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A pressure-robust numerical scheme for the Stokes equations based on the WOPSIP DG approach

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ABSTRACT

In this paper, we propose and analyze a new weakly over-penalized symmetric interior penalty (WOPSIP) discontinuous Galerkin (DG) scheme for the Stokes equations. The primary approach involves modifying the right-hand term and replacing the pressure–velocity coupling term $c_h(\cdot,\cdot)$ by incorporating a weak divergence instead of the divergence operator. These modifications allow for pressure-robustness in the scheme. We establish optimal order error estimates for the velocity $\boldsymbol{u}_h$ in discrete energy norm and $\boldsymbol{L}^2$ norm, as well as for the pressure p_h in L^2 norm. We also provide numerical results to validate the effectiveness of the proposed scheme.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded polyhedral domain with boundary $\partial\Omega$. Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$, we consider the following Stokes model problem:

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{1.2}$$

$$\boldsymbol{u} = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where $\nu > 0$ is the viscosity parameter, $\boldsymbol{u}$ is the velocity field, p is the pressure, and $\boldsymbol{f}$ stands for a body force.

An essential aspect of finite element methods (FEMs) applied to the Stokes equations is the verification of the inf-sup condition [1,2]. In the last few decades, various inf-sup stable FEMs have been designed and analyzed, for example, MINI element [3], Crouzeix–Raviart (CR) element [4], Bernardi–Raugel element [5]. However, these conventional FEM schemes only provide a priori error estimates that rely on the pressure variable, that is (see [6]),

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,h} \le C_1 \inf_{\boldsymbol{v}\in\boldsymbol{V}_h} \|\boldsymbol{u}-\boldsymbol{v}_h\| + \frac{C_2}{\nu}\inf_{q_h\in P_h}\|p-q_h\|_0.$$

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This inequality implies that the velocity error may become large when either the viscosity ν is small or the pressure error is large.

The question of how to develop a pressure-robust scheme that eliminates the dependence on pressure has garnered significant attention in recent decades. One approach to achieve pressure-robustness is to employ divergence-free mixed elements [7–11], although this often requires additional degrees of freedom or imposes restrictions on mesh partitions. Another method is to utilize grad-div stabilization [12,13], which mitigates the lack of pressure robustness but does not completely eliminate it. Recently, a popular approach for designing a pressure-robust scheme involves the use of a velocity reconstruction operator [14]. Specifically, this technique modifies the right-hand side by introducing a $\boldsymbol{H}(\mathrm{div})$ conforming operator that establishes L^2-orthogonality between the mapped test velocities and the gradient fields. This approach has been successfully applied to various numerical methods [6,15–31]. We also refer the interested reader to [32], wherein a ***curl-curl*** weak formulation for velocities is proposed to achieve pressure-robustness.

In addition to classical finite element methods (FEMs), discontinuous Galerkin methods (DG) [33] have proven to be effective in simulating Stokes flow [34]. DG methods allow for the use of piecewise polynomial functions that are totally discontinuous across the triangulation, providing flexibility in handling highly nonuniform and unstructured meshes. They are also well-suited for dealing with inhomogeneous boundary conditions and hp-adaptive computation. However, standard DG methods [34] do not possess pressure-robustness. One remedy for this limitation is to employ $\boldsymbol{H}(\mathrm{div})$ conforming elements with penalization in DG discretization [35,36]. Another approach, developed in recent work, extends the velocity reconstruction techniques in [14] to interior penalty DG (IPDG) and conforming DG methods [26]. And recently, it was also extended to enriched Galerkin methods [37]. In this work, we further explore this issue within the context of weakly over-penalized symmetric interior penalty (WOPSIP) DG methods. The WOPSIP method was initially proposed and analyzed in [38] for solving second-order elliptic equations. Its main idea involves utilizing a piecewise constant $\boldsymbol{L}^2$-projection on mesh boundaries for weak penalization. Compared to standard IPDG methods, WOPSIP DG methods offer the advantages of having a simple bilinear form and being suitable for parallel computation [39]. Additionally, in WOPSIP DG schemes, there is no need to select an excessively large interior penalty parameter to achieve favorable properties. Due to these advantages, WOPSIP DG approaches have been explored for solving the biharmonic equation [40], Stokes equations [41], Reissner–Mindlin plate problem [42], and variational inequalities [43]. Moreover, the higher order version of WOPSIP DG method for second order elliptic equations was addressed in [44]. However, the original WOPSIP method for the Stokes equation, as explored in [41], is not pressure-robust. In this work, our objective is to extend this approach to achieve pressure-robustness. To accomplish this goal, we introduce a $\boldsymbol{H}(\mathrm{div})$ conforming operator to modify the right-hand side. Furthermore, we observe that the divergence operator in the pressure–velocity coupling term $c_h(\cdot,\cdot)$ (see (2.7) below) needs to be replaced with a weak divergence operator to attain pressure-robustness (see Remark 4.7 below). We also establish pressure-robust error estimates. It is worth mentioning that WOPSIP method is a non-consistent method since it remove some terms inherited from standard IPDG scheme. This means that the corresponding convergence analysis is more involved than that for standard IPDG method, since the consistent error is also needed to be estimated. In fact, WOPSIP method has some interesting connections with CR FEM (see [45]), thus its error estimate can be resorted by some techniques related to CR FE function space.

The rest of our paper is organized as follows. In the next section, we present the pressure-robust scheme, which involves modifying the right-hand term and replacing the divergence operator in $c_h(\cdot,\cdot)$ with a weak divergence. In Section 3, we establish the well-posedness of the numerical schemes, while Section 4 focuses on proving the pressure-robustness of error estimates. To support the theoretical analysis, we provide several numerical tests in Section 5. Finally, in Section 6, we draw conclusions based on our findings.

2. The pressure-robust WOPSIP method

Throughout the paper, we adopt certain standard notation. For a bounded domain $\mathcal{D}\subset\mathbb{R}^d,\ (d=2,3)$, we denote $H^s(\mathcal{D})(s\geq 0)$ by the standard Sobolev space with its norm $\|\cdot\|_{s,\mathcal{D}}$ and seminorm $|\cdot|_{s,\mathcal{D}}$. When $s=0$, $H^0(\mathcal{D})$ is the Lebesgue space $L^2(\mathcal{D})$. In addition, for functions $w,\ v\in L^2(\mathcal{D})$ we denote the inner product by $(w,v)_{\mathcal{D}}=\int_{\mathcal{D}} wv\,dx$ and $\langle w,v\rangle_{\partial\mathcal{D}}=\int_{\partial\mathcal{D}} wv\,ds$. We shall drop the subscript $\mathcal{D}$ when $\mathcal{D}=\Omega$. Additionally, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ that has vanishing trace on $\partial\Omega$, i.e., $H_0^1(\Omega)=\{v\in H^1(\Omega):v|_{\partial\Omega}=0\}$. By convention, for the vector-valued analogs, we shall use boldface type: $\boldsymbol{H}^m(\mathcal{D})=[H^m(\mathcal{D})]^d$. We also use the same symbol for the inner product in $\boldsymbol{L}^2(\mathcal{D})$ and $\underline{\boldsymbol{L}}^2(\mathcal{D})=[L^2(\mathcal{D})]^{d\times d}$. More precisely, $(\boldsymbol{w},\boldsymbol{v})_{\mathcal{D}}=\sum_{i=1}^d(\boldsymbol{w}_i,\boldsymbol{v}_i)_{\mathcal{D}}$ for $\boldsymbol{w},\ \boldsymbol{v}\in\boldsymbol{L}^2(\mathcal{D})$ and $(\boldsymbol{\zeta},\boldsymbol{\eta})_{\mathcal{D}}=\sum_{i=1}^d\sum_{i=1}^d(\boldsymbol{\zeta}_{ij},\boldsymbol{\eta}_{ij})_{\mathcal{D}}$ for $\boldsymbol{\zeta},\ \boldsymbol{\eta}\in\underline{\boldsymbol{L}}^2(\mathcal{D})$. Moreover, we introduce the Hilbert space $\boldsymbol{H}(\mathrm{div};\Omega)=\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega):\nabla\cdot\boldsymbol{v}\in L^2(\Omega)\}$ that is endowed with graph norm $\|\boldsymbol{v}\|_{\mathrm{div}}=\left(\|\boldsymbol{v}\|_0^2+\|\nabla\cdot\boldsymbol{v}\|_0^2\right)^{1/2}$. $\boldsymbol{H}_0(\mathrm{div};\Omega)$ is the subspace of $\boldsymbol{H}(\mathrm{div};\Omega)$ with vanishing normal trace on $\partial\Omega$, that is, $\boldsymbol{H}_0(\mathrm{div};\Omega)=\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div};\Omega):\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial\Omega}=0\}$.

Let $\boldsymbol{V}=\boldsymbol{H}_0^1(\Omega)$ and $Q=L_0^2(\Omega)$, the weak formulation of the Stokes problem (1.1)–(1.3) is to find $(\boldsymbol{u},p)\in\boldsymbol{V}\times Q$ such that

$$\nu a(\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},p)=(\boldsymbol{f},\boldsymbol{v})\quad\forall\boldsymbol{v}\in\boldsymbol{V},\tag{2.1}$$

$$b(\boldsymbol{u},q)=0\qquad\forall q\in Q,\tag{2.2}$$

where

$$a(\boldsymbol{u},\boldsymbol{v})=(\nabla\boldsymbol{u},\nabla\boldsymbol{v}),$$

$$b(\boldsymbol{u},q)=-(q,\nabla\cdot\boldsymbol{u}),$$

and

$$L_0^2(\Omega)=\{q\in L^2(\Omega):\int_\Omega q\,dx=0\}.$$

It is well know that the spaces $\boldsymbol{V}$ and Q satisfy the inf-sup condition (see [1])

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{b(\boldsymbol{v},q)}{|\boldsymbol{v}|_1} \geq \beta_c \|q\|_0 \quad \forall q\in Q, \tag{2.3}$$

which means that the problem (2.1)–(2.2) has a unique solution. After introducing the divergence free function space $\boldsymbol{X} = \{\boldsymbol{v} \in \boldsymbol{V} : \nabla\cdot\boldsymbol{v} = 0\}$, the velocity $\boldsymbol{u}$ in (2.1)–(2.2) has an equivalent formulation: Find $\boldsymbol{u} \in \boldsymbol{X}$ such that

$$\nu a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall \boldsymbol{v}\in\boldsymbol{X}. \tag{2.4}$$

For simplicity, we only consider the model problem (1.1)–(1.3) in two dimensions. The extension to three dimensions can be done with straightforward modifications.

Consider a family of conforming shape-regular meshes $\mathcal{T}_h$ that partition the domain Ω into triangle elements $\{T\}$. Denote by $h_T = \mathrm{diam}(T)$ and $h = \max_{T\in\mathcal{T}_h} h_T$. We use $\boldsymbol{n}_T$ to denote the outward unit normal vector of each element T. Additionally, let $\mathcal{E}_h^I$ be the set of interior edges of $\mathcal{T}_h$, and $\mathcal{E}_h^\partial$ be the set of boundary edges on $\partial\Omega$. Thus, the set of all edges $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^\partial$. For each edge $e\in\mathcal{E}_h$, its length is denoted by h_e. In particular, the set of edges of an element T is stated as $\mathcal{E}_h^T$, that is, $\mathcal{E}_h^T = \{e\in\mathcal{E}_h : e\subset\partial T\}$. For $e\in\mathcal{E}_h$, we define $\mathcal{T}_e = \{T\in\mathcal{T}_h : e\subset\partial T\}$. We use $P_k(D)$ to denote the space of polynomials of degree at most k on D, similarly, $\boldsymbol{P}_k(D)$ denotes the vector-valued case. Furthermore, for $s\geq 1$, we define the broken Sobolev space

$$H^s(\mathcal{T}_h) = \left\{ v\in L^2(\Omega) : v|_T \in H^s(T), \quad \forall T\in\mathcal{T}_h \right\}.$$

Throughout the paper, all generic constants (with or without subscripts) are independent of h and the parameter ν, but may depend on the shape regularity of $\mathcal{T}_h$ and the polynomial degree k.

Let $e\in\mathcal{E}_h^I$ be an interior edge, which is shared by two adjacent elements T^+ and T^-. For convenience, the global index of T^+ is assumed to be smaller than that of T^-. For a piecewise smooth scalar, vector or tensor function v with $v^\pm = v|_{T^\pm}$, we define their averages and jumps by

$$\{v\} = \frac{1}{2}(v^+ + v^-), \qquad [v] = v^+ - v^-.$$

When restricted to a boundary edge $e\in\mathcal{E}_h^\partial\cap\partial T^\pm$, we set $\{v\} = v^\pm$ and $[v] = v^\pm$. Moreover, we associate each $e\in\mathcal{E}_h^I$ with the unit normal vector as $\boldsymbol{n}_e = \boldsymbol{n}_{T^+}|_e = -\boldsymbol{n}_{T^-}|_e$. Similarly, for $e\in\mathcal{E}_h^\partial$, its outward unit normal vector $\boldsymbol{n}_e$ is defined along $\partial\Omega$ restricted to e.

Now we introduce the two finite element spaces $\boldsymbol{V}_h$ and P_h. More precisely, the fluid velocity is approximated by discontinuous $\boldsymbol{P}_1$ finite element spaces. While the pressure is discretized by piecewise constant finite element space P_h, that is,

$$\boldsymbol{V}_h = \left\{\boldsymbol{v}\in\boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_T\in\boldsymbol{P}_1(T),\ \forall T\in\mathcal{T}_h\right\},$$
$$P_h = \left\{q\in L_0^2(\Omega) : q|_T\in P_0(T),\ \forall T\in\mathcal{T}_h\right\}.$$

The bilinear form of the weakly over-penalized symmetric interior penalty method is defined by (see Ref. [38])

$$a_h(\boldsymbol{w},\boldsymbol{v}) = \sum_{T\in\mathcal{T}_h}(\nabla\boldsymbol{w},\nabla\boldsymbol{v})_T + \sum_{e\in\mathcal{E}_h} h_e^{-3}\langle \Pi_e^0[\boldsymbol{w}], \Pi_e^0[\boldsymbol{v}]\rangle_e \quad \forall \boldsymbol{w},\boldsymbol{v}\in\boldsymbol{V}_h,$$

where Π_e^0 is the $\boldsymbol{L}^2$ projection from $\boldsymbol{L}^2(e)$ onto $\boldsymbol{P}_0(e)$, that is,

$$\Pi_e^0\boldsymbol{v} = \frac{1}{h_e}\int_e \boldsymbol{v}\,ds. \tag{2.5}$$

The standard WOPSIP method for Stokes Eqs. (1.1)–(1.3) is to find that $(\boldsymbol{u}_h, p_h)\in\boldsymbol{V}_h\times P_h$ satisfying (see [41])

$$\nu a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + c_h(\boldsymbol{v}_h,p_h) = (\boldsymbol{f},\boldsymbol{v}_h) \quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.6}$$

$$c_h(\boldsymbol{u}_h,q_h) = 0 \quad \forall q_h\in P_h, \tag{2.7}$$

with $c_h(\boldsymbol{v}_h,q_h) = -\sum_{T\in\mathcal{T}_h}(q_h,\nabla\cdot\boldsymbol{v}_h)_T$.

The above method is not pressure-robust, for a remedy, motivated by [26], we modified the right-hand term by introducing a $\boldsymbol{H}(\mathrm{div})$ conforming interpolation π_h. Additionally, we have replaced the divergence operator in the pressure–velocity coupling term $c_h(\cdot,\cdot)$ with a weak divergence. Consequently, the modified WOPSIP method for solving problem (1.1)–(1.3) can be stated as follows: Find $(\boldsymbol{u}_h,p_h)\in\boldsymbol{V}_h\times P_h$ such that

$$\nu a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h,p_h) = (\boldsymbol{f},\pi_h\boldsymbol{v}_h) \quad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.8}$$

$$b_h(\boldsymbol{u}_h,q_h) = 0 \quad \forall q_h\in P_h. \tag{2.9}$$

Here, $b_h(\boldsymbol{v}_h,q_h) = -\sum_{T\in\mathcal{T}_h}(q_h,\nabla_w\cdot\boldsymbol{v}_h)_T$, the term $\nabla_w\cdot\boldsymbol{u}_h$ is inspired by weak Galerkin method [46], further information regarding this can be found in Definition 2.1 provided below. In the above, π_h can take the form of π^{RT} or π^{BDM}, and additional details can be found in Definitions 2.2 and 2.3 respectively.

Definition 2.1. For a piecewise smooth vector function $\boldsymbol{v}$ on $\mathcal{T}_h$, its weak divergence $\nabla_w\cdot\boldsymbol{v}|_T\in P_0(T)$ is defined by

$$(\nabla_w\cdot\boldsymbol{v},q)_T = -(\boldsymbol{v},\nabla q)_T + \langle\{\boldsymbol{v}\}\cdot\boldsymbol{n}_T,q\rangle_{\partial T} \quad \forall q\in P_0(T). \tag{2.10}$$

Definition 2.2. For any $\boldsymbol{v} \in \boldsymbol{V} + \boldsymbol{V}_h$, $\pi^{\mathrm{RT}}\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega) \cap \mathrm{RT}_0$ is defined as: for each $e \in \partial T$, it satisfies

$$\langle \pi^{\mathrm{RT}}\boldsymbol{v} \cdot \boldsymbol{n}_T, q\rangle_e = \langle \{\boldsymbol{v}\} \cdot \boldsymbol{n}_T, q\rangle_e \quad \forall q \in P_0(e), \tag{2.11}$$

where $\mathrm{RT}_0 = \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}_h|_T \in \mathrm{RT}_0(T),\ \forall T \in \mathcal{T}_h\}$, with $\mathrm{RT}_0(T) = \boldsymbol{P}_0(T) + \boldsymbol{x}P_0(T)$ (see [47]).

Definition 2.3. For any $\boldsymbol{v} \in \boldsymbol{V} + \boldsymbol{V}_h$, $\pi^{\mathrm{BDM}}\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega) \cap \mathrm{BDM}_1$ is defined as: for each $e \in \partial T$, it satisfies

$$\langle \pi^{\mathrm{BDM}}\boldsymbol{v} \cdot \boldsymbol{n}_T, q\rangle_e = \langle \{\boldsymbol{v}\} \cdot \boldsymbol{n}_T, q\rangle_e \quad \forall q \in P_1(e), \tag{2.12}$$

where $\mathrm{BDM}_1 = \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}_h|_T \in \mathrm{BDM}_1(T),\ \forall T \in \mathcal{T}_h\}$ (see [48]), with $\mathrm{BDM}_1(T) = \boldsymbol{P}_1(T)$.

Using (2.10) and integration by parts, we have

$$b_h(\boldsymbol{v}_h, q_h) = -\sum_{T\in\mathcal{T}_h} (q_h, \nabla \cdot \boldsymbol{v}_h)_T + \sum_{e\in\mathcal{E}_h} \langle \{q_h\}, [\boldsymbol{v}_h \cdot \boldsymbol{n}_e]\rangle_e. \tag{2.13}$$

The following approximation properties are well known (see [1])

$$\|\boldsymbol{v} - \pi^{\mathrm{RT}}\boldsymbol{v}\|_0 \le Ch\|\boldsymbol{v}\|_{1,h} \quad \forall \boldsymbol{v} \in \boldsymbol{V} + \boldsymbol{V}_h, \tag{2.14}$$

$$\|\boldsymbol{v} - \pi^{\mathrm{BDM}}\boldsymbol{v}\|_0 \le Ch\|\boldsymbol{v}\|_{1,h} \quad \forall \boldsymbol{v} \in \boldsymbol{V} + \boldsymbol{V}_h. \tag{2.15}$$

Here, $\|\boldsymbol{v}\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} \|\nabla \boldsymbol{v}\|_{0,T}^2$.

By introducing the discretely divergence-free function space

$$\boldsymbol{X}_h = \{\boldsymbol{v}_h \in \boldsymbol{V}_h : b_h(\boldsymbol{v}_h, q_h) = 0 \ \ \forall q_h \in P_h\}, \tag{2.16}$$

we then show that $\boldsymbol{u}_h$ in (2.8)–(2.9) satisfies $\boldsymbol{u}_h \in \boldsymbol{X}_h$ and

$$\nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\boldsymbol{f}, \pi_h \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{X}_h. \tag{2.17}$$

Remark 2.4 (*Comparison with CR FEM*)**.** When we restrict discrete velocity on Crouzeix–Raviart FEM space, that is, choosing

$$\boldsymbol{V}_h^{\mathrm{CR}} = \{\boldsymbol{v}_h \in \boldsymbol{V}_h : \int_e [\boldsymbol{v}_h] ds = 0 \ \forall e \in \mathcal{E}_h\},$$

the numerical method (2.8)–(2.9) is reduced to the modified $CR - P_0$ FEM scheme (see [20,23]): Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h^{\mathrm{CR}} \times P_h$ such that

$$\nu \sum_{T\in\mathcal{T}_h} (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h) - \sum_{T\in\mathcal{T}_h} (\nabla \cdot \boldsymbol{v}_h, p_h) = (\boldsymbol{f}, \pi_h \boldsymbol{v}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.18}$$

$$-\sum_{T\in\mathcal{T}_h} (\nabla \cdot \boldsymbol{u}_h, q_h) = 0 \quad \forall q_h \in P_h. \tag{2.19}$$

3. Well-posedness

We first define two norms $\|\cdot\|_{\mathrm{WP}}$ and $\|\cdot\|_{\mathrm{DG}}$ on $\boldsymbol{H}^1(\mathcal{T}_h)$:

$$\|\boldsymbol{v}\|_{\mathrm{WP}} = \left(\|\boldsymbol{v}\|_{1,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-3}\|\Pi_e^0[\boldsymbol{v}]\|_{0,e}^2\right)^{1/2}, \tag{3.1}$$

$$\|\boldsymbol{v}\|_{\mathrm{DG}} = \left(\|\boldsymbol{v}\|_{1,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\boldsymbol{v}]\|_{0,e}^2\right)^{1/2}. \tag{3.2}$$

Then, using the Cauchy–Schwarz inequality, we can obtain the following lemma.

Lemma 3.1. *It holds that*

$$|a_h(\boldsymbol{w}, \boldsymbol{v})| \le \|\boldsymbol{w}\|_{\mathrm{WP}}\|\boldsymbol{v}\|_{\mathrm{WP}} \quad \forall \boldsymbol{w},\ \boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h). \tag{3.3}$$

Moreover, it follows from the definition of $\|\cdot\|_{\mathrm{WP}}$ *that*

$$a_h(\boldsymbol{v}, \boldsymbol{v}) = \|\boldsymbol{v}\|_{\mathrm{WP}}^2 \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h). \tag{3.4}$$

Then, we define the Crouzeix–Raviart interpolation operator $\Pi_T^{\mathrm{CR}} : \boldsymbol{H}^1(T) \to \boldsymbol{P}_1(T)$ by (see [4])

$$\int_e \Pi_T^{\mathrm{CR}}\boldsymbol{v} ds = \int_e \boldsymbol{v} ds \quad \forall e \in \mathcal{E}_h^T. \tag{3.5}$$

Standard error estimate implies that (see [4])

$$h_T^{-1}\|\boldsymbol{v} - \Pi_T^{\mathrm{CR}}\boldsymbol{v}\|_{0,T} + |\boldsymbol{v} - \Pi_T^{\mathrm{CR}}\boldsymbol{v}|_{1,T} \le Ch_T^m|\boldsymbol{v}|_{m+1,T}, \quad m = 0, 1. \tag{3.6}$$

Then, the global Crouzeix–Raviart interpolation operator $\Pi_h^{\mathrm{CR}} : \boldsymbol{H}^1(\Omega) \to \boldsymbol{V}_h$ can be constructed by

$$(\Pi_h^{\mathrm{CR}} \boldsymbol{v})|_T = \Pi_T^{\mathrm{CR}}(\boldsymbol{v}|_T) \quad \forall T \in \mathcal{T}_h.$$

Using (3.5) and integration by parts, for $\boldsymbol{v} \in \boldsymbol{H}^1(T)$ we have

$$(\nabla \cdot (\Pi_T^{\mathrm{CR}} \boldsymbol{v}), q_h)_T = (\nabla \cdot \boldsymbol{v}, q_h)_T \quad \forall q_h \in P_0(T). \tag{3.7}$$

Moreover, it follows from (3.5) that

$$\Pi_e^0[\Pi_h^{\mathrm{CR}} \boldsymbol{v}] = \frac{1}{h_e} \int_e [\Pi_h^{\mathrm{CR}} \boldsymbol{v}] ds = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V} = \boldsymbol{H}_0^1(\Omega), \tag{3.8}$$

which along with (3.6) yields

$$\|\Pi_h^{\mathrm{CR}} \boldsymbol{v}\|_{\mathrm{WP}} = \Big(\sum_{T \in \mathcal{T}_h} |\Pi_h^{\mathrm{CR}} \boldsymbol{v}|_{1,T}^2 \Big)^{1/2} \le C_\alpha |\boldsymbol{v}|_1 \quad \forall \boldsymbol{v} \in \boldsymbol{V} = \boldsymbol{H}_0^1(\Omega). \tag{3.9}$$

We also need the following trace inequality (see [33])

$$\|\boldsymbol{w}\|_{0,e}^2 \le C\big(h_e^{-1} \|\boldsymbol{w}\|_{0,T}^2 + h_e \|\nabla \boldsymbol{w}\|_{0,T}^2\big) \qquad \forall \boldsymbol{w} \in \boldsymbol{H}^1(T), \tag{3.10}$$

and the inverse inequality (see [49,50])

$$\|\boldsymbol{v}\|_{0,e} \le C h_T^{-1/2} \|\boldsymbol{v}\|_{0,T} \qquad \forall \boldsymbol{v} \in \boldsymbol{P}_1(T),\ \forall e \in \mathcal{E}_h^T. \tag{3.11}$$

To proceed, we recall the following useful result that shows the relation between the jumps across edges and the norm $\|\cdot\|_{\mathrm{WP}}$ (see Lemma 3.1 in [38]).

Lemma 3.2. *For all $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h)$, it holds that*

$$\sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\boldsymbol{v}]\|_{0,e}^2 \le C_1 \|\boldsymbol{v}\|_{\mathrm{WP}}^2. \tag{3.12}$$

This along with (3.1) and (3.2) yields

$$\|\boldsymbol{v}\|_{\mathrm{DG}} \le C_* \|\boldsymbol{v}\|_{\mathrm{WP}} \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h).$$

Next, we shall establish the following discrete inf-sup condition.

Theorem 3.3. *It holds that*

$$\sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{\mathrm{WP}}} \ge \beta_1 \|q_h\|_0 \tag{3.13}$$

for any $q_h \in P_h$.

Proof. For $\boldsymbol{v} \in \boldsymbol{V} = \boldsymbol{H}_0^1(\Omega)$, in light of (2.13), (3.7) and (3.8), we arrive at

$$\begin{aligned} b_h(\Pi_h^{\mathrm{CR}} \boldsymbol{v}, q_h) &= -\sum_{T \in \mathcal{T}_h} (q_h, \nabla \cdot \Pi_h^{\mathrm{CR}} \boldsymbol{v})_T + \sum_{e \in \mathcal{E}_h} \langle \{q_h\}, [\Pi_h^{\mathrm{CR}} \boldsymbol{v} \cdot \boldsymbol{n}_e] \rangle_e \\ &= -\sum_{T \in \mathcal{T}_h} (q_h, \nabla \cdot \Pi_h^{\mathrm{CR}} \boldsymbol{v})_T = -\sum_{T \in \mathcal{T}_h} (q_h, \nabla \cdot \boldsymbol{v})_T = b_h(\boldsymbol{v}, q_h) \quad \forall q_h \in P_h. \end{aligned}$$

This together with (2.3) and (3.9) implies that

$$\begin{aligned} \beta_c \|q_h\|_0 \le \sup_{\boldsymbol{v} \in \boldsymbol{V}} \frac{b_h(\boldsymbol{v}, q_h)}{|\boldsymbol{v}|_1} &= \sup_{\boldsymbol{v} \in \boldsymbol{V}} \frac{b_h(\Pi_h^{\mathrm{CR}} \boldsymbol{v}, q_h)}{|\boldsymbol{v}|_1} \\ &\le C_\alpha \sup_{\Pi_h^{\mathrm{CR}} \boldsymbol{v} \in \boldsymbol{V}_h} \frac{b_h(\Pi_h^{\mathrm{CR}} \boldsymbol{v}, q_h)}{\|\Pi_h^{\mathrm{CR}} \boldsymbol{v}\|_{\mathrm{WP}}} \\ &\le C_\alpha \sup_{\boldsymbol{w}_h \in \boldsymbol{V}_h} \frac{b_h(\boldsymbol{w}_h, q_h)}{\|\boldsymbol{w}_h\|_{\mathrm{WP}}}. \end{aligned}$$

Then, the desired assertion (3.13) is satisfied by taking $\beta_1 = \beta_c C_\alpha^{-1}$. □

Theorem 3.4. *The modified WOPSIP method (2.8)–(2.9) has a unique solution.*

Proof. It is enough to show that $\boldsymbol{f} = \mathbf{0}$ implies that $\boldsymbol{u}_h = \mathbf{0}$ and $p_h = 0$. In this case, taking $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (2.8) and $q_h = p_h$ in (2.9), and then subtracting (2.9) from (2.8) yields

$$\nu\|\boldsymbol{u}_h\|_{\mathrm{WP}}^2 = \nu a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) = 0.$$

It follows that $\boldsymbol{u}_h = \mathbf{0}$. This together (2.8) yields $b_h(\boldsymbol{v}_h, p_h) = 0$ for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$. It follows from the discrete inf-sup condition (3.13) that $p_h = 0$. The proof is completed. □

4. Error estimates

This section is devoted to deriving optimal order error estimates for the velocity $\boldsymbol{u}_h$ in $\|\cdot\|_{\mathrm{WP}}$ norm, and for the pressure p_h in L^2 norm. The main results are stated in Theorem 4.4. To obtain the desired estimates, we need Lemmas 4.1–4.3.

Lemma 4.1. *For any $\boldsymbol{\varphi} \in \boldsymbol{V} + \boldsymbol{V}_h$, on $T \in \mathcal{T}_h$, it holds that*

$$\nabla \cdot \pi_h \boldsymbol{\varphi} = \nabla_w \cdot \boldsymbol{\varphi}. \tag{4.1}$$

Proof. For any $q \in P_0(T)$, noting that $\nabla q|_T = \mathbf{0}$, it then follows from (2.10)–(2.12) that

$$\begin{aligned}(\nabla \cdot \pi_h \boldsymbol{\varphi}, q)_T &= -(\pi_h \boldsymbol{\varphi}, \nabla q)_T + \langle \pi_h \boldsymbol{\varphi} \cdot \boldsymbol{n}_T, q\rangle_{\partial T}\\ &= -(\boldsymbol{\varphi}, \nabla q)_T + \langle \{\boldsymbol{\varphi}\} \cdot \boldsymbol{n}_T, q\rangle_{\partial T}\\ &=(\nabla_w \cdot \boldsymbol{\varphi}, q)_T,\end{aligned}$$

which proves the desired result (4.1). □

Lemma 4.2. *For any $\phi \in H^1(\Omega)$, $\boldsymbol{v}_h \in \boldsymbol{V}_h$, there holds*

$$\sum_{T\in\mathcal{T}_h} (\nabla\phi, \pi_h \boldsymbol{v}_h)_T = -\sum_{T\in\mathcal{T}_h} (\pi_0\phi, \nabla_w \cdot \boldsymbol{v}_h)_T, \tag{4.2}$$

where π_0 is the L^2 projection from Q onto P_h.

Proof. Since $\phi \in H^1(\Omega)$, then $[\phi] = 0$ on any $e \in \mathcal{E}_h^I$. Observed that $\pi_h \boldsymbol{v}_h \in \boldsymbol{H}_0(\mathrm{div}; \Omega)$ implies $[\pi_h \boldsymbol{v}_h \cdot \boldsymbol{n}_e] = 0$ on any $e \in \mathcal{E}_h$. These facts together with (4.1) imply that

$$\begin{aligned}&\sum_{T\in\mathcal{T}_h} (\nabla\phi, \pi_h \boldsymbol{v}_h)_T\\ &= -\sum_{T\in\mathcal{T}_h} (\phi, \nabla \cdot \pi_h \boldsymbol{v}_h)_T + \sum_{T\in\mathcal{T}_h} \langle \phi, \pi_h \boldsymbol{v}_h \cdot \boldsymbol{n}_T\rangle_{\partial T}\\ &= -\sum_{T\in\mathcal{T}_h} (\phi, \nabla \cdot \pi_h \boldsymbol{v}_h)_T + \sum_{e\in\mathcal{E}_h^I} \big\langle [\phi], \{\pi_h \boldsymbol{v}_h \cdot \boldsymbol{n}_e\}\big\rangle_e\\ &\quad + \sum_{e\in\mathcal{E}_h} \big\langle \{\phi\}, [\pi_h \boldsymbol{v}_h \cdot \boldsymbol{n}_e]\big\rangle_e\\ &= -\sum_{T\in\mathcal{T}_h} (\phi, \nabla \cdot \pi_h \boldsymbol{v}_h)_T\\ &= -\sum_{T\in\mathcal{T}_h} (\pi_0\phi, \nabla \cdot \pi_h \boldsymbol{v}_h)_T\\ &= -\sum_{T\in\mathcal{T}_h} (\pi_0\phi, \nabla_w \cdot \boldsymbol{v}_h)_T,\end{aligned}$$

which is the desired assertion (4.2). □

Lemma 4.3. *Let $(\boldsymbol{u}, p)$ be the solutions of (2.1)–(2.2), assume that $p \in H^1(\Omega)$, there holds*

$$\frac{1}{\nu} \sup_{\boldsymbol{w}_h\in\boldsymbol{X}_h} \frac{|\nu a_h(\boldsymbol{u}, \boldsymbol{w}_h) - (\boldsymbol{f}, \pi_h \boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_{\mathrm{WP}}} \leq Ch\|\boldsymbol{u}\|_2. \tag{4.3}$$

Proof. For any $\boldsymbol{w}_h \in \boldsymbol{X}_h$, using (4.2) to obtain

$$\sum_{T\in\mathcal{T}_h} (\nabla p, \pi_h \boldsymbol{w}_h)_T = -\sum_{T\in\mathcal{T}_h} (\pi_0 p, \nabla_w \cdot \boldsymbol{w}_h)_T = b_h(\boldsymbol{w}_h, \pi_0 p). \tag{4.4}$$

It then follows from (2.16) that $b_h(\boldsymbol{w}_h, \pi_0 p) = 0$. This together with the Stokes Eq. (1.1) implies that

$$\frac{1}{\nu}|\nu a_h(\boldsymbol{u}, \boldsymbol{w}_h) - (\boldsymbol{f}, \pi_h \boldsymbol{w}_h)|$$

$$
\begin{aligned}
&=\frac{1}{\nu}\Big|\sum_{T\in\mathcal{T}_h}(\nu\nabla\boldsymbol{u},\nabla\boldsymbol{w}_h)_T-\sum_{T\in\mathcal{T}_h}(\boldsymbol{f},\pi_h\boldsymbol{w}_h)_T\Big|\\
&=\frac{1}{\nu}\Big|\sum_{T\in\mathcal{T}_h}(\nu\nabla\boldsymbol{u},\nabla\boldsymbol{w}_h)_T+\sum_{T\in\mathcal{T}_h}(\nu\Delta\boldsymbol{u}-\nabla p,\pi_h\boldsymbol{w}_h)_T\Big|\\
&=\frac{1}{\nu}\Big|\sum_{T\in\mathcal{T}_h}(\nu\nabla\boldsymbol{u},\nabla\boldsymbol{w}_h)_T+\sum_{T\in\mathcal{T}_h}(\nu\Delta\boldsymbol{u},\pi_h\boldsymbol{w}_h)_T\Big|\\
&=\Big|\sum_{T\in\mathcal{T}_h}(\nabla\boldsymbol{u},\nabla\boldsymbol{w}_h)_T+\sum_{T\in\mathcal{T}_h}(\Delta\boldsymbol{u},\pi_h\boldsymbol{w}_h)_T\Big|\\
&=\Big|\Big(\sum_{T\in\mathcal{T}_h}(\nabla\boldsymbol{u},\nabla\boldsymbol{w}_h)_T+\sum_{T\in\mathcal{T}_h}(\Delta\boldsymbol{u},\boldsymbol{w}_h)_T\Big)+\sum_{T\in\mathcal{T}_h}(\Delta\boldsymbol{u},\pi_h\boldsymbol{w}_h-\boldsymbol{w}_h)_T\Big|\\
&\triangleq\Big|\mathbb{A}_1+\mathbb{A}_2\Big|.
\end{aligned}
\tag{4.5}
$$

For the term $\mathbb{A}_1$, integrating by parts leads to

$$
\begin{aligned}
\mathbb{A}_1&=\sum_{T\in\mathcal{T}_h}(\nabla\boldsymbol{u},\nabla\boldsymbol{w}_h)_T+\sum_{T\in\mathcal{T}_h}(\Delta\boldsymbol{u},\boldsymbol{w}_h)_T\\
&=\sum_{T\in\mathcal{T}_h}\Big\langle(\nabla\boldsymbol{u})\boldsymbol{n}_T,\boldsymbol{w}_h\Big\rangle_{\partial T}\\
&=\sum_{e\in\mathcal{E}_h^I}\Big\langle[(\nabla\boldsymbol{u})\boldsymbol{n}_e],\{\boldsymbol{w}_h\}\Big\rangle_e+\sum_{e\in\mathcal{E}_h}\Big\langle\{(\nabla\boldsymbol{u})\boldsymbol{n}_e\},[\boldsymbol{w}_h]\Big\rangle_e\\
&=\sum_{e\in\mathcal{E}_h}\Big\langle\{(\nabla\boldsymbol{u})\boldsymbol{n}_e\},[\boldsymbol{w}_h]\Big\rangle_e\\
&=\sum_{e\in\mathcal{E}_h}\Big\langle(\nabla\boldsymbol{u}-\Pi_{\mathcal{T}_e}^{\mathrm{CR}}\boldsymbol{u})\boldsymbol{n}_e,[\boldsymbol{w}_h]\Big\rangle_e+\sum_{e\in\mathcal{E}_h}\Big\langle(\nabla\Pi_{\mathcal{T}_e}^{\mathrm{CR}}\boldsymbol{u})\boldsymbol{n}_e,[\boldsymbol{w}_h]\Big\rangle_e\\
&\triangleq\mathbb{A}_{11}+\mathbb{A}_{12}.
\end{aligned}
\tag{4.6}
$$

Combining the approximation property in (3.6), the trace inequality (3.10), (3.12) and the Cauchy–Schwarz inequality, we can estimate $\mathbb{A}_{11}$ by

$$
\begin{aligned}
|\mathbb{A}_{11}|&=\Big|\sum_{e\in\mathcal{E}_h}\Big\langle(\nabla\boldsymbol{u}-\Pi_{\mathcal{T}_e}^{\mathrm{CR}}\boldsymbol{u})\boldsymbol{n}_e,[\boldsymbol{w}_h]\Big\rangle_e\Big|\\
&\leq\Big(\sum_{e\in\mathcal{E}_h}h_e\|\nabla(\boldsymbol{u}-\Pi_{\mathcal{T}_e}^{\mathrm{CR}}\boldsymbol{u})\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}h_e^{-1}\|[\boldsymbol{w}_h]\|_{0,e}^2\Big)^{1/2}\\
&\leq Ch|\boldsymbol{u}|_2\|\boldsymbol{w}_h\|_{\mathrm{WP}}.
\end{aligned}
\tag{4.7}
$$

Using (2.5), (3.9), (3.11) and the Cauchy–Schwarz inequality, we bound $\mathbb{A}_{12}$ by

$$
\begin{aligned}
|\mathbb{A}_{12}|&=\Big|\sum_{e\in\mathcal{E}_h}\Big\langle(\nabla\Pi_{\mathcal{T}_e}^{\mathrm{CR}}\boldsymbol{u})\boldsymbol{n}_e,[\boldsymbol{w}_h]\Big\rangle_e\Big|\\
&=\Big|\sum_{e\in\mathcal{E}_h}\Big\langle(\nabla\Pi_{\mathcal{T}_e}^{\mathrm{CR}}\boldsymbol{u})\boldsymbol{n}_e,\Pi_e^0[\boldsymbol{w}_h]\Big\rangle_e\Big|\\
&\leq\Big(\sum_{e\in\mathcal{E}_h}h_e^3\|\nabla(\Pi_{\mathcal{T}_e}^{\mathrm{CR}}\boldsymbol{u})\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}h_e^{-3}\|\Pi_e^0[\boldsymbol{w}_h]\|_{0,e}^2\Big)^{1/2}\\
&\leq C\Big(\sum_{e\in\mathcal{E}_h}h_e^2\sum_{T\in\mathcal{T}_e}\|\nabla(\Pi_T^{\mathrm{CR}}\boldsymbol{u})\|_{0,T}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}h_e^{-3}\|\Pi_e^0[\boldsymbol{w}_h]\|_{0,e}^2\Big)^{1/2}\\
&\leq Ch\Big(\sum_{T\in\mathcal{T}_h}\|\nabla(\Pi_h^{\mathrm{CR}}\boldsymbol{u})\|_{0,T}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_h}h_e^{-3}\|\Pi_e^0[\boldsymbol{w}_h]\|_{0,e}^2\Big)^{1/2}\\
&\leq Ch|\boldsymbol{u}|_1\|\boldsymbol{w}_h\|_{\mathrm{WP}}.
\end{aligned}
\tag{4.8}
$$

Applying the approximation property in (2.14) and (2.15), we bound $\mathbb{A}_2$ by

$$
\begin{aligned}
|\mathbb{A}_2|&=\Big|\sum_{T\in\mathcal{T}_h}(\Delta\boldsymbol{u},\pi_h\boldsymbol{w}_h-\boldsymbol{w}_h)_T\Big|\\
&\leq\Big(\sum_{T\in\mathcal{T}_h}\|\Delta\boldsymbol{u}\|_{0,T}^2\Big)^{1/2}\Big(\sum_{T\in\mathcal{T}_h}\|\pi_h\boldsymbol{w}_h-\boldsymbol{w}_h\|_{0,T}^2\Big)^{1/2}\\
&\leq Ch|\boldsymbol{u}|_2\|\boldsymbol{w}_h\|_{1,h}\\
&\leq Ch|\boldsymbol{u}|_2\|\boldsymbol{w}_h\|_{\mathrm{WP}}.
\end{aligned}
\tag{4.9}
$$

Plugging (4.6)–(4.9) into (4.5) gives the desired estimate (4.3). □

We are now in a position to prove the error estimate for the numerical scheme (2.8)–(2.9).

Theorem 4.4. *Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be the solutions of (2.1)–(2.2) and (2.8)–(2.9), respectively. Assume that $(\boldsymbol{u}, p) \in \boldsymbol{H}^2(\Omega) \times H^1(\Omega)$, then it holds that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{WP}} \le Ch\|\boldsymbol{u}\|_2, \tag{4.10}$$

$$\|\pi_0 p - p_h\|_0 \le Ch\nu\|\boldsymbol{u}\|_2, \tag{4.11}$$

$$\|p - p_h\|_0 \le Ch(\nu\|\boldsymbol{u}\|_2 + |p|_1). \tag{4.12}$$

Proof. For any $\boldsymbol{v}_h \in \boldsymbol{X}_h$, denote by $\boldsymbol{w}_h = \boldsymbol{u}_h - \boldsymbol{v}_h$, it follows from (2.17) that

$$\begin{aligned}
\nu\|\boldsymbol{w}_h\|_{\mathrm{WP}}^2 =& \nu a_h(\boldsymbol{w}_h, \boldsymbol{w}_h)\\
=& \nu a_h(\boldsymbol{u}_h - \boldsymbol{v}_h, \boldsymbol{w}_h)\\
=& \nu a_h(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{w}_h) + \nu a_h(\boldsymbol{u}_h, \boldsymbol{w}_h) - \nu a_h(\boldsymbol{u}, \boldsymbol{w}_h)\\
=& \nu a_h(\boldsymbol{u} - \boldsymbol{v}_h, \boldsymbol{w}_h) + (\boldsymbol{f}, \pi_h \boldsymbol{w}_h) - \nu a_h(\boldsymbol{u}, \boldsymbol{w}_h)\\
\le& \nu\|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{WP}}\|\boldsymbol{w}_h\|_{\mathrm{WP}} + |\nu a_h(\boldsymbol{u}, \boldsymbol{w}_h) - (\boldsymbol{f}, \pi_h \boldsymbol{w}_h)|.
\end{aligned} \tag{4.13}$$

On the other hand, the triangle inequality implies that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{WP}} \le \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{WP}} + \|\boldsymbol{w}_h\|_{\mathrm{WP}}. \tag{4.14}$$

Combining the above two formulations in (4.13) and (4.14) to arrive at

$$\begin{aligned}
\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{WP}} \le& 2 \inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{WP}}\\
&+ \frac{1}{\nu} \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h} \frac{|\nu a_h(\boldsymbol{u}, \boldsymbol{w}_h) - (\boldsymbol{f}, \pi_h \boldsymbol{w}_h)|}{\|\boldsymbol{w}_h\|_{\mathrm{WP}}}.
\end{aligned} \tag{4.15}$$

Taking $\boldsymbol{v}_h = \Pi_h^{\mathrm{CR}} \boldsymbol{u}_h$, it follows from (3.6) and (3.8) that

$$\inf_{\boldsymbol{v}_h \in \boldsymbol{X}_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{WP}} \le \|\boldsymbol{u} - \Pi_h^{\mathrm{CR}} \boldsymbol{u}\|_{\mathrm{WP}} = \|\boldsymbol{u} - \Pi_h^{\mathrm{CR}} \boldsymbol{u}\|_{1,h} \le Ch\|\boldsymbol{u}\|_2. \tag{4.16}$$

Substituting (4.16), (4.3) into (4.15) gives the desired assertion (4.10).

Next, we shall prove the pressure estimates (4.11) and (4.12). First, we use the Pythagoras theorem to obtain

$$\|p - p_h\|_0^2 = \|p - \pi_0 p\|_0^2 + \|\pi_0 p - p_h\|_0^2. \tag{4.17}$$

Standard error estimates leads to (see [49,50])

$$\|p - \pi_0 p\|_0 \le Ch|p|_1, \tag{4.18}$$

it leaves us to estimate $\|\pi_0 p - p_h\|_0$. In view of the discrete inf-sup condition (3.13), we have

$$\|\pi_0 p - p_h\|_0 \le \frac{1}{\beta_1} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b_h(\boldsymbol{v}_h, \pi_0 p - p_h)}{\|\boldsymbol{v}_h\|_{\mathrm{WP}}}. \tag{4.19}$$

For any $\boldsymbol{v}_h \in \boldsymbol{V}_h$, it follows from (1.1), (4.2), and integration by parts that

$$\begin{aligned}
& b_h(\boldsymbol{v}_h, \pi_0 p - p_h)\\
=& b_h(\boldsymbol{v}_h, \pi_0 p) - b_h(\boldsymbol{v}_h, p_h)\\
=& -\sum_{T \in \mathcal{T}_h} (\pi_0 p, \nabla_w \cdot \boldsymbol{v}_h)_T - \sum_{T \in \mathcal{T}_h} (\boldsymbol{f}, \pi_h \boldsymbol{v}_h)_T + \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)\\
=& -\sum_{T \in \mathcal{T}_h} (\pi_0 p, \nabla_w \cdot \boldsymbol{v}_h)_T + \sum_{T \in \mathcal{T}_h} (\nu \Delta \boldsymbol{u} - \nabla p, \pi_h \boldsymbol{v}_h)_T + \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)\\
=& -\sum_{T \in \mathcal{T}_h} (\pi_0 p, \nabla_w \cdot \boldsymbol{v}_h)_T + \sum_{T \in \mathcal{T}_h} (\pi_0 p, \nabla_w \cdot \boldsymbol{v}_h)_T\\
&+ \sum_{T \in \mathcal{T}_h} (\nu \Delta \boldsymbol{u}, \pi_h \boldsymbol{v}_h)_T + \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)\\
=& \sum_{T \in \mathcal{T}_h} (\nu \Delta \boldsymbol{u}, \pi_h \boldsymbol{v}_h)_T + \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h)\\
=& \sum_{T \in \mathcal{T}_h} (\nu \Delta \boldsymbol{u}, \pi_h \boldsymbol{v}_h)_T + \nu a_h(\boldsymbol{u}, \boldsymbol{v}_h) + \nu a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h)\\
=& \Big(\sum_{T \in \mathcal{T}_h} (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v}_h)_T + \sum_{T \in \mathcal{T}_h} (\nu \Delta \boldsymbol{u}, \pi_h \boldsymbol{v}_h)_T \Big) + \nu a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h)
\end{aligned}$$

$$\triangleq \mathbb{A}_3 + \mathbb{A}_4. \tag{4.20}$$

Similar to $\mathbb{A}_1$, the term $\mathbb{A}_3$ can be bounded by

$$\begin{aligned}\mathbb{A}_3 &= \sum_{T\in\mathcal{T}_h}(\nu\nabla \boldsymbol{u}, \nabla \boldsymbol{v}_h)_T + \sum_{T\in\mathcal{T}_h}(\nu\Delta \boldsymbol{u}, \pi_h \boldsymbol{v}_h)_T \\ &\le Ch\nu\|\boldsymbol{u}\|_2\|\boldsymbol{v}_h\|_{\mathrm{WP}}.\end{aligned} \tag{4.21}$$

On the other hand, we have

$$\begin{aligned}\mathbb{A}_4 &= \nu a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}_h) \\ &\le \nu\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{WP}}\|\boldsymbol{v}_h\|_{\mathrm{WP}}.\end{aligned} \tag{4.22}$$

Combining (4.19)–(4.22) implies that

$$\|\pi_0 p - p_h\|_0 \le \nu\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{WP}} + Ch\nu\|\boldsymbol{u}\|_2. \tag{4.23}$$

This along with (4.10) yields to

$$\|\pi_0 p - p_h\|_0 \le Ch\nu\|\boldsymbol{u}\|_2, \tag{4.24}$$

which is the desired estimate (4.11). Substituting (4.24) and (4.18) into (4.17) leads to the desired conclusion (4.12). □

Next, we shall establish the optimal $\boldsymbol{L}^2$ error estimate for the discrete velocity $\boldsymbol{u}_h$. The main techniques are followed by [21,51], therein they address the modified CR FEMs. We first recall the following result which is based on the duality argument (see Lemma 4.1 in [51]).

Lemma 4.5. *Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions of (2.1)–(2.2) and (2.8)–(2.9), respectively. Given $\boldsymbol{s} \in \boldsymbol{L}^2(\Omega)$, let $\boldsymbol{u}_s \in \boldsymbol{X}$ satisfies*

$$\nu a(\boldsymbol{u}_s, \boldsymbol{v}) = (\boldsymbol{s}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X} \tag{4.25}$$

and let $\boldsymbol{u}_{s,h} \in \boldsymbol{X}_h$ denote the solution of

$$\nu a_h(\boldsymbol{u}_{s,h}, \boldsymbol{v}) = (\boldsymbol{s}, \pi_h \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_h. \tag{4.26}$$

Then, it holds that

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \le \sup_{\boldsymbol{s}\in \boldsymbol{L}^2(\Omega), \|\boldsymbol{s}\|_0 = 1} \Big\{ &\nu\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{WP}}\|\boldsymbol{u}_s - \boldsymbol{u}_{s,h}\|_{\mathrm{WP}} \\ &+ \left|\nu a_h(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_s) - (\boldsymbol{s}, \pi_h(\boldsymbol{u} - \boldsymbol{u}_h))\right| \\ &+ \left|\nu a_h(\boldsymbol{u}, \boldsymbol{u}_s - \boldsymbol{u}_{s,h}) - (\boldsymbol{f}, \pi_h(\boldsymbol{u}_s - \boldsymbol{u}_{s,h}))\right| \\ &+ \left|(\boldsymbol{s}, (\boldsymbol{u} - \boldsymbol{u}_h) - \pi_h(\boldsymbol{u} - \boldsymbol{u}_h))\right| \\ &+ \left|(\boldsymbol{f}, \boldsymbol{u}_s - \pi_h \boldsymbol{u}_s)\right| \Big\}.\end{aligned} \tag{4.27}$$

Based on the above lemma, we now can prove the corresponding $\boldsymbol{L}^2$ error estimate.

Theorem 4.6. *Let $(\boldsymbol{u}, p)$ and $(\boldsymbol{u}_h, p_h)$ be the solutions of (2.1)–(2.2) and (2.8)–(2.9), respectively.*

(I) If $\pi_h = \pi^{\mathrm{RT}}$ in (2.8), then it holds

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \le Ch^2(\|\boldsymbol{u}\|_2 + \|\Delta \boldsymbol{u}\|_2). \tag{4.28}$$

(II) If $\pi_h = \pi^{\mathrm{BDM}}$ in (2.8), then it holds

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_0 \le Ch^2\|\boldsymbol{u}\|_2. \tag{4.29}$$

Proof. We only give the proof for (4.28), which is mainly based on [21]. The corresponding estimate (4.29) can be addressed by using similar techniques [51]. Under the condition that Ω is convex, the standard regularity results for Stokes equations imply that (see [52])

$$\nu\|\boldsymbol{u}_s\|_2 \le C\|\boldsymbol{s}\|_0. \tag{4.30}$$

We then estimate five different terms in (4.27) to obtain the desired assertion (4.28).

It follows from (4.10) and (4.30) that

$$\begin{aligned}\nu\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{WP}}\|\boldsymbol{u}_s - \boldsymbol{u}_{s,h}\|_{\mathrm{WP}} &\le \nu(Ch\|\boldsymbol{u}\|_2)(Ch\|\boldsymbol{u}_s\|_2) \\ &\le Ch^2\|\boldsymbol{u}\|_2\|\boldsymbol{s}\|_0.\end{aligned} \tag{4.31}$$

Using similar arguments as for (4.5), and applying (4.10) and (4.30), we obtain

$$\begin{aligned}\left|\nu a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{u}_s)-(\boldsymbol{s},\pi^{\mathrm{RT}}(\boldsymbol{u}-\boldsymbol{u}_h))\right| &\leq Ch\nu\|\boldsymbol{u}_s\|_2\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{WP}}\\ &\leq Ch\nu\|\boldsymbol{u}_s\|_2(Ch\|\boldsymbol{u}\|_2)\\ &\leq Ch^2\|\boldsymbol{u}\|_2\|\boldsymbol{s}\|_0. \end{aligned}\tag{4.32}$$

Analogously, we also have

$$\begin{aligned}\left|\nu a_h(\boldsymbol{u},\boldsymbol{u}_s-\boldsymbol{u}_{s,h})-(\boldsymbol{f},\pi_h(\boldsymbol{u}_s-\boldsymbol{u}_{s,h}))\right| &\leq Ch\nu\|\boldsymbol{u}\|_2\|\boldsymbol{u}_s-\boldsymbol{u}_{s,h}\|_{\mathrm{WP}}\\ &\leq Ch\nu\|\boldsymbol{u}\|_2(Ch\|\boldsymbol{u}_s\|_2)\\ &\leq Ch^2\|\boldsymbol{u}\|_2\|\boldsymbol{s}\|_0. \end{aligned}\tag{4.33}$$

Combining (2.14) and (4.10) yields

$$\begin{aligned}&\left|(\boldsymbol{s},(\boldsymbol{u}-\boldsymbol{u}_h)-\pi^{\mathrm{RT}}(\boldsymbol{u}-\boldsymbol{u}_h))\right|\\ \leq& Ch\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{WP}}\|\boldsymbol{s}\|_0\leq Ch^2\|\boldsymbol{u}\|_2\|\boldsymbol{s}\|_0. \end{aligned}\tag{4.34}$$

Since $(\nabla p,\boldsymbol{u}_s-\pi^{\mathrm{RT}}\boldsymbol{u}_s)=0$, we arrive at

$$\begin{aligned}&\left|(\boldsymbol{f},\boldsymbol{u}_s-\pi^{\mathrm{RT}}\boldsymbol{u}_s)\right|\\ =&\left|\nu(\boldsymbol{\Delta u},\boldsymbol{u}_s-\pi^{\mathrm{RT}}\boldsymbol{u}_s)\right|\\ \leq&\left|\nu(\boldsymbol{\Delta u}-\Pi_0\boldsymbol{\Delta u},\boldsymbol{u}_s-\pi^{\mathrm{RT}}\boldsymbol{u}_s)\right|+\left|\nu(\Pi_0\boldsymbol{\Delta u},\boldsymbol{u}_s-\pi^{\mathrm{RT}}\boldsymbol{u}_s)\right|, \end{aligned}\tag{4.35}$$

where Π_0 is the $\boldsymbol{L}^2$ projection into piecewise constants. Using the standard approximation error estimates for Π_0 and (2.14) to find that

$$\begin{aligned}&|\nu(\boldsymbol{\Delta u}-\Pi_0\boldsymbol{\Delta u},\boldsymbol{u}_s-\pi^{\mathrm{RT}}\boldsymbol{u}_s)|\\ \leq& C\nu h^2\|\nabla(\boldsymbol{\Delta u})\|_0\|\boldsymbol{u}_s\|_1\leq Ch^2\|\nabla(\boldsymbol{\Delta u})\|_0\|\boldsymbol{s}\|_0. \end{aligned}\tag{4.36}$$

Using similar techniques developed as those in Theorem 3.11 in [21], we have

$$|\nu(\Pi_0\boldsymbol{\Delta u},\boldsymbol{u}_s-\pi^{\mathrm{RT}}\boldsymbol{u}_s)|\leq Ch^2\|\boldsymbol{\Delta u}\|_2\|\boldsymbol{s}\|_0.\tag{4.37}$$

Substituting (4.31)–(4.37) into (4.27) gives the desired estimate (4.28). □

Remark 4.7. The key step in designing the pressure-robust scheme (2.8)–(2.9) is to modify the right-hand by $(\boldsymbol{f},\pi_h\boldsymbol{v}_h)$, and replace the term $\sum_{T\in\mathcal{T}_h}(\nabla\cdot\boldsymbol{v}_h,q_h)$ with $\sum_{T\in\mathcal{T}_h}(\nabla_w\cdot\boldsymbol{v}_h,q_h)$. In this case, the result (4.2) in Lemma 4.2 holds true. Thus, we have (4.4), which means that the error estimate for $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{WP}}$ is pressure-robust (see lines 3–4 in (4.5)). In addition, it follows from (4.2) that $\sum_{T\in\mathcal{T}_h}(\nabla p,\pi_h\boldsymbol{v}_h)_T-\sum_{T\in\mathcal{T}_h}(\pi_0 p,\nabla_w\cdot\boldsymbol{v}_h)_T=0$, which means $\|\pi_0 p-p_h\|_0$ is pressure-robust (see lines 5–6 in (4.20)).

Remark 4.8. In this work, we only consider the lowest order WOPSIP method. The extension to second order and odd higher order schemes in 2D on triangular meshes is straightforward, since the inf-sup condition (3.13) holds for the corresponding P_{k+1}^{nc}-P_k^{dis} mixed finite element. For the even-order ($k>2$) extension in 2D, we have minor restrictions on the triangular mesh as the inf-sup condition holds for $k\geq 4$ Scott-Vogelius element [53]. In 3D, the method can be extended only to the second order scheme since the P_2^{nc}-P_1^{dis} mixed finite element is stable [54]. But for $k>2$, neither the P_k^{nc}-P_{k-1}^{dis} element nor the P_k^{c}-P_{k-1}^{dis} element is stable on tetrahedral meshes [55]. We may think of not using these two types of inf-sup stability. But the over-penalty in WOPSIP prevents us from using the inf-sup stability of BDM elements, unlike the other DG methods. Some concluding remarks on higher order WOPSIP schemes for the Stokes equations can be found in Section 6 in [41].

5. Numerical experiments

In this section, we present numerical results to validate our theoretical analysis. We solve the Stokes problem (1.1)–(1.3) on the domain $\Omega=(0,1)\times(0,1)$ using two different values of viscosity, namely $\nu=1$ and $\nu=10^{-6}$. The source term $\boldsymbol{f}$ is selected in such a way that the exact solution remains the same for both values of ν:

$$\begin{aligned}\boldsymbol{u}&=\begin{pmatrix}-2^8(2x-6x^2+4x^3)y^2(1-y)^2\\ 2^8(2y-6y^2+4y^3)x^2(1-x)^2\end{pmatrix},\\ p&=2^8(2x-6x^2+4x^3)(2y-6y^2+4y^3).\end{aligned}\tag{5.1}$$

The first three levels of mesh grids are displayed in Fig. 1. The higher-level grids are obtained by nested refinement.

In Table 1, we present the computational results for both methods when $\nu=1$. Here and in the following table, Π_1 denotes the $\boldsymbol{L}^2$ projection into the piecewise linear polynomial space. For the pressure approximation, we employ the reduced integration (2.8) with $\pi_h=RT_0$. It can be observed that optimal order convergence is achieved in all cases. Consequently, for $\nu=1$, both methods

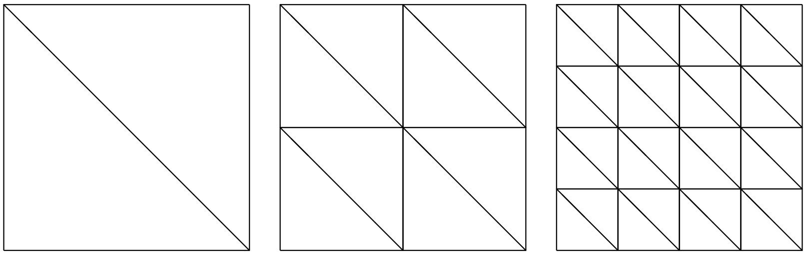

Fig. 1. The first three levels of uniform grids for computing (5.1) in Tables 1–2.

Table 1
Error profiles for solution (5.1), when $\nu = 1$.

$h/\sqrt{2}$	$\|\Pi_1 \boldsymbol{u} - \boldsymbol{u}_h\|_0$	Rate	$\|\nabla(\Pi_1 \boldsymbol{u} - \boldsymbol{u}_h)\|_0$	Rate	$\|\pi_0 p - p_h\|_0$	Rate
	By the standard WOPSIP method (2.6)–(2.7)					
1/4	0.6802E+00		0.6689E+01		0.2374E+01	
1/8	0.1842E+00	1.88	0.3729E+01	0.84	0.1259E+01	0.91
1/16	0.4819E−01	1.93	0.1940E+01	0.94	0.5877E+00	1.10
1/32	0.1223E−01	1.98	0.9816E+00	0.98	0.2731E+00	1.11
1/64	0.3065E−02	2.00	0.4924E+00	1.00	0.1320E+00	1.05
	By the pressure-robust WOPSIP method (2.8)–(2.9) with $\pi_h = RT_0$					
1/4	0.1215E+01		0.1302E+02		0.2913E+01	
1/8	0.3450E+00	1.82	0.7212E+01	0.85	0.1428E+01	1.03
1/16	0.9040E−01	1.93	0.3720E+01	0.96	0.6064E+00	1.24
1/32	0.2289E−01	1.98	0.1876E+01	0.99	0.2735E+00	1.15
1/64	0.5734E−02	2.00	0.9401E+00	1.00	0.1319E+00	1.05

Table 2
Error profiles for solution (5.1), when $\nu = 10^{-6}$.

$h/\sqrt{2}$	$\|\Pi_1 \boldsymbol{u} - \boldsymbol{u}_h\|_0$	Rate	$\|\nabla(\Pi_1 \boldsymbol{u} - \boldsymbol{u}_h)\|_0$	Rate	$\|\pi_0 p - p_h\|_0$	Rate
	By the standard WOPSIP method (2.6)–(2.7)					
1/4	0.1861E+06		0.2099E+07		0.6326E+00	
1/8	0.6962E+05	1.42	0.1409E+07	0.58	0.5031E+00	0.33
1/16	0.2099E+05	1.73	0.7736E+06	0.86	0.2230E+00	1.17
1/32	0.5638E+04	1.90	0.3986E+06	0.96	0.7124E−01	1.65
1/64	0.1440E+04	1.97	0.2010E+06	0.99	0.1967E−01	1.86
	By the pressure-robust WOPSIP method (2.8)–(2.9) with $\pi_h = RT_0$					
1/4	0.1215E+01		0.1302E+02		0.2911E−05	
1/8	0.3450E+00	1.82	0.7212E+01	0.85	0.1430E−05	1.02
1/16	0.9040E−01	1.93	0.3720E+01	0.96	0.6079E−06	1.23
1/32	0.2289E−01	1.98	0.1876E+01	0.99	0.2779E−06	1.13
1/64	0.5735E−02	2.00	0.9401E+00	1.00	0.1395E−06	0.99

yield comparable performance. However, in terms of velocity approximation, the velocity errors of pressure-robust WOPSIP method are approximately twice as large as that of standard WOPSIP method. Nevertheless, as we will demonstrate later, the pressure-robust WOPSIP method exhibits significant advantages when dealing with small values of ν.

In Table 2, we present the computational results for both methods when $\nu = 10^{-6}$. Optimal order convergence is observed in all cases. However, the velocity errors of the standard WOPSIP method are approximately about $10^6/5$ times larger than those of the pressure-robust WOPSIP method. Also, for the pressure-robust WOPSIP method, the velocity errors are almost the same for both $\nu = 1$ and $\nu = 10^{-6}$. These observations align with the theoretical proof that the velocity approximation in the pressure-robust WOPSIP method is independent of ν. Additionally, the pressure approximation in the pressure-robust WOPSIP method performs even better, exhibiting an improvement of approximately 10^{-6} compared to the standard WOPSIP method in terms of L^2 approximation. This matches well with the theoretical analysis shown in (4.11).

6. Conclusions

In this study, we propose a pressure-robust weakly over-penalized symmetric interior penalty (WOPSIP) discontinuous Galerkin (DG) method for solving the Stokes equations. We assume that the exact solutions possess sufficient smoothness in order to derive an error estimate. However, it is also an intriguing area of research to extend the error analysis to scenarios with minimal regularity assumptions [56]. We plan to thoroughly investigate this topic in future work.

Data availability

Data will be made available on request.

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Further reading

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