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Graph coloring and semidefinite rank

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Abstract

This paper considers the interplay between semidefinite programming, matrix rank, and graph coloring. Karger et al. (J ACM 45(2):246–265, 1998) give a vector program in which a coloring of a graph can be encoded as a semidefinite matrix of low rank. By complementary slackness conditions of semidefinite programming, if an optimal dual solution has high rank, any optimal primal solution must have low rank. We attempt to characterize graphs for which we can show that the corresponding dual optimal solution must have rank high enough that the primal solution encodes a coloring. In the case of the original Karger, Motwani, and Sudan vector program, we show that any graph which is a k-tree has sufficiently high dual rank, and we can extract the coloring from the corresponding low-rank primal solution. We can also show that if a graph is not uniquely colorable, then no sufficiently high rank dual optimal solution can exist. This allows us to completely characterize the planar graphs for which dual optimal solutions have sufficiently high dual rank, since it is known that the uniquely colorable planar graphs are precisely the planar 3-trees. We then modify the semidefinite program to have an objective function with costs, and explore when we can create an objective function such that the optimal dual solution has sufficiently high rank. We show that it is always possible to construct such an objective function given the graph coloring. The construction of the objective function gives rise to heuristics for 4-coloring planar graphs. We enumerated all maximal planar graphs with an induced K_4 of up to 14 vertices; the heuristics successfully found a 4-coloring for 99.75% of

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them. Our research was motivated by trying to use semidefinite programming to prove the four-color theorem, which states that every planar graph can be colored with four colors. There is an intriguing connection of the Karger–Motwani–Sudan semidefinite program with the Colin de Verdière graph invariant (J Combin. Theory Ser B 50:11-21, 1990) (and a corresponding conjecture of Colin de Verdière), in which matrices that have some similarities to the dual feasible matrices of the semidefinite program must have high rank in the case that graphs are of a certain type; for instance, planar graphs have rank that would imply that the primal solution of the semidefinite program encodes a 4-coloring.

Keywords Semidefinite programming · Graph coloring · Optimization

Mathematics Subject Classification 90-06

1 Introduction

Given an undirected graph $G = (V, E)$, a *coloring* of G is an assignment of colors to the vertices V such that for each edge $(i, j) \in E$, i and j receive different colors. The *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colors used such that a coloring of G exists. The *clique number* of a graph G, denoted $\omega(G)$, is the size of the largest *clique* in the graph; a set $S \subseteq V$ of vertices is a clique if for every distinct pair $i, j \in S$, $(i, j) \in E$. It is easy to see that $\omega(G) \leq \chi(G)$. Graph colorings have been intensively studied for over a century. One of the most well-known theorems of graph theory, the *four-color theorem*, states that four colors suffice to color any planar graph G; the problem of four-coloring a planar graph can be traced back to the 1850 s, and the computer-assisted proof of the four-color theorem by Appel and Haken [2, 3] is considered a landmark in graph theory. Jensen and Toft [10] and Molloy and Reed [14] provide book-length treatments of graph coloring in general. Fritsch and Fritsch [7], Ore [15], and Wilson [18] provide book-length treatments of the four-color theorem in particular, and Robertson, Sanders, Seymour, and Thomas [16] give a simplified computer-assisted proof of the four-color theorem.

This paper considers the use of semidefinite programming in graph coloring. The connection between semidefinite programming and graph coloring was initiated by Lovász [13], who introduced the Lovász theta function, $\vartheta(\bar{G})$, which is computable via semidefinite programming; $\bar{G}$ is the complement of graph G, in which all edges of G are replaced by nonedges and vice versa. Lovász shows that $\omega(G) \leq \vartheta(\bar{G}) \leq \chi(G)$; a helpful overview of this result is given by Knuth [12].

Another use of semidefinite programming for graph coloring was introduced by Karger et al. [11] (KMS), who show how to color k-colorable graphs with $O(n^{1-3/(k+1)} \log^{1/2} n)$ colors in polynomial time using semidefinite programming, where n is the number of vertices in the graph. KMS define the following vector program, which KMS call the *strict vector chromatic number* (Definition 8.1 of [11]); the vector program can be solved via semidefinite programming:

$$\begin{aligned}
&\text{minimize } \alpha \\
&\text{subject to } v_i \cdot v_j = \alpha, \ \forall (i,j) \in E, \\
&\qquad\qquad\quad v_i \cdot v_i = 1, \ \ \forall i \in V, \\
&\qquad\qquad\quad v_i \in \mathbb{R}^n, \quad \ \forall i \in V.
\end{aligned} \tag{SVCN}$$

KMS observe that any k-colorable graph has a feasible solution to the vector program with $\alpha = -1/(k-1)$. Suppose that we let $C_1, \ldots, C_k$ be a partition of the vertices representing a feasible k-coloring, so that all the vertices in C_i are assigned the same color and for any edge (i, j), i and j are contained in different sets of the partition. KMS show how to construct k vectors $u_1, \ldots, u_k \in \mathbb{R}^n$ that span $\mathbb{R}^{k-1}$ such that $u_i \cdot u_j = -1/(k-1)$ for any $i, j, i \neq j$ and $u_i \cdot u_i = 1$. We can then give a feasible solution to the vector program above by setting $v_i = u_j$ whenever $i \in C_j$; that is, each vector u_j represents a different color class, and we set the vector v_i for all vertices i receiving the jth color to the vector u_j. It is important for the following discussion that the u_i lie in a $(k-1)$-dimensional space, so that this solution of vectors v_i also lies in a $(k-1)$-dimensional space.

KMS also observe that there is a natural connection between the strict vector chromatic number and the Lovász theta function. In particular, for the optimal value α of the vector program above, it is possible to show that $\alpha = -1/(1 - \vartheta(\bar{G}))$ (see [11, Theorem 8.2]). If a graph G has an induced k-clique K_k and is k-colorable, then by Lovász's theorem, $\vartheta(\bar{G}) = k$, and so that a feasible solution of vectors v_i as given above with $\alpha = -1/(k-1)$ is an optimal solution. It is also possible to argue directly that a graph with an induced K_k must have $\alpha \geq -1/(k-1)$, again proving that the feasible solution given above is an optimal solution. We will call the feasible solution above in which there are k distinct unit vectors $u_1, \ldots, u_k \in \mathbb{R}^n$ with span $\mathbb{R}^{k-1}$, $u_i \cdot u_j = -1/(k-1)$ for any $i, j, i \neq j$, and each vector v_i is equal to one of these u_j the *reference solution*.

The goal of this paper is to explore situations in which the reference solution is the unique optimal solution of a semidefinite program (SDP), either the SDP corresponding to the strict vector chromatic number (SVCN) given above, or another that we will give shortly. To do this, we will use complementary slackness conditions for semidefinite programs. Consider the primal and dual SDPs shown in standard form below, where the constraint that X is a positive semidefinite matrix is represented by $X \succeq 0$, and we take the outer product of matrices, so that $C \bullet X$, for instance, denotes $\sum_{i=1}^{\ell} \sum_{j=1}^{\ell} c_{ij} x_{ij}$.

$$\begin{array}{llll}
\text{minimize} & C \bullet X & \text{maximize} & b^T y \\
\text{subject to} & A_i \bullet X = b_i \text{ for } i = 1, \ldots, m, & \text{subject to} & S = C - \sum_{i=1}^{m} y_i A_i, \\
(P) & X \succeq 0, & (D) & S \succeq 0, \\
 & X \in \mathbb{R}^{\ell \times \ell}, & & S \in \mathbb{R}^{\ell \times \ell}.
\end{array}$$

Duality theory for semidefinite programs (e.g. Alizadeh [1]) shows that for any feasible primal solution X and any feasible dual solution y, $C \bullet X \geq b^T y$. Furthermore if $C \bullet X = b^T y$, so that the solutions are optimal, then it must be the case that $\text{rank}(X) + \text{rank}(S) \leq \ell$, and $XS = 0$. Thus if we want to show that any optimal primal solution has rank at most r, it suffices to show the existence of an optimal dual solution of rank at least $\ell - r$. Turning back to the strict vector chromatic number vector program, the corresponding dual vector program (from KMS [11, Theorem 8.2]) is

$$\begin{aligned} &\text{maximize } -\textstyle\sum_i v_i \cdot v_i \\ &\text{subject to } \textstyle\sum_{i \neq j} v_i \cdot v_j \geq 1, \\ &\qquad\qquad v_i \cdot v_j = 0, \qquad \forall (i, j) \notin E, i \neq j, \\ &\qquad\qquad v_i \in \mathbb{R}^n, \qquad \forall i \in V. \end{aligned} \tag{SVCND}$$

Thus, given a k-colorable graph G with an induced K_k, if we can show the existence of a dual feasible solution of objective function value $-1/(k-1)$ and rank $n - k + 1$, then we know that the primal solution must have rank at most $k - 1$.[1] We will show that for graphs in which we can prove this, the primal solution is then the reference solution, and we can then recover a coloring of the graph. We will for shorthand say that there is an optimal dual solution of *sufficiently high rank*.

Our first result is to partially characterize the set of graphs for which the reference solution is the unique optimal solution to the strict vector chromatic number vector program (SVCN). To state our results, we need to define a $(k-1)$-*tree* and what it means for a graph to be *uniquely colorable*. A $(k-1)$-tree is a graph constructed by starting with a complete graph on k vertices. We then iteratively add vertices v; for each new vertex v, we add $k - 1$ edges from v to previously added vertices such that v together with these $k - 1$ neighbors form a clique. A k-colorable graph is uniquely colorable if it has only one possible coloring up to a permutation of the colors. The $(k-1)$-tree graphs are easily shown to be uniquely colorable. In particular, we can show that if a graph is a $(k-1)$-tree, then the reference solution is the unique optimal solution to (SVCN). In the opposite direction, if a graph is not uniquely colorable, then no optimal dual solution has sufficiently high rank, and there exist optimal primal solutions that are not the reference solution and are at least k-dimensional. In the case of planar graphs with induced K_4s, these results imply a complete characterization of the graphs for which the optimal solution is the reference solution, since it is known that the uniquely 4-colorable planar graphs are exactly the planar 3-trees, also known as the Apollonian networks [6]. We argue that it is not surprising that graphs which are not uniquely k-colorable do not have the reference solution as the sole optimal solution; we show that one can find a convex combination of the two different reference solutions corresponding to the two different colorings that gives an optimal solution to (SVCN) of dimension at least k.

To get around the issue of unique colorability, we instead look for minimum-cost feasible SDP solutions. That is, given a cost matrix C, we look to find optimal solutions to the primal SDP

[1] There are some subtleties here we are glossing over in the interest of getting across the main idea. In particular, the SDP corresponding to the strict vector chromatic number vector program has dimension $n + 1$, not n; we explain why and why that doesn't matter for our purposes in Sect. 3.

$$\begin{array}{ll} \text{minimize} & C \bullet X \\ \text{subject to} & X_{ij} = -1/(k-1), \ \forall (i,j) \in E, \\ & X_{ii} = 1, \qquad\qquad\quad \forall i \in V, \\ & X \succeq 0. \end{array} \tag{CP}$$

The corresponding dual SDP is

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} y_i - \frac{2}{k-1} \sum_{e \in E} z_e \\ \text{subject to} & S = C - \sum_{i=1}^{n} y_i E_{ii} - \sum_{e \in E} z_e E_e, \\ & S \succeq 0, \end{array} \tag{CD}$$

where E_{ii} is the matrix with a 1 at position ii and 0 elsewhere and for $e = (i, j)$, E_e is the matrix with 1 at positions ij and ji and 0 elsewhere. Once again, the reference solution is a feasible solution to the primal SDP using $X_{ij} = v_i \cdot v_j$. The goal now is to find a cost matrix C such that there is an optimal dual solution of sufficiently high rank (here rank $n-k+1$), so that the reference solution is the unique optimal solution to the primal SDP. We show that it is always possible to find a cost matrix C such that there exists a dual optimal solution of sufficiently high rank. Our construction of C depends on a coloring of the graph; however, we do show that such a C exists.

Furthermore, the construction of C suggests a heuristic for finding a coloring of a graph, and we show that the heuristic works well for planar graphs. By the four-color theorem, we know that any planar graph is 4-colorable, and if a planar graph contains an induced K_4, then it requires four colors. We enumerate all maximal planar graphs of up to 14 vertices containing an induced K_4. The heuristics successfully find a 4-coloring of all graphs of up to 11 vertices, and at least 99.75% of all graphs on 12, 13, and 14 vertices. The heuristics involve repeatedly solving semidefinite programs, and thus are not practical for large graphs (although they still run in polynomial time). However, we view them as a proof of concept that it might be possible to use our framework to reliably 4-color planar graphs.

Our research was motivated by trying to use semidefinite programming to prove the four-color theorem. There is an intriguing connection between the dual semidefinite program (CD) and the Colin de Verdière graph invariant [5] (and a corresponding conjecture of Colin de Verdière), in which matrices that have some similarities to dual solutions to (CD) must have high rank in the case that graphs have certain structure; for instance, if solutions for (CD) for planar graphs met the definition of the Colin de Verdière invariant, then the dual optimal would have sufficiently high rank, and the primal solution would correspond to a 4-coloring, which would imply the four-color theorem. While we were unable to show the connection, we still think it is an intriguing direction to explore. We explain the Colin de Verdière invariant [5] and the potential connection to (CD) in Sect. 6.1 in the conclusion.

The rest of this paper is structured as follows. In Sect. 2, we give some preliminary results on semidefinite programming. In Sect. 3, we show our results for the strict vector chromatic number SDP, and show that $(k-1)$-trees imply dual solutions of sufficiently high rank, while graphs that are not uniquely colorable imply that such dual solutions cannot exist. In Sect. 4, we turn to the SDP with cost matrix C, and show that for any k-colorable graph with an induced k-clique, a cost matrix C exists that gives rise to an optimal dual solution of sufficiently high rank. In Sect. 5, we give two heuristics for coloring planar graphs based on our construction of a cost matrix C, and show a case where the heuristic fails to find a 4-coloring of a planar graph. Finally, we turn to some further thoughts and remaining open questions in Sect. 6.

2 Preliminaries

In this section, we recall some basic facts about semidefinite matrices and semidefinite programs that we will use in subsequent sections.

Recall the primal and dual semidefinite programs given in the introduction, which we labelled by (P) and (D) respectively. We always have weak duality for semidefinite programs, so that the following holds.

Fact 2.1 *Given any feasible X for (P) and y for (D), $C \bullet X \geq b^T y$.*

Thus if we can produce a feasible X for (P) and a feasible y for (D) such that $C \bullet X = b^T y$, then X must be optimal for (P) and y optimal for (D).

The following is also known, and is the semidefinite programming version of complementary slackness conditions for linear programming.

Fact 2.2 [1, Theorem 2.10, Corollary 2.11] *For optimal X for (P) and y for (D), $XS = 0$ and $rank(X) + rank(S) \leq \ell$.*

Semidefinite programs and vector programs (such as the strict vector chromatic vector program) are equivalent because a symmetric $X \in \mathbb{R}^{n \times n}$ is positive semidefinite if and only if $X = QDQ^T$ for a real matrix $Q \in \mathbb{R}^{n \times n}$ and diagonal matrix D in which the entries of D are the eigenvalues of X, and the eigenvalues are all nonnegative. We can then consider $D^{1/2}$, the diagonal matrix in which each diagonal entry is the square root of the corresponding entry of D. Then $X = (QD^{1/2})(QD^{1/2})^T$. If we let $v_i \in \mathbb{R}^n$ be the ith row of $QD^{1/2}$, then $x_{ij} = v_i \cdot v_j$, and similarly, given the vectors v_i, we can construct a semidefinite matrix X with $x_{ij} = v_i \cdot v_j$. We also make the following observation based on this decomposition.

Observation 2.3 *Given a semidefinite matrix $X = QDQ^T \in \mathbb{R}^{n \times n}$ and vectors $v_i \in \mathbb{R}^n$ with v_i the ith row of $QD^{1/2}$, $rank(X) = d$ if and only if the vectors v_i are supported on just d coordinates.*

Throughout the rest of this paper, we will refer to vector programs and semidefinite programs interchangeably, and may do so without confusion because of the equivalence given above.

Recall from the introduction that we defined the *reference solution* to be a solution to (SVCN) such that each vector v_ℓ in the solution equals one of k distinct unit vectors $u_1, \ldots, u_k \in \mathbb{R}^n$ which span $\mathbb{R}^{k-1}$ and such that $u_i \cdot u_j = -1/(k-1)$ for any $i, j, i \neq j$. We note that by Observation 2.3 that the corresponding positive semidefinite matrix $X = WW^T$ (with v_ℓ the ℓth row of W) has rank $k-1$ and that any entry $x_{ij} \in \{1, -1/(k-1)\}$ for any i, j.

Lemma 2.4 *For the reference solution $u_1, \ldots, u_k \in \mathbb{R}^n$, $u_1 + \cdots + u_k = 0$, and any collection of $k-1$ of these vectors are linearly independent.*

Proof Since $u_1, \ldots, u_k$ span $\mathbb{R}^{k-1}$, they must be linearly dependent, and $a_1u_1 + \cdots + a_ku_k = 0$ for some $a_1, \ldots a_k \in \mathbb{R}$ not all 0. Let $A = a_1 + \cdots + a_k$. Since $u_i \cdot u_j = -\frac{1}{k-1}$ for $i \neq j$, for any fixed i,

$$\begin{aligned} 0 &= u_i \cdot (a_1u_1 + \cdots + a_ku_k) \\ &= \sum_{j=1}^{k} a_j(u_i \cdot u_j) \\ &= a_i - \frac{A - a_i}{k-1}. \end{aligned}$$

Therefore, $a_i = \frac{A}{k}$ for $i = 1, \ldots, k$ and $\frac{A}{k}(u_1 + \cdots + u_k) = 0$, so $u_1 + \cdots + u_k = 0$.

It follows that for each i, $u_i = -\sum_{j \neq i} u_j$. We claim any collection of $k-1$ of the u_i are linearly independent. Suppose not, and suppose $u_1, \ldots, u_{k-1}$ are linearly dependent, so that there exist $a_1, \ldots, a_{k-1} \in \mathbb{R}$ not all 0 so that $a_1u_1 + \cdots + a_{k-1}u_{k-1} = 0$. Then following the logic above, we have that $u_1 + \cdots + u_{k-1} = 0$. But then $u_k = -\sum_{j \neq k} u_j = 0$, which contradicts the fact that u_k is a unit vector in $\mathbb{R}^n$. □

We will then say that the positive semidefinite X is the *reference solution* if it has rank at most $k-1$ and for some W with $X = WW^T$, W has exactly k distinct rows $u_1, \ldots, u_k \in \mathbb{R}^n$ with $u_i \cdot u_j = -1/(k-1)$ for any $i, j, i \neq j$. We observe that the $k \times k$ submatrix of X induced by the indices of k distinct rows of W is $\frac{1}{k-1}(kI - J)$ (where I is the identity and J the all-ones matrix), which has rank $k-1$, and therefore any such X has rank exactly $k-1$.

Given a reference solution $X = (x_{ij})$ and corresponding W, we can easily define a k-coloring of the graph as long as $x_{ij} = -1/(k-1)$ for any $(i, j) \in E$: for the rows w_i of W, and k distinct rows $u_1, \ldots, u_k$, we let $C_\ell = \{j \in V : w_j = u_\ell\}$ for $\ell \in [k]$, and we color the vertices in C_ℓ with color ℓ. Then for any edge $(i, j) \in E$, i and j must receive different colors since $x_{ij} = w_i \cdot w_j = -1/(k-1)$; that is, $w_i \neq w_j$.

3 The strict vector chromatic number SDP

Recall the strict vector chromatic number vector program given in the introduction:

$$\begin{aligned} &\text{minimize } \alpha \\ &\text{subject to } v_i \cdot v_j = \alpha, \ \forall (i, j) \in E, \\ &\qquad\qquad\quad v_i \cdot v_i = 1, \ \ \forall i \in V, \\ &\qquad\qquad\quad v_i \in \mathbb{R}^n, \qquad \forall i \in V. \end{aligned} \tag{SVCN}$$

In this section, we give a partial characterization of graphs for which the reference solution is the unique optimal solution to (SVCN).

First, we observe that (SVCN) is equivalent to the following semidefinite program:

$$
\begin{aligned}
\text{minimize } & -\beta \\
\text{subject to } & x_{ij} + \beta = 0, && \forall (i,j) \in E, \\
& x_{ii} = 1, && \forall i \in V, \\
& Z = \left[\begin{array}{c|ccc} \beta & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & X & \\ 0 & & & \end{array}\right] \succeq 0, \\
& X = (x_{ij}) \in \mathbb{R}^{n \times n}.
\end{aligned}
\tag{SVCN-P}
$$

The dual of this SDP is

$$
\begin{aligned}
\text{maximize } & -\textstyle\sum_{i \in V} s_{ii} \\
\text{subject to } & \gamma = -1 + \textstyle\sum_{i \neq j} s_{ij}, \\
& s_{ij} = 0, && \forall (i,j) \notin E, i \neq j, \\
& W = \left[\begin{array}{c|ccc} \gamma & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & S & \\ 0 & & & \end{array}\right] \succeq 0, \\
& S = (s_{ij}) \in \mathbb{R}^{n \times n}.
\end{aligned}
\tag{SVCN-D}
$$

In what follows, we will want to relate the rank of the primal submatrix $X = (x_{ij})_{i,j \in V}$ to the rank of the dual submatrix $S = (s_{ij})_{i,j \in V}$; that is, we want to look at the submatrices that don't contain the 0th row and column of the primal solution (corresponding to the variable α in (SVCN)) and the corresponding 0th row and column of the dual solution.

Lemma 3.1 *Given an optimal primal solution Z to (SVCN-P) and optimal dual solution W to (SVCN-D), we have that $rank(X) + rank(S) \leq n$.*

Proof If for optimal dual solution W, the submatrix $S = (s_{ij})_{i \in V}$ has rank at least $n - \ell$, then the optimal dual solution W has rank at least $n - \ell$. Then by Fact 2.2, any optimal primal solution Z to (SVCN-P) must have rank at most $(n+1) - (n - \ell) = \ell + 1$. Let $Z = YY^T$, and let v_i be the ith row of Y. By Observation 2.3, the dimension of the vectors v_i must be at most $\ell + 1$. But we note that by the condition that $z_{i0} = v_i \cdot v_0 = 0$ for all $i \in V$, it must be the case that all vectors v_i for $i \in V$ are orthogonal to v_0, so that the vectors v_i for $i \in V$ lie in dimension at most ℓ. Then by Observation 2.3 the rank of X is at most ℓ, giving the desired inequality.

Similarly, if the rank of X is at least ℓ, then because $z_{00} = \beta$ is positive, the rank of Z must be at least $\ell + 1$. Then by Fact 2.2, the rank of W must be at most $(n+1) - (\ell + 1) = n - \ell$, so that the rank of S is at most $n - \ell$. □

Because the values of Z and W are determined by the submatrices X and S, we will for the rest of the section refer to primal solutions X and dual solutions S.

Our main result for this section is about graphs that are *k-trees*.

Definition 3.2 A $(k-1)$-tree with n vertices is an undirected graph constructed by beginning with the complete graph on k vertices and repeatedly adding vertices in such a way that each new vertex, v, has $k-1$ neighbors that, together with v, form a k-clique.

An easy inductive argument shows that these graphs are k-colorable. Also, $(k-1)$-trees are known to be uniquely k-colorable, where uniquely colorable means every coloring produces the same vertex partitioning. Once k colors are assigned to the initial complete graph with k vertices, the color of each new vertex is uniquely determined by its $k-1$ neighbors. This partitioning into color classes is unique up to permuting the colors. Note that by construction, a $(k-1)$-tree contains a K_k (a clique on k vertices).

Recall that Karger et al. [11] show that the solution to (SVCN) is $-1/(\vartheta(\bar{G})-1)$ where $\vartheta(\bar{G})$ is the Lovász theta function. Lovász [13] proved that $\omega(G) \le \vartheta(\bar{G}) \le \chi(G)$ where $\omega(G)$ and $\chi(G)$ are the clique and chromatic numbers of G respectively. In particular, if a graph is c-colorable, the optimal solution to this vector program is at most $-1/(c-1)$. Note that as previously remarked, $(k-1)$-trees contain K_k cliques and are k-colorable. As a result, the optimal value of (SVCN) for a $(k-1)$-tree will be exactly $-1/(k-1)$.

Our goal is to show there is an optimal solution to the dual of (SVCN) with high rank. In particular, given a $(k-1)$-tree with n vertices, we show the existence of a dual optimal solution to (SVCN-D) with rank at least $n-k+1$. This ensures that any primal optimal solution has rank at most $k-1$; we show that the reference solution is the unique optimal primal solution to (SVCN). This is formalized in the following theorem.

Theorem 3.3 *Given a $(k-1)$-tree G with n vertices, there is an optimal dual solution S to (SVCN-D) with rank at least $n-k+1$, and thus any optimal primal solution X to (SVCN-P) has rank at most $k-1$.*

We subsequently prove that the reference solution is indeed the unique optimal solution in this case.

Theorem 3.4 *The reference solution is the unique optimal primal solution to (SVCN-P) for a $(k-1)$-tree.*

To prove Theorem 3.3, we need a number of supporting lemmas. We begin with the following.

Lemma 3.5 *Let $tri(G)$ denote the number of triangles and $|E(G)|$ denote the number of edges in a $(k-1)$-tree G with n vertices. Then,*

$$|E(G)| = (2n-k)\frac{k-1}{2}, \tag{1}$$

$$tri(G) = \frac{(3n-2k)(k-1)(k-2)}{6}. \tag{2}$$

Proof We first prove (1) by induction. The smallest $(k-1)$-tree is the complete graph with $n=k$ vertices. This graph has $\binom{k}{2} = \frac{k(k-1)}{2}$ edges. We also have $(2n-k)\frac{k-1}{2} = (2k-k)\frac{k-1}{2} = \frac{(k-1)k}{2}$. Assume the claim is true for all $(k-1)$-trees with at most n

vertices. If we add an $n+1st$ vertex, we are also adding $k-1$ new edges. Our new graph will have $\frac{(2n-k)(k-1)}{2}+(k-1)=\frac{(2n-k)(k-1)+2(k-1)}{2}=\frac{(2(n+1)-k)(k-1)}{2}$ edges, as desired.

To count triangles, we begin with the complete graph on k vertices again. This graph has $\binom{k}{3}=\frac{k(k-1)(k-2)}{6}$ triangles. We also have $\frac{(3n-2k)(k-1)(k-2)}{6}=\frac{(3k-2k)(k-1)(k-2)}{6}=\frac{k(k-1)(k-2)}{6}$. Assume the claim is true for all $(k-1)$-trees with at most n vertices. If we add an $n+1st$ vertex, we are also adding $\binom{k-1}{2}$ new triangles. Then this new graph has $\frac{(3n-2k)(k-1)(k-2)}{6}+\frac{(k-1)(k-2)}{2}=\frac{(3n-2k)(k-1)(k-2)+3(k-1)(k-2)}{6}=\frac{(3(n+1)-2k)(k-1)(k-2)}{6}$ triangles, as desired. □

Consider a $(k-1)$-tree G with n vertices. For $v\in V$ we denote the neighborhood of v by $N(v)=\{u:(u,v)\in E\}$. We define the following matrix $S(G)\in\mathbb{R}^{n\times n}$ which may be referred to as S if G is clear from context.

$$S(G)_{ij}=\begin{cases}\dfrac{|N(i)|-(k-2)}{k(k-1)(n-k+1)}, & i=j,\\[2ex] \dfrac{|N(i)\cap N(j)|-(k-3)}{k(k-1)(n-k+1)}, & (i,j)\in E,\\[2ex] 0, & (i,j)\notin E, i\neq j.\end{cases}$$

We will show that $S(G)$ is an optimal dual solution to (SVCN-D) with rank $n-k+1$. First, we show $S(G)$ is a feasible solution to (SVCN-D) with help from the following lemma.

Lemma 3.6 *For a $(k-1)$-tree G with n vertices, $S(G)$ is positive semidefinite.*

Proof Observe that it suffices to show $S'(G)=k(k-1)(n-k+1)S(G)$ is positive semidefinite (PSD) since $k(k-1)(n-k+1)>0$ for $n\geq k$. We proceed by induction. First consider $(k-1)$-trees with k vertices. There is only one, $G=K_k$. Furthermore, $S'(K_k)$ is equal to the all-ones matrix which has eigenvalues k and 0 with multiplicity $k-1$ and thus is PSD.

Now assume there is some integer n such that for every $(k-1)$-tree, G, with at most n vertices, $S'(G)$ is PSD. Consider a $(k-1)$-tree G with $n+1$ vertices. Since it is a $(k-1)$-tree, it can be constructed from some smaller $(k-1)$-tree G' with n vertices by adding a vertex v and $(k-1)$ edges that form a k clique with the $k-1$ neighbors. By assumption, $S'(G')$ is PSD. Let I be the set of indices of the $k-1$ neighbors of v. Then we observe that $S'(G)=T+v_{n+1}v_{n+1}^T$ where

$$T=\left[\begin{array}{ccc|c} & & & 0\\ & S'(G') & & \vdots\\ & & & 0\\ \hline 0 & \cdots & 0 & 0\end{array}\right]$$

and

$$v_{n+1}(i) = \begin{cases} 1 & i \in I \cup \{n+1\} \\ 0 & \text{otherwise} \end{cases}.$$

Then $x^T S'(G)x = x^T T x + x^T v_{n+1} v_{n+1}^T x \geq x^T v_{n+1} v_{n+1}^T x = (v_{n+1}^T x)^2 \geq 0$ where the first inequality is due to T being PSD since $S'(G')$ is PSD. □

Lemma 3.7 *For a $(k-1)$-tree G with n vertices, $S(G)$ is a feasible dual solution to (SVCN-D).*

Proof Lemma 3.6 shows that $S(G)$ is PSD. To complete this claim, we must show that the dual constraints of (SVCN-D) are satisfied. That $S(G)_{ij} = 0$ for $(i, j) \notin E$ is clear by construction. The other constraint requires $\sum_{i \neq j} S(G)_{ij} \geq 1$. For $S(G)$,

$$\begin{aligned}
\sum_{i \neq j} S(G)_{ij} &= 2 \sum_{(i,j) \in E} \frac{|N(i) \cap N(j)| - (k-3)}{k(k-1)(n-k+1)} \\
&= \frac{-2(k-3)|E(G)| + 2\sum_{(i,j)\in E} |N(i) \cap N(j)|}{k(k-1)(n-k+1)} \\
&= \frac{-2(k-3)((2n-k)\frac{k-1}{2}) + 2\sum_{v \in V} (\text{\# of triangles in } G \text{ containing } v)}{k(k-1)(n-k+1)} \\
&= \frac{-(k-3)(k-1)(2n-k) + 6tri(G)}{k(k-1)(n-k+1)} \\
&= \frac{-(k-3)(k-1)(2n-k) + 6(\frac{(3n-2k)(k-1)(k-2)}{6})}{k(k-1)(n-k+1)} \\
&= \frac{(k-1)((k-2)(3n-2k) - (k-3)(2n-k))}{k(k-1)(n-k+1)} \\
&= \frac{(k-1)(3nk - 2k^2 - 6n + 4k - 2nk + k^2 + 6n - 3k)}{k(k-1)(n-k+1)} \\
&= \frac{k(k-1)(n-k+1)}{k(k-1)(n-k+1)} = 1
\end{aligned}$$

where we use both (1) and (2) from Lemma 3.5. □

We can now show that $S(G)$ is an optimal dual solution to (SVCN-D).

Theorem 3.8 *For a $(k-1)$-tree G with n vertices, $S(G)$ is an optimal dual solution to (SVCN-D).*

Proof We remarked earlier that optimal primal solutions to (SVCN-P) for a $(k-1)$-tree have objective value $-1/(k-1)$. Thus for $S(G)$ to be an optimal dual solution, it suffices to show that $-\sum_i S(G)_{ii} = -1/(k-1)$. Again using (1) from Lemma 3.5, we have

$$
\begin{aligned}
-\sum_{i=1}^{n} S(G)_{ii} &= -\sum_{i=1}^{n} \frac{|N(i)| - (k-2)}{k(k-1)(n-k+1)} \\
&= -\frac{1}{k(k-1)(n-k+1)} \left[-(k-2)n + \sum_{i=1}^{n} |N(i)| \right] \\
&= -\frac{-(k-2)n + 2|E|}{k(k-1)(n-k+1)} \\
&= -\frac{-(k-2)n + ((2n-k)(k-1))}{k(k-1)(n-k+1)} \\
&= -\frac{-nk + 2n + 2nk - 2n - k^2 + k}{k(k-1)(n-k+1)} = -1/(k-1).
\end{aligned}
$$

□

Finally, we want to show that for a $(k-1)$-tree G with n vertices, $S(G)$ has rank at least $n-k+1$. This guarantees that any primal solution has rank at most $k-1$.

Theorem 3.9 *For a $(k-1)$-tree G with n vertices, $S(G)$ has rank at least $n-k+1$.*

Proof It again suffices to show the claim is true for $S'(G) = k(k-1)(n-k+1)S(G)$. Proceeding by induction, for $n = k$ we have $rank(S'(G)) = rank(S'(K_k)) = 1 = k-(k-1)$ with $S'(K_k)$ equal to the all-ones matrix. Assuming the claim is true for all $(k-1)$-trees with at most n vertices, we consider a $(k-1)$-tree G with $n+1$ vertices. We again use the decomposition $S'(G) = T + v_{n+1}v_{n+1}^T$ where

$$
T = \left[\begin{array}{ccc|c} & & & 0 \\ & S'(G') & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right], \quad v_{n+1}(i) = \begin{cases} 1 & i \in I \cup \{n+1\} \\ 0 & \text{otherwise} \end{cases},
$$

and G' is a $(k-1)$-tree with n vertices acquired by removing vertex $n+1$ with exactly $k-1$ neighbors, $i \in I$, from G. Note $dim(ker(T)) = dim(ker(S'(G')) + 1 \leq k$ by assumption. Now assume $x \in ker(S'(G))$. Then

$$
0 = x^T S'(G)x = x^T T x + x^T v_{n+1}v_{n+1}^T x.
$$

Since T and $v_{n+1}v_{n+1}^T$ are both PSD, this implies $x^T T x = 0$ and $x^T v_{n+1}v_{n+1}^T x = 0$. Therefore $ker(S'(G)) = ker(T) \cap ker(v_{n+1}v_{n+1}^T)$. However, note that $x = (0, \cdots, 0, 1) \in ker(T)$, but $x \notin ker(v_{n+1}v_{n+1}^T)$. Then

$$
ker(S'(G)) = ker(T) \cap ker(v_{n+1}v_{n+1}^T) \subsetneq ker(T).
$$

This implies $dim(ker(S'(G)) < dim(ker(T)) \leq k$, so $rank(S'(G)) \geq (n+1)-k+1$.

□

We can now prove Theorem 3.3.

Proof of Theorem 3.3 Theorem 3.3 follows as an immediate consequence of Lemma 3.7, Theorems 3.8, and 3.9. □

We now turn to showing that the reference solution is indeed the optimal solution to (SVCN-P) in the case of $(k-1)$-trees.

Proof of Theorem 3.4 Let $X = YY^T$ be an optimal solution to (SVCN-P) for a graph G with an induced k-clique K with rank at most $k-1$. We claim that there are only k distinct rows $w_1, \ldots, w_k$ of Y, such that, if y_i is the row of Y corresponding to vertex i, $y_i \in \{w_1, \ldots, w_k\}$ for each i and $w_\ell \cdot w_j = -\frac{1}{k-1}$ for $\ell \neq j$, so X is the reference solution, as desired.

Consider the smallest $(k-1)$-tree, the complete graph on k vertices, K_k. Clearly in any optimal solution $X = YY^T$ to (SVCN-P) for K_k with rank at most $k-1$, we have $y_i \neq y_j$ for $i \neq j$ since each y_i is a unit vector and $y_i \cdot y_j = -\frac{1}{k-1}$ for $i \neq j$ by the constraints of (SVCN-P).

Now, assume the claim is true for all $(k-1)$-trees on n vertices. Consider a $(k-1)$-tree, G, with $n+1$ vertices and an optimal primal solution $X = YY^T$ to (SVCN-P) with rank at most $k-1$. G is constructed from a $(k-1)$-tree, G', with n vertices by attaching an additional vertex v with edges to all vertices in an induced $(k-1)$-clique, K_{k-1}, of G'. Let Y_{-v} be the submatrix of Y given by removing the row y corresponding to v. Then $X_{-v} = Y_{-v}Y_{-v}^T$ is X with the row and column corresponding to v removed and is an optimal primal solution to (SVCN-P) for G' with rank at most $k-1$. By induction, we know Y_{-v} only has k distinct rows $w_1, \ldots, w_k \in \mathbb{R}^{k-1}$ with $w_i \cdot w_j = -\frac{1}{k-1}$ for $i \neq j$. It remains to show that $y \in \{w_1, \ldots, w_k\}$.

Let $x_1, \ldots, x_{k-1}$ be the rows of Y corresponding to the vertices of the K_{k-1}. They must be distinct and linearly independent: by Lemma 2.4, any collection of $k-1$ distinct w_i must be linearly independent. Therefore we may assume $\{x_1, \ldots x_{k-1}\} = \{w_1, \ldots, w_{k-1}\}$, so we know $w_i \cdot y = -1/(k-1)$ for $i = 1, \ldots, k-1$ since there is an edge from v to each of the vertices of the K_{k-1}. Furthermore, for a given i, the set of solutions to $w_i \cdot y = -\frac{1}{k-1}$ is represented by a hyperplane H_i in the $(k-1)$-dimensional vector space spanned by $\{w_1, \ldots, w_{k-1}\}$. Therefore, a satisfying vector w must lie in $H_1 \cap \cdots \cap H_{k-1}$. Because $w_1, \ldots, w_{k-1}$ are linearly independent, $dim(H_1 \cap \cdots \cap H_{k-1}) = 0$. Thus there is a unique vector that satisfies the given equations. Since w_k satisfies all $k-1$ equations, we find that $y = w_k$ as desired. Therefore, X is the reference solution. □

Theorem 3.4 shows that we can partition the vertices of a $(k-1)$-tree into k sets with each set associated to a different vector assigned in the low rank primal solution. Since vertices u, v are only in the same set in the partition if they were assigned the same vector in the primal solution, it is not possible for neighbors to be in the same set. We can then produce a valid coloring of the vertices by associating one color to each set in the partition.

We now turn to characterizing cases in which we cannot find optimal dual solutions of sufficiently high rank by looking at potential solutions of vector colorings for graphs without unique colorings. In particular, we restrict our attention to graphs that have multiple distinct k-colorings and contain an induced k-clique. These assumptions provide information about the optimal objective function values.

Theorem 3.10 *Let G be a graph with n vertices, multiple distinct k-colorings, and an induced k-clique. There exists an optimal primal solution to the strict vector chromatic number program for G with rank greater than $k-1$, and thus by Fact 2.2 the rank of any optimal dual solution must be less than $n-k+1$.*

Proof Let c_1, c_2 be functions from the vertices of G to $\{1, \ldots, k\}$ mapping each vertex to a number i corresponding to its color for two distinct k-colorings of G. Let K be an induced k-clique in G. Begin with a reference solution of rank $k-1$ and assign each color class a corresponding vector in such a way that $c_1(i) = c_2(i)$ for $i \in K$. This fixes the color labelling for the vertices in K. Then, we can represent these colorings by the PSD matrices C_1 and C_2, respectively, where $C_p(ij) = 1$ if $c_p(i) = c_p(j)$ and $C_p(ij) = -1/(k-1)$ if $c_p(i) \neq c_p(j)$ for $p = 1, 2$ and $i, j \in [n]$. Note then, for $\alpha \in (0, 1)$, $X = \alpha C_1 + (1-\alpha)C_2$ is also a valid solution to (SVCN-P). It suffices to prove that X has rank greater than $k-1$ for some $\alpha \in (0, 1)$.

Because C_1 and C_2 are PSD, for any value of $\alpha \in (0, 1)$, $ker(X) = ker(C_1) \cap ker(C_2)$. We will show there is a vector $x \in ker(C_1)$ such that $x \notin ker(C_2)$ from which the result directly follows.

Let v be a vertex whose color changes, i.e. $c_1(v) \neq c_2(v)$. Then v cannot be in K. Let $s \in K$ be such that $c_1(s) = c_1(v)$ and thus $c_2(s) \neq c_2(v)$. Let $i_1, i_2, \ldots, i_{k-2} \in K$ such that $c_2(i_j) \neq c_2(v)$ for $j = 1, \ldots k-2$. Also note that $c_1(i_j) \neq c_1(v)$ since $c_1(s) = c_1(v)$ and $(i_j, s) \in E$ for $j = 1, \ldots, k-2$. Because $dim(ker(C_1)) = n-(k-1) = n-k+1$, there exists $x \in ker(C_1)$ such that $x \neq 0$ but $x(i) = 0$ for $i \neq v, s, i_1, \ldots, i_{k-2}$. Assume $x \in ker(C_2)$; we show this leads to a contradiction. Then,

$$\begin{aligned}(C_1x)(v) &= C_1(vv)x(v) + C_1(vs)x(s) + \sum_{j=1}^{k-2} C_1(vi_j)x(i_j)\\ &= x(v) + x(s) - \frac{1}{k-1}\sum_{j=1}^{k-2} x(i_j) = 0\end{aligned}$$

and

$$\begin{aligned}(C_2x)(v) &= C_2(vv)x(v) + C_2(vs)x(s) + \sum_{j=1}^{k-2} C_2(vi_j)x(i_j)\\ &= x(v) - \frac{1}{k-1}x(s) - \frac{1}{k-1}\sum_{j=1}^{k-2} x(i_j) = 0\end{aligned}$$

from which we can conclude that $x(s) = 0$. Similarly,

$$(C_1x)(s) = C_1(sv)x(v) + \sum_{j=1}^{k-2} C_1(si_j)x(i_j) = x(v) - \frac{1}{k-1}\sum_{j=1}^{k-2} x(i_j) = 0$$

and

$$(C_2x)(s) = C_2(sv)x(v) + \sum_{j=1}^{k-2} C_2(vi_j)x(i_j) = -\frac{1}{k-1}x(v) - \frac{1}{k-1}\sum_{j=1}^{k-2} x(i_j) = 0$$

imply that $x(v) = 0$. By considering row i_j for $j = 1, \ldots, k-2$, we see that the $x(i_j)$ satisfy

$$A[x(i_1), x(i_2), \ldots, x(i_{k-2})]^T = 0$$

where A is the $(k-2) \times (k-2)$ matrix with 1 along the diagonal and $-1/(k-1)$ everywhere else. We can write A as $A = -\frac{1}{k-1}J + \frac{k}{k-1}I$ where J is the all 1 s matrix. Then A has eigenvalues $2/(k-1)$ with multiplicity 1 and $k/(k-1)$ with multiplicity $k-3$, and thus has trivial nullspace. Therefore $[x(i_1), x(i_2), \ldots, x(i_{k-2})] = 0$ which contradicts that $x \neq 0$. Then $x \notin ker(C_2)$, so $rank(X) > k-1$. □

While we have shown that $(k-1)$-trees have solutions with sufficiently high dual rank for the standard vector chromatic number SDP, it would be nice if we could completely characterize which graphs have solutions with sufficiently high dual rank. A reasonable guess would be that a k-colorable graph G containing an induced k-clique has an optimal solution with high dual rank if and only if it is uniquely colorable. This assertion is true for the important special case of planar graphs.

Corollary 3.11 *A planar graph with n vertices has an optimal solution to (SVCN-D) with rank at least $n-3$ if and only if it is uniquely colorable.*

Proof Fowler [6] shows that uniquely-colorable planar graphs are exactly the set of planar 3-trees. By Theorem 3.3 we know such graphs have dual rank at least $n-3$. Furthermore, Theorem 3.10 shows that graphs with multiple colorings have optimal primal solutions to (SVCN-P) with rank more than 3 and therefore do not have dual solutions with rank $n-3$. □

Unfortunately, the following example in Fig. 1 shows unique colorability is not sufficient in general for a sufficiently high dual rank. Hillar and Windfeldt [9, Fig. 2] presented the uniquely 3-colorable graph in Fig. 1 excluding vertex 25 which adds a triangle. Computing the primal and dual SDPs of this graph returns solutions with objective value -0.5, primal rank of 24, and dual rank of 1. If the claim were true, we would expect all dual solutions to have rank at least 23.

Thus it remains an interesting open question to characterize in general cases in which graphs have sufficiently high dual rank and have the reference solution as the optimal primal solution to (SVCN-P).

4 A semidefinite program with costs

Unfortunately, Theorem 3.10 seems to indicate that this method of looking for graphs that have high dual rank with the standard vector chromatic number SDP cannot be

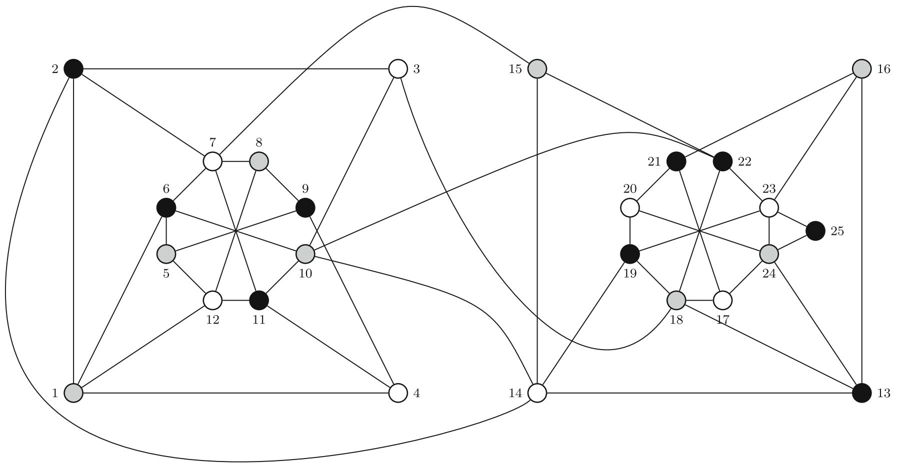

Fig. 1 Uniquely 3-colorable graph with an induced K_3 which does not have any dual optimal solution of sufficiently high dual rank

generalized to graphs with multiple colorings. To extend our method, we consider a modified SDP described next. The new program utilizes a new objective function. Here, we introduce the notion of a cost matrix $C(G)$. The goal is to identify a $C(G)$ such that minimizing $C(G) \bullet X$ forces X to have our desired rank. In particular, we consider the SDP given by

$$\begin{array}{ll} \text{minimize} & C(G) \bullet X \\ \text{subject to} & X_{ij} = -1/(k-1), \ \forall (i,j) \in E, \\ & X_{ii} = 1, \qquad\qquad\quad \forall i \in V, \\ & X \succeq 0. \end{array} \tag{CP}$$

We observe that the solutions to (SVCN) with $\alpha = -1/(k-1)$ are exactly the feasible solutions to (CP). The corresponding dual SDP is

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} y_i - \frac{2}{k-1} \sum_{e \in E} z_e \\ \text{subject to} & S = C - \sum_{i=1}^{n} y_i E_{ii} - \sum_{e \in E} z_e E_e, \\ & S \succeq 0 \end{array} \tag{CD}$$

where E_{ii} is the matrix with a 1 at position ii and 0 elsewhere and for $e = (i, j)$, E_e is the matrix with 1 at positions ij and ji and 0 elsewhere.

To demonstrate how this cost matrix influences the behavior of $rank(X)$, assume that $G = (V, E)$ is a k-colorable graph with an induced k-clique, but is not a $(k-1)$-tree. We still know there is a solution to the strict vector chromatic number program with $\alpha = -1/(k-1)$, and thus it is possible to find an X satisfying our modified vector program constraints. Now fix $c : V \to [k]$ to be a valid k-coloring of G. With this coloring, we can define an associated matrix $C(G)$ in the following way:

$$C(G)_{ij} = \begin{cases} -1 & i < j, c(i) = c(j), \forall \ell \text{ such that } i < \ell < j, c(i) \neq c(\ell) \\ -1 & i > j, c(i) = c(j), \forall \ell \text{ such that } i > \ell > j, c(i) \neq c(\ell) \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, the reference solution corresponding to the coloring given by c is the solution that will minimize total cost since we'll look for a solution X with $X_{ij} = 1$ exactly when $C(G)_{ij} = -1$; for such entries, we'll have the same vectors corresponding to vertices i and j. But we can show additionally that there is a dual optimal solution for cost matrix $C(G)$ that has sufficiently high rank.

Theorem 4.1 *For G and $C(G)$ as described, there is an optimal solution to (CD) with rank at least $n - k + 1$, so that any optimal solution to (CP) has rank at most $k - 1$.*

Let K be an induced k-clique in our k-colorable graph G. Let s_i denote the sum of entries in column i of $C(G)$. Consider the assignment of dual variables given by $y_i = s_i$ for $i \notin K$, $y_i = s_i - 1$ for $i \in K$, $z_e = -1$ for $e = (i, j), i, j \in K, i \neq j$, and $z_e = 0$ otherwise. We denote this assignment by (y, z).

Lemma 4.2 *The dual matrix S constructed with (y, z) is positive semidefinite.*

Proof Consider the complete graph on k vertices given by $G = K_k$ (as this is the smallest possible k-colorable graph containing an induced k-clique). Observe that $C(K_k)$ is the matrix of all 0 s as no two vertices can be colored the same. Thus $s_i = 0$ for all $i \in K_k$. Furthermore, (y, z) assigns $y_i = -1$ for all $i \in K_k$ and $z_e = -1$ for all $e \in K_k$. Then S is the all-ones matrix with eigenvalues k and 0 with multiplicity $k - 1$, and thus is positive semidefinite.

Now assume the claim is true for all k-colorable graphs containing an induced k-clique that have at most n vertices. Let $G = (V, E)$ such that $|V| = n + 1$, G is k-colorable, and G has an induced k-clique, K. Then G can be constructed by adding a vertex v_{n+1} and its adjacent edges to some graph $G' = (V', E')$ such that $|V'| = n$, G' is k-colorable, and G' contains K. By assumption, the matrix S' corresponding to G', $\{y'_i\}_{i=1}^{n}$, and $\{z'_e\}_{e\in E'}$ is positive semidefinite.

Let v_m be the largest-indexed vertex that is the same color as v_{n+1} after it is added to G. We consider how the addition of v_{n+1} affects $C(G')$, $\{y_i\}_{i=1}^{n+1}$, and $\{z_e\}_{e\in E}$. For $i, j \neq n+1$, $C(G)_{ij} = C(G')_{ij}$. We also observe $C(G)_{m(n+1)} = C(G)_{(n+1)m} = -1$ and $C(G)_{i(n+1)} = C(G)_{(n+1)i} = 0$ for $i \neq m$. Furthermore, for $i \neq m, n+1$, $y_i = y'_i$, while $y_m = y'_m - 1$ and $y_{n+1} = -1$. Finally, $z_e = -1$ for $e \in K$ and 0 otherwise.

With this update, we see that

$$S(G) = \left[\begin{array}{ccc|c} & & & 0 \\ & S'(G') & & \vdots \\ & & & 0 \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right] + vv^T$$

where $v^T = [0, \dots, 0, -1, 0, \dots, 0, 1]$. It follows that $S(G)$ is positive semidefinite since $S'(G')$ is PSD by assumption and for all $x \in \mathbb{R}^{n+1}$, $x^T vv^T x = (v^T x)^2 \geq 0$. □

Theorem 4.3 *The assignment* (y, z) *is an optimal dual solution to (CD), and the reference solution is an optimal primal solution to (CP).*

Proof The previous lemma shows (y, z) satisfies the constraints. Thus to prove this claim, it suffices to show (y, z) maximizes the objective function. We demonstrate this by showing that the reference solution has the same objective function value.

First, we consider the dual objective function value of (CD) for (y, z). We have

$$\begin{aligned}\sum_{i=1}^{n} y_i - \frac{2}{k-1}\sum_{e\in E} z_e &= \left(\sum_{i=1}^{n} s_i\right) - k + \frac{2}{k-1}\binom{k}{2}\\ &= \left(\sum_{1\leq i,j\leq n} C(G)_{ij}\right) - k + k = \sum_{1\leq i,j\leq n} C(G)_{ij}.\end{aligned}$$

Now, let X be the matrix given by the reference solution:

$$X_{ij} = \begin{cases} 1 & c(i) = c(j) \\ -\frac{1}{k-1} & c(i) \neq c(j) \end{cases}$$

where $c : V \to [k]$ is the fixed coloring used to generate $C(G)$. Note that X satisfies the constraints of the primal SDP (CP). The objective function value is given by

$$\begin{aligned}C(G) \bullet X &= \sum_{i,j:c(i)=c(j)} C(G)_{ij} - \frac{1}{k-1}\sum_{i,j:c(i)\neq c(j)} C(G)_{ij}\\ &= \sum_{i,j:c(i)=c(j)} C(G)_{ij} = \sum_{1\leq i,j,\leq n} C(G)_{ij}.\end{aligned}$$

Since the primal and dual objective function values are equal, the corresponding solutions must be optimal. □

Finally, we restate and prove Theorem 4.1.

Theorem 4.1. *For G and C(G) as described, there is an optimal solution to (CD) with rank at least n-k+1, so that any optimal solution to (CP) has rank at most k-1.*

Proof Again begin by considering the complete graph on k vertices given by K_k. As previously discussed, the matrix S determined by (y, z) is simply the all-ones matrix. It is straightforward to see this has rank $1 = k - k + 1$.

Assume the claim holds for all k-colorable graphs with an induced k-clique and at most n vertices. Let G be a k-colorable graph with an induced k-clique and $n + 1$ vertices. Following the same decomposition used previously, we can write

$$S(G) = T + vv^T$$

where

$$v(i) = \begin{cases} -1 & i = m \\ 1 & i = n+1 \\ 0 & otherwise \end{cases}$$

for v_m the largest-valued vertex colored the same as v_{n+1} and

$$T = \left[\begin{array}{c|c} & 0 \\ S'(G') & \vdots \\ & 0 \\ \hline 0 \quad \cdots \quad 0 & 0 \end{array}\right].$$

Since $S(G)$ is a sum of positive semidefinite matrices, $ker(S(G)) = ker(T) \cap ker(vv^T)$. Observe that $x = (0, \ldots, 0, 1) \in ker(T)$. However, $x \notin ker(vv^T)$. Thus $dim(ker(S(G))) = dim(ker(T) \cap ker(vv^T)) < dim(ker(T)) \leq k$ as desired. □

Theorem 4.4 *For G and $C(G)$ as described, the reference solution is the unique optimal primal solution to (CP).*

Proof Theorem 4.3 tells us that the reference solution is an optimal primal solution to (CP), while Theorem 4.1 tells us that any optimal primal solution to (CP) has rank at most $k-1$. Therefore it suffices to show that any rank $k-1$ optimal primal solution to (CP) is in fact the reference solution.

From the proof of Theorem 4.3, we know that the optimal objective function value is $\sum_{1 \leq i,j \leq n} C(G)_{ij}$. Furthermore, any primal feasible X satisfying $C(G) \bullet X = \sum_{1 \leq i,j \leq n} C(G)_{ij}$ must have $X_{ij} = 1$ whenever $C(G)_{ij} = -1$ since $X_{ij} \leq 1$ for all $1 \leq i, j \leq n$ and each entry of $C(G)$ is either 0 or -1. Let K be an induced K_k in G, $X = YY^T$ be a rank $k-1$ primal optimal solution to (CP) and $c : V \rightarrow [k]$ be the k-coloring used to construct $C(G)$. Define k sets by $S_i = \{v \in V : c(v) = i\}$ for $i = 1, \ldots, k$ and let $n_i = |S_i|$.

We claim that for each $i = 1, \ldots, k$ the rows of Y corresponding to the vertices in S_i are the same; in particular, there are only k distinct rows of Y. For each S_i, sort the vertices in S_i from smallest label to largest so that $S_i = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{n_i}}\}$ where $i_1 < i_2 < \cdots < i_{n_i}$. By construction of $C(G)$, we have that $C_{i_1 i_2} = C_{i_2 i_3} = \cdots = C_{i_{n_i - 1} i_{n_i}} = -1$ implying $X_{i_1 i_2} = X_{i_2 i_3} = \cdots = X_{i_{n_i - 1} i_{n_i}} = 1$, so that the rows of Y corresponding to $v_{i_1}, v_{i_2}, \ldots, v_{i_{n_i}}$ are all the same vector which we call w_i. We know that there cannot be multiple vertices of K in any S_i, so exactly one member of K must be in each S_i for $i = 1, \ldots, k$. Therefore, $w_i \cdot w_j = -1/(k-1)$ for all $i, j, i \neq j$, and this is the reference solution. □

5 Experimental results

Our results above show that we can find a cost matrix C such that the corresponding dual SDP has an optimal solution of sufficiently high rank given that we know the

Table 1 This table depicts the number of times the heuristic algorithms failed on maximally planar graphs with between 5 and 14 vertices

# Nodes	# Maximally planar graphs with K_4	# Heuristic 1 failures	# Heuristic 2 failures
5	1	0	0
6	1	0	0
7	4	0	0
8	12	0	0
9	45	0	0
10	222	0	0
11	1219	0	0
12	7485	18 ($\sim$.24%)	18 ($\sim$.24%)
13	49,149	108 ($\sim$.22%)	116 ($\sim$.24%)
14	337,849	619 ($\sim$.18%)	811 ($\sim$.24%)

coloring. In order to understand if it is possible to construct the cost matrix C without knowing the coloring in advance, especially in the case of planar graphs, we turned to implementing heuristics to find 4-colorings for planar graphs. Note that an algorithm that could provably find a 4-coloring for planar graphs would give an alternate proof to the four-color theorem (assuming the algorithm did not itself rely on the current proof of the four-color theorem).

Two heuristics have been implemented and experimentally demonstrated success returning low-rank primal solutions to (CP) for planar graphs. Neither algorithm assumes knowledge of a graph coloring. We tested these heuristics on all maximal planar graphs of up to 14 vertices with an induced K_4; note that for such graphs a 4-coloring exists and at least 4 colors are required. These graphs were generated via the planar graph generator plantri due to Brinkmann and McKay [4] found at https://users.cecs.anu.edu.au/~bdm/plantri/. The '-a' switch was used to produce graphs written in ascii format. The code was implemented in Python using the MOSEK Optimizer as the SDP solver. Both the graph data files and algorithm implementation can be found at https://github.com/rmirka/four-coloring.git. Our results are shown in Table 1. The heuristics successfully found a 4-coloring for all graphs with up to 11 vertices, and successfully found a 4-coloring for 99.75% of the graphs of 12–14 vertices. We do not record the running time of the heuristics; because the heuristics involve repeatedly solving semidefinite programs, they are not competitive with other greedy or local search style heuristics. Our primary reason for studying these heuristics was to find whether we could reliably find a cost matrix C giving rise to a 4-coloring for planar graphs.

At a high-level, both heuristics follow the same procedure. At each step, they solve the vector program (CP) given in Sect. 4. If the returned solution does not have the desired rank, the cost matrix C is updated and the process is repeated. The heuristics differ in how the cost matrix is updated.

Before presenting the details of the heuristics, we provide some intuition for them. Clearly, in any 4-coloring of a graph with an induced K_4, each of the vertices in the

K_4 must be assigned a different color. This also means the 4 vectors assigned to the vertices of the K_4 in any solution to the vector programming formulation of (CP) must be unique and motivates our desire to use these 4 unique vectors to define color classes. For any vertex i that is not part of the K_4, we consider it to be in one of the four color classes if and only if $X_{ij} = 1$ for some j in the K_4. In this case, i and j must be assigned the same vector in the solution and therefore are members of the same color class. We consider vertices unassigned a color class to be *uncolored*. Furthermore, if $X_{ij} = -1/3$ for some j in the K_4, we assume i and j should not be in the same color class, even if i is currently uncolored. By this notion of color classes, it is not guaranteed that all vertices will be assigned to a color class in every optimal solution to (CP). In order to work towards obtaining the reference solution and extracting a 4-coloring, we design cost matrices that encourage the SDP to assign at least one currently uncolored vertex v to a color class in the next iteration, while also maintaining the existing color classes. We now proceed with describing the heuristics.

The first heuristic (Algorithm 1) is based on the coloring-dependent cost matrix discussed in Sect. 4. The algorithm first identifies an induced $K_4 = \{k_i\}_{i=1}^4$ and finds an initial solution with $C = 0$. If the primal solution does not have low enough rank, the returned solution is used to update the cost matrix. Let $S_i = \{v \in V : X_{vk_i} = 1\}$ for $i = 1, 2, 3, 4$; these S_i represent the current color classes, but there will be some vertices not contained in any. Let v be a vertex in $V \backslash (\cup_{i=1}^4 S_i)$. Then there must exist $i^* \in \{1, 2, 3, 4\}$ such that $X_{vk_{i^*}} \neq 1$ and $X_{vk_{i^*}} \neq -1/3$. This indicates that the color class of k_{i^*} is a candidate color class for v, so we update S_{i^*} by adding v to it. Now, C is constructed based on the S_i, $i = 1, 2, 3, 4$. In particular, for $i = 1, 2, 3, 4$, if n_i denotes the number of vertices in S_i, then for $j = 1, \dots, n_i - 1$, we set $C_{rs} = C_{sr} = -1$ where r and s are the jth and $j + 1$st vertices in S_i. This new cost matrix C is used to compute an updated solution $\hat{X}$. Note that because v was added to S_{i^*}, there will be some v' already in the color class of k_{i^*} such that $C_{v'v} = C_{vv'} = -1$. Based on the objective function of (CP), this entry encourages the next solution to have $X_{vv'} = 1$ if possible, which would successfully color v (and hopefully more vertices). If $\hat{X}$ is of the desired rank, the algorithm terminates. If not, we first check to see if $\hat{X}_{vk_{i^*}} = 1$, i.e. if our selected vertex from the previous iteration was successfully colored. If yes, we repeat the process beginning with our solution $\hat{X}$ and selecting a currently uncolored vertex to try and assign a color to. If v was not successfully colored, we remove the entry in the cost matrix corresponding to this assignment from the previous iteration and resolve the SDP while adding k_{i^*} to a list of 'bad' colors for v; in particular, we no longer consider the color class of k_{i^*} to be a feasible color for v. We now repeat the process by selecting a new feasible color class for v (following the same rules as previously in addition to requiring it not be in the list of 'bad' colors for v) and constructing S_i, $i = 1, 2, 3, 4$ and C accordingly.

The second heuristic (Algorithm 2) is motivated by similar ideas but distinct cost-matrix updates. Again, the algorithm first identifies an induced $K_4 = \{k_i\}_{i=1}^4$ and finds an initial solution with $C = 0$. Now, if X is a primal solution with greater rank than desired, let $S = \{v \in V : \exists k \in K_4 \text{ such that } X_{vk} = 1\}$. Intuitively, S is the set of vertices that are currently assigned to some color class. Now, choose a single vertex $v \in V \setminus S$. Again, there must exist $k \in K_4$ such that $X_{vk} \neq 1$ or $-1/3$ meaning the color class of k is a candidate color class for v, so C is updated such

Algorithm 1:

```
input  : Planar G = (V, E) with a K_4
output: X ⪰ 0 which can be used to extract a coloring of G
Find a clique K = {k_1, k_2, k_3, k_4} ;
C = 0;
i = 0, j = 1;
badcolors = [];
good = False;
/* call the Mosek optimizer to solve the SDP on graph G with cost
   matrix C and return the primal and dual matrices (X, S,
   respectively) and ranks(r, p, respectively)                    */
X, S, r, p = solveModified(G, C);
while r > 3 do
    if X_{ik_j} ≠ 1 and good then
        /* Undo our previous cost matrix assignment if the candidate
           color class for i didn't work and remove the color class
           from future options for i                                */
        badcolors = badcolors ∪ {k_j};
        C_{iS_j[length(S_j)−2]} = C_{S_j[length(S_j)−2]i} = 0;
        X, S, r, p = solveModified(G, C);
        good = False
    else
        /* Search for a currently uncolored vertex and a candidate
           color class then update the cost matrix                  */
        good = True;
        S_s = {v ∈ V : X_{vk_s} = 1}, s = 1, 2, 3, 4;
        colored = ∪_{s=1}^4 S_s;
        stillLooking = True;
        while stillLooking do
            for q = 1, ..., 4 do
                if stillLooking and i ∉ colored and k_q ∉ badcolors and X_{ik_q} ≠ 1, −1/3 then
                    S_q = S_q ∪ {i};
                    stillLooking = False;
                    j = q;
                end
            end
            if stillLooking then
                i = i + 1  mod n;
                badcolors = []
            end
        end
        C = 0;
        for q = 1, ..., 4 do
            for s = 0, ..., length(S_q) − 2 do
                C_{S_q[s],S_q[s+1]} = C_{S_q[s+1],S_q[s]} = −1;
            end
        end
        X, S, r, p = solveModified(G, C)
    end
end
return X
```

Algorithm 2:

input : Planar $G = (V, E)$ with a K_4
output: $X \succeq 0$ which can be used to extract a coloring of G
Find a clique $K = \{k_1, k_2, k_3, k_4\}$;
$C = 0$;
/* call the Mosek optimizer to solve the SDP on graph G with cost matrix C and return the primal and dual matrices $(X, S$, respectively) and ranks$(r, p$, respectively) */
$X, S, r, p = solveModified(G, C)$;
$v^* = k^* = k_1$;
$badcolors = []$;
$good = False$;
while $r > 3$ **do**
  if $X_{v^*,k^*} \neq 1$ *and good* **then**
    /* Undo our previous cost matrix assignment if the candidate color class for v^* didn't work and remove the color class from future options for v^* */
    $C_{v^*,k^*} = C_{k^*,v^*} = 0$;
    $X, S, r, p = solveModified(G, C)$;
    $badcolors = badcolors \cup \{k^*\}$;
    $good = False$;
  else
    /* Search for a currently uncolored vertex and a candidate color class then update the cost matrix */
    $good = True$;
    $S_s = \{v \in V : X_{vk_s} = 1\}, s = 1, 2, 3, 4$;
    $colored = \cup_{s=1}^{4} S_s$;
    $stillLooking = True$;
    while *stillLooking* **do**
      for $q = 1, \ldots, 4$ **do**
        if $stillLooking$ *and* $v^* \notin colored$ *and* $k_q \notin badcolors$ *and* $X_{v^* k_q} \neq 1, -1/3$ **then**
          $C_{v^* k_q} = C_{k_q v^*} = -1$;
          stillLooking = False;
          $k^* = k_q$;
        end
      end
      if $stillLooking$ **then**
        $v^* = v^* + 1 \mod n$;
        $badcolors = []$
      end
    end
    $X, S, r, p = solveModified(G, C)$
  end
end
return X

that $C_{vk} = C_{kv} = -1$. Now the vector program is run again, and the value of X_{vk} in the new solution is immediately checked. If $X_{vk} = 1$ now, the algorithm proceeds as usual. However if $X_{vk} \neq 1$, the cost matrix is updated so that $C_{vk} = 0$ again and a different entry is chosen to update. In particular, if there is another k' in the K_4 whose color class has not been ruled out for v, we try setting $C_{vk'} = -1$, otherwise we find a different uncolored vertex v' and a k^* in the K_4 corresponding to a candidate color

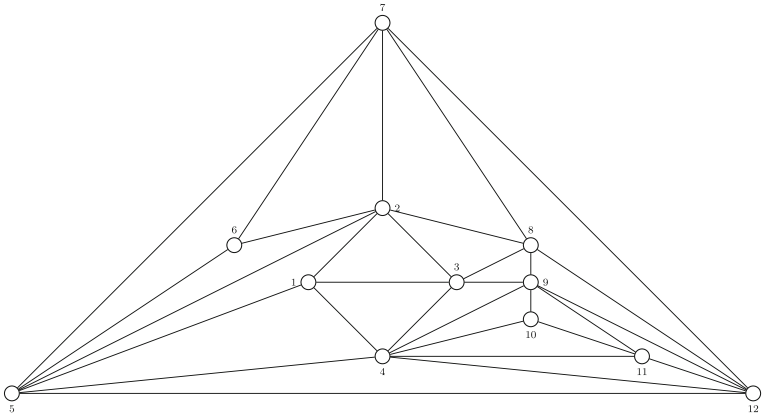

Fig. 2 Algorithm obstacle: $K_4 = \{2, 5, 6, 7\}$

class for v' and set $C_{v'k^*} = -1$. Again, this process is repeated until the desired primal rank is achieved.

In both heuristics, the success termination condition is that the primal rank is equal to 3, but this doesn't necessarily guarantee that the dual rank is $n - 3$. If instead one wanted to guarantee high dual rank, one could run the algorithm one more time, i.e. once the low-rank primal solution is achieved, extract the coloring and construct the corresponding C matrix as previously described in Theorem 4.1.

The example in Fig. 2 causes both heuristics to fail without coloring the graph. First we note the induced $K_4 = \{2, 5, 6, 7\}$. In the first iteration of the heuristic, these are the only four vertices that are assigned colors. In the second iteration, both heuristics successfully color vertex 1 to match vertex 6. However, afterwards each heuristic is unable to color any more vertices (it tries and fails on all other possible colors for the remaining vertices).

We considered whether our heuristics get stuck on graphs that also contained vertices resulting in irrevocable Kempe chain tangles. Irrevocable Kempe chain tangles occur when Kempe's local-search method of recoloring Kempe chains fails to make a color available for the vertex of interest; see Gethner et al. [8] for a computational and empirical analysis of Kempe's method and irrevocable Kempe chain tangles. As such, finding an irrevocable Kempe chain tangle in a graph indicates that Kempe's method will fail to color the graph. We tested two graphs known to contain vertices that often result in irrevocable Kempe chain tangles and slightly modified them to ensure they contained an induced K_4. The graphs are given in Figs. 3 and 4. Our heuristics did successfully color these graphs, indicating that the class of graphs for which our algorithm does not terminate is different than the ones for which coloring with Kempe chains does not work.

6 Further thoughts, open questions, and conclusions

In this section, we explain how our work is motivated by the Colin de Verdière graph parameter, give a possible strengthening of our results, and conclude by posing some open questions.

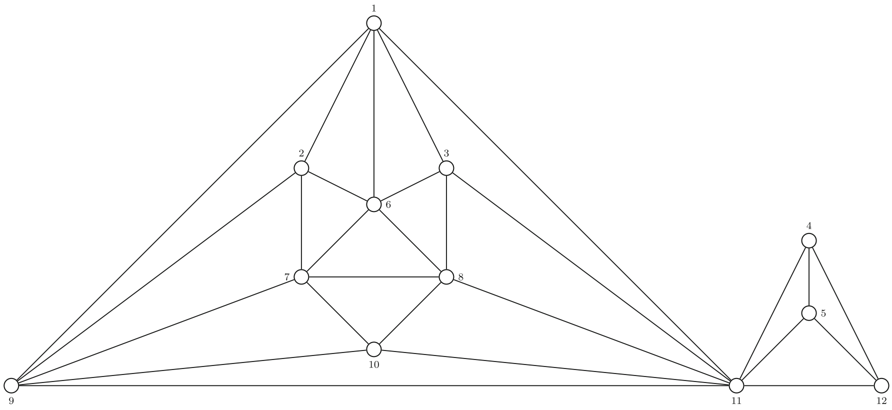

Fig. 3 A graph for which at least one vertex results in an irrevocable Kempe chain tangle for at least one labeling

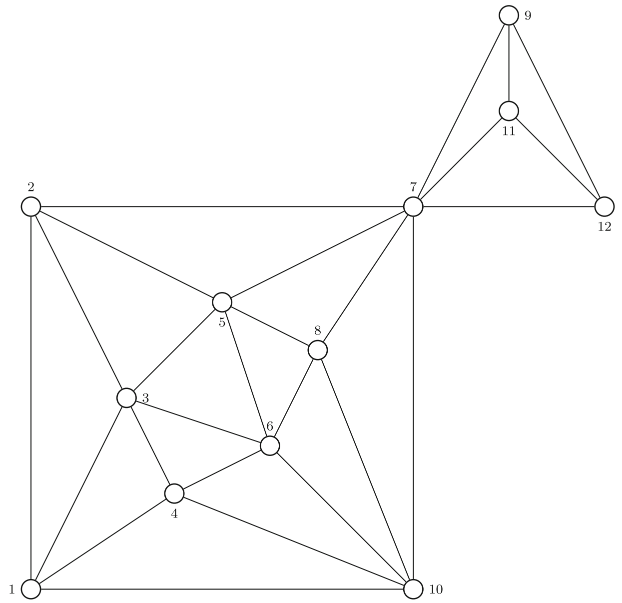

Fig. 4 A second graph for which at least one vertex results in an irrevocable Kempe chain tangle for at least one labeling

6.1 Connections with the Colin de Verdière graph parameter

As mentioned in the introduction, the research in this paper was prompted by an attempt to prove the four-color theorem and a connection between the Colin de Verdière graph parameter [5] (see also [17] for a useful survey of the invariant) and the dual semidefinite program (CD). A *generalized Laplacian* $L = (\ell_{ij})$ of graph G is a matrix such that the entries $\ell_{ij} < 0$ when $(i, j) \in E$, and $\ell_{ij} = 0$ when $(i, j) \notin E$ and $i \neq j$. The Colin de Verdière invariant, $\mu(G)$, is defined as follows.

Definition 6.1 The Colin de Verdière invariant $\mu(G)$ is the largest corank of a generalized Laplacian L of G such that:

1. L has exactly one negative eigenvalue of multiplicity one;
2. there is no nonzero matrix $X = (x_{ij})$ such that $LX = 0$ and such that $x_{ij} = 0$ whenever $i = j$ or $\ell_{ij} \neq 0$.

Colin de Verdière shows that $\mu(G) \leq 3$ if and only if G is planar; in other words, *any* generalized Laplacian of G with exactly one negative eigenvalue of multiplicity 1 will have rank at least $n - 3$ (modulo the second condition on the invariant, which we will ignore for the moment). Other results show that G is outerplanar if and only if $\mu(G) \leq 2$, and G is a collection of paths if and only if $\mu(G) \leq 1$. Colin de Verdière [5] conjectures that $\chi(G) \leq \mu(G) + 1$; this result is known to hold for $\mu(G) \leq 4$.

We note if G is planar that the part of the dual matrix for (CD), $S - C = -\sum_{i=1}^{n} y_i E_{ii} - \sum_{e \in E} z_e E_e$ is indeed a generalized Laplacian L of a planar graph when the $z_e \geq 0$ for all $e \in E$, and that if the set of edges e for which $z_e > 0$ is connected, then the y_i can be adjusted so that this matrix has a single negative eigenvalue of multiplicity one. Thus $S - C$, under these conditions, must have sufficiently high rank, as desired to verify that the optimal primal solution to (CP) is the reference solution. This would show that if the graph G has a clique on $\mu(G) + 1$ vertices, then indeed $\chi(G) = \mu(G) + 1$. So, for example, this would prove that any planar graph with an induced K_4 can be four-colored, leading to a non-computer assisted proof of the four-color theorem. However, we do not know how to find the corresponding cost matrix C or show that the dual S we find is optimal; indeed, for the dual we construct in Sect. 4, $z_e < 0$. Still, we view our heuristics as a step towards finding a way to construct the cost matrix C without knowledge of the coloring, and without reliance on the machinery of the proofs of the four-color theorem that have been developed thus far.

6.2 Coloring-independent cost matrix

The method given in Sect. 4 has a significant impediment. The C matrix defined previously assumes knowledge of a coloring for a graph. Ideally, for this method to have greater impact, we would like to find a definition of $C(G)$ and a corresponding dual assignment (y, z) based solely on the structure of an input graph and independent of a specific coloring, but still requiring the optimal primal solution to (CP) to be our desired low-rank solution.

Fortunately, there is a formal way of thinking about what any possible $C(G)$ must look like. Let us again assume for a moment that G is a k-colorable graph with an induced k-clique and define X based on a specific coloring, c, of G as described above. Then if S is an optimal dual solution and X is an optimal primal solution, $XS = 0$. Then for any $i, j \in [n]$, $\sum_{p=1}^{n} X_{ip} S_{pj} = 0$. If X is the reference solution for the coloring c, this implies $\sum_{p=1}^{n} X_{ip} S_{pj} = \sum_{p:c(p)=c(i)} S_{pj} - \frac{1}{k-1} \sum_{p:c(p)\neq c(i)} S_{pj}$ $= 0$. In particular, let $r_1, \ldots r_k$ be representatives of the k color classes. Since the above is true for any i, j, fixing j and iterating through $i = r_1, r_2, \ldots r_k$ shows $\sum_{p:c(p)=c(r_1)} S_{pj} = \cdots = \sum_{p:c(p)=c(r_k)} S_{pj}$.

This is slightly problematic as it seems to indicate either $C(G)$ or (y, z) will require knowledge of a specific coloring to guarantee this relationship. However, we can at least say something using the fact that a graph with an induced k-clique must use k different colors for these vertices alone. We show below a cost matrix C and an optimal dual solution for which any feasible primal solution (including the reference solution) is optimal for (CP).

For a graph G with an induced k-clique, consider $C(G)$ given by $C(G)_{ij} = 1$ if $i = j$ or $(i, j) \in E$ and 0 otherwise. Denote the number of induced K_ks in G by $\mathcal{K}$, the number of induced K_ks containing $i \in V$ by k_i, and the number of induced K_ks containing $(i, j) \in E$ by k_{ij}. Finally, for the assignment (y, z), set $y_i = C(G)_{ii} - k_i$ and $z_{ij} = C(G)_{ij} - k_{ij}$.

Recall that the dual matrix S is given by $S = C(G) - \sum_{i\in V} y_i E_{ii} - \sum_{e=(i,j)\in E} z_{ij} E_e$. Thus using (y, z), $S_{ij} = k_{ij}$ for $i \neq j$ while $S_{ii} = k_i$.

Lemma 6.2 *The matrix S obtained using the assignment (y, z) is positive semidefinite.*

Proof Assume G has only one induced K_k composed from the vertices $v_1, \ldots, v_k$. Then S has one eigenvalue of k with corresponding eigenvector $x_k(i) = 1$ if $i \in \{v_1, \ldots, v_k\}$ and 0 otherwise. S also has 0 as an eigenvalue with multiplicity $n - 1$ corresponding to $n - k$ elementary unit vectors $\{e_i : i \notin K_k\}$ and $k - 1$ basis vectors for the set $\{x : x_{v_1} + \cdots + x_{v_k} = 0\}$. Therefore S is PSD. Now if G contains p induced K_ks, we can write S as a sum of p PSD matrices where each corresponds to one of G's induced K_ks. Thus S is PSD. □

Lemma 6.3 *The matrix S obtained using the assignment (y, z) is optimal for (CD).*

Proof Consider the dual objective function of (CD). We have

$$\begin{aligned}
\sum_{i\in V} y_i - \frac{2}{k-1}\sum_{(i,j)\in E} z_{ij} &= \sum_{i\in V}(C(G)_{ii} - k_i) - \frac{2}{k-1}\sum_{(i,j)\in E}(C(G)_{ij} - k_{ij}) \\
&= \left(\sum_{i\in V} C(G)_{ii}\right) - k\mathcal{K} - \left(\frac{2}{k-1}\sum_{(i,j)\in E} C(G)_{ij}\right) + \frac{2}{k-1}\binom{k}{2}\mathcal{K} \\
&= \sum_{i\in V} C(G)_{ii} - \frac{2}{k-1}\sum_{(i,j)\in E} C(G)_{ij} \\
&= C(G) \bullet X
\end{aligned}$$

for any primal feasible X. □

6.3 Open questions

We close with several open questions. We were unable to give a complete characterization of the k-colorable graphs with an induced K_k for which the strict vector chromatic number (SVCN) has a unique primal solution of the reference solution. Such graphs must be uniquely colorable, but clearly some further restriction is needed.

When we know the coloring, we can produce a cost matrix C for the semidefinite program (CP) such that the reference solution is the unique optimal solution and it must have rank $k-1$. We wondered whether one could use (CP) in a greedy coloring scheme, by incrementally constructing the matrix C; the graph in Fig. 2 shows that our desired scheme does not work in a straightforward manner. Possibly one could consider an algorithm with a limited amount of backtracking, as long as one could show that the algorithm continued to make progress against some metric.

Another open question is whether one can somehow directly produce a cost matrix C leading to a dual solution of sufficiently high rank that does not need knowledge of the coloring. And we conclude with the open question that first motivated this work: is it possible to use the Colin de Verdière parameter to produce this matrix C?

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