

The Circlet Inequalities: A New, Circulant-Based, Facet-Defining Inequality for the TSP

Samuel C. Gutekunst,^{a,*} David P. Williamson^b

^aDepartments of Computer Science and of Mathematics, Bucknell University, Lewisburg, Pennsylvania 17837; ^bSchool of Operations Research and Information Engineering, Cornell University, Ithaca, New York 14850

*Corresponding author

Contact: sg040@bucknell.edu,  <https://orcid.org/0000-0002-3516-2996> (SCG); dpw@cs.cornell.edu (DPW)

Received: December 22, 2020

Revised: December 2, 2021

Accepted: February 12, 2022

Published Online in Articles in Advance:
July 12, 2022

MSC2020 Subject Classification: Primary:
90C27; secondary: 90C10, 05C45, 90C35

<https://doi.org/10.1287/moor.2022.1265>

Copyright: © 2022 INFORMS

Abstract. Facet-defining inequalities of the symmetric traveling salesman problem (TSP) polytope play a prominent role in both polyhedral TSP research and state-of-the-art TSP solvers. In this paper, we introduce a new class of facet-defining inequalities, the *circlet inequalities*. These inequalities were first conjectured in Gutekunst and Williamson [Gutekunst SC, Williamson DP (2019) Characterizing the integrality gap of the subtour LP for the circulant traveling salesman problem. *SIAM J. Discrete Math.* 33(4):2452–2478] when studying the circulant TSP, and they provide a bridge between polyhedral TSP research and number-theoretic investigations of Hamiltonian cycles stemming from a conjecture from Marco Buratti in 2007. The circlet inequalities exhibit circulant symmetry by placing the same weight on all edges of a given length; our main proof exploits this symmetry to prove the validity of the circlet inequalities. We then show that the circlet inequalities are facet-defining and compute their strength following Goemans [Goemans MX (1995) Worst-case comparison of valid inequalities for the TSP. *Math. Programming* 69:335–349]; they achieve the same worst case strength as the similarly circulant crown inequalities of Naddef and Rinaldi [Naddef D, Rinaldi G (1992) The crown inequalities for the symmetric traveling salesman polytope. *Math. Oper. Res.* 17(2):308–326] but are generally stronger.

Funding: This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program [Grant DGE-1650441] and by the National Science Foundation Division of Computing and Communications Foundations [Grant CCF-1908517].

Keywords: mathematics • combinatorics • sets • polyhedra • networks/graphs • traveling salesman • integer programming • algorithms • cutting plane/facet

1. Introduction and Outline

The symmetric traveling salesman problem (TSP) is a fundamental problem in combinatorial optimization, combinatorics, and theoretical computer science. An instance consists of a set $[n] := \{1, 2, 3, \dots, n\}$ of n cities and, for each pair of distinct cities $i, j \in [n]$, an associated cost or distance $c_{ij} = c_{ji} \geq 0$ reflecting the cost or distance of traveling from city i to city j . The TSP is then to find a minimum-cost tour visiting each city exactly once. Treating the cities as vertices of the complete, undirected graph K_n with vertex set $V = [n]$ and edge set E , and treating an edge $\{i, j\}$ of K_n as having cost c_{ij} , the TSP is equivalent to finding a minimum-cost Hamiltonian cycle on K_n .

Let $STSP(n)$ denote the symmetric traveling salesman problem polytope on n cities: the convex hull of the edge-incidence vectors of Hamiltonian cycles on K_n . That is,

$$STSP(n) = \text{conv}\{\chi_H : H \text{ is a Hamiltonian cycle on } K_n\} \subset \mathbb{R}^{|E|}.$$

A substantial theme of TSP research is polyhedral with a particular emphasis on *facet-defining inequalities*. These inequalities play an important role in developing TSP algorithms: despite the TSP being a fundamental NP-hard problem,¹ the state-of-the-art TSP solver Concorde (Applegate et al. [1]) has been able to successfully solve structured instances with nearly 100,000 vertices! (See, e.g., Applegate et al. [3], which certifies the optimality of pla85900 from TSPLIB (Reinelt [31]).)

Formally, an inequality $ax \geq a_0$ is *valid* for $STSP(n)$ if every $x \in STSP(n)$ satisfies $ax \geq a_0$. Facet-defining inequalities are valid inequalities that define a *facet* of $STSP(n)$ (i.e., letting $S = \{x \in STSP(n) : ax = a_0\}$, the inequality $ax \geq a_0$ is facet-defining when $\dim(S) = \dim(STSP(n)) - 1$ so that the inequality induces a facet of $STSP(n)$). A nice survey of the prominent role of facet-defining inequalities in TSP research and computation can be found in Applegate et al.

[2, chapter 5]. See also Grötschel and Padberg [18] and Naddef [26]. Goemans [15] surveys many facet-defining inequalities for the TSP and provides a way to evaluate the strength of such inequalities. Well-known inequalities for the TSP include the clique-tree inequalities (Grötschel and Pulleyblank [19]), the comb inequalities (Chvátal [7], Grötschel and Padberg [17]), the crown inequalities (Naddef and Rinaldi [28]), the path inequalities (Cornuéjols et al. [9]), the path-tree inequalities (Naddef and Rinaldi [27]), the binested inequalities (Naddef [25]), and the rank inequalities (Grötschel [16]).

The main result of this paper is a new facet-defining inequality arising from the *circulant TSP*, a special case of the TSP for which relatively little is known. Circulant TSP instances are those whose edge costs can be described by a symmetric, *circulant matrix*, a condition that imposes substantial symmetry: the cost of edge $\{i, j\}$ only depends on $(i - j) \bmod n$. The cost of an edge, then, depends only on its *length*, which we define as

$$\ell_{i,j} = \min\{|i - j|, n - |i - j|\}.$$

In analogue to a circulant TSP instance, we define a circulant inequality for the TSP as a valid inequality $ax \geq a_0$, where the coefficients of a are circulant (i.e., $a_{i,j} = a_{i',j'}$ whenever $\ell_{i,j} = \ell_{i',j'}$). Of the well-known facet-defining inequalities for $STSP(n)$, the crown inequalities of Naddef and Rinaldi [28] are circulant.

For a vector $x \in \mathbb{R}^{|E|}$, let

$$t_i = \sum_{\{s,t\} \in E: \ell_{s,t}=i} x_{s,t}$$

denote the total weight of edges of length i . A circulant inequality $ax \geq a_0$ can be rewritten as

$$\sum_{i=1}^d c_i t_i \geq a_0, \quad (1)$$

where $d = \lfloor \frac{n}{2} \rfloor$ and $c_i = a_{0,i}$ is the cost of any edge of length i .

Notice that any valid inequality of the form of Equation (1) expresses some requirement about edge lengths in a valid tour. For example, suppose that n is divisible by four. Then, $t_4 \leq n - 4$ is a simple inequality stating that you cannot use more than $n - 4$ edges of length 4: edges of length 4 decompose the graph of K_n into four distinct cycles, and a valid Hamiltonian cycle must use at most $n/4 - 1$ edges from any of these cycles. Circulant inequalities, thus, offer a way to bridge together polyhedral investigations of the TSP in combinatorial optimization with related questions in number theory. For example, the following conjecture dates to Marco Buratti in 2007 and conjectures conditions for a Hamiltonian path using prescribed edge lengths (see Buratti and Merola [5] for an initial statement, Horak and Rosa [22] for a generalization, and Pasotti and Pellegrini [30] for a rephrasal).

Conjecture 1.1 (Buratti): Let L be a multiset of size $n - 1$ consisting of edge lengths in $1, 2, \dots, \lfloor \frac{n}{2} \rfloor$. There exists a Hamiltonian path in K_n using edge lengths L if and only if, for every q that divides n ,

$$\#\{e \in L : q | e\} \leq n - q.$$

Here, $\{e \in L : q | e\}$ is taken as a multiset consisting of all edge lengths in L that are a multiple of q . In the case in which $n = 8$, for example, this condition says that L must contain at most $8 - 2 = 6$ edges of even length. Any analogue of Buratti's condition for Hamiltonian paths would give rise to a circulant inequality for the TSP. Costa et al. [10], for example, explicitly leave as open finding necessary and sufficient conditions for a graph to have a Hamiltonian cycle using prescribed edge lengths.

The circulant facet-defining inequality we propose (the *circlet inequality*) here takes the following form: Suppose that n is divisible by four and let $d = \frac{n}{2}$. Then, for any $x \in \mathbb{R}^{|E|}$ that is a feasible TSP input,

$$\sum_{i=1}^d c_i t_i \geq n - 2, \quad c_i = \begin{cases} i, & i \text{ odd} \\ d - i, & i \text{ even}. \end{cases} \quad (2)$$

Note that this inequality has an “alternating” structure based on parity: for i odd, the coefficient of t_i grows with i , whereas for i even, the coefficient of t_i decreases with i . When $n = 12$, for example, the inequality is

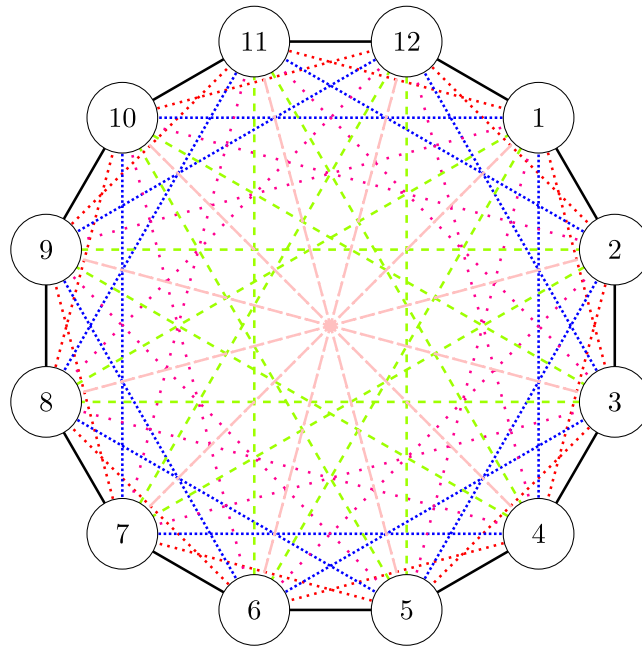
$$t_1 + 4t_2 + 3t_3 + 2t_4 + 5t_5 + 0t_6 \geq 10;$$

here, for example, t_2 expands as

$$t_2 = x_{1,3} + x_{2,4} + x_{3,5} + \dots + x_{10,12} + x_{11,1} + x_{12,2}.$$

Note that this inequality is valid for any arbitrary labeling of the vertices. Hence, for any TSP instance, there are on the order of $n!$ possible versions that can be applied. Consider a point $x \in \mathbb{R}^{|E|}$ that may or may not be in

Figure 1. (Color online) Circulant symmetry. Edges of a fixed length are indistinguishable. For example, all edges of the form $\{v, v + 1\}$ (where $v + 1$ is taken mod n to lie in $[n]$) have the same appearance.



$STSP(n)$. One can arbitrarily relabel the vertices with any permutation of $1, \dots, n$, determine the “edge lengths” and t_i values based on that new labeling, and apply Inequality (2). If x is not feasible for Inequality (2) under that relabeling, then x is not in $STSP(n)$.

In addition to the number theoretic connections and interpretations of the circlet inequalities, we highlight three additional properties of it. First, Inequality (2) was originally conjectured in Gutekunst and Williamson [20] and motivated by the circulant TSP. Gutekunst and Williamson [20] characterize the *integrality gap* of the prototypical LP relaxation of the TSP, the subtour LP. Specifically, Gutekunst and Williamson [20] show that the worst case ratio of the subtour LP relative to the TSP is exactly two on circulant TSP instances. Gutekunst and Williamson [20] consider several avenues to improving the integrality gap on circulant TSP instances. They note that many facet-defining inequalities eliminate the specific solution used to show an integrality gap of two, including the ladder, chain, and crown inequalities (see Boyd and Cunningham [4], Padberg and Hong [29], and Naddef and Rinaldi [28]). However, none of these inequalities is robust to a small modification of the specific solution used. Gutekunst and Williamson [20] conjecture Inequality (2) and note that, if valid, adding Inequality (2) robustly eliminates the specific solution; see Section 2 for more details. Thus, this paper resolves the circulant TSP conjecture of Gutekunst and Williamson [20].

Second, Inequality (2) can itself be considered as defining a circulant, nonmetric TSP instance. One places a cost c_i on each edge of length i and verifies if the minimum cost solution to that TSP instance costs at least $n - 2$; see Figure 1 for an example of the symmetry of such an instance. Indeed, when investigating this TSP inequality, we first experimentally tried to verify its validity using Concorde (Applegate et al. [1]). Our instances present potential computational novelty: despite having solved instances with nearly 100,000 vertices, Concorde struggled to verify the circlet inequality on even tiny instances.²

Finally, we hope that this paper motivates a search for other circulant facet-defining inequalities. Such inequalities are intimately connected to number theory and may provide a new approach to the TSP: decades of research have still not resolved many questions about $STSP(n)$, and one might wonder if a more number-theoretic take—an understanding of the combinations of edge lengths that can constitute a Hamiltonian cycle—might provide new insights. We specifically consider the projection of $STSP(n)$ to the variables $t_1, \dots, t_d$. Let

$$EL(n) := \text{conv} \left\{ (t_1, \dots, t_d) : x \in STSP(n), t_i = \sum_{\{s,t\} \in E: \ell_{s,t}=i} x_{s,t} \right\}$$

denote the edge-length TSP polytope. One might ask if it is possible to characterize $EL(n)$ and if valid inequalities for $EL(n)$ are useful in solving TSP instances.

1.1. Outline

We begin by providing brief background on the circulant TSP and context for our results in Section 2. Then, we are able to present the two main theorems in this paper: Theorem 3.1, which proves that Inequality (2) is valid, and Theorem 5.1, which proves that it is facet-defining.

Sections 3 and 4 present the proof of Theorem 3.1, which uses two lemmas that provide contrasting conditions on potential counterexamples to Inequality (2). Recall that this inequality is of the form $\sum_{i=1}^d c_i t_i \geq n - 2$, where $c_i, t_i \geq 0$ and $\sum_{i=1}^d t_i = n$. On one hand, $c_1 = 1, c_d = 0$ and $c_i \geq 2$ otherwise. Hence, any possible counterexample to Inequality (2) requires t_1 and t_d to be large: for $\sum_{i=1}^d c_i t_i < n - 2$, the cost of an average edge must be strictly less than one, and all other edges of length $i \notin \{1, d\}$ cost at least twice that. We formalize this observation in our first lemma, Lemma 3.1.

On the other hand, our more technical lemma, Lemma 3.2, argues that a minimal counterexample cannot be “dense” in edges of length 1 and d : any “window” of four vertices $v, v + 1, v + d, v + d + 1$ can include at most two such edges. By arguing that the conditions of Lemmas 3.1 and 3.2 are mutually incompatible, we can quickly deduce our main result. We do so in Section 3, in which we provide the proof up to Lemma 3.2. Then, in Section 4, we prove Lemma 3.2. Lemma 3.2 is considerably more involved than Lemma 3.1, and to prove it, we carefully consider what happens when we contract any window using three edges of length 1 and d . Doing so involves careful bookkeeping on how edge costs change under contraction in addition to combinatorial observations about the different types of edges a tour can contain.

In Section 5, we turn to our second main theorem. By presenting a relatively small set of tours and exploiting symmetry, we can quickly show that Inequality (2) is tight. Finally, in Section 6, we turn toward analyzing the strength of Inequality (2). We compute its strength following Goemans [15], and we show that the strength of Inequality (2) is

$$\frac{n^2 - 2n - 4}{n^2 - 3n} \leq \frac{11}{10}.$$

It is equal to $\frac{11}{10}$ when $n = 8$. For comparison, we note that the bound of $\frac{11}{10}$ is also attained when $n = 8$ by the crown inequality; otherwise, our circlet inequality is marginally stronger than the crown inequality (Naddef and Rinaldi [28]).

2. Background

2.1. Circulant TSP

As noted, circulant TSP instances are those whose edge costs can be described by a symmetric, circulant matrix. Because the cost of edge $\{i, j\}$ only depends on $(i - j) \bmod n$, the cost matrix is in terms of $\lfloor \frac{n}{2} \rfloor$ parameters:

$$C = (c_{(j-i) \bmod n})_{i,j=1}^n = \begin{pmatrix} 0 & c_1 & c_2 & c_3 & \cdots & c_1 \\ c_1 & 0 & c_1 & c_2 & \cdots & c_2 \\ c_2 & c_1 & 0 & c_1 & \ddots & c_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & c_4 & \cdots & 0 \end{pmatrix}, \quad (3)$$

with $c_0 = 0$ and $c_i = c_{n-i}$ for $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$. Importantly, in the circulant TSP, there is not necessarily an assumption that the edge costs are metric.

The circulant TSP initially arose from questions of minimizing wallpaper waste in Garfinkel [13] and reconfigurable network design in Medova [24]. One of the reasons that the circulant TSP remains so compelling is that circulant instances seem to provide just enough structure to make an ambiguous set of instances: it is unclear whether a given combinatorial optimization problem should remain hard or become easy when restricted to circulant instances. Some classic combinatorial optimization problems become easy when restricted to circulant instances. In the late 1980s, Burkard and Sandholzer [6] show that the decidability question for whether a symmetric circulant graph is Hamiltonian can be solved in polynomial time and show that bottleneck TSP is polynomial-time solvable on symmetric circulant graphs. Bach, Luby, and Goldwasser (cited in Gilmore et al. [14]) show that one can find minimum-cost Hamiltonian paths in (not necessarily symmetric) circulant graphs in polynomial time. In contrast, Codenotti et al. [8] show that max clique and graph coloring remain NP-hard when restricted to circulant graphs and do not admit constant-factor approximation algorithms unless $P = NP$.

Gutkunst and Williamson [20] analyze the prototypical LP relaxation of $STSP(n)$ on circulant TSP instances. This LP relaxation is the *subtour elimination linear program* (also referred to as the Dantzig–Fulkerson–Johnson relaxation (Dantzig et al. [12]) and the Held–Karp bound (Held and Karp [21]) and which we refer to as the subtour LP and whose feasible region we abbreviate as $SP(n)$). The subtour LP has a variable x_e associated to each

edge. For $S \subset V$, we denote the set of edges with exactly one endpoint in S by

$$\delta(S) := \{e = \{i, j\} : |\{i, j\} \cap S| = 1\}$$

and let $\delta(v) := \delta(\{v\})$. The subtour LP is then

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in \delta(v)} x_e = 2, \quad v = 1, \dots, n \\ & \sum_{e \in \delta(S)} x_e \geq 2, \quad S \subset V : S \neq \emptyset, S \neq V \\ & 0 \leq x_e \leq 1, \quad e \in E. \end{aligned} \tag{4}$$

Given a Hamiltonian cycle $\mathcal{C}$, there is a feasible solution to the subtour LP attained by setting $x_e = 1$ for each $e \in \mathcal{C}$ and $x_e = 0$ otherwise. When edge costs are metric, Wolsey [34], Cunningham [11], and Shmoys and Williamson [32] show that solutions to this linear program are within a factor of $\frac{3}{2}$ of the optimal, integer solution to the TSP.

Theorem 2.1 (Wolsey [34], Cunningham [11], and Shmoys and Williamson [32]). *The integrality gap of the subtour LP on metric TSP instances is at most $\frac{3}{2}$. That is, for any input to the TSP with metric edge costs, the following is true:*

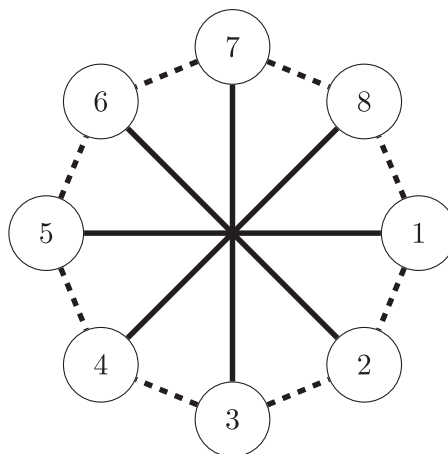
$$\frac{\text{Cost of Optimal TSP solution}}{\text{Cost of Optimal LP Solution}} \leq \frac{3}{2}.$$

It is conjectured that the integrality gap of the subtour LP on metric TSP instances is at most $\frac{4}{3}$, and one motivation for this conjecture stems from the definition of strength in Goemans [15]. The $\frac{3}{2}$ bound, however, remains state of the art.

Gutekunst and Williamson [20] show that the integrality gap of the subtour LP on circulant instances—the worst case ratio of the subtour LP relative to the TSP—is exactly two. Figure 2 describes the circulant (but non-metric) TSP instances and corresponding subtour LP solution used to show that the integrality gap on circulant instances is at least two. These instances have $n = 2^{k+1}$ vertices. Edges of length d have cost 0, edges of length 1 have cost 1, and all other edges have arbitrarily large costs. Gutekunst and Williamson [20] argue that the cheapest possible tour costs $2^{k+1} - 2$. In contrast, the solution shown in Figure 2 places weight $1/2$ on all edges of length 1 and weight 1 on all edges of length d . It is feasible for the subtour LP, and the cost of such a subtour LP solution is only 2^k .

Gutekunst and Williamson [20] note that many facet-defining inequalities eliminate the specific subtour LP solution indicated in Figure 2, including the ladder, chain, and crown inequalities (see Boyd and Cunningham [4], Padberg and Hong [29], and Naddef and Rinaldi [28]): any of these inequalities can be added to the subtour LP to potentially strengthen its integrality gap on circulant TSP instances. Indeed, the crown inequalities of Naddef and Rinaldi [28] are also motivated by the exact same subtour LP solution as shown in Figure 2!

Figure 2. An example of a class of instances showing that the integrality gap of the subtour LP restricted to circulant instances is at least two. The dashed edges have weight $1/2$ and cost 1, whereas the full edges have weight 1 and cost 0. All other edges have arbitrarily large costs.



However, Gutekunst and Williamson [20] also note that a cursory modification to these subtour LP weights—marginally increasing the weight on length 1 edges and decreasing the weight on length d edges—yields edge weights that (1) are feasible for the subtour LP; (2) obey the ladder, chain, and crown inequalities; and (3) still show that the integrality gap of the subtour LP is two on circulant instances.

More specifically, consider solutions that place a weight of λ on every edge of length 1 and a weight of $2 - 2\lambda$ on every edge of length d . Such a solution is only in $STSP(n)$ if

$$1 - \frac{2}{n} \leq \lambda \leq 1.$$

However, adding the crown inequalities, for example, only imposes that

$$\lambda \geq \frac{1}{2} + \frac{2}{3n} + \frac{1}{3(n-6)}.$$

Gutekunst and Williamson [20] conjectured Inequality (2) as a way to eliminate this entire family of bad instances. Consider instances on $n = 2^{k+1}$ vertices and potential solutions that place a weight of λ on every edge of length 1 and a weight of $2 - 2\lambda$ on every edge of length d . Then, Inequality (2) directly implies

$$n\lambda + 0\left(\frac{n}{2}\right)(2 - 2\lambda) \geq n - 2.$$

That is, that

$$\lambda \geq 1 - \frac{2}{n}.$$

Inequality (2), thus, takes a canonically bad family of subtour LP solutions and eliminates every single instance in that family that is outside $STSP(n)$.

3. Theorem 3.1

3.1. Preliminaries: Notation and Lemmas

We briefly recall our notation. Let $n = 2d$, where d is even. For a TSP instance on $[n]$, let $\ell_{i,j}$ denote the length of edge $\{i, j\}$, so that

$$\ell_{i,j} = \min\{|i - j|, n - |i - j|\}.$$

Let x_e denote the standard TSP variables associated to each edge $e \in E$ of the complete graph. For a vector $x \in \mathbb{R}^{|E|}$ and for $1 \leq i \leq d$, let

$$t_i = \sum_{\{s,t\} \in E: \ell_{s,t}=i} x_{s,t}$$

denote the total weight of edges of length i . The following is our main theorem, which restates Inequality (2).

Theorem 3.1. *For n divisible by four and $1 \leq i \leq d$, let*

$$c_i = \begin{cases} i, & i \text{ odd} \\ d - i, & i \text{ even}. \end{cases}$$

Then, for any $x \in \mathbb{R}^{|E|}$ that is a feasible TSP input,

$$\sum_{i=1}^d c_i t_i \geq n - 2.$$

That is, the circlet inequalities are valid.

Figure 3 shows some instances for which Inequality (2) is tight.

Our proof of Theorem 3.1 uses three ingredients. First, Claim 3.1 provides a base case.

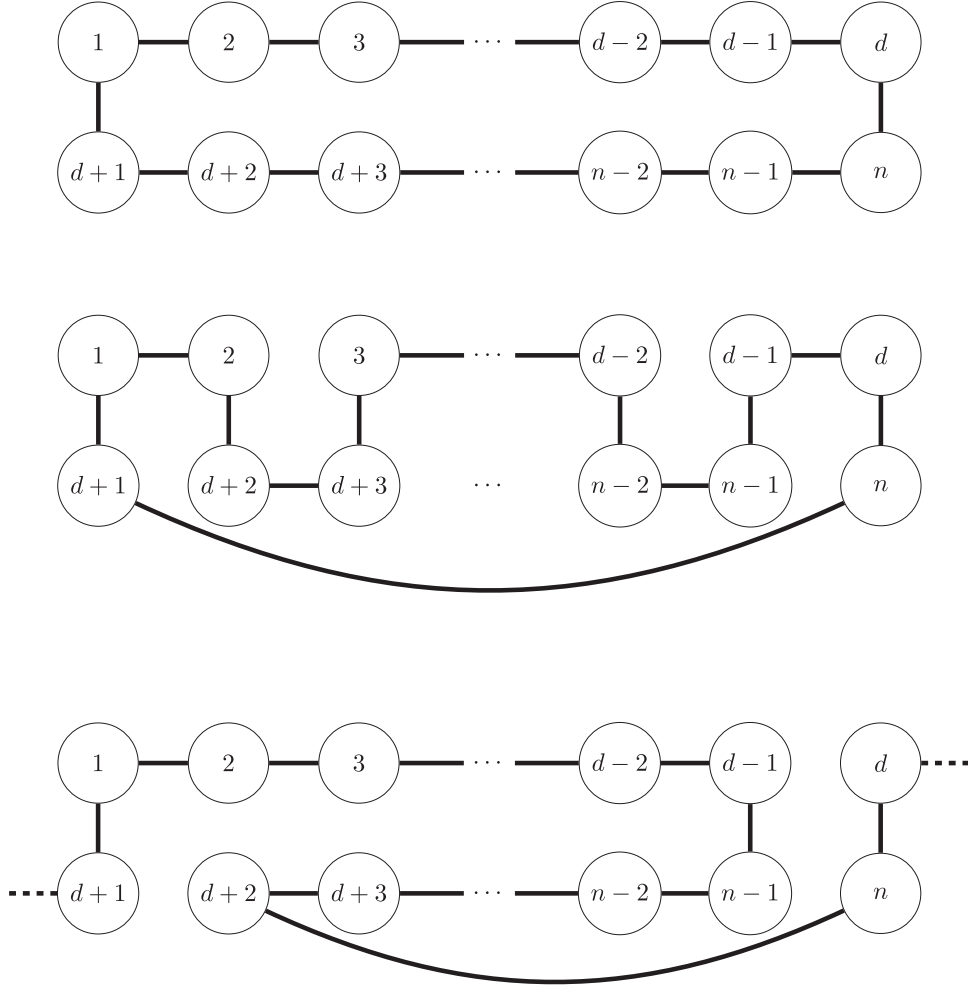
Claim 3.1. *Inequality (2) is valid for $n = 4$.*

Proof. When $n = 4$, the inequality becomes

$$t_1 + 0t_2 \geq 2.$$

Any Hamiltonian cycle must use four edges, the only possible edge lengths are 1 and 2, and there are only two distinct edges of length 2. Hence, the claim holds. $\square$

Figure 3. Instances for which Inequality (2) is tight. Any present edges indicate a weight of 1 on the corresponding LP variable. All other edges have weight 0. The first figure shows a tour using two edges of length d and $n - 2$ edges of length 1. The second tour using all d edges of length d , $d - 1$ edges of length 1, and one edge of length $d - 1$. The third tour (which includes the edge $\{d, d + 1\}$ indicated by a dashed line) uses three edges of length d , $n - 4$ edges of length 1, and one edge of length $d - 2$.



Our next lemma argues that any counterexample requires many cheap edges: edges of length 1 and d .

Lemma 3.1. *Let $n > 4$ be divisible by four (so that $d \geq 4$ and d is even). Suppose that we have a valid TSP instance in which*

$$\sum_{i=1}^d c_i t_i < n - 2.$$

Then,

$$t_1 + 2t_d > n + 2.$$

Proof. In a counterexample,

$$\begin{aligned} n - 2 &> \sum_{i=1}^d c_i t_i \\ &\geq t_1 + 0t_d + 2 \sum_{i=2}^{d-1} t_i \\ &= t_1 + 0 + 2(n - t_1 - t_d) \\ &= 2n - t_1 - 2t_d. \end{aligned}$$

Rearranging

$$n - 2 > 2n - t_1 - 2t_d$$

yields the desired inequality. $\square$

Finally, we state our technical lemma. Its proof is deferred to Section 4. In the notation of the lemma—and throughout this paper—vertex labels are implicitly assumed to be taken mod n (e.g., for a vertex $v \in [n]$, we write $v + k$ to denote $v + k \bmod n$ so that $v + k$ lies in $[n]$).

Lemma 3.2. *Suppose that we have a valid TSP instance in which*

$$\sum_{i=1}^d c_i t_i < n - 2,$$

and consider an instance that is minimal with respect to n . Then, the counterexample cannot have any of the structures shown in Figure 4. That is, for any $u \in [n]$, a minimal counterexample cannot contain the three edges $\{u + d, u\}, \{u, u + 1\}, \{u + 1, u + 1 + d\}$; a minimal counterexample cannot contain the three edges $\{u + 1, u\}, \{u, u + d\}, \{u + d, u + 1 + d\}$; and a minimal counterexample cannot contain the three edges $\{u, u + 1\}, \{u + 1, u + 1 + d\}, \{u + 1 + d, u + d\}$.

We note that we do not need separate type A1 and A2 cases (as we do with type B1 and B2 in Figure 4). That is, we do not need a statement about three edges $\{v, v + d\}, \{v + d, v + d + 1\}, \{v + d + 1, v + 1\}$ in Lemma 3.2: this statement is already covered by $\{u + d, u\}, \{u, u + 1\}, \{u + 1, u + 1 + d\}$ by taking $u = v + d = v - d$.

3.2. Proof of Theorem 3.1

We can now prove Theorem 3.1. We consider a hypothetical minimal counterexample to Theorem 3.1, that is, a Hamiltonian cycle in which

$$\sum_{i=1}^d c_i t_i < n - 2.$$

By Claim 3.1, $n \geq 8$. By Lemma 3.2, our Hamiltonian cycle cannot contain any of the structures shown in Figure 4. By Lemma 3.1,

$$t_1 + 2t_d > n + 2.$$

We specifically contradict this claim: we argue that the lack of structures in Figure 4 forces $t_1 + 2t_d$ to be small.

To compute $t_1 + 2t_d$, we look at induced subgraphs of our Hamiltonian cycle on vertices $u, u + 1, u + d, u + d + 1$. We call such a subgraph of four vertices a *window*. Figure 5 shows two natural ways to order the vertices and view a window. Notice that we can move from one window (e.g., $u = 1$) to the next (e.g., $u = 2$) by rotating (in the left picture) or sliding horizontally (on the right picture).

We count the number of edges of length 1 and d by moving through windows. If we count the total number of edges in each window with $u = 1, 2, 3, \dots, d$, we exactly count $t_1 + 2t_d$. The intuition for this process is shown in Figure 6, which shows exactly those d windows. Note that every possible length 1 edge is contained in exactly one window (e.g., $1 \sim 2$ is only in the window $1, 2, 7, 8$), whereas every length d edge is contained in exactly two windows (e.g., $1 \sim 7$ is contained in the windows $1, 2, 7, 8$ and $12, 1, 6, 7$).

Figure 4. Lemma 3.2 proves that none of these structures can occur in a minimal TSP instance (with respect to n), where $\sum_{i=1}^d c_i t_i < n - 2$.

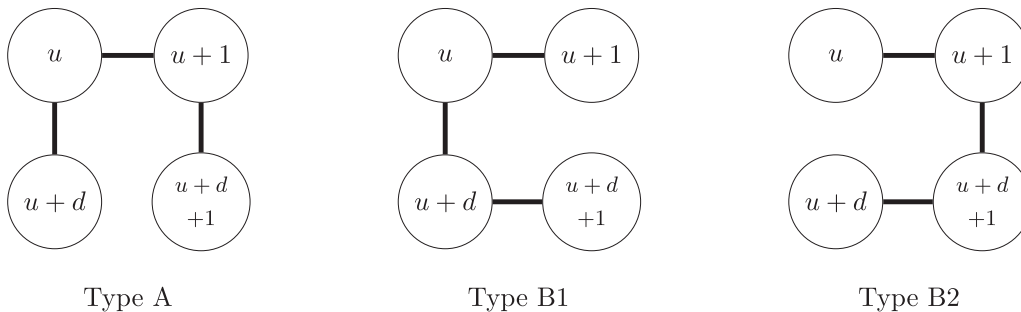
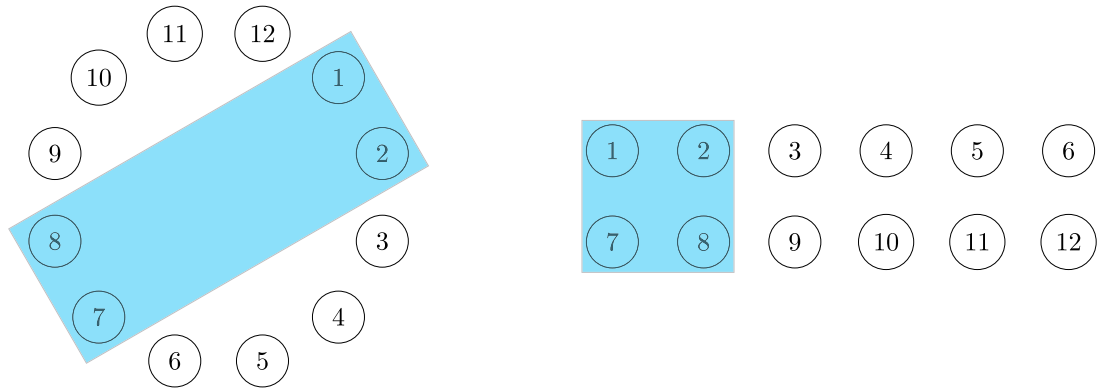


Figure 5. (Color online) Two views of a window: a collection of four vertices $u, u+1, u+d, u+d+1$. In the window shown, $u = 1$.



Proof of Theorem 3.1. To formalize our argument, we let W_u denote the window $u, u+1, u+d, u+1+d$. Figure 5, for example, shows the window $W_1 = W_7$ when $n = 12$. Let T_u denote the total number of length 1 and d edges within the window W_u :

$$T_u = \mathbb{1}_{u \sim u+1} + \mathbb{1}_{u+d \sim u+d+1} + \mathbb{1}_{u \sim u+d} + \mathbb{1}_{u+1 \sim u+1+d}.$$

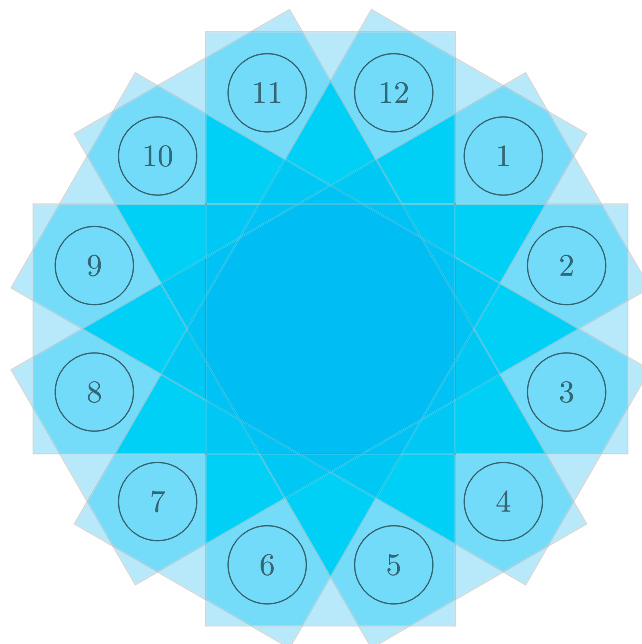
(Here, $\mathbb{1}_\circ$ denotes the indicator function if event $\circ$ occurs, so that, e.g., $\mathbb{1}_{u \sim u+1}$ is one if the edge $\{u, u+1\}$ is in our Hamiltonian cycle and zero otherwise.)

By Lemma 3.2,

$$T_u \leq 2.$$

Having $T_u = 4$ implies a subtour, and $T_u = 3$ leads to one of the bad structures in Figure 4.

Figure 6. (Color online) Rotating the window from $u = 1$ to $u = d$. Each possible edge of length 1 is in one window, whereas each possible edge of length d is in two.



Now, we consider rotating the window d times as in Figure 6. We note that

$$\begin{aligned} t_1 + 2t_d &= T_1 + T_2 + \dots + T_d \\ &\leq 2 + 2 + \dots + 2 \text{ (} d \text{ times)} \\ &= 2d \\ &= n. \end{aligned}$$

The fact that $t_1 + 2t_d = T_1 + T_2 + \dots + T_d$ follows by symmetry: any length 1 edge $i \sim i+1$ is included in either T_i (if $i \leq d$) or T_{i-d} . Because each of the d windows contains two length 1 edges and each of the n length 1 edges is included in exactly one window, each length 1 edge is counted exactly once. Each length d edge is analogously included in exactly two windows and counted twice.

This completes our proof as we argue that

$$t_1 + 2t_d \leq n < n + 2,$$

contradicting Lemma 3.1. $\square$

4. Proof of Technical Lemma

Recall our main technical lemma:

Lemma 3.2 (Repeated). *Suppose that we have a valid TSP instance in which*

$$\sum_{i=1}^d c_i t_i < n - 2,$$

and consider an instance that is minimal with respect to n . Then, the counterexample cannot have any of the structures shown in Figure 4. That is, for any $u \in [n]$, a minimal counterexample cannot contain the three edges $\{u + d, u\}, \{u, u + 1\}, \{u + 1, u + 1 + d\}$; a minimal counterexample cannot contain the three edges $\{u + 1, u\}, \{u, u + d\}, \{u + d, u + 1 + d\}$; and a minimal counterexample cannot contain the three edges $\{u, u + 1\}, \{u + 1, u + 1 + d\}, \{u + 1 + d, u + d\}$.

Proving this lemma includes involved casework, some of which is deferred to the appendix. First, we reduce our work by symmetry, arguing that we only need to consider instances of A and B2 from Figure 4.

Claim 4.1. *Suppose we have a counterexample to Theorem 3.1 on n vertices containing an instance of B2 from Figure 4. Then, there also exists a counterexample to Theorem 3.1 on n vertices containing an instance of B1 from Figure 4.*

Proof. Any counterexample with an instance of B1 can be viewed as a counterexample with an instance of B2 by relabeling the vertices. The circulant symmetry of the c_i means that we can relabel vertex i as $n - i$ for all $i = 1, \dots, n$, and then,

$$\ell_{i,j} = \min\{|i - j|, n - |i - j|\} = \min\{|(n - i) - (n - j)|, n - |(n - i) - (n - j)|\} = \ell_{n-i, n-j}.$$

Then, a counterexample with a B1 implies a counterexample with a B2 and vice versa. $\square$

Claim 4.2. *Suppose we have a counterexample to Theorem 3.1 on n vertices containing an instance of A or B2 from Figure 4. Without loss of generality, we may assume that $u = 1$.*

Proof. Circulant symmetry means that we can relabel vertex i as $i - (u - 1)$ for $i = 1, \dots, n$, and then,

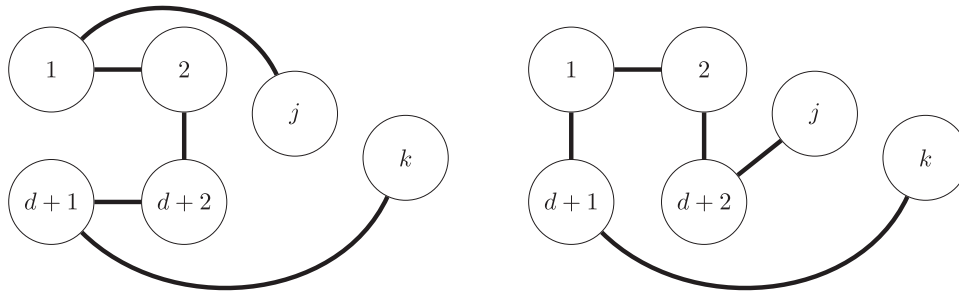
$$\ell_{i,j} = \min\{|i - j|, n - |i - j|\} = \min\{|(i - u + 1) - (j - u + 1)|, n - |(i - u + 1) - (j - u + 1)|\} = \ell_{i-u+1, j-u+1}. \quad \square$$

Hence, to prove Lemma 3.2, we need only show that a minimal counterexample cannot have either of the edge sequences shown in Figure 7.

The strategy for both is the same: we show that, if we contract the sequence of five edges from j to k , we attain a counterexample on four fewer vertices. Figure 8 shows this contraction process explicitly for the leftmost sequence of edges in Figure 7, which proceeds in two steps.

1. Delete the vertices $1, 2, d + 1, d + 2$ and replace the edges $\{j, 1\}, \{1, 2\}, \{2, d + 2\}, \{d + 2, d + 1\}, \{d + 1, k\}$ with a single edge $\{j, k\}$.

Figure 7. Proving Lemma 3.2 requires showing that neither of these sequences of edges can occur in a minimal counterexample to Theorem 3.1.



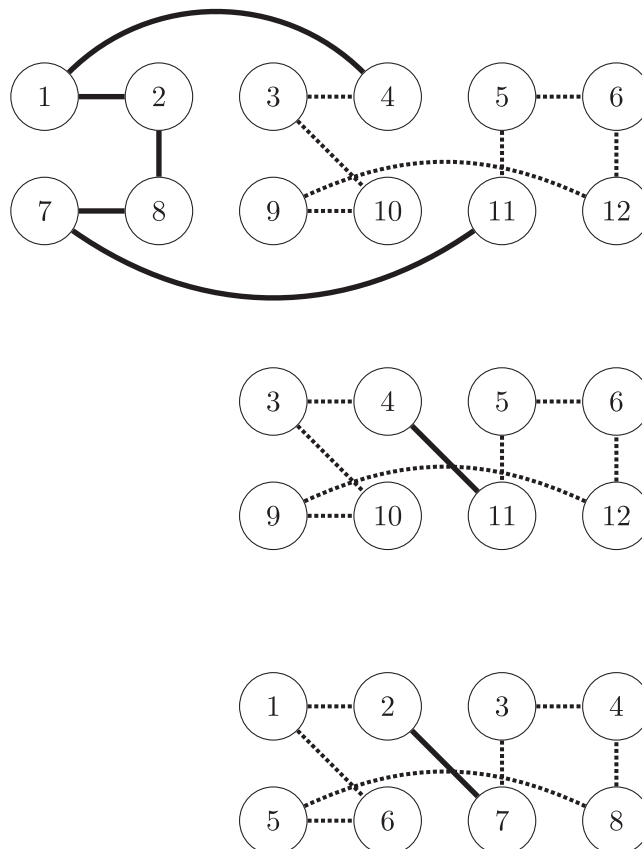
2. We relabel every other vertex s as

$$\begin{cases} s-2, & s \leq d \\ s-4, & s > d. \end{cases}$$

In the rightmost sequence of edges in Figure 7, we proceed analogously except replacing the edges $\{j, d+2\}, \{d+2, 2\}, \{2, 1\}, \{1, d+1\}, \{d+1, k\}$ with $\{j, k\}$ in the first step.

Note that, in both cases, we attain a feasible tour on four fewer vertices. To show that this smaller instance is also a counterexample requires showing, during contraction, we decrease the cost by at least four.

Figure 8. An example contraction used to prove Lemma 3.2 showing the original graph, the graph after edge deletions/additions and vertex deletions, and the final graph after relabeling vertices. Notice the addition of the edge $\{2, 7\}$ (which corresponds to an edge between vertices 4 and 11 precontraction). The dashed edges do not change.



Throughout, we denote by s' the new label of vertex s after contraction:

$$s' = \begin{cases} s-2, & 2 < s \leq d \\ s-4, & s > d+2. \end{cases}$$

We let

$$c_{i,j} := c_{\ell_{i,j}} = \begin{cases} \ell_{i,j}, & \ell_{i,j} \text{ odd} \\ d - \ell_{i,j}, & \ell_{i,j} \text{ even} \end{cases}$$

denote the cost of edge $\{i, j\}$, which has length $\ell_{i,j}$, before contraction. We denote the length of an edge $\{i, j\}$ in the contracted graph

$$\ell'_{i,j} = \min\{|i-j|, n-4-|i-j|\}$$

(where i and j are labels of vertices in the contracted graph). Similarly, we let

$$c'_{i,j} = c'_{\ell'_{i,j}} = \begin{cases} \ell'_{i,j}, & \ell'_{i,j} \text{ odd} \\ d-2-\ell'_{i,j}, & \ell'_{i,j} \text{ even} \end{cases}$$

denote the cost of edge $\{i, j\}$ in the contracted graph. Again, in this notation, i and j are labels of vertices in the contracted graph.

Note that, by design, any vertex s becomes a vertex s' of the same parity in the contracted graph and the length of an edge does not change significantly. This means, when we contract, we do not need to worry about an edge of odd length becoming an edge of even length or vice versa and, thus, potentially radically changing in cost. In particular, we first show that, after contraction, the cost of edges can only decrease. We use the notation $\equiv_2$ to denote equivalence mod 2 (i.e., $a \equiv_2 b$ means that a and b have the same parity; $a \not\equiv_2 b$ means that a and b have opposite parity).

Proposition 4.1. *Let $\{s, t\}$ be an edge in the uncontracted graph with $s, t \notin \{1, 2, d+1, d+2\}$. Then,*

$$\ell'_{s',t'} = \begin{cases} \ell_{s,t}, & 3 \leq s, t \leq d \text{ OR } d+3 \leq s, t \leq n \\ \ell_{s,t} - 2, & \text{otherwise.} \end{cases}$$

Moreover,

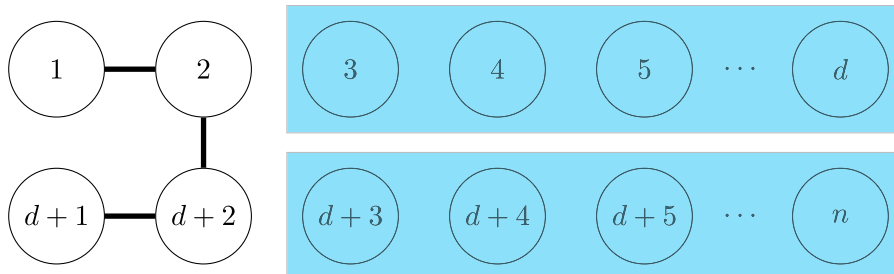
$$c'_{s',t'} = \begin{cases} c_{s,t}, & (s, t \leq d \text{ OR } d+3 \leq s, t) \text{ AND } s \not\equiv_2 t \\ c_{s,t}, & (s \leq d \leq d+3 \leq t \text{ OR } t \leq d \leq d+3 \leq s) \text{ AND } s \equiv_2 t \\ c_{s,t} - 2, & \text{else.} \end{cases}$$

In particular, any edge $\{s, t\}$ with $s, t \notin \{1, 2, d+1, d+2\}$ either gets cheaper or remains the same cost after the contraction process.

For this proof, it is helpful to consider Figure 9, in which the vertices are placed in two groups $\{3, \dots, d\}$ and $\{d+3, \dots, n\}$. The preceding proposition indicates that an edge within a group retains its cost if it is of odd length, and an edge between groups retains its cost if it is of even length. Other edges—those within groups and of even length or those between groups and of odd length—are lowered in cost by two.

Proof. The first part of Proposition 4.1 about edge length follows from Figure 9: If s, t are in the same group, then the edge between them does not change in length after contraction. If they are in different groups, then the edge between them “goes through” either $\{1, 2\}$ or $\{d+1, d+2\}$, and after contraction, gets shorter by two. More

Figure 9. (Color online) Placing the vertices into two groups.



formally, if s, t are in the same group, then $s' - s = t' - t$ so that $s' - t' = s - t$ and $|s - t| \leq d - 3$ so that

$$\ell'_{s',t'} = \min\{|s' - t'|, n - 4 - |s' - t'|\} = |s' - t'| = |s - t| = \min\{|s - t|, n - |s - t|\} = \ell_{s,t}.$$

Otherwise, without loss of generality, let s be in the top group so that $s < t, s' < t', s' = s - 2$, and $t' = t - 4$. Then,

$$|s' - t'| = t' - s' = (t - 4) - (s - 2) = t - s - 2 = |s - t| - 2,$$

and

$$n - 4 - |s' - t'| = n - 4 - (|s - t| - 2) = n - |s - t| - 2.$$

Thus,

$$\ell'_{s',t'} = \min\{|s' - t'|, n - 4 - |s' - t'|\} = \min\{|s - t| - 2, n - |s - t| - 2\} = \min\{|s - t|, n - |s - t|\} - 2 = \ell_{s,t} - 2.$$

The statement about costs then uses the new lengths and new cost equation. In two cases,

- If $s \not\equiv_2 t$, then

$$\begin{aligned} c'_{s',t'} &= \ell'_{s',t'} = \begin{cases} \ell_{s,t}, & 3 \leq s, t \leq d \text{ OR } d + 3 \leq s, t \leq n \\ \ell_{s,t} - 2, & s \leq d \leq d + 3 \leq t \text{ OR } t \leq d \leq d + 3 \leq s \end{cases} \\ &= \begin{cases} c_{s,t}, & 3 \leq s, t \leq d \text{ OR } d + 3 \leq s, t \leq n \\ c_{s,t} - 2, & s \leq d \leq d + 3 \leq t \text{ OR } t \leq d \leq d + 3 \leq s. \end{cases} \end{aligned}$$

- If $s \equiv_2 t$, then

$$\begin{aligned} c'_{s',t'} &= d - 2 - \ell'_{s',t'} = \begin{cases} d - 2 - \ell_{s,t}, & 3 \leq s, t \leq d \text{ OR } d + 3 \leq s, t \leq n \\ d - 2 - (\ell_{s,t} - 2), & s \leq d \leq d + 3 \leq t \text{ OR } t \leq d \leq d + 3 \leq s \end{cases} \\ &= \begin{cases} c_{s,t} - 2, & 3 \leq s, t \leq d \text{ OR } d + 3 \leq s, t \leq n \\ c_{s,t}, & s \leq d \leq d + 3 \leq t \text{ OR } t \leq d \leq d + 3 \leq s. \end{cases} \end{aligned}$$

These cases give the claimed results. $\square$

The only other change, after contraction, is that we contract five edges into one edge $\{j', k'\}$. The following propositions account for the change in cost after these contractions.

Proposition 4.2. Consider the first case, in which we contract $\{j, 1\}, \{1, 2\}, \{2, d + 2\}, \{d + 2, d + 1\}, \{d + 1, k\}$ into $\{j', k'\}$. Then,

$$c_{j,1} + c_{1,2} + c_{2,d+2} + c_{d+2,d+1} + c_{d+1,k} - c'_{j',k'} \geq 4. \quad (5)$$

Hence, the cost of the tour resulting from the contraction goes down in cost by at least four.

Proposition 4.3. Consider the second case, in which we contract $\{j, d + 2\}, \{d + 2, 2\}, \{2, 1\}, \{1, d + 1\}, \{d + 1, k\}$ into $\{j', k'\}$. Then,

$$c_{j,d+2} + c_{d+2,2} + c_{2,1} + c_{1,d+1} + c_{d+1,k} - c'_{j',k'} \geq 2 \quad (6)$$

and is even. Moreover, if

$$c_{j,d+2} + c_{d+2,2} + c_{2,1} + c_{1,d+1} + c_{d+1,k} - c'_{j',k'} = 2,$$

then there must have been at least one edge $\{s, t\}$ in the cycle with $s, t \notin \{1, 2, d + 1, d + 2\}$ such that $c'_{s',t'} = c_{s,t} - 2$. In either case, the cost of the tour resulting from the contraction goes down in cost by at least four.

Propositions 4.2 and 4.3 imply that a tour with any of the structures indicated in Lemma 3.2 and Figure 4 can be contracted to attain a tour on four fewer vertices and whose aggregate cost is at least four cheaper. They, thus, complete the proof of Lemma 3.2.

For the sake of full precision, we provide analytic formulas for $c_{j,1} + c_{1,2} + c_{2,d+2} + c_{d+2,d+1} + c_{d+1,k} - c'_{j',k'}$ and $c_{j,d+2} + c_{d+2,2} + c_{2,1} + c_{1,d+1} + c_{d+1,k} - c'_{j',k'}$. Doing so, however, involves substantial casework. We, thus, defer it to the appendix. Up to that casework, the propositions follow quickly.

Proof of Proposition 4.2. Note that

$$c_{j,1} + c_{1,2} + c_{2,d+2} + c_{d+2,d+1} + c_{d+1,k} - c'_{j',k'} = c_{j,1} + c_{d+1,k} - c'_{j',k'} + 2.$$

Casework in the appendix shows that

$$c_{j,1} + c_{d+1,k} - c'_{j',k'} \geq 2,$$

which completes the proof. $\square$

Proof of Proposition 4.3. Note that

$$c_{j,d+2} + c_{d+2,2} + c_{2,1} + c_{1,d+1} + c_{d+1,k} - c'_{j',k'} = c_{j,d+2} + c_{d+1,k} - c'_{j',k'} + 1.$$

To show that this equation evaluates to an even integer, we want to show that $c_{j,d+2} + c_{d+1,k} - c'_{j',k'}$ is odd. This follows because

$$c_{j,d+2} \equiv 2c_{d+1,k} \text{ if and only if } c'_{j',k'} \equiv 21.$$

(This can also be seen by considering the four cases depending on the parities of j and k .)

Casework in the appendix shows that

$$c_{j,d+2} + c_{d+1,k} - c'_{j',k'} \geq 1,$$

and equality can hold in exactly four cases:

- j, k are even and $j, k \leq d$.
- j, k are odd and $j, k \geq d+3$.
- j is odd, k is even, $j \geq d+3$, and $k \leq d$.
- j is even, k is odd, $j \leq d$, and $k \geq d+3$.

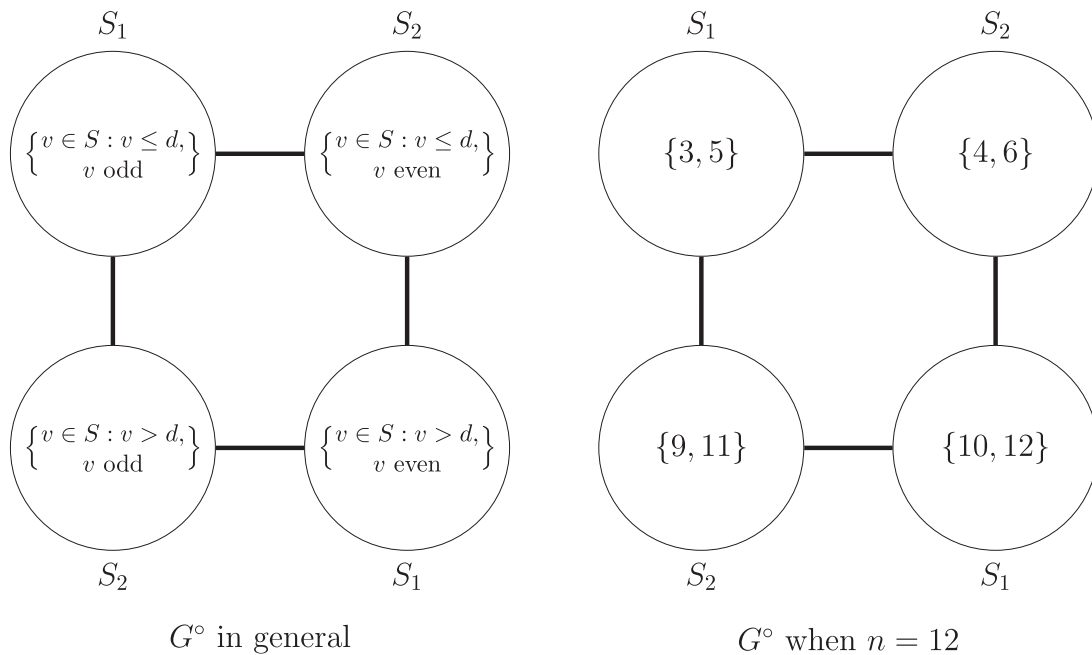
Recall from Proposition 4.1 that, for an edge $\{s, t\}$ with $s, t \notin \{1, 2, d+1, d+2\}$, either $c'_{s',t'} = c_{s,t}$ or $c'_{s',t'} = c_{s,t} - 2$. The existence of a single edge at which the cost decreases is sufficient to complete the proof of Proposition 4.3. Hence, we are concerned only with the case in which $c'_{s',t'} = c_{s,t}$ for every edge $\{s, t\}$ in the tour (outside of $\{j, d+2\}, \{d+2, 2\}, \{2, 1\}, \{1, d+1\}, \{d+1, k\}$). From Proposition 4.1, these are exactly the edges $\{s, t\}$, where

- $(s, t \leq d \text{ OR } d+3 \leq s, t) \text{ AND } s \not\equiv_2 t$.
- $(s \leq d \leq d+3 \leq t \text{ OR } t \leq d \leq d+3 \leq s) \text{ AND } s \equiv_2 t$.

Referring again to the groups indicated in Figure 9, these are exactly the within-group edges of odd length and the across-group edges of even length. We now break the vertices $S = \{3, 4, \dots, d, d+3, d+4, \dots, n-1, n\}$ into four groups as indicated in Figure 10:

$$\{v \in S : v \leq d, v \text{ odd}\}, \{v \in S : v \leq d, v \text{ even}\}, \{v \in S : v > d, v \text{ odd}\}, \text{ and } \{v \in S : v > d, v \text{ even}\}.$$

Figure 10. Placing the vertices into four groups to attain the graph G° : in general (on the left) and explicitly when $n = 12$ (on the right).



We let S_1 denote the groups in which $\{v \in S : v \leq d, v \text{ odd}\}$ and $\{v \in S : v > d, v \text{ even}\}$. We let S_2 denote the other groups. Let G° be the graph shown in Figure 10, in which there is one vertex for each group and edges between groups in opposite sets S_1, S_2 .

Note that all four groups have equal size $(n-4)/4$. The edges between groups in G° , moreover, indicate exactly the cases in which $c_{s,t} = c'_{s',t'}$. For example, consider an edge $\{s, t\}$, where $s, t \leq d$ and $s \equiv_2 t$. Using the edge $\{s, t\}$ in G corresponds to using the edge between $\{v \in S : v \leq d, v \text{ odd}\}$ and $\{v \in S : v \leq d, v \text{ even}\}$ in G° , that is, the top horizontal edge in Figure 10.

Note also that the only cases in which

$$c_{j,d+2} + c_{d+1,k} - c'_{j',k'} = 1$$

correspond to cases in which both $j, k \in S_2$. Suppose there exists a Hamiltonian tour (in the original, precontracted graph on n vertices) in which $c_{j,d+2} + c_{d+1,k} - c'_{j',k'} = 1$ and $c'_{s',t'} = c_{s,t}$ for every other edge. That is, consider the only potential type of tour whose cost does not go down by at least four after contraction.

We trace the $n-4$ vertices that connect j to k through S in such a tour. Tracing the vertices we visit in the graph G° corresponds to a walk in G° that

1. Starts and ends at one of the S_2 vertices (because $c_{j,d+2} + c_{d+1,k} - c'_{j',k'} = 1$).
2. Visits each of the four nodes in G° an equal number of times (because each node in G° has an equal number of vertices).

3. Only uses the four edges in G° . That is, the walk never uses an edge between the two S_2 vertices or an edge between the two S_1 vertices (because we only use edges for which $c_{s,t} = c'_{s',t'}$, which are exactly the edges in G°).

No such walk can exist: the first and third criteria indicate that the traced walk looks like $S_2, S_1, S_2, S_1, \dots, S_2, S_1, S_2$. Hence, if it visits vertices in S_1 k times, it visits vertices in S_2 $k+1$ times. Thus, we never satisfy the second criteria that we visit exactly as many S_1 vertices as we visit S_2 vertices.

5. Inequality (2) Is Facet-Defining

In the previous two sections, we show that the circlet inequalities are valid for the TSP. We now also prove that they are facet-defining for the symmetric traveling salesman polytope $STSP(n)$. Recall that $\chi_H \in \{0, 1\}^{|E|}$ denotes the incidence vector of a Hamiltonian cycle H on K_n . In terms of these incidence vectors, $STSP(n)$ is

$$STSP(n) = \text{conv}\{\chi_H : H \text{ is a Hamiltonian cycle on } K_n\} \subset \mathbb{R}_{\geq 0}^{|E|}.$$

To show that a valid TSP inequality is facet-defining for $STSP(n)$, we follow Naddef and Rinaldi [28, theorem 3.7]: the dimension of $STSP(n)$ is $|E| - |V| = \frac{n(n-3)}{2}$; to show that a valid inequality is facet-defining for the TSP, we must, thus, find $n(n-3)/2$ Hamiltonian cycles for which the circlet inequality is tight and whose incidence vectors are linearly independent.

To show that this inequality is facet-defining, we consider $d-1$ distinct types of tours. These tours contain edges of length 1 and d and up to one edge of a different length k . We index the tours by this extra edge length and count the number of linearly independent tours of each type. For intuition, consider the case when $n=8$. We consider three types of tours shown in Figure 11.

By construction, the circlet inequality is tight for all tours in Figure 11. We can attain $n=8$ versions of the second and third tour type by “rotating” the vertex labels (replacing every vertex v with $v+1 \pmod 8$ constitutes one rotation, as shown in Figure 12); by circulant symmetry, each rotated tour is again tight for the circlet inequality. We can similarly get $n/2 = 4$ copies of the first tour (as, after four rotations, we return to the original labeling). In total, this gives

$$4 + 2 \cdot 8 = 20 = \frac{8 \cdot 5}{2} = \frac{n(n-3)}{2}$$

tours for which the inequality is tight.

Figure 11. Three types of tours when $n=8$: a tour that uses only edges of length 1 and d , a tour that uses one edge of length 3, and a tour that uses one edge of length 2. Note that we can rotate the labels of each tour (e.g., replacing every vertex v with $v+1 \pmod 8$) to attain distinct tours of the same type. Doing so traces out four distinct tours of the leftmost type, eight distinct tours of the middle type, and eight distinct tours of the rightmost type.

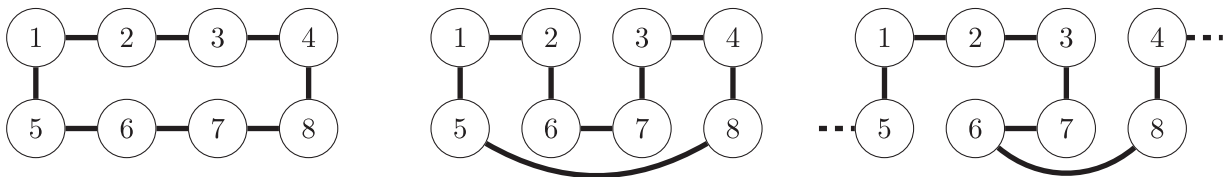
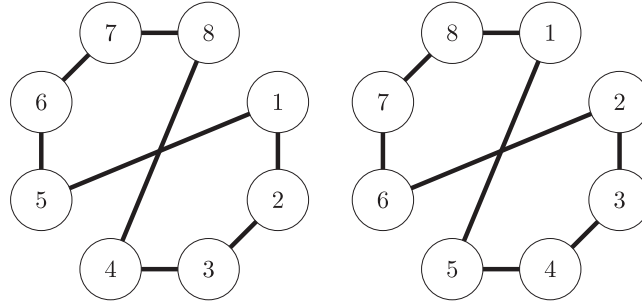


Figure 12. One version of the leftmost tour from Figure 11 and one rotation, for which each vertex label v is replaced with $v + 1 \bmod 8$. This corresponds to rotating the labels 45° counterclockwise.



When we generalize this argument, we again consider $n/2$ tours of the leftmost type (with two edges of length d and $n - 2$ edges of length 1). Then, for each of the $\frac{n}{2} - 2$ possible values of $k \in \{2, 3, \dots, \frac{n}{2} - 1\}$, we analogously find a tour using exactly one edge of cost k (and all other edges of length 1 or d).

Theorem 5.1. Let $4|n$. Then, the circlet inequalities are facet-defining for $STSP(n)$.

Proof. We follow the intuition outlined earlier:

- There are $d = \frac{n}{2}$ tours with two edges of length d and $n - 2$ edges of length 1: one such tour uses the two length- d edges $\{1, d + 1\}$ and $\{d, n\}$, and connects them via $\{1, 2\}, \{2, 3\}, \dots, \{d - 1, d\}$ and $\{d + 1, d + 2\}, \{d + 2, d + 3\}, \dots, \{n - 1, n\}$. This tour costs

$$0(2) + (n - 2) = n - 2,$$

and is tight for the circlet inequality. We then can rotate this tour by adding a constant m to the label of every vertex for $m = 1, 2, \dots, d - 1$ and, by circulant symmetry, attain another tight tour.

- For $k \in \{3, \dots, d\}$, we find n tours that each have a unique edge of cost $k - 1$. These, moreover, have k edges of length d and $n - k - 1$ edges of length 1. Their total cost is

$$(k - 1) + 0(k) + (n - k - 1) = n - 2,$$

and they are indeed tight for the circlet inequality. To find these n tours, we follow the previous process: we start with a tour with a single edge of cost $k - 1$, and we rotate it. For these tours, however, we are able to rotate them by adding any constant $m = 1, 2, \dots, n - 1$ to the label of every vertex (as we trace every edge of cost $k - 1$).

- When k is even, we use the tour type shown in Figure 13. Note that, by construction, it uses k edges of length d and $n - k - 1$ edges of length 1. The remaining edge is of length $k - 1$ and, because k is even, it is of cost $k - 1$.

- When k is odd, we use the tour type shown in Figure 14. Again, by construction, it uses k edges of length d and $n - k - 1$ edges of length 1. The remaining edge is of length $d - (k - 1)$ and, because k is odd, it is of cost $d - (d - (k - 1)) = k - 1$.

Figure 13. (Color online) Tight tours using exactly one edge of length k when k is even. In this example, $k = 4$.

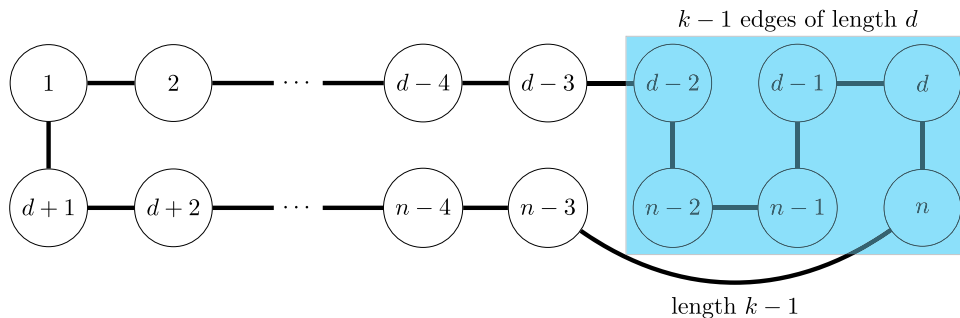
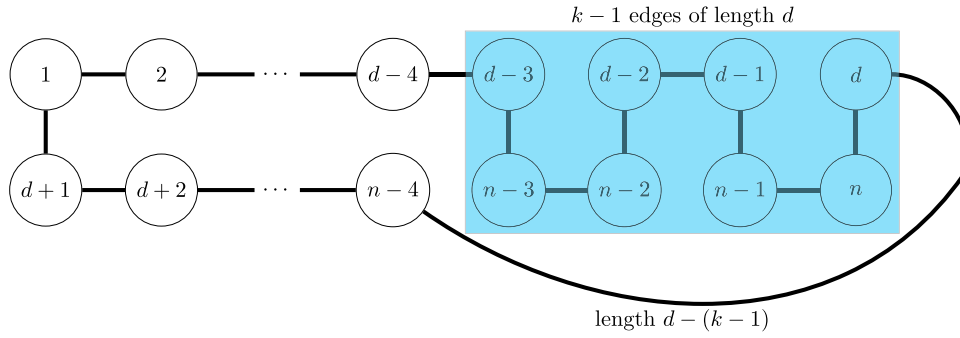


Figure 14. (Color online) Tight tours using exactly one edge of length k when k is odd. In the figure, $k = 5$.



In total, these give

$$\frac{n}{2} + \left(\frac{n}{2} - 2\right)n = \frac{n}{2} + \frac{n(n-4)}{2} = \frac{n(n-3)}{2}$$

tight tours. Their characteristic vectors are linearly independent: suppose that there is a nontrivial linear dependency and some nontrivial linear combination of these characteristic vectors added to zero. First, none of the tours with a unique edge of cost $k - 1$ can be used in such a nontrivial linear dependency: they contain an edge e that is not used by any other tour in this set of tight tours. From there, the remaining tours with two edges of length d and $n - 2$ edges of length 1 also cannot be used in any nontrivial linear dependency as each contains a unique edge e of length d not used by any of the remaining tours. Hence, any linear dependency must be trivial.

6. The Strength of the Circlet Inequalities

Goemans [15] provides a way of evaluating the strength of facet-defining inequalities for the TSP. This notion is with respect to the graphic traveling salesman problem $GTSP(n)$. Whereas $STSP(n)$ is the convex hull of incidence vectors of Hamiltonian cycles on K_n , $GTSP(n)$ is the convex hull of incidence vectors of Eulerian sub(multi)graphs on K_n : vectors $\chi_S \in \mathbb{N}^{|E|}$, where S is a multiset of edges in K_n such that the multigraph $([n], S)$ is connected and every vertex has even degree. Unlike $STSP(n)$, $GTSP(n)$ is full-dimensional. Similarly, $\chi_H \in GTSP(n)$ for any Hamiltonian cycle H on K_n .

Goemans' [15] definition of strength of a TSP inequality is relative to the prototypical TSP relaxation, the subtour LP, whose feasible region we abbreviate as $SP(n)$. Goemans [15, theorem 2.11] shows that any nontrivial inequality in *tight triangular form* that is facet-defining for $STSP(n)$ defines a facet of $GTSP(n)$. An inequality $fx \geq f_0$ is in tight triangular form if

- $f_{ij} + f_{jk} \geq f_{ik}$ for all distinct triples i, j, k .
- For all $j \in V$, there exist some $i, k \in V \setminus \{j\}$ such that $f_{i,j} + f_{j,k} = f_{i,k}$.

See Goemans [15, section 2]; one converts an inequality to tight triangular form by adding/subtracting multiples of the degree constraints.

Given an inequality $ax \geq b$ in tight triangular form, its *strength* relative to the subtour elimination polyhedron $SP(n)$ is

$$\frac{b}{\min\{ax : x \in SP(n)\}}.$$

In tight triangular form, the circlet inequalities can be stated as

$$fx \geq \frac{n^2}{2} - n - 2, f_e = \begin{cases} d - 2 + \ell_e, & \ell_e \text{ odd} \\ d - 2 + (d - \ell_e), & \ell_e \text{ even.} \end{cases}$$

We can rewrite this in the form of Inequality (2) as

$$\sum_{i=1}^d f_i t_i \geq \frac{n^2}{2} - n - 2, f_i = \begin{cases} d - 2 + i, & i \text{ odd} \\ d - 2 + (d - i), & i \text{ even.} \end{cases} \quad (7)$$

These are obtained from Inequality (2) by adding $\frac{1}{2}(d - 2)$ copies of each degree constraint: for any solution $\chi_H \in STSP(n)$, we have that $\sum_{e \in \delta(v)} (\chi_H)_e = 2$ for any $v \in V$ by the degree constraints. Every edge e is incident to exactly

two vertices, so by adding the degree constraints over all $v \in V$, we obtain

$$2 \sum_{e \in E} (\chi_H)_e = 2n.$$

Multiplying this by $\frac{1}{2}(d-2)$ yields that

$$(d-2) \sum_{e \in E} (\chi_H)_e = n(d-2).$$

Because this equality is satisfied by every $\chi_H \in STSP(n)$, we can add it to Inequality (2). Doing so yields Inequality (7): we add $d-2$ to the coefficient of every edge and add $n(d-2)$ to the right-hand side:

$$n-2+n(d-2) = n-2+\frac{n}{2}(n-4) = \frac{2n-4+n(n-4)}{2} = \frac{n^2-2n-4}{2} = \frac{n^2}{2} - n - 2.$$

Hence, Inequality (7) remains valid and facet-defining. It remains to show that Inequality (7) is in tight triangular form.

Lemma 6.1. *Inequality (7) is in tight triangular form.*

Proof. First, we argue that $f_{ij} + f_{jk} \geq f_{ik}$ for all distinct triples i, j, k . Note that, for any edge e , $d-2 \leq f_e \leq d-2+(d-1)$. If either $f_{ij} \geq d-1$ or $f_{jk} \geq d-1$ (or both), we have that

$$f_{ik} \leq d-2+(d-1) \leq f_{ij} + f_{jk}.$$

Hence, $f_{ij} + f_{jk} \geq f_{ik}$ except possibly in the case in which $f_{ij} = f_{jk} = d-2$. This case, however, requires that both $\{i, j\}$ and $\{j, k\}$ be edges of length $d = n/2$ so that i and k are not distinct. Thus, $f_{ij} + f_{jk} \geq f_{ik}$.

We must also show that, for each $j \in V$, there exist some $i, k \in V \setminus \{j\}$ such that $f_{ij} + f_{jk} = f_{i,k}$. By circulant symmetry, if this holds for some $j \in V$, it holds for all $j \in V$. Without loss of generality, take $j = d+1$. Then, we take $i = 1$ and $k = d$ so that $\{1, d\}$ is an edge of length $d-1$:

$$f_{i,k} = f_{1,d} = d-2+(d-1) = f_{1,d+1} + f_{d+1,d} = f_{i,j} + f_{j,k}.$$

Thus, Inequality (7) is in tight triangular form. $\square$

We can now readily compute the strength of our inequality.

Theorem 6.1. *The strength of Inequality (7) is*

$$\frac{n^2 - 2n - 4}{n^2 - 3n} \leq \frac{11}{10}.$$

It is equal to $\frac{11}{10}$ when $n = 8$.

Proof. By Goemans [15, theorem 2.11], its strength relative to the subtour elimination polyhedron $SP(n)$ is

$$\frac{\frac{n^2}{2} - n - 2}{\min\{fx : x \in SP(n)\}}.$$

Gutekunst and Williamson [20, theorems 3.1 and 4.1] show that $\min\{fx : x \in SP(n)\}$ is attained by

$$x_e = \begin{cases} \frac{1}{2}, & \ell_e = 1 \\ 1, & \ell_e = d \\ 0, & \text{else.} \end{cases}$$

In the notation of Inequality (7), we find that $f_1 = d-2+1 = d-1$, whereas $f_d = d-2$. This solution places a total weight d across all edges of length d , a total weight of d across all edges of length 1, and a total weight of zero on all other edges. Hence,

$$\begin{aligned} \min\{fx : x \in SP(n)\} &= d(d-1) + d(d-2) \\ &= d(2d-3) \\ &= \frac{n}{2}(n-3). \end{aligned}$$

Thus, the strength of the circlet inequality is

$$\frac{\frac{n^2}{2} - n - 2}{\min\{fx : x \in SP(n)\}} = \frac{\frac{n^2}{2} - n - 2}{\frac{n^2}{2} - \frac{3n}{2}} = \frac{n^2 - 2n - 4}{n^2 - 3n}. \quad \square$$

We note that the circlet inequality appears to be marginally stronger than that of the crown inequality of Naddef and Rinaldi [28], which was also motivated by the subtour LP solution placing $1/2$ weight on every length 1 edge and 1 weight on every length d edge. The crown inequality has strength

$$\frac{3d\left(\frac{d}{2}-1\right)-1}{3d\left(\frac{d}{2}-1\right)-\frac{d}{2}} = \frac{3n(n-4)-8}{3n(n-4)-2n} = \frac{3n^2-12n-8}{3n^2-14n} \leq \frac{11}{10}.$$

The bound of $\frac{11}{10}$ is also attained when $n = 8$ by the crown inequality; otherwise, ours is marginally stronger. For comparison, Goemans [15] also gives the strength of the comb inequalities ($\frac{10}{9}$) and the clique tree inequalities ($\frac{8}{7}$).

7. Conclusions

The main results of this paper introduce a new facet-defining inequality for the TSP. This inequality is comparable to the crown inequalities of Naddef and Rinaldi [28] in terms of the standard definition of strength. We note, however, that this standard notation of strength is not as applicable to the circulant TSP. If edge costs are metric, then a minimum-cost Hamiltonian cycle costs exactly the same as a minimum-cost Eulerian sub(multi)graph: because a Hamiltonian cycle is itself Eulerian,

$$\text{Cost of cheapest Eulerian sub(multi)graph} \leq \text{Cost of minimum-cost Hamiltonian cycle}.$$

When edge costs are metric, any Eulerian sub(multigraph) can, however, also be shortcut (see Williamson and Shmoys [33, section 2.4] for details of shortcutting) to obtain a Hamiltonian cycle of no greater cost. Hence,

$$\text{Cost of cheapest Eulerian sub(multi)graph} \geq \text{Cost of minimum-cost Hamiltonian cycle}.$$

On typical metric instances, then, optimizing over $STSP(n)$ and optimizing over $GTSP(n)$ yields the same solution. That is not the case with circulant instances.

For example, consider the circulant TSP instances motivating Inequality (2): instances in which $4|n$, edges of length 1 cost 1, and edges of length d cost 0. As argued in Gutekunst and Williamson [20], the optimal TSP solution (i.e., the minimum cost Hamiltonian cycle) uses two length- d edges and, hence, costs $n - 2$. In contrast, Figure 15 shows a minimum-cost Eulerian sub(multi)graph of cost $d = n/2$. Hence, on this circulant instance, optimizing over $GTSP(n)$ instead of $STSP(n)$ is nearly a factor of two off.

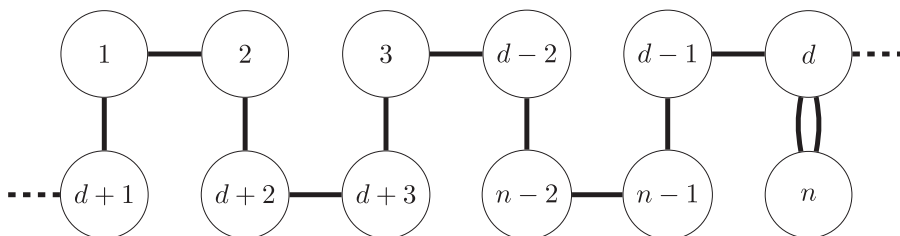
It is this discrepancy between $STSP(n)$ and $GTSP(n)$ on circulant instances that lends to the circlet inequality's weak strength: the circulant TSP is primarily concerned with understanding what combinations of edge lengths lead to a Hamiltonian tour, and shortcutting a Eulerian sub(multi)graph fundamentally changes edge lengths.

We end this paper by asking three questions. First, given a solution to the subtour LP, can we efficiently determine whether it violates a circlet inequality (and, if so, the labeling of the nodes that gives rise to the violated inequality)? Second, what is the right analogue of strength for circulant TSP inequalities? Finally, the crown inequalities are also circulant. What other circulant facet-defining inequalities are there for the TSP? Such inequalities may help define the edge-length polytope of the TSP:

$$EL(n) := \text{conv} \left\{ (t_1, \dots, t_d) : x \in STSP(n), t_i = \sum_{\{s,t\} \in E: \ell_{s,t}=i} x_{s,t} \right\}.$$

We hope that further polyhedral results on $EL(n)$ might open the door to new TSP results more generally and bridge connections between combinatorial optimization and number theory.

Figure 15. A minimum-cost Eulerian sub(multi)graph when length 1 edges cost 1, length d edges cost 0, and all other edge costs are arbitrarily large. The dashed edge “wraps around,” connecting d to $d + 1$.



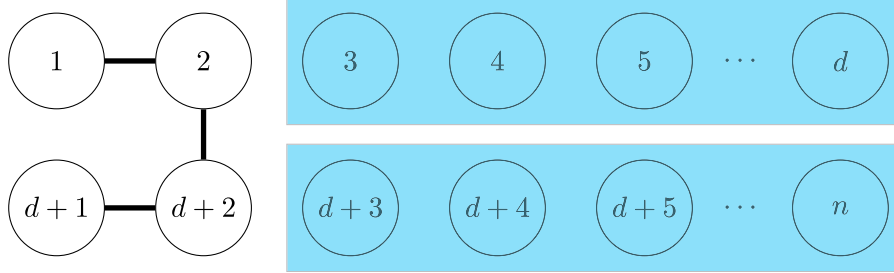
Acknowledgments

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. The authors thank the referees for their helpful feedback.

Appendix

A.1. A Casework for Propositions 4.2 and 4.3

Throughout the casework, we refer to the groups indicated in the block picture.



The top group consists of vertices s at which $3 \leq s \leq d$, whereas the bottom group consists of vertices s at which $d+3 \leq s \leq n$.

A.1.1. Casework for Proposition 4.2.

We first consider casework for Proposition 4.2. We want to show that

$$c_{j,1} + c_{d+1,k} - c'_{j,k'} \geq 2,$$

and so we show

$$c_{j,1} + c_{d+1,k} - c'_{j,k'} - 2 \geq 0.$$

To do so, we consider the possible parities of j and k that determine, for example, if $c_{j,1} = \ell_{j,1}$ or $c_{j,1} = d - \ell_{j,1}$.

Case 1: the values j and k are both even. In this case,

$$c_{j,1} + c_{d+1,k} - c'_{j,k'} - 2 = \ell_{j,1} + \ell_{d+1,k} - (d - 2 - \ell'_{j,k'}) - 2. \quad (\text{A.1})$$

- If j, k are in the same group, then $\ell'_{j,k'} = \ell_{j,k} = |j - k|$. Without loss of generality, suppose that j, k are in the top group. Then, Equation (A.1) becomes

$$c_{j,1} + c_{d+1,k} - c'_{j,k'} - 2 = (j - 1) + (d + 1 - k) - (d - 2 - |j - k|) - 2 = j - k + |j - k| \geq 0,$$

because $x + |x| \geq 0$. (If they were in the bottom group, it would be $k - j + |k - j| \geq 0$.)

- If j is on top and k is on bottom, then

$$\ell'_{j,k'} = \ell_{j,k} - 2 = \begin{cases} (k - j) - 2, & k < d + j \\ n - (k - j) - 2, & \text{else.} \end{cases}$$

Then, Equation (A.1) becomes

$$\begin{aligned} c_{j,1} + c_{d+1,k} - c'_{j,k'} - 2 &= (j - 1) + (k - (d + 1)) - (d - 2 - \ell'_{j,k'}) - 2 \\ &= j + k - n - 2 + \ell'_{j,k'} \\ &= j + k - n - 2 + \begin{cases} (k - j) - 2, & k < d + j \\ n - (k - j) - 2, & \text{else} \end{cases} \\ &= \begin{cases} 2k - n - 4, & k < d + j \\ 2j - 4, & \text{else.} \end{cases} \end{aligned}$$

Both are nonnegative as $j \geq 3$ and $k \geq d + 3$.

- If k is on top and j is on bottom, then

$$\ell'_{j,k'} = \ell_{j,k} - 2 = \begin{cases} (j - k) - 2, & j < d + k \\ n - (j - k) - 2, & \text{else.} \end{cases}$$

Then, Equation (A.1) becomes

$$\begin{aligned} c_{j,1} + c_{d+1,k} - c'_{j,k'} - 2 &= (n - j + 1) + (d + 1 - k) - (d - 2 - \ell'_{j,k'}) - 2 \\ &= n - j - k + 2 + \ell'_{j,k'} \\ &= n - j - k + 2 + \begin{cases} (j - k) - 2, & j < d + k \\ n - (j - k) - 2, & \text{else} \end{cases} \\ &= \begin{cases} n - 2k, & j < d + k \\ 2n - 2j, & \text{else.} \end{cases} \end{aligned}$$

Both are nonnegative as $k \leq d$ and $j \leq n$.

Case 2: j and k are both odd.

In this case,

$$c_{j,1} + c_{d+1,k} - c'_{j',k'} - 2 = d - \ell_{1,j} + d - \ell_{d+1,k} - (d - 2 - \ell'_{j',k'}) - 2 = d - \ell_{1,j} - \ell_{d+1,k} + \ell'_{j',k'}. \quad (\text{A.2})$$

- If j, k are in the same group, then $\ell'_{j',k'} = \ell_{j,k} = |j - k|$. Without loss of generality, suppose that j, k are in the top group. Then, Equation (A.2) becomes

$$d - \ell_{1,j} - \ell_{d+1,k} + \ell'_{j',k'} = d - (j - 1) - (d + 1 - k) + |j - k| = k - j + |k - j| \geq 0,$$

because $x + |x| \geq 0$. (If they were in the bottom group, it would be $j - k + |j - k| \geq 0$.)

- If j is on top and k is on bottom, then

$$\ell'_{j',k'} = \ell_{j,k} - 2 = \begin{cases} (k - j) - 2, & k < d + j \\ n - (k - j) - 2, & \text{else.} \end{cases}$$

Equation (A.2) becomes

$$\begin{aligned} d - \ell_{1,j} - \ell_{d+1,k} + \ell'_{j',k'} &= d - (j - 1) - (k - (d + 1)) + \ell'_{j',k'} \\ &= n + 2 - j - k + \ell'_{j',k'} \\ &= n + 2 - j - k + \begin{cases} (k - j) - 2, & k < d + j \\ n - (k - j) - 2, & \text{else} \end{cases} \\ &= \begin{cases} n - 2j, & k < d + j \\ 2n - 2k, & \text{else.} \end{cases} \end{aligned}$$

Both are nonnegative as $j \leq d$ and $k \leq n$.

- If k is on top and j is on bottom, then

$$\ell'_{j',k'} = \ell_{j,k} - 2 = \begin{cases} (j - k) - 2, & j < d + k \\ n - (j - k) - 2, & \text{else.} \end{cases}$$

Equation (A.2) becomes

$$\begin{aligned} d - \ell_{1,j} - \ell_{d+1,k} + \ell'_{j',k'} &= d - (n - j + 1) - (d + 1 - k) + \ell'_{j',k'} \\ &= j + k - n - 2 + \ell'_{j',k'} \\ &= j + k - n - 2 + \begin{cases} (j - k) - 2, & j < d + k \\ n - (j - k) - 2, & \text{else} \end{cases} \\ &= \begin{cases} 2j - n - 4, & j < d + k \\ 2k - 4, & \text{else.} \end{cases} \end{aligned}$$

Both are nonnegative as $j \geq d + 3$ and $k \geq 3$.

Case 3: j and k have opposite parity.

Without loss of generality, we let j be odd and k be even. Then,

$$c_{j,1} + c_{d+1,k} - c'_{j',k'} - 2 = d - \ell_{1,j} + \ell_{d+1,k} - \ell'_{j',k'} - 2. \quad (\text{A.3})$$

- If j, k are both in the top group, then $\ell'_{j',k'} = \ell_{j,k} = |j - k|$. Then, Equation (A.3) becomes

$$d - \ell_{1,j} + \ell_{d+1,k} - \ell'_{j',k'} - 2 = d - (j - 1) + (d + 1 - k) - |j - k| - 2 = n - j - k - |j - k| = \begin{cases} n - 2j, & j > k \\ n - 2k, & \text{else.} \end{cases}$$

Both are nonnegative because $j, k \leq d$.

- If j, k are both in the bottom group, then $\ell'_{j',k'} = \ell_{j,k} = |j - k|$. Equation (A.3) becomes

$$d - \ell_{1,j} + \ell_{d+1,k} - \ell'_{j',k'} - 2 = d - (n - j + 1) + (k - (d + 1)) - |j - k| - 2 = j + k - |j - k| - n - 4 = \begin{cases} 2k - n - 4, & j > k \\ 2j - n - 4, & \text{else,} \end{cases}$$

which is nonnegative because $j, k \geq d + 3$.

- If j is on top and k is on bottom, then

$$\ell'_{j',k'} = \ell_{j,k} - 2 = \begin{cases} (k - j) - 2, & k < d + j \\ n - (k - j) - 2, & \text{else.} \end{cases}$$

Equation (A.3) becomes

$$\begin{aligned} d - \ell_{1,j} + \ell_{d+1,k} - \ell'_{j,k'} - 2 &= d - (j - 1) + (k - (d + 1)) - \ell'_{j,k'} - 2 \\ &= k - j - 2 - \ell'_{j,k'} \\ &= k - j - 2 - \begin{cases} (k - j) - 2, & k < d + j \\ n - (k - j) - 2, & \text{else} \end{cases} \\ &= \begin{cases} 0, & k < d + j \\ 2k - 2j - n & \text{else.} \end{cases} \end{aligned}$$

Both are nonnegative as in the second case $2k - 2j \geq 2d = n$.

- If k is on top and j is on bottom, then

$$\ell'_{j,k'} = \ell_{j,k} - 2 = \begin{cases} (j - k) - 2, & j < d + k \\ n - (j - k) - 2, & \text{else.} \end{cases}$$

Equation (A.3) becomes

$$\begin{aligned} d - \ell_{1,j} + \ell_{d+1,k} - \ell'_{j,k'} - 2 &= d - (n - j + 1) + (d + 1 - k) - \ell'_{j,k'} - 2 \\ &= j - k - 2 - \ell'_{j,k'} \\ &= j - k - 2 - \begin{cases} (j - k) - 2, & j < d + k \\ n - (j - k) - 2, & \text{else} \end{cases} \\ &= \begin{cases} 0, & j < d + k \\ 2j - 2k - n, & \text{else.} \end{cases} \end{aligned}$$

Both are nonnegative as in the second case $2j - 2k \geq 2d = n$.

Up to symmetry, this shows that, in all possible cases,

$$c_{j,1} + c_{d+1,k} - c'_{j,k'} - 2 \geq 0.$$

For completeness, we include formulas for when j is even and k is odd. These are

- If j, k are in the top, $j + k - |j - k| - 4$.
- If j, k are in the bottom, $2n - j - k - |j - k|$.
- If j is on the top and k is on the bottom,

$$\begin{cases} 0, & k > d + j \\ n + 2j - 2k, & \text{else.} \end{cases}$$

- If j is on the bottom and k is on the top,

$$\begin{cases} 0, & j > d + k \\ n - 2j + 2k, & \text{else.} \end{cases}$$

A.2. A Casework for Proposition 4.3

The computations for Proposition 4.3 follow as before. We want to show that

$$c_{j,d+2} + c_{d+1,k} - c'_{j,k'} - 1 \geq 0,$$

and compute the left-hand side. Here, we just write the results, highlighting those when the inequality can be tight.

Case 1: j and k are both even. In this case,

$$c_{j,d+2} + c_{d+1,k} - c'_{j,k'} - 1 = d - \ell_{j,d+2} + \ell_{d+1,k} - (d - 2 - \ell'_{j,k'}) - 1. \quad (\text{A.4})$$

- If j, k are in the top, $\ell'_{j,k'} = \ell_{j,k} = |j - k|$. Then, Equation (A.4) becomes

$$d - \ell_{j,d+2} + \ell_{d+1,k} - (d - 2 - \ell'_{j,k'}) - 1 = d - (d + 2 - j) + (d + 1 - k) - (d - 2 - |j - k|) - 1 = j - k + |j - k|.$$

This is zero when $j < k$.

- If j, k are in the bottom, then Equation (A.4) becomes

$$d - \ell_{j,d+2} + \ell_{d+1,k} - (d - 2 - \ell'_{j,k'}) - 1 = d - (j - (d + 2)) + (k - (d + 1)) - (d - 2 - |j - k|) - 1 = 2 + |j - k| - (j - k).$$

This is never zero because $|x| - x \geq 0$.

- If j is on the top and k is on the bottom, then

$$\ell'_{j',k'} = \ell_{j,k} - 2 = \begin{cases} (k-j) - 2, & k < d+j \\ n - (k-j) - 2, & \text{else.} \end{cases}$$

Equation (A.4) becomes

$$\begin{aligned} d - \ell_{j,d+2} + \ell_{d+1,k} - (d-2 - \ell'_{j',k'}) - 1 &= d - (d+2-j) + (k - (d+1)) - (d-2 - \ell'_{j',k'}) - 1 \\ &= j + k - 2 - n + \ell'_{j',k'} \\ &= \begin{cases} j + k - n + (k-j) - 4, & k < d+j \\ j + k - n + n - (k-j) - 4, & \text{else} \end{cases} \\ &= \begin{cases} 2k - n - 4, & k < d+j \\ 2j - 4, & \text{else.} \end{cases} \end{aligned}$$

Because $j \geq 3$ and $k \geq d+3$, these are never zero.

- If j is on the bottom and k is on the top, then

$$\ell'_{j',k'} = \ell_{j,k} - 2 = \begin{cases} (j-k) - 2, & j < d+k \\ n - (j-k) - 2, & \text{else.} \end{cases}$$

Equation (A.4) becomes

$$\begin{aligned} d - \ell_{j,d+2} + \ell_{d+1,k} - (d-2 - \ell'_{j',k'}) - 1 &= d - (j - (d+2)) + (d+1-k) - (d-2 - \ell'_{j',k'}) - 1 \\ &= n - j - k + 4 + \ell'_{j',k'} \\ &= \begin{cases} n - 2k + 2, & j < d+k \\ 2n - 2j + 2, & \text{else.} \end{cases} \end{aligned}$$

Because $j \leq n$ and $k \leq d$, these are never zero.

Case 2: j and k are both odd.

In this case,

$$c_{j,d+2} + c_{d+1,k} - c'_{j',k'} - 1 = \ell_{j,d+2} + (d - \ell_{d+1,k}) - (d-2 - \ell'_{j',k'}) - 1 = \ell_{j,d+2} - \ell_{d+1,k} + \ell'_{j',k'} + 1. \quad (\text{A.5})$$

- If j, k are in the top, $\ell'_{j',k'} = \ell_{j,k} = |j-k|$. Then, Equation (A.5) becomes

$$\ell_{j,d+2} - \ell_{d+1,k} + \ell'_{j',k'} + 1 = (d+2-j) - (d+1-k) + |j-k| + 1 = 2 + |k-j| + (k-j).$$

This is never zero because $|x| + x \geq 0$.

- If j, k are in the bottom, $\ell'_{j',k'} = \ell_{j,k} = |j-k|$. Then, Equation (A.5) becomes

$$\ell_{j,d+2} - \ell_{d+1,k} + \ell'_{j',k'} + 1 = (j - (d+2)) - (k - (d+1)) + |k-j| + 1 = |j-k| + j - k.$$

This is zero when $j < k$.

- If j is on the top and k is on the bottom,

$$\ell'_{j',k'} = \ell_{j,k} - 2 = \begin{cases} (k-j) - 2, & k < d+j \\ n - (k-j) - 2, & \text{else.} \end{cases}$$

Then, Equation (A.5) becomes

$$\begin{aligned} \ell_{j,d+2} - \ell_{d+1,k} + \ell'_{j',k'} + 1 &= (d+2-j) - (k - (d+1)) + \ell'_{j',k'} + 1 \\ &= n + 4 - j - k + \ell'_{j',k'} \\ &= n + 4 - j - k + \begin{cases} (k-j) - 2, & k < d+j \\ n - (k-j) - 2, & \text{else} \end{cases} \\ &= \begin{cases} n + 2 - 2j, & k < d+j \\ 2n + 2 - 2k, & \text{else.} \end{cases} \end{aligned}$$

Because $k \leq n$ and $j \leq d$, these are never zero.

- If j is on the bottom and k is on the top,

$$\ell'_{j',k'} = \ell_{j,k} - 2 = \begin{cases} (j-k) - 2, & j < d+k \\ n - (j-k) - 2, & \text{else.} \end{cases}$$

Then, Equation (A.5) becomes

$$\begin{aligned}\ell_{j,d+2} - \ell_{d+1,k} + \ell'_{j,k'} + 1 &= (j - (d+2)) - (d+1-k) + \ell'_{j,k'} + 1 \\ &= j + k - n - 2 + \ell'_{j,k'} \\ &= j + k - n - 2 + \begin{cases} (j-k) - 2, & j < d+k \\ n - (j-k) - 2, & \text{else} \end{cases} \\ &= \begin{cases} 2j - n - 4, & j < d+k \\ 2k - 4, & \text{else.} \end{cases}\end{aligned}$$

Because $k \geq 3$ and $j \geq d+3$, these are never zero.

Case 3: j is odd and k is even.

In this case,

$$c_{j,d+2} + c_{d+1,k} - c'_{j,k'} - 1 = \ell_{j,d+2} + \ell_{d+1,k} - \ell'_{j,k'} - 1. \quad (\text{A.6})$$

- If j, k are in the top, then $\ell'_{j,k'} = \ell_{j,k} = |j-k|$. Equation (A.6) becomes

$$\ell_{j,d+2} + \ell_{d+1,k} - \ell'_{j,k'} - 1 = (d+2-j) + (d+1-k) - |j-k| - 1 = n+2-j-k-|j-k| = \begin{cases} n+2-2j, & j > k \\ n+2-2k, & \text{else.} \end{cases}$$

These are never zero because $j, k \leq d$.

- If j, k are in the bottom, then $\ell'_{j,k'} = \ell_{j,k} = |j-k|$. Equation (A.6) becomes

$$\ell_{j,d+2} + \ell_{d+1,k} - \ell'_{j,k'} - 1 = j - (d+2) + k - (d+1) - |j-k| - 1 = j+k-|j-k|-n-4 = \begin{cases} 2k-n-4, & j > k \\ 2j-n-4, & \text{else.} \end{cases}$$

These are never zero because $j, k \geq d+3$.

- If j is on the top and k is on the bottom, then

$$\ell'_{j,k'} = \ell_{j,k} - 2 = \begin{cases} (k-j) - 2, & k < d+j \\ n - (k-j) - 2, & \text{else.} \end{cases}$$

Equation (A.6) becomes

$$\begin{aligned}\ell_{j,d+2} + \ell_{d+1,k} - \ell'_{j,k'} - 1 &= (d+2-j) + (k - (d+1)) - \ell'_{j,k'} - 1 \\ &= k - j - \ell'_{j,k'} \\ &= k - j - \begin{cases} (k-j) - 2, & k < d+j \\ n - (k-j) - 2, & \text{else} \end{cases} \\ &= \begin{cases} 2 & k < d+j \\ 2k - 2j - n + 2 & \text{else.} \end{cases}\end{aligned}$$

Because the second case occurs only when $k \geq d+j \Rightarrow 2k - 2j - n \geq 0$, these are never zero.

- If j is on the bottom and k is on the top, then

$$\ell'_{j,k'} = \ell_{j,k} - 2 = \begin{cases} (j-k) - 2, & j < d+k \\ n - (j-k) - 2, & \text{else.} \end{cases}$$

Equation (A.6) becomes

$$\begin{aligned}\ell_{j,d+2} + \ell_{d+1,k} - \ell'_{j,k'} - 1 &= (j - (d+2)) + (d+1-k) - \ell'_{j,k'} - 1 \\ &= j - k - 2 - \ell'_{j,k'} \\ &= j - k - 2 - \begin{cases} (j-k) - 2, & j < d+k \\ n - (j-k) - 2, & \text{else} \end{cases} \\ &= \begin{cases} 0, & j < d+k \\ 2j - 2k - n, & \text{else.} \end{cases}\end{aligned}$$

The second case is not zero whenever $j > d+k$. Hence, the equation is zero when $j \leq d+k$.

Case 4: j is even and k is odd.

In this case,

$$c_{j,d+2} + c_{d+1,k} - c'_{j,k'} - 1 = (d - \ell_{j,d+2}) + (d - \ell_{d+1,k}) - \ell'_{j,k'} - 1 = n - \ell_{j,d+2} - \ell_{d+1,k} - \ell'_{j,k'} - 1. \quad (\text{A.7})$$

- If j, k are in the top, then $\ell'_{j,k'} = \ell_{j,k} = |j - k|$. Equation (A.7) becomes

$$n - \ell_{j,d+2} - \ell_{d+1,k} - \ell'_{j,k'} - 1 = n - (d + 2 - j) - (d + 1 - k) - |j - k| - 1 = j + k - |j - k| - 4 = \begin{cases} 2k - 4, & j > k \\ 2j - 4, & \text{else.} \end{cases}$$

These are never zero because $j, k \geq 3$.

- If j, k are in the bottom, then $\ell'_{j,k'} = \ell_{j,k} = |j - k|$. Equation (A.7) becomes

$$n - \ell_{j,d+2} - \ell_{d+1,k} - \ell'_{j,k'} - 1 = n - (j - (d + 2)) - (k - (d + 1)) - |j - k| - 1 = 2n + 2 - j - k - |j - k|$$

and

$$2n + 2 - j - k - |j - k| = \begin{cases} 2n + 2 - 2j, & j > k \\ 2n + 2 - 2k, & \text{else.} \end{cases}$$

These are never zero because $j, k \leq n$.

- If j is on the top and k is on the bottom, then

$$\ell'_{j,k'} = \ell_{j,k} - 2 = \begin{cases} (k - j) - 2, & k < d + j \\ n - (k - j) - 2, & \text{else.} \end{cases}$$

Equation (A.7) becomes

$$\begin{aligned} n - \ell_{j,d+2} - \ell_{d+1,k} - \ell'_{j,k'} - 1 &= n - (d + 2 - j) - (k - (d + 1)) - \ell'_{j,k'} - 1 \\ &= n + j - k - 2 - \ell'_{j,k'} \\ &= n + j - k - 2 - \begin{cases} (k - j) - 2, & k < d + j \\ n - (k - j) - 2, & \text{else} \end{cases} \\ &= \begin{cases} n + 2j - 2k, & k < d + j \\ 0, & \text{else.} \end{cases} \end{aligned}$$

The first case is not zero because $d + j - k > 0$. Hence, the equation is zero when $k \geq d + j$.

- If j is on the bottom and k is on the top, then

$$\ell'_{j,k'} = \ell_{j,k} - 2 = \begin{cases} (j - k) - 2, & j < d + k \\ n - (j - k) - 2, & \text{else.} \end{cases}$$

Equation (A.7) becomes

$$\begin{aligned} n - \ell_{j,d+2} - \ell_{d+1,k} - \ell'_{j,k'} - 1 &= n - (j - (d + 2)) - (d + 1 - k) - \ell'_{j,k'} - 1 \\ &= n - j + k - \ell'_{j,k'} \\ &= n - j + k - \begin{cases} (j - k) - 2, & j < d + k \\ n - (j - k) - 2, & \text{else} \end{cases} \\ &= \begin{cases} n + 2 - 2j + 2k, & j < d + k \\ 2, & \text{else.} \end{cases} \end{aligned}$$

The first case is not zero because $j < d + k$ implies $0 < n + 2k - 2j$.

In all cases,

$$c_{j,d+2} + c_{d+1,k} - c'_{j,k'} - 1 \geq 0,$$

with equality only when

- $j, k \leq d$, both j, k even, and $j < k$.
- $j, k \geq d + 3$, both j, k odd, and $j < k$.
- $j \geq d + 3$ and odd, $k \leq d$ and even, and $j \leq d + k$.
- $j \leq d$ and even, $k \geq d + 3$ and odd, and $k \geq d + j$.

This means that equality can hold only in those cases mentioned in the proof of Proposition 4.3.

Endnotes

¹ For results on hardness see, for example, Williamson and Shmoys [33, theorem 2.9]. Even with more restrictive assumptions, such as that the edge costs are metric (i.e., $c_{ij} \leq c_{ik} + c_{kj}$ for all distinct $i, j, k \in [n]$), it is known to be NP-hard to approximate TSP solutions in polynomial time to within any constant factor $\alpha < \frac{123}{122}$ (see Karpinski et al. [23]).

² Bill Cook, one of the authors of Concorde, generously ran our instances on Concorde and it took more than 40 hours to verify the circlet inequality when $n = 32$. He noted that he sometimes found difficult small instances for Concorde to solve, “but [that] 32 nodes might be a record.”

References

- [1] Applegate DL, Bixby R, Chvátal V, Cook W (2006) Concorde TSP solver.
- [2] Applegate DL, Bixby RE, Chvátal V, Cook WJ (2006) *The Traveling Salesman Problem: A Computational Study* (Princeton University Press, Princeton, NJ).
- [3] Applegate DL, Bixby RE, Chvátal V, Cook W, Espinoza DG, Goycoolea M, Helsgaun K (2009) Certification of an optimal TSP tour through 85,900 cities. *Oper. Res. Lett.* 37(1):11–15.
- [4] Boyd SC, Cunningham WH (1991) Small traveling salesman polytopes. *Math. Oper. Res.* 16(2):259–271.
- [5] Buratti M, Merola F (2013) Dihedral Hamiltonian cycle systems of the cocktail party graph. *J. Combinatorial Designs* 21(1):1–23.
- [6] Burkard R, Sandholzer W (1991) Efficiently solvable special cases of bottleneck traveling salesman problems. *Discrete Appl. Math.* 32(1):61–76.
- [7] Chvátal V (1973) Edmonds polytopes and weakly Hamiltonian graphs. *Math. Programming* 5(1):29–40.
- [8] Codenotti B, Gerace I, Vigna S (1998) Hardness results and spectral techniques for combinatorial problems on circulant graphs. *Linear Algebra Appl.* 285(1):123–142.
- [9] Cornuéjols G, Fonlupt J, Naddef D (1985) The traveling salesman problem on a graph and some related integer polyhedra. *Math. Programming* 33(1):1–27.
- [10] Costa S, Morini F, Pasotti A, Pellegrini MA (2018) A problem on partial sums in Abelian groups. *Discrete Math.* 341(3):705–712.
- [11] Cunningham WH (1986) On bounds for the metric TSP. Manuscript. School of Mathematics and Statistics, Carleton University, Ottawa.
- [12] Dantzig G, Fulkerson R, Johnson S (1954) Solution of a large-scale traveling-salesman problem. *J. Oper. Res. Soc. Amer.* 2(4):393–410.
- [13] Garfinkel RS (1977) Minimizing wallpaper waste, part 1: A class of traveling salesman problems. *Oper. Res.* 25(5):741–751.
- [14] Gilmore PC, Lawler EL, Shmoys DB (1985) Well-solved special cases. Lawler EL, Lenstra JK, Rinnooy Kan AHG, Shmoys DB, eds. *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization* (John Wiley and Sons, New York), 87–143.
- [15] Goemans MX (1995) Worst-case comparison of valid inequalities for the TSP. *Math. Programming* 69:335–349.
- [16] Grötschel M (1980) On the monotone symmetric travelling salesman problem: Hypohamiltonian/hypotractable graphs and facets. *Math. Oper. Res.* 5(2):285–292.
- [17] Grötschel M, Padberg MW (1979) On the symmetric travelling salesman problem I: Inequalities. *Math. Programming* 16(1):265–280.
- [18] Grötschel M, Padberg MW (1985) Polyhedral theory. Lawler EL, Lenstra J, Kan AHGR, Shmoys DB, eds. *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization* (John Wiley & Sons, New York), 251–306.
- [19] Grötschel M, Pulleyblank WR (1986) Clique tree inequalities and the symmetric travelling salesman problem. *Math. Oper. Res.* 11(4):537–569.
- [20] Gutekunst SC, Williamson DP (2019) Characterizing the integrality gap of the subtour LP for the circulant traveling salesman problem. *SIAM J. Discrete Math.* 33(4):2452–2478.
- [21] Held M, Karp RM (1970) The traveling-salesman problem and minimum spanning trees. *Oper. Res.* 18(6):1138–1162.
- [22] Horak P, Rosa A (2009) On a problem of Marco Buratti. *Electron. J. Combin.* 16(1):R20.
- [23] Karpinski M, Lampis M, Schmied R (2015) New inapproximability bounds for TSP. *J. Comput. System Sci.* 81(8):1665–1677.
- [24] Medova E (1993) Using QAP bounds for the circulant TSP to design reconfigurable networks. Pardalos PM, Wolkowicz H, eds. *Quadratic Assignment Related Problems, Proc. DIMACS Workshop*, Rutgers University, vol. 16 (American Mathematical Society), 275–292.
- [25] Naddef D (1992) The binested inequalities for the symmetric traveling salesman polytope. *Math. Oper. Res.* 17(4):882–900.
- [26] Naddef D (2007) Polyhedral theory and branch-and-cut algorithms for the symmetric TSP. Gutin G, Punnen AP, eds. *The Traveling Salesman Problem and Its Variations* (Springer, New York), 29–116.
- [27] Naddef D, Rinaldi G (1991) The symmetric traveling salesman polytope and its graphical relaxation: Composition of valid inequalities. *Math. Programming* 51(1–3):359–400.
- [28] Naddef D, Rinaldi G (1992) The crown inequalities for the symmetric traveling salesman polytope. *Math. Oper. Res.* 17(2):308–326.
- [29] Padberg MW, Hong S (1980) On the symmetric traveling salesman problem: A computational study. Padberg MW, ed. *Combinatorial Optimization* (Springer, Berlin, Heidelberg), 78–107.
- [30] Pasotti A, Pellegrini MA (2014) A new result on the problem of Buratti, Horak and Rosa. *Discrete Math.* 319(1):1–14.
- [31] Reinelt G (1991) TSPLIB: A traveling salesman problem library. *ORSA J. Comput.* 3(4):376–384.
- [32] Shmoys DB, Williamson DP (1990) Analyzing the Held-Karp TSP bound: A monotonicity property with application. *Inform. Processing Lett.* 35(6):281–285.
- [33] Williamson DP, Shmoys DB (2011) *The Design of Approximation Algorithms* (Cambridge University Press, New York).
- [34] Wolsey LA (1980) Heuristic analysis, linear programming and branch and bound. Rayward-Smith VJ, ed. *Combinatorial Optimization II* (Springer, Berlin, Heidelberg), 121–134.