A Multidimensional Opinion Evolution Model with Confirmation Bias

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***Abstract*— In this paper, we present a novel opinion dynamics model in vector spaces. We define individuals and information sources, where individuals' opinions evolve over time and are represented by vectors. The information sources are characterized by constant opinion vectors, while confirmation bias is explicitly incorporated into the model to account for individuals' selective information acquisition from information sources that align with their existing opinions. Drawing inspiration from the Friedkin-Johnsen model, this mechanism combines an individual's innate opinion, the opinions of their social network neighbors, and the influence of information sources. The social network operates on trust and remains static, while the information sources introduce dynamics through weighted norms based on the distance between an individual's and the information source's opinion vector. We characterize the convergence conditions for the proposed dynamics and present approximation and exact computation methods for steady-state values under both affine and non-linear state-dependent confirmation bias weight functions.**

I. Introduction

Empirical studies in social sciences indicate that individuals form opinions on different issues correlatedly, i.e., the opinion components evolve as a vector, see e.g. , [1]. In recent years, there has been a significant interest in expanding the scalar models to multidimensional settings in order to reflect the complexities of opinion evolution in real-world circumstances where individuals frequently have several interconnected opinions on various topics [2]–[7]. Individuals are represented as vectors of opinions in the multidimensional model, with each vector dimension corresponding to a specific issue. Extending scalar models to multidimensional settings requires developing new mathematical tools to describe the dynamics of opinion formation. One of the critical challenges is how to account for the interdependence between different issues and how individuals update their opinions on one issue based on their opinions on other issues. Another challenge is incorporating the effect of confirmation bias, where individuals tend to seek out and favor information sources that confirm their existing beliefs into the model. In [8], the authors proposed scalar evolution dynamics based on the well-known Friedkin-Johnsen model in [9], which explicitly considers the aforementioned confirmation bias.

In this paper, we extend the opinion evolution model in [8] which explicitly considers confirmation bias, to vector spaces. Beyond a simple concatenation of independent scalar dynamics, this nontrivial extension enables modeling the interactions between opinion components.

The structure of this paper is as follows. Section 2 introduces the notation and outlines the model for cyber-social networks. In Section 3, we delve into the analysis of convergence dynamics and the estimation and exact computation of the unique equilibrium point. Finally, Section 4 concludes the paper, summarizing key findings.

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II. Notation and Network Model

A. Notation

In this paper, we utilize the following notation:

$\mathbb{R}^n$	The set of n-dimensional real vectors;
$\mathbb{R}^+$	The set of non-negative real numbers;
$\mathbb{R}^{m\times n}$	The set of $m \times n$-dimensional real matrices
$\mathbb{I}^n$	The set of n-dimensional vector with all components within $[0, 1]$ interval
$\mathbb{I}^{m\times n}$	The set of $m \times n$-dimensional matrices with all components within $[0, 1]$ interval
$\mathbb{N}(\mathbb{N}_0)$	Set of positive integers(including zero)
$\top$	Matrix transposition
$\mathbf{x} \leq \mathbf{y}$	Element-wise inequality between vectors $\mathbf{x}$ and $\mathbf{y}$
$A \leq B$	Element-wise inequality between matrices A and B
diag$\{\mathbf{x}\}$	Diagonal matrix with appropriate dimensions whose diagonal elements are given elements of vector $\mathbf{x}$
sign$(\cdot)$	Sign function
I	Identity matrix
$\mathbf{0}$	Zero matrix
$\mathbf{1}_n$	n-dimensional vector of all ones
$\mathbf{0}_n$	n-dimensional vector of all zeros
$\lvert\cdot\rvert$	Absolute value of a real number, element-wise absolute value of a vector or mtrix, and the cardinality of a set
$\mathbf{x}(r)$	The r-th component of vector $\mathbf{x}$
$A(r, a)$	Component in r-th row and a-th column of matrix A
$\mathbf{x}[t]$	The time dependence of vector x in a discrete domain, where t is the time parameter

Scalars are represented using lowercase letters, vectors are denoted by bold lowercase letters, and matrices are indicated by capital letters. In the rest of this section, we provide definitions that are extensively utilized throughout this paper.

Definition 1 (l_1-norm and l_1-distance)**.** *For $\mathbf{x} \in \mathbb{R}^n$, the l_1-norm is defined as:*

$$||\mathbf{x}|| = \sum_{r=1}^{n} |\mathbf{x}(r)|.$$

The l_1-distance between two vectors $\mathbf{x}$ and $\mathbf{y}$ both in $\mathbb{R}^n$ is defined as:

$$d(\mathbf{x},\ \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|.$$

Definition 2 (Induced l_1-norm)**.** *For a square matrix $A \in \mathbb{R}^{n\times n}$, the induced l_1-norm is defined as:*

$$||A|| = \{\sup \frac{||A\mathbf{x}||}{||\mathbf{x}||} : \mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}_n\} = \max_a \sum_{r=1}^{n} |A(r, a)|.$$

Definition 3 (Weighted l_1-norm and distance)**.** *Weighted l_1-norm of vectors $\mathbf{x} \in \mathbb{R}^n$ is defined as:*

$$||\mathbf{x}||_{\mathbf{c}} = \sum_{r=1}^{n} \mathbf{c}(r)|\mathbf{x}(r)|.$$

Weighted l_1-distance between vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as:

$$d_{\mathbf{c}}(\mathbf{x},\ \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_{\mathbf{c}}.$$

Definition 4 (stochastic and sub-stochastic matrix)**.** *A matrix $A \in \mathbb{I}^{n\times m}$ is row-stochastic if:*

$$\sum_{a=1}^{m} A(r,a) = 1 \quad \forall r \in \{1,2,\dots,n\},$$

and sub-row-stochastic if:

$$\sum_{a=1}^{m} A(r,a) \le 1 \quad \forall r \in \{1,2,\dots,n\}.$$

Similarly $A \in \mathbb{I}^{n\times m}$ is called column-stochastic if:

$$\sum_{r=1}^{n} A(r,a) = 1 \quad \forall a \in \{1,2,\dots,m\},$$

and sub-column-stochastic if:

$$\sum_{r=1}^{n} A(r,a) \le 1 \quad \forall a \in \{1,2,\dots,m\}.$$

If a matrix is both row-stochastic and column-stochastic, we refer to it as a row-column-stochastic matrix. This sentence applies to sub-stochastic matrices as well.

B. Social Network Model

We investigate a network consisting of two layers: Social layer and information layer. In the social layer, there are $n \in \mathbb{N}$ individuals, each denoted as v_i. Every individual holds an expressed opinion vector $\mathbf{x}_i[t] \in \mathbb{I}^q$ at time t, along with an innate opinion vector $\mathbf{s}_i \in \mathbb{I}^q$ regarding $q \in \mathbb{N}$ interconnected issues. In the information layer, there are $m \in \mathbb{N}_0$ distinct information sources, denoted as u_k. Each of these sources has an associated information vector $\mathbf{y}_k \in \mathbb{I}^q$, all addressing the same q topics.

The interactions among individuals are modeled through a directed graph $\mathbb{G} = (\mathbb{V},\mathbb{E})$, where $\mathbb{V} = \{\mathrm{v}_1,\dots,\mathrm{v}_n\}$ signifies the individual vertices, and $\mathbb{E} \subseteq \mathbb{V}\times\mathbb{V}$ denotes the set of edges representing influence relationships. Self-loops are permitted, meaning that for certain $\mathrm{v}_i \in \mathbb{V}$, $(\mathrm{v}_i,\mathrm{v}_i) \in \mathbb{E}$. The transmission of information from sources to individuals is represented by a bipartite directed graph $\mathbb{H} = (\mathbb{V}\cup\mathbb{U},\mathbb{B})$, where $\mathbb{U} = \{\mathrm{u}_1,\dots,\mathrm{u}_m\}$ corresponds to the vertices representing information sources, and $\mathbb{B} \subseteq \mathbb{V}\times\mathbb{U}$ denotes the edge set. Consequently, each individual is associated with a set of neighboring individuals $N_i \subseteq \mathbb{V}$ and a set of neighboring information sources $Q_i \subseteq \mathbb{U}$.

Note: The vector $\mathbf{x}_i[t]$ represents the state at time t and is inherently time-dependent. However, for the sake of simplicity, in certain instances within this paper, we omit the explicit use of the time argument t.

We consider the following model which is adopted from [8]:

$$\mathbf{x}_i[t+1] = (I-A_i)\mathbf{s}_i + A_i(I-\Lambda_i)\sum_{j\in N_i} W_{i,j}\mathbf{x}_j[t] + A_i\Lambda_i\sum_{k\in Q_i} V_{i,k}(\mathbf{x}_i)\boldsymbol{\delta}_{i,k} \tag{1}$$

where

1) We define $\boldsymbol{\delta}_{i,k}$ as follows:

$$\boldsymbol{\delta}_{i,k} = \mathbf{y}_k - \mathbf{s}_i$$

2) Let $\boldsymbol{\alpha}_i \in \mathbb{I}^q$ be a vector containing parameters representing the social influence of individual $\mathrm{v_i}$. We define the matrix $A_i \in \mathbb{I}^{q\times q}$ as follows:

$$A_i = \mathrm{diag}\{\boldsymbol{\alpha}_i\}$$

3) Consider $\boldsymbol{\lambda}_i \in \mathbb{I}^q$ as a vector containing parameters that represent the information influence of individual $\mathrm{v_i}$. We define the matrix $\Lambda_i \in \mathbb{I}^{q\times q}$ as follows:

$$\Lambda_i = \mathrm{diag}\{\boldsymbol{\lambda}_i\}$$

4) $W_{i,j} \in \mathbb{R}^{+q\times q}$ represents the weighted influence matrix of individual $\mathrm{v_j}$ on individual $\mathrm{v_i}$.We emphasize that $W_{i,j}$ remains constant over time, and $W_{i,i}$ represents the interconnection between different components of the opinion vector $\mathbf{x}_i[t]$.
5) We introduce $\boldsymbol{\beta}_{i,k}(\cdot) : \mathbb{I}^q \to \mathbb{I}^q$, which is defined as follows:

$$\boldsymbol{\beta}_{i,k}(\mathbf{x}_i) \triangleq C_{i,k}|\mathbf{x}_i\ -\ \mathbf{y}_k| \tag{2}$$

The weight matrix $C_{i,k} \in \mathbb{I}^{q\times q}$ is a constant, stochastic matrix. Where i refers to individual $\mathrm{v_i}$ and k refers to information source $\mathrm{u_k}$.
6) Define the function $v^r_{i,k}(\cdot) : \mathbb{I}^q \to \mathbb{R}^+$ as:

$$v^r_{i,k}(\mathbf{x}_i) = g^r_{i,k}(\boldsymbol{\beta}_{i,k}(r)(\mathbf{x}_i)),$$

where $g^r_{i,k}(\cdot) : \mathbb{I} \to \mathbb{R}^+$ is a decreasing function. This function $v^r_{i,k}(\mathbf{x}_i)$ quantifies the confirmation bias of the r-th component of the opinion vector of individual $\mathrm{v_i}$ towards information from source $\mathrm{u_k}$.
7) $V_{i,k}(\mathbf{x}_i) \in \mathbb{R}^{+q\times q}$ is the weighted influence matrix of information source $\mathrm{u_k}$ on individual $\mathrm{v_i}$, which is defined as follows:

$$V_{i,k}(\mathbf{x}_i) \triangleq b_{i,k}\mathrm{diag}\Big\{\big[v^1_{i,k}(\mathbf{x}_i),\cdots,v^q_{i,k}(\mathbf{x}_i)\big]^\top\Big\} \tag{3}$$

where $b_{i,k} = 1$ if information source $\mathrm{u_k} \in Q_i$, and $b_{i,k} = 0$ otherwise.

Remark 1. *The definition of $C_{i,k}$ in* (2) *indicates that the confirmation bias within the r-th component of the opinion vector $\mathbf{x_i}$ of individual $\mathrm{v_i}$ is computed by averaging the components of $|\mathbf{x}_i - \mathbf{y}_k|$, thereby representing a weighted distance as follows:*

$$\boldsymbol{\beta}_{i,k}(\mathbf{x}_i)(r) = d_{C_{i,k}(r,:)}(\mathbf{x}_i,\ \mathbf{y}_k)$$

where $C_{i,k}(r,:)$ denotes the r-th row of $C_{i,k}$.

Remark 2. *Λ_i and A_i are specific characteristics of individual $\mathrm{v_i}$, independent of their connection to information sources. The connection of an individual $\mathrm{v_i}$ to an information source $\mathrm{u_k}$ is specified by $b_{i,k}$.*

We represent the social dynamic given by (1) in concatenated form as follows:

$$\mathbf{x}[t+1] = (I-A)\mathbf{s} + A((I-\Lambda)W\mathbf{x}[t] + \Lambda V(\mathbf{x})\boldsymbol{\delta}), \tag{4}$$

where we define the following variables:

$$\mathbf{x} \triangleq \left[\mathbf{x}_1^\top, \mathbf{x}_2^\top, \dots, \mathbf{x}_n^\top\right]^\top \in \mathbb{I}^{nq}, \tag{5}$$

$$\mathbf{s} \triangleq \left[\mathbf{s}_1^\top, \mathbf{s}_2^\top, \dots, \mathbf{s}_n^\top\right]^\top \in \mathbb{I}^{nq}, \tag{6}$$

$$\boldsymbol{\delta}_i \triangleq \left[\mathbf{y}_1^\top - \mathbf{s}_i^\top, \dots, \mathbf{y}_m^\top - \mathbf{s}_i^\top\right]^\top \in \mathbb{I}^{mq}, \tag{7}$$

$$\boldsymbol{\delta} \triangleq \left[\boldsymbol{\delta}_1^\top, \dots, \boldsymbol{\delta}_n^\top\right]^\top \in \mathbb{I}^{nmq}, \tag{8}$$

$$A \triangleq \mathrm{diag}\{A_1,\dots,A_n\} \in \mathbb{I}^{nq\times nq}, \tag{9}$$

$$\Lambda \triangleq \mathrm{diag}\{\Lambda_1,\dots,\Lambda_n\} \in \mathbb{I}^{nq\times nq}, \tag{10}$$

$$W \triangleq \begin{bmatrix} W_{1,1} & \dots & W_{1,n} \\ W_{2,1} & \dots & W_{2,n} \\ \vdots & \vdots & \vdots \\ W_{n,1} & \dots & W_{n,n} \end{bmatrix} \in \mathbb{I}^{nq\times nq}, \tag{11}$$

$$V_i(\mathbf{x}_i) \triangleq [V_{i,1}(\mathbf{x}_i), \dots V_{i,m}(\mathbf{x}_i)] \in \mathbb{I}^{q\times mq}, \tag{12}$$

$$V(\mathbf{x}) \triangleq \text{diag}\{V_1(\mathbf{x}_1), \dots, V_n(\mathbf{x}_n)\} \in \mathbb{I}^{nq\times nmq}. \tag{13}$$

Assumption 1 (upper and lower acceptance parameter)**.** *We introduce the upper-acceptance vector $\overline{\boldsymbol{\phi}}_i \in \mathbb{I}^q$ and the lower-acceptance vector $\underline{\boldsymbol{\phi}}_i \in \mathbb{I}^q$. The bound for $v^r_{i,k}(\mathbf{x}_i)$ is expressed as follows:*

$$(1-\boldsymbol{\alpha}_i(r))\underline{\boldsymbol{\phi}}_i(r) < v^r_{i,k}(\mathbf{x}_i) < (1-\boldsymbol{\alpha}_i(r))\overline{\boldsymbol{\phi}}_i(r)$$

Assumption 2. *Matrix $(I-\Lambda)W$ is a stochastic matrix.*

Assumption 3. *Matrix $A(\Lambda V(\mathbf{x})+I)$ is row-sub-stochastic matrix for any $\mathbf{x} \in \mathbb{I}^{nq}$.*

Assumption 4. *For any $\mathbf{x}$ and $\mathbf{z} \in \mathbb{I}^q$ weight function $v^r_{i,k}(\cdot)$ satisfies*

$$|v^r_{i,k}(\mathbf{x}) - v^r_{i,k}(\mathbf{z})| \le \boldsymbol{\mu}_{i,k}(r)||\mathbf{x}-\mathbf{z}||$$

for $\forall i \in \mathbb{V}$, $\forall k \in \mathbb{U}$, $\forall r \le q$ and some constant $\boldsymbol{\mu}_{i,k} \in \mathbb{R}^{+q}$.

Assumption 5. *The following condition holds:*

$$\max_{j\in\mathbb{V}} \max_{a\in\mathbb{Q}} \left\{ \sum_{i\in\mathbb{V}} \sum_{r\in\mathbb{Q}} \boldsymbol{\alpha}_i(r)(1-\boldsymbol{\lambda}_i(r))W_{i,j}(r,a) \right\} + \max_{i\in\mathbb{V}} \left\{ \sum_{k\in Q_i} \sum_{r=1}^{q} \boldsymbol{\alpha}_i(r)\boldsymbol{\lambda}_i(r)\boldsymbol{\mu}_{i,k}(r)|\boldsymbol{\delta}_{i,k}(r)| \right\} < 1.$$

Remark 3. *Using $\boldsymbol{\delta}_{i,k} = \mathbf{y}_k - \mathbf{s}_i$, we rewrite equation* (1) *as follows:*

$$\mathbf{x}_i[t+1] = (I - A_i - A_i\Lambda_i \sum_{k\in Q_i} V_{i,k}(\mathbf{x}_i))\mathbf{s}_i + A_i(I-\Lambda_i)\sum_{j\in N_i} W_{i,j}\mathbf{x}_j[t] + A_i\Lambda_i \sum_{k\in Q_i} V_{i,k}(\mathbf{x}_i)\mathbf{y}_k. \tag{14}$$

As evident in equation (14)*, given that the matrices $V_{i,k}(\mathbf{x}_i)$ are diagonal, an increase in the value of $v^r_{i,k}(\mathbf{x}_i)$ for the r-th component of the opinion vector $\mathbf{x}_i(r)$ implies a decrease in the influence of $\mathbf{s}_i(r)$ on the dynamics. Accordingly, Assumption 1 employs the concepts of upper-acceptance and lower-acceptance vectors to establish a boundary for $v^r_{i,k}(\mathbf{x}_i)$.*

Remark 4. *As evident from Assumption 1, if $\boldsymbol{\alpha}_i(r) = 1$, then $v^r_{i,k}(\mathbf{x}_i) = 0$. This implies that if Individual v_i does not consider its innate opinion, the information sources do not have an effect on its expressed opinion. Therefore, in this paper, when $\boldsymbol{\alpha}_i(r) = 1$, we consider $\boldsymbol{\lambda}_i(r) = 0$.*

Remark 5. *Due to the fact that $\mathbf{x}_i$, $\mathbf{s}_i$, and $\mathbf{y}_k$ are in $\mathbb{I}^q$,* (1) *and* (4) *represent averaging equations to ensure that $\mathbf{x}_i[t+1]$ remains in $\mathbb{I}^q$. This is guaranteed by Assumption 2 and 3.*

Remark 6. *The Lipschitz condition, as expressed in Assumption 4, establishes that the value of $v^r_{i,k}(\mathbf{x}_i)$ is amenable to control through a linear weight function. Consequently, certain nonlinear weight functions governed by Assumption 4 can be approximated as linear weight functions. As seen in works such as [10], [11], in control theory, Assumption 4 is a common technical requirement used for the stabilization of nonlinear systems.*

In the following example, we illustrate the significance of multidimensional modeling, where a multidimensional modeling approach yields significantly divergent outcomes compared to one-dimensional modeling.

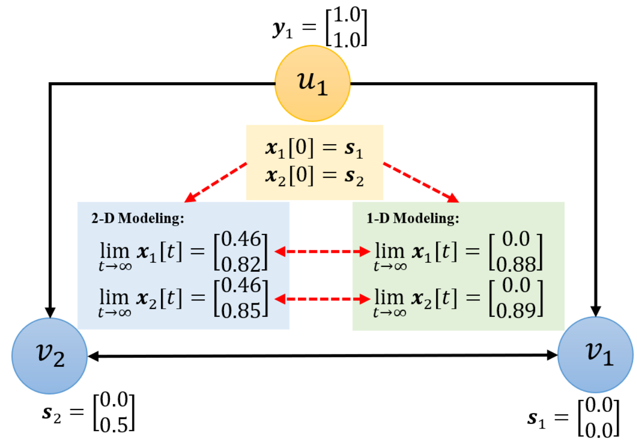

Fig. 1: A network with two individuals (v_1 and v_1) and one information source u_1. The equilibrium point of the system is obtained under two conditions: firstly, with two separate 1-D models for each issue. second: with one two-dimensional model. This setup allows us to analyze the dynamics of the system and observe how the equilibrium point shifts in response to different modeling approaches.

Example 1. *In this example, a network structure with two individuals v_1 and v_2, and an information source u_1, is employed. The opinion vector consists of two components, where the first component is called Issue 1, and the second component corresponds to Issue 2. The network structure is illustrated in Figure 1. We assume the weight influence matrices $W_{i,j}$ $(i, j \in \{1,\ 2\})$ to be diagonal as follows:*

$$W_{1,1} = W_{1,2} = W_{2,1} = W_{2,2} = \begin{bmatrix} 2.5 & 0.0 \\ 0.0 & 2.5 \end{bmatrix}.$$

The diagonal weight matrices $W_{i,j}$ $(i, j \in \{1,\ 2\})$ mean that when they discuss Issue 1 with each other, their beliefs about Issue 2 have no impact on Issue 1, and vice versa. This property also holds for the self-loops $W_{1,1}$ and $W_{2,2}$, which indicates that within themselves, v_1 and v_2 do not have interconnections for Issue 1 and Issue 2.

Social and information influence matrices are given by:

$$A_1 = A_2 = \Lambda_1 = \Lambda_2 = \text{diag}\{0.8, 0.8\},$$

A relatively large value of the social and information influence parameters indicates that both individuals possess a strong inclination toward the information source. Innate opinions of individuals and opinions of information sources are given by:

$$\mathbf{s}_1 = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}, \quad \mathbf{s}_2 = \begin{bmatrix} 0.0 \\ 0.5 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}.$$

Additionally, $v^r_{i,1}$, $i \in \{1, 2\}$ and $r \in \{1, 2\}$ are defined as:

$$v^r_{i,1} = g^r_{i,1}(\boldsymbol{\beta}_{i,1}(r)) = 0.3(1.0 - \boldsymbol{\beta}_{i,1}(r)),$$

where $\boldsymbol{\beta}_{2,1}(\mathbf{x}_i)$ is defined in (2)*. We use two different values for $C_{i,k} \forall i \in \{1, 2\}$ and $k = 1$ in* (2) *. In the first and second scenarios, $C_{i,k}$ is represented as C_1 and C_2 respectively ($\forall i \in \{1, 2\}$ and $k = 1$), and is explained as follows:*

$$C_1 = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}.$$

These two matrices appear quite similar. If we regard C_2 as the actual system matrix and C_1 as an approximated version of C_2, it might appear that the influences of non-diagonal factors have been neglected, and the resultant matrix has been normalized. The simulation results of the dynamics in the first and second scenarios are depicted in Figure 2. In the following, we discuss how a small change in $C_{i,k}$ has resulted in such a substantial difference in the outcomes.

The first scenario leads to two decoupled dynamical systems for each issue. For Issue 1, the dynamics are as follows:

$$\begin{cases} \mathbf{x}_1(1)[t+1] & = 0.2\mathbf{s}_1(1) + 0.4(\mathbf{x}_1(1)[t] + \mathbf{x}_2(1)[t]) \\ & +0.3(1 - |\mathbf{x}_1(1)[t] - \mathbf{y}_1(1)|)\boldsymbol{\delta}_{1,1}(1) \\ \\ \mathbf{x}_2(1)[t+1] & = 0.2\mathbf{s}_2(1) + 0.4(\mathbf{x}_1(1)[t] + \mathbf{x}_2(1)[t]) \\ & +0.3(1 - |\mathbf{x}_2(1)[t] - \mathbf{y}_2(1)|)\boldsymbol{\delta}_{2,1}(1) \end{cases}$$

Similarly, for Issue 2, the dynamics are as follows:

$$\begin{cases} \mathbf{x}_1(2)[t+1] & = 0.2\mathbf{s}_1(2) + 0.4(\mathbf{x}_1(2)[t] + \mathbf{x}_2(2)[t]) \\ & +0.3(1 - |\mathbf{x}_1(2)[t] - \mathbf{y}_1(2)|)\boldsymbol{\delta}_{1,1}(2) \\ \\ \mathbf{x}_2(2)[t+1] & = 0.2\mathbf{s}_2(2) + 0.4(\mathbf{x}_1(2)[t] + \mathbf{x}_2(2)[t]) \\ & +0.3(1 - |\mathbf{x}_2(2)[t] - \mathbf{y}_2(2)|)\boldsymbol{\delta}_{2,1}(2) \end{cases}$$

In the first scenario, where the initial conditions for simulating the dynamical systems are set to the innate opinions, $\mathbf{s}_1$ and $\mathbf{s}_2$, two notable observations emerge. For Issue 1, due to disagreement between individuals and the information source, the confirmation bias terms $v^1_{i,1},\ i \in \{1, 2\}$ become zero, and the difference equations associated with Issue 1 remain zero. Conversely, Issue 2 displays distinct dynamics. Here, the confirmation bias term $v^1_{2,1}$ is nonzero, prompting the individual to align with the information source, driven by a significant information influence parameter. This leads the belief of v_2 to converge with the information source's viewpoint. Even though the initial confirmation bias $v^1_{1,1}$ is zero at the initial condition, the impact of v_2 induces a gradual change in v_1's belief. Given v_1's inclination toward the information source due to a substantial information influence parameter, its belief progressively converges with that of the information source. Figure 2 confirms the analyses.

In contrast to the first scenario, the second scenario can no longer be modeled using separate sets of differential equations for each issue. Here, a coupled dynamical system for the entire network is derived as follows:

$$\begin{cases} \mathbf{x}_1(1)[t+1] & = 0.2\mathbf{s}_1(1) + 0.4(\mathbf{x}_1(1)[t] + \mathbf{x}_2(1)[t]) \\ & +0.3(1 - 0.95|\mathbf{x}_1(1)[t] - \mathbf{y}_1(1)| \\ & -0.05|\mathbf{x}_1(2)[t] - \mathbf{y}_1(2)|)\boldsymbol{\delta}_{1,1}(1) \\ \\ \mathbf{x}_1(2)[t+1] & = 0.2\mathbf{s}_1(2) + 0.4(\mathbf{x}_1(2)[t] + \mathbf{x}_2(2)[t]) \\ & +0.3(1 - 0.05|\mathbf{x}_1(1)[t] - \mathbf{y}_1(1)| \\ & -0.95|\mathbf{x}_1(2)[t] - \mathbf{y}_1(2)|)\boldsymbol{\delta}_{1,1}(2) \\ \\ \mathbf{x}_2(1)[t+1] & = 0.2\mathbf{s}_2(1) + 0.4(\mathbf{x}_1(1)[t] + \mathbf{x}_2(1)[t]) \\ & +0.3(1 - 0.95|\mathbf{x}_2(1)[t] - \mathbf{y}_1(1)| \\ & -0.05|\mathbf{x}_2(2)[t] - \mathbf{y}_1(2)|)\boldsymbol{\delta}_{2,1}(1) \\ \\ \mathbf{x}_2(2)[t+1] & = 0.2\mathbf{s}_2(2) + 0.4(\mathbf{x}_1(2)[t] + \mathbf{x}_2(2)[t]) \\ & +0.3(1 - 0.05|\mathbf{x}_2(1)[t] - \mathbf{y}_1(1)| \\ & -0.95|\mathbf{x}_2(2)[t] - \mathbf{y}_1(2)|)\boldsymbol{\delta}_{2,1}(2) \end{cases}$$

The results depicted in Figure 2 demonstrate a very strong alignment between the two scenarios in Issue 2; however, there are stark differences in the outcomes for Issue 1. This suggests that the nature of this dynamic is multi-dimensional, and two separate one-dimensional approximations cannot capture all the network's characteristics.

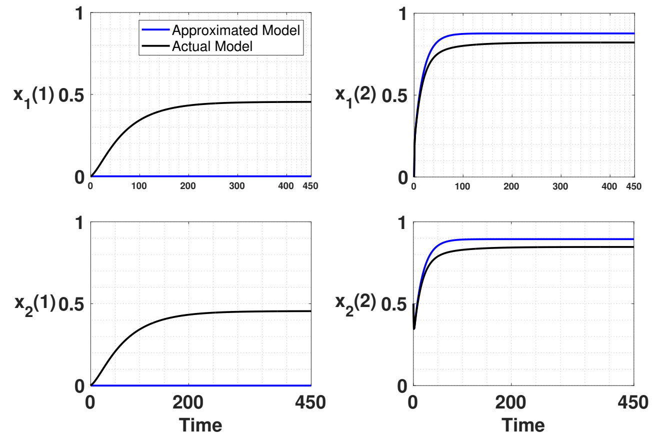

Fig. 2: The simulation results are generated based on two distinct scenarios. In the first scenario, the system is modeled using two separate one-dimensional models for each issue. In the second scenario, the system is represented by a single two-dimensional model. This setup lets us observe how Issue 1 varies in response to different modeling methodologies.

III. Convergence Analysis

In this section, we analyze the convergence behavior and equilibrium points of (1) and (4). In the following theorem, whose proof is presented in Appendix I, we establish the existence of a unique equilibrium point.

Theorem 1. *The dynamics described by* (1) *(and* (4)*) converge to a unique equilibrium point $\mathbf{x}^e$ that satisfy:*

$$\mathbf{x}^e = (I - A)\mathbf{s} + A(I - \Lambda)W\mathbf{x}^e + A\Lambda V(\mathbf{x}^e)\boldsymbol{\delta}$$

regardless of the initial conditions.

In the following example, we observe how the model converges to a unique equilibrium point regardless of the initial conditions.

Example 2. *This example illustrates a network with three individuals, each expressing opinions on two interconnected issues. The opinion of each individual is represented by a vector with two components, the first corresponding to Issue 1 and the second to Issue 2. The network structure is illustrated in Figure 3. The weight influence matrices $W_{i,j}$ ($i, j \in \{1,\ 2,\ 3\}$) are defined as:*

$$W_{1,1} = \begin{bmatrix} 0.6 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}, \quad W_{1,2} = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix},$$
$$W_{1,3} = \begin{bmatrix} 0.1 & 0.05 \\ 0.05 & 0.1 \end{bmatrix}, \ W_{2,1} = W_{2,2} = \begin{bmatrix} 2.0 & 0.5 \\ 0.2 & 0.425 \end{bmatrix},$$
$$W_{2,3} = W_{3,1} = \mathbf{0} \qquad W_{3,2} = \begin{bmatrix} 0.7 & 0.1 \\ 0.0 & 0.0 \end{bmatrix},$$
$$W_{3,3} = \begin{bmatrix} 0.55 & 0.0 \\ 0.625 & 0.625 \end{bmatrix}.$$

Social and information influence matrices are given by:

$$A_1 = \Lambda_1 = \Lambda_3 = \text{diag}\{0.2, 0.2\},$$
$$A_2 = \Lambda_2 = \text{diag}\{0.8, 0.2\}, \quad A_3 = \text{diag}\{0.8, 0.8\}.$$

Innate opinions of individuals and opinions of information sources are given by:

$$\mathbf{s}_1 = \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix}, \quad \mathbf{s}_2 = \mathbf{s}_3 = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}.$$

Additionally, $v_{i,k}^r(\mathbf{x}_i)$ *are defined as:*

$$v_{i,1}^r = 0; i \in \{1,\ 3\} \text{ and } r \in \{1,\ 2\}$$
$$v_{2,1}^r = g_{2,1}^r(\boldsymbol{\beta}_{2,1}(r)) = 0.5(1 - \boldsymbol{\beta}_{2,1}(r)); r \in \{1,\ 2\}.$$

Where $\boldsymbol{\beta}_{2,1}(\mathbf{x}_i)$ *is defined as:*

$$\boldsymbol{\beta}_{2,1}(\mathbf{x}_i) = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} |\mathbf{x}_2(1) - \mathbf{y}_1(1)| \\ |\mathbf{x}_2(2) - \mathbf{y}_1(2)| \end{bmatrix}.$$

The network parameters and matrices are defined in a manner that complies with Theorem 1. Figure 4 depicts the evolution of the opinions held by all individuals, starting from random initial conditions, across 500 simulations. The simulations demonstrate that the dynamics converge to a unique equilibrium point regardless of the initial conditions.

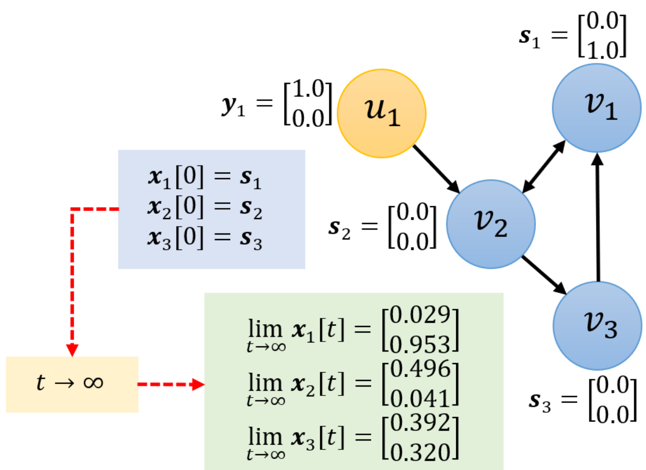

Fig. 3: A social network with three individuals and one information source.

IV. Equilibrium Analysis

In this section, we address equilibrium point characteristics in dynamical system (1). Initially, within the framework of Corollary 1 (whose proof is presented in Appendix II), we demonstrate the existence of an equilibrium point in the general state described by equation (1). However, its computation leads to an explosive search in the space $\mathbb{I}^q$. These problems are effectively addressed using constrained optimization techniques. Subsequently, in Section A, we demonstrate that by utilizing affine functions for $v_{i,k}^r(\mathbf{x}_i)$s, the equilibrium point can be determined through a search in a finite space. In subsection B we show that under conditions where the network structure is specified and $v_{i,k}^r(\mathbf{x}_i)$s are bounded, the equilibrium point of the dynamic system (5) is approximated, and under conditions stated in Section C, the system admits a closed-form solution.

Corollary 1. *The network system* (1) *has a unique equilibrium point that satisfies the nonlinear equation* (15)*:*

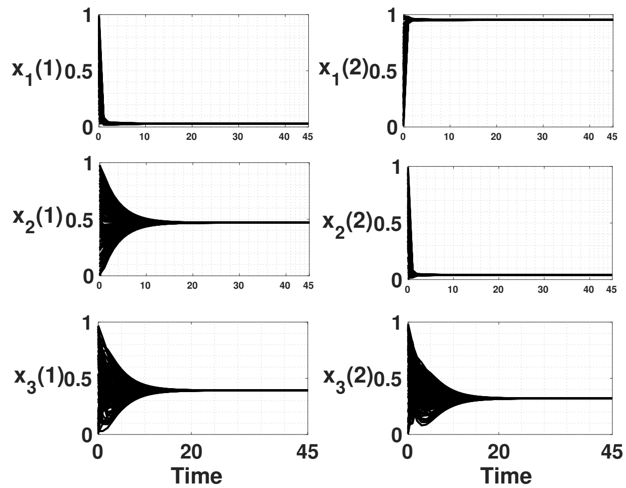

Fig. 4: Evolution of opinion of individuals in the social network, showing convergence to a single equilibrium point over 500 simulations.

$$\mathbf{x}^e = (I - A(I - \Lambda)W)^{-1}((I - A)\mathbf{s} + A\Lambda V(\mathbf{x}^e)\delta). \quad (15)$$

Remark 7. *Assumptions 2 and 5 imply that* $A(I - \Lambda)W$ *is a sub-row-column-stochastic matrix.*

Due to the dependence of V on $\mathbf{x}^e$, obtaining $\mathbf{x}^e$ via (15) frequently entails solving a constrained optimization problem.

A. Equilibrium in Presence of Affine Weight Functions

Our attention is directed towards affine weight functions represented as:

$$g_{i,k}^r = \boldsymbol{\omega}_i(r) - \boldsymbol{\gamma}_i(r)\boldsymbol{\beta}_{i,k}(\mathbf{x}_i)(r), \quad (16)$$

where, $\boldsymbol{\omega}_i \in \mathbb{I}^q$ and $\boldsymbol{\gamma}_i \in \mathbb{I}^q$ are non-negative vectors.

Remark 8. *By making reference to Remark 2, we infer that* $0 \leq g_{i,k}^r$*. Additionally, from its definition which is utilized in the proof of Theorem 1, we deduce that it is a decreasing function. Therefore, inequalities* $0 \leq \boldsymbol{\gamma}_i \leq \boldsymbol{\omega}_i$ *and* $\mathbf{0} \leq C_{i,k}$ *for all* $i,\ j \in \mathbb{V}$ *and* $k \in \mathbb{U}$ *are satisfied.*

Remark 9. *In* (16)*, the multiplication of* $\boldsymbol{\gamma}_i(r)$ *by* $\boldsymbol{\beta}_{i,k}(\mathbf{x}_i)(r)$ *introduces flexibility in the choice of* $\boldsymbol{\gamma}_i(r)$*. This flexibility is subject to the condition that the inequalities in Remark 8 remain valid. Consequently, it is reasonable to assume that* $C_{i,k}$ *is a stochastic matrix, without loss of generality.*

If $v_{i,k}^r(\mathbf{x}_i)$ takes the form of affine functions as described in (16), the dynamical system described in equation (4) is transformed into the subsequent equation:

$$\mathbf{x}[t+1] = A_a\mathbf{s} + V_a(\mathbf{x})\mathbf{y} + W_a(\mathbf{x})\mathbf{x}[t], \quad (17)$$

Where A_a, $V_a(\mathbf{x})$ and $W_a(\mathbf{x})$ defined in (37) to (39). The following theorem states that computing equilibrium point $\mathbf{x}^e$ in a known network structure in conjunction with affine weight functions, involves searching through a finite set. The proof is presented in Appendix III.

Theorem 2. *If* $v_{i,k}^r(\mathbf{x}_i)$ *takes form* (16)*, Algorithm 1 converges to the equilibrium point in a finite number of steps.*

Algorithm 1 Computation of Equilibrium Point

Input: The ordered set $\mathbb{P}$ encompasses all possible values of $\boldsymbol{\theta_{ik}}$ as defined in (35). For each $\boldsymbol{\theta_{ik}} \in \mathbb{P}$, there exist corresponding values for $\underline{M}$ and $\overline{M}$ as given in (44) and (50), respectively, initial index $p = 1$.
Output: Equilibrium point $\mathbf{x}^{\mathrm{e}}$.

- **while** $p \leq \|\mathbb{P}\|$ **do**
  - Compute: $\underline{M}$, $\overline{M}$ from (44) and (50), and corresponding values W_a and V_a.
  - Compute: $\mathbf{x}^{\mathrm{e}}{}_{\text{new}} \leftarrow (I - W_a)^{-1}(A_a\mathbf{s} + V_a\mathbf{y})$.
  - **if** b_{ik} sign$(\mathbf{x}_{i}{}^{\mathrm{e}}_{\text{new}} - \mathbf{y}_k) = \boldsymbol{\theta_{ik}}$, for $\forall i \in \mathbb{V}$, $\forall k \in \mathbb{U}$ **then**
    - Output equilibrium point: $\mathbf{x}^{\mathrm{e}} \leftarrow \mathbf{x}^{\mathrm{e}}{}_{\text{new}}$.
    - **Break**.
  - **else**
    - Update index: $p \leftarrow p + 1$.

B. Equilibrium Point Approximation

According to Assumption 1, $v^r_{i,k}(\mathbf{x}_i)$ is bounded. In the subsequent theorem, we demonstrate that it is possible to determine a bound for $\mathbf{x}^{\mathrm{e}}$. The proof is presented in Appendix IV.

Theorem 3. *Dynamics* (4) *converge to a unique equilibrium point* $\mathbf{x}^e$ *that satisfies inequality* (18)*,*

$$\mathbf{x}^{\mathrm{e}}_{\mathrm{L}} \leq \mathbf{x}^{e} \leq \mathbf{x}^{\mathrm{e}}_{\mathrm{U}}. \tag{18}$$

where:

$$\mathbf{x}^{\mathrm{e}}_{\mathrm{L}} \triangleq \max\big(Z\big(\mathbf{s} + A\Lambda\big(\underline{\Phi}B\mathbf{y} - \overline{\Phi}SB\mathbf{1}_{mq}\big)\big), \mathbf{0}_{nq}\big), \tag{19}$$

$$\mathbf{x}^{\mathrm{e}}_{\mathrm{U}} \triangleq \min\big(Z\big(\mathbf{s} + A\Lambda\big(\overline{\Phi}B\mathbf{y} - \underline{\Phi}SB\mathbf{1}_{mq}\big)\big), \mathbf{1}_{nq}\big), \tag{20}$$

$$\underline{\Phi} \triangleq \text{diag}\left\{\left[\underline{\boldsymbol{\phi}}_1^\top,\ \underline{\boldsymbol{\phi}}_2^\top, \ldots,\ \underline{\boldsymbol{\phi}}_n^\top\right]^\top\right\} \in \mathbb{R}^{nq\times nq}, \tag{21}$$

$$\overline{\Phi} \triangleq \text{diag}\left\{\left[\overline{\boldsymbol{\phi}}_1^\top,\ \overline{\boldsymbol{\phi}}_2^\top, \ldots,\ \overline{\boldsymbol{\phi}}_n^\top\right]^\top\right\} \in \mathbb{R}^{nq\times nq}, \tag{22}$$

$$Z \triangleq (I - A(I-\Lambda)W)^{-1}(I - A) \in \mathbb{R}^{nq\times nq}, \tag{23}$$

$$S \triangleq \text{diag}\{\mathbf{s}\} \in \mathbb{R}^{nq\times nq}. \tag{24}$$

Example 3. *In Example 2 if we replace $v^1_{2,1}$ and $v^2_{2,1}$ as follows:*

$$v^1_{2,1} = 0.75ln(2 - 0.7|x^1_2 - \mathbf{y}_1| - 0.3|x^2_2 - \mathbf{y}_2|)$$

$$v^2_{2,1} = 0.3(1 - sin(0.3|x^1_2 - \mathbf{y}_1| + 0.7|x^2_2 - \mathbf{y}_2|))$$

the upper bounds for $v^1_{2,1}$ and $v^2_{2,1}$ are straightforward to compute:

$$0 \leq v^1_{2,1} \leq (1 - \boldsymbol{\alpha}_2(1))1.04,$$

$$0.052 \leq v^2_{2,1} \leq (1 - \boldsymbol{\alpha}_2(2))0.33.$$

In Figure 5 we plot the equilibrium point, as well as its corresponding lower and upper bounds computed using (18)*.*

As depicted in Figure 5, minimal disparities are observed between the upper and lower bands at $\mathbf{x}_1(1)$, $\mathbf{x}_1(2)$, and $\mathbf{x}_2(2)$.

Remark 10. *If (18) is satisfied with equality for the $(iq + r - q)$-th entry, i.e, $\mathbf{x}^{\mathrm{e}}_{\mathrm{L}}(iq + r - q) = \mathbf{x}^{\mathrm{e}}_{\mathrm{U}}(iq + r - q)$, then the estimation of r-th component of individual $\mathrm{v_i}$'s opinion at steady state is exact:*

$$\mathbf{x}^e_i(r) = \mathbf{x}^{\mathrm{e}}_{\mathrm{L}}(iq + r - q) = \mathbf{x}^{\mathrm{e}}_{\mathrm{U}}(iq + r - q).$$

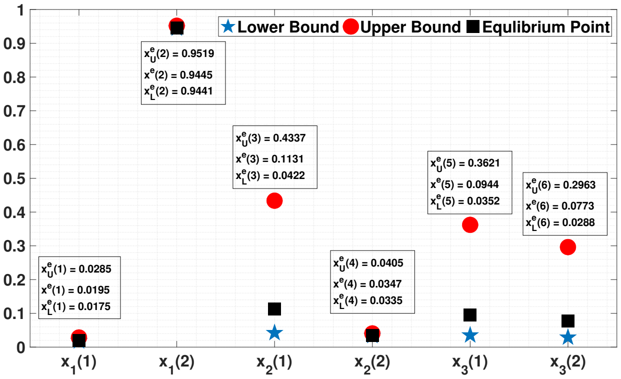

Fig. 5: This plot shows the opinion values of three individuals on two issues, along with upper and lower bounds, and an equilibrium point. The x-axis is labeled with each issue of each individual, while the y-axis represents the value of opinion.

C. Close-form Solutions in Presence of Affine Weight Functions

In the preceding subsection, we discussed that finding the equilibrium point involves a finite search among the at most $3^{(nmq)^2}$ possible cases. In the following corollary, we introduce a case in which there is a close-form solution. The proof is presented in Appendix V.

Corollary 2. *Given that $g^r_{i,k}$ is affine as described in* (16) *for all $i \in \mathbb{V}$, $k \in \mathbb{U}$ and $r \leq q$, if the following condition is met for $a = iq + r - q$:*

$$\mathbf{y}(a) > \mathbf{x}^{\mathrm{e}}_{\mathrm{U}}(a) \quad or \quad \mathbf{y}(a) < \mathbf{x}^{\mathrm{e}}_{\mathrm{L}}(a) \tag{25}$$

the equilibrium point of (1) *has a closed-form solution that satisfies:*

$$\mathbf{x}^e = (I - W_a)^{-1}(A_a\mathbf{s} + V_a\mathbf{y}).$$

Example 4. *In Example 2 we used $v^1_{2,1}$ and $v^2_{2,1}$ as follows:*

$$v^1_{2,1} = 0.5 - 0.5(0.8|\mathbf{x}_2(1) - \mathbf{y}_1(1)| + 0.2|\mathbf{x}_2(2) - \mathbf{y}_1(2)|),$$

$$v^2_{2,1} = 0.5 - 0.5(0.2|\mathbf{x}_2(1) - \mathbf{y}_1(1)| + 0.8|\mathbf{x}_2(2) - \mathbf{y}_1(2)|),$$

where

$$\boldsymbol{\omega}_1 = \boldsymbol{\gamma}_1 = \ [0.5,\ 0.5]^\top$$

The upper bounds for $v^1_{2,1}$ and $v^2_{2,1}$ are straightforward to derive:

$$0 \leq v^1_{2,1} \leq (1 - \boldsymbol{\alpha}_2(1)),\ and\ 0 \leq v^2_{2,1} \leq (1 - \boldsymbol{\alpha}_2(2))0.56.$$

The bounds of $\mathbf{x}_2$ calculated using (18)*, which gives:*

$$0.04 \leq \mathbf{x}_2(1) \leq 0.5,\ and\ 0.03 \leq \mathbf{x}_2(2) \leq 0.042$$

Using (56)*, we determine that $\boldsymbol{\theta_{ik}}(1) = -1$ and $\boldsymbol{\theta_{ik}}(2) = 1$, allowing us to simplify the system such that the cardinality of order set $\mathbb{P}$ in Algorithm 1 reduces to 1. As a result, we analytically obtain the equilibrium point using Theorem 2 as:*

$$\mathbf{x}^e = [0.029,\ 0.953,\ 0.496,\ 0.041,\ 0.392,\ 0.320]^\top.$$

V. Conclusion

In this paper, we presented a novel model of multidimensional opinion dynamics in social networks, with a special emphasis on confirmation bias. Along with model, we provided sufficient conditions for convergence of dynamics to a unique equilibrium point that is independent of initial conditions. We examined this equilibrium point for both linear and nonlinear confirmation bias functions: for nonlinear functions, we provided upper and lower bounds and for linear functions, we presented exact computation method of the aforementioned point, following the steps introduced in [8] for scalar systems.

Appendix I
Proof of Theorem 1

We reproduce the following well-known Banach fixed-point theorem that we use in proving Theorem 1.

Theorem 4 (Banach Fixed-Point Theorem [12])**.** *Let $(\mathcal{X}, d)$ be a complete metric space, and let $f : \mathcal{X} \to \mathcal{X}$ be a mapping. If there exists a constant $0 < c < 1$ such that for all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, we have*

$$d(f(\mathbf{x}), f(\mathbf{x}')) \leq c d(\mathbf{x}, \mathbf{y})$$

then f has a unique fixed point $\mathbf{x}^e$ in $\mathcal{X}$, i.e., there exists exactly one point $\mathbf{x}^e \in \mathcal{X}$ such that $f(\mathbf{x}^e) = \mathbf{x}^e$.

To prove Theorem 1, we consider the concatenated form (4) of the dynamics (1). We define the function

$$\mathbf{f}(\mathbf{x}) \triangleq (I - A)\mathbf{s} + A((I - \Lambda)W\mathbf{x} + \Lambda V(\mathbf{x})\boldsymbol{\delta}).$$

If $\mathbf{f}(\mathbf{x})$ satisfies Theorem 4 with l_1-norm, then the dynamical system defined by (4) has a unique equilibrium point. Thus, we relate norm of difference between $\mathbf{f}(\cdot)$ evaluated at $\mathbf{x}, \mathbf{x}' \in \mathbb{R}^{nd}$ as follows:

$$\begin{aligned} ||\mathbf{f}(\mathbf{x}') - \mathbf{f}(\mathbf{x})|| \leq\, & ||A(I - \Lambda)W(\mathbf{x}' - \mathbf{x})|| \\ & + ||A\Lambda\big(V(\mathbf{x}') - V(\mathbf{x})\big)\boldsymbol{\delta}||. \end{aligned} \tag{26}$$

suppose $\|\mathbf{x}' - \mathbf{x}\| \neq \mathbf{0}_q$:

$$\begin{aligned} & \|A(I - \Lambda)W\|\|\mathbf{x}' - \mathbf{x}\| \\ =& \{\sup \frac{||A(I - \Lambda)W\mathbf{y}||}{||\mathbf{y}||} : \mathbf{y} \in \mathbb{R}^q, \mathbf{y} \neq \mathbf{0}_q\}\|\mathbf{x}' - \mathbf{x}\| \\ \geq& \frac{||A(I - \Lambda)W(\mathbf{x}' - \mathbf{x}(t))||}{\|\mathbf{x}' - \mathbf{x}\|}\|\mathbf{x}' - \mathbf{x}\| \\ =& \|A(I - \Lambda)W(\mathbf{x}' - \mathbf{x})\| \end{aligned}$$

which indicates

$$\|A(I - \Lambda)W(\mathbf{x}' - \mathbf{x})\| \leq \|A(I - \Lambda)W\|\|\mathbf{x}' - \mathbf{x}\|. \tag{27}$$

It is evident that when $\|\mathbf{x}' - \mathbf{x}\| = \mathbf{0}_q$ the (27) still holds, which implies that (27) holds for all values of $\|\mathbf{x}' - \mathbf{x}\|$.

From (2) and recalling that $0 \leq |\boldsymbol{\delta}_{i,k}(r)| \leq 1$, we conclude that:

$$\begin{aligned} & ||A\Lambda\big(V(\mathbf{x}') - V(\mathbf{x})\big)\boldsymbol{\delta}|| \\ \leq & \sum_{i=1}^{n}\sum_{k\in Q_i}\sum_{r=1}^{q}\boldsymbol{\alpha}_i(r)\boldsymbol{\lambda}_i(r)|v^r_{i,k}(\mathbf{x}'_i(r)) - v^r_{i,k}(\mathbf{x}_i(r))||\boldsymbol{\delta}_{i,k}(r)|. \end{aligned} \tag{28}$$

By using Assumption 4 in right hand side of (28) we have:

$$\begin{aligned} & \sum_{i=1}^{n}\sum_{k\in Q_i}\sum_{r=1}^{q}\boldsymbol{\alpha}_i(r)\boldsymbol{\lambda}_i(r)|v^r_{i,k}(\mathbf{x}'_i(r)) - v^r_{i,k}(\mathbf{x}_i(r))||\boldsymbol{\delta}_{i,k}(r)| \\ \leq & \sum_{i=1}^{n}\sum_{k\in Q_i}\sum_{r=1}^{q}\boldsymbol{\alpha}_i(r)\boldsymbol{\lambda}_i(r)\boldsymbol{\mu}_{i,k}(r)|\mathbf{x}'_i(r) - \mathbf{x}_i(r)||\boldsymbol{\delta}_{i,k}(r)| \\ \leq & \|\mathbf{x}' - \mathbf{x}\|\max_{i\in\mathbb{V}}\left\{\sum_{k\in Q_i}\sum_{r=1}^{q}\boldsymbol{\alpha}_i(r)\boldsymbol{\lambda}_{i,k}(r)\boldsymbol{\mu}_{i,k}(r)|\boldsymbol{\delta}_{i,k}(r)|\right\} \end{aligned} \tag{29}$$

and hence,

$$\begin{aligned} & ||A\Lambda\big(V(\mathbf{x}') - V(\mathbf{x})\big)\boldsymbol{\delta}|| \\ \leq & \|\mathbf{x}' - \mathbf{x}\|\max_{i\in\mathbb{V}}\left\{\sum_{k\in Q_i}\sum_{r=1}^{q}\boldsymbol{\alpha}_i(r)\boldsymbol{\lambda}_i(r)\boldsymbol{\mu}_{i,k}(r)|\boldsymbol{\delta}_{i,k}(r)|\right\} \end{aligned} \tag{30}$$

Combining (26) with (27) and (30), we conclude that if Assumption 5 holds, $\mathbf{f}(\mathbf{x})$ satisfies Theorem 4. Hence, the (1) converges to a unique equilibrium point for any initial opinion $x(0) \in \mathbb{I}^q$.

Appendix II
Proof of Corollary 1

According to Theorem 1 and Assumption 4, an equilibrium point that satisfies Theorem 1 exists. We note that (15) can be derived directly from Theorem 1, provided that the matrix $(I - A(I - \Lambda)W)$ is non-singular. Based on Remark 7, $A(I - \Lambda)W$ is sub-row-stochastic, which implies that the spectral radius of this matrix, denoted by $\rho(A(I - \Lambda)W)$, is smaller than or equal to 1 (i.e., $\rho(A(I - \Lambda)W) \leq 1$).

We also deduce from Assumption 5 that the norm of $A(I-\Lambda)W$, denoted by $||A(I - \Lambda)W||$, is less than 1. It is well-known that the spectral radius of a matrix is always less than or equal to its norm, i.e., $\rho(A(I - \Lambda)W) \leq ||A(I - \Lambda)W||$. By combining these two inequalities, we obtain $\rho(A(I-\Lambda)W) \leq 1$ and $||A(I-\Lambda)W|| < 1$, which in turn implies that $\rho(A(I - \Lambda)W) < 1$.

This insight suggests that the matrix $A(I - \Lambda)W$ is stable. As a result, the matrix $(I - A(I - \Lambda)W)$ is non-singular.

Appendix III
Proof of Theorem 2

By substituting the affine weight function (16) into $V_{i,k}(\mathbf{x}_i)\boldsymbol{\delta}_{i,k}$ in (1), we obtain:

$$\begin{aligned} V_{i,k}(\mathbf{x}_i)\boldsymbol{\delta}_{i,k} =\, & (\Omega_i + \Gamma_i\Delta_{i,k}C_{i,k}\Theta_{i,k})\mathbf{y}_k - \Omega_i\mathbf{s}_i \\ & - \Gamma_i\Delta_{i,k}C_{i,k}\Theta_{i,k}\mathbf{x}_i, \end{aligned} \tag{31}$$

where

$$\Omega_i \triangleq \text{diag}\{\boldsymbol{\omega}_i\} \in \mathbb{R}^{q\times q}, \tag{32}$$

$$\Gamma_i \triangleq \text{diag}\{\boldsymbol{\gamma}_i\} \in \mathbb{R}^{q\times q}, \tag{33}$$

$$\Delta_{i,k} \triangleq \text{diag}\{\boldsymbol{\delta}_{i,k}\} \in \mathbb{R}^{q\times q}, \tag{34}$$

$$\boldsymbol{\theta_{i,k}} \triangleq \text{sign}(\mathbf{x}_i - \mathbf{y}_k), \tag{35}$$

$$\Theta_{i,k} \triangleq \text{diag}\{\boldsymbol{\theta_{i,k}}\}, \tag{36}$$

By substituting the (31) into (1), and expressing the obtained equation in matrix form (17), we have

$$A_a \triangleq I - A\big(I + \Lambda\Omega\overline{B}\big), \tag{37}$$

$$V_a(\mathbf{x}) \triangleq A\Lambda(\Omega B + M(\mathbf{x})), \tag{38}$$

$$W_a(\mathbf{x}) \triangleq A(I - \Lambda)W - A\Lambda\Gamma\overline{M}(\mathbf{x}), \tag{39}$$

$$\mathbf{y} \triangleq \left[\mathbf{y}_1^\top, \mathbf{y}_2^\top, \ldots, \mathbf{y}_m^\top\right]^\top \in \mathbb{I}^{mq}, \tag{40}$$

$$\Omega \triangleq \text{diag}\left\{\left[\boldsymbol{\omega}_1{}^\top, \boldsymbol{\omega}_2{}^\top, \ldots, \boldsymbol{\omega}_n{}^\top\right]^\top\right\}, \tag{41}$$

$$\Gamma \triangleq \text{diag}\left\{\left[\boldsymbol{\gamma}_1{}^\top, \boldsymbol{\gamma}_2{}^\top, \ldots, \boldsymbol{\gamma}_n{}^\top\right]^\top\right\}, \tag{42}$$

$$M_{i,k} \triangleq b_{i,k}\Delta_{i,k}C_{i,k}\Theta_{i,k},$$

$$M \triangleq \begin{bmatrix} M_{1,1} & \ldots & M_{1,m} \\ \vdots & \vdots & \vdots \\ M_{n,1} & \ldots & M_{n,m} \end{bmatrix}, \tag{44}$$

$$B_{i,k} \triangleq \text{diag}\{b_{i,k}\mathbf{1}_q\} \in \mathbb{R}^{q\times q}, \tag{45}$$

$$B \triangleq \begin{bmatrix} B_{1,1} & \dots & B_{1,m} \\ \vdots & \vdots & \vdots \\ B_{n,1} & \dots & B_{n,m} \end{bmatrix}, \tag{46}$$

$$B_i \triangleq \begin{bmatrix} B_{i,1} & \dots & B_{i,m} \end{bmatrix}, \tag{47}$$

$$\overline{B} \triangleq \text{diag}\{B_1, \dots, B_n\}, \tag{48}$$

$$M_i \triangleq \sum_{k \in \mathbb{U}} M_{i,k}, \tag{49}$$

$$\overline{M} \triangleq \text{diag}\{M_1, \dots, M_n\}, \tag{50}$$

Using convergence definition for (17), we obtain:

$$\mathbf{x}^e = (I - W_a(\mathbf{x}))^{-1}(A_a \mathbf{s} + V_a(\mathbf{x})\mathbf{y}). \tag{51}$$

In the context of linear weight functions, (16) introduces a dependence of $\boldsymbol{\theta}_{i,k}$ on $\mathbf{x}^e$ under steady-state conditions. Consequently, both M and $\overline{M}$ become functions of $\mathbf{x}^e$. It is important to note that $\boldsymbol{\theta}_{i,k}(r)$ takes one of three values: 1, 0, or -1. This limitation implies that the total number of potential choices for M and $\overline{M}$ at steady state is bounded by $3^{(nmq)^2}$ at most. We denote this set of all possible choices as $\mathbb{P}$. Considering the constraints imposed by (15), only one of these choices satisfies the constraints and corresponds to the equilibrium point. Therefore, the task of finding the equilibrium point involves a finite search among the aforementioned set of at most $3^{(nmq)^2}$ possible cases. Algorithm 1 exploits this property to obtain this equilibrium point using (51).

Appendix IV
Proof of Theorem 3

Theorem 3 is established through the derivation of bounds for $\mathbf{g} = V(\mathbf{x}^e)\boldsymbol{\delta}$ in (15). Element of $\mathbf{g}$ corresponds to the r-th component of the opinions of individual $\mathrm{v_i}$ obtained according to

$$\mathbf{g}(iq + r - q) = \sum_{k \in \mathbb{U}} v_{i,k}^r(\mathbf{x}_i^{\mathrm{e}})\mathbf{y}_k(r) - \sum_{k \in \mathbb{U}} v_{i,k}^r(\mathbf{x}_i^{\mathrm{e}})\mathbf{s}_i(r) \tag{52}$$

Using Assumption 1, we have:

$$\mathbf{g}(iq + r - q) \le (1 - \boldsymbol{\alpha}_i(r)) \left(\overline{\boldsymbol{\phi}}_i(r) \sum_{k \in \mathbb{U}} b_{i,k}\mathbf{y}_k(r) - \mathbf{s}_i(r)\underline{\boldsymbol{\phi}}_i(r) \sum_{k \in \mathbb{U}} b_{i,k} \right) \tag{53}$$

The right side of (53) corresponds to the $(iq + r - q)$-th entry of the matrix $\mathbf{h}$ derived from the expression:

$$\mathbf{h} = (I - A)\overline{\Phi}B\mathbf{y} - (I - A)\underline{\Phi}SB\mathbf{1}_{md}$$

Thus, we obtain:

$$V(\mathbf{x}^e)\boldsymbol{\delta} \le (I - A)\overline{\Phi}B\mathbf{y} - (I - A)\underline{\Phi}SB\mathbf{1}_{md} \tag{54}$$

Similarly, we derive:

$$V(\mathbf{x}^e)\boldsymbol{\delta} \ge (I - A)\underline{\Phi}B\mathbf{y} - (I - A)\overline{\Phi}SB\mathbf{1}_{md} \tag{55}$$

By employing (54) and (55) within (15) and considering that $\mathbf{0}_{nd} \le \mathbf{x}^e \le \mathbf{1}_{nd}$, we derive (19) and (20). The expressions for $\underline{\Phi}$, $\overline{\Phi}$, B, and S are provided by (21)-(24) respectively.

Appendix V
Proof of Corollary 2

As proposed in the study by [8], it is possible to reduce the number of available choices by incorporating Theorem 3 and (18) into the following form:

$$\boldsymbol{\theta_{i,k}}(r) = \begin{cases} -1, & \mathbf{x}_U^e(a) < \mathbf{y}(a) \\ 1, & \mathbf{x}_L^e(a) > \mathbf{y}(a) \\ -1, 0, & \mathbf{x}_U^e(a) = \mathbf{y}(a) \\ 1, 0, & \mathbf{x}_L^e(a) = \mathbf{y}(a) \\ -1, 1, 0, & \text{otherwise} \end{cases} \tag{56}$$

The conditions outlined in (25) align with the first and second conditions presented in (56). Consequently, only one possible value for $\boldsymbol{\theta_{i,k}}(r)$ exists, implying that $|\mathbb{P}| = 1$. This, in turn, signifies that Algorithm 1 converges within a single step. As a result, (51) stands as a close-form solution for (1).

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