum_{e\in E\setminus M} w_e\right) \\
&= \frac{1}{2}W_{tot} + \frac{1}{4}\left(2W_{tot} - 4\sum_{e\in E\setminus M} w_e\right) \\
&= W_{tot} - \sum_{e\in E\setminus M} w_e = \sum_{e\in M} w_e.
\end{aligned}$$

Since the dual objective value for S matches the primal objective value for the reference solution, both must be optimal. □

Clearly if the reference solution is the unique optimal solution, then the matrix S from Theorem 6 is still PSD (where we do not assume that the reference solution is the unique optimal solution). However, while Theorem 6 guarantees the optimality of the reference solution to (MC-P) (if S is PSD), it does *not* necessarily guarantee the existence of a rank $(n-1)$ optimal dual solution to (MC-D) or the uniqueness of the optimality of the reference solution. Consider a triangle with edge weights $w_1 = 1$ and $w_2 = w_3 = 2$. In this case,

$$S = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

which is PSD (showing the optimality of the reference solution) but itself only has rank $(n-2)$. This example illustrates why we may not be able to immediately conclude the reference solution is the unique primal solution even if S is PSD; however, we conjecture this is the case:

Conjecture 7. *Let $G = (V, E)$ be a graph with a unique max cut, W its weight matrix, and $M \subset E$ the set of edges in its max cut. For $v \in V$, let $\gamma_v = \sum_{(v,j)\in M} w_{vj} - \sum_{(v,j)\in E\setminus M} w_{vj}$. If the rank 1 reference solution corresponding to the max cut is an optimal primal solution or, equivalently, $S = W + diag(\gamma)$ is PSD, then the reference solution is the* ***unique*** *optimal primal solution.*

5. Conclusion

In this work, we have partially characterized graphs for which we can find a rank $n-1$ solution to (MC-D). This goal was motivated by the desire to classify instances when the reference solution of a max cut is the unique optimal solution to (MC-P). We show in Theorem 4 that this is always the case for connected bipartite graphs. In order to classify further, we had to first explore when the reference solution is an optimal solution to (MC-P). Theorem 5 shows that uniqueness of a max cut is a necessary condition, and Theorem 6 shows that given the uniqueness of a max cut, whether the reference solution is an optimal primal solution is equivalent to whether a matrix with entries depending only on the edge-weights of G and its max cut is positive semidefinite. These results do not completely characterize when there is a unique optimal solution with rank 1, but we conjecture this is the case when the conditions of Theorem 6 are met. A deeper understanding of classes of graphs when the matrix S given in Theorem 6 is PSD could provide useful insights into proving Conjecture 7.

In addition to settling Conjecture 7, there are several potential directions of future study which we present now. Conjecture 7 claims for a general graph that if the GW-relaxation is exact and the graph has a unique max cut, then the reference solution corresponding to the max cut is the unique optimizer of the relaxation, and Theorem 4 proves this for connected bipartite graphs. In general, we do not know when the GW-relaxation is exact, but we do know it remains exact for bipartite graphs even if they are not connected. However, in this case, the max cuts are not unique and therefore the optimal solution to (MC-P) is not unique either. One might explore whether the bipartiteness can still be exploited to analyze the optimum solutions in this setting. In particular, one set of optimal solutions is given by selecting a unit vector (or reference solution) for each connected component of the bipartite graph. A reasonable assertion might be that all optimum solutions are then convex combinations of solutions of this type. One could also investigate an analogous question about the rank of optimal solutions and how it potentially corresponds to the number of connected components. For general graphs, if the GW-relaxation is exact, but there are several max cuts, can we understand the form these optimal solutions to (MC-P) take? Are they always convex combinations of reference solutions?

Additionally, while Theorem 6 provides one condition for the reference solution being an optimal solution to (MC-P) in general graphs, more can be said in the case of planar graphs if the max cut SDP (MC-P) is modified slightly.

Barahona and Mahjoub [2] consider the SDP given by adding the following constraints to (MC-P): for all $i, j, k \in V$,

$$\begin{aligned}
v_i \cdot v_j + v_i \cdot v_k + v_j \cdot v_k &\geq -1 \\
-v_i \cdot v_j - v_i \cdot v_k + v_j \cdot v_k &\geq -1 \\
-v_i \cdot v_j + v_i \cdot v_k - v_j \cdot v_k &\geq -1 \\
v_i \cdot v_j - v_i \cdot v_k - v_j \cdot v_k &\geq -1.
\end{aligned}$$

We will refer to (MC-P) with these additional constraints as (MC2-P). They show that (MC2-P) has optimal objective value equal to that of the max cut for any planar graph instance. Furthermore, we can observe that the reference solution of a max cut given by $u_1, u_2 \in \mathbb{R}^n$ is still a feasible solution to (MC2-P); all constraints are satisfied if $v_i, v_j, v_k \in \{u_1, u_2\}$ for all $i, j, k \in V$. Thus, since the objective value of (MC2-P) is equal to the value of the max cut for planar graphs, and the reference solution is a feasible solution achieving the value of the max cut, it must be an optimal solution to (MC2-P). Therefore, we can immediately focus on the uniqueness of the reference solution as an optimal solution to (MC2-P).

To do so, we can again consider finding sufficiently high rank solutions to the dual program of (MC2-P). The new dual (MC2-D) has the following form:

$$\begin{aligned} &\text{minimize} \quad \Bigg[\frac{1}{2}W_{tot} + \frac{1}{4}\sum_{i=1}^{n}\gamma_i \\ &\qquad + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n}\left(\alpha_{ijk} + \beta_{ijk} + \delta_{ijk} + \gamma_{ijk}\right)\Bigg] \\ (\text{MC2-D}) \quad &\text{subject to} \quad W + diag(\gamma) + T \succeq 0, \end{aligned}$$

where T is the symmetric matrix such that $T_{ii} = 0$ for all i and

$$\begin{aligned} T_{ij} = &\sum_{k<i}\left(-\alpha_{kij} - \beta_{kij} + \delta_{kij} + \lambda_{kij}\right) \\ &+ \sum_{i<k<j}\left(-\alpha_{ikj} + \beta_{ikj} - \delta_{ikj} + \lambda_{ikj}\right) \\ &+ \sum_{j<k}\left(-\alpha_{ijk} + \beta_{ijk} + \delta_{ijk} - \lambda_{ijk}\right) \end{aligned}$$

for $i < j$.

Unfortunately, the large number of dual variables adds significant complexity to the analysis of potential high-rank dual solutions. While we believe the reference solution should be the unique optimal solution to (MC2-P) for planar graphs, we were unable to find a general form of a sufficiently high rank solution to (MC2-D) which would prove this. It would be an interesting direction of research to understand more about optimal solutions to (MC2-D). In particular, can this SDP be used to classify when the reference solution is the unique optimal primal solution for planar graphs?

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