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Market Impacts of Pooling Intermittent Spectrum

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***Abstract*—Temporal sharing of spectrum as in the CBRS system provides wireless service providers (SPs) with spectrum that is intermittently available. This intermittency can decrease the value of the spectrum to a SP. In this paper we consider a setting where a SP can *pool* multiple intermittent bands of spectrum with independent availability. We find that pooling can achieve a higher spectrum efficiency in terms of the congestion incurred by users compared to using a single intermittent band (with the same total bandwidth). We show that this efficiency gain can be achieved with a relatively small pool of bands and it quickly converges to the optimal case as the number of bands increases. We also observe that pooled intermittency has a lesser impact on bids if spectrum is auctioned.**

***Index Terms*—wireless spectrum sharing, game theory, network pricing**

I. Introduction

Spectrum sharing has emerged as a promising technology to meet the ever-growing demands for wireless spectrum. Sharing provides new spectral resources for commercial service providers (SPs) without the necessity of relocating existing services, which can be cost-prohibitive and incur large delays. However, spectrum sharing also incurs costs that must be weighed against these gains [1]. In particular, for temporal-based sharing as in the recent CBRS system [2], one cost to commercial users from sharing is that they have a lower priority to access the spectrum than federal incumbents [1]. This can make the spectrum *intermittently available* to a commercial SP, which in turn can reduce the value of that band of spectrum. This is a point made in a recent report on CBRS commissioned by the CTIA, which notes that "Federal preemption of commercial spectrum rights are a barrier to applications that require guaranteed levels of service" [3].

In this paper, we consider an approach to mitigate the impact of intermittency through *pooling* multiple intermittent bands of spectrum, where each band's availability is independent of the others. The main idea is that by pooling such bands a SP can dynamically shift their traffic to the available bands at any time and so better maintain service in the face of intermittency.

Our objective is to study the market impacts of pooling intermittent spectrum. We are interested in how much benefit SPs can get from pooling and how it affects the congestion level incurred by users. We also study how many bands are needed to achieve a considerable pooling gain and how it affects the bidding result if the spectrum is auctioned.

Our analysis builds on the framework in [4] where SPs compete via Cournot competition with a single licensed shared band of spectrum (similar to PAL spectrum in the CBRS system) and a single band of exclusively licensed spectrum. Here, we instead consider a pool of shared bands for each SP. Our main results are summarized as follows:

- Pooling intermittent spectrum always achieves higher spectrum efficiency in terms of the congestion incurred by users compared to using a single intermittent band with the same bandwidth. In other words, from the perspective of SPs, the revenue loss due to congestion is reduced by pooling multiple bands.
- The revenue gain is bounded, and the maximum gain is achieved by pooling an infinite number of intermittent bands.
- The convergence rate of the revenue gain to the extreme case is $\Theta(1/n)$, where n is the number of pooled bands. This means that the revenue gain converges quickly and a considerable gain can be achieved with a relatively small pool of bands.
- If a pool of spectrum is auctioned, the intermittency of each individual band has a lesser impact on the final bids submitted by the SPs.

These results suggest that a regulator may want to design allocation mechanisms (e.g. auctions) so that SPs can create packages of different bands of shared spectrum with independent availability (e.g. by having bands that are shared with different incumbent users). For example, suppose that there are two distinct bands of spectrum each with bandwidth W and with different incumbent users. Then, if these bands were being allocated to two SPs, it may be preferred to allocate $W/2$ of each band to each SP as opposed to allocating one band to one SP and one to another.[1]

In terms of related work, this paper lies in a stream of work that treats wireless spectrum as congestible resources and analyzes the market impact of sharing policies, including [4]–[8]. Much of this work has focused on models where spectrum is not intermittent, unlike our approach here. These types of models have also been used to study other aspects

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[1]Of course these gains of pooling must be weighted against other costs from using multiple bands as opposed to a single band.

of spectrum markets such as security implications [9], the bundling of content with service [10], or markets of spectrum measurements [11]. Note that our work is different from work on spectrum pooling as in [8] in which the same spectrum is shared by multiple SPs to leverage statistical multiplexing due to the randomness in their traffic. In this work, we consider a setting in which a SP possesses multiple intermittent bands of spectrum, thus the randomness lies in the spectrum itself instead of the traffic. This work is more related to the field of spectrum partitioning and there is a large body of work on either the technical side [12], [13] or the policy side [14], [15]. Note that the spectrum partitioning in [12] is used to increase the overall throughput by fixing the data rate on each partitioned band, which increases the SINR requirement for each band. In this work, we partition the spectrum in order to mitigate the impact of intermittency and we also show that the revenue gain can be achieved with a relatively small pool of bands. Thus if we consider enhancing the overall throughput by maintaining a high data rate on each band as a by-product of this partitioning, we still do not require a significant increase in SINR.

The rest of the paper is organized as follows: Section II introduces the pooling model. Section III examines the efficiency of pooling multiple intermittent bands in comparison to using a single intermittent band with the same bandwidth. Section IV characterizes the Nash equilibrium within a spectrum sharing market and discusses auctioning a pool of intermittent spectrum. Finally, Section V concludes the paper.

II. The Model

We consider a model in which there are N SPs competing for a common pool of non-atomic customers through Cournot competition as in [4]. Without loss of generality, we assume that the total amount of customers (i.e., the market size) is 1. Specifically, SPs announce the quantity of customers they want to serve. This in turn leads to a market-clearing price given by

$$p_d = P(X) = 1 - X, \tag{1}$$

where X is the total quantity of customers announced by the SPs, and $P(\cdot)$ is a linear demand function. Note that this linear demand function is chosen for tractability and the main results (i.e., Theorem 2, 3, and 4) of this paper are independent of the choice of demand functions. Here, p_d is the *delivered price*, which is the total cost incurred by users for using the wireless service. We assume that users are sensitive to congestion that is measured in terms of a latency cost. The latency cost per user when x users are served on a static[2] band with bandwidth B is given by

$$l(x, B) = \frac{x}{B}. \tag{2}$$

For intermittent bands, whenever a band is unavailable, the SP reallocates the users on this band to other available shared bands or its proprietary band.[3] For simplicity, we assume that there is no cost for user reallocation and so our results can be viewed as giving insight into the potential gains of pooling when reallocation costs are small. The latency cost depends on how SPs distribute users among their bands. We will derive the latency cost for intermittent bands under the optimal user allocation later.

The *service price* that a SP charges its users is given by the difference between p_d and the latency cost incurred by that SP's users. In other words, the latency experienced by users will cause a revenue loss to SPs, as they can not charge as much for a congested network.

Suppose each SP $i \in \{1 \dots N\}$ possesses a proprietary band with bandwidth B_i and a pool of $n_i \in \mathcal{N}$ licensed shared bands with aggregate bandwidth W_i. For analytical convenience, we assume that the bandwidth δ_i of these licensed shared bands are the same, i.e.,

$$\delta_i = \frac{W_i}{n_i}. \tag{3}$$

SPs have exclusive use of their own proprietary bands whereas they share their licensed shared bands with incumbent users who have higher priority, which makes these bands intermittent to the SPs. The intermittency of each shared band is modeled by a Bernoulli random variable which takes value 1 with probability $\alpha_i \in [0, 1]$ meaning the corresponding band is available. We assume that, for each SP, the Bernoulli random variables among different shared bands are mutually independent and identically distributed (IID). Therefore, the spectrum resources of SP i can be characterized by a tuple as $(B_i, W_i, n_i, \alpha_i)$.

Next, we derive the average latency cost of SP i when serving x_i users with spectrum resources $(B_i, W_i, n_i, \alpha_i)$. From the perspective of SP i, if x_i is given, it will try to maximize its revenue (which is equivalent to minimizing the loss due to the latency cost) by choosing an optimal user allocation strategy.

First, we consider a SP which possesses a collection of K static bands (i.e., the availability $\alpha_k = 1, \forall k$) with bandwidth $\mathbf{B} = (B_1, \dots, B_K)$. Given a vector of user allocations $\mathbf{y} = (y_1, \dots, y_K)$ and a delivered price p_d, the service price charged by this SP on band K is given by $p_d - \frac{y_k}{B_k}$. Thus the revenue of this SP is in turn given by

$$\sum_{k=1}^{K} y_k \left(p_d - \frac{y_k}{B_k} \right) = p_d \sum_{k=1}^{K} y_k - \sum_{k=1}^{K} \frac{y_k^2}{B_k}. \tag{4}$$

If we consider the second term in (4) as the revenue loss due to the latency experienced by its users, then we naturally define the revenue loss as:

$$L(\mathbf{y}) := \sum_{k=1}^{K} \frac{y_k^2}{B_k}. \tag{5}$$

If the total amount of users is given, then the optimal user allocation is characterized by the following lemma.

Lemma 1. *Consider a collection of K static bands with bandwidth $\mathbf{B} = (B_1, \dots, B_K)$ and $x = \sum_{k=1}^{K} y_k$ total*

[2] The term "static" means there is no intermittency.

[3] For example, 5G networks support traffic steering across multiple spectrum bands that could be used for this reallocation.

amount of users. Under the optimal allocation, the amount of users served on the k-th band is given by

$$y_k^\star = \frac{B_k}{\sum_{j=1}^K B_j} x, \tag{6}$$

which is proportional to the bandwidth B_k. The revenue loss in turn is given by

$$L(x, \mathbf{B}) := L(\mathbf{y}^\star) = \frac{x^2}{\sum_{k=1}^K B_k}. \tag{7}$$

Proof. Let $\mathbf{y} = (y_1, \ldots, y_K)$ be the vector of user allocation. For convenience, let $\rho_k = y_k/B_k$ be the user density on the k-th band, then with some abuse of notation, (5) can be written as

$$L(\boldsymbol{\rho}) = \sum_{k=1}^K B_k \rho_k^2. \tag{8}$$

The minimum revenue loss can then be formulated as follows:

$$\min_{\boldsymbol{\rho}} \ L(\boldsymbol{\rho}),$$

$$\text{s.t.} \sum_{k=1}^K B_k \rho_k = x; \tag{9}$$

$$\rho_k \geq 0, \ k = 1, \ldots, K. \tag{10}$$

Since the objective function (8) is convex and the feasible set is also convex, strong duality holds. The Lagrangian is

$$La(\boldsymbol{\rho}, \mu) = \sum_{k=1}^K B_k \rho_k^2 + \mu \left(\sum_{k=1}^K B_k \rho_k - x \right), \tag{11}$$

where μ is the Lagrangian multiplier of the equality constraint.

Let $g(\mu) = \min_{\boldsymbol{\rho}:\boldsymbol{\rho}\geq 0} La(\boldsymbol{\rho}, \mu)$. The solution to the dual problem $\max_\mu g(\mu)$ is given by:

$$\mu^\star = -\frac{2x}{\sum_{k=1}^K B_k}, \tag{12}$$

$$\rho_k^\star = -\frac{\mu^\star}{2} = \frac{x}{\sum_{k=1}^K B_k}, \tag{13}$$

$$L(\boldsymbol{\rho}^\star) = g(\mu^\star) = \frac{x^2}{\sum_{k=1}^K B_k}. \tag{14}$$

□

Note that Lemma 1 implies that a SP should spread its customers across a set of static bands so that the user density per unit bandwidth on each band is equal. An immediate result from Lemma 1 is that pooling K static bands with bandwidth $\mathbf{B} = (B_1, \ldots, B_K)$ is equivalent to having one single band with bandwidth $\bar{B} = \sum_{k=1}^K B_k$ and a revenue loss of

$$L(x, \bar{B}) = \frac{x^2}{\bar{B}}. \tag{15}$$

In other words, for the assumed model with static bands, having either one band or a pool of bands with the same bandwidth is equivalent. Next we will see that this is not the case with intermittent spectrum.

Assume a SP possesses a spectrum profile of $\tilde{\mathbf{B}} = (B, W, n, \alpha)$, which contains intermittent bands. Since there is no penalty for user reallocation, whenever k shared bands are available, the minimum loss is achieved when users are served on the k available bands as in Lemma 1, i.e., with an equivalent bandwidth of $\bar{B} = B + k\delta$ in (15). The expected revenue loss of this SP averaged over the possible realizations of channel availabilities is then given by

$$\mathbb{E}\left[L(x, \tilde{\mathbf{B}})\right] := \mathbb{E}\left[\frac{x^2}{B + \delta Y}\right] \tag{16}$$

$$= x^2 \sum_{k=0}^n \frac{\binom{n}{k}\alpha^k (1-\alpha)^{n-k}}{B + k\delta}, \tag{17}$$

where $\delta = W/n$, and Y is a binomial random variable such that $Y \sim Binomial(n, \alpha)$.

Next, we derive the revenue of SP $i \in \{1 \ldots N\}$ in a market of N SPs. Assume SP i possesses the spectrum of $\tilde{\mathbf{B}}_i = (B_i, W_i, n_i, \alpha_i)$. The revenue[4] of SP i is then given by

$$\begin{aligned} R_i &= x_i \left(p_d - x_i \sum_{k=0}^n \frac{\binom{n}{k}\alpha^k(1-\alpha)^{n-k}}{B + k\delta} \right) && (18) \\ &= p_d x_i - \mathbb{E}\left[L(x_i, \tilde{\mathbf{B}}_i)\right] \\ &= P\left(\sum_{k=1}^N x_k\right) x_i - \mathbb{E}\left[L(x_i, \tilde{\mathbf{B}}_i)\right] \\ &= x_i - \left(\sum_{k=1}^N x_k\right) x_i - \mathbb{E}\left[L(x_i, \tilde{\mathbf{B}}_i)\right], && (19) \end{aligned}$$

where $x_1, \ldots, x_N$ are the users served by the N SPs.

From (19) we can conclude that the revenue of each SP is coupled with that of other SPs through the delivered price p_d which is determined by the total demands $\sum_{k=1}^N x_k$. SPs suffer a revenue loss determined by the amount of users x_i they want to serve and the amount of resources $\tilde{\mathbf{B}}_i$ they have. Each SP wants to maximize its revenue and this coupling makes it a game among SPs.

III. Efficiency of Pooling Intermittent Spectrum

In this section, we study the efficiency of pooling multiple intermittent bands compared to using one single band with the same expected bandwidth. Specifically, we compare the revenue loss when $n \geq 2$ intermittent bands are pooled, i.e., $\tilde{\mathbf{B}}^n = (B, W, n, \alpha)$, to the case when $\tilde{\mathbf{B}}^1 = (B, W, 1, \alpha)$. By abuse of notation, we also use $\tilde{\mathbf{B}}^n$ and $\tilde{\mathbf{B}}^1$ to denote the random variables that represent the corresponding available

[4] As mentioned before, the service price charged by each SP is the difference between p_d and the latency cost incurred by its users. Thus (18) holds because the revenue of SP i is given by the product of the amount of users x_i and the corresponding service price.

bandwidth. Note that for a fair comparison, the expected bandwidth is the same in both cases:

$$\begin{aligned}\mathbb{E}\left[\tilde{\mathbf{B}}^n\right] &= B + n(\alpha\delta)\\ &= B + n(\alpha\frac{W}{n})\\ &= B + \alpha W\\ &= \mathbb{E}\left[\tilde{\mathbf{B}}^1\right].\end{aligned} \tag{20}$$

Theorem 2 (Revenue Loss Inequality)**.** *Given the same expected bandwidth of the intermittent bands, serving the same amount of users on a pool with $n \geq 2$ intermittent bands yields a smaller revenue loss compared to the case in which there is only one intermittent band. That is, the following inequality holds:*

$$\mathbb{E}\left[L(x, \tilde{\mathbf{B}}^n)\right] \leq \mathbb{E}\left[L(x, \tilde{\mathbf{B}}^1)\right], \tag{21}$$

for any $x \geq 0$, or equivalently,

$$\sum_{k=0}^{n} \frac{\binom{n}{k}\alpha^k(1-\alpha)^{n-k}}{B + k\frac{W}{n}} \leq \frac{\alpha}{B+W} + \frac{1-\alpha}{B}. \tag{22}$$

According to Theorem 2, pooling any number of intermittent bands can reduce the revenue loss compared to serving users on a single large intermittent band, indicating that pooling can achieve higher revenue. This suggests that to mitigate the impact of intermittency, regulators should consider providing SPs with multiple bands with independent (or less correlated) incumbent activities. One possible way to achieve this is to assign each SP multiple bands that are spread over a large span of spectrum instead of a single band that occupies a certain range of spectrum to reduce the correlation between bands.

Next, we study the asymptotic behavior of revenue loss as n goes to infinity. We want to examine the theoretical limit of the pooling efficiency. Consider the case in which there are infinite arbitrarily small intermittent bands with finite aggregate bandwidth W, i.e., $\tilde{\mathbf{B}}^\infty = (B, W, \infty, \alpha)$.

Theorem 3 (Pooling Infinite Intermittent Bands)**.** *Consider a spectrum profile of $\tilde{\mathbf{B}}^\infty = (B, W, \infty, \alpha)$, the revenue loss due to the latency incurred by users is given as follows*

$$\mathbb{E}\left[L(x, \tilde{\mathbf{B}}^\infty)\right] = \frac{x^2}{B + \alpha W}, \tag{23}$$

which also provides a lower bound of revenue loss, that is

$$\mathbb{E}\left[L(x, \tilde{\mathbf{B}}^\infty)\right] \leq \mathbb{E}\left[L(x, \tilde{\mathbf{B}}^n)\right], \forall n \in \{1, 2, \dots\}. \tag{24}$$

Proof. Consider a profile $\tilde{\mathbf{B}}^n$ with n intermittent bands. According to (16), the expected revenue loss is

$$\begin{aligned}\mathbb{E}\left[L(x, \tilde{\mathbf{B}}^n)\right] &= \mathbb{E}\left[\frac{x^2}{B + \delta Y}\right]\\ &= x^2\mathbb{E}\left[\frac{1}{B + \frac{Y}{n}W}\right].\end{aligned} \tag{25}$$

Since $Y \sim Binomial(n, \alpha)$, by the law of large numbers (LLN), Y/n converges to α as $n \to \infty$, which also implies the convergence in distribution. Then by weak convergence, (23) holds.

To prove (23) is a lower bound, define

$$g(y) := \frac{1}{B + \frac{y}{n}W}, \tag{26}$$

which is a convex function of y. Then by Jensen's inequality, we have

$$\begin{aligned}\mathbb{E}\left[g(Y)\right] &\geq g\left(\mathbb{E}\left[Y\right]\right)\\ &= g\left(\alpha n\right)\\ &= \frac{1}{B + \alpha W}.\end{aligned} \tag{27}$$

□

From Theorem 3, we can conclude that, with a large number of intermittent bands, the equivalent bandwidth approaches $B + \alpha W$ without intermittency as n goes to infinity. In other words, $\tilde{\mathbf{B}}^\infty$ is equivalent to a static band with bandwidth $B + \alpha W$ in terms of revenue loss. Also, (24) indicates that the gain of pooling is bounded as n increases.

Next, we examine how fast the lower bound in (24) is achieved by increasing the number of pooled bands.

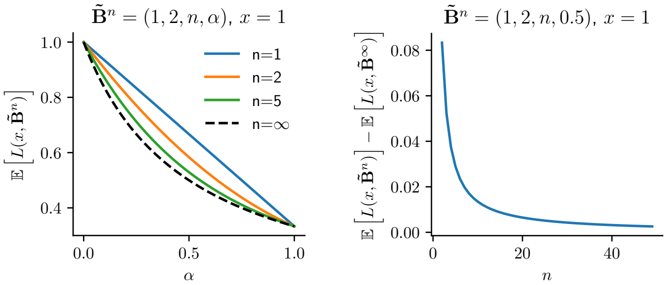
Theorem 4 (Order of Convergence)**.** *The revenue loss with finite pooled bands converges to the extreme case given by (23) with order $\Theta(1/n)$, that is*[5]

$$\mathbb{E}\left[L(x, \tilde{\mathbf{B}}^n)\right] - \mathbb{E}\left[L(x, \tilde{\mathbf{B}}^\infty)\right] = \Theta\left(\frac{1}{n}\right). \tag{28}$$

The convergence rate given by Theorem 4 is relatively fast. This suggests that it is not required to have an extremely large n to achieve a near-optimal pooling gain, proving pooling intermittent bands is an efficient strategy. This also means that in the meantime we can increase the overall throughput by fixing the data rate on each band without worrying too much about the increasing SINR requirement. The numerical result below will show that a relatively small n can achieve a considerable improvement in latency.

Fig. 1 gives a numerical example of this asymptotic behavior of $\mathbb{E}[L(x, \tilde{\mathbf{B}}^n)]$. As shown in Fig. 1a, pooling multiple intermittent bands (i.e., $n \geq 2$) always yields less revenue loss compared to using a single intermittent band (i.e., $n = 1$), as suggested by Theorem 2. Pooling infinite intermittent bands provides a lower bound of revenue loss, as suggested by Theorem 3. It is also worth noting that there is a significant reduction in revenue loss even when $n = 2$ compared to $n = 1$. Also, the revenue loss with $n = 5$ is already close to the lower bound, indicating that the lower bound is easy to approach with a relatively small number of intermittent bands. Also note that the largest gains from pooling occur with $\alpha = 0.5$, i.e. when 50% of the time, the channels are not available. If channels have larger (or smaller) availability, then the gain in increasing

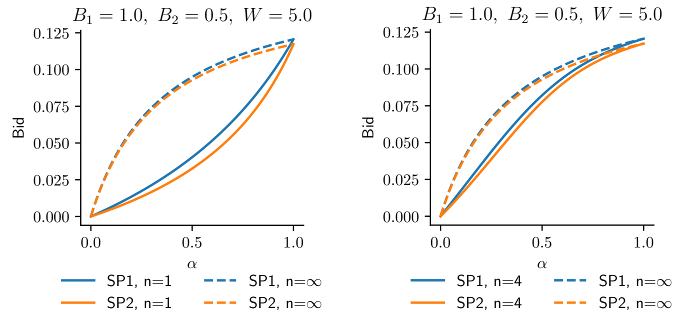
[5]Here, the notation that $f(n) = \Theta(g(n))$ means there are constants P and Q so that $\lim_{n\to\infty} \frac{f(n)}{g(n)} < P$ and $\lim_{n\to\infty} \frac{g(n)}{f(n)} < Q$.



(a) Revenue loss

(b) Difference

Fig. 1: (a) shows the revenue loss $\mathbb{E}[L(x, \tilde{\mathbf{B}}^n)]$ vs. α and number of partitions n. (b) shows the difference in (28) vs. n with fixed $\alpha = 0.5$.



(a) $n = 1$ vs. $n = \infty$

(b) $n = 4$ vs. $n = \infty$

Fig. 2: Bids vs. availability α with different numbers of partitions n in a second price auction.

beyond $n = 2$ reduces. Fig. 1b shows the difference in (28) for $\alpha = 0.5$ versus n. This has the shape of $\Theta(1/n)$, as suggested by Theorem 4.

IV. Equilibrium and Auctions

A. Nash Equilibrium

Consider a market of N SPs and each SP $i \in \{1 \dots N\}$ possesses bandwidth $\tilde{\mathbf{B}}_i = (B_i, W_i, n_i, \alpha_i)$. We assume a Cournot competition among these SPs in which SPs simultaneously announce the amount of users $(x_1, \dots, x_N)$ they plan to serve. Each SP i strategically chooses x_i to maximize its revenue R_i which is given by (19).

To simplify the notation, we define

$$T_i = \left(\sum_{k=0}^{n_i} \frac{\binom{n_i}{k} \alpha_i^k (1 - \alpha_i)^{n_i - k}}{B_i + k \frac{W_i}{n_i}} \right)^{-1}, \tag{29}$$

which can be considered as the equivalent bandwidth of $\tilde{\mathbf{B}}_i$.

With this definition, we rewrite (19) as

$$R_i = x_i - \left(\sum_{k=1}^{N} x_k \right) x_i - \frac{x_i^2}{T_i}. \tag{30}$$

The following theorem gives the Nash Equilibrium of this game.

Theorem 5 (Equilibrium)**.** *There is a unique Nash equilibrium given by*

$$x_i^\star = \frac{1}{1 + \frac{2}{T_i}} \cdot \frac{1}{1 + \sum_{k=1}^{N} \frac{1}{1 + \frac{2}{T_k}}}. \tag{31}$$

Proof. This theorem can be proved by solving $\partial R_i / \partial x_i = 0$ for all users. ☐

From (31), we observe that

$$\frac{x_i^\star}{x_j^\star} = \frac{1 + \frac{2}{T_j}}{1 + \frac{2}{T_i}}, \tag{32}$$

which means the more spectrum resources a SP possesses, the larger its market share will be at the equilibrium.

B. Auctions

Next, we study the effect of pooling intermittent spectrum on auction results. Consider a market of two SPs, i.e., $N = 2$, and each possesses B_i amount of proprietary spectrum and no shared spectrum. Now assume a pool of intermittent bands with aggregate bandwidth W, partition n, and availability α is to be allocated through a second price auction. Since in a second price auction, truthful bidding is an equilibrium, we assume that each SP bids the revenue difference between winning and losing.

Fig. 2 shows a numerical result of the bids offered by two SPs with proprietary bandwidths $B_1 = 1$ and $B_2 = 0.5$. Fig. 2a shows the bids versus α for $n = 1$ and $n = \infty$; Fig. 2b shows that case of $n = 4$ and $n = \infty$. In every case SP1 (the SP with the larger value of B_1) wins the auction. The reason behind this is that having more proprietary spectrum enables SP1 to better absorb traffic offloaded due to the intermittency. However, the difference between the bids becomes smaller as n increases. This suggests that should a regulator want to offer bidding credits to help a small SP compete,[6] the amount of credit needed would reduce with more pooling. Note also that when $n = 1$, the bid curves are a convex function of α, meaning that the spectrum needs to have a fairly high availability before the SPs are willing to submit large bids. However, when $n = 4$ (or $n = \infty$), these curves are concave in α. This means that the bids increase more rapidly in α showing that pooling spectrum with a lower availability is valued more.

V. Conclusions

We studied the market impacts of pooling intermittent spectrum and showed that pooling can boost spectrum efficiency in the sense that it can reduce the congestion experienced by users, leading to less revenue loss by SPs. The improvement in efficiency is an increasing function of the number of pooled intermittent bands but it is also bounded. The improvement converges to this upper bound on the order of $\Theta(1/n)$ as the number of pooled bands n increases. Numerical results show that a considerable efficiency improvement can be achieved

[6]For instance, the FCC permits certain qualified small business or rural providers to utilize bidding credits in the auction of CBRS band.

with a small pool of bands, indicating that pooling may be an efficient way to increase the value of intermittent spectrum. This is also illustrated in a simple auction model, where pooling is shown to reduce the impact of intermittency on bids. A natural next step for this work would be to model the costs of switching between different bands and understand how that may impact the benefits of pooling.

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APPENDIX

A. Proof of Theorem 2

We prove this theorem by induction.

Before we begin the proof, we first introduce the following Arithmetic mean-Harmonic mean (AM-HM) inequality:

$$\frac{a+b}{2} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}, \tag{33}$$

or equivalently,

$$\frac{1}{a}+\frac{1}{b} \geq \frac{2}{\frac{a+b}{2}}, \tag{34}$$

which holds for any positive a and b.

To prove (22), we first verify the case of $n=2$, which is

$$\frac{\alpha}{B+2\delta}+\frac{1-\alpha}{B} \geq \frac{\alpha^2}{B+2\delta}+\frac{2\alpha(1-\alpha)}{B+\delta}+\frac{(1-\alpha^2)}{B}, \tag{35}$$

where $\delta = W/2$.

By scaling, this is equivalent to the following inequality

$$\frac{\alpha}{m+2}+\frac{1-\alpha}{m} \geq \frac{\alpha^2}{m+2}+\frac{2\alpha(1-\alpha)}{m+1}+\frac{(1-\alpha^2)}{m}, \tag{36}$$

where $m = B/\delta$. We can simplify (36) as follows:

$$\begin{aligned} &\frac{\alpha-\alpha^2}{m+2}+\frac{(1-\alpha)-(1-\alpha)^2}{m} \geq \frac{2\alpha(1-\alpha)}{m+1} \\ &\Leftarrow \frac{\alpha(1-\alpha)}{m+2}+\frac{\alpha(1-\alpha)}{m} \geq \frac{2\alpha(1-\alpha)}{m+1} \\ &\Leftarrow \frac{1}{m+2}+\frac{1}{m} \geq \frac{2}{m+1}. \end{aligned} \tag{37}$$

By (34), (37) holds.

It is worth noting that, in the proof of $n=2$, we use the inequality (34) once. One can verify that the case of $n=3$ can be proved by applying (34) twice. Theoretically, for any larger n, the proof can be done by using (34) for $n-1$ times. Thus for any n, (22) would eventually boil down to (34) after some manipulations.

Back to the proof, it is obvious that (36) holds for any positive m. Thus, for the induction hypothesis, we assume the following inequality holds for any positive m:

$$\frac{\alpha}{m+n-1}+\frac{1-\alpha}{m} \geq \sum_{k=0}^{n-1} \frac{\binom{n-1}{k}\alpha^k(1-\alpha)^{n-1-k}}{m+k}, \tag{38}$$

from which we want to prove the following holds:

$$\frac{\alpha}{m+n}+\frac{1-\alpha}{m} \geq \sum_{k=0}^{n} \frac{\binom{n}{k}\alpha^k(1-\alpha)^{n-k}}{m+k}. \tag{39}$$

Starting from the left-hand side of (39), we split each term into two terms as follows:

$$\begin{aligned} LHS =& \left(\frac{\alpha^2}{m+n}+\frac{\alpha-\alpha^2}{m+n}\right) \\ &+\left(\frac{(1-\alpha)^2}{m}+\frac{(1-\alpha)-(1-\alpha)^2}{m}\right) \\ =&\alpha\left(\frac{\alpha}{m+n}\right)+(1-\alpha)\left(\frac{1-\alpha}{m}\right) \\ &+\frac{\alpha(1-\alpha)}{m+n}+\frac{\alpha(1-\alpha)}{m}. \end{aligned} \tag{40}$$

In order to apply the hypothesis (38), we add and subtract the term we want in the first and second terms, respectively:

$$\begin{aligned}LHS =&\alpha\left(\frac{\alpha}{m+n}+\frac{1-\alpha}{m+1}-\frac{1-\alpha}{m+1}\right)+(1-\alpha)\\&\left(\frac{\alpha}{m+n-1}+\frac{1-\alpha}{m}-\frac{\alpha}{m+n-1}\right)\\&+\frac{\alpha(1-\alpha)}{m+n}+\frac{\alpha(1-\alpha)}{m}\\=&\alpha\left(\frac{\alpha}{m+n}+\frac{1-\alpha}{m+1}\right)\\&+(1-\alpha)\left(\frac{\alpha}{m+n-1}+\frac{(1-\alpha)}{m}\right)\\&+\alpha(1-\alpha)C,\end{aligned} \tag{41}$$

where

$$C=\frac{1}{m+n}+\frac{1}{m}-\frac{1}{m+1}-\frac{1}{m+n-1}. \tag{42}$$

Then, we can apply the hypothesis to the first term as follows:

$$\begin{aligned}\frac{\alpha}{m+n}+\frac{1-\alpha}{m+1}=&\frac{\alpha}{(m+1)+n-1}+\frac{1-\alpha}{m+1}\\=&\frac{\alpha}{m'+n-1}+\frac{1-\alpha}{m'}\\\geq&\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}\alpha^k(1-\alpha)^{n-1-k}}{m'+k},\end{aligned} \tag{43}$$

where $m'=m+1$.

Similarly, we apply the hypothesis to the second term in (41), then we have

$$LHS \geq \alpha A+(1-\alpha)B+\alpha(1-\alpha)C, \tag{44}$$

where

$$A=\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}\alpha^k(1-\alpha)^{n-1-k}}{m+1+k}, \tag{45}$$

$$B=\sum_{k=0}^{n-1}\frac{\binom{n-1}{k}\alpha^k(1-\alpha)^{n-1-k}}{m+k}. \tag{46}$$

Next, we will prove that $\alpha A+(1-\alpha)B=RHS$ and $C\geq 0$. To see this, by pulling the last term in A from the summation and the first term in B from the summation, we have

$$\begin{aligned}\alpha A=&\frac{\alpha^n}{m+n}+\sum_{k=0}^{n-2}\frac{\binom{n-1}{k}\alpha^{k+1}(1-\alpha)^{n-1-k}}{m+1+k}\\=&\frac{\alpha^n}{m+n}+\sum_{k=1}^{n-1}\frac{\binom{n-1}{k-1}\alpha^k(1-\alpha)^{n-k}}{m+k},\end{aligned} \tag{47}$$

and

$$(1-\alpha)B=\sum_{k=1}^{n-1}\frac{\binom{n-1}{k}\alpha^k(1-\alpha)^{n-k}}{m+k}+\frac{(1-\alpha)^n}{m}. \tag{48}$$

Then, by Pascal's rule which is

$$\binom{n-1}{k-1}+\binom{n-1}{k}=\binom{n}{k}, \tag{49}$$

we have

$$\begin{aligned}&\alpha A+(1-\alpha)B\\=&\frac{\alpha^n}{m+n}+\sum_{k=1}^{n-1}\frac{\binom{n}{k}\alpha^k(1-\alpha)^{n-k}}{m+k}+\frac{(1-\alpha)^n}{m}\\=&\sum_{k=0}^{n}\frac{\binom{n}{k}\alpha^k(1-\alpha)^{n-k}}{m+k}\\=&RHS\end{aligned} \tag{50}$$

To prove $C\geq 0$, we rearrange (42) as follows:

$$\begin{aligned}C=&\left(\frac{1}{m+n}-\frac{1}{m+1}\right)-\left(\frac{1}{m+n-1}-\frac{1}{m}\right)\\=&(1-n)\left(\frac{1}{(m+1)(m+n)}-\frac{1}{m(m+n-1)}\right)\\\geq&\, 0,\end{aligned} \tag{51}$$

which holds for any $n=1,2,\ldots$.

Therefore, from (44), we have

$$\begin{aligned}LHS\geq&\, RHS+\alpha(1-\alpha)C\\\geq&\, RHS.\end{aligned} \tag{52}$$

B. Proof of Theorem 4

We first simplify the difference by ignoring the coefficient which does not affect the order of convergence:

$$\mathbb{E}\left[L(x,\tilde{\mathbf{B}}^n)\right]-\mathbb{E}\left[L(x,\tilde{\mathbf{B}}^\infty)\right]\propto\mathbb{E}\left[\frac{\alpha-\frac{Y}{n}}{B+\frac{Y}{n}W}\right]. \tag{53}$$

Then define a function $f:\mathcal{R}\to\mathcal{R}$

$$f(y):=\frac{y}{B+W(\alpha-y)}, \tag{54}$$

and let $y=\alpha-Y/n$ which takes values in $[\alpha-1,\alpha]$. Then, from (53), the difference is proportional to $\mathbb{E}\left[f(y)\right]$.

Now, it is equivalent to proving that

$$\lim_{n\to\infty}n\mathbb{E}\left[f(y)\right]\geq K_1, \tag{55}$$

$$\lim_{n\to\infty}n\mathbb{E}\left[f(y)\right]\leq K_2, \tag{56}$$

where K_1 and K_2 are some non-zero constants.

We start with the proof of (55). The k-th order derivative of $f(y)$ is given by

$$f^{(k)}(y)=\frac{k!W^{k-1}(B+\alpha W)}{(B+\alpha W-Wy)^{k+1}}, \tag{57}$$

with $y\in[\alpha-1,\alpha]$.

By Taylor's Theorem [16], we can write $f(y)$ as the summation of its $(n-1)$-th order Taylor approximation at $y=0$ and an error term as follows

$$f(y)=\sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}y^k+\frac{f^{(n)}(\hat{y})}{n!}y^n, \tag{58}$$

which holds for some $\hat{y}$ between y and 0.[7]

[7] $\hat{y}$ is a function of y.

Next, let $n = 4$, we have

$$\begin{aligned} f(y) =& \frac{y}{B+\alpha W} + \frac{W}{(B+\alpha W)^2} y^2 + \frac{W^2}{(B+\alpha W)^3} y^3 \\ &+ \frac{W^3(B+\alpha W)}{(B+\alpha W - W\hat{y})^5} y^4. \end{aligned} \tag{59}$$

Since $y \in [\alpha-1, \alpha]$, we have $\hat{y} \in [\alpha-1, \alpha]$. Then one can verify that the last term in (59) is non-negative, which implies that

$$f(y) \geq \frac{y}{B+\alpha W} + \frac{W}{(B+\alpha W)^2} y^2 + \frac{W^2}{(B+\alpha W)^3} y^3. \tag{60}$$

Note that

$$\mathbb{E}\left[y\right] = \mathbb{E}\left[\alpha - \frac{Y}{n}\right] = \alpha - \frac{n\alpha}{n} = 0; \tag{61}$$

$$\begin{aligned} \mathbb{E}\left[y^2\right] =& \mathbb{E}\left[\left(\alpha - \frac{Y}{n}\right)^2\right] = Var\left(\frac{Y}{n}\right) = \frac{1}{n^2} Var\left(Y\right) \\ =& \frac{\alpha(1-\alpha)}{n}. \end{aligned} \tag{62}$$

Then we have

$$\begin{aligned} &\lim_{n\to\infty} n\mathbb{E}[f(y)] \\ \geq& \lim_{n\to\infty} n\left(0 + \frac{W}{(B+\alpha W)^2}\frac{\alpha(1-\alpha)}{n} + \frac{W^2}{(B+\alpha W)^3}\mathbb{E}[y^3]\right) \\ =& \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2} + \frac{W^2}{(B+\alpha W)^3} \lim_{n\to\infty} n\mathbb{E}[y^3]. \end{aligned} \tag{63}$$

By the central limit theorem, the random variable $\sqrt{n}y = \sqrt{n}(\alpha - Y/n)$ converges in distribution to a normal $\mathcal{N}(0, \alpha(1-\alpha))$. Thus, $\mathbb{E}[(\sqrt{n}y)^3]$ will converge to the third moment of a zero-mean Gaussian, which implies that $\lim_{n\to\infty} \mathbb{E}[(\sqrt{n}y)^3] = 0$.

Continue from (63), we have

$$\begin{aligned} &\lim_{n\to\infty} n\mathbb{E}[f(y)] \\ \geq& \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2} + \frac{W^2}{(B+\alpha W)^3} \lim_{n\to\infty} \frac{n}{n\sqrt{n}}\mathbb{E}[(\sqrt{n}y)^3] \\ =& \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2} + \frac{W^2}{(B+\alpha W)^3} \lim_{n\to\infty} \frac{1}{\sqrt{n}}\mathbb{E}[(\sqrt{n}y)^3] \\ =& \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2}, \end{aligned} \tag{64}$$

which proves (55) with $K_1 = \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2}$.

To prove (56), we revisit the Taylor approximation in (59). We can conclude that the last term in (59) is finite for any $y \in [\alpha-1, \alpha]$, since $\hat{y} \in [\alpha-1, \alpha]$. Then there must exist a constant C such that the following equality holds:

$$\begin{aligned} f(y) \leq& \frac{y}{B+\alpha W} + \frac{W}{(B+\alpha W)^2} y^2 + \frac{W^2}{(B+\alpha W)^3} y^3 \\ &+ Cy^4. \end{aligned} \tag{65}$$

A simple choice of C could be

$$C = \inf_{y\in[\alpha-1,\alpha]} \frac{W^3(B+\alpha W)}{(B+\alpha W - W\hat{y}(y))^5} < \infty. \tag{66}$$

Then we have

$$\begin{aligned} &\lim_{n\to\infty} n\mathbb{E}[f(y)] \\ \leq& \lim_{n\to\infty} n\Big(0 + \frac{W}{(B+\alpha W)^2}\frac{\alpha(1-\alpha)}{n} + \frac{W^2}{(B+\alpha W)^3}\mathbb{E}[y^3] \\ &+ C\mathbb{E}[y^4]\Big) \\ =& \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2} + C\lim_{n\to\infty} n\mathbb{E}[y^4]. \end{aligned} \tag{67}$$

Recall that $y = \alpha - Y/n$.

$$\begin{aligned} \mathbb{E}[y^4] =& \mathbb{E}\left[\left(\alpha - \frac{Y}{n}\right)^4\right] \\ =& \frac{\mathbb{E}\left[\left(n\alpha - Y\right)^4\right]}{n^4} \\ \overset{(a)}{=}& \frac{n\alpha(1-\alpha)\left(1 + (3n-6)\alpha(1-\alpha)\right)}{n^4} \\ =& \frac{\alpha(1-\alpha)\left(1 + (3n-6)\alpha(1-\alpha)\right)}{n^3} \end{aligned} \tag{68}$$

where (a) holds because $\mathbb{E}\left[\left(n\alpha - Y\right)^4\right]$ is the 4-th central moment of $Y \sim Binomial(n, \alpha)$.

Continue from (67), we have

$$\begin{aligned} \lim_{n\to\infty} n\mathbb{E}[f(y)] \leq& \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2} \\ &+ C\lim_{n\to\infty} \frac{\alpha(1-\alpha)\left(1 + (3n-6)\alpha(1-\alpha)\right)}{n^2} \\ =& \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2}, \end{aligned} \tag{69}$$

which proves (56) with $K_2 = \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2}$.

Since $K_1 = K_2$, we can conclude that

$$\lim_{n\to\infty} n\mathbb{E}\left[f(y)\right] = \frac{\alpha(1-\alpha)W}{(B+\alpha W)^2}. \tag{70}$$