

# Fast Hausdorff Moment Transforms for Meta Distributions in Wireless Networks

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**Abstract**—In the analysis of wireless networks, the standard signal-to-interference (SIR) distribution does not capture the performance at the individual link level. The meta distribution (MD) of the SIR resolves this problem by separating different sources of randomness, such as fading and point process(es). While it allows for a much sharper performance characterization, it can in most cases only be calculated based on the moments of the underlying conditional distribution, i.e., by solving a Hausdorff moment problem. Several methods to reconstruct MDs from the moments have been proposed but a rigorous analysis, comparison of their performance, and practical implementations are missing. In addition, a standard is needed for a consistent and objective comparison. This paper addresses the above-mentioned important shortcomings, introduces a tweaking mapping for adjusting approximations, presents terminology to categorize the quality of approximations, proposes the use of the Fourier-Legendre method, which has not previously been applied to MDs, and provides the achievable lower and upper bounds on the MD given the first  $n$  moments. Further, to facilitate the use of MDs, we give comprehensive guidance on the selection of the best method to determine MDs, and we offer ready-to-use implementations of the proposed algorithms. This study fills an important gap in the literature by rigorously analyzing the MDs, comparing the performance of different methods, and offering user-friendly implementations for recovering MDs from moments.

**Index Terms**—Meta distribution, Hausdorff moment problem, stochastic geometry, signal fraction

## I. INTRODUCTION

### A. Motivation and Related Work

Due to the rapid development of smart devices, many advanced applications, e.g., augmented reality and interactive online gaming, demand strict reliability in wireless networks, and wireless users expect a consistent performance experience. However, while cellular and WiFi equipment manufacturers' and service providers' advertisements mostly focus on the peak performance, neither the peak performance nor the average performance is what a user actually experiences. Instead, it is necessary to consider the performance that 95%/50%/5% of the users can achieve. In the practical evaluation of wireless network performance, as in the evaluation for the recent 5G standards [1] and WiFi standards [2], the throughput that 95% of the users should achieve has been specified. However, analytical approaches usually focus on the distribution of the signal-to-interference ratio (SIR), which is the performance-limiting metric [3] and from which many other metrics can be

derived, such the data rate and the spectral efficiency [4], [5]. The SIR distribution lumps all the randomness (such as point process(es), fading, shadowing, and channel access schemes) together, which obfuscates the impact of the individual random elements. For example, in an ergodic model, it reveals the fraction of links that achieves a certain SIR at any time instant. But the composition of the successful links is time-varying, and it is possible that a significant fraction of links consistently suffer from bad connections even when the SIR distribution has a relatively high value at the target SIR. To address this issue, different from the traditional use of stochastic geometry to derive averages, [6], [7] introduce the concept of *deep stochastic geometry* that is based on meta distributions (MDs). MDs are distributions of conditional complementary cumulative distribution functions (ccdfs) that by design disentangle different sources of randomness, which allows an analysis of the effects of different sources of randomness and capture the disparity of the performance of individual links [4], [8]–[13]. The ccdfs are the complementary cumulative distribution functions (ccdfs) of the conditional random variables. Consequently, they are random variables themselves.

For most network models, the calculation of MDs is challenging. In some simplified cases, where signal power or SIR MDs with only the nearest interferer are considered, closed-form expressions of MDs can be obtained [6], [7]. For more complete models, e.g., the SIR MDs with all interferers in the Poisson cellular network model [8], it is quite likely impossible to directly obtain analytic (let alone closed-form) expressions. However, it is often possible to calculate moments of the underlying ccdfs [8], [9], [14]. This prompts us to study and develop moment transform methods to reconstruct MDs (distributions of ccdfs) from moments (of ccdfs). While MDs are two-argument functions, they can be viewed as a family of univariate distributions and thus in the literature, univariate approximations have been applied [4], [8], [13]. Depending on the type of moments that are used, there are two categories of methods. The first category consists of the Gil-Pelaez (GP) method which utilizes only imaginary moments, while the methods in the second category utilize only positive integer moments. The GP method is based on the Gil-Pelaez theorem [15] and entails an integral of infinite range which is too complicated to be analytically calculated.<sup>1</sup> Its numerical approximation requires a careful selection of integration upper limits and step sizes, and the integrand may contain a singularity. Furthermore, the accurate calculation of imaginary

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<sup>1</sup>In wireless applications, we are not aware of any case where the GP integral has an analytical solution. However, there are some special abstract cases where analytic solutions exist [16].

moments can be time-consuming. Thus, here we mainly focus on methods in the second category, where integer moments are used to recover the cumulative distribution function (cdf) or the ccdf.

In probability theory, such problems that ask for the existence of a distribution that matches a sequence of integer moments are known as *moment problems*. Due to the bounded support of the MDs, the one relevant to our work is the *Hausdorff moment problem* (HMP) where the support of the distribution is bounded. Hausdorff has shown that such a distribution exists if and only if the infinite sequence is completely monotonic (c.m.) and that the distribution is unique if it exists [17]. Although the solution to the HMP is unique (if exists), in practice, we are concerned with the truncated version where only a finite sequence of integer moments is available to recover/approximate the distribution. Instead of merely asking for the existence of solutions, the truncated HMP asks for a solution or an approximation. We call a method that finds or approximates a solution of the truncated HMP from integer moments a *Hausdorff moment transform* (HMT). Naturally, it also approximates the unique solution to the HMP.

A popular method in the second category is to approximate the MDs by a family of beta distributions [8], [10], [14], which only requires the first and second moments. It is convenient and efficient when computing moments is time-consuming [18].<sup>2</sup> However, it fails to capture the properties of the distribution determined by the higher-order moments, such as the skewness and kurtosis. Thus, the beta approximation is an approximation without any accuracy guarantee. To the best of our knowledge, the binomial mixture (BM) [19] and Fourier-Jacobi (FJ) [20] methods are the only two HMTs (other than the beta approximations) that have been applied to MDs [4], [13]. There are also some other HMTs in the literature that have not been applied to MDs yet, such as the Fourier-Legendre (FL) [21], [22], Fourier-Chebyshev (FC) [23], and Chebyshev-Markov (CM) [21] [23] methods. In [23], the advantages and disadvantages of the BM, FJ, FL, FC, and CM methods have been discussed. One critical shortcoming of the FJ method is that its coefficients vary significantly for different moment sequences and its convergence properties vary strongly as well. No method has been found to give a conclusive answer to whether the FJ approximations converge or not for general moment sequences [23]. Based on the conclusions in [23], here we focus on the BM and FL methods. The former computes very fast while the latter has good convergence properties.

## B. Contributions

- Different from the SIR that is commonly considered, here we consider the signal fraction (SF), which is the ratio of the signal power to the total received power. The SF is constrained to  $[0, 1]$  and has other advantages over the SIR for both analysis and visualization [24].

<sup>2</sup>Though the calculation of higher-order moments can be time-consuming, we are not aware of any cases where there are fundamental obstacles in calculating higher-order moments if the second moment can be calculated.

Since SFs are bounded in support and range, natural metrics, such as the distance between the MDs and the approximations, are also bounded. Thus, we can infer whether an approximation is accurate or not directly from the distance. Moreover, SF MDs can be plotted in the unit cube, and we can visually estimate their distance from the plot. In contrast, the SIR MDs have unbounded support, and there is no way to visually estimate how large the distance is between two of them.

- Our work provides a *rigorous* foundation to the reconstruction of MDs via integer moments and makes the analysis of MDs *practical*.

To make the reconstruction of MDs rigorous:

- We introduce terminology to categorize the quality of approximations by the total distance. The categories agree with the visual impression.
- We provide the infima and suprema of the MDs given a finite sequence of integer moments and an algorithm to determine the number of moments that are needed to obtain the smallest interval in which the distribution is guaranteed to lie at an arbitrary point of interest. The infima and suprema are tighter than the bounds in [8]. Their average naturally provides a good approximation.
- We provide the convergence properties of the BM and FL methods and analyze their computational complexity.

To make the analysis of MD reconstructions practical:

- In addition to approximating MDs as a family of univariate distributions, we also propose a method to approximate MDs in their entirety, i.e., as functions  $[0, 1]^2 \rightarrow [0, 1]$ . In particular, we approximate MDs of network models by abstract MDs. This method is simple but there is no accuracy guarantee. Its key advantage over the commonly used beta approximation is that it directly approximates the complete two-argument function.
- We compare the performance of the methods in terms of both the MD and its corresponding percentile functions. The percentile functions are a key tool in the practical analysis of the network performance as they quantify the user experienced performance. Besides, we focus on relevant SIR/SF thresholds and we numerically evaluate the computational complexity of the methods in wireless applications.
- We provide comprehensive guidelines for which HMT should be used depending on the objective, such as accuracy or computational complexity.
- We provide MATLAB implementations for the applications of the methods and the evaluations.

## C. Layout

The remaining sections of this paper are organized as follows. In Section II, we introduce the MDs considered, discuss adjustments to approximations, evaluate their accuracy, and highlight the inadequacy of the commonly used beta approximation. Section III introduces the CM inequalities,

which offer rigorous bounds for the MDs. In Section IV, we describe efficient approximations applicable to the MDs, specifically three one-dimensional methods and one two-dimensional method. Section V compares the performance of the beta approximation with that of the three one-dimensional methods. Moreover, we provide comprehensive guidelines for bounding or approximating MDs within this section. Section VI presents an example illustrating how to obtain rigorous bounds for MDs, explains the principle of choosing the desired number of moments, and demonstrates the application of the aforementioned methods. Finally, Section VII concludes the paper.

*Notation.*  $[n] \triangleq \{1, 2, \dots, n\}$  and  $[n]_0 \triangleq \{0\} \cup [n]$ .  $\mathbb{N}$  is the set of positive integers.  $\mathbb{N}_0$  is the set of non-negative integers.  $\mathbb{R}$  is the set of real numbers.  $\mathbb{R}_0^+$  is the set of non-negative real numbers.  $\mathbb{C}$  is the set of complex numbers.

## II. PRELIMINARIES

### A. MD of the Signal Fraction

We call the random variables, such as the SIR, that affect the user experience *performance random variables* (PRVs). This term is used to contrast them against performance *metrics*, which are deterministic quantities. To analyze the performance of wireless networks, it is natural to consider the distribution of the PRVs. However, due to the nature of wireless networks, the PRVs strongly depend on the user location and are also time-varying. To characterize their statistics and, in turn, the user experience, fully, we consider MDs that are conditioned on the point process of the transceiver locations, i.e., the distributions of the cccdfs

$$P_t \triangleq \mathbb{P}(X > t \mid \Phi), \quad t \in \mathbb{R}^+, \quad (1)$$

where  $X$  is a PRV, such as the signal power, SIR, SF, instantaneous rate, or energy, and  $\Phi$  is the point process. Since the probability measure here is that of the temporal randomness while the spatial randomness is held fixed, the MDs decompose the randomness of the network model into the spatial and temporal parts [6].

The MD is the ccdf of  $P_t$ , i.e.,

$$\bar{F}(x, t) \triangleq \mathbb{P}(P_t > x), \quad x \in [0, 1], t \in \mathbb{R}^+, \quad (2)$$

and the  $z$ -th moment of the cccdf  $P_t$  is

$$m_z(t) \triangleq \int_0^1 z x^{z-1} \bar{F}(x, t) dx, \quad z \in \mathbb{C}, \quad (3)$$

which we call the  $(z, t)$ -th moment of the MD. The characteristic function of the cccdf  $P_t$  is

$$\varphi_t(s) \triangleq 1 + js \int_0^1 \bar{F}(x, t) e^{jxs} dx, \quad s \in \mathbb{R}. \quad (4)$$

We also define the percentile functions for a partial monotone MD, i.e.,  $\bar{F}(\cdot, t)$  is monotone for any  $t \in \mathbb{R}^+$  and  $\bar{F}(x, \cdot)$  is monotone for any  $x \in [0, 1]$ , as follows:

**Definition 1** (Percentile functions). *The percentile function  $Q_1(\bar{F})$  of the PRV at a target cccdf value  $x_0$  is defined by*

$$Q_1(\bar{F})(x_0, y) = \min_{\bar{F}(x_0, t)=y} t, \quad y \in [0, 1]. \quad (5)$$

*The percentile function  $Q_2(\bar{F})$  of the cccdf random variable at a target metric  $t_0$  of the PRV is defined by*

$$Q_2(\bar{F})(y, t_0) = \min_{\bar{F}(x, t_0)=y} x, \quad y \in [0, 1]. \quad (6)$$

The percentile function  $Q_1(\bar{F})(x_0, y) = t$  can be interpreted as the threshold  $t$  of the PRV that a fraction  $y$  of links can achieve with cccdf value  $x_0$ . The percentile function  $Q_2(\bar{F})(y, t_0) = x$  can be interpreted as the cccdf value  $x$  that a fraction  $y$  of links can achieve with a threshold  $t_0$  of the PRV. For any  $t \in [0, 1]$ , the percentile function  $Q_2(\bar{F})(\cdot, t)$  is also the quantile function of the distribution of  $P_t$ . In Section III-B will discuss how to bound the quantile function given the moments of  $P_t$ .

In this paper, we focus on the MDs of the SF [24], which is defined as the ratio of the signal power to the total received power, i.e.,  $\text{SF} \triangleq S/(S+I)$ , where  $S$  is the signal power and  $I$  is the interference power. The cccdf of the SF is

$$P_t = \mathbb{P}(\text{SF} > t \mid \Phi), \quad t \in [0, 1], \quad (7)$$

and the SF MD is given in (2). Note that for each  $t \in [0, 1]$ ,  $\bar{F}(\cdot, t)$  is the ccdf of  $P_t$ , thus we can view  $(\bar{F}(\cdot, t))_{t \in [0, 1]}$  as a family of univariate cccdfs. Based on the definition, there are two basic properties of an SF MD:

**Remark 1.** *Properties of SF MDs*

- *Partial monotonicity:*  $\bar{F}(\cdot, t)$  is monotonically decreasing for all  $t \in [0, 1]$  and  $\bar{F}(x, \cdot)$  is monotonically decreasing for all  $x \in [0, 1]$ , i.e.,  $\bar{F}(x_1, t) \leq \bar{F}(x_2, t)$  for all  $x_1 \geq x_2$  and  $\bar{F}(x, t_1) \leq \bar{F}(x, t_2)$  for all  $t_1 \geq t_2$ .
- *Bounded domain and bounded range:*  $\bar{F} : [0, 1]^2 \mapsto [0, 1]$ .

In contrast to the SIR, defined as  $\text{SIR} \triangleq S/I \in [0, \infty)$ ,  $\text{SF} = \text{SIR}/(\text{SIR}+1)$  is supported on  $[0, 1]$ , thus all its positive moments always exist and no truncation is needed when visualizing its distribution. Moreover, the bounded support and range of SF MDs make it possible to compare the distance between different approximations of SF MDs. The bounded distance is critical to characterize how good the approximation is.

For a wireless network modeled by a stationary and ergodic point process, the SF MD  $\bar{F}(\cdot, \cdot)$  yields the distribution of the link success probabilities at a target SF, or, equivalently, the distribution of the SF threshold at a target reliability [5, Thm. 1]. So we can use the SF MD to directly derive the spectral efficiency MD and its percentile performance at a target value of the cccdf. By the Shannon–Hartley theorem and [5, Thm. 1], this is obtained from the inverse function with respect to the second argument of the spectral efficiency MD,  $\bar{F}_{\text{rate}}(x, r) = \bar{F}(x, 1 - 2^{-r})$ .

### B. Adjustments of Approximations and Distance Metrics

In this part, we first introduce our method to adjust approximations of MDs and then we define two distance metrics to measure how close two functions are to each other.

As mentioned in Section II-A, the SF MDs can be viewed as a family of univariate cccdfs, and many approximations

operate on one of the arguments. In most cases, an SF MD needs to be decomposed into many cccdfs with respect to the SF threshold  $t$ . Moreover, some of the threshold values are more important to analyze in practice. For example, in the 5G ultra-reliable low-latency communication (URLLC) scenario, certain signal-to-interference-plus-noise ratios (SINRs) are required to achieve a very small target block error rate.

For practical purposes, it is necessary to restrict the evaluation of MDs to a discrete and finite subset of its domain. Here, we consider the uniform set  $\mathcal{U}_{m,n} \triangleq \{\frac{i}{m}, i \in [m]_0\} \times \{\frac{j}{n}, j \in [n]_0\}$ . Another issue is that the approximations may not preserve properties of the SF MDs, such as the partial monotonicity and ranges restricted to  $[0, 1]$ . Definition 2 introduces a tweaking mapping to adjust the values of the approximations so that the adjusted approximations satisfy the properties of SF MDs.

**Definition 2** (Tweaking mapping). *Let  $u_{i,j} = (\frac{i}{m}, \frac{j}{n}), i \in [m]_0, j \in [n]_0$ , denote the elements in  $\mathcal{U}_{m,n}$ . We define  $T$  that maps  $F : \mathcal{U}_{m,n} \mapsto \mathbb{R}$  to  $\hat{F} : \mathcal{U}_{m,n} \mapsto [0, 1]$  such that*

- 1)  $T(F)(u_{i,j}) = 1, \forall u_{i,j} \in \{F(u_{i,j}) > 1\}$ , and  $T(F)(u_{i,j}) = 0, \forall u_{i,j} \in \{F(u_{i,j}) < 0\}$ .
- 2)  $T(F)(u_{0,j}) = 0, \forall j \in [n-1]_0, T(F)(u_{i,0}) = 0, \forall i \in [m-1]_0, T(F)(u_{i,n}) = 1, \forall i \in [m]_0$ , and  $T(F)(u_{m,j}) = 1, \forall j \in [n]_0$ .
- 3)  $T(F)(u_{0,0}) = F(u_{0,0})$  and  $T(F)(u_{i,j}) = \max(F(u_{i-1,j}), F(u_{i,j-1}), F(u_{i,j})), \forall i \in [m]$  and  $j \in [n]$ .

In order to avoid assigning values multiple times, the last two steps are done in a zigzag and diagonal order of all the elements in  $\mathcal{U}_{m,n}$ , i.e., in the order of  $u_{0,0}, u_{1,0}, u_{0,1}, u_{0,2}, \dots, u_{m,n}$ .

The tweaking mapping in Definition 2 eliminates the out-of-range points and restores the partial monotonicity. Let  $F : \mathcal{U}_{m,n} \mapsto \mathbb{R}$  be an approximation obtained for an SF MD. After tweaking, we obtain monotone and bounded approximations over a discrete set, denoted as  $\hat{F} = T(F)$ . Next, we interpolate the discrete points to obtain continuous functions. We first apply linear interpolation to  $\hat{F}(\cdot, t)$  for each  $t$  in the discrete subset and then we apply linear interpolation on  $t$ . Linear interpolation preserves monotonicity. For convenience, we refer to the function after interpolation as  $\hat{F}$  as well, and we call the original function  $F$  the *raw* function and the tweaked and interpolated  $\hat{F}$  the *polished* function. The polished function is partially monotone and has a bounded domain and range.

Next, we introduce two distance metrics to measure how close two functions are to each other. Let  $F_1, F_2 : U \mapsto Y$  be two bounded functions. For SF MDs,  $U \triangleq [0, 1]^2$  and  $Y \triangleq [0, 1]$ . This domain reflects the fact that the MD is inherently a two-argument function.

**Definition 3** (Maximum and total distances). *The maximum and total distances between functions  $F_1$  and  $F_2$  are defined*

as the  $\mathcal{L}_\infty$ -norm and  $\mathcal{L}_1$ -norm of  $F_1$  and  $F_2$  over  $U$ , i.e.,

$$d_M(F_1, F_2) \triangleq \|F_1 - F_2\|_{\mathcal{L}_\infty(U)} \quad (8)$$

$$= \max_{x \in U} |F_1(x) - F_2(x)|, \quad (9)$$

$$d(F_1, F_2) \triangleq \|F_1 - F_2\|_{\mathcal{L}_1(U)} \quad (10)$$

$$= \int_U |F_1(x) - F_2(x)| dx. \quad (11)$$

We mainly focus on the total distance and use the maximum distance as a secondary metric. Since  $U = [0, 1]^2$  and  $Y = [0, 1]$ , both  $d_M(F_1, F_2)$  and  $d(F_1, F_2) \in [0, 1]$ . Thanks to the bounded distance, we can directly infer the quality of the approximation simply by its value. Specifically, we declare  $F_1$  and  $F_2$  a *perfect match*, an *excellent match*, a *good match*, an *acceptable match*, a *mediocre match*, and a *bad match* if the total distance  $d(F_1, F_2)$  is in  $[0, 0.002)$ ,  $[0.002, 0.005)$ ,  $[0.005, 0.01)$ ,  $[0.01, 0.02)$ ,  $[0.02, 0.05)$ , and  $[0.05, 1]$  respectively. The correspondence is shown in Table I. This classification agrees well with the visual impression of the gap between two curves. More details about the classification and a dynamic visualization can be found in [25].

Figure 1 and Figure 2 give examples of  $|F_1(x) - F_2(x)|$  and  $|F_1(w, v) - F_2(w, v)|$  for total distances of the order of  $10^{-1}$  and  $10^{-2}$ , respectively.

We also calculate the maximum and total distance of the percentile functions, called the maximum and total horizontal distance. Since the total distances are the same, we only consider the maximum horizontal distance. The corresponding maximum horizontal distances of the percentile functions are defined as

$$d_{H_1}(F_1, F_2) \triangleq d_M(Q_1(F_1), Q_1(F_2)), \quad (12)$$

$$d_{H_2}(F_1, F_2) \triangleq d_M(Q_2(F_1), Q_2(F_2)), \quad (13)$$

respectively. We refer to the first one as *maximum horizontal distance 1* and the second one as *maximum horizontal distance 2*.

### C. Inadequacy of the Beta Approximation

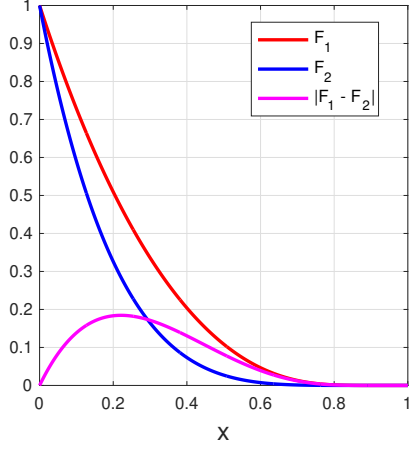
As a simple approach that only requires the first two moments, the beta approximation has been widely used to recover MDs from moments [8], [9], [14]. However, due to the nature of the beta approximation, it is unsuitable for certain distributions, such as multi-modal distributions and non-smooth distributions.

As an example, in Figure 3, we compare the beta approximations and the actual cdf of the cccdf  $P_t \triangleq \mathbb{P}(\text{SF} > t \mid \Phi)$  in a downlink Poisson cellular network where base stations (BSs) form a stationary PPP  $\Phi$  with nearest-BS association, power-law path loss  $\alpha$  and Nakagami- $m$  fading, and only the nearest interfering BS is considered [7].

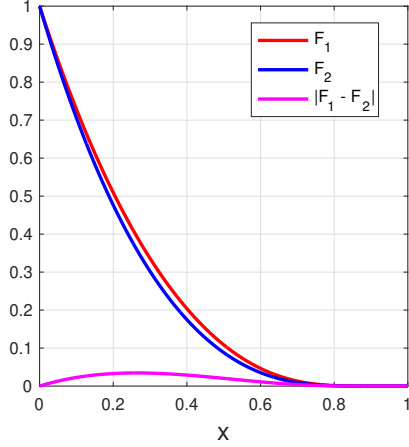
We can observe that the beta approximation cannot capture the non-smooth points and instead smooths out the entire distribution. Moreover, the total distance is 0.0502 and 0.0208 for  $m = 10$  and  $m = 1$ , which indicates that the beta approximation is a bad/mediocre match in these cases. Thus, it is necessary to consider other methods, including the rigorous bounding techniques discussed in the next section.

TABLE I  
TOTAL DISTANCE AND TERMINOLOGY FOR THE QUALITY OF THE APPROXIMATION.

Terminology	perfect	excellent	good	acceptable	mediocre	bad
$d(F_1, F_2) \in$	$[0, 0.002)$	$[0.002, 0.005)$	$[0.005, 0.01)$	$[0.01, 0.02)$	$[0.02, 0.05)$	$[0.05, 1]$



(a)  $d(F_1, F_2) = 0.077$  and  $d_M(F_1, F_2) = 0.185$ .  $F_1$  and  $F_2$  are a bad match.



(b)  $d(F_1, F_2) = 0.016$  and  $d_M(F_1, F_2) = 0.035$ .  $F_1$  and  $F_2$  are an acceptable match.

Fig. 1.  $F_1, F_2$  and  $|F_1 - F_2|$ . The type of match is obtained from Table I.

### III. RIGOROUS BOUNDING USING THE CHEBYSHEV-MARKOV INEQUALITIES

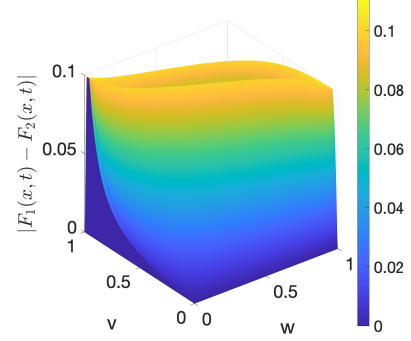
#### A. The Hausdorff Moment Problem and the CM Inequalities

The Hausdorff moment problem asks for the existence of a distribution  $F$  supported over  $[0, 1]$  such that its moments match a given infinite sequence  $(m_k)_{k=1}^{\infty}$ , i.e.,

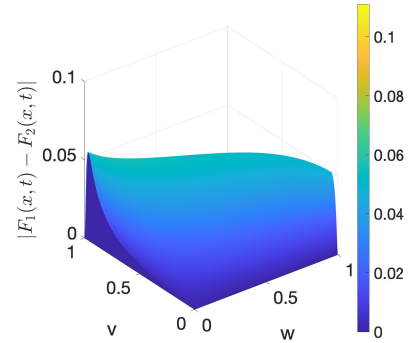
$$\int_0^1 x^k dF(x) = m_k, \quad \forall k \in \mathbb{N}. \quad (14)$$

The solution exists if and only if the infinite sequence is c.m., and the distribution is unique if it exists [17].

When trying to recover distributions from moments, the existence of the unique solution to the HMP can be determined directly by checking whether the infinite sequence is c.m. In most cases of the reconstruction of MDs, the sequences are c.m. since they are based on analytic expressions for the moments. However, our goal is to find a distribution whose



(a)  $d(F_1, F_2) = 0.058$  and  $d_M(F_1, F_2) = 0.1111$ .  $F_1$  and  $F_2$  are a bad match.



(b)  $d(F_1, F_2) = 0.029$  and  $d_M(F_1, F_2) = 0.0565$ .  $F_1$  and  $F_2$  are a mediocre match.

Fig. 2. 3D plot of  $|F_1 - F_2|$ . The type of match is obtained from Table I.

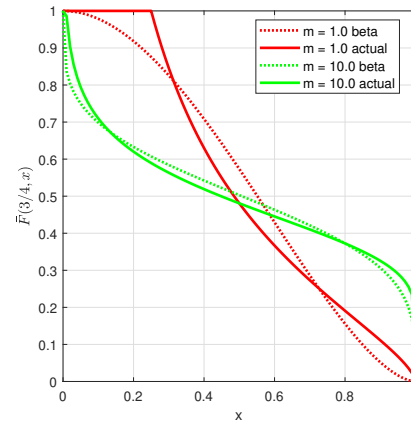


Fig. 3. The reconstructed cdfs of the ccdf  $P_t$  at threshold  $t = 3/4$  of the downlink Poisson cellular network with Nakagami- $m$  fading and power-law path loss  $\alpha = 3$  [7, Eq. (8)]. The total distance between the beta approximation and the original cdf is 0.0502 and 0.0208, which are a bad match and a mediocre match per Table I, respectively.

moments (closely) match the finite sequence, which leads to the truncated HMP [21]: given a finite sequence  $(m_k)_{k=1}^n$ ,  $n \in \mathbb{N}$ , find or approximate an  $F$  that solves

$$\int_0^1 x^k dF(x) = m_k, \quad \forall k \in [n]_0, \quad (15)$$

where  $F$  is right-continuous and increasing with  $F(0^-) = 0$  and  $F(1) = 1$ , i.e.,  $F$  is the cdf of a random variable supported on  $[0, 1]$ . Let  $\mathcal{F}_n$  denote the set of all possible  $F$  that solve (15). Assuming  $\mathcal{F}_n \neq \emptyset$ , it is natural to consider the infimum and supremum of  $F$  at the point of interest, i.e.,  $\inf_{F \in \mathcal{F}_n} F(x_0)$  and  $\sup_{F \in \mathcal{F}_n} F(x_0)$  for any  $x_0 \in [0, 1]$ .

Markov [26] provided a method to obtain the infimum and supremum for any  $x_0 \in [0, 1]$ . The most important step of the method is the construction of a discrete distribution in  $\mathcal{F}_n$  where the difference between the infimum and supremum, or, equivalently, the maximum mass [27, Cor. 2.5.4], is concentrated at  $x_0$  such that its moments are the prescribed ones. Let  $p_0$  denote this maximum mass that can be concentrated at  $x_0$  and  $F^*$  denote the discrete distribution where  $p_0$  is concentrated at  $x_0$ , i.e.,  $F^*(x_0) - F^*(x_0^-) = p_0$ . Then,  $\inf_{F \in \mathcal{F}_n} F(x_0) = F^*(x_0^-)$  and  $\sup_{F \in \mathcal{F}_n} F(x_0) = F^*(x_0)$ . In the following, we provide a step-by-step high-level description of how to construct  $F^*$ . For a discrete random variable, the jump locations correspond to the possible values of the discrete random variable, and the jump heights correspond to the probability mass assigned to each value. These two are sufficient to determine such a discrete distribution. Suppose that  $F^*$  is constructed by jumps at  $x_i$  with heights  $p_i$ ,  $1 \leq i \leq v$ ,  $v \in \mathbb{N}$ , and the first  $n$  moments of  $F^*$  are the prescribed ones.

- Step 1: Determine the jump locations. Finding the jump locations  $(x_i)_{i=1}^v$  requires finding the roots of an orthogonal polynomial [20]. We refer to [21], [28], [29] for the details. Note that  $x_0 \in \{x_i, i \in [v]\}$ .
- Step 2: Calculate the jump heights. After obtaining the jump locations  $(x_i)_{i=1}^v$ , the jump heights  $(p_j)_{j=1}^v$  can be obtained by solving

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_v \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n & x_2^n & \dots & x_v^n \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_v \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ \vdots \\ m_n \end{pmatrix}. \quad (16)$$

- Step 3: Equipped with jump locations  $(x_i)_{i=1}^v$  and jump heights  $(p_j)_{j=1}^v$ , the infimum and supremum at  $x_0$  are given by

$$\inf_{F \in \mathcal{F}_n} F(x_0) = F^*(x_0^-) = \sum_{j: x_j < x_0} p_j, \quad (17)$$

$$\sup_{F \in \mathcal{F}_n} F(x_0) = F^*(x_0) = \sum_{j: x_j \leq x_0} p_j. \quad (18)$$

The inequalities established by the infima and suprema obtained in the above procedure are the CM inequalities. Our MATLAB implementation of the CM inequalities **CM-Bounds.m** [30] takes the first  $n$  moments and the point of interest  $x_0 \in [0, 1]$  as input and produces the infimum and supremum of  $F(x_0)$  as output.

In the following, we provide two examples of the CM inequalities at  $n = 1$  and  $n = 2$ . Zelen gives the explicit expressions of the infima and suprema at  $n = 4$  [31].

**Example 1** ( $n = 1$ ). For  $n = 1$ , the infima and suprema are

$$\inf_{F \in \mathcal{F}_n} F(x_0) = \begin{cases} 0, & 0 \leq x_0 \leq m_1, \\ 1 - \frac{m_1}{x_0}, & m_1 < x_0 \leq 1, \end{cases} \quad (19)$$

$$\sup_{F \in \mathcal{F}_n} F(x_0) = \begin{cases} \frac{1-m_1}{1-x_0}, & 0 \leq x_0 \leq m_1, \\ 1, & m_1 < x_0 \leq 1. \end{cases} \quad (20)$$

The infima are equivalent to Markov's inequality.

**Example 2** ( $n = 2$ ). For  $n = 2$ , consider the case where all the Hankel determinants are positive, or, equivalently,  $m_0 m_2 > m_1^2$ . Thus  $\mu_1 \triangleq \frac{m_1 - m_2}{m_0 - m_1} < \frac{m_2}{m_1} \triangleq \mu_2$ . The infima and suprema are

$$\inf_{F \in \mathcal{F}_n} F(x_0) = \begin{cases} 0, & 0 \leq x_0 \leq \mu_1, \\ 1 - \frac{m_1 - m_2}{x_0 - x_0^2} - \frac{m_2 - m_1 x_0}{1 - x_0}, & \mu_1 < x_0 \leq \mu_2, \\ \frac{(1-m_1)(m_1 - x_0 m_0)}{(m_1 - m_2) - (m_0 - m_1)x_0}, & \mu_2 < x_0 \leq 1, \end{cases} \quad (21)$$

$$\sup_{F \in \mathcal{F}_n} F(x_0) = \begin{cases} \frac{m_2 - m_1^2}{(m_2 - m_1) + (m_0 - m_1)x_0}, & 0 \leq x_0 \leq \mu_1, \\ 1 - \frac{m_2 - m_1 x_0}{1 - x_0}, & \mu_1 < x_0 \leq \mu_2, \\ 1, & \mu_2 < x_0 \leq 1. \end{cases} \quad (22)$$

These infima and suprema are naturally tighter than the bounds considered in [8].

Figure 4 shows two examples of the infima and suprema for different  $n$ . The convergence of the infima and suprema as  $n$  increases is apparent.

### B. Bounding the Quantile Function at a Given Point

As discussed in the previous subsection, the CM inequalities are a powerful tool to bound a distribution with support  $[0, 1]$  based on its moments. They give the infimum and supremum of the distribution at each point of interest, and, in turn, the smallest interval in which the distribution is guaranteed to lie at the point of interest. An important application in wireless networks is to find the percentile performance. In an SF MD, it characterizes the reliability that a certain fraction of users achieve with a target SF threshold. Algorithm 1 returns the possible range of the quantile function at a certain point.

---

**Algorithm 1** Algorithm for bounding the quantile function at a given point of the cdf  $F$

---

**Input:** The moment sequence  $(m_k)_{k=1}^n$ , the point of interest  $y \in (0, 1)$ , and tolerance  $0 < \epsilon \ll 1$

**Output:** The possible range of  $F^{-1}(y)$  denoted as  $[z_0, z_1]$

**if**  $F(0)^+ \leq y$  **then**

Set  $x_0 = 0$  and  $x_1 = 1$ .

Obtain  $F(x_0)^+$  and  $F(x_1)^+$  from the CM inequalities described in Section III-A.

**while**  $|F(x_0)^+ - y| > \epsilon$  **do**

$x = (x_0 + x_1)/2$  and obtain  $F(x)^+$  from the CM inequalities.

**if**  $|F(x)^+ - y| \leq \epsilon$  **then**

Set  $x_0 = x$ .

**end if**

**if**  $F(x)^+ - y < -\epsilon$  **then**

Set  $x_0 = x$ .

**end if**

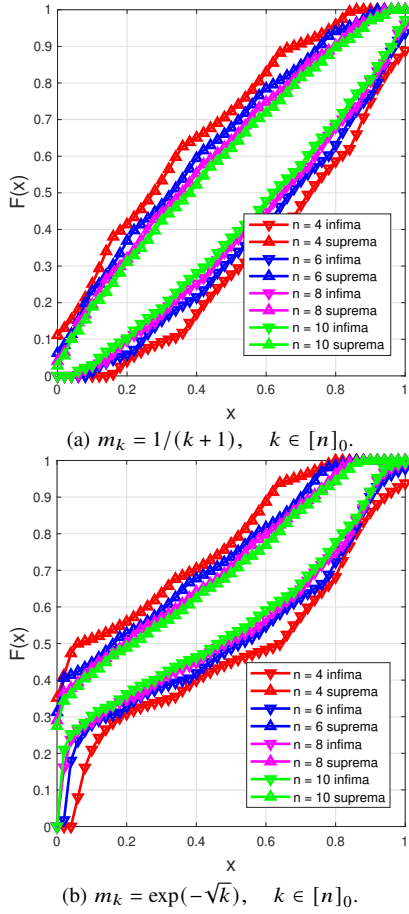


Fig. 4. The infima and suprema from the CM inequalities for  $n = 4, 6, 8, 10$ .  $x$  is discretized to  $\mathcal{U}_{50} = \{i/50, i \in [50]_0\}$ .

```

if  $F(x)^+ - y > \epsilon$  then
    Set  $x_1 = x$ .
end if
end while
Set  $z_0 = x_0$ ,  $x_0 = 0$ , and  $x_1 = 1$ .
else
    Set  $z_0 = 0$ ,  $x_0 = 0$ , and  $x_1 = 1$ .
end if
if  $y \leq F(1)^-$  then
    Obtain  $F(x_0)^-$  and  $F(x_1)^-$  from the CM inequalities de-
    scribed in Section III-A.
    while  $|F(x_0)^- - y| > \epsilon$  do
         $x = (x_0 + x_1)/2$  and obtain  $F(x)^-$  from the CM inequalities.
        if  $|F(x)^- - y| \leq \epsilon$  then
            Set  $x_0 = x$ .
        end if
        if  $F(x)^- - y < -\epsilon$  then
            Set  $x_0 = x$ .
        end if
        if  $F(x)^- - y > \epsilon$  then
            Set  $x_1 = x$ .
        end if
    end while
    Set  $z_1 = x_0$ .
else
    Set  $z_1 = 1$ .
end if

```

If a certain accuracy is required on this interval, such as  $|x_1 - x_0| < \eta$ , where  $\eta$  is the tolerance,  $n$  can be increased

until the condition is satisfied.

#### IV. EFFICIENT APPROXIMATIONS

In SF MDs, for a fixed  $t$ , the function  $F(\cdot, t)$  is a cdf of a conditional distribution. Naturally, there are two ways to approximate the MDs: with a family of one-dimensional approximations or with one two-dimensional approximation.

For simplicity, we refer to the  $n$ -th moment of the distribution  $F(\cdot, t)$  as the  $(n, t)$ -th moment of  $F$ , denoted as  $m_n(t)$ . For SF MDs,  $t \in [0, 1]$ , and for SIR MDs,  $t \in \mathbb{R}_0^+$ .

##### A. One-dimensional Methods: the CM, BM, and FL Methods

As discussed in [23], there are several different ways to approximate the distribution. In this paper, we will only discuss the methods recommended in [23], namely the CM, BM, and FL method. We first apply them separately to  $F(\cdot, t)$  with different  $t$  and then we interpolate across  $t$  to obtain the overall approximation. More details on this procedure can be found in Section II-B and Section V-A.

1) *Chebyshev-Markov (CM)*: As the infima and suprema obtained by the CM inequalities naturally converges as the number of moments increases, it is sensible to consider their average as an approximation and we call it the CM method. For any  $n \in \mathbb{N}$  and  $x, t \in [0, 1]$ , the approximation by the CM method is

$$F_{\text{CM},n}(x, t) \triangleq \frac{1}{2} \left( \sup_{F \in \mathcal{F}_n} F(x, t) + \inf_{F \in \mathcal{F}_n} F(x, t) \right), \quad (23)$$

where  $\sup_{F \in \mathcal{F}_n} F(x, t)$  and  $\inf_{F \in \mathcal{F}_n} F(x, t)$  are given by the CM inequalities with the moment sequence  $(m_k(t))_{k=1}^n$ . It is worth noting that by averaging, the error can be bounded.

2) *Binomial mixture (BM)*: A piecewise approximation of the cdf based on binomial mixtures is proposed in [19]. For any  $n \in \mathbb{N}$  and  $t \in [0, 1]$ , the approximation by the BM method is

$$F_{\text{BM},n}(x, t) \triangleq \begin{cases} \sum_{k=0}^{\lfloor nx \rfloor} h_k(t), & x \in (0, 1], \\ 0, & x = 0, \end{cases} \quad (24)$$

where  $h_k(t) \triangleq \sum_{i=k}^n \binom{n}{i} \binom{i}{k} (-1)^{i-k} m_i(t)$ . As  $n \rightarrow \infty$ ,  $F_{\text{BM},n}(x, t) \rightarrow F(x, t)$  for each  $x$  at which  $F(x, t)$  is continuous [32], [33].

According to [4],  $\mathbf{h}(t) \triangleq (h_k(t))_{k=0}^n$  can be written as a linear transform of  $\mathbf{m}(t) \triangleq (m_k(t))_{k=0}^n$ , i.e.,

$$\mathbf{h}(t) = \mathbf{A}\mathbf{m}(t), \quad (25)$$

where the transform matrix  $\mathbf{A} \in \mathbb{Z}^{(n+1) \times (n+1)}$  is given by

$$A_{ik} \triangleq \binom{n}{k} \binom{k}{i} (-1)^{k-i} \mathbb{1}(k \geq i), \quad i, k \in [n]_0, \quad (26)$$

and  $\mathbb{1}$  is the indicator function. As the transform matrix needs to be calculated only once for the number of moments  $n$  and thus can be done offline, the matrix-vector multiplication requires  $\frac{(n+1)(n+2)}{2}$  multiplications.

3) *Fourier-Legendre (FL)*: The shifted Legendre polynomials

$$R_n(x) = \sum_{j=0}^n \binom{n}{j} \binom{n}{n-j} (x-1)^{n-j} x^j \quad (27)$$

are a class of orthogonal polynomials defined on  $[0, 1]$ . They are orthogonal w.r.t. the weight function 1, i.e.,

$$\int_0^1 R_n(x) R_m(x) dx = \frac{\delta_{mn}}{2n+1}, \quad (28)$$

where  $\delta_{mn}$  is the Kronecker delta function.

For any  $n \in \mathbb{N}$  and  $t \in [0, 1]$ , the approximation obtained by the  $n$ -th partial sum of the FL expansion of the cdf  $F(\cdot, t)$  is

$$F_{FL,n}(x, t) \triangleq \sum_{l=0}^n c_l(t) R_l(x), \quad x \in [0, 1], \quad (29)$$

where

$$\begin{aligned} c_l(t) &= (2l+1) \int_0^1 F(x, t) R_l(x) dx \\ &= (2l+1) \sum_{k=0}^l (1 - m_{l-k+1}(t)) \\ &\quad \left( \frac{(-1)^k}{l-k+1} \sum_{j=0}^{l-k} \binom{l}{j} \binom{l}{l-j} \binom{l-j}{k} \right). \end{aligned} \quad (30)$$

For any  $t$ , pointwise convergence holds for the FL expansion of  $F(\cdot, t)$  as  $n \rightarrow \infty$ .

According to [23], we can write  $\mathbf{c}(t) = (c_l(t))_{l=0}^n$  as a linear transformation of  $\hat{\mathbf{m}}(t) = (m_l(t))_{l=1}^{n+1}$ , i.e.,

$$\mathbf{c}(t) = \hat{\mathbf{A}}(\mathbf{1} - \hat{\mathbf{m}}(t)), \quad (32)$$

where  $\mathbf{c}(t)$  and  $\hat{\mathbf{m}}(t)$  are understood as column vectors,  $\mathbf{1}$  is the 1-vector of size  $(n+1)$ , and the transform matrix  $\hat{\mathbf{A}} \in \mathbb{Z}^{(n+1) \times (n+1)}$  is given by

$$\begin{aligned} \hat{A}_{kl} &\triangleq \frac{(-1)^{(l-k)}}{k+1} \\ &\quad \sum_{j=0}^k \binom{l}{j} \binom{l}{l-j} \binom{l-j}{l-k} \mathbb{1}(k \leq l), \quad k, l \in [n]_0. \end{aligned} \quad (33)$$

Similar to the BM method, the above matrix-vector multiplication requires  $\frac{(n+1)(n+2)}{2}$  multiplications. Besides, the evaluation of the shifted Legendre polynomial  $R_l(x)$  can be facilitated via the recursion formula, i.e.,

$$R_l(x) = \frac{2l-1}{l} (2x-1) R_{l-1}(x) - \frac{l-1}{l} R_{l-2}(x), \quad (34)$$

where  $R_0(x) = 0$  and  $R_1(x) = 2x-1$ . If the finite set of  $x$  is denoted as  $\mathcal{X}$ , the calculation of  $\mathbf{R} \triangleq [R_l(x)]_{l \in [n]_0, x \in \mathcal{X}}$  can be done offline. Then the matrix-vector multiplication of  $\mathbf{R}\mathbf{c}(t)$  requires  $(n+1)|\mathcal{X}|$  multiplications, where  $|\mathcal{X}|$  is the cardinality of the finite set  $\mathcal{X}$ .

For the BM and FL methods, we use a series of moment sequences of length  $n$  to reconstruct the MDs. For the BM method, we consider  $F_{BM,n}|u_{n+1,m}$ , and for the FL method, we consider  $F_{FL,n-1}|u_{n,m}$ . The remaining steps are the same as in Section II-B. In total, The BM method requires  $\frac{(n+1)(n+2)(m+1)}{2}$  multiplications, and the FL method requires  $\frac{3(n+1)(n+2)(m+1)}{2}$  multiplications.

*B. A Two-dimensional Method: Matching Moments to those of Abstract MDs*

In the previous sections, we use HMTs or other approximation methods based on univariate distributions to approximate the MDs by univariate distributions. It is also possible that we use a two-argument function to directly approximate the MDs. First, we introduce a class of two-argument functions in Definition 4 that serve as abstract MDs.

**Definition 4** (A class of two-argument functions).

$$\begin{aligned} \bar{F}_{\text{abs}}(x, t) &\triangleq g\left(\left(\frac{x}{1-x}\right)^p \cdot \left(\frac{t}{1-t}\right)^q\right), \\ (x, t) &\in [0, 1]^2, p, q > 0, \end{aligned} \quad (35)$$

where

$$g(u) = \frac{b^d e^{-au}}{(b+cu)^d}, \quad a, c, d \geq 0, b > 0, (a, c), (a, d) \neq (0, 0).$$

To make  $\bar{F}_{\text{abs}}(\cdot, t)$  right-continuous, we define  $\bar{F}_{\text{abs}}(x, 1) = 0$ ,  $x \in [0, 1]$  and  $\bar{F}_{\text{abs}}(1, t) = 0$ ,  $t \in [0, 1]$ .

Next, we explore the possibility of approximating MDs directly by (35). To find parameters  $(a, b, c, d, p, q)$  such that  $F_{\text{abs}}$  closely approximates the MD, we match the moments of  $F_{\text{abs}}$  with moments of the MD. We denote the  $(z, t)$ -th moments of the abstract MD as  $\tilde{m}_z(t)$ ,  $z \in \mathbb{C}$ .

There are two ways to proceed with this process: finding parameters that exactly match six moments, or finding the parameters that minimize the distance between the moments of  $F_{\text{abs}}$  and moments of the MD. In terms of the choice of moments, there are two options. First, we choose the first few moments for the same  $t$ , i.e., the 1st to  $n$ -th moments for the same  $t$ . Second, we choose  $0 < t \leq 1/2$ , since in practice, we are mainly interested in small values of  $t$ .

To find parameters that exactly match six moments, we formulate the following problem.

**Problem 1** (Exactly matching six moments). Find  $(a, b, c, d, p, q)$  such that

$$\sum_{n=1}^N \sum_{k=1}^{6/N} \|m_n(t_{nk}) - \tilde{m}_n(t_{nk})\| = 0, \quad (36)$$

where  $N \in \{1, 2, 3, 6\}$ ,  $t_{nk} \in [0, 1]$  and  $t_{nk_1} \neq t_{nk_2}, k_1 \neq k_2$ .

Since Problem 1 is non-linear and non-convex, it is difficult to determine whether a solution exists or not, and if it does, the solution can be found numerically only when proper initial points for the algorithm are chosen. Further, the accuracy of the abstract MD is not guaranteed. To address this issue, we run MATLAB's `fsolve` multiple times, each time starting from a randomly chosen initial value.

The second problem asks for the parameters that minimize the gap between the moments of  $\bar{F}_{\text{abs}}$  and moments of the MD.

**Problem 2** (Minimization).

$$\begin{aligned} &(a, b, c, d, p, q) \\ &= \arg \min \sum_{n=1}^N \sum_{k=1}^K \|m_n(t_{nk}) - \tilde{m}_n(t_{nk})\|_2^2, \end{aligned} \quad (37)$$



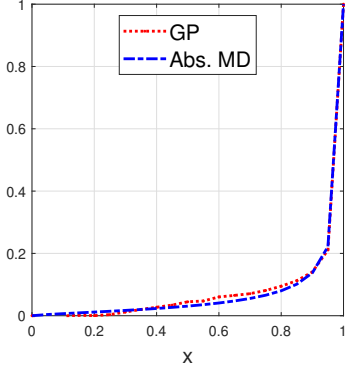


Fig. 5. The reconstructed cdfs  $F(x)$  of the conditional success probability at threshold  $t = 0.2$  of a Poisson bipolar network and the GP approximation. Their total distance at  $t = 0.2$  is 0.0096, which is a good match per Table I.

where  $NK \geq 6$ ,  $t_{nk} \in [0, 1]$  and  $t_{nk_1} \neq t_{nk_2}$ ,  $k_1 \neq k_2$ .

If  $NK = 6$ , Problem 2 provides a solution to Problem 1 if the minimum is 0. Involving different  $t$  facilitates the process of finding  $q$  because  $q$  is the exponent of  $t/(1-t)$  in  $\hat{F}_{\text{abs}}$ .

We provide an example of approximating the SF MD of a Poisson bipolar network model ( $\lambda = 1, p = 2/3, R = 1/5, \delta = 1/2$ ) by the two-dimensional method (minimization) at  $t = 0, 0.2, 0.4, 0.6, 0.8, 1$  and  $N = 6$ . The total distance is 0.015, which indicates that the abstract MD approximation is an acceptable match in this case.

## V. PERFORMANCE EVALUATION OF APPROXIMATIONS

In this section, we mainly focus on the performance evaluation of the one-dimensional methods. To make comparisons, moment sequences and a reference function serving as the ground truth are needed. We would like to evaluate/compare the methods based on moment sequences that are likely to occur in wireless applications.<sup>3</sup>

Accordingly, we focus on the generation of moments from real network models where analytic expressions of the moments have been derived (see Section V-A) and where analytic expressions of MDs have been derived (see Section V-B). The reference function of the first kind is obtained by the GP method. The randomization is done over the parameters. This way, moment sequences relevant for wireless applications are obtained.

### A. Comparison Based on MDs with Known Analytic Expressions of Moments

In this subsection, the expressions of the moments are known but the underlying MDs are unknown. We focus on the Poisson bipolar network, where the transmitters form a homogeneous Poisson point process (PPP)  $\Phi$  of intensity  $\lambda$

<sup>3</sup>More broadly, we may want to evaluate their performance via randomly generated MDs. However, no method exists for generating uniformly distributed random MDs. The method in [34] of generating uniformly randomly distributed moment sequences of the univariate distributions cannot be extended to the MDs. Also, the generation of uniformly random moment sequences may distract from the practical scenarios, where certain sequences are more likely to occur.

and each transmitter has a dedicated receiver at distance  $R$  in a random orientation [35, Def. 5.8]. In each time slot, nodes in  $\Phi$  independently transmit with probability  $p$ , and all channels are subject to Rayleigh fading and power-law path loss with exponent  $\alpha$ . The  $(b, t)$ -th moment of the SF MD of this network model is derived from the  $(b, t/(1-t))$ -th moment of the corresponding SIR MD [8, Theorem 1] as<sup>4</sup>

$$m_b(t) = \exp\left(-C \left(\frac{t}{1-t}\right)^\delta p b {}_2F_1(1-b, 1-\delta; 2; p)\right), \quad (38)$$

where  $\delta \triangleq 2/\alpha$  and  $C \triangleq \lambda \pi R^2 \Gamma(1-\delta) \Gamma(1+\delta)$ .

As discussed earlier, we use MDs obtained by the GP method as the reference and we consider the approximation on a finite set  $\mathcal{U}_{50,10}$ . Denote the raw function as  $F_{\text{GP}}|_{\mathcal{U}_{50,10}}$  and the polished one as  $\hat{F}_{\text{GP}}$ . For the beta approximation, at each value of  $t$  considered, the first two moments  $m_1(t)$  and  $m_2(t)$  have been applied to reconstruct the cdf of the cccdf at threshold  $t$  and the moments are numerically evaluated by (40). We use reconstruction over the finite set  $\mathcal{U}_{50,10}$  for the beta approximation. For the CM, BM, and FL method, at each value of  $t$  considered, a moment sequence of length  $n$ , i.e.,  $(m_k(t))_{k=1}^n$ , has been applied to reconstruct the cdf of the cccdf at threshold  $t$  and the  $(k, t)$ -th moment is numerically evaluated by (40). We use reconstruction over the finite sets  $\mathcal{U}_{10,10}$ ,  $\mathcal{U}_{n+1,10}$  and  $\mathcal{U}_{n,10}$ , respectively. Denote the raw functions as  $F_{\text{CM}}|_{\mathcal{U}_{10,10}}$ ,  $F_{\text{BM}}|_{\mathcal{U}_{n+1,10}}$  and  $F_{\text{FL}}|_{\mathcal{U}_{n,10}}$ , respectively. The polished ones are denoted as  $\hat{F}_{\text{CM}}$ ,  $\hat{F}_{\text{BM}}$ , and  $\hat{F}_{\text{FL}}$ , respectively. Due to the high complexity of the CM method, we only consider it when  $n \leq 6$ . Also, because of the limitation of the standard beta approximation, we can only use 2 moments even when there are more moments available.

Since there are only two parameters  $\delta$  and  $p$  in (38), we set  $\delta = (V+1)/2$ , where  $V \sim \text{Beta}(2, 2)$  and  $p \sim \text{Beta}(2, 2)$ . In Figure 6, we plot the average of the total distance, maximum vertical distance, maximum horizontal distance 1 and 2 versus the number of moments for the beta approximation and the CM, BM, and FL method. The average is taken over 100 randomized versions of (38). From Figure 6, we observe that the average total distance decreases as the number of moments increases for the CM, BM, and FL method. But that of the CM and FL method is much smaller than that of the BM method. The beta approximation achieves an acceptable match. The CM method achieves a mediocre match with 6 moments. The FL method achieves an excellent match with only 15 moments and achieves a perfect match with 30 moments while the BM method stays at a mediocre/acceptable match from 10 to 30 moments. In contrast, the CM, BM, and FL method do not perform well with respect to the maximum vertical distance, but they all show a decreasing trend.

In terms of the maximum horizontal distance 1 and 2, since the CM, BM, and FL method only guarantee convergence in the vertical direction, there is no guarantee in the horizontal direction. But the results show the decreasing trend of the maximum horizontal distance; here, the FL method performs better than the BM method and the beta approximation.

<sup>4</sup>In [23], moment sequences have been categorized into six types based on their tail behavior. This class of moment sequences is rich enough to produce at least 2 out of 6 decay types introduced in [23, Def. 4.3].

The computation time of the two methods will be discussed later in Section V-B.

### B. Comparison Based on MDs with Known Analytic Expressions of Distributions

In this subsection, we consider a class of cases where the underlying MDs are known. We consider the the downlink cellular network where BSs form a stationary PPP  $\Phi$  with nearest-BS association, power-law path loss  $\alpha$  and Rayleigh fading. Only the nearest interfering BS is considered. This is a special case ( $m = 1$ ) of the model considered in Figure 3. The SF MD is [7, Cor. 3]

$$\bar{F}(x, t) = \min \left\{ 1, \left( \frac{(1-x)(1-t)}{xt} \right)^\delta \right\} \quad (39)$$

where  $\delta = 2/\alpha$ . The corresponding  $(b, t)$ -th moment is

$$m_b(t) = \delta \left( \frac{1-t}{t} \right)^\delta B^*(1-t; b-\delta, \delta), \quad (40)$$

where  $B^*(z; a, b) = \int_z^1 x^{a-1} (1-x)^{b-1} dx$  is the upper incomplete beta function.

**Remark 2** (Abstract MDs). *The SF MDs in (39) can also be replaced by the more general abstract MDs. However, the abstract MDs may lead to sequences that are unlikely to occur in any practical scenarios.*

We use the SF MD evaluated over a finite set  $\mathcal{U}_{50,10}$  as the reference and denote it as  $F|_{\mathcal{U}_{50,10}}$ . For the beta approximation, at each value of  $t$  considered, the first two moments  $m_1(t)$  and  $m_2(t)$  have been applied to reconstruct the cdf of the cccdf at threshold  $t$  and the moments are numerically evaluated by (40). We use reconstruction over the finite set  $\mathcal{U}_{50,10}$  for the beta approximation. For the CM, BM, and FL method, at each value of  $t$  considered, a moment sequence of length  $n$ , i.e.,  $(m_k(t))_{k=1}^n$ , has been applied to reconstruct the cdf of the cccdf at threshold  $t$  and the  $(k, t)$ -th moment is numerically evaluated by (40). We use reconstruction over the finite sets  $\mathcal{U}_{10,10}$ ,  $\mathcal{U}_{n+1,10}$  and  $\mathcal{U}_{n,10}$ , respectively. Denote the raw functions as  $F_{CM}|_{\mathcal{U}_{10,10}}$ ,  $F_{BM}|_{\mathcal{U}_{n+1,10}}$  and  $F_{FL}|_{\mathcal{U}_{n,10}}$ , respectively. The polished ones are denoted as  $\hat{F}_{CM}$ ,  $\hat{F}_{BM}$  and  $\hat{F}_{FL}$ , respectively. Due to the high complexity of the CM method, we only consider it when  $n \leq 6$ . Also, because of the limitation of the standard beta approximation, we can only use 2 moments even when there are more moments available.

Since the only parameter in (39) is the path-loss exponent  $\alpha$ , we set  $\delta = (V+1)/2$ , where  $V \sim \text{Beta}(2, 2)$  as before. In Figure 7, we plot the average of the total distance, maximum horizontal distance 1 and 2, and computation time versus the number of moments for the beta approximation and the CM, BM, and FL method. The averaging is taken over 100 randomized versions of (39). From Figure 7, we observe that the average total distance decreases as the number of moments increases for the CM, BM, and FL method, while the beta approximation stays at a mediocre match. The CM method achieves an acceptable match with only 4 moments. The FL method achieves an excellent match with only 15 moments and achieves a perfect match with 30 moments while the

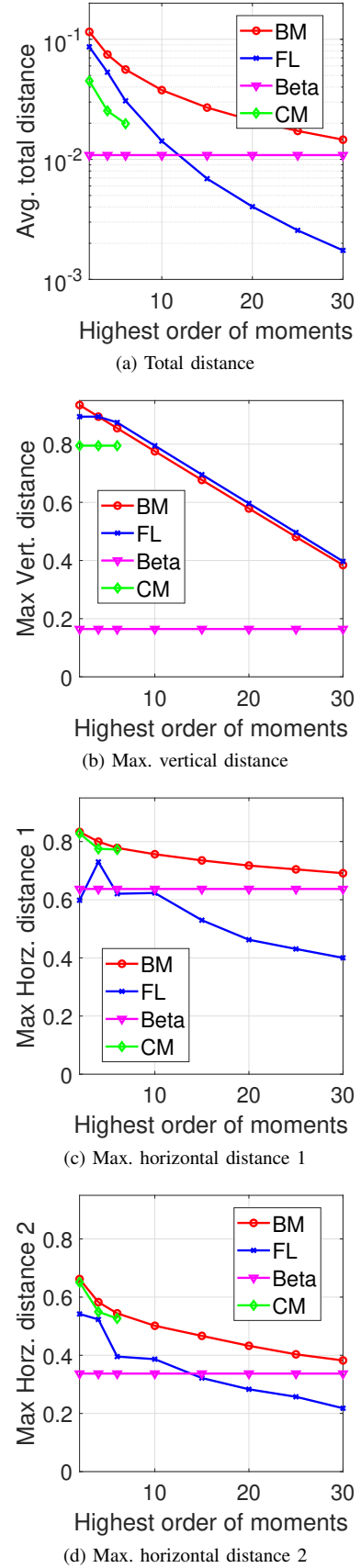


Fig. 6. The average of the total distance, maximum vertical distance, maximum horizontal distance 1, and maximum horizontal distance 2 versus the number of moments for the BM and FL method. Averaging is performed over 100 randomized versions of (38).

BM method stays at an acceptable match from 10 to 30 moments. The average of the maximum vertical distance is not shown since it is very similar to the one in Figure 6b. These decreasing trends of the CM, BM, and FL method regarding the vertical distance are consistent with our observations in Section V-A since all these three methods guarantee convergence. For the same reason mentioned in Section V-A, the BM and FL method do not perform well regarding the maximum distance; however, different from Figure 6, the BM method generally performs better than the FL method which probably results from the fact the MD is 1 for  $x \in [0, 1 - t]$ .

Since the computation time in Section V-A is very similar to the one in Section V-B, we only show the latter. From Figure 7d, the computation time of the CM is more than 50 times of the computation time of the other methods. Also, we observe that the computation time of the BM and FL method grows quadratically as the number of moments increases. Up to 30 moments, it is close to 3 s for the FL method and close to 0.3 s for the BM method (each of which is operated for 9 rounds in order to reconstruct the MD). Besides, the computation time of the BM method is about 1/10 of that of the FL method. Even though the two methods are both linear transforms, there are some differences resulting the different computational cost. First, the FL method requires three times of the number of multiplications of the BM method. Second, the largest entry in the transform matrix of the FL method is much larger than the one of the BM method, therefore, the digit requirement of the FL method is about twice of that of the BM method. Last, the Legendre polynomials in the FL method takes a longer time to compute than the step function in the BM method.

Based on the results presented in Section V-A and Section V-B, it can be concluded that the beta approximation and the CM, BM, and FL method have their own advantages and disadvantages. The beta approximation and BM method demonstrates efficiency in terms of computation time, but their accuracy is not as good as the CM and FL method. Conversely, the CM and FL method take longer to run but produce more accurate reconstructions. The choice of the best method ultimately depends on the specific objective and the desired trade-off between computational cost and accuracy. The computational complexity is determined by two tasks: the calculation of moments and the method itself. This choice may also be affected by the information we know about the distribution. We divide the entire process into three parts: identifying the bottleneck of the computational cost which could be either the moments or the reconstruction methods; determining the maximum allowable order of the moments; and selecting the appropriate method.

To include all the cases, the complete guidelines in choosing the best method are as follows:

- If rigorous bounds are required: the CM inequalities needs to be calculated.
- Otherwise, we identify the bottleneck of the computational cost by considering the computation time of the moments and Figure 7d. If the bottleneck is the reconstruction methods, the maximum allowable order of the moments is set to be a large number, such as 1000;

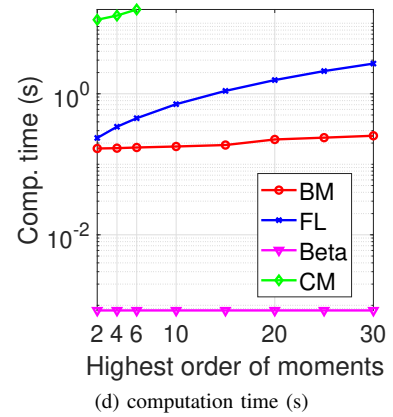
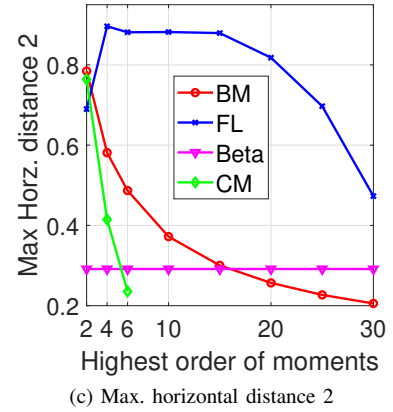
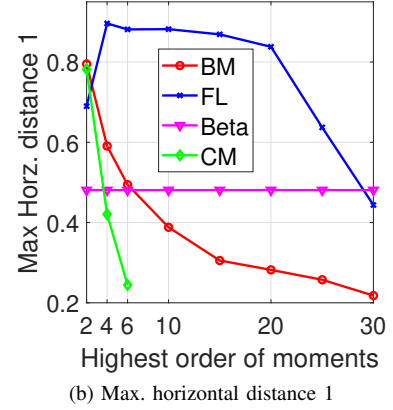
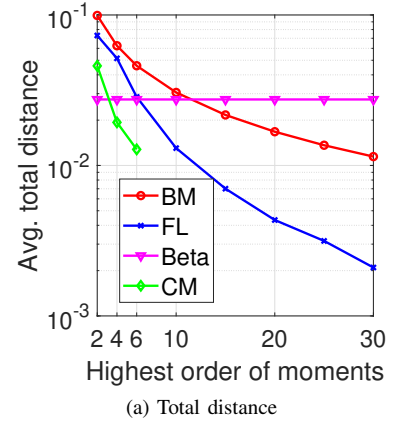


Fig. 7. The average of the total distance, maximum horizontal distance 1, maximum horizontal distance 2 and computation time versus the number of moments for the BM and FL method. Averaging is performed over 100 randomized versions of (39).

otherwise, we determine the maximum order based on the computational cost limit.

- If the maximum order is less than 10: we will choose between the CM method and the beta approximation. The final choice depends on the trade-off between the accuracy and computational complexity, which can be found in Figure 6 and Figure 7. Specifically, if we can only afford calculating 2 moments or the distribution is known to be uni-modal and smooth, the beta approximation is recommended.
- If the maximum order is greater than or equal to 10: we will choose between the BM method or beta approximation (if the distribution is known to be uni-modal and smooth) and the FL method. The final choice depends on the trade-off between the accuracy and computational complexity, which can be found in Figure 6 and Figure 7. Generally, we suggest the BM method/beta approximation for low complexity and we recommend the FL method for high accuracy. Moreover, if the BM/FL method is chosen, we suggest calculating 15 to 20 moments, which usually produces a good match.

## VI. APPLICATION TO POISSON CELLULAR NETWORKS

In this section, we focus on a specific wireless network model and show how we utilize the above-mentioned methods, i.e., the CM, BM, and FL method, to infer the number of moments required, to derive the tightest possible bounds for the distribution, and to recover the distribution. The performance of the methods are shown in one particular case, and the results agree with our comparison in Section V-A and Section V-B. At the end of the section, we show how the reconstructions can be used to infer and further improve the network performance. We consider the standard downlink Poisson cellular model with nearest-base station association, power-law path loss and Rayleigh fading. The  $(n, t)$ -th moment is given by [8]

$$m_n(t) = \frac{1}{(1-t)^n {}_2F_1(n, 1; 1-\delta, t)}, \quad t \in [0, 1), \quad (41)$$

where  ${}_2F_1$  is the Gauss hypergeometric function and  $2/\delta$  is the path loss exponent. Here we consider the distribution of the ccdf at the SF thresholds  $t = 0.387, 0.5, 0.613$  (corresponding to the SIR thresholds  $-2$  dB,  $0$  dB,  $2$  dB) and  $\delta = 0.25, 0.5, 0.7$ . We derive the number of moments needed, calculate the CM inequalities, and apply the recommended CM, BM and FL method, to recover the distribution of the considered ccdf.

First, we find the number of moments that are reasonable for reconstruction with acceptable accuracy, i.e., at least a mediocre match. Figure 8 shows the total distance between the infima and suprema as a function of the number of moments at different SF thresholds and  $\delta$ . At  $n = 20$ , the total distance is less than 0.1, which means that the total distance between the average of the infima and suprema and the original cdf is less than 0.05, and thus the average is at least a mediocre match. From Figure 8a we observe that the curves flatten after  $n$  approaches 30, which indicates that increasing the number of moments do not offer much benefit for the single MD (with fixed  $\delta$ ). Similarly, from Figure 8b where  $\delta = 0.25$ , the curve

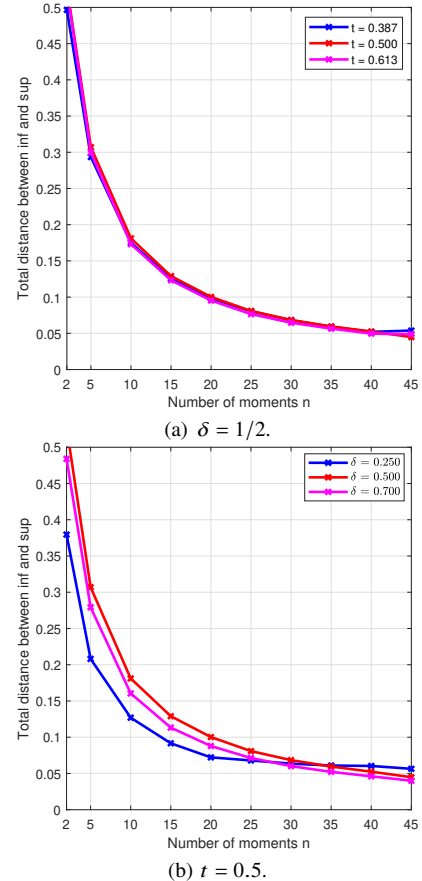


Fig. 8. The total distance between the infima and suprema as a function of the number of moments for different  $t$  and  $\delta$ .

flattens after  $n = 20$ . However, the two curves for  $\delta = 0.5$  and  $0.7$  keep promising gradients even after  $n$  approaches 40, which indicates that under a small path-loss exponent setting, increasing the number of moments is still helpful. That said, we have to keep in mind that the more moments, the digit requirements (both for moments and for each method) become more strict. Also, with proper interpolations, we can infer that at least 100 or more moments are required to guarantee the average to be a perfect match for the original cdf.

Next, we focus on the case where  $t = 0.387$  and  $\delta = 0.5$ . The corresponding infima and suprema at  $n = 20$  are shown in Figure 9. Since the actual cdf of the ccdf is not available, we use the (time-consuming) approximation by the GP method as a reference.

We apply the beta approximation and the CM, BM, and FL method to recover the distribution, using 20 moments for reconstruction. The absolute difference and the total distance are shown in Figure 10 and Table II, respectively. We can observe that the CM, BM, and FL method outperform the beta approximation. In the meantime, the CM, BM, and FL method take longer time to compute than the beta approximation. But the computation of the FL and BM method takes much less time than computing the CM method. Please note that in Table II, we only apply the methods for  $t = 0.387$ , while in Figure 6 and Figure 7, we apply the methods for nine different values of  $t$ . Therefore, the computation time in Figure 6 and



TABLE II  
TOTAL DISTANCE, THE TYPE OF MATCH AND COMPUTATION TIME OF  
EACH METHOD

Method	FL	BM	CM avg.	beta
Total distance	$7.09 \times 10^{-4}$	0.0065	0.0018	0.0101
Match	perfect	good	perfect	acceptable
Comp. time (s)	0.2488	0.0469	22.52	0.0052

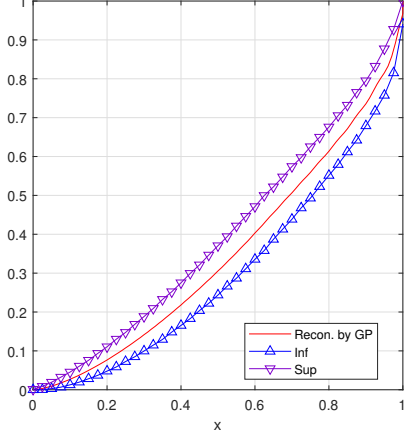


Fig. 9. The reconstructed cdfs of the ccdf  $P_t$  at threshold  $t = 0.387$  of the downlink Poisson cellular network via the GP approximation and the inf. and sup. via the CM inequalities at  $n = 20$ .

Figure 7 is expected to be about nine times of the computation time in Table II.

Finally, we show how the reconstructions can be used to infer and optimize the network performance. Recall that for an SF MD  $F(x, t) = y$ , it can be interpreted as the reliability  $x$  at the threshold SF value  $t$  that a fraction  $1 - y$  of users achieves. For example, choosing  $t$  and finding the pairs  $(t, x)$  for  $F(x, t) = 0.05$  yields the reliability  $x$  given a target SF value  $t$  that 5% of users cannot achieve, i.e., the pairs of the reliability and the SF threshold value at the 5-th percentile. In other words, it is the reliability and the SF threshold value that 95% of users achieve. In the following discussion, we call the pairs  $(t, x)$  for  $F(x, t) = y$  as the pairs for the  $(100y)$ -th user percentile. To make it more easy to understand, we show the

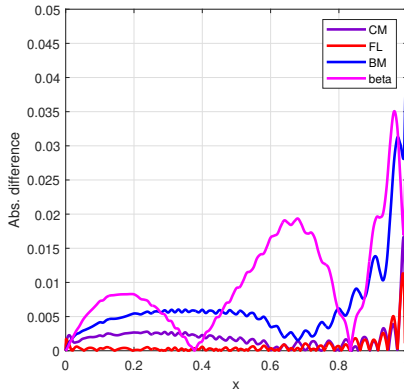
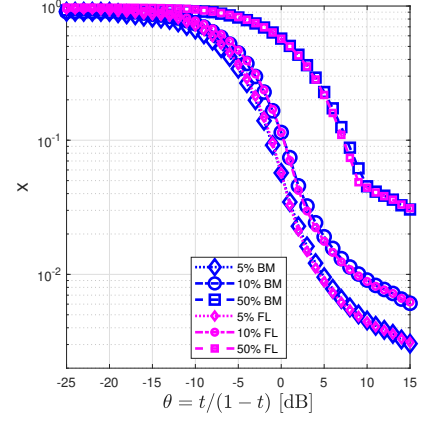
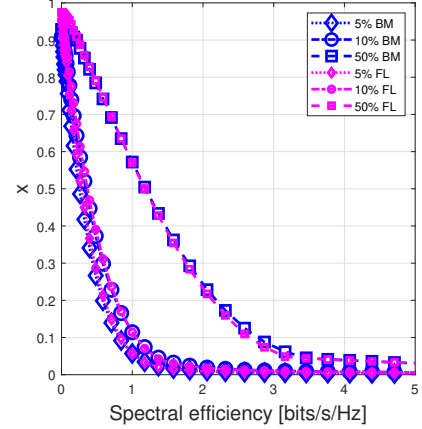


Fig. 10. The abs. diff. between the reconstruction via the average of the inf. and sup. of the CM inequalities, the FL and BM method, and the beta approximation and the reference by the GP method of the downlink Poisson cellular network.



(a) Pairs  $(\theta, x)$ .



(b) Rate-reliability tradeoff.

Fig. 11. Pairs  $(\theta, x)$  and  $(\log_2(1+\theta), x)$  where  $F(x, \theta/(1+\theta)) = 0.05, 0.1$  and  $0.5$  for the BM and FL method at  $n = 20$ .

pairs  $(\theta, x)$  instead of  $(t, x)$ , where  $\theta = t/(1-t)$  is the SIR threshold value corresponding to the SF threshold value  $t$ .

Figure 11 shows the pairs  $(\theta, x)$  and the spectral efficiency-reliability tradeoff  $(\log_2(1+\theta), x)$  via the BM and FL method for the 5-th, 10-th, and 50-th user percentiles, respectively. By incorporating these tradeoff curves, readers can gain valuable insights into the relationship between spectral efficiency and reliability, allowing them to make informed decisions to optimize wireless network performance based on specific requirements. For instance, given a target spectral efficiency that 95% of users must achieve, they can determine the corresponding reliability using the curves provided and optimize the network performance based on this procedure.

## VII. CONCLUSION

To recover the MDs from moments, the widely used beta distribution is simple but inadequate in accurately approximating many distributions, such as multi-modal or non-smooth ones. Alternative approaches may not preserve the properties of MDs and possess their own advantages and disadvantages, which requires a practical tweaking mapping and a comprehensive comparison for choosing the right one. Our work provides a rigorous foundation and practical solutions in reconstructing MDs. A tweaking mapping for adjusting approximations and a categorization for a consistent and

objective comparison of the quality of approximations are presented. Based on our findings, for each MD at a specified threshold value  $t$  of the PRV, we suggest using 20 moments (if possible) via the proposed/improved BM or FL method to recover the distribution  $F(\cdot, t)$ . The resulting approximation is likely to be at least a good match to the exact one. Also, with the given moments, the infima and suprema of the distribution are obtained via the CM inequalities. Further, the tightest range of the quantile function  $Q_2(\bar{F})$  at a certain point  $(y, t)$  can be found by Algorithm 1. Depending on the objective, the beta approximation, the CM, BM, or FL method is chosen for reconstruction. Specifically, with 10 or more moments calculated, the FL method is preferred for accuracy, while the BM method is preferred for low computational cost. With less than 10 moments calculated and knowing that the distribution is uni-modal and smooth, the beta approximation is recommended. When only 2 moments are calculated, the beta approximation is preferred as well. Otherwise, we recommend the CM method. When moments for different values of  $t$  are given, we can apply the above procedures to them and obtain the complete MD by linear interpolation.

## APPENDIX

### A. Code

We provide solutions to six key problems in MATLAB that occur in the use of MDs. These programs can be found at [30]. The pseudo codes and corresponding scenarios are provided below.

- Algorithm 2: Use the CM inequalities to find infima and suprema of a moment sequence.
- Algorithm 3: Produce uniformly random moment sequences.
- Algorithm 4: Compare the FL method and the BM method by a random moment sequence with specific decay rate.
- Algorithm 5: Use a moment sequence with an analytic expression to recover the cdf.
- Algorithm 6: Find the number of moments needed such that the total distance between infima and suprema is less than  $\epsilon = 1/100$ .
- Algorithm 7: Having  $N > 1$  moments available, plot the difference of the supremum and the infimum at the 95% point (of the ccdf) as a function of the number of moments, from 1 to  $N$  moments.

---

**Algorithm 2** Algorithm for finding the infima and suprema of all possible cdfs from a moment sequence

---

**Input:** The moment sequence  $(m_k)_{k=1}^n$ , the point of interest  $y \in (0, 1)$ .

**Output:** The infimum  $\inf_{F \in \mathcal{F}_n} F(y)$  and supremum  $\sup_{F \in \mathcal{F}_n} F(y)$ .  
 1: Use the CM inequalities (CMClass) to calculate  $\inf_{F \in \mathcal{F}_n} F(y)$  and  $\sup_{F \in \mathcal{F}_n} F(y)$ .

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**Algorithm 3** Algorithm for producing uniformly random moment sequences

---

**Input:** Length of the moment sequence  $n$ .

**Output:** Randomly distributed moment sequence  $(m_k)_{k=1}^n$ .

- 1: Use the canonical moment method [23, Alg. 1] (canonicalMomentClass) to generate  $(m_k)_{k=1}^n$ .
- 

---

**Algorithm 4** Algorithm for comparing the FL and BM method via a random moment sequence with specific decay rate

---

**Input:** Length of the moment sequence  $n$ , decay type, decay rate and corresponding decay parameters.

**Output:** Reconstructions via the FL and BM methods.

- 1: Use [23, Alg. 2] to generate the moment sequence with desired properties.
  - 2: Use the FL method (FLClass) and the BM method (BMClass) to reconstruct the cdf via the moment sequence, respectively.
- 

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**Algorithm 5** Algorithm for recovering the accurate cdf via a moment sequence with an analytic expression

---

**Input:** The analytic expression for the moment sequence, and the length of integer moments  $n$ .

**Output:** Reconstructed cdf  $F$ .

- 1: Use the GP method (GPClass) to recover the cdf  $F_{GP}$  and set  $F = F_{GP}$ .
  - 2: **if**  $F_{GP}$  does not follow the basic properties of a cdf **then**
  - 3:   Use the FL method (FLClass) to reconstruct the cdf  $F_{FL}$  via the moment sequence of length  $n$  and set  $F = F_{FL}$ .
  - 4: **end if**
- 

---

**Algorithm 6** Algorithm to find the number of moments needed for bounding cdf

---

**Input:** The accurate moments  $(m_k)_{k=1}^N$ , points of interest  $(y_k)_{k=1}^j$ , and tolerance  $\epsilon$ .

**Output:** The number of moments needed  $n$

- 1: Set  $n = 1$  and  $d = \max_{1 \leq k \leq j} \sup_{F \in \mathcal{F}_n} F(y_k) - \inf_{F \in \mathcal{F}_n} F(y_k)$ , where  $\sup_{F \in \mathcal{F}_n}$  and  $\inf_{F \in \mathcal{F}_n}$  are obtained from the CM inequalities with moment sequence  $(m_k)_{k=1}^n$ .
  - 2: **while**  $d > \epsilon$  or  $n < N$  **do**
  - 3:   Set  $n = n + 1$  and  $d = \max_{1 \leq k \leq j} (\sup_{F \in \mathcal{F}_n} F(y_k) - \inf_{F \in \mathcal{F}_n} F(y_k))$ , where  $\sup_{F \in \mathcal{F}_n}$  and  $\inf_{F \in \mathcal{F}_n}$  are obtained from the CM inequalities with moment sequence  $(m_k)_{k=1}^n$ .
  - 4: **end while**
- 

---

**Algorithm 7** Algorithm for plotting the difference of the supremum and the infimum at the 95% point as a function of the number of moments

---

**Input:** The accurate moments  $(m_k)_{k=1}^N$ , selected points  $(y_k)_{k=1}^j$ .

**Output:** The difference of the 95% point  $(d_k)_{k=1}^j$ .

- 1: Set  $n = 1$ .
  - 2: **while**  $n \leq N$  **do**
  - 3:   Obtain  $\sup_{F \in \mathcal{F}_n} F(y_k)$  and  $\inf_{F \in \mathcal{F}_n} F(y_k)$  from the CM inequalities with moment sequence  $(m_k)_{k=1}^n$ .
  - 4:   Do linear interpolations for  $\left(\sup_{F \in \mathcal{F}_n} F(y_k)\right)_{k=1}^j$  and  $\left(\inf_{F \in \mathcal{F}_n} F(y_k)\right)_{k=1}^j$ .
  - 5:   Find the values of the 5% point, set  $d_n$  as their difference, and set  $n = n + 1$ .
  - 6: **end while**
-

## REFERENCES

- [1] "Minimum requirements related to technical performance for IMT-2020 radio interface(s)," [https://www.itu.int/dms\\_pub/itu-r/rep/REP-M.2410-2017-PDF-E.pdf](https://www.itu.int/dms_pub/itu-r/rep/REP-M.2410-2017-PDF-E.pdf), accessed: 2022-05-31.
- [2] "11ax Evaluation Methodology," <https://mentor.ieee.org/802.11/dcn/14/11-14-0571-12-00ax-evaluation-methodology.docx>, accessed: 2022-04-13.
- [3] M. Haenggi, J. G. Andrews, F. Baccelli, O. Dousse, and M. Franceschetti, "Stochastic Geometry and Random Graphs for the Analysis and Design of Wireless Networks," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 7, pp. 1029–1046, 2009.
- [4] M. Haenggi, "Efficient Calculation of Meta Distributions and the Performance of User Percentiles," *IEEE Wireless Communications Letters*, vol. 7, no. 6, pp. 982–985, 2018.
- [5] S. S. Kalamkar and M. Haenggi, "Per-link reliability and rate control: Two facets of the SIR meta distribution," *IEEE Wireless Communications Letters*, vol. 8, no. 4, pp. 1244–1247, 2019.
- [6] M. Haenggi, "Meta distributions—part 1: Definition and examples," *IEEE Communications Letters*, vol. 25, no. 7, pp. 2089–2093, 2021.
- [7] —, "Meta distributions—part 2: Properties and interpretations," *IEEE Communications Letters*, vol. 25, no. 7, pp. 2094–2098, 2021.
- [8] —, "The Meta Distribution of the SIR in Poisson Bipolar and Cellular Networks," *IEEE Transactions on Wireless Communications*, vol. 15, no. 4, pp. 2577–2589, 2016.
- [9] Y. Wang, Q. Cui, M. Haenggi, and Z. Tan, "On the SIR Meta Distribution for Poisson Networks With Interference Cancellation," *IEEE Wireless Communications Letters*, vol. 7, no. 1, pp. 26–29, 2018.
- [10] Y. Wang, M. Haenggi, and Z. Tan, "SIR Meta Distribution of K-Tier Downlink Heterogeneous Cellular Networks With Cell Range Expansion," *IEEE Transactions on Communications*, vol. 67, no. 4, pp. 3069–3081, 2019.
- [11] K. Feng and M. Haenggi, "Separability, Asymptotics, and Applications of the SIR Meta Distribution in Cellular Networks," *IEEE Transactions on Wireless Communications*, vol. 19, no. 7, pp. 4806–4816, 2020.
- [12] J. P. Jeyaraj, M. Haenggi, A. H. Sakr, and H. Lu, "The Transdimensional Poisson Process for Vehicular Network Analysis," *IEEE Transactions on Wireless Communications*, vol. 20, no. 12, pp. 8023–8038, 2021.
- [13] S. Guruacharya and E. Hossain, "Approximation of Meta Distribution and Its Moments for Poisson Cellular Networks," *IEEE Wireless Communications Letters*, vol. 7, no. 6, pp. 1074–1077, 2018.
- [14] X. Tang, X. Xu, and M. Haenggi, "Meta Distribution of the SIR in Moving Networks," *IEEE Transactions on Communications*, vol. 68, no. 6, pp. 3614–3626, 2020.
- [15] J. Gil-Pelaez, "Note on the inversion theorem," *Biometrika*, vol. 38, no. 3-4, pp. 481–482, 1951.
- [16] K. A. Penson, K. Górski, A. Horzela, and G. H. E. Duchamp, "Hausdorff moment problem for combinatorial numbers of Brown and Tutte: exact solution," 2022. [Online]. Available: <https://arxiv.org/abs/2209.06574>
- [17] F. Hausdorff, "Summationsmethoden und Momentfolgen. I," *Mathematische Zeitschrift*, vol. 9, no. 1-2, pp. 74–109, 1921.
- [18] M. Shi, X. Gao, K. Yang, D. Niyato, and Z. Han, "Meta Distribution of the SINR for mmWave Cellular Networks With Clusters," *IEEE Transactions on Communications*, vol. 69, no. 10, pp. 6956–6970, 2021.
- [19] R. Mnatsakanov and F. Ruymgaart, "Some properties of moment-empirical cdf's with application to some inverse estimation problems," *Mathematical Methods of Statistics*, vol. 12, no. 4, pp. 478–495, 2003.
- [20] G. Szeg, *Orthogonal polynomials*. American Mathematical Soc., 1939, vol. 23.
- [21] J. Shohat and J. Tamarkin, *The Problem of Moments*, ser. Mathematical surveys and monographs. American Mathematical Society, 1950.
- [22] I. Schoenberg, "Remarks Concerning a Numerical Inversion of the Laplace Transform Due to Bellman, Kalaba and Lockett," *Journal of Mathematical Analysis and Applications*, vol. 43, no. 3, pp. 823–828, 1973.
- [23] X. Wang and M. Haenggi, "Hausdorff Moment Transforms and Their Performance," 2022. [Online]. Available: <https://arxiv.org/abs/2212.12622>
- [24] M. Haenggi, "SIR Analysis Via Signal Fractions," *IEEE Communications Letters*, vol. 24, no. 7, pp. 1358–1362, 2020.
- [25] "How well do distributions match? A case for the MH distance," <https://stogblog.net/2022/06/15/how-well-do-distributions-match-a-case-for-the-mh-distance>, accessed: 2022-12-16.
- [26] A. Markoff, "Démonstration de certaines inégalités de M. Tchébychef," *Mathematische Annalen*, vol. 24, no. 2, pp. 172–180, 1884.
- [27] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*. SIAM, 2020.
- [28] K. A. Possé, *Sur quelques applications des fractions continues algébriques*. L'Académie Impériale des Sciences, 1886.
- [29] X. Wang and M. Haenggi, "The Chebyshev-Markov inequalities," Department of Electrical Engineering, University of Notre Dame, Tech. Rep., 2022. [Online]. Available: [http://www.nd.edu/~mhaenggi/pubs/techreport\\_cm22.pdf](http://www.nd.edu/~mhaenggi/pubs/techreport_cm22.pdf)
- [30] —, "Useful programs for recovering meta distribution," 2023. [Online]. Available: <https://github.com/ewa-haenggi/meta-distribution>
- [31] M. Zelen, "Bounds on a distribution function that are functions of moments to order four," *Journal of Research of the National Bureau of Standards*, vol. 53, no. 6, pp. 377–81, 1954.
- [32] R. Mnatsakanov and F. Ruymgaart, "Some properties of moment-empirical CDF's with application to some inverse estimation problems," *Mathematical Methods of Statistics*, vol. 12, no. 4, pp. 478–495, 2003.
- [33] R. M. Mnatsakanov, "Hausdorff moment problem: reconstruction of distributions," *Statistics & Probability Letters*, vol. 78, no. 12, pp. 1612–1618, 2008.
- [34] F.-C. Chang, J. Kemperman, and W. Studden, "A Normal Limit Theorem for Moment Sequences," *The Annals of Probability*, vol. 21, no. 3, pp. 1295–1309, 1993.
- [35] M. Haenggi, *Stochastic geometry for wireless networks*. Cambridge University Press, 2012.



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