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RESEARCH

Estimates of discrete time derivatives for the parabolic-parabolic Robin-Robin coupling method

Erik Burman[1] · Rebecca Durst[2,3] · Miguel A. Fernández[4,5] · Johnny Guzmán[6] · Sijing Liu[7]

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Abstract

We consider a loosely coupled, non-iterative Robin-Robin coupling method proposed and analyzed in Burman et al. (J. Numer. Math. **31**(1):59–77, 2023) for a parabolic-parabolic interface problem and prove estimates for the discrete time derivatives of the scalar field in different norms. When the interface is flat and perpendicular to two of the edges of the domain we prove error estimates in the H^2-norm. Such estimates are key ingredients to analyze a defect correction method for the parabolic-parabolic interface problem. Numerical results are shown to support our findings.

Keywords Parabolic-parabolic interface problem · Loosely coupled scheme · Robin conditions · Stability estimates

Mathematics Subject Classification (2010) 65M12

1 Introduction

Time splitting methods are popular for fluid-structure interaction (FSI) problems (see, e.g., [2, 4, 8–10, 17, 18, 25, 26, 29, 31])). One of the first stable splitting methods was proposed by Burman and Fernández [17]. Later the same authors [18] developed a related method which was coined the genuine Robin-Robin splitting method. Both these methods however suffered from a coupling between the space and time discretization parameters that reduced the accuracy. This constraint was lifted by eliminating the mesh dependence of the Robin parameter in [16]. The resulting method has been analyzed in [11, 13, 16]. A very similar method was developed and analyzed by Bukač and Seboldt [30]. In [13] it was proved for the FSI problems that the method converges as $\mathcal{O}(\sqrt{\Delta t})$ where Δt is the time step. Numerical evidence suggested that those estimates were not sharp, and nearly first-order accuracy was proved (mod possibly a logarithmic factor) in [11] for the analogue method applied to the parabolic-parabolic and hyperbolic-parabolic problems. The analysis was extended to the FSI problem in [14, 23]. For parabolic-parabolic couplings, there is a rich liter-

Extended author information available on the last page of the article ature on splitting schemes motivated by models of ocean-atmosphere interaction. In these models, friction forces on the interface render the physical coupling dissipative through a Robin-type coupling condition, as discussed in [27]. This aspect has been successfully exploited in the design of splitting methods [20–22, 28, 32–35]. In our case, the coupling conditions consist of continuity of both the primal variables and the fluxes across the interface. This coupling is conservative, and hence the approach suggested in the above references fails. Instead, the splitting method uses a Robin condition for the computational coupling, which turns out to lead to an unconditionally stable algorithm. An approach for conservative fluid-fluid coupling problems was proposed in [24], using Nitsche or Robin type couplings similar to those introduced in [18]. In a domain decomposition framework, a splitting method based on subcycling, i.e., iterative solution, was proposed and analyzed in [5, 6]. In a similar spirit, but focusing on a multi-timestep approach, a Robin-Robin coupling for time-dependent advection–diffusion was introduced in [19], with numerical investigation of the stability.

It is well known that discrete time differences for time stepping methods (e.g., backward Euler method) superconverge. In particular, when the backward Euler method is applied to a parabolic problem, the first-time difference of the errors converge with order $\mathcal{O}((\Delta t)^2)$. In this paper, we prove similar results for the splitting method [11] applied to an interface problem. In particular, we will prove second-order convergence for the scalar fields living on the two sub-domains in the L^2-norm. Moreover, in special configurations (i.e., the interface is horizontal and perpendicular to two sides of the domain), we will be able to prove second-order convergence in the H^2-norm. Numerically, the H^2 second-order convergence rates seem to hold on more general configurations. This appears to be the first work where error estimates in stronger norms for splitting methods applied to interface problems have been considered.

Error estimates of derivatives of the solution are, of course, of interest in their own right in many applications. However, our main motivation for this work is the application of these estimates in the analysis of a prediction-correction method [12] that we are concurrently developing. The aim is to improve the first-order convergence of [11] to second-order convergence in time through a defect-correction procedure [7]. The method in [12] uses a prediction step, which is exactly the splitting method we analyze here. The second-order accuracy of the correction step depends on the second-order accuracy of time differences of the prediction step, which is precisely the subject of this paper.

Our overall goal is to propose and analyze a second-order loosely coupled scheme for FSI problems. This paper represents the first step towards that goal. Based on our past experiences (see [11, 13, 15]), the analysis of FSI is very similar to those of hyperbolic-parabolic and parabolic-parabolic interface problems. Therefore, we begin by considering the simplest problem, namely, the parabolic-parabolic interface problem, to ensure that the proofs are more transparent while maintain some of the difficulties of FSI. In this paper, all the estimates, including the H^2 estimate, are crucial components in analyzing a second-order convergent correction method, which is included in [12]. We believe it is beneficial to fully understand the parabolic-parabolic interface problem first before progressing to the more complex hyperbolic-parabolic interface problem and FSI problems. Note that while the problem we are considering can be formulated using a unified approach with discontinuous coefficients, our focus is on developing a loosely coupled scheme, as we mentioned, which is why we prefer the partitioned setting.

As for a single parabolic problem, the idea to prove higher convergence for time differences is to use that the time differences satisfy a similar discrete equation with new right-hand sides that have time differences themselves. Then, one uses the error analysis for the original method to proceed. In our case, we will use the analysis provided in [11] to do this. The main difference here is that we consider a problem with Neumann boundary conditions on two of the sides of a square instead of pure Dirichlet boundary conditions, which were considered in [11]. This, in fact, simplifies the analysis slightly, and additionally, one can remove the logarithmic factor that appears in [11]. It should be mentioned that we only consider the time discrete case. The fully discrete case is more involved.

The rest of the paper is organized as follows. In Section 2, we introduce a parabolic-parabolic interface problem and the corresponding Robin-Robin coupling method. In Section 3, we present stability results for a Robin-Robin method. Section 4 is devoted to the error estimates. Finally, we provide some numerical results in Section 5 and end with some concluding remarks in Section 6.

2 The parabolic-parabolic interface problem and Robin-Robin coupling method

Let $\Omega = (0, 1)^2$ and suppose that $\Omega = \Omega_f \cup \Omega_s \cup \Sigma$. The interface Σ is assumed to be a line segment that intersects Ω on the two side edges; see Fig. 1. We let Γ_{Ne} denote the two side edges of Ω and we let Γ_D be the bottom and top edges of Ω. We let $\Gamma_{Ne}^i = \Gamma_{Ne} \cap \partial\Omega_i$ for $i = s, f$.

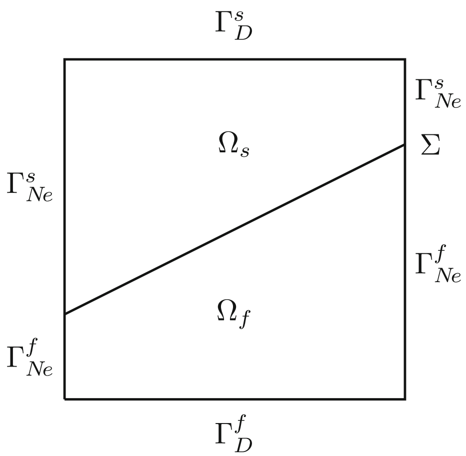

Fig. 1 The domains Ω_f and Ω_s with interface Σ and Neumann boundaries

2.1 The parabolic-parabolic problem

We consider the interface problem

$$\begin{aligned}
\partial_t \mathsf{u} - \nu_f \Delta \mathsf{u} =& 0, && \text{in}[0,T]\times\Omega_f, \\
\mathsf{u}(0,x) =& \mathsf{u}_0(x), && \text{in } \Omega_f, && \text{(2.1a)}\\
\mathsf{u} =& 0, && \text{on } [0,T]\times\Gamma_D^f, \\
\partial_{\boldsymbol{n}} \mathsf{u} =& 0, && \text{on } [0,T]\times\Gamma_{Ne}^f, \\
\partial_t \mathsf{w} - \nu_s \Delta \mathsf{w} =& 0, && \text{in } [0,T]\times\Omega_s, \\
\mathsf{w}(0,x) =& \mathsf{w}_0(x), && \text{in } \Omega_s, && \text{(2.1b)}\\
\mathsf{w} =& 0, && \text{on } [0,T]\times\Gamma_D^s, \\
\partial_{\boldsymbol{n}} \mathsf{w} =& 0, && \text{on } [0,T]\times\Gamma_{Ne}^s, \\
\mathsf{w} - \mathsf{u} =& 0, && \text{on } [0,T]\times\Sigma, && \text{(2.1c)}\\
\nu_s \nabla \mathsf{w}\cdot\boldsymbol{n}_s + \nu_f \nabla \mathsf{u}\cdot\boldsymbol{n}_f =& 0, && \text{on } [0,T]\times\Sigma, && \text{(2.1d)}
\end{aligned}$$

where $\boldsymbol{n}_f$ and $\boldsymbol{n}_s$ are the outward facing normal vectors for Ω_f and Ω_s, respectively. We assume that the initial data is smooth and that u and w are smooth on Ω_f and Ω_s, respectively.

2.2 Variational form

Let $(\cdot,\cdot)_i$ be the L^2-inner product on Ω_i for $i = f, s$. Moreover, let $\langle \cdot,\cdot \rangle$ be the L^2-inner product on Σ. Let $N > 0$ be an integer, and define $\Delta t := \frac{T}{N}$, and let $\mathsf{u}^n := \mathsf{u}(t_n,\cdot)$, where $t_n := n\Delta t$ for $n \in \{0,1,2,\ldots,N\}$. We consider the spaces

$$\begin{aligned}
V_f :=& \{v \in H^1(\Omega_f) : v = 0 \text{ on } \Gamma_D^f\}, && \text{(2.2a)}\\
V_s :=& \{v \in H^1(\Omega_s) : v = 0 \text{ on } \Gamma_D^s\}, && \text{(2.2b)}\\
V_g :=& L^2(\Sigma). && \text{(2.2c)}
\end{aligned}$$

By setting $\mathsf{l}^{n+1} := \nu_f \partial_{\boldsymbol{n}_f} \mathsf{u}^{n+1}$ and assuming that $\mathsf{l}^{n+1} \in L^2(\Sigma)$ for all n, the solution to (2.1) also satisfies the following variational formulation, for $n = 0,\ldots,N-1$:

$$\begin{aligned}
(\partial_t \mathsf{w}^{n+1}, z)_s + \nu_s(\nabla \mathsf{w}^{n+1}, \nabla z)_s + \langle \mathsf{l}^{n+1}, z\rangle =& 0, && z \in V_s, && \text{(2.3a)}\\
(\partial_t \mathsf{u}^{n+1}, v)_f + \nu_f(\nabla \mathsf{u}^{n+1}, \nabla v)_f - \langle \mathsf{l}^{n+1}, v\rangle =& 0, && v \in V_f, && \text{(2.3b)}\\
\langle \mathsf{w}^{n+1} - \mathsf{u}^{n+1}, \mu\rangle =& 0, && \mu \in V_g. && \text{(2.3c)}
\end{aligned}$$

2.3 Robin-Robin coupling: time discrete method

We define the discrete time derivatives:

$$\partial_{\Delta t} v^{n+1} = \frac{v^{n+1} - v^n}{\Delta t},$$

and

$$\partial^2_{\Delta t} v^{n+1} = \frac{v^{n+1} - 2v^n + v^{n-1}}{(\Delta t)^2}.$$

The Robin-Robin method solves sequentially:

$$\begin{aligned}
\partial_{\Delta t} w^{n+1} - \nu_s \Delta w^{n+1} &= 0, && \text{in } \Omega_s, && (2.4a)\\
w^{n+1} &= 0, && \text{on } \Gamma^s_D, && (2.4b)\\
\partial_{\boldsymbol{n}} w^{n+1} &= 0, && \text{on } \Gamma^s_{Ne}, && (2.4c)\\
\alpha w^{n+1} + \nu_s \partial_{\boldsymbol{n}_s} w^{n+1} &= \alpha u^n - \nu_f \partial_{\boldsymbol{n}_f} u^n && \text{on } \Sigma. && (2.4d)
\end{aligned}$$

$$\begin{aligned}
\partial_{\Delta t} u^{n+1} - \nu_f \Delta u^{n+1} &= 0, && \text{in } \Omega_f, && (2.5a)\\
u^{n+1} &= 0, && \text{on } \Gamma^f_D, && (2.5b)\\
\partial_{\boldsymbol{n}} u^{n+1} &= 0, && \text{on } \Gamma^f_{Ne}, && (2.5c)\\
\alpha u^{n+1} + \nu_f \partial_{\boldsymbol{n}_f} u^{n+1} &= \alpha w^{n+1} + \nu_f \partial_{\boldsymbol{n}_f} u^n && \text{on } \Sigma. && (2.5d)
\end{aligned}$$

We let $\lambda^{n+1} = \nu_f \partial_{\boldsymbol{n}_f} u^{n+1}$. Then, the time semi-discrete solution solves the following: Find $w^{n+1} \in V_s$, $u^{n+1} \in V_f$, and $\lambda^{n+1} \in V_g$ such that, for $n \geq 0$,

$$(\partial_{\Delta t} w^{n+1}, z)_s + \nu_s (\nabla w^{n+1}, \nabla z)_s + \alpha \langle w^{n+1} - u^n, z \rangle + \langle \lambda^n, z \rangle = 0, \qquad z \in V_s, \tag{2.6a}$$

$$(\partial_{\Delta t} u^{n+1}, v)_f + \nu_f (\nabla u^{n+1}, \nabla v)_f - \langle \lambda^{n+1}, v \rangle = 0, \qquad v \in V_f, \tag{2.6b}$$

$$\langle \alpha (u^{n+1} - w^{n+1}) + (\lambda^{n+1} - \lambda^n), \mu \rangle = 0, \qquad \mu \in V_g, \tag{2.6c}$$

with $u^0 = \mathsf{u}_0(x)$ and $w^0 = \mathsf{w}_0(x)$.

One can rewrite (2.6c) as

$$\alpha (u^{n+1} - w^{n+1}) = \lambda^n - \lambda^{n+1}, \quad \text{on } \Sigma. \tag{2.7}$$

3 Stability result

In this section we will prove stability results for the Robin-Robin method with a more general right-hand side: Find $w^{n+1} \in V_s$, $u^{n+1} \in V_f$, such that, for $n \geq 0$,

$$\partial_{\Delta t} w^{n+1} - \nu_s \Delta w^{n+1} = b_1^{n+1}, \quad \text{in } \Omega_s, \tag{3.1a}$$
$$w^{n+1} = 0, \quad \text{on } \Gamma_D^s, \tag{3.1b}$$
$$\partial_{\boldsymbol{n}} w^{n+1} = 0, \quad \text{on } \Gamma_{Ne}^s, \tag{3.1c}$$
$$\alpha w^{n+1} + \nu_s \partial_{\boldsymbol{n}_s} w^{n+1} = \alpha u^n - \nu_f \partial_{\boldsymbol{n}_f} u^n + \varepsilon_1^{n+1} \quad \text{on } \Sigma. \tag{3.1d}$$

$$\partial_{\Delta t} u^{n+1} - \nu_f \Delta u^{n+1} = b_2^{n+1}, \quad \text{in } \Omega_f, \tag{3.2a}$$
$$u^{n+1} = 0, \quad \text{on } \Gamma_D^f, \tag{3.2b}$$
$$\partial_{\boldsymbol{n}} u^{n+1} = 0, \quad \text{on } \Gamma_{Ne}^f, \tag{3.2c}$$
$$\alpha u^{n+1} + \nu_f \partial_{\boldsymbol{n}_f} u^{n+1} = \alpha w^{n+1} + \nu_f \partial_{\boldsymbol{n}_f} u^n + \varepsilon_2^{n+1} \quad \text{on } \Sigma, \tag{3.2d}$$

with
$$u^0 = w^0 = b_i^0 = \varepsilon_i^0 = 0 \quad i = 1, 2, \tag{3.3}$$

where b_1^{n+1}, b_2^{n+1}, ε_1^{n+1} and ε_2^{n+1} are general right-hand side terms that are sufficiently smooth in space and time. We assumed zero initial conditions for simplicity and this will be enough for the error analysis below. When analyzing the error of the Robin-Robin method the terms $\{b_i^n\}$, $\{\varepsilon_i^n\}$ will be the residual terms.

Remark 3.1 (Regularity) We assume that $u^{n+1} \in H^2(\Omega_f)$ and $w^{n+1} \in H^2(\Omega_s)$ for the remainder of this paper. The primary reason we assume this H^2 regularity and subsequently prove an H^2 estimate in Corollary 3.11 is that, in the analysis of the correction methods, we apply the trace inequality to the Lagrange multiplier on the interface Σ. Then the H^2 regularity assumption is useful in analyzing the resulting quantities. It would be interesting to remove this assumption, but we currently do not have the means to do so.

Remark 3.2 (Notations) For any negative superscripts, we set the values of the terms on the right-hand side to be zero, i.e.,

$$u^j = w^j = b_i^j = \varepsilon_i^j = 0 \quad j < 0, \; i = 1, 2. \tag{3.4}$$

We let $\lambda^{n+1} = \nu_f \partial_{\mathbf{n}_f} u^{n+1}$ and we see that the above solution satisfies, for $n \geq 0$,

$$(\partial_{\Delta t} w^{n+1}, z)_s + \nu_s (\nabla w^{n+1}, \nabla z)_s + \alpha \langle w^{n+1} - u^n, z \rangle + \langle \lambda^n, z \rangle = L_1(z), \qquad z \in V_s, \tag{3.5a}$$

$$(\partial_{\Delta t} u^{n+1}, v)_f + \nu_f (\nabla u^{n+1}, \nabla v)_f - \langle \lambda^{n+1}, v \rangle = L_2(v), \qquad v \in V_f, \tag{3.5b}$$

$$\langle \alpha (u^{n+1} - w^{n+1}) + (\lambda^{n+1} - \lambda^n), \mu \rangle = L_3(\mu), \qquad \mu \in V_g, \tag{3.5c}$$

where

$$\begin{aligned} L_1(z) &:= (b_1^{n+1}, z)_s + \langle \varepsilon_1^{n+1}, z \rangle, \\ L_2(v) &:= (b_2^{n+1}, v)_f, \\ L_3(\mu) &:= \langle \varepsilon_2^{n+1}, \mu \rangle. \end{aligned}$$

For convenience we rewrite (3.5c) as:

$$\alpha (u^{n+1} - w^{n+1}) = \lambda^n - \lambda^{n+1} + \varepsilon_2^{n+1}, \quad \text{on } \Sigma. \tag{3.6}$$

We will need to define the following quantities for the stability estimates.

$$\begin{aligned} Z^{n+1}(\psi, \phi, \theta) :=& \frac{1}{2} \|\phi^{n+1}\|^2_{L^2(\Omega_f)} + \frac{1}{2} \|\psi^{n+1}\|^2_{L^2(\Omega_s)} + \frac{\Delta t \alpha}{2} \|\phi^{n+1}\|^2_{L^2(\Sigma)} + \frac{\Delta t}{2\alpha} \|\theta^{n+1}\|^2_{L^2(\Sigma)}, \\ S^{n+1}(\psi, \phi, \theta) :=& \Delta t (\nu_f \|\nabla \phi^{n+1}\|^2_{L^2(\Omega_f)} + \nu_s \|\nabla \psi^{n+1}\|^2_{L^2(\Omega_s)}) + \frac{1}{2} (\|\psi^{n+1} - \psi^n\|^2_{L^2(\Omega_s)} \\ &+ \|\phi^{n+1} - \phi^n\|^2_{L^2(\Omega_f)}) \\ &+ \frac{\alpha \Delta t}{2} \|\phi^{n+1} - \phi^n + \frac{1}{\alpha} (\theta^{n+1} - \theta^n)\|^2_{L^2(\Sigma)}. \end{aligned}$$

We first state a preliminary result.

Lemma 3.3 *Let w, u solve (3.1) and (3.2) then the following identity holds.*

$$Z^{n+1}(w, u, \lambda) + S^{n+1}(w, u, \lambda) = Z^n(w, u, \lambda) + \Delta t F^{n+1}(w, u) + \frac{\Delta t}{\alpha} \langle \varepsilon_2^{n+1}, \lambda^{n+1} \rangle, \tag{3.7}$$

where

$$\begin{aligned} F^{n+1}(w, u) := & (b_1^{n+1}, w^{n+1})_s + (b_2^{n+1}, u^{n+1})_f + \langle \varepsilon_1^{n+1} + \varepsilon_2^{n+1}, w^{n+1} \rangle \\ &+ \langle u^{n+1} - u^n, \varepsilon_2^{n+1} \rangle. \end{aligned}$$

Proof To begin, we set $z = \Delta t\, w^{n+1}$ in (3.5a) and $v = \Delta t\, u^{n+1}$ in (3.5b) to get

$$\begin{aligned}
&\frac{1}{2}\|w^{n+1}\|^2_{L^2(\Omega_s)} + \frac{1}{2}\|u^{n+1}\|^2_{L^2(\Omega_f)} + \frac{1}{2}\|w^{n+1} - w^n\|^2_{L^2(\Omega_s)} + \frac{1}{2}\|u^{n+1} - u^n\|^2_{L^2(\Omega_f)} \\
&+ \nu_s \Delta t \|\nabla w^{n+1}\|^2_{L^2(\Omega_s)} + \nu_f \Delta t \|\nabla u^{n+1}\|^2_{L^2(\Omega_f)} \\
&= \frac{1}{2}\|w^n\|^2_{L^2(\Omega_s)} + \frac{1}{2}\|u^n\|^2_{L^2(\Omega_f)} + \Delta t \mathsf{J}^{n+1},
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\mathsf{J}^{n+1} :=& -\alpha\langle w^{n+1} - u^n, w^{n+1}\rangle - \langle \lambda^n, w^{n+1}\rangle \\
&+ \langle \lambda^{n+1}, u^{n+1}\rangle + L_1(w^{n+1}) + L_2(u^{n+1}).
\end{aligned} \tag{3.9}$$

Manipulating the first three terms in (3.9) and using (3.6), we obtain

$$\begin{aligned}
&-\alpha\langle w^{n+1} - u^n, w^{n+1}\rangle - \langle \lambda^n, w^{n+1}\rangle + \langle \lambda^{n+1}, u^{n+1}\rangle \\
&\qquad = \mathbb{J}^{n+1} + \frac{1}{\alpha}\langle \varepsilon_2^{n+1}, \lambda^{n+1}\rangle + \langle \varepsilon_2^{n+1}, w^{n+1}\rangle \\
&\qquad\quad - \langle u^n - u^{n+1}, \varepsilon_2^{n+1}\rangle,
\end{aligned} \tag{3.10}$$

with

$$\mathbb{J}^{n+1} := \alpha\langle u^n - u^{n+1}, u^{n+1}\rangle + \frac{1}{\alpha}\langle \lambda^n - \lambda^{n+1}, \lambda^{n+1}\rangle - \langle u^n - u^{n+1}, \lambda^n - \lambda^{n+1}\rangle.$$

One can easily show that

$$\begin{aligned}
\mathbb{J}^{n+1} =& \frac{\alpha}{2}(\|u^n\|^2_{L^2(\Sigma)} - \|u^{n+1}\|^2_{L^2(\Sigma)}) + \frac{1}{2\alpha}(\|\lambda^n\|^2_{L^2(\Sigma)} - \|\lambda^{n+1}\|^2_{L^2(\Sigma)}) \\
&- \frac{\alpha}{2}\|(u^n - u^{n+1}) + \frac{1}{\alpha}(\lambda^n - \lambda^{n+1})\|^2_{L^2(\Sigma)}.
\end{aligned}$$

By combining this identity with (3.9) and (3.10), we arrive at

$$\begin{aligned}
\mathsf{J}^{n+1} =& \frac{\alpha}{2}(\|u^n\|^2_{L^2(\Sigma)} - \|u^{n+1}\|^2_{L^2(\Sigma)}) + \frac{1}{2\alpha}(\|\lambda^n\|^2_{L^2(\Sigma)} - \|\lambda^{n+1}\|^2_{L^2(\Sigma)}) \\
&- \frac{\alpha}{2}\|(u^n - u^{n+1}) + \frac{1}{\alpha}(\lambda^n - \lambda^{n+1})\|^2_{L^2(\Sigma)} \\
&+ L_1(w^{n+1}) + L_2(u^{n+1}) + \frac{1}{\alpha}\langle \varepsilon_2^{n+1}, \lambda^{n+1}\rangle + \langle \varepsilon_2^{n+1}, w^{n+1}\rangle \\
&+ \langle \varepsilon_2^{n+1}, u^{n+1} - u^n\rangle.
\end{aligned}$$

If we plug in these results to (3.8) we arrive at the identity.

□

We can now state an identity for the last term in (3.7).

Lemma 3.4 *Let w, u solve (3.1) and (3.2) and assuming that $\varepsilon_2^{n+1} \in V_f$ then the following identity holds*

$$
\begin{aligned}
\frac{\Delta t}{\alpha}\sum_{n=0}^{N-1}\langle \varepsilon_2^{n+1}, \lambda^{n+1}\rangle = &-\frac{\Delta t}{\alpha}\sum_{n=1}^{N-1}(u^n, \partial_{\Delta t}\varepsilon_2^{n+1})_f + \frac{\Delta t}{\alpha}\sum_{n=0}^{N-1}\Big(\nu_f(\nabla u^{n+1}, \nabla\varepsilon_2^{n+1})_f - (b_2^{n+1}, \varepsilon_2^{n+1})_f\Big)\\
&+\frac{1}{\alpha}(u^N, \varepsilon_2^N)_f.
\end{aligned}
$$

Proof We first take $v = \Delta t\varepsilon_2^{n+1}$ in (3.5b) to obtain

$$
\Delta t\langle \varepsilon_2^{n+1}, \lambda^{n+1}\rangle = (u^{n+1} - u^n, \varepsilon_2^{n+1})_f + \Delta t\nu_f(\nabla u^{n+1}, \nabla\varepsilon_2^{n+1})_f - \Delta t(b_2^{n+1}, \varepsilon_2^{n+1})_f.
$$

If we take the sum over $n = 0, \ldots, N-1$ and use summation by parts, we get

$$
\begin{aligned}
\Delta t\sum_{n=0}^{N-1}\langle \varepsilon_2^{n+1}, \lambda^{n+1}\rangle = &-\Delta t\sum_{n=1}^{N-1}(u^n, \partial_{\Delta t}\varepsilon_2^{n+1})_f + \Delta t\sum_{n=0}^{N-1}\Big(\nu_f(\nabla u^{n+1}, \nabla\varepsilon_2^{n+1})_f - (b_2^{n+1}, \varepsilon_2^{n+1})_f\Big)\\
&+(u^N, \varepsilon_2^N)_f - (u^0, \varepsilon_2^1).
\end{aligned}
$$

We conclude the proof by using (3.3).

□

To state the stability estimate we need the next definition.

$$
\begin{aligned}
\Xi^N(m_1, m_2, s_1, s_2) :=&\Delta t\sum_{n=0}^{N-1}\left[\frac{1}{\nu_s}\|m_1^{n+1}\|^2_{L^2(\Omega_s)} + \left(\frac{1}{\nu_f} + \frac{1}{\alpha}\right)\|m_2^{n+1}\|^2_{L^2(\Omega_f)}\right]\\
&+\Delta t\sum_{n=0}^{N-1}\left(\frac{1}{\nu_f\alpha^2}\|\partial_{\Delta t}s_2^{n+1}\|^2_{L^2(\Omega_f)} + \frac{\nu_f}{\alpha^2}\|\nabla s_2^{n+1}\|^2_{L^2(\Omega_f)} + \frac{1}{\alpha}\|s_2^{n+1}\|^2_{L^2(\Omega_f)}\right)\\
&+\Delta t\sum_{n=0}^{N-1}\left(\frac{1}{\nu_s}\|s_1^{n+1} + s_2^{n+1}\|^2_{L^2(\Sigma)} + \frac{1}{\nu_f}\|s_2^{n+1}\|^2_{L^2(\Sigma)}\right) + \frac{1}{\alpha^2}\|s_2^N\|^2_{L^2(\Omega_f)}.
\end{aligned}
$$

Theorem 3.5 *Let w, u solve (3.1) and (3.2) and assuming that $\varepsilon_2^{n+1} \in V_f$ then the following estimate holds*

$$
Z^N(w, u, \lambda) + \sum_{n=0}^{N-1} S^{n+1}(w, u, \lambda) \le C\,\Xi^N(b_1, b_2, \varepsilon_1, \varepsilon_2).
$$

Proof Using Lemma 3.3 and taking the sum we get

$$Z^N(w,u,\lambda)+\sum_{n=0}^{N-1}S^{n+1}(w,u,\lambda)=Z^0(w,u,\lambda)+\Delta t\sum_{n=0}^{N-1}F^{n+1}(w,u)$$
$$+\frac{\Delta t}{\alpha}\sum_{n=0}^{N-1}\langle\varepsilon_2^{n+1},\lambda^{n+1}\rangle.$$

Using the Poincaré and trace inequalities, we easily obtain the following bound

$$\Delta t\sum_{n=0}^{N-1}F^{n+1}(w,u)\leq\frac{1}{8}\sum_{n=0}^{N-1}S^{n+1}(w,u,\lambda)+C\Delta t\sum_{n=0}^{N-1}(\frac{1}{\nu_s}\|b_1^{n+1}\|^2_{L^2(\Omega_s)}$$
$$+\frac{1}{\nu_f}\|b_2^{n+1}\|^2_{L^2(\Omega_f)})$$
$$+C\Delta t\sum_{n=0}^{N-1}\Big(\frac{1}{\nu_s}\|\varepsilon_1^{n+1}+\varepsilon_2^{n+1}\|^2_{L^2(\Sigma)}+\frac{1}{\nu_f}\|\varepsilon_2^{n+1}\|^2_{L^2(\Sigma)}\Big).$$

Using Lemma 3.4 and the Poincaré inequality we have

$$\frac{\Delta t}{\alpha}\sum_{n=0}^{N-1}\langle\varepsilon_2^{n+1},\lambda^{n+1}\rangle\leq\frac{1}{4}\|u^N\|^2_{L^2(\Omega_f)}+\frac{1}{8}\sum_{n=0}^{N-1}S^{n+1}(w,u,\lambda)$$
$$+C\Delta t\sum_{n=0}^{N-1}\Big(\frac{1}{\nu_f\alpha^2}\|\partial_{\Delta t}\varepsilon_2^{n+1}\|^2_{L^2(\Omega_f)}+\frac{\nu_f}{\alpha^2}\|\nabla\varepsilon_2^{n+1}\|^2_{L^2(\Omega_f)}$$
$$+\frac{1}{\alpha}\|\varepsilon_2^{n+1}\|^2_{L^2(\Omega_f)}\Big)$$
$$+C\frac{\Delta t}{\alpha}\sum_{n=0}^{N-1}\|b_2^{n+1}\|^2_{L^2(\Omega_f)}+C\frac{1}{\alpha^2}\|\varepsilon_2^N\|^2_{L^2(\Omega_f)}.$$

It follows from (3.3) and the definition of Z^0 that $Z^0(w,u,\lambda)=0$. We finish the proof by combining the above estimates.

□

Due to Remark 3.2, the discrete time derivative of u and w solves (3.1) and (3.2) with discrete time derivatives of the data as the right-hand sides, we immediately get the following.

Corollary 3.6 *Let w, u solve (3.1) and (3.2) and assuming that $\partial_{\Delta t}\varepsilon_2^{n+1}\in V_f$ then*

$$Z^N(\partial_{\Delta t}w,\partial_{\Delta t}u,\partial_{\Delta t}\lambda)+\sum_{n=0}^{N-1}S^{n+1}(\partial_{\Delta t}w,\partial_{\Delta t}u,\partial_{\Delta t}\lambda)$$
$$\leq C\,\Xi^N(\partial_{\Delta t}b_1,\partial_{\Delta t}b_2,\partial_{\Delta t}\varepsilon_1,\partial_{\Delta t}\varepsilon_2).$$

Similarly, the second-order discrete time derivative of u and w solves (3.1) and (3.2) with the second-order discrete time derivatives of the data as the right-hand sides, we also have the following.

Corollary 3.7 *Let w, u solve (3.1) and (3.2) and assuming that $\partial^2_{\Delta t}\varepsilon_2^{n+1} \in V_f$ then*

$$Z^N(\partial^2_{\Delta t}w, \partial^2_{\Delta t}u, \partial^2_{\Delta t}\lambda) + \sum_{n=0}^{N-1} S^{n+1}(\partial^2_{\Delta t}w, \partial^2_{\Delta t}u, \partial^2_{\Delta t}\lambda) \\ \leq C\,\Xi^N(\partial^2_{\Delta t}b_1, \partial^2_{\Delta t}b_2, \partial^2_{\Delta t}\varepsilon_1, \partial^2_{\Delta t}\varepsilon_2).$$

Remark 3.8 We can also analyze the problem (3.1)-(3.2) but replacing the homogeneous Neumann boundary conditions with homogeneous Dirichlet boundary conditions. In other words, the problem with pure Dirichlet boundary conditions that is $\Gamma^i_D = \partial\Omega_i\backslash\Sigma$ for $i = s, f$. In this case exactly the same estimates hold. Now of course, we assume $\varepsilon_2^n \in V_f$ where V_f has zero Dirichlet boundary conditions on $\Gamma^f_D = \partial\Omega_f\backslash\Sigma$.

3.1 H^2 stability in a special case

In this section we prove H^2 estimates for u in a special configuration. Again, we assume that $\Omega = (0, 1)^2$, and now assume Σ is parallel to the x-axis (see Fig. 2). We take advantage of the fact that the sides composing Γ_{Ne} are perpendicular to the x-axis and, moreover, we also notice that Σ is parallel to the x-axis.

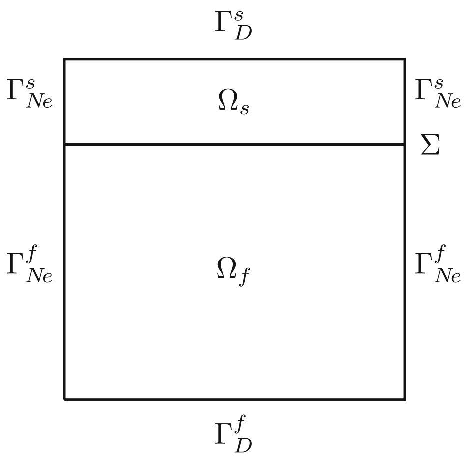

Fig. 2 The domains Ω_f and Ω_s with horizontal interface Σ

Then, in this particular case we have, for $n \geq 0$,

$$
\begin{aligned}
\partial_{\Delta t}\partial_x w^{n+1} - \nu_s \Delta \partial_x w^{n+1} &= \partial_x b_1^{n+1}, && \text{in} \Omega_s, && (3.11a)\\
\partial_x w^{n+1} &= 0, && \text{on } \Gamma_D^s, && (3.11b)\\
\partial_x w^{n+1} &= 0, && \text{on } \Gamma_{Ne}^s, && (3.11c)\\
\alpha \partial_x w^{n+1} + \nu_s \partial_{\mathbf{n}_s} \partial_x w^{n+1} &= \alpha \partial_x u^n - \nu_f \partial_{\mathbf{n}_f} \partial_x u^n + \partial_x \varepsilon_1^{n+1} && \text{on } \Sigma, && (3.11d)
\end{aligned}
$$

$$
\begin{aligned}
\partial_{\Delta t}\partial_x u^{n+1} - \nu_f \Delta \partial_x u^{n+1} &= \partial_x b_2^{n+1}, && \text{in } \Omega_f, && (3.12a)\\
\partial_x u^{n+1} &= 0, && \text{on } \Gamma_D^f, && (3.12b)\\
\partial_x u^{n+1} &= 0, && \text{on } \Gamma_{Ne}^f, && (3.12c)\\
\alpha \partial_x u^{n+1} + \nu_f \partial_{\mathbf{n}_f} \partial_x u^{n+1} &= \alpha \partial_x w^{n+1} + \nu_f \partial_{\mathbf{n}_f} \partial_x u^n + \partial_x \varepsilon_2^{n+1} && \text{on } \Sigma, && (3.12d)
\end{aligned}
$$

with

$$\partial_x w^0 = \partial_x u^0 = \partial_x b_i^0 = \partial_x \varepsilon_i^0 = 0 \quad i = 1, 2 \tag{3.13}$$

according to (3.3).

Remark 3.9 The assumption that the interface is perpendicular to the sides and parallel to the x-direction is crucial. First, notice that, in this case, the outer normal direction $\boldsymbol{n}$ to two sides aligns with the x-direction, leading to $\partial_x w^{n+1} = \partial_{\boldsymbol{n}} w^{n+1} = 0$ on Γ_{Ne}^s and $\partial_x u^{n+1} = \partial_{\boldsymbol{n}} u^{n+1} = 0$ on Γ_{Ne}^f where we use (3.1c) and (3.2c). Secondly, given (3.1d) and (3.2d), the conditions (3.11d) and (3.12d) are only valid in the case where the interface is perpendicular to the sides. Unfortunately, we have not yet found a general method to prove the H^2 estimates for the non-horizontal case which has proved to be quite challenging. However, we believe this technique could be useful for the analysis of fluid-structure interaction (FSI) problems, given the similarity of the coupling conditions.

We then get an immediate Corollary from Remark 3.8 and Theorem 3.5.

Corollary 3.10 *Suppose that Σ is perpendicular to the two sides of Γ_{Ne} as in* Fig. 2. *Let w, u solve* (3.1) *and* (3.2) *and assuming that $\partial_x \varepsilon_2^n \in \{v \in H^1(\Omega_f) : v = 0$ on $\partial\Omega_f \backslash \Sigma\}$ on Γ_{Ne}^f then the following estimate holds*

$$Z^N(\partial_x w, \partial_x u, \partial_x \lambda) + \sum_{n=0}^{N-1} S^{n+1}(\partial_x w, \partial_x u, \partial_x \lambda) \leq C\,\Xi^N(\partial_x b_1, \partial_x b_2, \partial_x \varepsilon_1, \partial_x \varepsilon_2).$$

Corollary 3.11 *Under the hypothesis of* Corollaries 3.10 and 3.6 *we have*

$$\begin{aligned}\Delta t\sum_{n=0}^{N-1}\nu_f\|D^2u^{n+1}\|^2_{L^2(\Omega_f)} \leq & C\Big(\Xi^N(\partial_{\Delta t}b_1,\partial_{\Delta t}b_2,\partial_{\Delta t}\varepsilon_1,\partial_{\Delta t}\varepsilon_2)\\ &+\Xi^N(\partial_x b_1,\partial_x b_2,\partial_x\varepsilon_1,\partial_x\varepsilon_2)\Big)\\ &+C\Delta t\sum_{n=0}^{N-1}\nu_f\|b_2^{n+1}\|^2_{L^2(\Omega_f)}.\end{aligned}$$

Proof From Corollary 3.10 we get

$$\Delta t\sum_{n=0}^{N-1}\nu_f\|\nabla\partial_x u^{n+1}\|^2_{L^2(\Omega_f)}\leq C\,\Xi^N(\partial_x b_1,\partial_x b_2,\partial_x\varepsilon_1,\partial_x\varepsilon_2).$$

Moreover, using (3.2a) and Corollary 3.6, we have

$$\begin{aligned}\Delta t\sum_{n=0}^{N-1}\nu_f\|\Delta u^{n+1}\|^2_{L^2(\Omega_f)} =& \Delta t\sum_{n=0}^{N-1}\nu_f\|\partial_{\Delta t}u^{n+1}-b_2^{n+1}\|^2_{L^2(\Omega_f)}\\ \leq & C\,\Xi^N(\partial_{\Delta t}b_1,\partial_{\Delta t}b_2,\partial_{\Delta t}\varepsilon_1,\partial_{\Delta t}\varepsilon_2)\\ &+2\Delta t\sum_{n=0}^{N-1}\nu_f\|b_2^{n+1}\|^2_{L^2(\Omega_f)}.\end{aligned}$$

Finally, using the following estimate,

$$\Delta t\sum_{n=0}^{N-1}\nu_f\|\partial_y^2u^{n+1}\|^2_{L^2(\Omega_f)}\leq 2\Delta t\sum_{n=0}^{N-1}\nu_f\Big(\|\partial_x^2u^{n+1}\|^2_{L^2(\Omega_f)}+\|\Delta u^{n+1}\|^2_{L^2(\Omega_f)}\Big),$$

and combining the above estimates we obtain the result. □

4 Error estimates of the Robin-Robin method

In this section we apply the stability results of the previous sections to obtain error estimates of the Robin-Robin splitting method (2.4)-(2.5) applied to (2.1). We use the following notation for the errors :

$$U^n:=\mathsf{u}^n-u^n,\quad W^n:=\mathsf{w}^n-w^n,\quad \Lambda^n:=\mathsf{l}^n-\lambda^n.$$

We use the convention $u^j=\mathsf{u}^j=\mathsf{u}_0$ and $w^j=\mathsf{w}^j=\mathsf{w}_0$ for $j<0$.

Then the error equations read, for $n \geq 0$,

$$\begin{aligned} \partial_{\Delta t} W^{n+1} - \nu_s \Delta W^{n+1} &= -h_1^{n+1}, && \text{in } \Omega_s, && (4.1\text{a})\\ W^{n+1} &= 0, && \text{on } \Gamma_D^s, && (4.1\text{b})\\ \partial_{\boldsymbol{n}} W^{n+1} &= 0, && \text{on } \Gamma_{Ne}^s, && (4.1\text{c})\\ \alpha W^{n+1} + \nu_s \partial_{\boldsymbol{n}_s} W^{n+1} &= \alpha U^n - \nu_f \partial_{\boldsymbol{n}_f} U^n + \alpha g_1^{n+1} - g_2^{n+1} && \text{on } \Sigma. && (4.1\text{d}) \end{aligned}$$

$$\begin{aligned} \partial_{\Delta t} U^{n+1} - \nu_f \Delta U^{n+1} &= -h_2^{n+1}, && \text{in } \Omega_f, && (4.2\text{a})\\ U^{n+1} &= 0, && \text{on } \Gamma_D^f, && (4.2\text{b})\\ \partial_{\boldsymbol{n}} U^{n+1} &= 0, && \text{on } \Gamma_{Ne}^f, && (4.2\text{c})\\ \alpha U^{n+1} + \nu_f \partial_{\boldsymbol{n}_f} U^{n+1} &= \alpha W^{n+1} + \nu_f \partial_{\boldsymbol{n}_f} U^n + g_2^{n+1} && \text{on } \Sigma. && (4.2\text{d}) \end{aligned}$$

where

$$\begin{aligned} h_1^{n+1} &:= \partial_t \mathsf{w}^{n+1} - \partial_{\Delta t} \mathsf{w}^{n+1}, & g_1^{n+1} &:= \mathsf{u}^{n+1} - \mathsf{u}^n,\\ h_2^{n+1} &:= \partial_t \mathsf{u}^{n+1} - \partial_{\Delta t} \mathsf{u}^{n+1}, & g_2^{n+1} &:= \mathsf{l}^{n+1} - \mathsf{l}^n. \end{aligned}$$

The equations (4.1)-(4.2) are well-defined and we also have

$$W^0 = U^0 = h_i^0 = g_i^0 = 0, \; i = 1, 2, \tag{4.3}$$

$$\partial_{\Delta t} W^0 = \partial_{\Delta t} U^0 = \partial_{\Delta t} h_i^0 = \partial_{\Delta t} g_i^0 = 0, \; i = 1, 2, \tag{4.4}$$

$$\partial_{\Delta t}^2 W^0 = \partial_{\Delta t}^2 U^0 = \partial_{\Delta t}^2 h_i^0 = \partial_{\Delta t}^2 g_i^0 = 0, \; i = 1, 2. \tag{4.5}$$

We then extend l to Ω_f in a natural way. We let $\tilde{\mathsf{l}} = \phi \nu_f \nabla \mathsf{u} \cdot \mathbf{n}_f$ where ϕ is a function that is one on Σ and vanishes on Γ_D^f. Then, we define $\tilde{g}_2^{n+1} = \tilde{\mathsf{l}}^{n+1} - \tilde{\mathsf{l}}^n$. By construction $\tilde{g}_2^{n+1} \in V_f$ and $\tilde{g}_2^n = g_2^n$ on Σ. We immediately get the following result if we apply Theorem 3.5 and Corollary 3.6.

Corollary 4.1 *Let* u, w *solve* (2.1) *and* w, u *solve* (2.4) *and* (2.5) *then*

$$Z^N(W, U, \Lambda) + \sum_{n=0}^{N-1} S^{n+1}(W, U, \Lambda) \leq C \, \Xi^N(h_1, h_2, \alpha g_1 - g_2, \tilde{g}_2), \tag{4.6}$$

$$\begin{aligned} Z^N(\partial_{\Delta t} W, \partial_{\Delta t} U, \partial_{\Delta t} \Lambda) + \sum_{n=0}^{N-1} S^{n+1}(\partial_{\Delta t} W, \partial_{\Delta t} U, \partial_{\Delta t} \Lambda)\\ \leq C \, \Xi^N(\partial_{\Delta t} h_1, \partial_{\Delta t} h_2, \alpha \partial_{\Delta t} g_1 - \partial_{\Delta t} g_2, \partial_{\Delta t} \tilde{g}_2), \end{aligned} \tag{4.7}$$

and

$$Z^N(\partial^2_{\Delta t}W, \partial^2_{\Delta t}U, \partial^2_{\Delta t}\Lambda) + \sum_{n=0}^{N-1} S^{n+1}(\partial^2_{\Delta t}W, \partial^2_{\Delta t}U, \partial^2_{\Delta t}\Lambda)$$
$$\leq C\,\Xi^N(\partial^2_{\Delta t}h_1, \partial^2_{\Delta t}h_2, \alpha\partial^2_{\Delta t}g_1 - \partial^2_{\Delta t}g_2, \partial^2_{\Delta t}\tilde{g}_2). \tag{4.8}$$

Then, it is quite straightforward to get a convergence rate by estimating the right-hand sides. The proof of the following Corollary can be found in Appendix A.

Corollary 4.2 *Let* u, w *solve* (2.1) *and* w, u *solve* (2.4) *and* (2.5) *then*

$$Z^N(W, U, \Lambda) + \sum_{n=0}^{N-1} S^{n+1}(W, U, \Lambda) \leq C(\Delta t)^2 \mathsf{Y}, \tag{4.9}$$

$$Z^N(\partial_{\Delta t}W, \partial_{\Delta t}U, \partial_{\Delta t}\Lambda) + \sum_{n=0}^{N-1} S^{n+1}(\partial_{\Delta t}W, \partial_{\Delta t}U, \partial_{\Delta t}\Lambda) \leq C(\Delta t)^2 \mathcal{Y}, \tag{4.10}$$

and

$$Z^N(\partial^2_{\Delta t}W, \partial^2_{\Delta t}U, \partial^2_{\Delta t}\Lambda) + \sum_{n=0}^{N-1} S^{n+1}(\partial^2_{\Delta t}W, \partial^2_{\Delta t}U, \partial^2_{\Delta t}\Lambda) \leq C(\Delta t)^2 \mathfrak{Y}, \tag{4.11}$$

where Y *is defined as*

$$\begin{aligned}\mathsf{Y} :=&\ \frac{1}{\nu_s}\|\partial_t^2\mathsf{w}\|^2_{L^2(0,T;L^2(\Omega_s))} + (\frac{1}{\nu_f} + \frac{1}{\alpha})\|\partial_t^2\mathsf{u}\|^2_{L^2(0,T;L^2(\Omega_f))}\\ &+ (\frac{\nu_f}{\alpha^2} + \frac{\nu_f^2}{\alpha})\|\partial_t\mathsf{u}\|^2_{L^2(0,T;H^1(\Omega_f))} + \frac{(\nu_f)^3}{\alpha^2}\|\partial_t\mathsf{u}\|^2_{L^2(0,T;H^2(\Omega_f))}\\ &+ \frac{\alpha^2}{\nu_s}\|\partial_t\mathsf{u}\|^2_{L^2(0,T;L^2(\Sigma))} + \frac{1}{\nu_f}\|\partial_t\mathsf{l}\|^2_{L^2(0,T;L^2(\Sigma))} + \frac{\nu_f^2}{\alpha^2}\|\partial_t\mathsf{u}\|^2_{L^\infty(0,T;H^1(\Omega_f))},\end{aligned} \tag{4.12}$$

$\mathcal{Y}$ *is defined as*

$$\begin{aligned}\mathcal{Y} :=&\ \frac{1}{\nu_s}\|\partial_t^3\mathsf{w}\|^2_{L^2(0,T;L^2(\Omega_s))} + (\frac{1}{\nu_f} + \frac{1}{\alpha})\|\partial_t^3\mathsf{u}\|^2_{L^2(0,T;L^2(\Omega_f))}\\ &+ (\frac{\nu_f}{\alpha^2} + \frac{\nu_f^2}{\alpha})\|\partial_t^2\mathsf{u}\|^2_{L^2(0,T;H^1(\Omega_f))} + \frac{(\nu_f)^3}{\alpha^2}\|\partial_t^2\mathsf{u}\|^2_{L^2(0,T;H^2(\Omega_f))}\\ &+ \frac{\alpha^2}{\nu_s}\|\partial_t^2\mathsf{u}\|^2_{L^2(0,T;L^2(\Sigma))} + \frac{1}{\nu_f}\|\partial_t^2\mathsf{l}\|^2_{L^2(0,T;L^2(\Sigma))} + \frac{\nu_f^2}{\alpha^2}\|\partial_t^2\mathsf{u}\|^2_{L^\infty(0,T;H^1(\Omega_f))},\end{aligned} \tag{4.13}$$

and $\mathfrak{Y}$ *is defined as*

$$\mathfrak{Y} := \frac{1}{\nu_s}\|\partial_t^4 \mathsf{w}\|^2_{L^2(0,T;L^2(\Omega_s))} + (\frac{1}{\nu_f} + \frac{1}{\alpha})\|\partial_t^4 \mathsf{u}\|^2_{L^2(0,T;L^2(\Omega_f))}$$
$$+ (\frac{\nu_f}{\alpha^2} + \frac{\nu_f^2}{\alpha})\|\partial_t^3 \mathsf{u}\|^2_{L^2(0,T;H^1(\Omega_f))} + \frac{(\nu_f)^3}{\alpha^2}\|\partial_t^3 \mathsf{u}\|^2_{L^2(0,T;H^2(\Omega_f))}$$
$$+ \frac{\alpha^2}{\nu_s}\|\partial_t^3 \mathsf{u}\|^2_{L^2(0,T;L^2(\Sigma))} + \frac{1}{\nu_f}\|\partial_t^3 \mathsf{l}\|^2_{L^2(0,T;L^2(\Sigma))} + \frac{\nu_f^2}{\alpha^2}\|\partial_t^3 \mathsf{u}\|^2_{L^\infty(0,T;H^1(\Omega_f))}. \tag{4.14}$$

4.1 H^2 error estimates in a special case

Here we assume that we have the configuration as in Fig. 1. Then, we see that $\partial_x \tilde{g_2}^{n+1}$ vanishes on Γ^f_{Ne} and Γ^f_D and hence belongs to V_f. Hence, Corollary 3.11 gives the following corollary.

Corollary 4.3 *Suppose that* Σ *is perpendicular to the two sides of* Γ_{Ne} *as in* Fig. 2. *Let* u, w *solve* (2.1) *and* w, u *solve* (2.4) *and* (2.5) *then*

$$\Delta t \sum_{n=0}^{N-1} \nu_f \|D^2(U^{n+1})\|^2_{L^2(\Omega_f)} \leq C\,\Xi^N(\partial_{\Delta t} h_1, \partial_{\Delta t} h_2, \partial_{\Delta t}(\alpha g_1 - g_2), \partial_{\Delta t}\tilde{g}_2)$$
$$+ C\,\Xi^N(\partial_x h_1, \partial_x h_2, \partial_x(\alpha g_1 - g_2), \partial_{\Delta t}\partial_x \tilde{g}_2)$$
$$+ C\Delta t \sum_{n=0}^{N-1} \nu_f \|h_2^{n+1}\|^2_{L^2(\Omega_f)}.$$

Since $\partial_{\Delta t} U$ and $\partial_{\Delta t} W$ satisfy the same equations as U, W with time difference right-hand sides, we have

Corollary 4.4 *Suppose that* Σ *is perpendicular to the two sides of* Γ_{Ne} *as in* Fig. 2. *Let* u, w *solve* (2.1) *and* w, u *solve* (2.4) *and* (2.5) *, then*

$$\Delta t \sum_{n=0}^{N-1} \nu_f \|D^2(\partial_{\Delta t} U^{n+1})\|^2_{L^2(\Omega_f)} \leq C\ \Xi^N(\partial^2_{\Delta t} h_1, \partial^2_{\Delta t} h_2, \partial^2_{\Delta t}(\alpha g_1 - g_2), \partial^2_{\Delta t}\tilde{g}_2)$$
$$+ C\,\Xi^N(\partial_{\Delta t}\partial_x h_1, \partial_{\Delta t}\partial_x h_2, \partial_{\Delta t}\partial_x(\alpha g_1 - g_2), \partial_{\Delta t}\partial_x \tilde{g}_2)$$
$$+ C\Delta t \sum_{n=0}^{N-1} \nu_f \|\partial_{\Delta t} h_2^{n+1}\|^2_{L^2(\Omega_f)}. \tag{4.15}$$

Then, it is straight-forward to obtain the convergence rate by estimate the right-hand sides. See Appendix B for a proof for the following Corollary.

Corollary 4.5 *Suppose that* Σ *is perpendicular to the two sides of* Γ_{Ne} *as in* Fig. 2. *Let* u, w *solve* (2.1) *and* w, u *solve* (2.4) *and* (2.5) *then*

$$\Delta t \sum_{n=0}^{N-1} \nu_f \|D^2(\partial_{\Delta t} U^{n+1})\|^2_{L^2(\Omega_f)} \le C(\Delta t)^2(\mathfrak{Y} + \mathbb{Y} + \nu_f \|\partial_t^3 \mathsf{u}\|^2_{L^2((0,T),L^2(\Omega_f))})$$

where $\mathfrak{Y}$ *is defined in* Corollary 4.2 *and* $\mathbb{Y}$ *is defined as,*

$$\begin{aligned}
\mathbb{Y} := &\frac{1}{\nu_s}\|\partial_x \partial_t^3 \mathsf{w}\|^2_{L^2(0,T;L^2(\Omega_s))} + (\frac{1}{\nu_f} + \frac{1}{\alpha})\|\partial_x \partial_t^3 \mathsf{u}\|^2_{L^2(0,T;L^2(\Omega_f))} \\
&+ (\frac{\nu_f}{\alpha^2} + \frac{\nu_f^2}{\alpha})\|\partial_x \partial_t^2 \mathsf{u}\|^2_{L^2(0,T;H^1(\Omega_f))} + \frac{(\nu_f)^3}{\alpha^2}\|\partial_x \partial_t^2 \mathsf{u}\|^2_{L^2(0,T;H^2(\Omega_f))} \\
&+ \frac{\alpha^2}{\nu_s}\|\partial_x \partial_t^2 \mathsf{u}\|^2_{L^2(0,T;L^2(\Sigma))} + \frac{1}{\nu_f}\|\partial_x \partial_t^2 \mathsf{l}\|^2_{L^2(0,T;L^2(\Sigma))} \\
&+ \frac{\nu_f^2}{\alpha^2}\|\partial_x \partial_t^2 \mathsf{u}\|^2_{L^\infty(0,T;H^1(\Omega_f))}.
\end{aligned} \tag{4.16}$$

5 Numerical experiments

In this section we provide numerical experiments that agree with our theoretical results. We let

$$\begin{aligned}
e_u &= \|U^N\|_{L^2(\Omega_f)}, \quad e_{1,u} = \|U^N - U^{N-1}\|_{L^2(\Omega_f)}, \\
e_{2,u} &= \|(U^N - U^{N-1}) - (U^{N-1} - U^{N-2})\|_{L^2(\Omega_f)}, \\
e_\lambda &= \|\Lambda^N\|_{L^2(\Sigma)}, \quad e_{1,\lambda} = \|\Lambda^N - \Lambda^{N-1}\|_{L^2(\Sigma)} \\
e_{1,u,2} &= \|U^N - U^{N-1}\|_{H^2(\Omega_f)}.
\end{aligned}$$

All numerical experiments are performed using FEniCS and multiphenics [1, 3]. Although we only analyze the semi-discrete method, here we present the results for a fully discrete method where we use the piecewise linear finite element method for the spatial discretization, except for the computation of $e_{1,u,2}$, where we use the piecewise quadratic finite element method because piecewise linear function does not approximate a function well in H^2-norm. In addition, we also present convergence rates for the Lagrange multiplier.

Example 5.1 We consider the domain $\Omega = (0, 1)^2$, $\Omega_f = (0, 1) \times (0, 0.75)$ and $\Omega_s = (0, 1) \times (0.75, 1)$. See Fig. 2 for an illustration. We take $\nu_f = 1 = \nu_s$ and take

Table 1 Errors and convergence rates of U^N for Example 5.1

Δt	e_u	Rates	$e_{1,u}$	Rates	$e_{2,u}$	Rates
$(1/2)^2$	7.65e-02	–	7.73e-02	–	6.57e+01	–
$(1/2)^3$	3.72e-02	1.04	5.87e-02	0.40	1.55e-01	8.73
$(1/2)^4$	1.74e-02	1.10	1.47e-02	1.99	7.91e-03	4.29
$(1/2)^5$	7.95e-03	1.13	3.58e-03	2.04	1.27e-03	2.64
$(1/2)^6$	3.52e-03	1.17	8.41e-04	2.09	1.75e-04	2.85
$(1/2)^7$	1.62e-03	1.12	1.96e-04	2.10	2.15e-05	3.03
$(1/2)^8$	7.70e-04	1.07	4.69e-05	2.06	2.62e-06	3.04
$(1/2)^9$	3.75e-04	1.04	1.14e-05	2.04	3.22e-07	3.02

the solution of (2.3) to be

$$\mathsf{w} = \mathsf{u} = e^{-2\pi^2 t} \cos(\pi x_1) \sin(\pi x_2).$$

We take $h = \Delta t$, $T = 0.25$ and $\alpha = 4$ where h is the mesh size of the triangulation.

As we can see from Tables 1-2, the L^2 error at the final step, e_u is of order (Δt) whereas the difference of two consecutive errors, $e_{1,u}$ is of order $(\Delta t)^2$ and the second difference, $e_{2,u}$ is of order $(\Delta t)^3$. The L^2 error of the Lagrange multiplier at the final time, e_λ is of order Δt and the difference $e_{1,\lambda}$, is of order $(\Delta t)^2$. It is also clear that the H^2 error of the difference of U^N, $e_{1,u,2}$ is of order $(\Delta t)^2$ as we proved in Corollary 4.5.

Example 5.2 In this example, we test our algorithm for a non-horizontal interface problem. We consider the domain $\Omega = (0, 1)^2$ and we let Σ be defined as the straight line connecting $(0, 0.25)$ and $(1, 0.75)$. We then define Ω_s as the region above Σ and Ω_f as the region below Σ. We take $\nu_f = 1 = \nu_s$ and take the solution of (2.3) to be

$$\mathsf{w} = \mathsf{u} = e^{-2\pi^2 t} \cos(\pi x_1) \sin(\pi x_2).$$

Table 2 Error and convergence rates of Λ^N for Example 5.1

Δt	e_λ	Rates	$e_{1,\lambda}$	Rates	$e_{1,u,2}$	Rates
$(1/2)^2$	2.55e-01	–	2.55e-01	–	1.33e+01	–
$(1/2)^3$	9.73e-02	1.39	2.41e-01	0.08	1.11e+01	0.26
$(1/2)^4$	3.06e-02	1.67	4.41e-02	2.45	2.61e-01	2.08
$(1/2)^5$	1.62e-02	0.92	6.69e-03	2.72	5.76e-02	2.18
$(1/2)^6$	9.20e-03	0.82	2.06e-03	1.70	1.40e-02	2.04
$(1/2)^7$	4.55e-03	1.01	5.28e-04	1.97	3.32e-03	2.07
$(1/2)^8$	2.23e-03	1.03	1.31e-04	2.01	8.03e-04	2.05
$(1/2)^9$	1.10e-03	1.02	3.26e-05	2.01	1.97e-04	2.03

Table 3 Errors and convergence rates of U^N for Example 5.2

Δt	e_u	Rates	$e_{1,u}$	Rates	$e_{2,u}$	Rates
$(1/2)^2$	8.05e-02	–	9.64e-02	–	4.87e+01	–
$(1/2)^3$	5.10e-02	0.66	4.23e-02	1.19	1.36e-01	8.48
$(1/2)^4$	2.46e-02	1.05	1.57e-02	1.43	4.94e-03	4.78
$(1/2)^5$	9.20e-03	1.42	4.07e-03	1.95	1.47e-03	1.74
$(1/2)^6$	3.52e-03	1.38	8.46e-04	2.27	1.82e-04	3.02
$(1/2)^7$	1.52e-03	1.22	1.86e-04	2.18	2.11e-05	3.11
$(1/2)^8$	7.00e-04	1.11	4.33e-05	2.10	2.49e-06	3.08
$(1/2)^9$	3.36e-04	1.06	1.04e-05	2.06	3.02e-07	3.04

Other parameters are identical to those of Example 5.1.

We report the convergence results in Tables 3-4. We again observe expected convergence rates for both U^N and Λ^N. It indicates that our methods also work for a more general interface problem.

6 Concluding remarks

We analyzed the Robin-Robin coupling methods [11] for parabolic-parabolic interface problems and proved higher convergence rates in time for the first-order and second-order discrete time derivatives. We also prove H^2 estimates of the discrete time derivatives in a special case. All the estimates in this work are key ingredients in proving that a prediction correction method [12] produces a $O((\Delta t)^2)$ convergence rate.

Table 4 Error and convergence rates of Λ^N for Example 5.2

Δt	e_λ	Rates	$e_{1,\lambda}$	Rates	$e_{1,u,2}$	Rates
$(1/2)^2$	8.51e-01	–	8.54e-01	–	2.04e+01	–
$(1/2)^3$	5.01e-01	0.76	3.81e-01	1.16	9.67e-01	1.08
$(1/2)^4$	1.94e-01	1.37	1.45e-01	1.39	3.66e-01	1.40
$(1/2)^5$	5.34e-02	1.86	2.44e-02	2.58	9.08e-02	2.01
$(1/2)^6$	1.84e-02	1.54	4.38e-03	2.48	1.86e-02	2.29
$(1/2)^7$	7.49e-03	1.30	9.10e-04	2.27	4.04e-03	2.20
$(1/2)^8$	3.39e-03	1.14	2.07e-04	2.13	9.31e-04	2.11
$(1/2)^9$	1.61e-03	1.07	4.95e-05	2.07	2.23e-04	2.06

7 Appendix A: Sketch of proof: Corollary 4.2

Proof of (4.9) ***in Corollary*** 4.2. To prove (4.9), it is suffice to bound the term $\Xi^N(h_1, h_2, \alpha g_1 - g_2, \tilde{g}_2)$. Note that since $g_2^{n+1} = \tilde{g}_2^{n+1}$ on Σ, we have

$$
\begin{aligned}
\Xi^N(h_1, h_2, \alpha g_1 - g_2, \tilde{g}_2) :=& \Delta t \sum_{n=0}^{N-1} \big(\frac{1}{\nu_s}\|h_1^{n+1}\|^2_{L^2(\Omega_s)} + (\frac{1}{\nu_f} + \frac{1}{\alpha})\|h_2^{n+1}\|^2_{L^2(\Omega_f)}\big) \\
&+ \Delta t \sum_{n=0}^{N-1} \Big(\frac{1}{\nu_f \alpha^2}\|\partial_{\Delta t}\tilde{g}_2^{n+1}\|^2_{L^2(\Omega_f)} + \frac{\nu_f}{\alpha^2}\|\nabla \tilde{g}_2^{n+1}\|^2_{L^2(\Omega_f)} + \frac{1}{\alpha}\|\tilde{g}_2^{n+1}\|^2_{L^2(\Omega_f)}\Big) \\
&+ \Delta t \sum_{n=0}^{N-1} \Big(\frac{1}{\nu_s}\|\alpha g_1^{n+1}\|^2_{L^2(\Sigma)} + \frac{1}{\nu_f}\|\tilde{g}_2^{n+1}\|^2_{L^2(\Sigma)}\Big) + \frac{1}{\alpha^2}\|\tilde{g}_2^N\|^2_{L^2(\Omega_f)} \\
&= T_1 + T_2 + \ldots + T_8.
\end{aligned}
$$

All the terms in $\Xi^N(h_1, h_2, \alpha g_1 - g_2, \tilde{g}_2)$ can be easily estimated by (A.2b) except the T_3 and T_4. For T_3, it follows from (A.5) that,

$$
\begin{aligned}
\Delta t \sum_{n=0}^{N-1} \frac{1}{\nu_f \alpha^2}\|\partial_{\Delta t}\tilde{g}_2^{n+1}\|^2_{L^2(\Omega_f)} &= (\Delta t)^3 \sum_{n=0}^{N-1} \frac{1}{\nu_f \alpha^2}\|\partial^2_{\Delta t}\tilde{\mathsf{l}}^{n+1}\|^2_{L^2(\Omega_f)} \\
&\le C\frac{(\Delta t)^2}{\nu_f \alpha^2}\|\partial_t^2 \tilde{\mathsf{l}}\|^2_{L^2(0,T;L^2(\Omega_f))} \\
&\le C(\Delta t)^2 \frac{\nu_f}{\alpha^2}\|\partial_t^2 \mathsf{u}\|^2_{L^2(0,T;H^1(\Omega_f))}.
\end{aligned} \tag{A.1}
$$

The term T_4 can be bounded as follows,

$$
\begin{aligned}
\Delta t \frac{\nu_f}{\alpha^2} \sum_{n=0}^{N-1} \|\nabla \tilde{g}_2^{n+1}\|^2_{L^2(\Omega_f)} \le C \Delta t \frac{(\nu_f)^3}{\alpha^2} \sum_{n=0}^{N-1} \Big(&\|\nabla(\mathsf{u}^{n+1} - u^n)\|^2_{L^2(\Omega_f)} \\
&+ \|D^2(\mathsf{u}^{n+1} - \mathsf{u}^n)\|^2_{L^2(\Omega_f)}\Big).
\end{aligned}
$$

Here $D^2\mathsf{u}$ denotes the Hessian of u. Using (A.2a) we get

$$
\Delta t \frac{\nu_f}{\alpha^2} \sum_{n=0}^{N-1} \|\nabla \tilde{g}_2^{n+1}\|^2_{L^2(\Omega_f)} \le C(\Delta t)^2 \frac{(\nu_f)^3}{\alpha^2}\|\partial_t \mathsf{u}\|^2_{L^2(0,T;H^2(\Omega_f))}.
$$

The estimates above imply (4.9). ◻

Before we prove the estimate (4.10), we need the following preliminary results. We use the Bochner norms $\|v\|_{L^2(a,b;X)} = \left(\int_a^b \|v(\cdot,s)\|_X^2 ds\right)^{1/2}$ and $\|v\|_{L^\infty(a,b;X)} = \text{ess}\sup_{a\leq s\leq b}\|v(\cdot,s)\|_X$. For a Sobolev space X, it is well known that

$$\|v^{n+1} - v^n\|_X^2 \leq C\Delta t \int_{t_n}^{t_{n+1}} \|\partial_t v(\cdot,s)\|_X^2 ds, \tag{A.2a}$$

$$\|\partial_{\Delta t} v^{n+1} - \partial_t v^{n+1}\|_X^2 \leq C\Delta t \int_{t_n}^{t_{n+1}} \|\partial_t^2 v(\cdot,s)\|_X^2 ds, \tag{A.2b}$$

$$\int_a^b \|v(\cdot,s)\|_X^2 ds \leq (b-a)\|v\|_{L^\infty(a,b;X)}^2. \tag{A.2c}$$

The following identities can easily be shown

$$\partial_{\Delta t}^2 v^n = \frac{1}{(\Delta t)^2}\int_{-\Delta t}^{\Delta t}(\Delta t - |s|)\partial_t^2 v(\cdot, t_n + s) ds, \tag{A.3}$$

and

$$\begin{aligned}\frac{\partial_{\Delta t}^2 v^n - \partial_{\Delta t}^2 v^{n-1}}{\Delta t} &= \frac{1}{(\Delta t)^3}\int_{-\Delta t}^{\Delta t}(\Delta t - |s|)\big(\partial_t^2 v(\cdot, t_n + s) - \partial_t^2 v(\cdot, t_{n-1} + s)\big) ds \\ &= \frac{1}{(\Delta t)^3}\int_{-\Delta t}^{\Delta t}(\Delta t - |s|)\int_{t_{n-1}+s}^{t_n+s}\partial_t^3 v(\cdot, r) dr ds. \end{aligned}\tag{A.4}$$

From these we can show that

$$\|\partial_{\Delta t}^2 v^n\|_X^2 \leq \frac{C}{\Delta t}\int_{t_{n-1}}^{t_{n+1}} \|\partial_t^2 v(\cdot,s)\|_X^2 ds, \tag{A.5}$$

and

$$\left\|\frac{\partial_{\Delta t}^2 v^n - \partial_{\Delta t}^2 v^{n-1}}{\Delta t}\right\|_X^2 \leq \frac{C}{\Delta t}\int_{t_{n-2}}^{t_{n+1}} \|\partial_t^3 v(\cdot,r)\|_X^2 dr. \tag{A.6}$$

Proof of (4.10) ***in Corollary*** 4.2. It is similar to prove (4.10). Indeed, let us define, for $j = 1, 2$

$$G_j^{n+1} = \partial_{\Delta t} g_j^{n+1}, \quad \tilde{G}_2^{n+1} = \partial_{\Delta t}\tilde{g}_2^{n+1}, \quad H_j^{n+1} = \partial_{\Delta t} h_j^{n+1}.$$

We then need to bound the term

$$
\begin{aligned}
&\Xi^N(H_1, H_2, \alpha G_1 - G_2, \tilde{G}_2)\\
=&\Delta t \sum_{n=0}^{N-1} \left(\frac{1}{\nu_s} \|H_1^{n+1}\|^2_{L^2(\Omega_s)} + \left(\frac{1}{\nu_f} + \frac{1}{\alpha}\right) \|H_2^{n+1}\|^2_{L^2(\Omega_f)}\right)\\
&+ \Delta t \sum_{n=0}^{N-1} \left(\frac{1}{\nu_f \alpha^2} \|\partial_{\Delta t} \tilde{G}_2^{n+1}\|^2_{L^2(\Omega_f)} + \frac{\nu_f}{\alpha^2} \|\nabla \tilde{G}_2^{n+1}\|^2_{L^2(\Omega_f)} + \frac{1}{\alpha} \|\tilde{G}_2^{n+1}\|^2_{L^2(\Omega_f)} \right)\\
&+ \Delta t \sum_{n=0}^{N-1} \left(\frac{1}{\nu_s} \|\alpha G_1^{n+1}\|^2_{L^2(\Sigma)} + \frac{1}{\nu_f} \|\tilde{G}_2^{n+1}\|^2_{L^2(\Sigma)} \right) + \frac{1}{\alpha^2} \|\tilde{G}_2^N\|^2_{L^2(\Omega_f)}\\
=&R_1 + R_2 + \ldots + R_8.
\end{aligned}
$$

For R_1, it follows from the definitions of H_1^{n+1} and h_1^{n+1} that,

$$
\begin{aligned}
\|H_1^{n+1}\|^2_{L^2(\Omega_s)} &= \int_{\Omega_s} \left(\frac{h_1^{n+1} - h_1^n}{\Delta t} \right)^2 dx\\
&= \int_{\Omega_s} \left(\frac{\partial_t \mathrm{w}^{n+1} - \partial_{\Delta t} \mathrm{w}^{n+1} - \partial_t \mathrm{w}^n + \partial_{\Delta t} \mathrm{w}^n}{\Delta t} \right)^2 dx\\
&= \int_{\Omega_s} \left(\partial_{\Delta t} (\partial_t \mathrm{w})^{n+1} - \partial^2_{\Delta t} \mathrm{w}^n \right)^2 dx\\
&= \int_{\Omega_s} \left(\partial_{\Delta t} (\partial_t \mathrm{w})^{n+1} - \partial_t^2 \mathrm{w}^n + \partial_t^2 \mathrm{w}^n - \partial^2_{\Delta t} \mathrm{w}^{n+1} \right)^2 dx\\
&\leq C \int_{\Omega_s} \left(\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} (\partial_t^2 \mathrm{w}(t_n) - \partial_t^2 \mathrm{w}(s)) ds \right)^2 dx\\
&+ C \int_{\Omega_s} \left(\frac{1}{(\Delta t)^2} \int_{t_{n-1}}^{t_{n+1}} \big(\Delta t - |s - t_n|\big) \big(\partial_t^2 \mathrm{w}(s) - \partial_t^2 \mathrm{w}(t_n)\big) ds \right)^2 dx\\
&\leq C \Delta t \|\partial_t^3 \mathrm{w}\|^2_{L^2((t_{n-1}, t_{n+1}), L^2(\Omega_s))}.
\end{aligned} \tag{A.7}
$$

Therefore, we obtain

$$
\Delta t \sum_{n=0}^{N-1} \frac{1}{\nu_s} \|H_1^{n+1}\|^2_{L^2(\Omega_s)} \leq C(\Delta t)^2 \frac{1}{\nu_s} \|\partial_t^3 \mathrm{w}\|^2_{L^2((0,T), L^2(\Omega_s))} \tag{A.8}
$$

The estimate of R_2 is similar to that of R_1. The term R_3 can be estimated as follow by (A.6),

$$\begin{aligned}\Delta t\sum_{n=0}^{N-1}\frac{1}{\nu_f\alpha^2}\|\partial_{\Delta t}\tilde{G}_2^{n+1}\|^2_{L^2(\Omega_f)} &= (\Delta t)^3\sum_{n=2}^{N-1}\frac{1}{\nu_f\alpha^2}\|\frac{\partial^2_{\Delta t}\tilde{\mathsf{l}}^n-\partial^2_{\Delta t}\tilde{\mathsf{l}}^{n-1}}{\Delta t}\|^2_{L^2(\Omega_f)}\\ &\le C\frac{(\Delta t)^2}{\nu_f\alpha^2}\|\partial_t^3\tilde{\mathsf{l}}\|^2_{L^2(0,T;L^2(\Omega_f))}\\ &\le C(\Delta t)^2\frac{\nu_f}{\alpha^2}\|\partial_t^3\mathsf{u}\|^2_{L^2(0,T;H^1(\Omega_f))}.\end{aligned}\tag{A.9}$$

For R_4, it follows from the definition of $\tilde{\mathsf{l}}$ and the fact $\|\nabla\phi\|_{L^2(\Omega_f)}\le C$ that,

$$\begin{aligned}\|\nabla\tilde{G}_2^{n+1}\|^2_{L^2(\Omega_f)} &= \int_{\Omega_f}\left(\frac{\nabla\tilde{\mathsf{l}}^{n+1}-2\nabla\tilde{\mathsf{l}}^n+\nabla\tilde{\mathsf{l}}^{n-1}}{\Delta t}\right)^2dx\\ &= \int_{\Omega_f}\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_{n+1}}\big(\Delta t-|s-t_n|\big)\partial_t^2\nabla\tilde{\mathsf{l}}(s)\,ds\right)^2dx\\ &\le C\int_{\Omega_f}\left(\int_{t_{n-1}}^{t_{n+1}}|\partial_t^2\nabla\tilde{\mathsf{l}}(s)|\,ds\right)^2dx\\ &\le C\Delta t\int_{\Omega_f}\int_{t_{n-1}}^{t_{n+1}}(\partial_t^2\nabla\tilde{\mathsf{l}}(s))^2\,ds\,dx\\ &\le C\nu_f^2\Delta t\|\partial_t^2\mathsf{u}\|^2_{L^2(t_{n-1},t_{n+1};H^2(\Omega_f))},\end{aligned}\tag{A.10}$$

and hence,

$$\Delta t\sum_{n=0}^{N-1}\frac{\nu_f}{\alpha^2}\|\nabla\tilde{G}_2^{n+1}\|^2_{L^2(\Omega_f)}\le C(\Delta t)^2\frac{(\nu_f^3)}{\alpha^2}\|\partial_t^2\mathsf{u}\|^2_{L^2(0,T;H^2(\Omega_f))}.\tag{A.11}$$

The remaining terms R_5 to R_8 can be estimated similarly, we give the estimate of R_6 here:

$$\begin{aligned}\|G_1^{n+1}\|^2_{L^2(\Sigma)} &= \int_\Sigma\left(\frac{g_1^{n+1}-g_1^n}{\Delta t}\right)^2d\sigma\\ &= \int_\Sigma\left(\frac{\mathsf{u}^{n+1}-2\mathsf{u}^n+\mathsf{u}^{n-1}}{\Delta t}\right)^2d\sigma\\ &= \int_\Sigma\left(\frac{1}{\Delta t}\int_{t_{n-1}}^{t_{n+1}}\big(\Delta t-|s-t_n|\big)\partial_t^2\mathsf{u}(s)\,ds\right)^2d\sigma\\ &\le C\Delta t\|\partial_t^2\mathsf{u}\|^2_{L^2((t_{n-1},t_{n+1}),L^2(\Sigma))},\end{aligned}\tag{A.12}$$

and therefore, we obtain

$$\Delta t \sum_{n=0}^{N-1} \frac{1}{\nu_s} \|\alpha G_1\|^2_{L^2(\Sigma)} \leq C \frac{\alpha^2}{\nu_s} \|\partial_t^2 \mathsf{u}\|^2_{L^2((0,T),L^2(\Sigma))}. \tag{A.13}$$

We finish the proof by combining all the estimates above. □

In order to prove (4.11), we define the following quantities for $j = 1, 2$,

$$\mathcal{G}_j^{n+1} = \partial_{\Delta t} G_j^{n+1}, \quad \mathcal{H}_j^{n+1} = \partial_{\Delta t} H_j^{n+1}, \tilde{\mathcal{G}}_2^{n+1} = \partial_{\Delta t} \tilde{G}_2^{n+1}. \tag{A.14}$$

Note that we also have the following

$$\tilde{\mathcal{G}}_2^{n+1} = \frac{\tilde{g}_2^{n+1} - 2\tilde{g}_2^n + \tilde{g}_2^{n-1}}{(\Delta t)^2} = \frac{\tilde{\mathsf{l}}^{n+1} - 3\tilde{\mathsf{l}}^n + 3\tilde{\mathsf{l}}^{n-1} - \tilde{\mathsf{l}}^{n-2}}{(\Delta t)^2}. \tag{A.15}$$

Denote

$$\partial_{\Delta t}^3 v^{n+1} = \frac{v^{n+1} - 3v^n + 3v^{n-1} - v^{n-2}}{(\Delta t)^3}, \tag{A.16}$$

thus we see that

$$\frac{\tilde{\mathcal{G}}_2^{n+1} - \tilde{\mathcal{G}}_2^n}{(\Delta t)^2} = \frac{\partial_{\Delta t}^3 \tilde{\mathsf{l}}^{n+1} - \partial_{\Delta t}^3 \tilde{\mathsf{l}}^n}{\Delta t}. \tag{A.17}$$

Moreover, we see that

$$\begin{aligned}
\frac{\partial_{\Delta t}^3 v^n - \partial_{\Delta t}^3 v^{n-1}}{\Delta t} =& \frac{1}{(\Delta t)^4} \Big(\int_{-\Delta t}^{\Delta t} (\Delta t - |s|)\big(\partial_t^2 v(\cdot, t_n + s) - \partial_t^2 v(\cdot, t_{n-1} + s)\big) ds \\
&- \int_{-\Delta t}^{\Delta t} (\Delta t - |s|)\big(\partial_t^2 v(\cdot, t_{n-1} + s) - \partial_t^2 v(\cdot, t_{n-2} + s)\big) ds \Big) \\
=& \frac{1}{(\Delta t)^4} \Big(\int_{-\Delta t}^{\Delta t} (\Delta t - |s|)\big(\partial_t^2 v(\cdot, t_n + s) - 2\partial_t^2 v(\cdot, t_{n-1} + s) + \partial_t^2 v(\cdot, t_{n-2} + s)\big) ds \Big) \\
=& \frac{1}{(\Delta t)^4} \int_{-\Delta t}^{\Delta t} (\Delta t - |s|) \int_{t_{n-2}+s}^{t_n+s} (\Delta t - |r - t_{n-1}|) \partial_t^4 v(\cdot, r) dr ds.
\end{aligned}$$

Therefore, we obtain the following estimate

$$\|\frac{\partial_{\Delta t}^3 v^n - \partial_{\Delta t}^3 v^{n-1}}{\Delta t}\|_X^2 \leq \frac{C}{\Delta t} \int_{t_{n-3}}^{t_{n+1}} \|\partial_t^4 v(\cdot, r)\|_X^2 dr. \tag{A.18}$$

Proof of (4.11) ***in Corollary*** 4.2. We now bound the term $\Xi^N(\partial_{\Delta t}^2 h_1, \partial_{\Delta t}^2 h_2, \partial_{\Delta t}^2 (\alpha g_1 - g_2), \partial_{\Delta t}^2 \tilde{g}_2)$ which is $\Xi^N(\mathcal{H}_1, \mathcal{H}_2, (\alpha \mathcal{G}_1 - \mathcal{G}_2), \tilde{\mathcal{G}}_2)$ according to (A.14). The techniques are similar to those mentioned above. We present the key estimates involving $\mathcal{H}_1$ and $\tilde{\mathcal{G}}_2$ first. It follows from the definition of $\mathcal{H}_1^{n+1}$ that,

$$
\begin{aligned}
\|\mathcal{H}_1^{n+1}\|^2_{L^2(\Omega_s)} &= \int_{\Omega_s} \left(\frac{h_1^{n+1} - 2h_1^n + h_1^{n-1}}{(\Delta t)^2} \right)^2 dx \\
&= \int_{\Omega_s} \left(\frac{\partial_{\Delta t}(\partial_t \mathsf{w})^{n+1} - \partial^2_{\Delta t}\mathsf{w}^{n+1} - (\partial_{\Delta t}(\partial_t \mathsf{w})^n - \partial^2_{\Delta t}\mathsf{w}^n)}{\Delta t} \right)^2 dx \\
&= \int_{\Omega_s} \left(\partial^2_{\Delta t}(\partial_t \mathsf{w})^{n+1} - \partial^3_{\Delta t}\mathsf{w}^{n+1} \right)^2 dx \\
&= \int_{\Omega_s} \left(\partial^2_{\Delta t}(\partial_t \mathsf{w})^{n+1} - \partial^3_t \mathsf{w}^n + \partial^3_t \mathsf{w}^n - \partial^3_{\Delta t}\mathsf{w}^{n+1} \right)^2 dx \\
&\leq C \int_{\Omega_s} \left(\frac{1}{(\Delta t)^2} \int_{t_{n-1}}^{t_{n+1}} \left(\Delta t - |s - t_n|\right)\left(\partial^3_t \mathsf{w}(s) - \partial^3_t \mathsf{w}(t_n)\right) ds \right)^2 dx \\
&+ C \int_{\Omega_s} \left(\frac{1}{(\Delta t)^3} \int_{-\Delta t}^{\Delta t} (\Delta t - |s|) \int_{t_{n-1}+s}^{t_n+s} \partial^3_t \mathsf{w}(t_n) - \partial^3_t \mathsf{w}(\cdot, r)\, ds \right)^2 dx \\
&\leq C\Delta t \|\partial^4_t \mathsf{w}\|^2_{L^2((t_{n-2},t_{n+1}),L^2(\Omega_s))}.
\end{aligned} \tag{A.19}
$$

We also have, according to the definition of $\tilde{\mathsf{l}}$ and the fact $\|\nabla\phi\|_{L^2(\Omega_f)} \leq C$ that,

$$
\begin{aligned}
\|\nabla \tilde{\mathcal{G}}_2^{n+1}\|^2_{L^2(\Omega_f)} &= \int_{\Omega_f} \left(\frac{\frac{\nabla\tilde{\mathsf{l}}^{n+1} - 2\nabla\tilde{\mathsf{l}}^n + \nabla\tilde{\mathsf{l}}^{n-1}}{\Delta t} - \frac{\nabla\tilde{\mathsf{l}}^n - 2\nabla\tilde{\mathsf{l}}^{n-1} + \nabla\tilde{\mathsf{l}}^{n-2}}{\Delta t}}{\Delta t} \right)^2 dx \\
&= \int_{\Omega_f} \left(\frac{1}{(\Delta t)^2} \int_{-\Delta t}^{\Delta t} (\Delta t - |s|)\left(\partial^2_t \nabla\tilde{\mathsf{l}}(\cdot, t_n + s) - \partial^2_t \nabla\tilde{\mathsf{l}}(\cdot, t_{n-1} + s)\right) ds \right)^2 dx \\
&\leq C \int_{\Omega_f} \left(\frac{1}{\Delta t} \int_{-\Delta t}^{\Delta t} \left| \int_{t_{n-1}+s}^{t_n+s} \partial^3_t \nabla\tilde{\mathsf{l}}(\cdot, r) dr \right| ds \right)^2 dx \\
&\leq C\Delta t \int_{\Omega_f} \int_{t_{n-2}}^{t_{n+1}} (\partial^3_t \nabla\tilde{\mathsf{l}}(\cdot, r))^2\, dr\, dx \\
&\leq C\nu_f^2 \Delta t \|\partial^3_t \mathsf{u}\|^2_{L^2(t_{n-2},t_{n+1};H^2(\Omega_f))}.
\end{aligned} \tag{A.20}
$$

It follows from (A.18) that,

$$
\begin{aligned}
\Delta t \sum_{n=0}^{N-1} \frac{1}{\nu_f \alpha^2} \|\partial_{\Delta t}\tilde{\mathcal{G}_2}^{n+1}\|^2_{L^2(\Omega_f)} &= (\Delta t)^3 \sum_{n=0}^{N-1} \frac{1}{\nu_f \alpha^2} \left\| \frac{\partial^3_{\Delta t}\tilde{\mathsf{l}}^n - \partial^3_{\Delta t}\tilde{\mathsf{l}}^{n-1}}{\Delta t} \right\|^2_{L^2(\Omega_f)} \\
&\leq C \frac{(\Delta t)^2}{\nu_f \alpha^2} \|\partial^4_t \tilde{\mathsf{l}}\|^2_{L^2(0,T;L^2(\Omega_f))} \\
&\leq C(\Delta t)^2 \frac{\nu_f}{\alpha^2} \|\partial^4_t \mathsf{u}\|^2_{L^2(0,T;H^1(\Omega_f))}.
\end{aligned} \tag{A.21}
$$

Then we follow the same idea as before and obtain the following bound:

$$\Xi^N(\mathcal{H}_1, \mathcal{H}_2, (\alpha\mathcal{G}_1 - \mathcal{G}_2), \tilde{\mathcal{G}}_2) \leq C(\Delta t)^2 \mathfrak{Y}, \tag{A.22}$$

where $\mathfrak{Y}$ is defined in (4.14). □

8 Appendix B: Sketch of proof: Corollary 4.5

Proof According to Corollary 4.4, we need to bound the terms on the right-hand side of the inequality (4.15). The first term in (4.15) is bounded in Corollary 4.2 while the last term in (4.15) can be easily bounded similar to (A.7). At last, the analysis to bound the term $\Xi^N(\partial_{\Delta t}\partial_x h_1, \partial_{\Delta t}\partial_x h_2, \partial_{\Delta t}\partial_x(\alpha g_1 - g_2), \partial_{\Delta t}\partial_x \tilde{g}_2)$ is almost identical to that of the term $\Xi^N(H_1, H_2, \alpha G_1 - G_2, \tilde{G}_2)$ except the additional partial derivative which does not affect the techniques. Therefore, we obtain the bound

$$\Xi^N(\partial_{\Delta t}\partial_x h_1, \partial_{\Delta t}\partial_x h_2, \partial_{\Delta t}\partial_x(\alpha g_1 - g_2), \partial_{\Delta t}\partial_x \tilde{g}_2) \leq C(\Delta t)^2 \mathbb{Y}, \tag{B.1}$$

where $\mathbb{Y}$ is defined in (4.16). This finishes the proof.

□

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Declarations

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Authors and Affiliations

Erik Burman[1] · Rebecca Durst[2,3] · Miguel A. Fernández[4,5] · Johnny Guzmán[6] · Sijing Liu[7]

✉ Sijing Liu
sijing_liu@brown.edu

Erik Burman
e.burman@ucl.ac.uk

Rebecca Durst
rebecca_durst@alumni.brown.edu; rdurst@anl.gov

Miguel A. Fernández
miguel.fernandez@inria.fr

Johnny Guzmán
johnny_guzman@brown.edu

1 Department of Mathematics, University College London, UK-WC1E 6BT London, United Kingdom

2 Department of Mathematics, University of Pittsburgh, 15261 Pittsburgh, Pennsylvania, USA

3 Present Address: Argonne National Laboratory, Lemont, IL 60439, USA

4 Sorbonne Université & CNRS, UMR 7598 LJLL, 75005 Paris, France

5 Inria, 75012 Paris, France

6 Division of Applied Mathematics, Brown University, 182 George Street, 02912 Providence, RI, USA

7 The Institute for Computational and Experimental Research in Mathematics, Brown University, 121 South Main Street, 02912 Providence, RI, USA