HAMILTONIAN MECHANICS AND LIE ALGEBROID CONNECTIONS

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Abstract. We develop a new, coordinate-free formulation of Hamiltonian mechanics on the dual of a Lie algebroid. Our approach uses a connection, rather than coordinates in a local trivialization, to obtain global expressions for the horizontal and vertical dynamics. We show that these dynamics can be obtained in two equivalent ways: (1) using the canonical Lie–Poisson structure, expressed in terms of the connection; or (2) using a novel variational principle that generalizes Hamilton's phase space principle.

1. Introduction

The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ has a canonical (up to sign) *Lie–Poisson structure*, defined by the ($\pm$) Lie–Poisson brackets

$$\{F,G\}_\pm(\mu) \coloneqq \pm\left\langle \mu, \left[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\right]\right\rangle. \tag{1}$$

Here, $F,G\colon \mathfrak{g}^* \to \mathbb{R}$, $\mu \in \mathfrak{g}^*$, $\langle\cdot,\cdot\rangle$ is the duality pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, $[\cdot,\cdot]$ is the Lie bracket on $\mathfrak{g}$, and $\delta F/\delta\mu \in \mathfrak{g}$ is the variational derivative defined by $\langle \delta\mu, \delta F/\delta\mu\rangle = \lim_{\epsilon\to 0}\bigl[F(\mu+\epsilon\,\delta\mu) - F(\mu)\bigr]/\epsilon$ for all $\delta\mu \in \mathfrak{g}^*$. Using the ($\pm$) bracket, a Hamiltonian $H\colon \mathfrak{g}^* \to \mathbb{R}$ gives rise to the dynamics

$$\dot{\mu} = \mp\,\mathrm{ad}^*_{\delta H/\delta\mu}\,\mu, \tag{2}$$

which are called the *Lie–Poisson equations* (Marsden and Ratiu [16]). These arise in diverse applications ranging from rigid body mechanics to incompressible fluid dynamics.

More generally, one may define a Lie–Poisson structure on the dual of a *Lie algebroid* (Courant [5], Weinstein [18]). This includes as special cases not only (1) but also the canonical Poisson structure on a cotangent bundle or its quotient by a Lie group action. As such, Lie algebroid duals provide a rich setting for Hamiltonian mechanics and reduction. However, in contrast to the global, coordinate-free form of (2), Hamilton's equations in this more general setting have only been expressed with respect to coordinates in a local trivialization: cf. Weinstein [18], Martínez [17], de León, Marrero, and Martínez [8].

In this paper, we develop a new, coordinate-free formulation of Hamiltonian mechanics on the dual of a Lie algebroid. Our approach uses a connection, rather than coordinates in a local trivialization, to obtain global expressions for the horizontal and vertical dynamics. In particular, the Lie–Poisson equations (2) are obtained as a special case, with ad^* arising from the connection and $\delta H/\delta\mu$ corresponding to the vertical part of $\mathrm{d}H$. This can be viewed as a Hamiltonian counterpart to the paper Li, Stern, and Tang [13], which provides the Lagrangian side of this story.

The paper is organized as follows:

- Section 2 reviews the basic ideas of Lie algebroids, previous work on generalized Lie–Poisson dynamics in local coordinates, and Lie algebroid connections.
- Section 3 develops a coordinate-free expression for the Lie–Poisson structure on a Lie algebroid dual relative to a connection, and uses this to derive the horizontal and vertical dynamics arising from a Hamiltonian.

- Section 4 introduces a novel variational principle for Hamiltonian dynamics on Lie algebroid duals, which generalizes Hamilton's phase space principle in the case of a cotangent bundle. Using a connection, it is shown that paths satisfying the variational principle are precisely solutions to the horizontal and vertical Lie–Poisson equations derived in the preceding section. As special cases, this also generalizes the Lie–Poisson and Hamilton–Poincaré variational principles of Cendra et al. [1].
- Section 5 concludes the paper with a discussion of the relationship between the Hamiltonian and Lagrangian formalisms, linking the results of the present paper with those of Li, Stern, and Tang [13].

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2. Lie algebroid preliminaries

2.1. **Lie algebroids.** We begin with the definition of a Lie algebroid, along with a few standard examples. See Mackenzie [14] for a comprehensive reference.

Definition 2.1. A *Lie algebroid* is a real vector bundle $\tau\colon A \to Q$ equipped with a Lie bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(A)$ and a bundle map $\rho\colon A \to TQ$, called the *anchor*, satisfying the Leibniz rule

$$[X, fY] = \rho(X)[f]Y + f[X, Y],$$

for all $f \in C^\infty(Q)$ and $X, Y \in \Gamma(A)$.

Remark 2.2. Here and henceforth, we use the notation $v[f] \coloneqq \langle \mathrm{d}f, v\rangle$ for a tangent vector or vector field acting as a derivation on smooth functions.

Example 2.3. The tangent bundle $\tau\colon TQ \to Q$ is a Lie algebroid, where $[\cdot,\cdot]$ is the Jacobi–Lie bracket on vector fields and $\rho\colon TQ \to TQ$ is the identity map.

Example 2.4. A Lie algebra $\mathfrak{g}$ can be interpreted as a Lie algebroid over a single point $\tau\colon \mathfrak{g} \to \bullet$, where $[\cdot,\cdot]$ is the bracket on $\mathfrak{g}$ and ρ is trivial.

Example 2.5. Given a principal G-bundle $Q \to Q/G$, one may form the *Atiyah algebroid* $\tau\colon TQ/G \to Q/G$, where $[\cdot,\cdot]$ agrees with the Jacobi–Lie bracket of G-invariant vector fields on Q, and where ρ agrees with the (G-invariant) tangent map of the principal bundle projection. Examples 2.3 and 2.4 are the special cases $G = \{e\}$ and $Q = G$, respectively.

We are often interested in a particular class of paths in Lie algebroids, especially for applications in geometric mechanics.

Definition 2.6. Let $a\colon I \to A$ be a path in A, where I is an interval (interpreted as time), and let $q = \tau \circ a\colon I \to Q$ be the corresponding base path in Q. We say that a is an *A-path* over q if it satisfies $\dot{q}(t) = \rho\big(a(t)\big)$ for all $t \in I$.

For example, TQ-paths are just the tangent prolongations of paths in Q. (In the context of geometric mechanics, Yoshimura and Marsden [21, 22] call this the "second-order condition.") On the other hand, every path in $\mathfrak{g}$ is a $\mathfrak{g}$-path, since the condition to be satisfied is trivial.

2.2. **The generalized Lie–Poisson structure on A^*.** Let $\pi\colon A^* \to Q$ denote the dual bundle of a Lie algebroid A.

2.2.1. *The* ($\pm$) *Lie–Poisson brackets.* In order to define the Lie–Poisson structure on A^*, we first introduce some useful notation adopted from Marle [15].

Definition 2.7. Given a section $X \in \Gamma(A)$, let $\Phi_X \in C^\infty(A^*)$ denote the fiberwise-linear function $\Phi_X(p) = \langle p, X\rangle$, where $\langle \cdot, \cdot \rangle$ is the duality pairing between A^* and A.

Definition 2.8. The ($\pm$) *Lie–Poisson bracket* on A^* is the unique bracket satisfying

$$\{\Phi_X, \Phi_Y\}_\pm = \pm \Phi_{[X,Y]}, \tag{3a}$$

for all $X, Y \in \Gamma(A)$.

For any $f, g \in C^\infty(Q)$, replacing X and Y in (3a) by fX and gY shows that, in order for $\{\cdot,\cdot\}_\pm$ to satisfy the Leibniz rule, it must also satisfy

$$\{\Phi_X, g \circ \pi\}_\pm = \pm \rho(X)[g] \circ \pi, \tag{3b}$$

$$\{f \circ \pi, g \circ \pi\}_\pm = 0. \tag{3c}$$

Since every covector in $T_p^* A^*$ can be written as $\mathrm{d}(\Phi_X + f \circ \pi)(p)$ for some $X \in \Gamma(A)$ and $f \in C^\infty(Q)$ [15, Lemma 6.3.2], this completely defines the Lie–Poisson structure on A^*. Courant [5, Theorem 2.1.4] shows that the converse also holds: any fiberwise-linear Poisson structure on a dual vector bundle $E^* \to Q$ determines a Lie algebroid structure on $E \to Q$.

2.2.2. *Local coordinates and Hamiltonian dynamics.* Let q^i be local coordinates for Q and $\{e_I\}$ be a local basis of sections of A. Let C_{IJ}^K and ρ_I^i be the local-coordinate representations of $[\cdot,\cdot]$ and ρ, defined by

$$[e_I, e_J] = C_{IJ}^K e_K, \qquad \rho(e_I) = \rho_I^i \frac{\partial}{\partial q^i}.$$

(We use the Einstein index convention, where there is an implicit sum over repeated indices.) In the special case of a Lie algebra, C_{IJ}^K are known as *structure constants*; in the general case, they depend smoothly on $q \in Q$ and are called *structure functions*.

It follows that q^i and $\mu_I = \Phi_{e_I}$ are local coordinate functions on A^*, where μ_I gives the coefficient in the dual basis $\{e^I\}$ of sections of A^*. We can now express the bracket relations (3), in terms of these local coordinate functions, as

$$\begin{aligned} \{\mu_I, \mu_J\}_\pm &= \pm C_{IJ}^K \mu_K, \\ \{\mu_I, q^i\}_\pm &= \pm \rho_I^i, \\ \{q^i, q^j\}_\pm &= 0. \end{aligned}$$

(We commit a slight abuse of notation by identifying q^i with $q^i \circ \pi$.) It follows that

$$\{F, G\}_\pm = \pm C_{IJ}^K \frac{\partial F}{\partial \mu_I} \frac{\partial G}{\partial \mu_J} \mu_K \pm \rho_I^i \left(\frac{\partial F}{\partial \mu_I} \frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial \mu_I} \right), \tag{4}$$

for arbitrary functions $F, G \in C^\infty(A^*)$.

Now, given a Hamiltonian $H \in C^\infty(A^*)$, we recall that Hamilton's equations are defined by $\dot{F} = \{F, H\}_\pm$ for all $F \in C^\infty(A^*)$. Taking F to be the coordinate functions q^i and μ_I, we obtain

$$\dot{q}^i = \mp \rho_I^i \frac{\partial H}{\partial \mu_I}, \tag{5a}$$

$$\dot{\mu}_I = \pm C_{IJ}^K \frac{\partial H}{\partial \mu_J} \mu_K \pm \rho_I^i \frac{\partial H}{\partial q^i}, \tag{5b}$$

which are local-coordinate expressions of the generalized Lie–Poisson equations on A^*.

Example 2.9. Let $\tau\colon TQ \to Q$ be the tangent bundle of Q and $\pi\colon T^*Q \to Q$ be the cotangent bundle. Local coordinates q^i on Q give rise to a local basis of sections $\partial/\partial q^i$ of TQ. From this, we get so-called canonical coordinates q^i, p_i on T^*Q, where $p_i = \Phi_{\partial/\partial q^i}$ in the notation used above. (Due to the correspondence between base and fiber coordinates, we use lowercase indices i, j, k for both.) In these coordinates, we have

$$C^k_{ij} = 0, \qquad \rho^i_j = \delta^i_j,$$

where δ^i_j is the Kronecker delta. Thus, (4) gives the $(\pm)$ brackets on T^*Q,

$$\{F, G\}_\pm = \pm\biggl(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}\biggr), \tag{6}$$

with respect to which Hamilton's equations (5) are

$$\dot{q}^i = \mp\frac{\partial H}{\partial p_i}, \tag{7a}$$

$$\dot{p}_i = \pm\frac{\partial H}{\partial q^i}. \tag{7b}$$

From this, we see that the $(-)$ bracket gives the usual form of the canonical Poisson structure and Hamilton's equations on T^*Q, while $(+)$ gives the opposite sign convention.

Example 2.10. Let $\tau\colon \mathfrak{g} \to \bullet$ be a Lie algebra and $\pi\colon \mathfrak{g}^* \to \bullet$ its dual, considered as vector bundles over a single point. Given a basis $\{e_I\}$ of $\mathfrak{g}$, the coefficients C^K_{IJ} are the structure constants of $\mathfrak{g}$, and the anchor is trivial. In these coordinates, the $(\pm)$ brackets (4) on $\mathfrak{g}^*$ are

$$\{F, G\}_\pm = \pm C^K_{IJ}\frac{\partial F}{\partial \mu_I}\frac{\partial G}{\partial \mu_J}\mu_K,$$

with respect to which Hamilton's equations (5) are

$$\dot{\mu}_I = \pm C^K_{IJ}\frac{\partial H}{\partial \mu_J}\mu_K.$$

These are precisely local-coordinate expressions for the $(\pm)$ Lie–Poisson bracket (1) and Lie–Poisson equations (2), respectively.

2.3. **Connections and variations of A-paths.** We recall the notion of connection on a Lie algebroid due to Crainic and Fernandes [6] (see also Fernandes [9]), along with the important role that such connections play in calculus of variations for A-paths.

Definition 2.11. Given a Lie algebroid $A \to Q$ an *A-connection* on a vector bundle $E \to Q$ is an $\mathbb{R}$-bilinear map $\nabla\colon \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(X, s) \mapsto \nabla_X s$, that is $C^\infty(Q)$-linear in the first argument and satisfies a Leibniz rule in the second, i.e.,

$$\nabla_{fX} s = f \nabla_X s, \qquad \nabla_X (fs) = \rho(X)[f]s + f\nabla_X s,$$

for all $f \in C^\infty(Q)$.

An A-connection on a vector bundle naturally induces an A-connection on the dual bundle.

Definition 2.12. Given an A-connection ∇ on E, the *dual connection* ∇^* on E^* is given by

$$\langle \nabla^*_X \sigma, s\rangle = \rho(X)\bigl[\langle \sigma, s\rangle\bigr] - \langle \sigma, \nabla_X s\rangle,$$

where $X \in \Gamma(A)$, $\sigma \in \Gamma(E^*)$, and $s \in \Gamma(E)$.

For example, a TQ-connection is the usual notion of a connection on a vector bundle. Given a TQ-connection ∇ on A, there are two induced A-connections on A, which are denoted by ∇ and $\overline{\nabla}$:

$$\nabla_X Y \coloneqq \nabla_{\rho(X)} Y, \qquad \overline{\nabla}_X Y \coloneqq \nabla_{\rho(Y)} X + [X, Y].$$

In particular, if ∇ is the trivial $T\bullet$-connection on $\mathfrak{g} \to \bullet$, then the induced $\mathfrak{g}$-connections are

$$\nabla_\xi \eta = 0, \qquad \overline{\nabla}_\xi \eta = [\xi, \eta] = \operatorname{ad}_\xi \eta,$$

for $\xi, \eta \in \mathfrak{g}$. For A-connections on A itself, such as these, we have a notion of torsion.

Definition 2.13. The *torsion* of an A-connection ∇ on A is defined to be

$$T(X, Y) \coloneqq \nabla_X Y - \nabla_Y X - [X, Y].$$

We note that $T \colon \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is $C^\infty(Q)$-bilinear, since the Leibniz-rule terms from the connection cancel with those from the bracket, so we may treat T as a tensor. Furthermore, the torsion of ∇ may be written $T(X, Y) = \nabla_X Y - \overline{\nabla}_X Y$, which is seen to be tensorial since it is the difference of two connections.

In addition to using an A-connection ∇ to differentiate along sections, we will also use it to differentiate along paths.

Definition 2.14. Suppose ∇ is an A-connection on E. Given an A-path a over a base path q, choose a time-dependent section ξ of A such that $\xi\big(t, q(t)\big) = a(t)$. Similarly, given a path u in E over the same base path q, choose a time-dependent section η of E such that $\eta\big(t, q(t)\big) = u(t)$. We then define

$$\nabla_a u(t) \coloneqq \partial_t \eta\big(t, q(t)\big) + \nabla_\xi \eta\big(t, q(t)\big),$$

which is independent of the choice of ξ and η.

Finally, we recall the notion of admissible variations of A-paths from Crainic and Fernandes [6], which can be readily expressed in terms of a connection. As in [6], let $\widetilde{P}(A)$ denote the Banach manifold of C^1 paths $I \to A$ having C^2 base paths $I \to Q$, and let $P(A) \subset \widetilde{P}(A)$ denote the Banach submanifold of A-paths.

Definition 2.15. An *admissible variation* of $a \in P(A)$ is a tangent vector $X_{b,a} \in T_a P(A)$, where $b \in \widetilde{P}(A)$ covers the same base path and vanishes at the endpoints of I, such that the vertical and horizontal components relative to a TQ-connection ∇ on A are

$$X_{b,a}^{\mathrm{ver}} = \overline{\nabla}_a b, \qquad X_{b,a}^{\mathrm{hor}} = \rho(b).$$

By vertical and horizontal, we mean the components in the splitting $T_a A \cong A_q \oplus T_q Q$ induced by ∇ for each $t \in I$. Note that $X_{b,a}^{\mathrm{hor}} \in T_q Q$ is independent of the choice of connection.

Example 2.16. Recall that TQ-paths are tangent prolongations of base paths $q \colon I \to Q$. Since the anchor of TQ is the identity, admissible variations are completely determined by $X_{b,a}^{\mathrm{hor}} = b$, which is an arbitrary variation of the base path (usually written δq).

Example 2.17. For a Lie algebra, recall that $\mathfrak{g}$-paths are arbitrary paths. Taking ∇ to be the trivial connection on $\mathfrak{g}$, we see that admissible variations of ξ have the restricted form

$$\delta \xi = \dot{\eta} + \operatorname{ad}_\xi \eta,$$

where η is arbitrary. In the context of Lagrangian mechanics on Lie algebras, these restrictions on admissible variations are called *Lin constraints* (Marsden and Ratiu [16, Chapter 13]).

3. Lie–Poisson structure and equations relative to a connection

3.1. **Splitting of T^*A^*.** As in the previous section, let $\tau\colon A \to Q$ be a Lie algebroid and $\pi\colon A^* \to Q$ its dual. A TQ-connection ∇ on A gives a splitting of TA into horizontal and vertical subbundles, which induces a natural splitting of T^*A^* as well.

The following lemma gives a convenient coordinate-free expression for this splitting. This result seems like it ought to be standard, and it may be known to experts, but we were unable to find it stated explicitly in the literature. We remark that although the result is stated for a Lie algebroid, which is our application of interest, it holds for connections on vector bundles more generally.

Lemma 3.1. *A TQ-connection ∇ on A induces a splitting $T_p^*A^* \cong A_q \oplus T_q^*Q$, where $p \in A_q^*$. This splitting is characterized by the following condition: For all $X \in \Gamma(A)$ and $f \in C^\infty(Q)$,*

$$\mathrm{d}(\Phi_X + f\circ\pi)^{\mathrm{ver}}(p) = X(q), \tag{8a}$$

$$\mathrm{d}(\Phi_X + f\circ\pi)^{\mathrm{hor}}(p) = \langle p, \nabla X\rangle + \mathrm{d}f(q). \tag{8b}$$

By $\langle p, \nabla X\rangle \in T_q^*Q$, we mean the covector whose pairing with $v \in T_qQ$ is $\langle p, \nabla_v X\rangle$.

Proof. Given $p \in A_q^*$, recall that the vertical lift $V_p\colon A_q^* \to T_pA^*$ is defined by

$$V_p r \coloneqq \frac{\mathrm{d}}{\mathrm{d}\epsilon}(p + \epsilon r)\Big|_{\epsilon=0}.$$

Associated to the dual connection ∇^* on A^*, the horizontal lift $H_p\colon T_qQ \to T_pA^*$ is given by

$$H_p v = T\mu(v) - V_p(\nabla_v^*\mu),$$

where $\mu \in \Gamma(A^*)$ is any section satisfying $\mu(q) = p$. This is independent of the choice of section, and together the vertical and horizontal lifts define the splitting

$$V_p \oplus H_p\colon A_q^* \oplus T_qQ \xrightarrow{\cong} T_pA^*.$$

See, for instance, Wendl [19, §3.3] and Kolář, Michor, and Slovák [12, §17.9]. By taking the dual of these vertical and horizontal lifts, we obtain projections

$$V_p^*\colon T_p^*A^* \to A_q, \qquad \alpha \mapsto \alpha^{\mathrm{ver}},$$
$$H_p^*\colon T_p^*A^* \to T_q^*Q, \qquad \alpha \mapsto \alpha^{\mathrm{hor}},$$

and thus a dual splitting

$$V_p^* \oplus H_p^*\colon T_p^*A^* \xrightarrow{\cong} A_q \oplus T_q^*Q.$$

It remains to show that α^{ver} and α^{hor} have the claimed expressions for $\alpha = \mathrm{d}(\Phi_X + f\circ\pi)(p)$.

First, observe that for all $r \in A_q^*$, we have

$$\langle \mathrm{d}\Phi_X, V_p r\rangle = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\langle p + \epsilon r, X\rangle\Big|_{\epsilon=0} = \langle r, X\rangle.$$

Furthermore,

$$\big\langle \mathrm{d}(f\circ\pi), V_p r\big\rangle = \frac{\mathrm{d}}{\mathrm{d}\epsilon}(f\circ\pi)(p+\epsilon r)\Big|_{\epsilon=0} = 0,$$

since $\pi(p + \epsilon r) = q$ is constant in ϵ. Together, these establish (8a). Next, for all $v \in T_qQ$,

$$\begin{aligned}\langle \mathrm{d}\Phi_X, H_p v\rangle &= \big\langle \mathrm{d}\Phi_X, T\mu(v) - V_p(\nabla_v^*\mu)\big\rangle\\ &= v\big[\langle \mu, X\rangle\big] - \langle \nabla_v^*\mu, X\rangle\\ &= \langle \mu, \nabla_v X\rangle,\end{aligned}$$

where the last equality is the defining property of the dual connection ∇^*. Finally,

$$\begin{aligned}\big\langle \mathrm{d}(f\circ\pi), H_p v\big\rangle &= \big\langle \mathrm{d}(f\circ\pi), T\mu(v) - V_p(\nabla^*_v\mu)\big\rangle\\ &= \big\langle \mathrm{d}(f\circ\pi\circ\mu), v\big\rangle - 0,\\ &= \langle \mathrm{d}f, v\rangle.\end{aligned}$$

Together, these establish (8b), which completes the proof. □

3.2. **Lie–Poisson structure.** We now use the splitting in Lemma 3.1 to express the Lie–Poisson structure on A^* in terms of the connection ∇. The following result is not new; we first learned of it from Rui Loja Fernandes, and it appears as an exercise in Crainic, Fernandes, and Mărcuţ [7, Exercise 13.73]. However, the suggested proof in [7] compares with a local coordinate expression for the Poisson tensor, and we believe the following coordinate-free proof is new.

Lemma 3.2. *Let ∇ be a TQ-connection on A, and let $T\colon A\otimes A\to A$ be the torsion tensor of the associated A-connection. The $(\pm)$ Lie–Poisson tensor $\Pi_\pm\colon T^*A^*\otimes T^*A^*\to\mathbb{R}$ satisfies*

$$\Pi_\pm(\alpha,\beta) = \pm\Big[\big\langle \beta^{\mathrm{hor}}, \rho(\alpha^{\mathrm{ver}})\big\rangle - \big\langle \alpha^{\mathrm{hor}}, \rho(\beta^{\mathrm{ver}})\big\rangle - \big\langle p, T(\alpha^{\mathrm{ver}}, \beta^{\mathrm{ver}})\big\rangle\Big],$$

for all $\alpha,\beta\in T^*_pA^*$.

Proof. It suffices to show that $\{F,G\}_\pm = \Pi_\pm(\mathrm{d}F,\mathrm{d}G)$ satisfies the bracket relations (3). First, if $\alpha = \mathrm{d}\Phi_X(p)$ and $\beta = \mathrm{d}\Phi_Y(p)$, then Lemma 3.1 gives

$$\begin{aligned}\Pi_\pm\big(\mathrm{d}\Phi_X(p), \mathrm{d}\Phi_Y(p)\big) &= \pm\Big[\langle p, \nabla_X Y\rangle - \langle p, \nabla_Y X\rangle - \big\langle p, T(X,Y)\big\rangle\Big]\\ &= \pm\big\langle p, [X,Y]\big\rangle,\end{aligned}$$

verifying (3a). Next, if $\alpha = \mathrm{d}\Phi_X(p)$ and $\beta = \mathrm{d}(g\circ\pi)(p)$, then the two terms involving $\beta^{\mathrm{ver}} = 0$ vanish, leaving only

$$\begin{aligned}\Pi_\pm\big(\mathrm{d}\Phi_X(p), \mathrm{d}(g\circ\pi)(p)\big) &= \pm\Big\langle \mathrm{d}g(q), \rho\big(X(q)\big)\Big\rangle\\ &= \pm\rho(X)[g](q),\end{aligned}$$

verifying (3b). Finally, if $\alpha = \mathrm{d}(f\circ\pi)(p)$ and $\beta = \mathrm{d}(g\circ\pi)(p)$, then all three terms vanish, since $\alpha^{\mathrm{ver}} = \beta^{\mathrm{ver}} = 0$. Therefore,

$$\Pi_\pm\big(\mathrm{d}(f\circ\pi)(p), \mathrm{d}(g\circ\pi)(p)\big) = 0,$$

verifying (3c) and completing the proof. □

3.3. **Lie–Poisson equations.** We are now ready to state our main result, expressing the $(\pm)$ Lie–Poisson equations on A^* relative to a given TQ-connection ∇ on A.

Theorem 3.3. *Given $H\in C^\infty(A^*)$, a path p in A^* with base path q is an integral curve of the Hamiltonian vector field, with respect to the $(\pm)$ Lie–Poisson structure, if and only if*

$$\dot q = \rho(a), \tag{9a}$$

$$\overline{\nabla}^*_a p = \pm\rho^*\big(\mathrm{d}H^{\mathrm{hor}}(p)\big), \tag{9b}$$

where $a = \mp\mathrm{d}H^{\mathrm{ver}}(p)$. The first equation says that a is an A-path, so it is valid to write $\overline{\nabla}^*_a$.

Proof. By definition, p is an integral curve of H if and only if $\frac{\mathrm{d}}{\mathrm{d}t}F(p) = \{F,H\}_\pm(p)$ for all $F\in C^\infty(A^*)$. It suffices to consider $F = f\circ\pi$ for $f\in C^\infty(Q)$ and $F = \Phi_X$ for $X\in\Gamma(A)$.

First, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}(f\circ\pi)(p) = \big\langle \mathrm{d}f(q), \dot q\big\rangle,$$

while Lemmas 3.1 and 3.2 imply

$$\{f\circ\pi, H\}_\pm(p) = \mp\Bigl\langle \mathrm{d}f(q), \rho\bigl(\mathrm{d}H^{\mathrm{ver}}(p)\bigr)\Bigr\rangle = \bigl\langle \mathrm{d}f(q), \rho(a)\bigr\rangle.$$

These are equal for all $f\in C^\infty(Q)$ if and only if (9a) holds, i.e., $a=\mp\mathrm{d}H^{\mathrm{ver}}(p)$ is an A-path.

Next, if μ is a time-dependent section of A^* such that $\mu\bigl(t,q(t)\bigr)=p(t)$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_X(p) &= \frac{\mathrm{d}}{\mathrm{d}t}\langle \mu, X\rangle(q)\\ &= \langle \partial_t\mu, X\rangle + \rho(a)\bigl[\langle\mu,X\rangle\bigr]\\ &= \langle \partial_t\mu + \overline{\nabla}{}^*_a\mu, X\rangle + \langle \mu, \overline{\nabla}_a X\rangle\\ &= \langle \overline{\nabla}{}^*_a p, X\rangle + \langle p, \overline{\nabla}_a X\rangle,\end{aligned}$$

while Lemmas 3.1 and 3.2 imply

$$\begin{aligned}\{\Phi_X, H\}_\pm(p) &= \pm\Bigl[\bigl\langle \mathrm{d}H^{\mathrm{hor}}(p), \rho(X)\bigr\rangle - \langle p, \nabla_{\mathrm{d}H^{\mathrm{ver}}(p)}X\rangle - \Bigl\langle p, T\bigl(X, \mathrm{d}H^{\mathrm{ver}}(p)\bigr)\Bigr\rangle\Bigr]\\ &= \pm\Bigl[\bigl\langle \mathrm{d}H^{\mathrm{hor}}(p), \rho(X)\bigr\rangle - \langle p, \overline{\nabla}_{\mathrm{d}H^{\mathrm{ver}}(p)}X\rangle\Bigr]\\ &= \pm\Bigl\langle \rho^*\bigl(\mathrm{d}H^{\mathrm{hor}}(p)\bigr), X\Bigr\rangle + \langle p, \overline{\nabla}_a X\rangle,\end{aligned}$$

using the fact that torsion is the difference between ∇ and $\overline{\nabla}$. These two expressions are equal for all $X\in\Gamma(A)$ if and only if (9b) holds. □

We now relate the coordinate-free form of the Lie–Poisson equations in Theorem 3.3 to the local-coordinate formulation (5). Given local coordinates q^i for Q and a local basis of sections $\{e_I\}$ of A, choose the locally trivial TQ-connection defined by $\nabla_{\partial/\partial q^i}e_J=0$. With respect to this connection, the vertical and horizontal components of $\mathrm{d}H$ are given by

$$\mathrm{d}H^{\mathrm{ver}} = \frac{\partial H}{\partial \mu_I}e_I, \qquad \mathrm{d}H^{\mathrm{hor}} = \frac{\partial H}{\partial q^i}\,\mathrm{d}q^i.$$

It follows that $a=\mp\mathrm{d}H^{\mathrm{ver}}(p)$ is an A-path if and only if

$$\dot q^i = \mp\rho^i_I\frac{\partial H}{\partial\mu_I},$$

which is (5a). Next, if μ is a time-dependent section of A^* such that $\mu\bigl(t,q(t)\bigr)=p(t)$, then

$$\langle\overline{\nabla}{}^*_a p, e_I\rangle = \langle \partial_t\mu + \overline{\nabla}{}^*_a\mu, e_I\rangle = \dot\mu_I \mp C^K_{IJ}\frac{\partial H}{\partial\mu_J}\mu_K.$$

Setting this equal to $\pm\bigl\langle \mathrm{d}H^{\mathrm{hor}}(p), \rho(e_I)\bigr\rangle$ gives

$$\dot\mu_I \mp C^K_{IJ}\frac{\partial H}{\partial\mu_J}\mu_K = \pm\rho^i_I\frac{\partial H}{\partial q^i},$$

which rearranges to (5b).

Remark 3.4. To illustrate that these equations are independent of the TQ-connection chosen, suppose more generally that $\nabla_{\partial/\partial q^i}e_J=\Gamma^K_{iJ}e_K$, where Γ^K_{iJ} are Christoffel symbols. Then

$$\mathrm{d}H^{\mathrm{ver}} = \frac{\partial H}{\partial \mu_I}e_I, \qquad \mathrm{d}H^{\mathrm{hor}} = \biggl(\frac{\partial H}{\partial q^i} + \Gamma^K_{iJ}\frac{\partial H}{\partial\mu_J}\mu_K\biggr)\,\mathrm{d}q^i.$$

Since $\mathrm{d}H^{\mathrm{ver}}$ is the same as above, once again (9a) becomes (5a). Meanwhile, (9b) becomes

$$\dot\mu_I \pm (\rho^i_I\Gamma^K_{iJ} - C^K_{IJ})\frac{\partial H}{\partial\mu_J}\mu_K = \pm\rho^i_I\biggl(\frac{\partial H}{\partial q^i} + \Gamma^K_{iJ}\frac{\partial H}{\partial\mu_J}\mu_K\biggr),$$

and the connection-dependent terms cancel to give (5b).

3.4. **Examples.** We now illustrate the results of this section by showing how they give the canonical structures and dynamics on Lie algebra duals and cotangent bundles.

Example 3.5. If $A = \mathfrak{g} \to \bullet$ is a Lie algebra and ∇ is the trivial $T\bullet$-connection, then we have the associated A-connections $\nabla = 0$, $\overline{\nabla} = \operatorname{ad}$, and $\overline{\nabla}^* = -\operatorname{ad}^*$.

Given $F \in C^\infty(\mathfrak{g}^*)$, we have $\mathrm{d}F^{\mathrm{ver}} = \delta F/\delta\mu$ and $\mathrm{d}F^{\mathrm{hor}} = 0$, and likewise for $G \in C^\infty(\mathfrak{g}^*)$. Since the horizontal components vanish and $T(\cdot,\cdot) = -[\cdot,\cdot]$, applying Lemma 3.2 gives

$$\{F,G\}_\pm(\mu) = \mp\Big\langle \mu, T\big(\mathrm{d}F^{\mathrm{ver}}(\mu), \mathrm{d}G^{\mathrm{ver}}(\mu)\big)\Big\rangle = \pm\bigg\langle \mu, \bigg[\frac{\delta F}{\delta\mu}, \frac{\delta G}{\delta\mu}\bigg]\bigg\rangle,$$

recovering the $(\pm)$ Lie–Poisson brackets on $\mathfrak{g}^*$ from (1).

Applying Theorem 3.3 to $H \in C^\infty(\mathfrak{g}^*)$, the horizontal components vanish, and we have $a = \mp\delta H/\delta\mu$. Identifying p with the time-dependent section $\mu(t,\bullet) = p(t)$, we get

$$\dot{\mu} \pm \operatorname{ad}^*_{\delta H/\delta\mu}\mu = \dot{\mu} + \overline{\nabla}^*_a \mu = \overline{\nabla}^*_a p = 0,$$

so (9a) is trivial and (9b) is equivalent to the Lie–Poisson equations (2) on $\mathfrak{g}^*$.

Example 3.6. Suppose $A = TQ \to Q$ is a tangent bundle. As in Example 2.9, let q^i be local coordinates on Q, $\partial/\partial q^i$ be the corresponding local basis of sections of TQ, and q^i, p_i be canonical coordinates for T^*Q. Take the locally trivial connection $\nabla_{\partial/\partial q^i}(\partial/\partial q^j) = 0$. (This may not be globally trivial, but Remark 3.4 shows that there is no loss of generality.) Since $[\partial/\partial q^i, \partial/\partial q^j] = 0$, it follows that $\overline{\nabla}$ and $\overline{\nabla}^*$ are also trivial in local coordinates.

Given $F \in C^\infty(T^*Q)$, the vertical and horizontal components of $\mathrm{d}F$ are

$$\mathrm{d}F^{\mathrm{ver}} = \frac{\partial F}{\partial p_i}\frac{\partial}{\partial q^i}, \qquad \mathrm{d}F^{\mathrm{hor}} = \frac{\partial F}{\partial q^i}\,\mathrm{d}q^i,$$

and likewise for $G \in C^\infty(T^*Q)$. Since ρ is the identity on TQ and ∇ is torsion-free, Lemma 3.2 gives

$$\{F,G\}_\pm = \pm\big[\langle \mathrm{d}G^{\mathrm{hor}}, \mathrm{d}F^{\mathrm{ver}}\rangle - \langle \mathrm{d}F^{\mathrm{hor}}, \mathrm{d}G^{\mathrm{ver}}\rangle\big] = \pm\bigg(\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}\bigg),$$

recovering the previous expression (6) for the $(\pm)$ brackets. Again, we note that the $(-)$ bracket agrees with the commonly-used sign convention for the canonical Poisson structure on a cotangent bundle, e.g., as in Marsden and Ratiu [16].

Applying Theorem 3.3 to $H \in C^\infty(T^*Q)$, we have

$$a = \mp\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}.$$

Since ρ is the identity and $\overline{\nabla}^*$ is trivial in local coordinates, (9) gives

$$\begin{aligned}\dot{q}^i &= \mp\frac{\partial H}{\partial p_i},\\ \dot{p}_i &= \pm\frac{\partial H}{\partial q^i},\end{aligned}$$

recovering the $(\pm)$ Hamilton's equations (7).

We give one more example, on the cotangent bundle, where we apply Theorem 3.3 to a Hamiltonian arising from a metric on Q, obtaining coordinate-free dynamics in terms of the Levi-Civita connection.

Example 3.7. Let $g\colon TQ \otimes TQ \to \mathbb{R}$ be a (pseudo-)Riemannian metric on Q. Let $g^\flat \colon TQ \to T^*Q$ denote the bundle map $v \mapsto g(v, \cdot)$ and $g^\sharp \coloneqq (g^\flat)^{-1} \colon T^*Q \to TQ$. Now, suppose we have a Hamiltonian in the form

$$H(q,p) = \frac{1}{2}\langle p, g^\sharp(p)\rangle + U(q), \tag{10}$$

where $U \in C^\infty(Q)$ is a potential energy function.

Equip T^*Q with the canonical $(-)$ Poisson structure, and let ∇ be the Levi-Civita connection on Q. It follows that

$$\mathrm{d}H^{\mathrm{ver}}(q,p) = g^\sharp(p), \qquad \mathrm{d}H^{\mathrm{hor}}(q,p) = \mathrm{d}U(q),$$

where the kinetic-energy part of the Hamiltonian does not contribute to the horizontal part due to the metric-compatibility of ∇. Furthermore, since ∇ is torsion-free, we have $\nabla = \overline{\nabla}$. Since ρ is the identity and $a = g^\sharp(p)$, we see that (9) gives the equations

$$\begin{aligned} \dot{q} &= g^\sharp(p), \\ \nabla^*_{\dot{q}} p &= -\mathrm{d}U(q). \end{aligned}$$

In particular, the case $U = 0$ corresponds to geodesic flow. Note that applying $g^\sharp$ to both sides of the second equation, and substituting the first, gives the second-order dynamics

$$\nabla_{\dot{q}}\dot{q} = -\operatorname{grad} U(q).$$

3.5. **Systems arising from Lie algebroid metrics.** We now generalize Example 3.7 to the case where A is an arbitrary Lie algebroid over Q, equipped with a bundle metric $g\colon A \otimes A \to \mathbb{R}$, and (10) is a kinetic-plus-potential Hamiltonian on A^*. Our approach extends that of Martínez [17] for the case of action algebroids; see also Cortés and Martínez [4] for a related approach in the context of Lagrangian control systems.

Some caution is required: while there is a suitable notion of Levi-Civita (metric-compatible, torsion-free) A-connection [4, 3, 10], this A-connection generally *does not* arise from a TQ-connection ∇. For example, when $A = \mathfrak{g} \to \bullet$, the trivial connection is the unique $T\bullet$-connection, but its associated A-connection has torsion $-[\cdot,\cdot]$, which is nonvanishing unless $\mathfrak{g}$ is abelian. Instead, since g is a bundle metric, we may only assume the existence (but not necessarily uniqueness) of a metric-compatible TQ-connection ∇, whose associated A-connection may have nonvanishing torsion. (See the remark following Proposition III.1.5 in Kobayashi and Nomizu [11] on the existence of bundle-metric-compatible connections.)

Metric-compatibility of ∇ gives the same splitting of $\mathrm{d}H$ into vertical and horizontal parts as in Example 3.7. Taking the $(-)$ Lie–Poisson structure, we therefore have $a = g^\sharp(p)$ and

$$\dot{q} = \rho(a), \tag{11a}$$

$$\overline{\nabla}^*_a p = -\rho^*\big(\mathrm{d}U(q)\big). \tag{11b}$$

Here, $\overline{\nabla}$ generally differs from ∇ and need not be metric-compatible, so applying $g^\sharp$ to (11b) is not as simple as before. We first define an additional A-connection

$$\overline{\nabla}^\dagger_X Y \coloneqq g^\sharp\big(\overline{\nabla}^*_X g^\flat(Y)\big), \qquad X, Y \in \Gamma(A),$$

which is directly verified to satisfy Definition 2.11. Next, as in Cortés and Martínez [4], we define the gradient of U with respect to the Lie algebroid metric to be the section

$$\operatorname{grad} U \coloneqq g^\sharp \circ \rho^* \circ \mathrm{d}U \in \Gamma(A).$$

Therefore, applying $g^\sharp$ to both sides of (11b) gives

$$\overline{\nabla}^\dagger_a a = -\operatorname{grad} U(q), \tag{12}$$

and the case $U = 0$ gives geodesic flow with respect to the A-connection $\overline{\nabla}^\dagger$.

Although $\overline{\nabla}^\dagger$ is generally distinct from the Levi-Civita A-connection ∇^g, we next show that in fact $\overline{\nabla}^\dagger_a a = \nabla^g_a a$. Thus $\overline{\nabla}^\dagger$ and ∇^g are interchangeable in (12), and in particular they have the same geodesic flow.

Lemma 3.8. *Let A be a Lie algebroid over Q equipped with a bundle metric g. For any bundle-metric-compatible TQ-connection ∇, the Levi-Civita connection ∇^g satisfies*

$$\nabla^g_X Y = \frac{1}{2}\big(\overline{\nabla}^\dagger_X Y + \overline{\nabla}^\dagger_Y X + [X,Y]\big),$$

for all $X, Y \in \Gamma(A)$. That is, $\nabla^g = \frac{1}{2}\big(\overline{\nabla}^\dagger + \overline{\overline{\nabla}^\dagger}\big)$.

Proof. From [4, Proposition 2.6], the Levi-Civita connection is determined by

$$\begin{aligned}2g(\nabla^g_X Y, Z) = {} & \rho(X)\big[g(Y,Z)\big] + \rho(Y)\big[g(X,Z)\big] - \rho(Z)\big[g(X,Y)\big] \\ & - g\big(X,[Y,Z]\big) - \quad g\big(Y,[X,Z]\big) + \quad g\big(Z,[X,Y]\big),\end{aligned}$$

for all $X, Y, Z \in \Gamma(A)$, which is a generalization of the usual tangent bundle formula. Now, by the definitions of $\overline{\nabla}^\dagger$, the dual connection $\overline{\nabla}^*$, and $\overline{\nabla}$ itself, we have

$$\begin{aligned}\rho(X)\big[g(Y,Z)\big] &= g(\overline{\nabla}^\dagger_X Y, Z) + g(Y, \overline{\nabla}_X Z) \\ &= g(\overline{\nabla}^\dagger_X Y, Z) + g(Y, \nabla_Z X + [X,Z]),\end{aligned}$$

and likewise,

$$\rho(Y)\big[g(X,Z)\big] = g(\overline{\nabla}^\dagger_Y X, Z) + g(X, \nabla_Z Y + [Y,Z])$$

On the other hand, metric-compatibility of ∇ gives

$$\rho(Z)\big[g(X,Y)\big] = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

Substituting these into the formula for ∇^g and canceling terms, we obtain

$$2g(\nabla^g_X Y, Z) = g\big(\overline{\nabla}^\dagger_X Y + \overline{\nabla}^\dagger_Y X + [X,Y], Z\big),$$

which completes the proof. □

Corollary 3.9. *For all $X \in \Gamma(A)$, we have $\nabla^g_X X = \overline{\nabla}^\dagger_X X$. In particular, an A-path a satisfies* (12) *if and only if*

$$\nabla^g_a a = -\operatorname{grad} U(q).$$

Example 3.10. When ∇ is the Levi-Civita connection on $A = TQ \to Q$, we have $\overline{\nabla}^\dagger = \overline{\nabla} = \nabla$ and thus recover the equations obtained in Example 3.7.

Example 3.11. The special case where A is an action algebroid recovers the equations of Martínez [17], as we now show. An (infinitesimal) action of a Lie algebra $\mathfrak{g}$ on Q is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(Q)$, $\xi \mapsto \xi_Q$. The associated *action algebroid* is the trivial bundle $A = Q \times \mathfrak{g} \to Q$, where $\rho(q,\xi) = \xi_Q(q)$. The Lie bracket on $\Gamma(A)$ is determined by requiring that it agree with the bracket of $\mathfrak{g}$ on constant sections, where $\xi \in \mathfrak{g}$ is identified with the constant section $q \mapsto (q,\xi)$, and extended to arbitrary sections by the Leibniz rule. We take the standard connection on a trivial bundle, where $\nabla\xi = 0$ on constant sections.

Now, an inner product on $\mathfrak{g}$ gives a bundle metric on A that is constant with respect to the basepoint, i.e., $g_q(\xi,\eta) = g(\xi,\eta)$, so ∇ is a metric-compatible connection. It follows that $\overline{\nabla}_\xi \eta = \operatorname{ad}_\xi \eta$, $\overline{\nabla}^*_\xi \mu = -\operatorname{ad}^*_\xi \mu$, and $\overline{\nabla}^\dagger_\xi \eta = -\operatorname{ad}^\dagger_\xi \eta$ for constant sections $\xi, \eta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. Writing $a(t) = \big(q(t), \xi(t)\big)$ and $p(t) = \big(q(t), \mu(t)\big)$, we have $\xi = g^\sharp(\mu)$, and (11) becomes

$$\begin{aligned}\dot q &= \xi_Q(q), \\ \dot\mu - \operatorname{ad}^*_\xi \mu &= -\rho^*\big(\mathrm{d}U(q)\big).\end{aligned}$$

Applying $g^\sharp$ to both sides of the second equation gives

$$\dot{\xi} - \operatorname{ad}_\xi^\dagger \xi = - \operatorname{grad} U(q),$$

as in Martínez [17]. Lemma 3.8 recovers the formula $\nabla^g_\xi \eta = \frac{1}{2}\bigl(- \operatorname{ad}_\xi^\dagger \eta - \operatorname{ad}_\eta^\dagger \xi + [\xi, \eta] \bigr)$ for constant sections $\xi, \eta \in \mathfrak{g}$, as in Cortés and Martínez [4, Section 7].

Remark 3.12. See Grabowska, Urbański, and Grabowski [10] for an application of generalized geodesic flow on a Lie algebroid to Wong's equations [20], which are a classic example in reduction theory. See also Li, Stern, and Tang [13, Example 3.18] for a discussion of this example from the Lagrangian point of view.

4. Generalization of Hamilton's phase space principle

4.1. **The variational principle.** In this section, we establish a variational principle for a Hamiltonian $H \in C^\infty(A^*)$ whose critical paths are solutions to the generalized Lie–Poisson dynamics (9). We begin by describing the admissible paths and variations.

Definition 4.1. An $(A \oplus A^*)$-*path* is a C^1 path $(a, p) \colon I \to A \oplus A^*$ over a C^2 base path $q = \tau \circ a = \pi \circ p \colon I \to Q$ such that $\dot{q} = \rho(a)$, i.e., a is an A-path. Let $P(A \oplus A^*) \subset \widetilde{P}(A \oplus A^*)$ denote the Banach submanifold of $(A \oplus A^*)$-paths, among all C^1 paths with C^2 base paths.

An *admissible variation* of $(a, p) \in P(A \oplus A^*)$ is a tangent vector $(\delta a, \delta p) \in T_{(a,p)} P(A \oplus A^*)$ such that $\delta a = X_{b,a} \in T_a P(A)$ is admissible in the sense of Definition 2.15. Relative to a TQ-connection ∇ on A, we have

$$\delta a^{\mathrm{ver}} = \overline{\nabla}_a b, \qquad \delta p^{\mathrm{ver}} = r, \qquad \delta a^{\mathrm{hor}} = \delta p^{\mathrm{hor}} = \rho(b),$$

where $(b, r) \in \widetilde{P}(A \oplus A^*)$ covers the same base path q, where b vanishes at the endpoints of I, and where b and r may otherwise be arbitrary.

Note that $\delta a^{\mathrm{hor}} = \delta p^{\mathrm{hor}}$ is necessary for $(\delta a, \delta p)$ to be tangent to $P(A \oplus A^*)$, which we can see by differentiating the condition $\tau \circ a = \pi \circ p$.

Remark 4.2. An alternative perspective is that $A \oplus A^* \to Q$ is in fact a Lie algebroid, where the anchor is $(a, p) \mapsto \rho(a)$ and the Lie bracket is just that for the A-component. From this point of view, $(A \oplus A^*)$-paths and admissible variations are a special case of the earlier definitions in Section 2. In particular, we may make use of the results in Crainic and Fernandes [6, Section 4.2] regarding the Banach manifold structure of $P(A \oplus A^*)$.

The usual techniques of calculus of variations may now be applied to $P(A \oplus A^*)$. Given a functional $S \colon P(A \oplus A^*) \to \mathbb{R}$, we denote

$$\delta S(a, p) \coloneqq \bigl\langle \mathrm{d} S(a, p), (\delta a, \delta p) \bigr\rangle,$$

where $(\delta a, \delta p)$ is an admissible variation of (a, p). If $\epsilon \mapsto (a_\epsilon, p_\epsilon) \in P(A \oplus A^*)$ is a curve, i.e., a homotopy of $(A \oplus A^*)$-paths, such that $\frac{\mathrm{d}}{\mathrm{d}\epsilon} (a_\epsilon, p_\epsilon) \bigr\rvert_{\epsilon = 0} = (\delta a, \delta p)$, then we have $\delta S(a, p) = \frac{\mathrm{d}}{\mathrm{d}\epsilon} S(a_\epsilon, p_\epsilon) \bigr\rvert_{\epsilon=0}$. In particular, when S has the form

$$S(a, p) = \int_I \mathcal{L}\bigl(a(t), p(t)\bigr) \, \mathrm{d}t,$$

for some $\mathcal{L} \colon A \oplus A^* \to \mathbb{R}$, then differentiating under the integral sign gives

$$\delta S(a, p) = \int_I \frac{\partial}{\partial \epsilon} \mathcal{L}\bigl(a_\epsilon(t), p_\epsilon(t)\bigr) \Bigr\rvert_{\epsilon = 0} \, \mathrm{d}t = \int_I \Bigl\langle \mathrm{d}\mathcal{L}\bigl(a(t), p(t)\bigr), \bigl(\delta a(t), \delta p(t)\bigr) \Bigr\rangle \, \mathrm{d}t.$$

For notational simplicity, we now suppress the dependence of the integrand on t. The TQ-connection ∇ can be used to expand the integrand into vertical and horizontal parts,

$$\bigl\langle \mathrm{d}\mathcal{L}(a, p), (\delta a, \delta p) \bigr\rangle = \bigl\langle \mathrm{d}\mathcal{L}^{\mathrm{ver}}(a, p), (\overline{\nabla}_a b, r) \bigr\rangle + \bigl\langle \mathrm{d}\mathcal{L}^{\mathrm{hor}}(a, p), \rho(b) \bigr\rangle.$$

Denoting $\mathrm{d}\mathcal{L}^{\mathrm{ver}} =: (\mathrm{d}\mathcal{L}_a^{\mathrm{ver}}, \mathrm{d}\mathcal{L}_p^{\mathrm{ver}})$, we therefore obtain the variational formula

$$\begin{aligned}\delta S(a,p) &= \int_I \Big(\big\langle \mathrm{d}\mathcal{L}_a^{\mathrm{ver}}(a,p), \overline{\nabla}_a b \big\rangle + \big\langle r, \mathrm{d}\mathcal{L}_p^{\mathrm{ver}}(a,p) \big\rangle + \big\langle \mathrm{d}\mathcal{L}^{\mathrm{hor}}(a,p), \rho(b) \big\rangle \Big)\, \mathrm{d}t \\ &= \int_I \Bigg(\Big\langle -\overline{\nabla}_a^* \mathrm{d}\mathcal{L}_a^{\mathrm{ver}}(a,p) + \rho^*\big(\mathrm{d}\mathcal{L}^{\mathrm{hor}}(a,p)\big), b \Big\rangle + \big\langle r, \mathrm{d}\mathcal{L}_p^{\mathrm{ver}}(a,p) \big\rangle \Bigg)\, \mathrm{d}t\end{aligned}$$

On the second line, we integrate by parts using the dual connection $\overline{\nabla}_a^*$ and the fact that b vanishes at the endpoints of I. Since b and r are otherwise arbitrary paths over q, we conclude that $\delta S(a,p) = 0$ for all admissible variations $(\delta a, \delta p)$ if and only if

$$\overline{\nabla}_a^* \mathrm{d}\mathcal{L}_a^{\mathrm{ver}}(a,p) = \rho^*\big(\mathrm{d}\mathcal{L}^{\mathrm{hor}}(a,p)\big), \qquad \mathrm{d}\mathcal{L}_p^{\mathrm{ver}}(a,p) = 0. \tag{13}$$

In light of Remark 4.2, this can be seen as a particular case of the Euler–Lagrange–Poincaré equations in Li et al. [13, Theorem 3.12], discussed further in Section 5, where $\mathcal{L}$ is a Lagrangian on $A \oplus A^*$ viewed as a Lie algebroid.

Note that, using the vertical lifts $V_a \colon A_q \to T_a A$ and $V_p \colon A_q^* \to T_p A^*$, we can write

$$\begin{aligned}\big\langle \mathrm{d}\mathcal{L}_a^{\mathrm{ver}}(a,p), \overline{\nabla}_a b \big\rangle = \big\langle \mathrm{d}\mathcal{L}(a,p), V_a(\overline{\nabla}_a b) \big\rangle &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \mathcal{L}(a + \epsilon \overline{\nabla}_a b, p) \Big|_{\epsilon=0} \\ \big\langle r, \mathrm{d}\mathcal{L}_p^{\mathrm{ver}}(a,p) \big\rangle = \big\langle V_p r, \mathrm{d}\mathcal{L}(a,p) \big\rangle &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \mathcal{L}(a, p + \epsilon r) \Big|_{\epsilon=0},\end{aligned}$$

so $\mathrm{d}\mathcal{L}_a^{\mathrm{ver}}$ and $\mathrm{d}\mathcal{L}_p^{\mathrm{ver}}$ are simply the fiber derivatives of $\mathcal{L}$ along the A and A^* fibers, respectively.

We now show that a particular choice of $\mathcal{L}$ gives a variational principle equivalent to the generalized Lie–Poisson equations for H.

Theorem 4.3. *Given a Hamiltonian $H \in C^\infty(A^*)$, an $(A \oplus A^*)$-path (a,p) satisfies the $(\pm)$ generalized Lie–Poisson equations (9) with $a = \mp \mathrm{d}H^{\mathrm{ver}}(p)$ if and only if*

$$\delta \int_I \big(\langle p, a \rangle \pm H(p) \big)\, \mathrm{d}t = 0,$$

with respect to admissible variations.

Proof. Equation (9a) is just the A-path condition for a, so it holds automatically for $(A \oplus A^*)$-paths. Let $\mathcal{L}(a,p) = \langle p, a \rangle \pm H(p)$, and let ∇ be a TQ-connection on A. Taking the fiber derivatives of $\mathcal{L}$, as above, we see that

$$\mathrm{d}\mathcal{L}_a^{\mathrm{ver}}(a,p) = p, \qquad \mathrm{d}\mathcal{L}_p^{\mathrm{ver}}(a,p) = a \pm \mathrm{d}H^{\mathrm{ver}}(p).$$

Next, to show that the horizontal derivative of $\phi(a,p) \coloneqq \langle a, p \rangle$ vanishes, we use the horizontal lifts $H_a \colon T_q Q \to T_a A$ of ∇ and $H_p \colon T_q Q \to T_p A^*$ of ∇^*. Similarly to the proof of Lemma 3.1, if $\xi \in \Gamma(A)$ and $\mu \in \Gamma(A^*)$ satisfy $\xi(q) = a$ and $\mu(q) = p$, and if $v \in T_q Q$, then

$$\begin{aligned}\big\langle \mathrm{d}\phi^{\mathrm{hor}}(a,p), v \big\rangle &= \big\langle \mathrm{d}\phi(a,p), (H_a v, H_p v) \big\rangle \\ &= \Big\langle \mathrm{d}\phi(a,p), \big(T\xi(v) - V_a(\nabla_v \xi), T\mu(v) - V_p(\nabla_v^* \mu) \big) \Big\rangle \\ &= \big\langle \mathrm{d}\phi(a,p), T(\xi,\mu)(v) \big\rangle - \big\langle \mathrm{d}\phi_a^{\mathrm{ver}}(a,p), \nabla_v \xi \big\rangle - \big\langle \nabla_v^* \mu, \mathrm{d}\phi_p^{\mathrm{ver}}(a,p) \big\rangle \\ &= v\big[\langle \xi, \mu \rangle \big] - \langle \mu, \nabla_v \xi \rangle - \langle \nabla_v^* \mu, \xi \rangle \\ &= 0,\end{aligned}$$

where the last equality is the defining property of ∇^*. (The preceding lines can be seen as the Leibniz rule for the covariant derivative of the tensor $\phi \colon A \otimes A^* \to \mathbb{R}$.) Thus,

$$\mathrm{d}\mathcal{L}^{\mathrm{hor}}(a,p) = \pm \mathrm{d}H^{\mathrm{hor}}(p).$$

We conclude that (13) holds if and only if (9b) and $a = \mp \mathrm{d}H^{\mathrm{ver}}(p)$ hold. □

Example 4.4. If $A = \mathfrak{g} \to \bullet$ is a Lie algebra, the variational principle of Theorem 4.3 can be restated as follows: Find $(\xi, \mu)\colon I \to \mathfrak{g} \oplus \mathfrak{g}^*$ such that

$$\delta \int_I \bigl(\langle \mu, \xi \rangle \pm H(\mu) \bigr) \, \mathrm{d}t = 0,$$

where $\delta\xi = \dot{\eta} + [\xi, \eta]$ for arbitrary $\eta\colon I \to \mathfrak{g}$ vanishing at the endpoints of I, and where $\delta\mu$ is arbitrary with no boundary conditions. This recovers the *Lie–Poisson variational principle* of Cendra et al. [1, Theorem 2.1], who present it for the $(-)$ Lie–Poisson equations.

Example 4.5. If $A = TQ \to Q$ is the tangent bundle, recall that the TQ-path condition is $a = \dot{q}$, since ρ is the identity map. Thus, TQ-paths are identified (via tangent prolongation) with paths in Q, and it follows that $(TQ \oplus T^*Q)$-paths are identified with paths in T^*Q. Hence, Theorem 4.3 gives the equivalence of the $(\pm)$ Hamilton's equations (7) with

$$\delta \int_I \bigl(\langle p, \dot{q} \rangle \pm H(q,p) \bigr) \, \mathrm{d}t = 0.$$

The $(-)$ case, which is the usual sign convention, recovers *Hamilton's phase space principle*.

4.2. **Special case: the Hamilton–Poincaré variational principle and equations.** If $A = TQ/G \to Q/G$ is the Atiyah algebroid of a principal bundle $Q \to Q/G$, we now show that Theorem 4.3 recovers the equivalence of the *Hamilton–Poincaré variational principle* and *Hamilton–Poincaré equations* of Cendra et al. [1, Theorem 8.1]. This is similar to the Lie algebroid approach to the Lagrange–Poincaré variational principle and Lagrange–Poincaré equations in Li, Stern, and Tang [13, Section 2.4], from which we adapt some of the details. Note that the base of this algebroid is Q/G rather than Q.

As in [13], we begin by observing that a principal connection is a right splitting of the Atiyah sequence

$$0 \to \widetilde{\mathfrak{g}} \to TQ/G \xrightarrow{\rho} T(Q/G) \to 0,$$

where $\widetilde{\mathfrak{g}}$ denotes the adjoint bundle $Q \times_G \mathfrak{g}$, and where a left splitting is a principal connection 1-form (Mackenzie [14, Chapter 5]). This gives a splitting of the Atiyah algebroid $A = TQ/G \cong T(Q/G) \oplus \widetilde{\mathfrak{g}}$, where ρ is the projection onto the first component. In terms of this splitting, the bracket of two sections $\xi = (X, \overline{\xi})$ and $\eta = (Y, \overline{\eta})$ is

$$\bigl[(X, \overline{\xi}), (Y, \overline{\eta}) \bigr] = \bigl([X,Y], \widetilde{\nabla}_X \overline{\eta} - \widetilde{\nabla}_Y \overline{\xi} + [\overline{\xi}, \overline{\eta}] - \widetilde{R}(X,Y) \bigr),$$

where $\widetilde{\nabla}$ is the covariant derivative and $\widetilde{R}$ the curvature form of the principal connection (Li et al. [13, Equation 3.4], Cendra et al. [2, Theorem 5.2.4], Mackenzie [14, Theorem 7.3.7]).

Given an A-path $a = (x, \dot{x}, \overline{v})$, where x is the base path in Q/G, and an arbitrary path $b = (x, \delta x, \overline{\eta})$ in A, it follows from a calculation in [13, Section 3.4] that admissible variations have horizontal component $\rho(b) = \delta x$ and vertical component

$$\overline{\nabla}_a b = \bigl(\overline{\nabla}_{\dot{x}} (\delta x), \widetilde{\nabla}_{\dot{x}} \overline{\eta} + [\overline{v}, \overline{\eta}] - \widetilde{R}(\dot{x}, \delta x) \bigr). \tag{14}$$

That is, admissible variations have the form $\delta a = (\delta x, \delta \dot{x}, \delta \overline{v})$, where

$$\delta \dot{x} = \overline{\nabla}_{\dot{x}} (\delta x), \qquad \delta \overline{v} = \widetilde{\nabla}_{\dot{x}} \overline{\eta} + [\overline{v}, \overline{\eta}] - \widetilde{R}(\dot{x}, \delta x),$$

and where δx and $\overline{\eta}$ both vanish at the endpoints of I. Finally, using the principal connection to split $A^* = T^*Q/G \cong T^*(Q/G) \oplus \widetilde{\mathfrak{g}}^*$, we can write $p = (x, y, \overline{\mu})$, whose variations have the form $\delta p = (\delta x, \delta y, \delta \overline{\mu})$, where δy and $\delta \overline{\mu}$ are arbitrary. It follows that the variational principle in Theorem 4.3 can be written as

$$\delta \int_I \bigl(\langle y, \dot{x} \rangle + \langle \overline{\mu}, \overline{v} \rangle \pm H(x, y, \overline{\mu}) \bigr) \, \mathrm{d}t = 0,$$

subject to the admissible variations above. The $(-)$ case is precisely the Hamilton–Poincaré variational principle of Cendra et al. [1, Theorem 8.1].

Let us now see how this corresponds to the generalized Lie–Poisson equations. First, observe that (9a) says that $a = (x, \dot{x}, \overline{v})$ is an A-path, so $a = \mp \mathrm{d}H^{\mathrm{ver}}(p)$ becomes

$$\dot{x} = \mp \frac{\partial H}{\partial y}, \tag{15a}$$

$$\overline{v} = \mp \frac{\partial H}{\partial \overline{\mu}}. \tag{15b}$$

Next, observe that, for $p = (x, y, \overline{\mu})$ and arbitrary $b = (x, \delta x, \overline{\eta})$, by (14) we have

$$\langle p, \overline{\nabla}_a b \rangle = \big\langle y, \overline{\nabla}_{\dot{x}}(\delta x) \big\rangle + \big\langle \overline{\mu}, \widetilde{\nabla}_{\dot{x}} \overline{\eta} + [\overline{v}, \overline{\eta}] - \widetilde{R}(\dot{x}, \delta x) \big\rangle,$$

from which it follows that

$$\overline{\nabla}_a^* p = \Big(\overline{\nabla}_{\dot{x}}^* y + \big\langle \overline{\mu}, \widetilde{R}(\dot{x}, \cdot) \big\rangle, \widetilde{\nabla}_{\dot{x}}^* \overline{\mu} - \mathrm{ad}_{\overline{v}}^* \overline{\mu} \Big).$$

Finally, equation (9b) says that this is equal to

$$\pm \rho^* \big(\mathrm{d}H^{\mathrm{hor}}(p) \big) = \Big(\pm \frac{\partial H}{\partial x}, 0 \Big).$$

That is,

$$\overline{\nabla}_{\dot{x}}^* y = \pm \frac{\partial H}{\partial x} - \big\langle \overline{\mu}, \widetilde{R}(\dot{x}, \cdot) \big\rangle, \tag{15c}$$

$$\widetilde{\nabla}_{\dot{x}}^* \overline{\mu} = \mathrm{ad}_{\overline{v}}^* \overline{\mu}. \tag{15d}$$

In the $(-)$ case, (15) is the coordinate-free form of the Hamilton–Poincaré equations in Cendra et al. [1], modulo small differences in notation, e.g., [1] write both covariant derivatives $\overline{\nabla}_{\dot{x}}$ and $\widetilde{\nabla}_{\dot{x}}$ as D/Dt.

5. Correspondence to the Lagrangian case

We conclude with a discussion of the relationship between Hamiltonian mechanics on A^* and Lagrangian mechanics on A, linking the results of the present paper to those of Li, Stern, and Tang [13].

Definition 5.1. We say $H \in C^\infty(A^*)$ is *hyperregular* if $\mathrm{d}H^{\mathrm{ver}} \colon A^* \to A$ is a diffeomorphism. Likewise, $L \in C^\infty(A)$ is hyperregular if $\mathrm{d}L^{\mathrm{ver}} \colon A \to A^*$ is a diffeomorphism.

Remark 5.2. Here, $\mathrm{d}H^{\mathrm{ver}}$ and $\mathrm{d}L^{\mathrm{ver}}$ are defined independently of a choice of connection, since they are simply the derivatives along fibers. These fiber derivatives are often denoted (especially in the case $A = TQ$) by $\mathbb{F}H$ and $\mathbb{F}L$ and called *Legendre transformations*.

If $H \in C^\infty(A^*)$ is a hyperregular Hamiltonian, and A^* is equipped with the $(\pm)$ generalized Lie–Poisson structure, we define the Lagrangian $L \in C^\infty(A)$ by

$$L(a) = \langle p, a \rangle \pm H(p), \tag{16a}$$

where $p \in A^*$ is defined implicitly by $a = \mp \mathrm{d}H^{\mathrm{ver}}(p)$. It follows by a short calculation using the chain rule that L is also hyperregular with

$$p = \mathrm{d}L^{\mathrm{ver}}(a). \tag{16b}$$

(See Marsden and Ratiu [16, Exercise 7.2-3] for the case of arbitrary vector bundles.) Conversely, given a hyperregular Lagrangian $L \in C^\infty(A)$, we may define $H \in C^\infty(A^*)$ by

$$H(p) = \mp \big(\langle p, a \rangle - L(a) \big), \tag{17a}$$

where $a \in A$ is defined implicitly by $p = \mathrm{d}L^{\mathrm{ver}}(a)$. A calculation similar to the one described above shows that H is also hyperregular with

$$a = \mp \mathrm{d}H^{\mathrm{ver}}(p). \tag{17b}$$

Thus, the hypothesis of hyperregularity allows one to start with either a Hamiltonian or Lagrangian and pass to the other.

The following theorem summarizes the main results of this paper, along with those of Li, Stern, and Tang [13], and establishes their relationship. Compare Cendra et al. [1, Theorem 8.1] for the special case of the Atiyah algebroid discussed in Section 4.2.

Theorem 5.3. *Let (a,p) be an $(A \oplus A^*)$-path, and let ∇ be a TQ-connection on A. Given a Hamiltonian $H \in C^\infty(A^*)$, the following are equivalent:*

(i) The variational principle

$$\delta \int_I \bigl(\langle p, a \rangle \pm H(p) \bigr) \,\mathrm{d}t = 0$$

holds with respect to admissible variations of $(A \oplus A^*)$-paths.

(ii) The generalized Lie–Poisson equations

$$\overline{\nabla}_a^* p = \pm \rho^* \bigl(\mathrm{d}H^{\mathrm{hor}}(p) \bigr)$$

hold with $a = \mp \mathrm{d}H^{\mathrm{ver}}(p)$.

Given a Lagrangian $L \in C^\infty(A)$, the following are equivalent:

(iii) The variational principle

$$\delta \int_I L(a) \,\mathrm{d}t = 0$$

holds with respect to admissible variations of A-paths, and $p = \mathrm{d}L^{\mathrm{ver}}(a)$.

(iv) The Euler–Lagrange–Poincaré equations

$$\overline{\nabla}_a^* p = \rho^* \bigl(\mathrm{d}L^{\mathrm{hor}}(a) \bigr)$$

hold with $p = \mathrm{d}L^{\mathrm{ver}}(a)$.

Under the hypothesis of hyperregularity, all of (i)–(iv) are equivalent.

Remark 5.4. In [13], $\overline{\nabla}_a^*$ does not denote the dual connection but rather the adjoint of $\overline{\nabla}_a$, similar to ad and ad^*. This is $-\overline{\nabla}_a^*$ in the notation of the present paper, leading to a change of sign in the Euler–Lagrange–Poincaré equations.

Proof. The equivalence of (i) and (ii) is Theorem 4.3, and that of (iii) and (iv) is [13, Theorem 3.12]. Under the hypothesis of hyperregularity, (i) implies (iii) by (16), and conversely, (iii) implies (i) by (17). □

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