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Low regularity error analysis for an H(div)-conforming discontinuous Galerkin approximation of Stokes problem

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ABSTRACT

In this paper, we derive an improved error estimate for the $\boldsymbol{H}(\mathrm{div})$-conforming discontinuous Galerkin (DG) approximation of the Stokes equations, assuming only minimal regularity on the exact solution. The estimate relies on both a priori and a posteriori analysis, and thus is called a medius error analysis. More precisely, we proved an optimal order error estimate under the assumption $(\boldsymbol{u}, p) \in \boldsymbol{H}^{1+s}(\Omega) \times H^s(\Omega)$ with any $s \in (0, 1]$. Extension to the standard interior penalty DG methods is also explored. Finally, numerical results are provided to verify our theoretical findings.

1. Introduction

The Stokes equations are used to model incompressible fluids. Since it is difficult to obtain the analytical solution of these equations, many researchers turn their attentions to numerical methods. Among them, finite element methods (FEMs) are a class of most commonly used numerical schemes for addressing such problems. As we known, the key step in designing mixed FEMs for Stokes problem is to check the inf-sup condition. For the classical conforming or nonconforming FEMs that satisfy this condition, we refer the reader to Girault and Raviart [1], and Brezzi et al. [2] for more detailed presentation. On the other hand, discontinuous Galerkin (DG) method [3] is another effective scheme for solving Stokes equations, see Schötzau et al. [4]. DG methods allow totally discontinuous functions of piecewise polynomials on the triangulation. Thus, they can easily deal with highly nonuniform and unstructured meshes. Moreover, they have flexibility in handling inhomogeneous boundary conditions and curved boundaries, and they are also suitable for hp-adaptive computation. Recently, combining interior penalty discontinuous Galerkin (IPDG) methods with $\boldsymbol{H}(\mathrm{div})$ mixed finite elements, the authors design a pressure-robust scheme for the Stokes problem [5–9]. Constructing pressure-robust numerical schemes to discretize for Stokes problem has drawn more and more attentions in recent years. We refer the reader to [10–22] for more details. It is also worth mentioning that $\boldsymbol{H}(\mathrm{div})$-conforming IPDG methods are effective numerical methods for addressing Darcy-Stokes problem [23–28].

For the above $\boldsymbol{H}(\mathrm{div})$ IPDG method, Wang and Ye [7] established a priori error estimate under the assumption that the exact solution is smooth enough. Later, a residual-based a posterior estimator was developed for the adaptive computation [29]. In this

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work, we shall further derive an improved error estimates for this method under minimal regularity assumption for the exact solution. To obtain the desired estimate, we utilize some technical tools developed by Gudi [30], which are based on both a priori and a posteriori error analysis, and thus is called a medius error analysis. Recently, some works have devoted to the convergence analysis of Stokes equations for various FEMs under minimal regularity assumption. We refer [31] for stable conforming FEMs, and [32] for nonconforming FEMs. In [32], Li et al. employed an enrichment operator to map the nonconforming FE functions to a conforming FE space. This approach has been further extended to the other pressure-robust schemes [10,12,16,17,22]. In the context of DG methods, an error analysis with minimal regularity assumption has been established by [33]. In this work, we shall further investigated the error estimates for the $\boldsymbol{H}(\text{div})$ IPDG method. It is worth mentioning that our analysis is different from the one stated in [33], in which the estimates for the velocities and pressure are processed separately. In particular, in [33], some more involved techniques based on $\boldsymbol{H}(\text{div})$ finite element are needed to derive pressure error estimate. While in this work, we provide the error analysis for the velocities and pressure via the stability result directly (see Theorem 3.3 below) and in a unified approach. We should note that similar idea has been explored for stable conforming FEMs [31]. Our work can be viewed an extended approach to the $\boldsymbol{H}(\text{div})$ IPDG method. Moreover, we point that our approach is not limited to $\boldsymbol{H}(\text{div})$ IPDG case, it can be also extended to the standard IPDG schemes (see Section 5 for more details). Naturally the method remains a pressure robust scheme in solving a low regularity Stokes problem, i.e., the velocity error is independent of pressure, as we use the $H(\text{div})$ finite elements.

The rest of our paper is organized as follows. In Section 2, we first introduce the model problem and then describe the $\boldsymbol{H}(\text{div})$ IPDG method. The stability result for the $\boldsymbol{H}(\text{div})$ IPDG method is presented in Section 3. Next, based on the results obtained in Section 3, we derive a medius error analysis for the $\boldsymbol{H}(\text{div})$ IPDG method under the minimal regularity assumption in Section 4. Section 5 mainly discusses how to extend the corresponding error analysis to the standard IPDG method. Some numerical tests are provided in Section 6 to validate the theory result. Finally, some conclusions are made in Section 7.

2. Preliminaries

2.1. The Stokes equations

We first introduce some notations. For a bounded domain $\mathcal{D} \subset \mathbb{R}^2$, we denote by $H^s(\mathcal{D})(s \geq 0)$ the standard Sobolev space endowed with norm $\|\cdot\|_{s,\mathcal{D}}$ and seminorm $|\cdot|_{s,\mathcal{D}}$. When $s = 0$, $H^0(\mathcal{D})$ is the Lebesgue space $L^2(\mathcal{D})$, and its inner product is denoted by $(\cdot,\cdot)_{\mathcal{D}}$. We shall drop the subscript $\mathcal{D}$ when $\mathcal{D} = \Omega$. Additionally, we introduce $H_0^1(\Omega)$ as the subspace of $H^1(\Omega)$, in which the functions vanish on $\partial\Omega$, i.e., $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$. By convention, for the vector-valued analogs, we shall use boldface type: $\boldsymbol{H}^m(\mathcal{D}) = [H^m(\mathcal{D})]^2$. Moreover, we introduce the Hilbert space $\boldsymbol{H}(\text{div};\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \nabla\cdot\boldsymbol{v} \in L^2(\Omega)\}$ with its graph norm $\|\boldsymbol{v}\|_{\text{div}} = \left(\|\boldsymbol{v}\|_0^2 + \|\nabla\cdot\boldsymbol{v}\|_0^2\right)^{1/2}$. Similarly, we denote by $\boldsymbol{H}_0(\text{div};\Omega)$ the subspace of $\boldsymbol{H}(\text{div};\Omega)$ with vanishing normal trace on $\partial\Omega$, namely, $\boldsymbol{H}_0(\text{div};\Omega) = \{\boldsymbol{v} \in \boldsymbol{H}(\text{div};\Omega) : \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega} = 0\}$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with boundary $\partial\Omega$. Given the body force $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, we consider the following Stokes problem

$$-\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{3}$$

where $\boldsymbol{u}$ is the velocity field and p is the pressure.

For simplicity, we only consider the model problem (1)–(3) in two dimensions. The corresponding results can be extended to three dimensions with straightforward modifications.

It is well known that the weak formulation of the above Stokes problem is to find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \tag{4}$$

$$b(\boldsymbol{u},q) = 0 \qquad \forall q \in L_0^2(\Omega), \tag{5}$$

where

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\,dx, \tag{6}$$

$$b(\boldsymbol{u},q) = -\int_\Omega q\nabla\cdot\boldsymbol{u}\,dx, \tag{7}$$

and

$$L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}.$$

2.2. $\boldsymbol{H}(\mathrm{div})$-conforming DG methods

Let $\mathcal{T}_h$ be a family of conforming and shape-regular triangulation of Ω. For each element T, we denote $h_T = \mathrm{diam}(T)$, and the mesh size $h = \max_{T\in\mathcal{T}_h} h_T$. We use $\boldsymbol{n}_T$ to stand for its unit outward normal vector. Additionally, we denote by $\mathcal{E}_h^I$ the set of interior edges of $\mathcal{T}_h$, and by $\mathcal{E}_h^\partial$ the set of boundary edges on $\partial\Omega$. Thus, the set of all edges $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^\partial$. For each $e \in \mathcal{E}_h$, its length is denoted by h_e. In particularly, we introduce $\mathcal{E}_h^T$ to denote the set of edges of an element T, that is, $\mathcal{E}_h^T = \{e \in \mathcal{E}_h : e \subset \partial T\}$. We also use $P_k(D)$ to denote the space of polynomials of degree at most k on D. Similarly, $\boldsymbol{P}_k(D)$ denotes the vector-valued case. In what follows, all generic constants (with or without subscripts) in this paper are independent of h but may depend on the shape regularity of $\mathcal{T}_h$ and the polynomial degree k.

Let $e \in \mathcal{E}_h^I$ be an interior edge, which is shared by two adjacent elements T^+ and T^-. For convenience, the global index of T^+ is assumed smaller than that of T^-. For a piecewise smooth scalar, vector or tensor function v with $v^\pm = v|_{T^\pm}$, we define their averages and jumps by

$$\{v\} = \frac{1}{2}(v^+ + v^-), \qquad [v] = v^+ - v^-.$$

When restricted to a boundary edge $e \in \mathcal{E}_h^\partial \cap \partial T$, we set $\{v\} = v$ and $[v] = v$. Moreover, we associate each $e \in \mathcal{E}_h^I$ with the unit normal vector as $\boldsymbol{n}_e = \boldsymbol{n}_{T^+}|_e = -\boldsymbol{n}_{T^-}|_e$. Similarly, for $e \in \mathcal{E}_h^\partial$, its outward unit normal vector $\boldsymbol{n}_e$ is defined along $\partial\Omega$ restricted to e.

Now we introduce the two finite element spaces $\boldsymbol{V}_h$ and P_h. More precisely, the fluid velocity is approximated by $\boldsymbol{BDM}$ [34] element functions, while the pressure is discretized by the piecewise polynomial functions, that is,

$$\boldsymbol{V}_h = \left\{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega) : \boldsymbol{v}|_T \in \boldsymbol{BDM}_k(T), \ \ \forall T \in \mathcal{T}_h\right\},$$
$$P_h = \left\{q \in L_0^2(\Omega) : q|_T \in P_{k-1}(T), \ \ \forall T \in \mathcal{T}_h\right\},$$

where $\boldsymbol{BDM}_k(T) = \boldsymbol{P}_k(T)$.

For the standard symmetric IPDG method, it is well known that its bilinear form is defined by (see [3])

$$\begin{aligned} a_h(\boldsymbol{w},\boldsymbol{v}) = &\sum_{T\in\mathcal{T}_h}\int_T \nabla\boldsymbol{w} : \nabla\boldsymbol{v}\,dx + \sum_{e\in\mathcal{E}_h}\Big(-\int_e \{\nabla\boldsymbol{w}\}\boldsymbol{n}_e\cdot[\boldsymbol{v}]ds \\ &- \int_e\{\nabla\boldsymbol{v}\}\boldsymbol{n}_e\cdot[\boldsymbol{w}]ds + \alpha_e h_e^{-1}\int_e[\boldsymbol{w}]\cdot[\boldsymbol{v}]ds\Big), \qquad \forall \boldsymbol{w},\boldsymbol{v}\in\boldsymbol{V}_h. \end{aligned}$$

For any $\boldsymbol{v} \in \boldsymbol{V}_h$, by direct computations, we can decompose $\boldsymbol{v}$ as its normal and tangential components $\boldsymbol{v}^{\boldsymbol{n}_e}$ and $\boldsymbol{v}^{\boldsymbol{t}_e}$, that is,

$$\boldsymbol{v}^{\boldsymbol{n}_e} = (\boldsymbol{v}\cdot\boldsymbol{n}_e)\boldsymbol{n}_e, \quad \boldsymbol{v}^{\boldsymbol{t}_e} = (\boldsymbol{v}\cdot\boldsymbol{t}_e)\boldsymbol{t}_e = \boldsymbol{v} - \boldsymbol{v}^{\boldsymbol{n}_e}.$$

Moreover, since $\boldsymbol{V}_h \subset \boldsymbol{H}_0(\mathrm{div};\Omega)$, we have $[\boldsymbol{v}^{\boldsymbol{n}_e}] = 0$, which yields

$$\begin{aligned} a_h(\boldsymbol{w},\boldsymbol{v}) = &\sum_{T\in\mathcal{T}_h}\int_T \nabla\boldsymbol{w} : \nabla\boldsymbol{v}\,dx + \sum_{e\in\mathcal{E}_h}\Big(-\int_e \{\nabla\boldsymbol{w}\}\boldsymbol{n}_e\cdot[\boldsymbol{v}^{\boldsymbol{t}_e}]ds \\ &- \int_e\{\nabla\boldsymbol{v}\}\boldsymbol{n}_e\cdot[\boldsymbol{w}^{\boldsymbol{t}_e}]ds + \alpha_e h_e^{-1}\int_e[\boldsymbol{w}^{\boldsymbol{t}_e}]\cdot[\boldsymbol{v}^{\boldsymbol{t}_e}]ds\Big) \qquad \forall \boldsymbol{w},\boldsymbol{v}\in\boldsymbol{V}_h. \end{aligned} \tag{8}$$

As a consequence, the corresponding $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming IPDG method for the problem (1)–(3) is: Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times P_h$ such that

$$a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = (\boldsymbol{f},\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{9}$$

$$b(\boldsymbol{u}_h, q_h) = 0 \qquad \forall q_h \in P_h. \tag{10}$$

3. Stability of DG methods

This section aims at establishing the stability property of numerical scheme (9)–(10). The main result is stated in (14), which will be used for the medius error analysis in Section 5. We begin by defining the following DG norm on $\boldsymbol{v} \in \boldsymbol{V}(h) = \boldsymbol{V}_h + \boldsymbol{H}_0^1(\Omega)$:

$$\|\boldsymbol{v}\|_{\mathrm{DG}}^2 = \|\boldsymbol{v}\|_{1,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{v}^{\boldsymbol{t}_e}]\|_{0,e}^2, \tag{11}$$

where $\|\boldsymbol{v}\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{v}\|_0^2$.

The following boundedness, coerciveness, and the inf-sup condition can be found in [7].

Lemma 3.1. *It holds that*

$$|a_h(\boldsymbol{w},\boldsymbol{v})| \le C_a\|\boldsymbol{w}\|_{\mathrm{DG}}\|\boldsymbol{v}\|_{\mathrm{DG}} \qquad \forall \boldsymbol{w},\boldsymbol{v}\in\boldsymbol{V}_h,$$
$$|b(\boldsymbol{v},q)| \le C_b\|\boldsymbol{v}\|_{\mathrm{DG}}\|q\|_0 \qquad \forall \boldsymbol{v}\in\boldsymbol{V}_h, \ q \in \boldsymbol{L}^2(\Omega)/\mathbb{R}.$$

Moreover, if the interior penalty parameter α_e is sufficiently large, we have

$$a_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge C_s\|\boldsymbol{v}_h\|_{\mathrm{DG}}^2 \ \ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Lemma 3.2. *There exists a constant $\vartheta > 0$ such that*

$$\sup_{\boldsymbol{v}_h\in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\mathrm{DG}}} \geq \vartheta \|q_h\|_0 \quad \forall q_h \in P_h. \tag{12}$$

Next, for any $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h$ and $p_h, q_h \in P_h$, according to the numerical scheme (9)–(10), we define the following bilinear form by

$$\mathbb{A}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) = a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) + b(\boldsymbol{u}_h, q_h). \tag{13}$$

Based on Lemmas 3.1 and 3.2, we can prove the following stability result which is the main result of this section.

Theorem 3.3. *Let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times P_h$ be the solution of (9)–(10), it holds that*

$$\sup_{(\boldsymbol{v}_h,q_h)\in \boldsymbol{V}_h\times P_h} \frac{\mathbb{A}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\mathrm{DG}} + \|q_h\|_0} \geq \beta\big(\|\boldsymbol{u}_h\|_{\mathrm{DG}} + \|p_h\|_0\big), \tag{14}$$

which also means that the discrete scheme (9)–(10) is well-posed.

Proof. In light of the inf-sup condition (12) stated in Lemma 3.2, for a given $p_h \in P_h$, there exists $\boldsymbol{w}_h \in \boldsymbol{V}_h$ such that

$$b(\boldsymbol{w}_h, p_h) \geq \vartheta\|p_h\|_0^2 \quad \text{and} \quad \|p_h\|_0 = \|\boldsymbol{w}_h\|_{\mathrm{DG}}.$$

Then, for any $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $p_h \in P_h$, by setting $\boldsymbol{v}_h = \boldsymbol{u}_h - \delta\boldsymbol{w}_h$ ($\delta > 0$) and $q_h = -p_h$, and using Lemma 3.1, we arrive at

$$\begin{aligned}
\mathbb{A}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) =& a_h(\boldsymbol{u}_h, \boldsymbol{u}_h - \delta\boldsymbol{w}_h) - b(\boldsymbol{u}_h - \delta\boldsymbol{w}_h, p_h) + b(\boldsymbol{u}_h, p_h)\\
\geq& C_s\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 - \delta a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) + \delta b(\boldsymbol{w}_h, p_h)\\
\geq& C_s\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 - C_a\delta\|\boldsymbol{u}_h\|_{\mathrm{DG}}\|\boldsymbol{w}_h\|_{\mathrm{DG}} + \delta b(\boldsymbol{w}_h, p_h)\\
\geq& C_s\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 - \frac{C_s}{4}\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 - \frac{C_a\delta^2}{C_s}\|\boldsymbol{w}_h\|_{\mathrm{DG}}^2 + \delta\vartheta\|p_h\|_0^2\\
=& \frac{3C_s}{4}\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 + (\delta\vartheta - \frac{C_a\delta^2}{C_s})\|p_h\|_0^2.
\end{aligned}$$

Then we choose $\delta = \frac{C_s\vartheta}{2C_a}$ to infer that

$$\begin{aligned}
\mathbb{A}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) \geq& \tfrac{3C_s}{4}\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 + \tfrac{C_s\vartheta^2}{4C_a}\|p_h\|_0^2\\
\geq& \eta_1\big(\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 + \|p_h\|_0^2\big),
\end{aligned} \tag{15}$$

where $\eta_1 = \min\{\frac{3C_s}{4}, \frac{C_s\vartheta^2}{4C_a}\}$. On the other hand, we have

$$\begin{aligned}
\big(\|\boldsymbol{v}_h\|_{\mathrm{DG}} + \|q_h\|_0\big)^2 =& \big(\|\boldsymbol{u}_h - \delta\boldsymbol{w}_h\|_{\mathrm{DG}} + \|-p_h\|_0\big)^2\\
\leq& 4\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 + 4\delta^2\|\boldsymbol{w}_h\|_{\mathrm{DG}}^2 + 2\|p_h\|_0^2\\
\leq& 4\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 + \Big(\frac{C_s^2\vartheta^2}{C_a^2} + 2\Big)\|p_h\|_0^2\\
\leq& \eta_2(\|\boldsymbol{u}_h\|_{\mathrm{DG}}^2 + \|p_h\|_0^2)
\end{aligned} \tag{16}$$

with $\eta_2 = \max\{2, \frac{C_s^2\vartheta^2}{2C_a^2} + 1\}$. Combining (15) with (16) implies that (14) holds with $\beta = \eta_1\eta_2^{-1/2}$.

Moreover, by using the Cauchy–Schwarz inequality, we have

$$\mathbb{A}_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) \leq C\big(\|\boldsymbol{u}_h\|_{\mathrm{DG}} + \|p_h\|_0\big)\big(\|\boldsymbol{v}_h\|_{\mathrm{DG}} + \|q_h\|_0\big). \tag{17}$$

Thus, we apply the inequalities (14) and (17) and the standard Babuška theory [35] to deduce that the numerical scheme (9)–(10) is well-posed. □

4. A medius error analysis

To provide an improved a priori error analysis for (9)–(10) under minimal regularity assumptions, we first introduce an error indicator for any $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times P_h$ as

$$\begin{aligned}
\eta^2(\boldsymbol{v}_h, q_h) =& \sum_{T\in\mathcal{T}_h} h_T^2\|\boldsymbol{f} + \Delta\boldsymbol{v}_h - \nabla q_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{v}_h^{t_e}]\|_{0,e}^2\\
&+ \sum_{e\in\mathcal{E}_h^I} h_e\|[\nabla\boldsymbol{v}_h + q_h\boldsymbol{I}]\boldsymbol{n}_e\|_{0,e}^2.
\end{aligned}$$

Utilizing similar bubble function techniques as in Section 4.2 of [29], we have the following lower bound.

Lemma 4.1. *Let $(\boldsymbol{u}, p)$ and be the solution of (4)–(5). For any $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times P_h$, it holds that*

$$\eta(\boldsymbol{v}_h, q_h) \le C\Big\{ \|\boldsymbol{u}-\boldsymbol{v}_h\|_{\mathrm{DG}}^2 + \|p-q_h\|_0^2 + \mathrm{osc}_h^2(\boldsymbol{f}) \Big\}^{1/2},$$

where $\mathrm{osc}_h^2(\boldsymbol{f}) = \sum_{T\in\mathcal{T}_h} h_T^2 \|\boldsymbol{f}-\boldsymbol{f}_h\|_{0,T}^2$, with $\boldsymbol{f}_h$ being the $\boldsymbol{L}^2$ projection of $\boldsymbol{f}$ onto $\boldsymbol{V}_h$.

To proceed, we first recall the following standard inverse inequality (see [36,37]):

$$\|\boldsymbol{v}\|_{0,e} \le C h_T^{-1/2} \|\boldsymbol{v}\|_{0,T} \quad \forall \boldsymbol{v} \in \boldsymbol{P}_k(T),\ \forall e \in \mathcal{E}_h^T. \tag{18}$$

Moreover, for any $\boldsymbol{w}_h \in \boldsymbol{V}_h$, there exists $E_h\boldsymbol{v}_h \in \boldsymbol{V}_h \cap \boldsymbol{H}_0^1(\Omega)$, such that (see [38])

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h} \Big(\|\nabla(\boldsymbol{w}_h - E_h\boldsymbol{w}_h)\|_{0,T}^2 + h_T^{-2}\|\boldsymbol{w}_h - E_h\boldsymbol{w}_h\|_{0,T}^2 \Big) \\ &\le C\Big(\sum_{e\in\mathcal{E}_h} h_e^{-1} \|[\boldsymbol{w}_h]\|_{0,e}^2 \Big) = C\Big(\sum_{e\in\mathcal{E}_h} h_e^{-1} \|[\boldsymbol{w}_h^{t_e}]\|_{0,e}^2 \Big), \end{aligned} \tag{19}$$

which together with the triangle inequality yields to

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h} \|\nabla(E_h\boldsymbol{w}_h)\|_{0,T}^2 \\ &\le 2\Big(\sum_{T\in\mathcal{T}_h} \|\nabla\boldsymbol{w}_h\|_{0,T}^2 + \sum_{T\in\mathcal{T}_h} \|\nabla(\boldsymbol{w}_h - E_h\boldsymbol{w}_h)\|_{0,T}^2 \Big) \\ &\le C\Big(\sum_{T\in\mathcal{T}_h} \|\nabla\boldsymbol{w}_h\|_{0,T}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{w}_h]\|_{0,e}^2 \Big) = C\|\boldsymbol{w}_h\|_{\mathrm{DG}}^2. \end{aligned} \tag{20}$$

In order to prove Theorem 4.6, we shall establish some preliminary results that are stated in Lemmas 4.2–4.5.

Lemma 4.2. *For any $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{V}_h$, it holds that*

$$a(\boldsymbol{u}, E_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, E_h\boldsymbol{w}_h) \le C\|\boldsymbol{u}-\boldsymbol{v}_h\|_{\mathrm{DG}}\|\boldsymbol{w}_h\|_{\mathrm{DG}}. \tag{21}$$

Proof. Since $E_h\boldsymbol{w}_h \in \boldsymbol{H}_0^1(\Omega)$, it follows from the definition of $a_h(\cdot,\cdot)$ that

$$\begin{aligned} &a_h(\boldsymbol{v}_h, E_h\boldsymbol{w}_h) \\ &= \sum_{T\in\mathcal{T}_h} \int_T \nabla\boldsymbol{v}_h : \nabla(E_h\boldsymbol{w}_h)dx - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla(E_h\boldsymbol{w}_h)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_h^{t_e}]ds \end{aligned}$$

and

$$\begin{aligned} a(\boldsymbol{u}, E_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, E_h\boldsymbol{w}_h) = &\sum_{T\in\mathcal{T}_h} \int_T \nabla(\boldsymbol{u}-\boldsymbol{v}_h) : \nabla(E_h\boldsymbol{w}_h)dx \\ &- \sum_{e\in\mathcal{E}_h} \int_e \{\nabla(E_h\boldsymbol{w}_h)\}\boldsymbol{n}_e \cdot [(\boldsymbol{u}-\boldsymbol{v}_h)^{t_e}]ds. \end{aligned} \tag{22}$$

For the first term, the Cauchy–Schwarz inequality yields

$$\begin{aligned} &\Big| \sum_{T\in\mathcal{T}_h} \int_T \nabla(\boldsymbol{u}-\boldsymbol{v}_h) : \nabla(E_h\boldsymbol{w}_h)dx \Big| \\ &\le \Big(\sum_{T\in\mathcal{T}_h} \|\nabla(\boldsymbol{u}-\boldsymbol{v}_h)\|_{0,T}^2 \Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} \|\nabla(E_h\boldsymbol{w}_h)\|_{0,T}^2 \Big)^{1/2}. \end{aligned} \tag{23}$$

For the second term, we use the Cauchy–Schwarz inequality and the inverse inequality (18) to derive that

$$\begin{aligned} &\Big| \sum_{e\in\mathcal{E}_h} \int_e \{\nabla(E_h\boldsymbol{w}_h)\}\boldsymbol{n}_e \cdot [(\boldsymbol{u}-\boldsymbol{v}_h)^{t_e}]ds \Big| \\ &\le \Big(\sum_{e\in\mathcal{E}_h} h_e^{-1} \|[(\boldsymbol{u}-\boldsymbol{v}_h)^{t_e}]\|_{0,e}^2 \Big)^{1/2} \Big(\sum_{e\in\mathcal{E}_h} h_e \|\{\nabla(E_h\boldsymbol{w}_h)\}\boldsymbol{n}_e\|_{0,e}^2 \Big)^{1/2} \\ &\le \Big(\sum_{e\in\mathcal{E}_h} h_e^{-1} \|[(\boldsymbol{u}-\boldsymbol{v}_h)^{t_e}]\|_{0,e}^2 \Big)^{1/2} \Big(\sum_{T\in\mathcal{T}_h} \|\nabla(E_h\boldsymbol{w}_h)\|_{0,T}^2 \Big)^{1/2}. \end{aligned} \tag{24}$$

Substituting (23) and (24) into (22), and using the estimate (20), we obtain the desired inequality (21). □

Lemma 4.3. *For any $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $s_h \in P_h$, we have*

$$b(\boldsymbol{u}, s_h) - b(\boldsymbol{v}_h, s_h) \le C\|\boldsymbol{u}-\boldsymbol{v}_h\|_{\mathrm{DG}}\|s_h\|_0.$$

Proof. It follows from the definition of $b(\cdot,\cdot)$ and the Cauchy–Schwarz inequality that

$$\begin{aligned} b(\boldsymbol{u}, s_h) - b(\boldsymbol{v}_h, s_h) &= -\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(\boldsymbol{u}-\boldsymbol{v}_h)s_h dx \\ &\le \|\nabla\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\|_0\|s_h\|_0 \\ &\le C\|\boldsymbol{u}-\boldsymbol{v}_h\|_{1,h}\|s_h\|_0 \\ &\le C\|\boldsymbol{u}-\boldsymbol{v}_h\|_{\mathrm{DG}}\|s_h\|_0. \end{aligned}$$

The proof is completed. □

Lemma 4.4. *For any $p\in L^2(\Omega)$, $\boldsymbol{w}_h\in\boldsymbol{V}_h$, $q_h\in P_h$, there holds that*

$$b(E_h\boldsymbol{w}_h, p) - b(E_h\boldsymbol{w}_h, q_h) \le C\|p-q_h\|_0\|\boldsymbol{w}_h\|_{\mathrm{DG}}.$$

Proof. Combining (7) and (20) implies that

$$\begin{aligned} b(E_h\boldsymbol{w}_h, p) - b(E_h\boldsymbol{w}_h, q_h) &= -\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(E_h\boldsymbol{w}_h)(p-q_h)dx \\ &\le \|p-q_h\|_0\|\nabla\cdot(E_h\boldsymbol{w}_h)\|_0 \\ &\le C\|p-q_h\|_0\|E_h\boldsymbol{w}_h\|_{1,h} \\ &= C\|p-q_h\|_0\|E_h\boldsymbol{w}_h\|_{\mathrm{DG}} \\ &\le C\|p-q_h\|_0\|\boldsymbol{w}_h\|_{\mathrm{DG}}, \end{aligned}$$

which is the desired assertion. □

Lemma 4.5. *Let $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega)$. For any $\boldsymbol{w}_h, \boldsymbol{v}_h\in\boldsymbol{V}_h$, $q_h\in P_h$, it holds that*

$$\begin{aligned} &(\boldsymbol{f}, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - b(\boldsymbol{w}_h - E_h\boldsymbol{w}_h, q_h) \\ \le& C\eta(\boldsymbol{v}_h, q_h)\|\boldsymbol{w}_h\|_{\mathrm{DG}}. \end{aligned}$$

Proof. Denote by $\boldsymbol{\varphi}_h = \boldsymbol{w}_h - E_h\boldsymbol{w}_h$, integrating by parts, we have

$$\begin{aligned} &\sum_{T\in\mathcal{T}_h}\int_T \nabla\boldsymbol{v}_h : \nabla\boldsymbol{\varphi}_h dx \\ =& -\sum_{T\in\mathcal{T}_h}\int_T \Delta\boldsymbol{v}_h\cdot\boldsymbol{\varphi}_h dx + \sum_{T\in\mathcal{T}_h}\int_{\partial T}(\nabla\boldsymbol{v}_h\boldsymbol{n}_T)\cdot\boldsymbol{\varphi}_h ds \\ =& -\sum_{T\in\mathcal{T}_h}\int_T \Delta\boldsymbol{v}_h\cdot\boldsymbol{\varphi}_h dx + \sum_{e\in\mathcal{E}_h}\int_e\{\nabla\boldsymbol{v}_h\}\boldsymbol{n}_e\cdot[\boldsymbol{\varphi}_h]ds \\ &+ \sum_{e\in\mathcal{E}_h^I}\int_e[\nabla\boldsymbol{v}_h]\boldsymbol{n}_e\cdot\{\boldsymbol{\varphi}_h\}ds \\ =& -\sum_{T\in\mathcal{T}_h}\int_T \Delta\boldsymbol{v}_h\cdot\boldsymbol{\varphi}_h dx + \sum_{e\in\mathcal{E}_h}\int_e\{\nabla\boldsymbol{v}_h\}\boldsymbol{n}_e\cdot[\boldsymbol{\varphi}_h^{t_e}]ds \\ &+ \sum_{e\in\mathcal{E}_h^I}\int_e[\nabla\boldsymbol{v}_h]\boldsymbol{n}_e\cdot\{\boldsymbol{\varphi}_h\}ds, \end{aligned} \tag{25}$$

and noting that $\boldsymbol{\varphi}_h\in\boldsymbol{H}_0(\mathrm{div};\Omega)$ implies

$$\begin{aligned} b(\boldsymbol{\varphi}_h, q_h) &= -\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot\boldsymbol{\varphi}_h q_h ds \\ &= \sum_{T\in\mathcal{T}_h}\int_T \nabla q_h\cdot\boldsymbol{\varphi}_h ds - \sum_{T\in\mathcal{T}_h}\int_{\partial T} q_h(\boldsymbol{\varphi}_h\cdot\boldsymbol{n}_T)ds \\ &= \sum_{T\in\mathcal{T}_h}\int_T \nabla q_h\cdot\boldsymbol{\varphi}_h ds - \sum_{e\in\mathcal{E}_h^I}\int_e[q_h\boldsymbol{I}]\boldsymbol{n}_e\cdot\{\boldsymbol{\varphi}_h\}ds. \end{aligned} \tag{26}$$

Recalling the definition of $a_h(\cdot,\cdot)$ in (8), and utilizing (25) and (26), we then obtain

$$\begin{aligned} &(\boldsymbol{f}, \boldsymbol{\varphi}_h) - a_h(\boldsymbol{v}_h, \boldsymbol{\varphi}_h) - b(\boldsymbol{\varphi}_h, q_h) \\ =& \sum_{T\in\mathcal{T}_h}\int_T(\boldsymbol{f} + \Delta\boldsymbol{v}_h - \nabla q_h)\cdot\boldsymbol{\varphi}_h dx - \sum_{e\in\mathcal{E}_h^I}\int_e[\nabla\boldsymbol{v}_h - q_h\boldsymbol{I}]\boldsymbol{n}_e\cdot\{\boldsymbol{\varphi}_h\}ds \end{aligned}$$

$$+\sum_{e\in\mathcal{E}_h}\int_e\{\nabla\boldsymbol{\varphi}_h\}\boldsymbol{n}_e\cdot[\boldsymbol{v}_h^{t_e}]ds-\sum_{e\in\mathcal{E}_h}\alpha_e h_e^{-1}\int_e[\boldsymbol{v}_h^{t_e}]\cdot[\boldsymbol{\varphi}_h^{t_e}]ds$$
$$=A_1+A_2+A_3+A_4. \tag{27}$$

Applying the Cauchy–Schwarz inequality, we infer for the first term A_1 that

$$|A_1|\le\Big(\sum_{T\in\mathcal{T}_h}h_T^2\|\boldsymbol{f}+\Delta\boldsymbol{v}_h-\nabla q_h\|_{0,T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|\boldsymbol{w}_h-E_h\boldsymbol{w}_h\|_{0,T}^2\Big)^{1/2}.$$

For the term A_2, it follows from the Cauchy–Schwarz inequality and the inverse inequality (18) that

$$|A_2|\le\left(\sum_{e\in\mathcal{E}_h^I}h_e\|[\nabla\boldsymbol{v}_h-q_h\boldsymbol{I}]\boldsymbol{n}_e\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h^I}h_e^{-1}\|\{\boldsymbol{w}_h-E_h\boldsymbol{w}_h\}\|_{0,e}^2\right)^{1/2}$$
$$\le C\left(\sum_{e\in\mathcal{E}_h^I}h_e\|[\nabla\boldsymbol{v}_h-q_h\boldsymbol{I}]\boldsymbol{n}_e\|_{0,e}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|\boldsymbol{w}_h-E_h\boldsymbol{w}_h\|_{0,T}^2\right)^{1/2}.$$

Similarly, the term A_3 can be bounded as follows:

$$|A_3|\le\left(\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[\boldsymbol{v}_h^{t_e}]\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h}h_e\|\{\nabla(\boldsymbol{w}_h-E_h\boldsymbol{w}_h)\}\boldsymbol{n}_e\|_{0,e}^2\right)^{1/2}$$
$$\le C\left(\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[\boldsymbol{v}_h^{t_e}]\|_{0,e}^2\right)^{1/2}\left(\sum_{T\in\mathcal{T}_h}\|\nabla(\boldsymbol{w}_h-E_h\boldsymbol{w}_h)\|_{0,T}^2\right)^{1/2}.$$

The term A_4 can be estimated by

$$|A_4|\le\left(\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[\boldsymbol{v}_h^{t_e}]\|_{0,e}^2\right)^{1/2}\left(\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[(\boldsymbol{w}_h-E_h\boldsymbol{w}_h)^{t_e}]\|_{0,e}^2\right)^{1/2}.$$

Plugging the above four estimates into (27), using the approximation property in (19) and the Cauchy–Schwarz inequality, we obtain the desired result. □

Now, we are in a position to prove the main result of this section.

Theorem 4.6. *Let $(\boldsymbol{u},p)$ and $(\boldsymbol{u}_h,p_h)$ be the solutions of (4)–(5) and (9)–(10), respectively. Then it holds that*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\rm DG}+\|p-p_h\|_0$$
$$\le C\Big\{\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{\rm DG}+\inf_{q_h\in P_h}\|p-q_h\|_0+{\rm osc}_h(\boldsymbol{f})\Big\}. \tag{28}$$

Furthermore, if $(\boldsymbol{u},p)\in\boldsymbol{H}^{s+1}(\Omega)\times H^s(\Omega)$ with $s>0$, we have

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\rm DG}+\|p-p_h\|_0\le C\Big(h^{\min(s,k)}\big(\|\boldsymbol{u}\|_{s+1}+\|p\|_s\big)+{\rm osc}_h(\boldsymbol{f})\Big). \tag{29}$$

Proof. Let $(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times P_h$ be arbitrary. By the stability estimate (14), there exits $(\boldsymbol{w}_h,s_h)\in\boldsymbol{V}_h\times P_h$ with

$$\|\boldsymbol{w}_h\|_{\rm DG}+\|s_h\|_0=1, \tag{30}$$

and

$$\|\boldsymbol{u}_h-\boldsymbol{v}_h\|_{\rm DG}+\|p_h-q_h\|_0\le C\mathbb{A}_h(\boldsymbol{u}_h-\boldsymbol{v}_h,p_h-q_h;\boldsymbol{w}_h,s_h).$$

Noting that $a(\boldsymbol{u}, E_h\boldsymbol{w}_h) + b(E_h\boldsymbol{w}_h, p) = (\boldsymbol{f}, E_h\boldsymbol{w}_h)$, by adding and subtracting terms in the above inequality, we can further obtain that

$$\begin{aligned}
&C^{-1}\left(\|\boldsymbol{u}_h - \boldsymbol{v}_h\|_{\mathrm{DG}} + \|p_h - q_h\|_0\right) \le \mathbb{A}_h(\boldsymbol{u}_h - \boldsymbol{v}_h, p_h - q_h; \boldsymbol{w}_h, s_h)\\
&= \mathbb{A}_h(\boldsymbol{u}_h, p_h; \boldsymbol{w}_h, s_h) - \mathbb{A}_h(\boldsymbol{v}_h, q_h; \boldsymbol{w}_h, s_h)\\
&= a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) + b(\boldsymbol{w}_h, p_h) + b(\boldsymbol{u}_h, s_h) - a_h(\boldsymbol{v}_h, \boldsymbol{w}_h)\\
&\quad - b(\boldsymbol{w}_h, q_h) - b(\boldsymbol{v}_h, s_h)\\
&= (\boldsymbol{f}, \boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, \boldsymbol{w}_h) - b(\boldsymbol{w}_h, q_h) + b(\boldsymbol{u}_h, s_h) - b(\boldsymbol{v}_h, s_h)\\
&= \left[(\boldsymbol{f}, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - b(\boldsymbol{w}_h - E_h\boldsymbol{w}_h, q_h)\right]\\
&\quad + \left[a(\boldsymbol{u}, E_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, E_h\boldsymbol{w}_h)\right] + \left[b(E_h\boldsymbol{w}_h, p) - b(E_h\boldsymbol{w}_h, q_h)\right]\\
&\quad + \left[b(\boldsymbol{u}_h, s_h) - b(\boldsymbol{v}_h, s_h)\right]\\
&= \left[(\boldsymbol{f}, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - b(\boldsymbol{w}_h - E_h\boldsymbol{w}_h, q_h)\right]\\
&\quad + \left[a(\boldsymbol{u}, E_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, E_h\boldsymbol{w}_h)\right] + \left[b(E_h\boldsymbol{w}_h, p) - b(E_h\boldsymbol{w}_h, q_h)\right]\\
&\quad + \left[b(\boldsymbol{u}, s_h) - b(\boldsymbol{v}_h, s_h)\right]\\
&= B_1 + B_2 + B_3 + B_4,
\end{aligned} \tag{31}$$

where in the last second line we have used the fact that $b(\boldsymbol{u}_h, s_h) = b(\boldsymbol{u}, s_h) = 0$.

By combining Lemmas 4.5 and 4.1, we bound B_1 as

$$\begin{aligned}
B_1 &= (\boldsymbol{f}, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - b(\boldsymbol{w}_h - E_h\boldsymbol{w}_h, q_h)\\
&\le C\eta(\boldsymbol{v}_h, q_h)\|\boldsymbol{w}_h\|_{\mathrm{DG}}\\
&\le C\left\{\|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{DG}} + \|p - q_h\|_0 + \mathrm{osc}_h(\boldsymbol{f})\right\}\|\boldsymbol{w}_h\|_{\mathrm{DG}}.
\end{aligned} \tag{32}$$

From Lemma 4.2, we have

$$B_2 = a(\boldsymbol{u}, E_h\boldsymbol{w}_h) - a_h(\boldsymbol{v}_h, E_h\boldsymbol{w}_h) \le C\|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{DG}}\|\boldsymbol{w}_h\|_{\mathrm{DG}}. \tag{33}$$

Additionally, it follows from Lemma 4.4 that

$$B_3 = b(E_h\boldsymbol{w}_h, p) - b(E_h\boldsymbol{w}_h, q_h) \le C\|p - q_h\|_0\|\boldsymbol{w}_h\|_{\mathrm{DG}}. \tag{34}$$

Moreover, from Lemma 4.3, we find that

$$B_4 = b(\boldsymbol{u}, s_h) - b(\boldsymbol{v}_h, s_h) \le C\|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{DG}}\|s_h\|_0. \tag{35}$$

Then, submitting (32)–(35) into (31) and using (30), we arrive at (28).

At last, combining (28) together with the standard finite element approximation estimates, yields (29). The proof is completed. □

5. Extension to the standard IPDG method

In this section, we shall utilize the approach used in the above section to establish the medius error estimate for the standard symmetric IPDG method. We first introduce the discrete space for velocities and pressure:

$$\begin{aligned}
\boldsymbol{V}_h^s &= \left\{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_T \in \boldsymbol{P}_k(T),\ \ \forall T \in \mathcal{T}_h\right\},\\
P_h &= \left\{q \in L_0^2(\Omega) : q|_T \in P_{k-1}(T),\ \ \forall T \in \mathcal{T}_h\right\}.
\end{aligned}$$

As in (11), to carry out the error analysis, we define the following $\mathbb{DG}$ norm on $\boldsymbol{v} \in \boldsymbol{V}^s(h) = \boldsymbol{V}_h^s + \boldsymbol{H}_0^1(\Omega)$

$$\|\boldsymbol{v}\|_{\mathbb{DG}}^2 = \|\boldsymbol{v}\|_{1,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{v}]\|_{0,e}^2.$$

Note that here use the symbol $\|\cdot\|_{\mathbb{DG}}$, which is different from the one $\|\cdot\|_{\mathrm{DG}}$ stated in (11).

Recall that the bilinear form of IPDG method, which is used to discretize the diffusion term, is defined as

$$\begin{aligned}
a_h^s(\boldsymbol{w}, \boldsymbol{v}) &= \sum_{T\in\mathcal{T}_h}\int_T \nabla\boldsymbol{w} : \nabla\boldsymbol{v}\,dx - \sum_{e\in\mathcal{E}_h}\int_e \{\nabla\boldsymbol{w}\}\boldsymbol{n}_e \cdot [\boldsymbol{v}]\,ds\\
&- \sum_{e\in\mathcal{E}_h}\int_e \{\nabla\boldsymbol{v}\}\boldsymbol{n}_e \cdot [\boldsymbol{w}]\,ds + \sum_{e\in\mathcal{E}_h}\alpha_e h_e^{-1}\int_e [\boldsymbol{w}]\cdot[\boldsymbol{v}]\,ds, \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}_h^s,
\end{aligned} \tag{36}$$

and the discretization of pressure–velocity coupling is defined by

$$c_h(\boldsymbol{v}, q) = -\sum_{T\in\mathcal{T}_h}\int_T q\nabla\cdot\boldsymbol{v}\,dx + \sum_{e\in\mathcal{E}_h}\int_e [\boldsymbol{v}]\cdot\boldsymbol{n}_e\{q\}\,ds.$$

Thus the standard IPDG approximation for Stokes equations (1)–(3) is to find $(\boldsymbol{u}_h^s, p_h^s) \in \boldsymbol{V}_h^s \times P_h$ such that

$$a_h^s(\boldsymbol{u}_h^s, \boldsymbol{v}_h) + c_h(\boldsymbol{v}_h, p_h^s) = (\boldsymbol{f}, \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^s, \tag{37}$$

$$c_h(\boldsymbol{u}_h^s, q_h) = 0 \quad \forall q_h \in P_h. \tag{38}$$

Analogous to (13), for any $\boldsymbol{u}_h^s, \boldsymbol{v}_h \in \boldsymbol{V}_h^s$ and $p_h^s, q_h \in P_h$, we define the following bilinear form induced by (37)–(38).

$$\mathbb{B}_h(\boldsymbol{u}_h^s, p_h^s; \boldsymbol{v}_h, q_h) = a_h^s(\boldsymbol{u}_h^s, \boldsymbol{v}_h) + c_h(\boldsymbol{v}_h, p_h^s) + c_h(\boldsymbol{u}_h^s, q_h). \tag{39}$$

We then have the stability result similar to Theorem 3.3.

Theorem 5.1. *Let $(\boldsymbol{u}_h^s, p_h^s) \in \boldsymbol{V}_h^s \times P_h$ be the solution of (37)–(38), there holds*

$$\sup_{(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h^s \times P_h} \frac{\mathbb{B}_h(\boldsymbol{u}_h^s, p_h^s; \boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\mathbb{DG}} + \|q_h\|_0} \geq \beta_s \left(\|\boldsymbol{u}_h^s\|_{\mathbb{DG}} + \|p_h^s\|_0 \right).$$

Proof. For simplicity, we only sketch the proof. We first have (see [39])

$$a_h^s(\boldsymbol{w}, \boldsymbol{v}) \leq C_\alpha \|\boldsymbol{w}\|_{\mathbb{DG}} \|\boldsymbol{v}\|_{\mathbb{DG}} \qquad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}_h^s,$$

$$c_h(\boldsymbol{v}, q) \leq C_\beta \|\boldsymbol{v}\|_{\mathbb{DG}} \|q\|_0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h^s,\ q \in \boldsymbol{L}^2(\Omega)/\mathbb{R}.$$

Additionally, when the penalty parameter α_e is sufficiently large, there holds (see [39])

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \geq C_\sigma \|\boldsymbol{v}_h\|_{\mathbb{DG}}^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^s.$$

Also, we have the following inf-sup condition (see [39])

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h^s} \frac{c_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\mathbb{DG}}} \geq \vartheta_s \|q_h\|_0 \quad \forall q_h \in P_h.$$

Using the above results, similar arguments as in Theorem 3.3 give the desired estimate. □

The error indicator used for (37)–(38) is stated as

$$\eta_s^2(\boldsymbol{v}_h^s, q_h^s) = \sum_{T \in \mathcal{T}_h} h_T^2 \|\boldsymbol{f} + \Delta \boldsymbol{v}_h^s - \nabla q_h^s\|_{0,T}^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\boldsymbol{v}_h^s]\|_{0,e}^2 + \sum_{e \in \mathcal{E}_h^I} h_e \|[\nabla \boldsymbol{v}_h^s + q_h^s \boldsymbol{I}]\boldsymbol{n}_e\|_{0,e}^2,$$

for $(\boldsymbol{v}_h^s, q_h^s) \in \boldsymbol{V}_h^s \times P_h$. It satisfies the following lower bound (see [33]).

Lemma 5.2. *Let $(\boldsymbol{u}, p)$ and be the solution of (4)–(5). For any $(\boldsymbol{v}_h^s, q_h^s) \in \boldsymbol{V}_h^s \times P_h$, it holds that*

$$\eta_s(\boldsymbol{v}_h^s, q_h^s) \leq C \left\{ \|\boldsymbol{u} - \boldsymbol{v}_h^s\|_{\mathbb{DG}}^2 + \|p - q_h^s\|_0^2 + \mathrm{osc}_h^2(\boldsymbol{f}) \right\}^{1/2},$$

where $\mathrm{osc}_h^2(\boldsymbol{f}) = \sum_{T \in \mathcal{T}_h} h_T^2 \|\boldsymbol{f} - \boldsymbol{f}_h\|_{0,T}^2$, with $\boldsymbol{f}_h$ being the $\boldsymbol{L}^2$ projection of $\boldsymbol{f}$ onto $\boldsymbol{V}_h^s$.

Next, in order to establish the medius error analysis, we shall prove some results that are parallel to Lemmas 4.2–4.5. One main difference between (9)–(10) and (37)–(38) is that the pressure–velocity coupling term is not the same. This makes Lemma 5.4 is more involved than that in Lemma 4.3.

Lemma 5.3. *For any $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ and $\boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{V}_h^s$, it holds that*

$$a(\boldsymbol{u}, E_h \boldsymbol{w}_h) - a_h^s(\boldsymbol{v}_h, E_h \boldsymbol{w}_h) \leq C \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathbb{DG}} \|\boldsymbol{w}_h\|_{\mathbb{DG}}.$$

Proof. Since $E_h \boldsymbol{w}_h \in \boldsymbol{H}_0^1(\Omega)$, we then have

$$a(\boldsymbol{u}, E_h \boldsymbol{w}_h) - a_h^s(\boldsymbol{v}_h, E_h \boldsymbol{w}_h) = \sum_{T \in \mathcal{T}_h} \int_T \nabla(\boldsymbol{u} - \boldsymbol{v}_h) : \nabla(E_h \boldsymbol{w}_h) dx - \sum_{e \in \mathcal{E}_h} \int_e \{\nabla(E_h \boldsymbol{w}_h)\}\boldsymbol{n}_e \cdot [(\boldsymbol{u} - \boldsymbol{v}_h)] ds.$$

Noting that (19) and (20) also hold for $\boldsymbol{v}_h \in \boldsymbol{V}_h^s$, similar arguments as in Lemma 4.2 yields the assertion. □

Lemma 5.4. *For any $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$, $\boldsymbol{v}_h \in \boldsymbol{V}_h^s$, $s_h \in P_h$, we have*

$$c_h(\boldsymbol{u}, s_h) - c_h(\boldsymbol{v}_h, s_h) \leq C \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathbb{DG}} \|\{s_h\}\|_0.$$

Proof. Combining the definition of $c_h(\cdot,\cdot)$, the Cauchy–Schwarz inequality and the inverse inequality (18), we have

$$\begin{aligned}
&c_h(\boldsymbol{u}, s_h) - c_h(\boldsymbol{v}_h, s_h)\\
&= -\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(\boldsymbol{u}-\boldsymbol{v}_h)s_h dx + \sum_{e\in\mathcal{E}_h}\int_e [\boldsymbol{u}-\boldsymbol{v}_h]\cdot\boldsymbol{n}_e\{s_h\}ds\\
&\le \sum_{T\in\mathcal{T}_h}\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\|_{0,T}\|s_h\|_{0,T} + \sum_{e\in\mathcal{E}_h} h_e^{-1/2}\|[\boldsymbol{u}-\boldsymbol{v}_h]\|_{0,e}h_e^{1/2}\|\{s_h\}\|_{0,e}\\
&\le \Big(\sum_{T\in\mathcal{T}_h}\|\nabla\cdot(\boldsymbol{u}-\boldsymbol{v}_h)\|_{0,T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|s_h\|_{0,T}^2\Big)^{1/2}\\
&\quad + \Big(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{u}-\boldsymbol{v}_h]\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h} h_e\|\{s_h\}\|_{0,e}^2\Big)^{1/2}\\
&\le C\|\boldsymbol{u}-\boldsymbol{v}_h\|_{\mathbb{DG}}\|\{s_h\}\|_0.
\end{aligned}$$

The proof is completed. □

Lemma 5.5. *For any $p\in L^2(\Omega)$, $\boldsymbol{w}_h\in \boldsymbol{V}_h^s$, $q_h\in P_h$, it is true that*

$$c_h(E_h\boldsymbol{w}_h, p) - c_h(E_h\boldsymbol{w}_h, q_h) \le C\|p-q_h\|_0\|\boldsymbol{w}_h\|_{\mathbb{DG}}.$$

Proof. Since $E_h\boldsymbol{w}_h \in \boldsymbol{H}_0^1(\Omega)$, we thus have

$$c_h(E_h\boldsymbol{w}_h, p) - c_h(E_h\boldsymbol{w}_h, q_h) = -\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot(E_h\boldsymbol{w}_h)(p-q_h)dx.$$

Similar arguments as in Lemma 4.4 conclude the proof. □

Lemma 5.6. *Let $\boldsymbol{f}\in \boldsymbol{L}^2(\Omega)$. For any $\boldsymbol{w}_h, \boldsymbol{v}_h\in \boldsymbol{V}_h^s$, $q_h\in P_h$, there have*

$$\begin{aligned}
&(\boldsymbol{f}, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - a_h^s(\boldsymbol{v}_h, \boldsymbol{w}_h - E_h\boldsymbol{w}_h) - c_h(\boldsymbol{w}_h - E_h\boldsymbol{w}_h, q_h)\\
&\le C\eta_s(\boldsymbol{v}_h, q_h)\|\boldsymbol{w}_h\|_{\mathbb{DG}}.
\end{aligned}$$

Proof. Denote by $\boldsymbol{\varphi}_h = \boldsymbol{w}_h - E_h\boldsymbol{w}_h$, integrating by parts, we have

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h}\int_T \nabla\boldsymbol{v}_h : \nabla\boldsymbol{\varphi}_h dx &= -\sum_{T\in\mathcal{T}_h}\int_T \Delta\boldsymbol{v}_h\cdot\boldsymbol{\varphi}_h dx\\
&\quad + \sum_{e\in\mathcal{E}_h}\int_e \{\nabla\boldsymbol{v}_h\}\boldsymbol{n}_e\cdot[\boldsymbol{\varphi}_h]ds + \sum_{e\in\mathcal{E}_h^I}\int_e [\nabla\boldsymbol{v}_h]\boldsymbol{n}_e\cdot\{\boldsymbol{\varphi}_h\}ds,
\end{aligned} \tag{40}$$

and

$$\begin{aligned}
c_h(\boldsymbol{\varphi}_h, q_h) &= -\sum_{T\in\mathcal{T}_h}\int_T \nabla\cdot\boldsymbol{\varphi}_h q_h ds + \sum_{e\in\mathcal{E}_h}\int_e [\boldsymbol{\varphi}_h]\cdot\boldsymbol{n}_e\{q_h\}ds\\
&= \sum_{T\in\mathcal{T}_h}\int_T \nabla q_h\cdot\boldsymbol{\varphi}_h ds - \sum_{T\in\mathcal{T}_h}\int_{\partial T} q_h(\boldsymbol{\varphi}_h\cdot\boldsymbol{n}_T)ds\\
&\quad + \sum_{e\in\mathcal{E}_h}\int_e [\boldsymbol{\varphi}_h]\cdot\boldsymbol{n}_e\{q_h\}ds\\
&= \sum_{T\in\mathcal{T}_h}\int_T \nabla q_h\cdot\boldsymbol{\varphi}_h ds - \sum_{e\in\mathcal{E}_h^I}\int_e [q_h\boldsymbol{I}]\boldsymbol{n}_e\cdot\{\boldsymbol{\varphi}_h\}ds.
\end{aligned} \tag{41}$$

Recalling the definition of $a_h^s(\cdot,\cdot)$ in (37), and utilizing (40) and (41), we then obtain

$$\begin{aligned}
&(\boldsymbol{f}, \boldsymbol{\varphi}_h) - a_h^s(\boldsymbol{v}_h, \boldsymbol{\varphi}_h) - c_h(\boldsymbol{\varphi}_h, q_h)\\
&= \sum_{T\in\mathcal{T}_h}\int_T (\boldsymbol{f} + \Delta\boldsymbol{v}_h - \nabla q_h)\cdot\boldsymbol{\varphi}_h dx - \sum_{e\in\mathcal{E}_h^I}\int_e [\nabla\boldsymbol{v}_h - q_h\boldsymbol{I}]\boldsymbol{n}_e\cdot\{\boldsymbol{\varphi}_h\}ds\\
&\quad + \sum_{e\in\mathcal{E}_h}\int_e \{\nabla\boldsymbol{\varphi}_h\}\boldsymbol{n}_e\cdot[\boldsymbol{v}_h]ds - \sum_{e\in\mathcal{E}_h}\alpha_e h_e^{-1}\int_e [\boldsymbol{v}_h]\cdot[\boldsymbol{\varphi}_h]ds\\
&= D_1 + D_2 + D_3 + D_4.
\end{aligned}$$

Direct imitations of the proofs in Lemma 4.5, we can give bounds for D_1–D_4 and then obtain the result asserted. □

Based on Theorem 5.1 and Lemmas 5.2–5.6, using the techniques as in Theorem 4.6, we have the following medius error analysis.

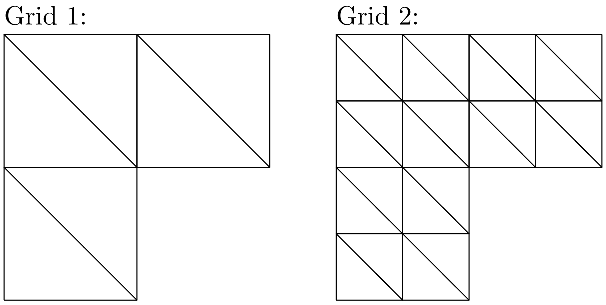

Fig. 1. The first two levels of triangular grids for Tables 1 and 2.

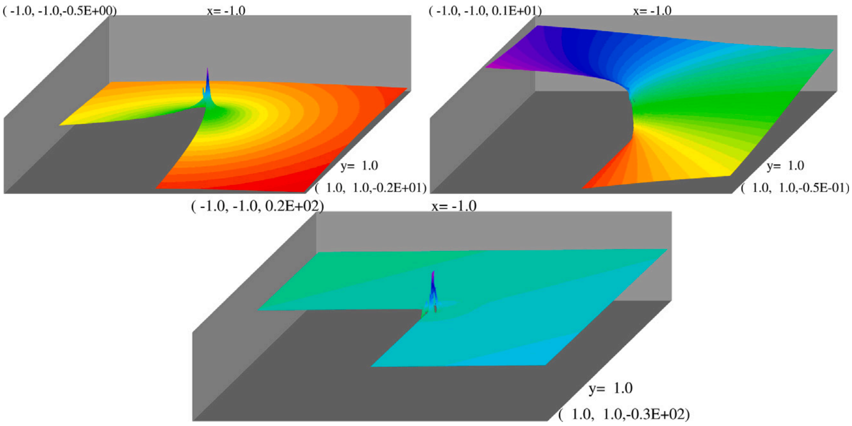

Fig. 2. The P_2^2-$P_1^{\rm dis}$ finite element solutions $u_{1,h}$ (top left), $u_{2,h}$ (top right) and p_h (bottom) for singular problem (43) on Grid 5, shown in Fig. 1.

Theorem 5.7. *Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h^s, p_h^s)$ be the solutions of (4)–(5) and (37)–(38), respectively. Then it holds that*

$$\begin{aligned}&\|\boldsymbol{u}-\boldsymbol{u}_h^s\|_{\mathbb{DG}}+\|p-p_h^s\|_0\\ \le\, & C\Big\{\inf_{\boldsymbol{v}_h\in\boldsymbol{V}_h^s}\|\boldsymbol{u}-\boldsymbol{v}_h\|_{\mathbb{DG}}+\inf_{q_h\in P_h}\|p-q_h\|_0+\operatorname{osc}_h(\boldsymbol{f})\Big\}.\end{aligned}$$

Furthermore, if $(\boldsymbol{u}, p)\in \boldsymbol{H}^{m+1}(\Omega)\times H^m(\Omega)$ with $m>0$, we have

$$\|\boldsymbol{u}-\boldsymbol{u}_h^s\|_{\mathbb{DG}}+\|p-p_h^s\|_0\le C\Big(h^{\min(m,k)}\big(\|\boldsymbol{u}\|_{m+1}+\|p\|_m\big)+\operatorname{osc}_h(\boldsymbol{f})\Big).$$

6. Numerical experiments

In this section, we report some numerical experiments, in which the exact solutions are smooth, or have some corner singularities, to validate our theoretical results. More precisely, we solve first the problem (1)–(3) in the L-shape domain $\Omega=(-1,1)^2\setminus[0,1)\times(-1,0]$, where the exact solution is chosen a smooth one as (similar to the examples in [32,40])

$$\boldsymbol{u}=\left(-2r^4\cos 4\theta 2r^4\sin 4\theta\right),\quad p=r\cos\theta+r\sin\theta, \tag{42}$$

where $r=\sqrt{x^2+y^2}$ and $\tan\theta=y/x$. In this case, the solution is smooth and we should obtain optimal orders of convergence for the finite element solutions.

In Table 1, we list the computational results for the smooth solution (42) on uniform triangular meshes shown in Fig. 1. As predicted by the standard theory, all finite element solutions converge at the optimal orders in all norms.

Next we solve the Stokes problem (1)–(3) in the L-shape domain $\Omega=(-1,1)^2\setminus[0,1)\times(-1,0]$ again, but with a singular solution

$$\boldsymbol{u}=\begin{pmatrix}-2r^{1/9}\cos(\theta/9)\\ 2r^{1/9}\sin(\theta/9)\end{pmatrix},\quad p=r\cos\theta+r\sin\theta. \tag{43}$$

In Table 2, the errors of the two finite element solutions are listed, when solving (43) on the uniform grids shown in Fig. 1. Since $\boldsymbol{u}\in\boldsymbol{H}^{1+1/9}$, the proved order of convergence in $\boldsymbol{H}^1$ norm is 0.11, which is verified by Table 2. Here the solution regularity is of $H^{1+1/9}$ and the domain regularity is of $H^{1+2/3}$. Thus the order of L^2-convergence should be $1/9+2/3=0.778$. The numerical

Table 1
Computed errors for (42) on uniform meshes shown in Fig. 1.

Grid	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	$O(h^r)$	$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$	$O(h^r)$	$\|p-p_h\|_0$	$O(h^r)$
	By the BDM P_1^2-P_0^{dis} finite element.					
4	0.4792E−01	1.84	0.1381E+01	1.28	0.5224E+01	1.26
5	0.1285E−01	1.90	0.5325E+00	1.38	0.2414E+01	1.11
6	0.3344E−02	1.94	0.1975E+00	1.43	0.1177E+01	1.04
	By the BDM P_2^2-P_1^{dis} finite element.					
3	0.5836E−02	2.96	0.2696E+00	1.85	0.3837E+00	2.14
4	0.7401E−03	2.98	0.7093E−01	1.93	0.9239E−01	2.05
5	0.9355E−04	2.98	0.1819E−01	1.96	0.2271E−01	2.02

Table 2
Computed errors for (43) on uniform meshes shown in Fig. 1.

Grid	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	$O(h^r)$	$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$	$O(h^r)$	$\|p-p_h\|_0$	$O(h^r)$
	By the BDM P_1^2-P_0^{dis} finite element.					
4	0.6606E−01	0.68	0.1517E+01	0.10	0.1271E+01	0.33
5	0.3931E−01	0.75	0.1400E+01	0.12	0.1097E+01	0.21
6	0.2299E−01	0.77	0.1287E+01	0.12	0.9767E+00	0.17
	By the BDM P_2^2-P_1^{dis} finite element.					
3	0.2989E−01	0.82	0.1109E+01	0.12	0.7899E+00	0.20
4	0.1679E−01	0.83	0.1017E+01	0.12	0.7061E+00	0.16
5	0.9498E−02	0.82	0.9312E+00	0.13	0.6385E+00	0.15

Table 3
Computed errors for (42) on irregular meshes shown in Fig. 3.

Grid	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	$O(h^r)$	$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$	$O(h^r)$	$\|p-p_h\|_0$	$O(h^r)$
	By the BDM P_1^2-P_0^{dis} finite element.					
4	0.1955E−01	2.06	0.2277E+01	1.09	0.1590E+01	0.91
5	0.4811E−02	2.02	0.1098E+01	1.05	0.8227E+00	0.95
6	0.1196E−02	2.01	0.5388E+00	1.03	0.4189E+00	0.97
	By the BDM P_2^2-P_1^{dis} finite element.					
3	0.2473E−01	2.98	0.1496E+01	1.95	0.7366E+00	2.16
4	0.3107E−02	2.99	0.3802E+00	1.98	0.1720E+00	2.10
5	0.3895E−03	3.00	0.9579E−01	1.99	0.4146E−01	2.05

Table 4
Computed errors for (43) on irregular meshes shown in Fig. 3.

Grid	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_0$	$O(h^r)$	$\|\nabla(\boldsymbol{u}-\boldsymbol{u}_h)\|_0$	$O(h^r)$	$\|p-p_h\|_0$	$O(h^r)$
	By the BDM P_1^2-P_0^{dis} finite element.					
4	0.7066E−01	0.77	0.3414E+01	0.12	0.1972E+01	0.19
5	0.3976E−01	0.83	0.3135E+01	0.12	0.1780E+01	0.15
6	0.2206E−01	0.85	0.2873E+01	0.13	0.1621E+01	0.14
	By the BDM P_2^2-P_1^{dis} finite element.					
3	0.2283E+00	1.00	0.1037E+02	0.12	0.6581E+01	0.18
4	0.1117E+00	1.03	0.9494E+01	0.13	0.5967E+01	0.14
5	0.5514E−01	1.02	0.8679E+01	0.13	0.5441E+01	0.13

results for the P_1 BDM match this order well in Table 2. The numerical order of L^2-convergence for the P_2 BDM is slightly higher. To view roughly the error in L^∞-norm, we plot the P_2^2-P_1^{dis} finite element solution on Grid 5 in Fig. 2.

We recompute above two problems on slightly irregular meshes, shown as in Fig. 3. For the smooth solution problem (42), the error and the order of convergence are listed in Table 3. We can see that the optimal order of convergence is achieved for both finite elements. Comparing the results with that from the computation on the uniform triangular meshes, the method is less accurate on the irregular meshes.

In Table 4, we list the results for the two finite elements for solving singular solution (43) on irregular meshes shown in Fig. 3. We expected the finite element solution behaves better this case as there are two more mesh lines at the singular points, comparing to the uniform triangulations shown in Fig. 1. It is surprising that the order of L^2-convergence on irregular meshes is even higher than that on the regular meshes. We get about orders 0.85 and 1.02 for the P_1 and P_2 solutions, respectively, while the theoretic order is $1/9+2/3=0.778$. We may guess that the extra mesh line at the re-entrant corner, cf. Fig. 3, can improve the L^2-convergence.

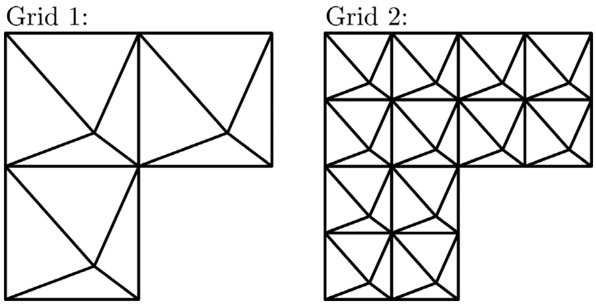

Fig. 3. The first two levels of triangular grids for Tables 3 and 4.

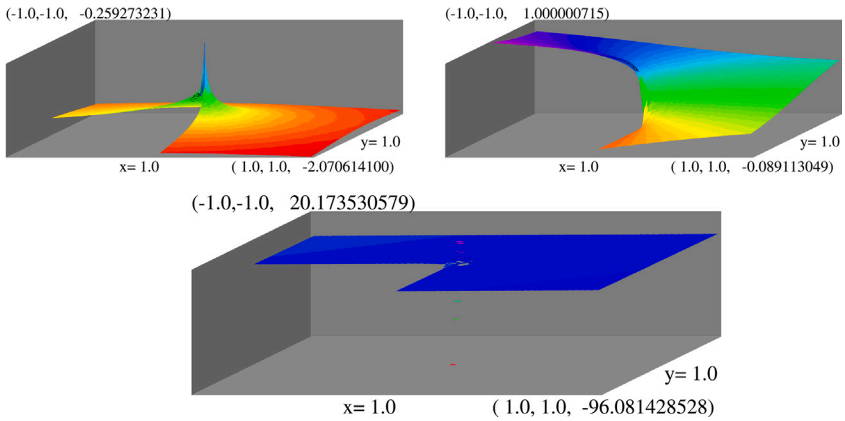

Fig. 4. The P_2^2-P_1^{dis} finite element solutions $\boldsymbol{u}_{1,h}$, $\boldsymbol{u}_{2,h}$ and p_h for the singular problem (43) on Grid 5 shown in Fig. 3.

Finally, in Fig. 4 we plot the P_2^2-P_1^{dis} finite element solution for the singular solution (43). Comparing to the solution in Fig. 2, the error near the singularity populates to a larger region and the error oscillates more.

7. Conclusions

We have established an improved a priori error estimate for an $\boldsymbol{H}(\mathrm{div})$-conforming IPDG method of Stokes equations. More precisely, we proved this numerical scheme is quasi-optimal up to higher-order data oscillations. How to extend this approach to a more complicated Darcy-Stokes singular perturbation problem [26] is an interesting topic, that will be carefully investigated in future work.

Data availability

No data was used for the research described in the article.

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