A nonconforming primal hybrid finite element method for the two-dimensional vector Laplacian

Mary Barker [1]
Shuhao Cao [2]
Ari Stern [3]

[1] Public Health Sciences Division, Fred Hutchinson Cancer Center
Email address: `marybarker103@gmail.com`
[2] Division of Computing, Analytics, and Mathematics, University of Missouri–Kansas City
Email address: `scao@umkc.edu`
[3] Department of Mathematics, Washington University in St. Louis
Email address: `stern@wustl.edu`.

Abstract. We introduce a nonconforming hybrid finite element method for the two-dimensional vector Laplacian, based on a primal variational principle for which conforming methods are known to be inconsistent. Consistency is ensured using penalty terms similar to those used to stabilize hybridizable discontinuous Galerkin (HDG) methods, with a carefully chosen penalty parameter due to Brenner, Li, and Sung [Math. Comp., 76 (2007), pp. 573–595]. Our method accommodates elements of arbitrarily high order and, like HDG methods, it may be implemented efficiently using static condensation. The lowest-order case recovers the P_1-nonconforming method of Brenner, Cui, Li, and Sung [Numer. Math., 109 (2008), pp. 509–533], and we show that higher-order convergence is achieved under appropriate regularity assumptions. The analysis makes novel use of a family of weighted Sobolev spaces, due to Kondrat'ev, for domains admitting corner singularities.

Keywords. nonconforming finite element methods, hybridization, hydridizable discontinuous Galerkin methods, vector Laplacian, weighted Sobolev spaces.

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1. Introduction

Given a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, $f \in \left[L^2(\Omega)\right]^2$, and $\alpha \in \mathbb{R}$, we consider the primal variational problem: Find $u \in H(\mathrm{div};\Omega) \cap \mathring{H}(\mathrm{curl};\Omega)$ such that, for all $v \in H(\mathrm{div};\Omega) \cap \mathring{H}(\mathrm{curl};\Omega)$,

$$(\nabla \cdot u, \nabla \cdot v)_\Omega + (\nabla \times u, \nabla \times v)_\Omega + \alpha(u,v)_\Omega = (f,v)_\Omega, \qquad (1.1)$$

where $(\cdot,\cdot)_\Omega$ denotes the L^2 inner product on Ω. Here, we recall the familiar spaces

$$H(\mathrm{div};\Omega) := \left\{v \in \left[L^2(\Omega)\right]^2 : \nabla \cdot v \in L^2(\Omega)\right\},$$
$$H(\mathrm{curl};\Omega) := \left\{v \in \left[L^2(\Omega)\right]^2 : \nabla \times v \in L^2(\Omega)\right\},$$
$$\mathring{H}(\mathrm{curl};\Omega) := \left\{v \in H(\mathrm{curl};\Omega) : v \times n = 0 \text{ on } \partial\Omega\right\},$$

where n is the outer unit normal vector field on the boundary. The strong form of (1.1) is the boundary value problem

$$-\nabla\nabla \cdot u + \nabla \times \nabla \times u + \alpha u = f \quad \text{in } \Omega, \tag{1.2a}$$

$$u \times n = 0 \quad \text{on } \partial\Omega, \tag{1.2b}$$

$$\nabla \cdot u = 0 \quad \text{on } \partial\Omega, \tag{1.2c}$$

noting that $-\nabla\nabla \cdot + \nabla \times \nabla \times = -\Delta$ is the negative vector Laplacian.

It is well established that conforming finite element methods for (1.1) have severe difficulties. For instance, a finite element approximation that is both div- and curl-conforming will also be H^1-conforming, but when Ω is non-convex, $\left[H^1(\Omega)\right]^2 \subsetneq H(\operatorname{div};\Omega) \cap H(\operatorname{curl};\Omega)$ is a proper closed subspace (see, e.g., [3, 20]). Consequently, such a method will fail to converge to solutions with reentrant corner singularities. Even when Ω is convex, other problems may arise, e.g., spurious modes for the eigenvalue problem [4, 1]. For this reason, the primal formulation (1.1) is often avoided in favor of mixed formulations of (1.2) that are *either* div- or curl-conforming, but not both.

Brenner, Cui, Li, and Sung [6] developed a P_1-*nonconforming* primal interior-penalty method that bypasses these difficulties. Let $\mathcal{T}_h$ be a conforming triangulation of Ω, and let $\mathcal{E}_h$ denote the set of edges, partitioned into interior edges $\mathcal{E}_h^\circ$ and boundary edges $\mathcal{E}_h^\partial$. Denote the broken L^2 inner products $(\cdot,\cdot)_{\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h}(\cdot,\cdot)_K$, $\langle\cdot,\cdot\rangle_{\mathcal{E}_h^\circ} := \sum_{e \in \mathcal{E}_h^\circ}\langle\cdot,\cdot\rangle_e$, and $\langle\cdot,\cdot\rangle_{\mathcal{E}_h^\partial} := \sum_{e \in \mathcal{E}_h^\partial}\langle\cdot,\cdot\rangle_e$. The method of [6] is based on the variational problem

$$(\nabla \cdot u_h, \nabla \cdot v_h)_{\mathcal{T}_h} + (\nabla \times u_h, \nabla \times v_h)_{\mathcal{T}_h} + \alpha(u_h, v_h)_{\mathcal{T}_h} + \left\langle \gamma[\![u_h]\!], [\![v_h]\!] \right\rangle_{\mathcal{E}_h^\circ} + \langle \gamma u_h \times n, v_h \times n\rangle_{\mathcal{E}_h^\partial}$$
$$= (f, v_h)_{\mathcal{T}_h}, \tag{1.3}$$

where $\gamma|_e := \gamma_e > 0$ is a penalty parameter on each $e \in \mathcal{E}_h$, to be detailed further in Section 2, and $[\![\cdot]\!]$ is the jump in both tangential and normal components across an interior edge. (See also Brenner, Li, and Sung [9, 10, 11, 12] for related work on curl-curl source problems and eigenproblems arising in Maxwell's equations.) In [6], u_h and v_h are linear vector fields continuous at the midpoint of each $e \in \mathcal{E}_h^\circ$ (i.e., both components live in the P_1-nonconforming space of Crouzeix and Raviart [17]) whose tangential components vanish at the midpoint of each $e \in \mathcal{E}_h^\partial$. Brenner and Sung [13] later developed a quadratic nonconforming element for this problem and conjectured that it could be generalized to higher degree, as well as to dimension three. The two-dimensional conjecture was subsequently proved by Mirebeau [27], who also gave a counterexample to the three-dimensional case. However, for $k > 1$, the order-k elements are not simply P_k vector fields: they are enriched by additional vector fields up to degree $2k - 1$ that are gradients of harmonic polynomials.

In this paper, we present a three-field primal hybridization of (1.3) in the following form: Find $(u_h, p_h, \hat{u}_h) \in V_h \times Q_h \times \hat{V}_h$ such that, for all $(v_h, q_h, \hat{v}_h) \in V_h \times Q_h \times \hat{V}_h$,

$$(\nabla \cdot u_h, \nabla \cdot v_h)_{\mathcal{T}_h} + (\nabla \times u_h, \nabla \times v_h)_{\mathcal{T}_h} + \alpha(u_h, v_h)_{\mathcal{T}_h} + \langle\hat{p}_h, v_h\rangle_{\partial\mathcal{T}_h} = (f, v_h)_{\mathcal{T}_h}, \tag{1.4a}$$

$$\langle u_h - \hat{u}_h, q_h\rangle_{\partial\mathcal{T}_h} = 0, \tag{1.4b}$$

$$\langle\hat{p}_h, \hat{v}_h\rangle_{\partial\mathcal{T}_h} = 0, \tag{1.4c}$$

where $\hat{p}_h := p_h + \gamma(u_h - \hat{u}_h)$ and $\langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h} := \sum_{K \in \mathcal{T}_h}\langle\cdot,\cdot\rangle_{\partial K}$. With appropriately chosen finite element spaces, as detailed in Section 2, this method has the following properties:

- The lowest-order case is a hybridization of the method of Brenner et al. [6].

- Arbitrarily high order may be obtained using standard polynomial finite elements. The more exotic Brenner–Sung–Mirebeau spaces and projections play a crucial role in the analysis but are not needed for implementation.

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- As with HDG methods [14], the hybrid formulation enables efficient local assembly and static condensation, where u_h and p_h may be eliminated to solve a smaller global system involving only the approximate trace $\hat{u}_h$.

In addition to these contributions, we also present a novel error analysis using weighted Sobolev spaces, cf. Costabel and Dauge [16]. This approach allows us to obtain error estimates on domains admitting corner singularities, without imposing the mesh-grading conditions on $\mathcal{T}_h$ required by [6].

The paper is organized as follows. In Section 2, we describe the method and discuss its fundamental properties. Next, in Section 3, we present the error analysis of the method. Finally, in Section 4, we present the results of numerical experiments, which demonstrate and confirm the analytically obtained convergence results.

2. The hybrid method

2.1. Description of the method

The proposed method is based on the variational problem (1.4), using the following finite element spaces. Given a positive integer k, define

$$V_h \coloneqq \prod_{K \in \mathcal{T}_h} \left[P_{2k-1}(K) \right]^2, \qquad Q_h \coloneqq \prod_{K \in \mathcal{T}_h} \prod_{e \subset \partial K} \left[P_{k-1}(e) \right]^2.$$

These are "broken" finite element spaces, with no inter-element continuity or boundary conditions imposed, so vector fields in these spaces are generally double-valued at interior edges. To (weakly) impose inter-element continuity and boundary conditions, we define

$$\hat{V}_h \coloneqq \left\{ \hat{v}_h \in \prod_{e \in \mathcal{E}_h} \left[P_{2k-1}(e) \right]^2 : \hat{v}_h \times n = 0 \text{ on } \mathcal{E}_h^{\partial} \right\},$$

whose elements are single-valued on edges.

The extra variables p_h and $\hat{u}_h$, and their role in the variational problem, may be understood as follows. From (1.4b), we see that p_h acts as a Lagrange multiplier, constraining the degree $\leq k-1$ moments of u_h and $\hat{u}_h$ to agree on $\mathcal{E}_h$. Consequently, u_h satisfies weak inter-element continuity and boundary conditions, and $\hat{u}_h$ may be seen as an approximate trace of u. Next, on each $K \in \mathcal{T}_h$, taking the inner product of the strong form (1.2a) with $v \in H(\mathrm{div}; K) \cap H(\mathrm{curl}; K)$ and integrating by parts implies that the solution to (1.1) satisfies

$$(\nabla \cdot u, \nabla \cdot v)_K + (\nabla \times u, \nabla \times v)_K + \alpha(u, v)_K - \langle \nabla \cdot u, v \cdot n \rangle_{\partial K} + \langle \nabla \times u, v \times n \rangle_{\partial K} = (f, v)_K. \quad (2.1)$$

Comparing with (1.4a) and writing $\langle \hat{p}_h, v_h \rangle_{\partial K} = \langle \hat{p}_h \cdot n, v_h \cdot n \rangle_{\partial K} + \langle \hat{p}_h \times n, v_h \times n \rangle_{\partial K}$, it follows that $\hat{p}_h \cdot n|_{\partial K}$ and $\hat{p}_h \times n|_{\partial K}$ can be seen as approximating $-\nabla \cdot u|_{\partial K}$ and $\nabla \times u|_{\partial K}$, respectively. Finally, (1.4c) shows that $\hat{u}_h$ also acts as a Lagrange multiplier, constraining $\hat{p}_h \cdot n$ and $\hat{p}_h \times n$ to be single-valued on interior edges and $\hat{p}_h \cdot n = 0$ on boundary edges. The latter may be seen as an approximation of the natural boundary condition (1.2c).

To ensure convergence of the method for solutions with corner singularities, the penalty γ must be chosen carefully. Here, we recall the penalty used by Brenner et al. [6], which is the same one that we will use. Denote the corners of Ω by $c_1, \ldots, c_L$, and let $r_\ell(x) \coloneqq |x - c_\ell|$ be the distance from $x \in \Omega$ to each corner. Given a multi-exponent $\lambda = (\lambda_1, \ldots, \lambda_L)$, we denote $r^\lambda \coloneqq \prod_{\ell=1}^L r_\ell^{\lambda_\ell}$. Now, at each corner c_ℓ, with interior angle ω_ℓ, choose a parameter μ_ℓ such that

$$\mu_\ell = 1 \quad \text{if } \omega_\ell \leq \frac{\pi}{2}, \qquad \mu_\ell < \frac{\pi}{2\omega_\ell} \quad \text{if } \omega_\ell > \frac{\pi}{2},$$

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and $\mu := (\mu_1, \ldots, \mu_L)$. For each $e \in \mathcal{E}_h$, whose midpoint is denoted m_e, we then define

$$\Phi_\mu(e) := r^{1-\mu}(m_e) = \prod_{\ell=1}^{L} |m_e - c_\ell|^{1-\mu_\ell}.$$

Finally, the penalty parameter on e is taken to be

$$\gamma_e := \frac{\left[\Phi_\mu(e)\right]^2}{|e|},$$

where $|e|$ is the length of e. This ensures that $\gamma_e \sim 1/|e|$ away from corners, while being appropriately weakened near corners to allow convergence to singular solutions, as we will see in Section 3.

2.2. The Brenner–Sung–Mirebeau element and projection

We now recall the nonconforming finite element developed in Brenner and Sung [13] and Mirebeau [27], which we call the Brenner–Sung–Mirebeau (BSM) element. While it is not used to implement the method described above, this element and its associated projection play an important role in the numerical analysis of the method—and will also make clear why we have taken polynomial spaces of degrees $2k - 1$ and $k - 1$.

Definition 2.1. Given a positive integer k, define the *Brenner–Sung–Mirebeau (BSM) space* on a triangle $K \subset \mathbb{R}^2$ to be

$$BSM_k(K) := \left[P_k(K)\right]^2 \oplus \nabla H_{k+2}(K) \oplus \cdots \oplus \nabla H_{2k}(K),$$

where $H_j(K)$ is the space of homogeneous harmonic polynomials of degree j on K. (By *harmonic*, we mean having vanishing Laplacian.)

We immediately see that $\left[P_k(K)\right]^2 \subset BSM_k(K) \subset \left[P_{2k-1}(K)\right]^2$, with equality if and only if $k = 1$. Indeed, since $\dim H_j(K) = 2$ for each j, it follows that $\dim BSM_k(K) = k(k+5)$. Brenner and Sung [13] conjectured, and Mirebeau [27] proved, that an element $v_h \in BSM_k(K)$ is uniquely determined by the $k(k+5)$ degrees of freedom

$$\langle v_h, q_h \rangle_{\partial K} \text{ for } q_h \in \prod_{e \subset \partial K} \left[P_{k-1}(e)\right]^2, \qquad (v_h, w_h)_K \text{ for } w_h \in \left[P_{k-2}(K)\right]^2.$$

Moreover, the canonical interpolation using these degrees of freedom naturally defines a projection $\Pi_h \colon \left[H^\sigma(K)\right]^2 \to BSM_k(K)$, for any $\sigma > \frac{1}{2}$, such that

$$\langle \Pi_h v, q_h \rangle_{\partial K} = \langle v, q_h \rangle_{\partial K}, \qquad (\Pi_h v, w_h)_K = (v, w_h)_K, \tag{2.2}$$

for all q_h and w_h as above. Letting $P_h \colon L^2(K) \to P_{k-1}(K)$ be the L^2-orthogonal projection for scalar fields, we obtain the following commuting-projection property; the proof is basically identical to that in Brenner and Sung [13].

Lemma 2.2. *For all $v \in \left[H^\sigma(K)\right]^2$ with $\sigma > \frac{1}{2}$ such that $\nabla \cdot v, \nabla \times v \in L^2(K)$, we have*

$$\nabla \cdot \Pi_h v = P_h(\nabla \cdot v), \qquad \nabla \times \Pi_h v = P_h(\nabla \times v).$$

Proof. For all $\phi_h \in P_{k-1}(K)$, integrating by parts using the divergence theorem gives

$$\left(\nabla \cdot (v - \Pi_h v), \phi_h\right)_K = \langle v - \Pi_h v, \phi_h n \rangle_{\partial K} - (v - \Pi_h v, \nabla \phi_h)_K = 0,$$

by (2.2) with $q_h = \phi_h n$ and $w_h = \nabla \phi_h$. This proves the first equality; the proof of the second is essentially the same, using Green's theorem instead of the divergence theorem. ∎

The solution to (1.1) satisfies the hypotheses of this lemma on each $K \in \mathcal{T}_h$, as a result of the regularity theory discussed in Section 3.1.

2.3. **Equivalence to reduced methods with jump terms**

We next show that the three-field hybrid method described in Section 2.1 may be reduced to a two-field or one-field method with jump terms. The coupling introduced by the jump terms prevents static condensation, so we generally prefer the three-field formulation for implementation. However, these reduced formulations will be useful analytically, and will help in relating our method to that of Brenner et al. [6].[1]

First, we introduce notation and definitions for the average and jump of a vector field across interior edges. Suppose $e \in \mathcal{E}_h^\circ$ is an interior edge shared by two triangles, K^+ and K^-, and let $n^\pm$ denote the unit normal to e pointing outward from $K^\pm$. If a vector field w takes values $w^\pm$ on the $K^\pm$ sides of e, we define the average and jump of w at e to be

$$\{\!\!\{w\}\!\!\}_e := \frac{1}{2}(w^+ + w^-), \qquad [\![w]\!]_e := w^+ \otimes n^+ + w^- \otimes n^-,$$

where $w \otimes n := w n^\top$ is the outer product. It is straightforward to see that the i-th row of $[\![w]\!]_e$ is the transpose of the usual scalar jump $[\![w_i]\!]_e = w_i^+ n^+ + w_i^- n^-$ for $i = 1, 2$. This definition of $[\![\cdot]\!]$ encodes the jump in both tangential and normal directions, without requiring a global orientation of the edges. It is then easily verified that the $\langle \cdot, \cdot \rangle_{\partial \mathcal{T}_h}$ inner product of vector fields (just as for scalar fields) may be expanded as

$$\langle w, v \rangle_{\partial \mathcal{T}_h} = 2 \langle \{\!\!\{w\}\!\!\}, \{\!\!\{v\}\!\!\} \rangle_{\mathcal{E}_h^\circ} + \frac{1}{2} \langle [\![w]\!], [\![v]\!] \rangle_{\mathcal{E}_h^\circ} + \langle w, v \rangle_{\mathcal{E}_h^\partial}, \tag{2.3}$$

where the inner product of the matrix-valued jumps is taken in the Frobenius sense. Since functions are single-valued on boundary edges, we leave average and jump undefined on $\mathcal{E}_h^\partial$.

Lemma 2.3. *If $(u_h, p_h, \hat{u}_h)$ satisfies* (1.4b)–(1.4c), *then*

$$\{\!\!\{u_h\}\!\!\} = \hat{u}_h, \qquad \{\!\!\{p_h\}\!\!\} = 0, \quad on \; \mathcal{E}_h^\circ,$$
$$u_h \cdot n = \hat{u}_h \cdot n, \qquad p_h \cdot n = 0, \quad on \; \mathcal{E}_h^\partial.$$

Proof. Let $q_h = \{\!\!\{p_h\}\!\!\}$ on $\mathcal{E}_h^\circ$ and $q_h = (p_h \cdot n)n$ on $\mathcal{E}_h^\partial$. Taking this as the test function in (1.4b) and applying the identity (2.3) gives

$$2 \langle \{\!\!\{u_h\}\!\!\} - \hat{u}_h, \{\!\!\{p_h\}\!\!\} \rangle_{\mathcal{E}_h^\circ} + \langle (u_h - \hat{u}_h) \cdot n, p_h \cdot n \rangle_{\mathcal{E}_h^\partial} = 0. \tag{2.4}$$

There are no interior jump-jump or tangential boundary terms, since this choice of q_h has $[\![q_h]\!] = 0$ on $\mathcal{E}_h^\circ$ and $q_h \times n = 0$ on $\mathcal{E}_h^\partial$. Similarly, $[\![\hat{v}_h]\!] = 0$ on $\mathcal{E}_h^\circ$ and $\hat{v}_h \times n = 0$ on $\mathcal{E}_h^\partial$ for all $\hat{v}_h \in \hat{V}_h$, so (1.4c) may be rewritten as

$$2 \langle \{\!\!\{p_h\}\!\!\} + \gamma(\{\!\!\{u_h\}\!\!\} - \hat{u}_h), \hat{v}_h \rangle_{\mathcal{E}_h^\circ} + \langle p_h \cdot n + \gamma(u_h - \hat{u}_h) \cdot n, \hat{v}_h \cdot n \rangle_{\mathcal{E}_h^\partial} = 0. \tag{2.5}$$

Now, take $\hat{v}_h = \{\!\!\{u_h\}\!\!\} - \hat{u}_h$ on $\mathcal{E}_h^\circ$ and $\hat{v}_h \cdot n = (u_h - \hat{u}_h) \cdot n$ on $\mathcal{E}_h^\partial$. The terms involving p_h vanish by (2.4), leaving

$$2 \langle \gamma(\{\!\!\{u_h\}\!\!\} - \hat{u}_h), \{\!\!\{u_h\}\!\!\} - \hat{u}_h \rangle_{\mathcal{E}_h^\circ} + \langle \gamma(u_h - \hat{u}_h) \cdot n, (u_h - \hat{u}_h) \cdot n \rangle_{\mathcal{E}_h^\partial} = 0.$$

Since $\gamma > 0$, it follows that $\{\!\!\{u_h\}\!\!\} = \hat{u}_h$ on $\mathcal{E}_h^\circ$ and $u_h \cdot n = \hat{u}_h \cdot n$ on $\mathcal{E}_h^\partial$, as claimed. Finally, substituting these equalities into (2.5) gives

$$2 \langle \{\!\!\{p_h\}\!\!\}, \hat{v}_h \rangle_{\mathcal{E}_h^\circ} + \langle p_h \cdot n, \hat{v}_h \cdot n \rangle_{\mathcal{E}_h^\partial} = 0,$$

so taking $\hat{v}_h = \{\!\!\{p_h\}\!\!\}$ on $\mathcal{E}_h^\circ$ and $\hat{v}_h \cdot n = p_h \cdot n$ on $\mathcal{E}_h^\partial$ completes the proof. ∎

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[1]This approach to reduction was inspired by some recent papers on "extended Galerkin" methods [22, 21].

Remark 2.4. Note that $\{\!\{w\}\!\}_e = 0$ can be rewritten as $w^+ = -w^-$. Since the outer normals satisfy $n^+ = -n^-$, it follows that $w^+ \times n^+ = w^- \times n^-$ and $w^+ \cdot n^+ = w^- \cdot n^-$, i.e., the tangential and normal components of w agree on both sides of e. Thus, Lemma 2.3 says that the tangential and normal components of p_h and $u_h - \hat{u}_h$ are single-valued, with normal components vanishing on boundary edges. In particular, the same is therefore true of $\hat{p}_h$, as previously remarked in Section 2.1.

Using Lemma 2.3 and the identity (2.3), observe that the edge terms in (1.4a) reduce to

$$\langle \hat{p}_h, v_h \rangle_{\partial \mathcal{T}_h} = \frac{1}{2} \langle [\![\hat{p}_h]\!], [\![v_h]\!] \rangle_{\mathcal{E}_h^\circ} + \langle \hat{p}_h \times n, v_h \times n \rangle_{\mathcal{E}_h^\partial}$$

$$= \frac{1}{2} \langle [\![p_h + \gamma u_h]\!], [\![v_h]\!] \rangle_{\mathcal{E}_h^\circ} + \langle (p_h + \gamma u_h) \times n, v_h \times n \rangle_{\mathcal{E}_h^\partial},$$

for all $v_h \in V_h$. Similarly, the edge terms in (1.4b) reduce to

$$\langle u_h - \hat{u}_h, q_h \rangle_{\partial \mathcal{T}_h} = \frac{1}{2} \langle [\![u_h]\!], [\![q_h]\!] \rangle_{\mathcal{E}_h^\circ} + \langle u_h \times n, q_h \times n \rangle_{\mathcal{E}_h^\partial},$$

for all $q_h \in Q_h$. This allows us to eliminate $\hat{u}_h$ and the equation (1.4c) from the variational problem. A two-field reduced formulation is defined as follows. Let

$$\mathring{Q}_h := \left\{ q_h \in Q_h : \{\!\{ q_h \}\!\} = 0 \text{ on } \mathcal{E}_h^\circ \text{ and } q_h \cdot n = 0 \text{ on } \mathcal{E}_h^\partial \right\},$$

and define the bilinear forms $a_h \colon V_h \times V_h \to \mathbb{R}$ and $b_h \colon V_h \times \mathring{Q}_h \to \mathbb{R}$ by

$$a_h(u_h, v_h) := (\nabla \cdot u_h, \nabla \cdot v_h)_{\mathcal{T}_h} + (\nabla \times u_h, \nabla \times v_h)_{\mathcal{T}_h} + \alpha(u_h, v_h)_{\mathcal{T}_h}$$

$$+ \frac{1}{2} \langle \gamma [\![u_h]\!], [\![v_h]\!] \rangle_{\mathcal{E}_h^\circ} + \langle \gamma u_h \times n, v_h \times n \rangle_{\mathcal{E}_h^\partial},$$

$$b_h(v_h, q_h) := \frac{1}{2} \langle [\![v_h]\!], [\![q_h]\!] \rangle_{\mathcal{E}_h^\circ} + \langle v_h \times n, q_h \times n \rangle_{\mathcal{E}_h^\partial}.$$

We then consider the problem: Find $(u_h, p_h) \in V_h \times \mathring{Q}_h$ such that, for all $(v_h, q_h) \in V_h \times \mathring{Q}_h$,

$$a_h(u_h, v_h) + b_h(v_h, p_h) = (f, v_h)_{\mathcal{T}_h}, \tag{2.6a}$$

$$b_h(u_h, q_h) = 0. \tag{2.6b}$$

This resembles a standard two-field hybrid method in saddle-point form, where $\mathring{Q}_h$ is the space of Lagrange multipliers. Compare with the nonconforming hybrid method of Raviart and Thomas [31] for the scalar Poisson equation.

Finally, we may reduce even further to a one-field formulation on

$$\mathring{V}_h := \left\{ v_h \in V_h : b_h(v_h, q_h) = 0 \text{ for all } q_h \in \mathring{Q}_h \right\},$$

consisting of degree-$(2k-1)$ vector fields whose degree $\leq k-1$ moments are continuous on $\mathcal{E}_h^\circ$ and have vanishing tangential component on $\mathcal{E}_h^\partial$. We then consider the problem: Find $u_h \in \mathring{V}_h$ such that, for all $v_h \in \mathring{V}_h$,

$$a_h(u_h, v_h) = (f, v_h)_{\mathcal{T}_h}. \tag{2.7}$$

This is precisely (1.3), modulo a constant factor of $\frac{1}{2}$ for the penalty on interior edges, and the lowest-order case $k = 1$ recovers the method of Brenner et al. [6].

We have thus shown the equivalence of the three-field, two-field, and one-field formulations, which we now state as a lemma.

Lemma 2.5. *The following are equivalent:*

(i) $(u_h, p_h, \hat{u}_h) \in V_h \times Q_h \times \hat{V}_h$ is a solution to (1.4);

(ii) $(u_h, p_h) \in V_h \times \mathring{Q}_h$ is a solution to (2.6), $\hat{u}_h = \{\!\{ u_h \}\!\}$ on $\mathcal{E}_h^\circ$, and $\hat{u}_h \cdot n = u_h \cdot n$ on $\mathcal{E}_h^\partial$;

(iii) $u_h \in \mathring{V}_h$ is a solution to (2.7), p_h satisfies (2.6a) for all $v_h \in V_h$, and $\hat{u}_h$ is as in (ii).

2.4. **Existence/uniqueness and static condensation**

The problem (1.1) is well-posed if and only if α is not an eigenvalue of the vector Laplacian on $H(\mathrm{div}; \Omega) \cap \mathring{H}(\mathrm{curl}; \Omega)$. In particular, the bilinear form

$$a(u, v) := (\nabla \cdot u, \nabla \cdot v)_\Omega + (\nabla \times u, \nabla \times v)_\Omega + \alpha(u, v)_\Omega$$

on $H(\mathrm{div}; \Omega) \cap \mathring{H}(\mathrm{curl}; \Omega)$ is coercive if $\alpha > 0$, and if the complement of Ω is connected (e.g., Ω is simply connected), then it is also coercive for $\alpha = 0$ by Friedrichs's inequality (cf. Monk [28]).

We are now ready to prove our main result on existence and uniqueness for the hybrid method.

Theorem 2.6. *For the problems (1.4), (2.6), and (2.7), existence and uniqueness of solutions holds—or fails to hold—simultaneously for all three. In particular, all three are uniquely solvable if $\alpha > 0$, and if the complement of Ω is connected, then this is also true for $\alpha = 0$.*

Proof. By Lemma 2.5, unique solvability of (1.4) is equivalent to that of (2.6), since $\hat{u}_h$ is uniquely determined by u_h, so it suffices to show equivalence of (2.6) and (2.7). Using classic saddle-point theory (cf. Boffi et al. [5, Theorem 3.2.1]), (2.6) is uniquely solvable if and only if $v_h \mapsto b_h(v_h, \cdot)$ is surjective and the restriction of $a_h(\cdot, \cdot)$ to its kernel is an isomorphism. The isomorphism-on-the-kernel condition is precisely the unique solvability of (2.7), so it remains to show that the surjectivity condition holds. In fact, we will show something slightly stronger, which is surjectivity of the map $B_h \colon V_h \to Q_h^*$, $\langle B_h v_h, q_h \rangle_{Q_h^* \times Q_h} := \langle v_h, q_h \rangle_{\partial \mathcal{T}_h}$, which agrees with $b_h(v_h, q_h)$ when $q_h \in \mathring{Q}_h$ by (2.3). Given $q_h \in Q_h$, on each $K \in \mathcal{T}_h$ there exists $v_h|_K \in BSM_k(K)$ whose degree $\leq k - 1$ moments agree with q_h on ∂K. Combining these into $v_h \in V_h$, it follows that $\langle v_h, q_h \rangle_{\partial \mathcal{T}_h} = \langle q_h, q_h \rangle_{\partial \mathcal{T}_h}$, which is strictly positive whenever q_h is nonzero. Hence, the transpose of B_h is injective, so B_h is surjective.

Finally, if $\alpha > 0$, then $a_h(\cdot, \cdot)$ is clearly positive-definite, so (2.7) is uniquely solvable. If $\alpha = 0$, then $a_h(\cdot, \cdot)$ is generally only positive-semidefinite. However, observe that $a_h(u_h, u_h) = 0$ implies $[\![u_h]\!] = 0$ on $\mathcal{E}_h^\circ$ and $u_h \times n = 0$ on $\mathcal{E}_h^\partial$, so $u_h \in H(\mathrm{div}; \Omega) \cap \mathring{H}(\mathrm{curl}; \Omega)$ with $a(u_h, u_h) = 0$. If the complement of Ω is connected, then $a(\cdot, \cdot)$ is coercive, so $u_h = 0$, and thus $a_h(\cdot, \cdot)$ is positive-definite. ∎

Next, we discuss the static condensation of the hybrid method, which eliminates the spaces V_h and Q_h from (1.4) to obtain a smaller global variational problem on $\hat{V}_h$ alone. We take a similar approach to that used for HDG methods in Cockburn, Gopalakrishnan, and Lazarov [14]. Observe that, given $\hat{u}_h$ and f, (1.4a)–(1.4b) state that $(u_h, p_h) \in V_h \times Q_h$ solves the local problems

$$(\nabla \cdot u_h, \nabla \cdot v_h)_K + (\nabla \times u_h, \nabla \times v_h)_K + \alpha(u_h, v_h)_K + \langle p_h + \gamma u_h, v_h \rangle_{\partial K} = (f, v_h)_K + \langle \gamma \hat{u}_h, v_h \rangle_{\partial K},$$

$$\langle u_h, q_h \rangle_{\partial K} = \langle \hat{u}_h, q_h \rangle_{\partial K},$$

for all $K \in \mathcal{T}_h$ and $(v_h, q_h) \in V_h \times Q_h$. On each $K \in \mathcal{T}_h$, define the local bilinear forms

$$a_K(u_h, v_h) := (\nabla \cdot u_h, \nabla \cdot v_h)_K + (\nabla \times u_h, \nabla \times v_h)_K + \alpha(u_h, v_h)_K + \langle \gamma u_h, v_h \rangle_{\partial K},$$

$$b_K(v_h, q_h) := \langle q_h, v_h \rangle_{\partial K}.$$

To separate the influence of $\hat{u}_h$ and f, we define two local solvers: Find $(\mathsf{U}\hat{u}_h, \mathsf{P}\hat{u}_h) \in V_h \times Q_h$ and $(\mathsf{U}f, \mathsf{P}f) \in V_h \times Q_h$ such that

$$a_K(\mathsf{U}\hat{u}_h, v_h) + b_K(v_h, \mathsf{P}\hat{u}_h) = \langle \gamma \hat{u}_h, v_h \rangle_{\partial K}, \qquad a_K(\mathsf{U}f, v_h) + b_K(v_h, \mathsf{P}f) = (f, v_h)_K, \qquad (2.8\mathrm{a})$$

$$b_K(\mathsf{U}\hat{u}_h, q_h) = \langle \hat{u}_h, q_h \rangle_{\partial K}, \qquad\qquad b_K(\mathsf{U}f, q_h) = 0, \qquad\qquad (2.8\mathrm{b})$$

for all $K \in \mathcal{T}_h$ and $(v_h, q_h) \in V_h \times Q_h$.

Lemma 2.7. *If $\alpha \geq 0$, then the local solvers are well-defined, i.e., (2.8) is uniquely solvable.*

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Proof. First, we show that $\sum_{K \in \mathcal{T}_h} a_K(\cdot, \cdot)$ is coercive. This is obvious when $\alpha > 0$; when $\alpha = 0$, $a_K(u_h, u_h) = 0$ implies that $u|_{\partial K} = 0$, so Friedrichs's inequality implies $u|_K = 0$. Finally, the surjectivity of $v_h \mapsto \sum_{K \in \mathcal{T}_h} b_K(v_h, \cdot) = B_h v_h$ has already been shown in the proof of Theorem 2.6. ∎

Assuming the local solvers are well-defined—which always holds for $\alpha \geq 0$, by Lemma 2.7—we now define $\hat{\mathsf{P}} \hat{u}_h := \mathsf{P} \hat{u}_h + \gamma(\mathsf{U} \hat{u}_h - \hat{u}_h)$ and $\hat{\mathsf{P}} f := \mathsf{P} f + \gamma \mathsf{U} f$. Substituting into (1.4c) and rearranging gives the condensed problem: Find $\hat{u}_h \in \hat{V}_h$ such that, for all $\hat{v}_h \in \hat{V}_h$,

$$- \langle \hat{\mathsf{P}} \hat{u}_h, \hat{v}_h \rangle_{\partial \mathcal{T}_h} = \langle \hat{\mathsf{P}} f, \hat{v}_h \rangle_{\partial \mathcal{T}_h}. \tag{2.9}$$

Since the local solvers may be computed element-by-element, in parallel if desired, the condensation from (1.4) to (2.9) is efficient to implement. The condensed bilinear form $\hat{a}_h(\hat{u}_h, \hat{v}_h) := -\langle \hat{\mathsf{P}} \hat{u}_h, \hat{v}_h \rangle_{\partial \mathcal{T}_h}$ on the left-hand side of (2.9) has the following useful symmetric expression.

Lemma 2.8. *For all $\hat{u}_h, \hat{v}_h \in \hat{V}_h$,*

$$\hat{a}_h(\hat{u}_h, \hat{v}_h) = (\nabla \cdot \mathsf{U} \hat{u}_h, \nabla \cdot \mathsf{U} \hat{v}_h)_{\mathcal{T}_h} + (\nabla \times \mathsf{U} \hat{u}_h, \nabla \times \mathsf{U} \hat{v}_h)_{\mathcal{T}_h} + \alpha(\mathsf{U} \hat{u}_h, \mathsf{U} \hat{v}_h)_{\mathcal{T}_h}$$
$$+ \left\langle \gamma(\mathsf{U} \hat{u}_h - \hat{u}_h), \mathsf{U} \hat{v}_h - \hat{v}_h \right\rangle_{\partial \mathcal{T}_h}.$$

Proof. We begin by writing

$$- \langle \hat{\mathsf{P}} \hat{u}_h, \hat{v}_h \rangle_{\partial \mathcal{T}_h} = -\langle \hat{\mathsf{P}} \hat{u}_h, \mathsf{U} \hat{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \hat{\mathsf{P}} \hat{u}_h, \mathsf{U} \hat{v}_h - \hat{v}_h \rangle_{\partial \mathcal{T}_h}.$$

For the first term, (2.8a) with $v_h = \mathsf{U} \hat{v}_h$ implies

$$- \langle \hat{\mathsf{P}} \hat{u}_h, \mathsf{U} \hat{v}_h \rangle_{\partial \mathcal{T}_h} = (\nabla \cdot \mathsf{U} \hat{u}_h, \nabla \cdot \mathsf{U} \hat{v}_h)_{\mathcal{T}_h} + (\nabla \times \mathsf{U} \hat{u}_h, \nabla \times \mathsf{U} \hat{v}_h)_{\mathcal{T}_h} + \alpha(\mathsf{U} \hat{u}_h, \mathsf{U} \hat{v}_h)_{\mathcal{T}_h}.$$

For the second term, (2.8b) implies $\langle \mathsf{P} \hat{u}_h, \mathsf{U} \hat{v}_h - \hat{v}_h \rangle_{\partial \mathcal{T}_h} = 0$, so

$$\langle \hat{\mathsf{P}} \hat{u}_h, \mathsf{U} \hat{v}_h - \hat{v}_h \rangle_{\partial \mathcal{T}_h} = \left\langle \gamma(\mathsf{U} \hat{u}_h - \hat{u}_h), \mathsf{U} \hat{v}_h - \hat{v}_h \right\rangle_{\partial \mathcal{T}_h},$$

which completes the proof. ∎

Theorem 2.9. *Assuming the local solvers are well-defined, $(u_h, p_h, \hat{u}_h) \in V_h \times Q_h \times \hat{V}_h$ is a solution of (1.4) if and only if $\hat{u}_h$ is a solution of (2.9) with $u_h = \mathsf{U} \hat{u}_h + \mathsf{U} f$ and $p_h = \mathsf{P} \hat{u}_h + \mathsf{P} f$. Consequently, (1.4) is uniquely solvable if and only if (2.9) is. In particular, $\hat{a}_h(\cdot, \cdot)$ is symmetric positive-definite if $\alpha > 0$, and if the complement of Ω is connected, then this is also true for $\alpha = 0$.*

Proof. The equivalence of (1.4) and (2.9) has already been demonstrated in the discussion above. When $\alpha \geq 0$, Lemma 2.7 states that the local solvers are well-defined, and Lemma 2.8 implies that $\hat{a}_h(\cdot, \cdot)$ is symmetric positive-semidefinite. Furthermore, if $\hat{a}_h(\hat{u}_h, \hat{u}_h) = 0$, then $\mathsf{U} \hat{u}_h = \hat{u}_h$ on $\mathcal{E}_h$, so $\mathsf{U} \hat{u}_h \in H(\operatorname{div}; \Omega) \cap \mathring{H}(\operatorname{curl}; \Omega)$ with $a(\mathsf{U} \hat{u}_h, \mathsf{U} \hat{u}_h) = 0$. Hence, as in the proof of Theorem 2.6, $\hat{a}_h(\cdot, \cdot)$ is positive-definite whenever $a(\cdot, \cdot)$ is. ∎

Remark 2.10. These results tell us that static condensation from (1.4) to (2.9) does not merely reduce the size of the global system. It also makes the system more amenable to efficient global solvers, such as the conjugate gradient method in the case where (2.9) is positive-definite.

3. Regularity and error analysis

3.1. Weighted Sobolev spaces and regularity

Costabel and Dauge [16] characterize the regularity of solutions to Maxwell's equations in two dimensions (as well as in three) using a family of weighted Sobolev spaces due to Kondrat'ev [23]. We now recall

these spaces and give corresponding regularity results for the problem (1.1), combining the approach used in [16] with that of Brenner et al. [6, Section 2]. For detailed treatments of Kondrat'ev spaces and elliptic regularity in domains with corners, we refer the reader to Nazarov and Plamenevsky [29] and Kozlov, Maz'ya, and Rossmann [24].

As in Section 2.1, let $r_\ell(x)$ denote the distance from $x \in \Omega$ to a corner c_ℓ and $r^\lambda := \prod_{\ell=1}^{L} r_\ell^{\lambda_\ell}$ for a multi-exponent $\lambda = (\lambda_1, \ldots, \lambda_L)$. Given a nonnegative integer m, define the weighted Sobolev space

$$V_\lambda^m(\Omega) := \left\{ \phi \in \mathcal{D}'(\Omega) : r^{\lambda - m + |\beta|} \partial^\beta \phi \in L^2(\Omega) \text{ for all } |\beta| \leq m \right\},$$

where β is a multi-index, equipped with the natural norm defined by

$$\|\phi\|_{m,\lambda}^2 := \sum_{|\beta| \leq m} \|r^{\lambda - m + |\beta|} \partial^\beta \phi\|_\Omega^2.$$

This space also has the following equivalent characterization: If $\Omega = \Omega_0 \cup \bigcup_{\ell=1}^{L} \Omega_\ell$, where $\overline{\Omega}_0$ contains none of the corners and $\overline{\Omega}_\ell$ contains only corner c_ℓ, then

$$V_\lambda^m(\Omega) = \left\{ \phi \in \mathcal{D}'(\Omega) : \phi|_{\Omega_0} \in H^m(\Omega_0), \text{ and } r_\ell^{\lambda_\ell - m + |\beta|} \partial^\beta \phi|_{\Omega_\ell} \in L^2(\Omega_\ell) \text{ for all } \ell \text{ and } |\beta| \leq m \right\},$$

since $r_\ell \sim 1$ on Ω_0 for all ℓ, and $r_{\ell'} \sim 1$ on Ω_ℓ for $\ell' \neq \ell$.

From the definitions, we immediately obtain the continuous inclusion

$$V_{\lambda+1}^{m+1}(\Omega) \subset V_\lambda^m(\Omega),$$

which may be interpolated to obtain fractional-order spaces. That is, if $s \geq 0$, then $V_\lambda^s(\Omega)$ may be defined by complex interpolation between $V_{\lambda-s+\lfloor s \rfloor+1}^{\lfloor s \rfloor+1}(\Omega)$ and $V_{\lambda-s+\lfloor s \rfloor}^{\lfloor s \rfloor}(\Omega)$, cf. [33, Section 2.4.5]. It follows that, more generally,

$$V_{\lambda+\epsilon}^{s+\epsilon}(\Omega) \subset V_\lambda^s(\Omega),$$

for all $s \geq 0$ and $\epsilon > 0$. Additionally, the continuous inclusions $V_{\lambda'}^m(\Omega) \subset V_\lambda^m(\Omega)$ for $\lambda' \leq \lambda$ and $V_0^m(\Omega) \subset H^m(\Omega)$ extend in the obvious way from nonnegative integer m to real $s \geq 0$.

Remark 3.1. Schneider [33] uses an alternative notation for Kondrat'ev spaces,

$$\mathcal{K}_{p,a}^m(\Omega) := \left\{ \phi \in \mathcal{D}'(\Omega) : r^{|\beta|-a} \partial^\beta \phi \in L^p(\Omega) \text{ for all } |\beta| \leq m \right\},$$

so $V_\lambda^m(\Omega) = \mathcal{K}_{2,m-\lambda}^m(\Omega)$. For example, the inclusion $\mathcal{K}_{p,a}^{m+1}(\Omega) \subset \mathcal{K}_{p,a}^m(\Omega)$ in the notation of [33] gives $V_{\lambda+1}^{m+1}(\Omega) \subset V_\lambda^m(\Omega)$ in our notation, since $p = 2$ and $a = m - \lambda = (m+1) - (\lambda+1)$. Fractional Kondrat'ev spaces are denoted in [33] by $\mathfrak{K}_{p,a}^s(\Omega)$, and similarly we have $V_\lambda^s(\Omega) = \mathfrak{K}_{2,s-\lambda}^s(\Omega)$.

Finally, we note that an intrinsic treatment of fractional weighted Sobolev spaces may be found in Dauge [18, Appendix A].

Suppose now that $u \in H(\text{div}; \Omega) \cap \mathring{H}(\text{curl}; \Omega)$ satisfies (1.1). We recall that $\nabla \cdot u \in \mathring{H}^1(\Omega)$, since it can be seen as the solution to the Dirichlet problem

$$-\Delta(\nabla \cdot u) = \nabla \cdot (f - \alpha u) \in H^{-1}(\Omega).$$

Likewise, $\nabla \times u \in H^1(\Omega)$, since it can be seen as the zero-mean solution to a Neumann problem. See Costabel and Dauge [15, Theorem 1.2] and similar arguments in Brenner et al. [6, Section 2]. Using this, we may now obtain a minimum weighted Sobolev regularity result for u itself.

Theorem 3.2. *If u satisfies (1.1), then $u \in \left[V_{2-2\mu+\epsilon}^2(\Omega) \right]^2$ for all $\epsilon > 0$. Furthermore, if (1.1) is well-posed, then we have the stability estimate $\|u\|_{2,2-2\mu+\epsilon} \lesssim \|f\|_\Omega$.*

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Proof. As in Brenner et al. [6, Section 2], it is sufficient to establish regularity and stability for Ω simply connected, since the general case follows by a partition of unity argument.

Assuming Ω is simply connected, we can express u in terms of its Helmholtz decomposition $u = \nabla\phi + \nabla \times \psi$, where $\phi \in \mathring{H}^1(\Omega)$ and $\psi \in H^1(\Omega)$ solve

$$-\Delta\phi = -\nabla \cdot u, \qquad -\Delta\psi = \nabla \times u,$$

with homogeneous Dirichlet and Neumann boundary conditions, respectively. For uniqueness, we take $\int_\Omega \psi = 0$. To determine the regularity of ϕ and ψ, we follow Chapter 2 of Nazarov and Plamenevsky [29], which characterizes the regularity of solutions to Dirichlet and Neumann problems in plane domains with corner points; similar results are also found in Kozlov et al. [24, §6.6.1–6.6.2].

Hardy's inequality gives $H^1(\Omega) \subset V_\epsilon^1(\Omega)$ for all $\epsilon > 0$, and $\mu \leq 1$ implies $V_\epsilon^1(\Omega) \subset V_{2-2\mu+\epsilon}^1(\Omega)$. Therefore, the right-hand sides $-\nabla \cdot u$ and $\nabla \times u$ are both in $V_{2-2\mu+\epsilon}^1(\Omega)$. Furthermore, since $0 < 2\mu_\ell - \epsilon < \pi/\omega_\ell$ holds for all ℓ when ϵ is sufficiently small, Theorem 3.1 in [29, Chapter 2] implies

$$\phi \in V_{2-2\mu+\epsilon}^3(\Omega), \qquad\qquad \|\phi\|_{3,2-2\mu+\epsilon} \lesssim \|\nabla \cdot u\|_{1,2-2\mu+\epsilon},$$

and Theorem 4.2 in [29, Chapter 2] implies

$$\psi \in V_{2-2\mu+\epsilon}^3(\Omega), \qquad\qquad \|\psi\|_{3,2-2\mu+\epsilon} \lesssim \|\nabla \times u\|_{1,2-2\mu+\epsilon}.$$

By $u = \nabla\phi + \nabla \times \psi$ and the continuity of the inclusion $H^1(\Omega) \subset V_{2-2\mu+\epsilon}^1(\Omega)$, we thus obtain

$$u \in \left[V_{2-2\mu+\epsilon}^2(\Omega)\right]^2, \qquad\qquad \|u\|_{2,2-2\mu+\epsilon} \lesssim \|\nabla \cdot u\|_1 + \|\nabla \times u\|_1, \qquad (3.1)$$

where $\|\cdot\|_1$ is the H^1 norm. This proves the first statement.

Now, observe that the strong form (1.2a) rearranges to

$$-\nabla\nabla \cdot u + \nabla \times \nabla \times u = f - \alpha u,$$

where the left-hand side is an L^2-orthogonal sum. Therefore, by the Pythagorean theorem,

$$|\nabla \cdot u|_1^2 + |\nabla \times u|_1^2 = \|f - \alpha u\|_\Omega^2 \implies |\nabla \cdot u|_1 + |\nabla \times u|_1 \lesssim \|f\|_\Omega + \|u\|_\Omega,$$

where $|\cdot|_1$ is the H^1 seminorm. Combining this with (3.1) gives

$$\|u\|_{2,2-2\mu+\epsilon} \lesssim \|f\|_\Omega + \|u\|_\Omega + \|\nabla \cdot u\|_\Omega + \|\nabla \times u\|_\Omega.$$

Finally, if (1.1) is well-posed, then $\|u\|_\Omega + \|\nabla \cdot u\|_\Omega + \|\nabla \times u\|_\Omega \lesssim \|f\|_\Omega$, which completes the proof. ∎

Most of the subsequent error analysis will use the following corollary of Theorem 3.2.

Corollary 3.3. *If $s < \mu_\ell$ for all ℓ, then $u \in \left[V_{1-\mu}^{s+1}(\Omega)\right]^2$ and $\nabla \cdot u, \nabla \times u \in V_{\mu-1}^s(\Omega)$. Furthermore, if (1.1) is well-posed, then we have the stability estimate*

$$\|u\|_{s+1,1-\mu} + \|\nabla \cdot u\|_{s,\mu-1} + \|\nabla \times u\|_{s,\mu-1} \lesssim \|f\|_\Omega.$$

Proof. Pick $\epsilon > 0$ such that $s < \mu_\ell - \epsilon$ for all ℓ. Then this follows by the continuous inclusions

$$V_{2-2\mu+\epsilon}^2(\Omega) = V_{1-\mu+(1-\mu+\epsilon)}^{s+1+(1-s)}(\Omega) \subset V_{1-\mu+(1-s)}^{s+1+(1-s)}(\Omega) \subset V_{1-\mu}^{s+1}(\Omega)$$

and

$$H^1(\Omega) \subset V_\epsilon^1(\Omega) = V_{\mu-1+(1-\mu+\epsilon)}^{s+(1-s)}(\Omega) \subset V_{\mu-1+(1-s)}^{s+(1-s)}(\Omega) \subset V_{\mu-1}^s(\Omega),$$

together with Theorem 3.2. ∎

Finally, we note that this also implies the following well-known unweighted Sobolev regularity result, cf. Assous, Ciarlet, and Sonnendrücker [2].

Corollary 3.4. *If $s < \min\left(1, \pi/(2\omega_\ell)\right)$ for all ℓ, then $u \in \left[V_0^{2s}(\Omega)\right]^2 \subset \left[H^{2s}(\Omega)\right]^2$. Furthermore, if (1.1) is well-posed, then we have the stability estimate $\|u\|_{2s} \lesssim \|f\|_\Omega$.*

Proof. Since μ_ℓ may be taken arbitrarily close to $\min\big(1, \pi/(2\omega_\ell)\big)$, choose μ_ℓ so that $s < \mu_\ell$ for all ℓ. Then Corollary 3.3 implies $u \in \big[V_{1-\mu}^{s+1}(\Omega)\big]^2$, and we have the continuous inclusions

$$V_{1-\mu}^{s+1}(\Omega) = V_{1-\mu}^{2s+(1-s)}(\Omega) \subset V_{1-s}^{2s+(1-s)}(\Omega) \subset V_0^{2s}(\Omega) \subset H^{2s}(\Omega),$$

which completes the proof. ∎

In particular, since $\omega_\ell < 2\pi$ for all ℓ, we may take $s > \frac{1}{4}$ in Corollary 3.4 to conclude that $u \in \big[H^\sigma(\Omega)\big]^2$ with $\sigma > \frac{1}{2}$.

3.2. **Preliminary estimates**

We now establish two weighted Sobolev norm approximation results that will be useful in the subsequent error analysis; compare Lemmas 5.2 and 5.3 in [9].

For the remainder of the paper, we assume that $\mathcal{T}_h$ is shape-regular, but we make no additional assumptions about quasi-uniformity or grading. Let h_K denote the diameter of $K \in \mathcal{T}_h$ and $h := \max_{K \in \mathcal{T}_h} h_K$. We denote the weighted Sobolev norm on $V_\lambda^s(\Omega)|_K$ by $\|\cdot\|_{s,\lambda,K}$ (with distances taken to the corners of Ω, not those of K) and the ordinary Sobolev seminorm on $H^s(K)$ by $|\cdot|_{s,K}$.

Lemma 3.5. If $v \in \big[H^\sigma(\Omega) \cap V_{1-\mu}^{s+1}(\Omega)\big]^2$ with $\sigma > \frac{1}{2}$ and $s \leq k$, then

$$\frac{\big[\Phi_\mu(e)\big]^2}{|e|}\|v - \Pi_h v\|_e^2 \lesssim h_K^{2s}\|v\|_{s+1,1-\mu,K}^2,$$

for all $K \in \mathcal{T}_h$ and $e \subset \partial K$.

Proof. If K does not have any of the corners c_ℓ as a vertex, then $v|_K \in \big[H^{s+1}(K)\big]^2$, so the trace inequality with scaling and Bramble–Hilbert lemma imply

$$|e|^{-1}\|v - \Pi_h v\|_e^2 \lesssim h_K^{-2}\|v - \Pi_h v\|_K^2 + |v - \Pi_h v|_{1,K}^2 \lesssim h_K^{2s}|v|_{s+1,K}^2.$$

By shape regularity, we have $\Phi_\mu(e) = r^{1-\mu}(m_e) \sim r^{1-\mu}(x)$ for all $x \in K$, and therefore

$$\frac{\big[\Phi_\mu(e)\big]^2}{|e|}\|v - \Pi_h v\|_e^2 \lesssim h_K^{2s}\big[\Phi_\mu(e)\big]^2|v|_{s+1,K}^2 \lesssim h_K^{2s}\|v\|_{s+1,1-\mu,K}^2.$$

On the other hand, if K has c_ℓ as a vertex, then the inclusions $V_{1-\mu_\ell}^{s+1}(K) \subset V_0^{s+\mu_\ell}(K) \subset H^{s+\mu_\ell}(K)$ imply $v|_K \in \big[H^{s+\mu_\ell}(K)\big]^2$. Hence, the trace inequality with scaling and Bramble–Hilbert give

$$|e|^{-1}\|v - \Pi_h v\|_e^2 \lesssim h_K^{-2}\|v - \Pi_h v\|_K^2 + h_K^{2\min(1,\sigma)-2}|v - \Pi_h v|_{\min(1,\sigma),K}^2$$
$$\lesssim h_K^{2(s+\mu_\ell)-2}|v|_{s+\mu_\ell,K}^2.$$

In this case, $\Phi_\mu(e) \sim r_\ell^{1-\mu_\ell}(m_e) \sim h_K^{1-\mu_\ell}$, and therefore

$$\frac{\big[\Phi_\mu(e)\big]^2}{|e|}\|v - \Pi_h v\|_e^2 \lesssim h_K^{2s}|v|_{s+\mu_\ell,K}^2 \lesssim h_K^{2s}\|v\|_{s+1,1-\mu,K}^2,$$

where the last inequality is due to the continuity of the inclusion $V_{1-\mu}^{s+1}(K) \subset H^{s+\mu_\ell}(K)$. ∎

Lemma 3.6. If $\eta \in H^1(\Omega) \cap V_{\mu-1}^s(\Omega)$ with $s \leq k$, then

$$\frac{|e|}{\big[\Phi_\mu(e)\big]^2}\|\eta - P_h\eta\|_e^2 \lesssim h_K^{2s}\|\eta\|_{s,\mu-1,K}^2,$$

for all $K \in \mathcal{T}_h$ and $e \subset \partial K$.

Proof. Since $\mu - 1 \leq 0$, we have $\eta \in V_{\mu-1}^s(\Omega) \subset V_0^s(\Omega) \subset H^s(\Omega)$. Thus, for all $K \in \mathcal{T}_h$ and $e \subset \partial K$, the trace inequality with scaling and Bramble–Hilbert lemma give

$$|e| \|\eta - P_h \eta\|_e^2 \lesssim \|\eta - P_h \eta\|_K^2 + h_K^2 |\eta - P_h \eta|_{1,K}^2 \lesssim h_K^{2s} |\eta|_{s,K}^2.$$

By shape regularity, $[\Phi_\mu(e)]^{-1} = r^{\mu-1}(m_e) \lesssim r^{\mu-1}(x)$ for all $x \in K$, and therefore

$$\frac{|e|}{[\Phi_\mu(e)]^2} \|\eta - P_h \eta\|_e^2 \lesssim h_K^{2s} [\Phi_\mu(e)]^{-2} |\eta|_{s,K}^2 \lesssim h_K^{2s} \|\eta\|_{s,\mu-1,K}^2,$$

which completes the proof. ∎

3.3. **Error estimates**

We now estimate the error $u - u_h$, where u satisfies (1.1) and u_h satisfies (2.7). The argument follows a similar general outline to that in Brenner et al. [6], but the details differ in several important respects—especially in the use of weighted Sobolev regularity hypotheses and higher-order polynomial approximation, and in the absence of mesh-grading assumptions.

As in [6], we will first estimate the error in the mesh-dependent energy norm

$$\|v\|_h^2 := \|v\|_\Omega^2 + \|\nabla \cdot v\|_{\mathcal{T}_h}^2 + \|\nabla \times v\|_{\mathcal{T}_h}^2 + \frac{1}{2} \langle \gamma \llbracket v \rrbracket, \llbracket v \rrbracket \rangle_{\mathcal{E}_h^\circ} + \langle \gamma v \times n, v \times n \rangle_{\mathcal{E}_h^\partial}.$$

If we extend $a_h(\cdot, \cdot)$ from $\mathring{V}_h$ to $H(\text{div}; \Omega) \cap \mathring{H}(\text{curl}; \Omega) + \mathring{V}_h$, then in the special case $\alpha = 1$, this is precisely the norm associated to $a_h(\cdot, \cdot)$ considered as an inner product.

For arbitrary α, we immediately see that $a_h(\cdot, \cdot)$ is bounded with respect to $\|\cdot\|_h$. For $\alpha > 0$, we have the coercivity condition

$$a_h(v, v) \geq \min(1, \alpha) \|v\|_h^2.$$

If the complement of Ω is connected, then we also have coercivity for $\alpha = 0$, by the argument in the proof of Theorem 2.6. In general, for $\alpha \leq 0$, we have a Gårding inequality (which is actually an equality),

$$a_h(v, v) + (|\alpha| + 1) \|v\|_\Omega^2 = \|v\|_h^2.$$

This implies the following Strang-type abstract estimates, whose proofs are identical to those of Lemma 3.5 and Lemma 3.6 in Brenner et al. [10].

Lemma 3.7. *If $\alpha > 0$, u is the solution to (1.1), and u_h is the solution to (2.7), then*

$$\|u - u_h\|_h \lesssim \inf_{v_h \in \mathring{V}_h} \|u - v_h\|_h + \sup_{0 \neq w_h \in \mathring{V}_h} \frac{a_h(u - u_h, w_h)}{\|w_h\|_h}, \tag{3.2}$$

and if the complement of Ω is connected, then this also holds for $\alpha = 0$. If $\alpha \leq 0$, u satisfies (1.1), and u_h satisfies (2.7), then

$$\|u - u_h\|_h \lesssim \inf_{v_h \in \mathring{V}_h} \|u - v_h\|_h + \sup_{0 \neq w_h \in \mathring{V}_h} \frac{a_h(u - u_h, w_h)}{\|w_h\|_h} + \|u - u_h\|_\Omega. \tag{3.3}$$

We will proceed by estimating the two terms on the right-hand side of (3.2), which correspond to approximation error and consistency error, respectively.

Lemma 3.8. *If $u \in [V_{1-\mu}^{s+1}(\Omega)]^2$ and $\nabla \cdot u, \nabla \times u \in V_{\mu-1}^s(\Omega)$ with $s \leq k$, then*

$$\inf_{v_h \in \mathring{V}_h} \|u - v_h\|_h \leq \|u - \Pi_h u\|_h \lesssim h^s \big(\|u\|_{s+1,1-\mu} + \|\nabla \cdot u\|_{s,\mu-1} + \|\nabla \times u\|_{s,\mu-1} \big).$$

Proof. The first inequality holds since the BSM projection maps $u \in \left[H^\sigma(\Omega)\right]^2$ with $\sigma > \frac{1}{2}$ to $\Pi_h u \in \mathring{V}_h$. Next, letting $\mu_{\min} \coloneqq \min_\ell \mu_\ell$, the continuous inclusion $V_{1-\mu}^{s+1}(\Omega) \subset H^{s+\mu_{\min}}(\Omega)$ implies that $u \in \left[H^{s+\mu_{\min}}(\Omega)\right]^2$, so polynomial approximation theory gives

$$\|u - \Pi_h u\|_K^2 \lesssim h_K^{2(s+\mu_{\min})}|u|_{s+\mu_{\min},K}^2 \lesssim h_K^{2(s+\mu_{\min})}\|u\|_{s+1,1-\mu,K}^2. \tag{3.4}$$

Furthermore, Lemma 2.2 and $V_{\mu-1}^s(\Omega) \subset H^s(\Omega)$ imply

$$\left\|\nabla \cdot (u - \Pi_h u)\right\|_K^2 = \left\|(\nabla \cdot u) - P_h(\nabla \cdot u)\right\|_K^2 \ \lesssim h_K^{2s}|\nabla \cdot u|_{s,K}^2 \ \lesssim h_K^{2s}\|\nabla \cdot u\|_{s,\mu-1,K}^2, \tag{3.5a}$$

$$\left\|\nabla \times (u - \Pi_h u)\right\|_K^2 = \left\|(\nabla \times u) - P_h(\nabla \times u)\right\|_K^2 \lesssim h_K^{2s}|\nabla \times u|_{s,K}^2 \lesssim h_K^{2s}\|\nabla \times u\|_{s,\mu-1,K}^2. \tag{3.5b}$$

It remains to estimate the contributions from the penalty terms. Observe that, for $e \in \mathcal{E}_h^\circ$,

$$\left\langle\gamma[\![u - \Pi_h u]\!], [\![u - \Pi_h u]\!]\right\rangle_e \le 2\Big(\left\langle\gamma(u - \Pi_h u), u - \Pi_h u\right\rangle_{e+} + \left\langle\gamma(u - \Pi_h u), u - \Pi_h u\right\rangle_{e-}\Big),$$

by the parallelogram identity. Therefore,

$$\left\langle\gamma[\![u - \Pi_h u]\!], [\![u - \Pi_h u]\!]\right\rangle_{\mathcal{E}_h^\circ} + \left\langle\gamma(u - \Pi_h u) \times n, (u - \Pi_h u) \times n\right\rangle_{\mathcal{E}_h^\partial} \le 2\left\langle\gamma(u - \Pi_h u), u - \Pi_h u\right\rangle_{\partial\mathcal{T}_h},$$

so it suffices to estimate the contribution from each $K \in \mathcal{T}_h$ and $e \subset \partial K$. By Lemma 3.5,

$$\left\langle\gamma(u - \Pi_h u), (u - \Pi_h u)\right\rangle_{\partial K} \lesssim h_K^{2s}\|u\|_{s+1,1-\mu,K}^2. \tag{3.6}$$

Finally, combining (3.4), (3.5), and (3.6) and summing over $K \in \mathcal{T}_h$ completes the proof. ∎

Lemma 3.9. *Suppose u satisfies* (1.1) *and u_h satisfies* (2.7). *If $\nabla \cdot u, \nabla \times u \in V_{\mu-1}^s(\Omega)$ with $s \le k$, then*

$$\sup_{0 \ne w_h \in \mathring{V}_h} \frac{a_h(u - u_h, w_h)}{\|w_h\|_h} \lesssim h^s\big(\|\nabla \cdot u\|_{s,\mu-1} + \|\nabla \times u\|_{s,\mu-1}\big).$$

Proof. Subtracting (2.7) from (2.1) with $v = v_h = w_h \in \mathring{V}_h$, we get

$$a_h(u - u_h, w_h) = \langle\nabla \cdot u, w_h \cdot n\rangle_{\partial\mathcal{T}_h} - \langle\nabla \times u, w_h \times n\rangle_{\partial\mathcal{T}_h}$$
$$= \left\langle\nabla \cdot u, [\![w_h \cdot n]\!]\right\rangle_{\mathcal{E}_h^\circ} - \left\langle\nabla \times u, [\![w_h \times n]\!]\right\rangle_{\mathcal{E}_h^\circ} - \left\langle\nabla \times u, w_h \times n\right\rangle_{\mathcal{E}_h^\partial}, \tag{3.7}$$

where we have denoted the normal and tangential jump components on $e \in \mathcal{E}_h^\circ$ by

$$[\![w_h \cdot n]\!]_e \coloneqq w_h^+ \cdot n^+ + w_h^- \cdot n^-, \qquad [\![w_h \times n]\!]_e \coloneqq w_h^+ \times n^+ + w_h^- \times n^-.$$

The condition $w_h \in \mathring{V}_h$ says that $[\![w_h \cdot n]\!]_e$ and $[\![w_h \times n]\!]_e$ are each L^2-orthogonal to $P_{k-1}(e)$ for $e \in \mathcal{E}_h^\circ$, and that $w_h \times n|_e$ is L^2-orthogonal to $P_{k-1}(e)$ for $e \in \mathcal{E}_h^\partial$. Therefore, letting P_h be projection onto either triangle $K^\pm$ containing $e \in \mathcal{E}_h^\circ$, since $P_h(\nabla \cdot u)|_e \in P_{k-1}(e)$, it follows that

$$\left\langle\nabla \cdot u, [\![w_h \cdot n]\!]\right\rangle_{\mathcal{E}_h^\circ} = \left\langle\nabla \cdot u - P_h(\nabla \cdot u), [\![w_h \cdot n]\!]\right\rangle_{\mathcal{E}_h^\circ}$$
$$= \left\langle\gamma^{-1/2}[\nabla \cdot u - P_h(\nabla \cdot u)], \gamma^{1/2}[\![w_h \cdot n]\!]\right\rangle_{\mathcal{E}_h^\circ}$$
$$\le \left\|\gamma^{-1/2}[\nabla \cdot u - P_h(\nabla \cdot u)]\right\|_{\mathcal{E}_h^\circ}\left\|\gamma^{1/2}[\![w_h \cdot n]\!]\right\|_{\mathcal{E}_h^\circ},$$

where the last step uses the Cauchy–Schwarz inequality. Applying Lemma 3.6 with $\eta = \nabla \cdot u$ to the first term and the definition of the energy norm to the second, we conclude that

$$\left\langle\nabla \cdot u, [\![w_h \cdot n]\!]\right\rangle_{\mathcal{E}_h^\circ} \lesssim h^s\|\nabla \cdot u\|_{s,\mu-1}\|w_h\|_h.$$

Similarly,

$$-\left\langle\nabla \times u, [\![w_h \times n]\!]\right\rangle_{\mathcal{E}_h^\circ} - \left\langle\nabla \times u, w_h \times n\right\rangle_{\mathcal{E}_h^\partial} \lesssim h^s\|\nabla \times u\|_{s,\mu-1}\|w_h\|_h,$$

and the result follows. ∎

Next, we use a duality argument to control the error in the L^2 norm.

Lemma 3.10. *Suppose u satisfies (1.1) and u_h satisfies (2.7). Suppose also that $\nabla \cdot u, \nabla \times u \in V_{\mu-1}^s(\Omega)$ with $s \leq k$, and let $t < \mu_{\min}$. If (1.1) is well-posed, then*

$$\|u - u_h\|_\Omega \lesssim h^{s+t}\big(\|\nabla \cdot u\|_{s,\mu-1} + \|\nabla \times u\|_{s,\mu-1}\big) + h^t \|u - u_h\|_h$$

Proof. Let $z \in H(\mathrm{div};\Omega) \cap \mathring{H}(\mathrm{curl};\Omega)$ be the solution to

$$(\nabla \cdot v, \nabla \cdot z)_\Omega + (\nabla \times v, \nabla \times z)_\Omega + \alpha(v,z)_\Omega = (v, u - u_h)_\Omega, \tag{3.8}$$

for all $v \in H(\mathrm{div};\Omega) \cap \mathring{H}(\mathrm{curl};\Omega)$. By Corollary 3.3, $z \in \big[V_{1-\mu}^{t+1}(\Omega)\big]^2$ and $\nabla \cdot z, \nabla \times z \in V_{\mu-1}^t(\Omega)$, and we have the stability estimate

$$\|z\|_{t+1,1-\mu} + \|\nabla \cdot z\|_{t,\mu-1} + \|\nabla \times z\|_{t,\mu-1} \lesssim \|u - u_h\|_\Omega. \tag{3.9}$$

Hence, Lemma 3.8 implies

$$\|z - \Pi_h z\|_h \lesssim h^t \|u - u_h\|_\Omega. \tag{3.10}$$

To express $\|u - u_h\|_\Omega^2$ in terms of z, we would like to take $v = u - u_h$ in (3.8), but we cannot do so since generally $u_h \notin H(\mathrm{div};\Omega) \cap \mathring{H}(\mathrm{curl};\Omega)$. Instead, integrating by parts as in (2.1) gives

$$\|u - u_h\|_\Omega^2 = a_h(u - u_h, z) + \langle u_h \cdot n, \nabla \cdot z \rangle_{\partial \mathcal{T}_h} - \langle u_h \times n, \nabla \times z \rangle_{\partial \mathcal{T}_h},$$
$$= a_h(u - u_h, z) + \big\langle [\![u_h \cdot n]\!], \nabla \cdot z \big\rangle_{\mathcal{E}_h^\circ} - \big\langle [\![u_h \times n]\!], \nabla \times z \big\rangle_{\mathcal{E}_h^\circ} - \langle u_h \times n, \nabla \times z \rangle_{\mathcal{E}_h^\partial}, \tag{3.11}$$

which we will estimate term-by-term.

For the first term of (3.11), we write

$$a_h(u - u_h, z) = a_h(u - u_h, z - \Pi_h z) + a_h(u - u_h, \Pi_h z).$$

By the boundedness of $a_h(\cdot, \cdot)$ in the energy norm and (3.10), we have

$$a_h(u - u_h, z - \Pi_h z) \lesssim h^t \|u - u_h\|_h \|u - u_h\|_\Omega.$$

Next, by (3.7) with $w_h = \Pi_h z \in \mathring{V}_h$, we have

$$a_h(u - u_h, \Pi_h z) = \big\langle \nabla \cdot u, [\![\Pi_h z \cdot n]\!] \big\rangle_{\mathcal{E}_h^\circ} - \big\langle \nabla \times u, [\![\Pi_h z \times n]\!] \big\rangle_{\mathcal{E}_h^\circ} - \langle \nabla \times u, \Pi_h z \times n \rangle_{\mathcal{E}_h^\partial}.$$

By a similar argument to that used in Lemma 3.9, along with the fact that $[\![z \cdot n]\!] = 0$,

$$\big\langle \nabla \cdot u, [\![\Pi_h z \cdot n]\!] \big\rangle_{\mathcal{E}_h^\circ} = \Big\langle \nabla \cdot u - P_h(\nabla \cdot u), [\![(\Pi_h z - z) \cdot n]\!] \Big\rangle_{\mathcal{E}_h^\circ}$$
$$= \Big\langle \gamma^{-1/2}[\nabla \cdot u - P_h(\nabla \cdot u)], \gamma^{1/2}[\![(\Pi_h z - z) \cdot n]\!] \Big\rangle_{\mathcal{E}_h^\circ}$$
$$\leq \Big\| \gamma^{-1/2}[\nabla \cdot u - P_h(\nabla \cdot u)] \Big\|_{\mathcal{E}_h^\circ} \Big\| \gamma^{1/2}[\![(z - \Pi_h z) \cdot n]\!] \Big\|_{\mathcal{E}_h^\circ}$$
$$\lesssim h^s \|\nabla \cdot u\|_{s,\mu-1} \|z - \Pi_h z\|_h$$
$$\lesssim h^{s+t} \|\nabla \cdot u\|_{s,\mu-1} \|u - u_h\|_\Omega,$$

where the last two lines use Lemma 3.6 with $\eta = \nabla \cdot u$ and (3.10). Similarly,

$$-\big\langle \nabla \times u, [\![\Pi_h z \times n]\!] \big\rangle_{\mathcal{E}_h^\circ} - \langle \nabla \times u, \Pi_h z \times n \rangle_{\mathcal{E}_h^\partial} \lesssim h^{s+t} \|\nabla \times u\|_{s,\mu-1} \|u - u_h\|_\Omega.$$

Thus, we have estimated the first term of (3.11) by

$$a_h(u - u_h, z) \lesssim \Big[h^{s+t}\big(\|\nabla \cdot u\|_{s,\mu-1} + \|\nabla \times u\|_{s,\mu-1}\big) + h^t \|u - u_h\|_h \Big] \|u - u_h\|_\Omega. \tag{3.12}$$

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For the remaining terms of (3.11), we use a similar argument to the one above to get

$$
\begin{aligned}
\left\langle [\![u_h \cdot n]\!], \nabla \cdot z \right\rangle_{\mathcal{E}_h^\circ} &= \left\langle [\![(u_h - u) \cdot n]\!], \nabla \cdot z - P_h(\nabla \cdot z) \right\rangle_{\mathcal{E}_h^\circ} \\
&= \left\langle \gamma^{1/2} [\![(u_h - u) \cdot n]\!], \gamma^{-1/2} \big[\nabla \cdot z - P_h(\nabla \cdot z) \big] \right\rangle_{\mathcal{E}_h^\circ} \\
&\leq \left\| \gamma^{1/2} [\![(u - u_h) \cdot n]\!] \right\|_{\mathcal{E}_h^\circ} \left\| \gamma^{-1/2} \big[\nabla \cdot z - P_h(\nabla \cdot z) \big] \right\|_{\mathcal{E}_h^\circ} \\
&\lesssim h^t \| u - u_h \|_h \| \nabla \cdot z \|_{t,\mu-1} \\
&\lesssim h^t \| u - u_h \|_h \| u - u_h \|_\Omega, \tag{3.13}
\end{aligned}
$$

where the last two lines use Lemma 3.6 with $\eta = \nabla \cdot z$ and (3.9). Likewise,

$$
- \left\langle [\![u_h \times n]\!], \nabla \times z \right\rangle_{\mathcal{E}_h^\circ} - \left\langle u_h \times n, \nabla \times z \right\rangle_{\mathcal{E}_h^\partial} \lesssim h^t \| u - u_h \|_h \| \nabla \times z \|_{t,\mu-1} \lesssim h^t \| u - u_h \|_h \| u - u_h \|_\Omega. \tag{3.14}
$$

Altogether, estimating (3.11) by combining (3.12), (3.13), and (3.14), we have

$$
\| u - u_h \|_\Omega^2 \lesssim \left[h^{s+t} \big(\| \nabla \cdot u \|_{s,\mu-1} + \| \nabla \times u \|_{s,\mu-1} \big) + h^t \| u - u_h \|_h \right] \| u - u_h \|_\Omega,
$$

which completes the proof. ∎

Finally, we are ready to state the main energy and L^2 error estimates.

Theorem 3.11. *Suppose u satisfies (1.1) with $u \in \big[V_{1-\mu}^{s+1}(\Omega) \big]^2$ and $\nabla \cdot u, \nabla \times u \in V_{\mu-1}^s(\Omega)$, where $s \leq k$, and let $t < \mu_{\min}$. If $\alpha > 0$, then the solution u_h to (2.7) satisfies the error estimates*

$$
\| u - u_h \|_h \lesssim h^s \big(\| u \|_{s+1,1-\mu} + \| \nabla \cdot u \|_{s,\mu-1} + \| \nabla \times u \|_{s,\mu-1} \big),
$$
$$
\| u - u_h \|_\Omega \lesssim h^{s+t} \big(\| u \|_{s+1,1-\mu} + \| \nabla \cdot u \|_{s,\mu-1} + \| \nabla \times u \|_{s,\mu-1} \big),
$$

and if the complement of Ω is connected, then this also holds for $\alpha = 0$. If $\alpha < 0$ and (1.1) is well-posed, then (2.7) is uniquely solvable for sufficiently small h, and the solution u_h satisfies these same estimates.

Proof. If $\alpha > 0$, or if $\alpha = 0$ with Ω having connected complement, then the proof is fairly immediate. The energy estimate follows from the abstract estimate (3.2) in Lemma 3.7, together with Lemmas 3.8 and 3.9, and the L^2 estimate follows by Lemma 3.10.

When $\alpha < 0$ is such that (1.1) is well-posed, we follow the approach in Brenner et al. [10, Theorem 4.5], which uses a technique for indefinite problems due to Schatz [32]. Suppose that u_h satisfies (2.7). From the abstract estimate (3.3) in Lemma 3.7, along with Lemmas 3.8, 3.9, and 3.10, we have

$$
\| u - u_h \|_h \leq C \left[h^s \big(\| u \|_{s+1,1-\mu} + \| \nabla \cdot u \|_{s,\mu-1} + \| \nabla \times u \|_{s,\mu-1} \big) + h^t \| u - u_h \|_h \right], \tag{3.15}
$$

where the constant C has been made explicit. Now, choose h_* small enough that $C h_*^t < 1$. It follows that, whenever $h \leq h_*$, we may subtract $C h^t \| u - u_h \|_h$ from both sides of (3.15) to obtain

$$
\| u - u_h \|_h \lesssim h^s \big(\| u \|_{s+1,1-\mu} + \| \nabla \cdot u \|_{s,\mu-1} + \| \nabla \times u \|_{s,\mu-1} \big).
$$

In particular, when $f = 0$, well-posedness of (1.1) gives the unique solution $u = 0$, and $\| u_h \|_h \lesssim 0$ implies that (2.7) has the unique solution $u_h = 0$. Hence, (2.7) is uniquely solvable and satisfies the energy estimate whenever $h \leq h_*$, and the L^2 estimate follows by another application of Lemma 3.10. ∎

Corollary 3.12 (minimum-regularity case)**.** *If $\alpha > 0$, u is the solution to (1.1), and u_h is the solution to (2.7), then for all $s < \mu_{\min}$, we have the error estimates*

$$
\| u - u_h \|_h \lesssim h^s \| f \|_\Omega,
$$
$$
\| u - u_h \|_\Omega \lesssim h^{2s} \| f \|_\Omega,
$$

	$k = 1$				$k = 2$				$k = 3$			
N	$\|u - u_h\|_h$	rate	$\|u - u_h\|_\Omega$	rate	$\|u - u_h\|_h$	rate	$\|u - u_h\|_\Omega$	rate	$\|u - u_h\|_h$	rate	$\|u - u_h\|_\Omega$	rate
2	$2.62e{-}3$		$4.10e{-}4$		$7.21e{-}4$		$8.79e{-}5$		$1.52e{-}4$		$1.61e{-}5$	
4	$1.30e{-}3$	1.00	$8.98e{-}5$	2.19	$2.02e{-}4$	1.84	$1.27e{-}5$	2.79	$2.02e{-}5$	2.92	$1.00e{-}6$	4.01
8	$6.39e{-}4$	1.03	$2.04e{-}5$	2.14	$5.31e{-}5$	1.93	$1.74e{-}6$	2.87	$2.51e{-}6$	3.01	$5.93e{-}8$	4.08
16	$3.16e{-}4$	1.02	$4.87e{-}6$	2.07	$1.35e{-}5$	1.97	$2.26e{-}7$	2.94	$3.11e{-}7$	3.01	$3.58e{-}9$	4.05
32	$1.57e{-}4$	1.01	$1.19e{-}6$	2.03	$3.41e{-}6$	1.99	$2.88e{-}8$	2.98	$3.86e{-}8$	3.01	$2.20e{-}10$	4.03
64	$7.83e{-}5$	1.00	$2.95e{-}7$	2.01	$8.54e{-}7$	2.00	$3.62e{-}9$	2.99	$4.82e{-}9$	3.00	$1.36e{-}11$	4.01

Table 1. Convergence to a smooth solution on a square domain.

and if the complement of Ω is connected, then this also holds for $\alpha = 0$. If $\alpha < 0$ and (1.1) is well-posed, then these estimates hold for sufficiently small h.

Proof. This is immediate from Corollary 3.3 and Theorem 3.11 with $s = t$. ∎

4. Numerical experiments

In this section, we present numerical experiments illustrating the convergence behavior of the method, showing how convergence is affected by the interior angles of Ω and by the regularity of the exact solution u, and relating these numerical results to the theoretical results of Section 3. For all numerical experiments, we take $\alpha = 1$.

All computations have been carried out using the Firedrake finite element library [30] (version 0.13.0+4959.gac22e4c5), and a Firedrake component called Slate [19] was used to implement the local solvers for static condensation and postprocessing.

4.1. Smooth solution on square domain

We begin by considering the square domain $\Omega = \left(0, \frac{1}{2}\right)^2$. Since all four corners of Ω are $\pi/2$, we have $\mu = (1, 1, 1, 1)$. Given $N \in \mathbb{N}$, we construct a uniform triangle mesh by partitioning Ω uniformly into $N \times N$ squares, then dividing each into two triangles.

Table 1 shows the result of applying our method to the problem whose exact solution is

$$u = \begin{bmatrix} (x^3/3 - x^2/4)(y^2 - y/2)\sin y \\ (y^3/3 - y^2/4)(x^2 - x/2)\cos x \end{bmatrix}.$$

(This is the same u that Brenner et al. [6] use for their numerical experiments on the square.) Since u is smooth, we observe convergence rates of k for the energy error and $k + 1$ for the L^2 error, consistent with Theorem 3.11.

4.2. Minimum-regularity solutions on L-shaped domain

We next consider the L-shaped domain $\Omega = \left(-\frac{1}{2}, \frac{1}{2}\right)^2 \setminus \left[0, \frac{1}{2}\right]^2$. This has a reentrant corner at the origin with angle $\omega_1 = 3\pi/2$, so we may take any $\mu_1 < 1/3$. The remaining corners c_ℓ have $\omega_\ell = \pi/2$ and thus $\mu_\ell = 1$ for $\ell = 2, \ldots, 6$. Given $N \in \mathbb{N}$, we construct a uniform triangle mesh of Ω by taking a uniform $2N \times 2N$ mesh of the square $\left(-\frac{1}{2}, \frac{1}{2}\right)^2$, as in Section 4.1, and removing the first quadrant.

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4.2.1. *Minimum-regularity singular harmonic vector field*

In polar coordinates (r, θ), we first consider the problem whose exact solution is

$$u = \nabla \times \left(r^{2/3} \cos\left[\frac{2}{3}\left(\theta - \frac{\pi}{2} \right) \right] \right),$$

which is a harmonic vector field with $\nabla \cdot u = 0$ and $\nabla \times u = 0$. We observe that $u \in \left[V_{m-1/3-\mu}^m(\Omega) \right]^2$ for all m, since the condition for $\partial^\beta u$ to be in the appropriate weighted L^2 space in a δ-neighborhood of the origin is

$$\int_0^\delta \left(r^{-1/3-\mu_1+|\beta|} r^{2/3-1-|\beta|} \right)^2 r \, \mathrm{d}r = \int_0^\delta r^{2(1/3-\mu_1)-1} \, \mathrm{d}r < \infty,$$

which holds since $\mu_1 < 1/3$. (Compare Costabel and Dauge [16, Theorem 6.1].) By interpolation, we get $u \in \left[V_{1-\mu}^{1/3+1}(\Omega) \right]^2$, so the hypotheses of the error estimates in Section 3.3 hold with $s = 1/3$.

Remark 4.1. Although u does not satisfy the homogeneous boundary condition $u \times n = 0$ on all of $\partial\Omega$, it does satisfy this condition on the boundary edges $\theta = \pi/2$ and $\theta = 2\pi$ adjacent to the reentrant corner. Thus, taking ϕ to be a smooth cutoff function supported in a small neighborhood of the origin, we may write $u = u\phi + u(1 - \phi)$, where $u\phi$ satisfies the homogeneous boundary condition and $u(1 - \phi)$ is a smooth extension of the inhomogeneous boundary condition. It follows that u and $u\phi$ have the same regularity, and standard arguments may be used to extend the numerical properties of the method from the homogeneous boundary value problem with exact solution $u\phi$ to the inhomogeneous boundary value problem with exact solution u.

Table 2 shows the results of applying our method to this problem, where the inhomogeneous boundary conditions are imposed on $\hat{u}_h \times n$ by interpolating $u \times n$ on $\mathcal{E}_h^\partial$. Since $s = 1/3$, we observe minimal convergence rates of approximately $1/3$ for the energy error and $2/3$ for the L^2 error for all k, consistent with Theorem 3.11.

4.2.2. *Minimum-regularity nonsingular vector field*

The next example shows that even a nonsingular vector field may have minimum regularity, owing to the conditions $\nabla \cdot u, \nabla \times u \in V_{\mu-1}^s(\Omega)$. Given arbitrarily small $\epsilon > 0$, consider the problem whose exact solution is

$$u = \nabla \times r^{2+\epsilon},$$

where the inhomogeneous boundary conditions may also be dealt with as in Remark 4.1. By a similar calculation as for the singular harmonic vector field, we have $u \in \left[V_{m-5/3-\mu}^m(\Omega) \right]^2$ for all m, and thus $u \in \left[V_{1-\mu}^{5/3+1}(\Omega) \right]^2$. However, we merely have $\nabla \times u \in V_{m-1/3+\mu-1}^m(\Omega)$ for all m, provided that $1/3 - \epsilon < \mu_1 < 1/3$, since

$$\int_0^\delta (r^{-1/3+\mu_1-1+|\beta|} r^{\epsilon-|\beta|})^2 r \, \mathrm{d}r = \int_0^\delta r^{2(\mu_1-1/3+\epsilon)-1} \, \mathrm{d}r < \infty.$$

and interpolation gives $\nabla \times u \in V_{\mu-1}^{1/3}(\Omega)$. Hence, even though u does not have a singularity at the origin, the regularity hypotheses of Theorem 3.11 hold merely with $s = 1/3$.

Table 3 shows the results of applying our method to this problem with $\epsilon = 0.001$. As with the previous example, since $s = 1/3$, we observe minimal convergence rates of approximately $1/3$ for the energy error and $2/3$ for the L^2 error for all k, consistent with Theorem 3.11.

	$k=1$				$k=2$				$k=3$			
N	$\|u-u_h\|_h$	rate	$\|u-u_h\|_\Omega$	rate	$\|u-u_h\|_h$	rate	$\|u-u_h\|_\Omega$	rate	$\|u-u_h\|_h$	rate	$\|u-u_h\|_\Omega$	rate
2	$2.23e{-}1$		$7.71e{-}2$		$1.60e{-}1$		$5.73e{-}2$		$1.45e{-}1$		$4.42e{-}2$	
4	$1.77e{-}1$	0.33	$5.02e{-}2$	0.62	$1.24e{-}1$	0.36	$3.69e{-}2$	0.64	$1.13e{-}1$	0.35	$2.82e{-}2$	0.65
8	$1.40e{-}1$	0.34	$3.23e{-}2$	0.64	$9.73e{-}2$	0.35	$2.35e{-}2$	0.65	$8.92e{-}2$	0.35	$1.79e{-}2$	0.66
16	$1.11e{-}1$	0.34	$2.07e{-}2$	0.65	$7.64e{-}2$	0.35	$1.49e{-}2$	0.66	$7.03e{-}2$	0.34	$1.13e{-}2$	0.66
32	$8.76e{-}2$	0.34	$1.31e{-}2$	0.65	$6.03e{-}2$	0.34	$9.40e{-}3$	0.66	$5.56e{-}2$	0.34	$7.13e{-}3$	0.66
64	$6.93e{-}2$	0.34	$8.32e{-}3$	0.66	$4.77e{-}2$	0.34	$5.93e{-}3$	0.66	$4.40e{-}2$	0.34	$4.50e{-}3$	0.66

Table 2. Convergence to the minimum-regularity singular harmonic on an L-shaped domain.

	$k=1$				$k=2$				$k=3$			
N	$\|u-u_h\|_h$	rate	$\|u-u_h\|_\Omega$	rate	$\|u-u_h\|_h$	rate	$\|u-u_h\|_\Omega$	rate	$\|u-u_h\|_h$	rate	$\|u-u_h\|_\Omega$	rate
2	$7.23e{-}3$		$3.65e{-}3$		$4.52e{-}3$		$1.94e{-}3$		$2.96e{-}3$		$1.04e{-}3$	
4	$6.06e{-}3$	0.26	$2.57e{-}3$	0.50	$3.71e{-}3$	0.28	$1.31e{-}3$	0.56	$2.40e{-}3$	0.30	$6.87e{-}4$	0.60
8	$4.98e{-}3$	0.28	$1.75e{-}3$	0.56	$3.02e{-}3$	0.30	$8.66e{-}4$	0.60	$1.93e{-}3$	0.31	$4.46e{-}4$	0.62
16	$4.05e{-}3$	0.30	$1.16e{-}3$	0.59	$2.43e{-}3$	0.31	$5.63e{-}4$	0.62	$1.55e{-}3$	0.32	$2.87e{-}4$	0.64
32	$3.26e{-}3$	0.31	$7.54e{-}4$	0.62	$1.95e{-}3$	0.32	$3.62e{-}4$	0.64	$1.24e{-}3$	0.33	$1.83e{-}4$	0.65
64	$2.61e{-}3$	0.32	$4.85e{-}4$	0.64	$1.55e{-}3$	0.33	$2.31e{-}4$	0.65	$9.85e{-}4$	0.33	$1.16e{-}4$	0.66

Table 3. Convergence to a minimum-regularity nonsingular solution on an L-shaped domain.

4.3. **Higher-regularity solutions on L-shaped domain**

Finally, we present numerical results for convergence to solutions with higher regularity on the L-shaped domain, observing improved convergence for larger k. As in Section 4.2, we consider both a harmonic and a non-harmonic example—here, both having $s = 7/3$—on the same family of uniform meshes, where inhomogeneous boundary conditions for $u \times n$ on $\partial\Omega$ are handled in the same way.

4.3.1. *Higher-regularity harmonic vector field*

Consider the problem whose exact solution is

$$u = \nabla \times \left(r^{8/3} \cos\left[\frac{8}{3}\left(\theta - \frac{\pi}{2}\right)\right]\right),$$

which is a harmonic vector field with $\nabla \cdot u = 0$ and $\nabla \times u = 0$. By a similar calculation to that in Section 4.2.1, we get $u \in \left[V^m_{m-7/3-\mu}(\Omega)\right]^2$ for all m. By interpolation, we see that $u \in \left[V^{7/3+1}_{1-\mu}(\Omega)\right]^2$, so the hypotheses of the error estimates in Section 3.3 hold with $s = 7/3$.

Table 4 shows the results of applying our method to this problem. Since $s \leq 3$, for $k = 3$ we observe the maximum convergence rates predicted by Theorem 3.11: roughly $7/3$ for the energy error and $8/3$ for the L^2 error. For $k = 2$, however, we *also* observe rates of approximately $7/3$ for the energy error and $8/3$ for the L^2 error. This is explained by the fact that u is the curl of a harmonic function, and $BSM_k(K)$ contains gradients (hence curls) of harmonic polynomials with degree $\leq 2k$. In this special case, the condition $s \leq k$ in the approximation estimate Lemma 3.8 improves to $s \leq 2k - 1$, while the consistency error in Lemma 3.9 vanishes due to $\nabla \cdot u = 0$ and $\nabla \times u = 0$. For $k = 1$, we observe the expected energy-norm convergence rate of 1, but the L^2-norm convergence rate of 2 is *better* than the duality-based estimate of $4/3$ in Theorem 3.11. We do not yet have a satisfying analytical explanation for this better-than-expected gap between the energy-norm and L^2-norm rates when $s > k$; see further discussion in the next example and in Section 5.

| | $k = 1$ | | | | $k = 2$ | | | | $k = 3$ | | | |
N	$\|u - u_h\|_h$	rate	$\|u - u_h\|_\Omega$	rate	$\|u - u_h\|_h$	rate	$\|u - u_h\|_\Omega$	rate	$\|u - u_h\|_h$	rate	$\|u - u_h\|_\Omega$	rate
2	$1.08e{-}1$		$1.79e{-}2$		$8.62e{-}4$		$2.28e{-}4$		$1.56e{-}4$		$3.84e{-}5$	
4	$5.50e{-}2$	0.97	$4.49e{-}3$	2.00	$1.72e{-}4$	2.32	$3.63e{-}5$	2.65	$3.07e{-}5$	2.34	$6.12e{-}6$	2.65
8	$2.79e{-}2$	0.98	$1.12e{-}3$	2.00	$3.42e{-}5$	2.33	$5.73e{-}6$	2.66	$6.06e{-}6$	2.34	$9.69e{-}7$	2.66
16	$1.40e{-}2$	0.99	$2.79e{-}4$	2.00	$6.77e{-}6$	2.34	$9.05e{-}7$	2.66	$1.20e{-}6$	2.34	$1.53e{-}7$	2.66
32	$7.04e{-}3$	0.99	$6.96e{-}5$	2.00	$1.34e{-}6$	2.34	$1.43e{-}7$	2.67	$2.37e{-}7$	2.34	$2.41e{-}8$	2.67
64	$3.53e{-}3$	1.00	$1.74e{-}5$	2.00	$2.66e{-}7$	2.34	$2.25e{-}8$	2.67	$4.69e{-}8$	2.34	$3.80e{-}9$	2.67

TABLE 4. Convergence to a higher-regularity harmonic vector field on an L-shaped domain.

| | $k = 1$ | | | | $k = 2$ | | | | $k = 3$ | | | |
N	$\|u - u_h\|_h$	rate	$\|u - u_h\|_\Omega$	rate	$\|u - u_h\|_h$	rate	$\|u - u_h\|_\Omega$	rate	$\|u - u_h\|_h$	rate	$\|u - u_h\|_\Omega$	rate
2	$2.67e{+}0$		$8.25e{-}1$		$5.33e{-}1$		$1.82e{-}1$		$6.84e{-}5$		$2.18e{-}5$	
4	$1.31e{+}0$	1.02	$2.03e{-}1$	2.02	$1.47e{-}1$	1.86	$3.29e{-}2$	2.47	$1.42e{-}5$	2.27	$3.58e{-}6$	2.61
8	$6.47e{-}1$	1.02	$5.02e{-}2$	2.01	$3.89e{-}2$	1.92	$5.62e{-}3$	2.55	$2.87e{-}6$	2.30	$5.80e{-}7$	2.63
16	$3.20e{-}1$	1.01	$1.24e{-}2$	2.01	$1.00e{-}2$	1.96	$9.28e{-}4$	2.60	$5.76e{-}7$	2.32	$9.30e{-}8$	2.64
32	$1.59e{-}1$	1.01	$3.09e{-}3$	2.01	$2.55e{-}3$	1.98	$1.50e{-}4$	2.63	$1.15e{-}7$	2.33	$1.48e{-}8$	2.65
64	$7.95e{-}2$	1.00	$7.68e{-}4$	2.01	$6.44e{-}4$	1.99	$2.41e{-}5$	2.64	$2.29e{-}8$	2.33	$2.35e{-}9$	2.66

TABLE 5. Convergence to a higher-regularity non-harmonic vector field on an L-shaped domain.

4.3.2. *Higher-regularity non-harmonic vector field*

Given arbitrarily small $\epsilon > 0$, consider the problem whose exact solution is

$$u = \nabla \times r^{4+\epsilon}.$$

By a similar calculation to that in Section 4.2.2, we have $u \in \left[V_{m-11/3-\mu}^m(\Omega)\right]^2$ for all m, and thus $u \in \left[V_{1-\mu}^{11/3+1}(\Omega)\right]^2$. However, we merely have $\nabla \times u \in V_{m-7/3+\mu-1}^m(\Omega)$ for all m, provided that $1/3 - \epsilon < \mu_1 < 1/3$, so interpolation gives $\nabla \times u \in V_{\mu-1}^{7/3}(\Omega)$. Hence, the regularity hypotheses of Theorem 3.11 hold merely with $s = 7/3$.

Table 5 shows the results of applying our method to this problem with $\epsilon = 0.001$. For all k, we observe a convergence rate of approximately $\min(k, 7/3)$ in the energy norm, consistent with Theorem 3.11. This also supports the argument that the improved energy error in Section 4.3.1, which is not observed here, was due to that exact solution being the curl of a harmonic function. For $k = 3$, we observe the expected L^2-norm convergence rate of approximately $8/3$. However, for $k = 2$ and $k = 1$, we observe better-than-expected rates of $8/3$ (rather than $7/3$) and 2 (rather than $4/3$), respectively, similar to what we saw with the $k = 1$ case in Section 4.3.1.

5. Conclusion

We have presented a nonconforming primal hybrid finite element method for the two-dimensional vector Laplacian that extends the P_1-nonconforming method of Brenner et al. [6] to arbitrary order k. The method uses only standard polynomial finite elements, although the more exotic BSM element and projection play a key role in the analysis, and the method may be implemented efficiently using static condensation. Using the weighted Sobolev spaces of Kondrat'ev for domains with corners, we have obtained error estimates that hold on general shape-regular meshes, without mesh-grading conditions.

These estimates establish the convergence of the method, even for minimum-regularity solutions with corner singularities, and the convergence rate improves with k to the extent regularity allows.

Let us conclude with a brief discussion of one area where the numerical experiments in Section 4 suggest possible room for improvement. Dropping the hypothesis that $s \leq k$, we may rewrite the estimates of Theorem 3.11 as

$$\|u - u_h\|_h \lesssim h^{\min(k,s)}\big(\|u\|_{s+1,1-\mu} + \|\nabla \cdot u\|_{s,\mu-1} + \|\nabla \times u\|_{s,\mu-1}\big), \tag{5.1}$$

$$\|u - u_h\|_\Omega \lesssim h^{\min(k,s)+t}\big(\|u\|_{s+1,1-\mu} + \|\nabla \cdot u\|_{s,\mu-1} + \|\nabla \times u\|_{s,\mu-1}\big). \tag{5.2}$$

From the numerical experiments, it appears that the energy estimate (5.1) is sharp, and in general one cannot relax the restriction $t < \mu_{\min}$ in the L^2 estimate (5.2). However, when $s > k$, it appears that a sharper estimate than (5.2) holds, which we now state as a conjecture.

Conjecture 5.1. *Under the conditions of Theorem 3.11 (except for the condition $s \leq k$),*

$$\|u - u_h\|_\Omega \lesssim h^{\min(k+1,s+t)}\big(\|u\|_{s+1,1-\mu} + \|\nabla \cdot u\|_{s,\mu-1} + \|\nabla \times u\|_{s,\mu-1}\big).$$

Establishing this would require some new analytical arguments, perhaps involving weighted-norm error estimates. Indeed, a duality estimate of the sort in Lemma 3.10 can only give an L^2 error estimate of the form (5.2), where the improvement over the energy rate is the same for all k, and the numerical experiments suggest that there is no way to sharpen this uniformly in k.

Finally, it is natural to ask how the two-dimensional method presented in this paper might be generalized to three dimensions. In contrast with some other approaches that are limited to dimension two—such as methods that use the Hodge decomposition to transform vector problems into scalar problems [7, 25, 26, 8]—the variational form of the hybrid method (1.4) extends naturally to the three-dimensional case. The main challenge is to choose suitable finite element spaces and a suitable penalty, and here there are two obstacles to overcome. First, when $k > 1$, we do not yet know a three-dimensional version of the BSM element and commuting projection, which would be needed to make the analysis work. (A naive extension to three dimensions fails to satisfy unisolvence when $k = 2$, as shown by Mirebeau [27].) Consequently, it is not clear what polynomial degrees would be needed for the finite element spaces V_h, Q_h, and $\hat{V}_h$. Second, the weighted Sobolev analysis in three-dimensional domains becomes more complicated, since singularities can form along boundary edges, as well as at corners where edges meet, cf. Costabel and Dauge [16]. Consequently, a penalty γ would need to be carefully constructed, likely involving the distances both to edges and to corners with suitable exponents.

Acknowledgements

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