Mixed finite element methods for nonlinear reaction-diffusion equations with interfaces

Xinran Jin[1], Jeonghun J. Lee[2]

[1,2] *Department of Mathematics, Baylor University, Waco, Texas, USA*

Abstract

We develop mixed finite element methods for nonlinear reaction-diffusion equations with interfaces which have Robin-type interface conditions. We introduce the velocity of chemicals as new variables and reformulate the governing equations. The stability of semidiscrete solutions, existence and the a priori error estimates of fully discrete solutions are proved by fixed point theorem and continuous/discrete Grönwall inequalities. Numerical results illustrating our theoretical analysis are included.

Keywords: reaction-diffusion equations, mixed finite element methods, interface conditions, error analysis
2010 MSC: Primary: 65N30, 65N15

1. Introduction

The reaction-diffusion equations are widely used to describe the diffusion of chemical substances with their reactions. Therefore, methods to numerically solve reaction-diffusion equations have also been studied for a very long time. Recently, a reaction-diffusion model interacting with other physical/chemical conditions has been actively studied, beyond the simple reaction-diffusion equations. An example of such extended reaction-diffusion equations is the reaction-diffusion model with a thin membrane in the domain. This model can be used to model the cases where a semi-permeable membrane is involved in reaction-diffusion processes of chemicals. The governing equations are a set of reaction-diffusion equations in which interface conditions on thin membranes are involved (cf. [1, 2]). Reaction-diffusion equations with such interface conditions have been studied in several previous studies. Well-posedness of several partial differential equation models were studied ([7, 8, 9, 10, 11, 13, 14, 16, 17]). Discontinuous Galerkin finite element methods for some diffusion, advection-diffusion, reaction-diffusion equations with possibly nonlinear interface conditions were studied in [3, 4, 5, 6, 15, 12].

Email addresses: xinran_jin1@baylor.edu (Xinran Jin[1]), jeonghun_lee@baylor.edu (Jeonghun J. Lee[2])

Preprint submitted to Journal of Computational and Applied Mathematics 31st August 2024

In this paper, we will study mixed finite element methods to solve nonlinear reaction-diffusion equations with interface conditions, particularly, for the models in [7]. In mixed finite element methods using the dual mixed form of diffusion equations (see, e.g., [18]), the velocity of each chemical is chosen as additional variable. Compared to the classical discontinuous Galerkin methods in the previous studies ([3, 4, 5, 6, 15, 12]), numerical solutions of the mixed finite element methods satisfy local mass conservation without additional post-processing of numerical solutions. Moreover, to the best of our knowledge, fast solver algorithms of discontinuous Galerkin methods for interface problems have not been explored. In contrast, scalable fast solver algorithms for the interface problems that we consider in this paper, have been developed with a solid theoretical basis (cf. [19]). Therefore, the mixed methods in this paper are advantageous for numerical simulations of large scale problems.

The paper is organized as follows. In Section 2 we introduce definitions, governing equations of the reaction-diffusion equations with membrane structures, and semidiscrete discretization with finite element methods. In Section 3 we define fully discrete scheme with the Crank–Nicolson method and prove well-posedness of fully discrete solutions for sufficiently small time step sizes. We prove the a priori error estimates of the fully discrete scheme in Section 4 and present numerical experiment results in Section 5. Conclusions and future research directions will be given in Section 6.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz continuous polygonal/polyhedral boundary. For finite element discretization we consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω with shape-regular triangles/tetrahedra and without hanging nodes. Here $h > 0$ is the maximum radius of triangles/tetrahedra in $\mathcal{T}_h$. The $(d-1)$-dimensional simplices in $\mathcal{T}_h$ will be called facets in the paper.

For $1 \leqslant r \leqslant \infty$, $L^r(\Omega)$ is the Lebesgue space with the norm

$$\|v\|_{L^r(\Omega)} = \begin{cases} \left(\int_\Omega |v(x)|^r \, dx\right)^{1/r}, & \text{if } 1 \leqslant r < \infty, \\ \operatorname{esssup}_{x \in \Omega}\{|v(x)|\}, & \text{if } r = \infty. \end{cases}$$

For a subdomain $D \subset \Omega$ with positive d-dimensional Lebesgue measure, let $L^2(D)$ and $L^2(D; \mathbb{R}^d)$ be the sets of $\mathbb{R}$- and $\mathbb{R}^d$-valued square integrable functions with inner products $(v, v')_D := \int_D vv' \, dx$ and $(\boldsymbol{v}, \boldsymbol{v}')_D := \int_D \boldsymbol{v} \cdot \boldsymbol{v}' \, dx$. In the paper $H^s(D)$, $s \geqslant 0$, denotes the Sobolev space based on the L^2-norm with s-differentiability on D. We refer to [20] for a rigorous definition of $H^s(D)$. The norm on $H^s(D)$ is denoted by $\| \cdot \|_{s,D}$ and D is omitted if $D = \Omega$.

For $T > 0$ and a separable Hilbert space $\mathcal{X}$, let $C^0([0,T]; \mathcal{X})$ denote the set of functions $f : [0,T] \to \mathcal{X}$ that are continuous in $t \in [0,T]$. For an integer $m \geqslant 1$, we define

$$C^m([0,T]; \mathcal{X}) = \{f \mid \partial_t^i f \in C^0([0,T]; \mathcal{X}),\ 0 \leqslant i \leqslant m\},$$

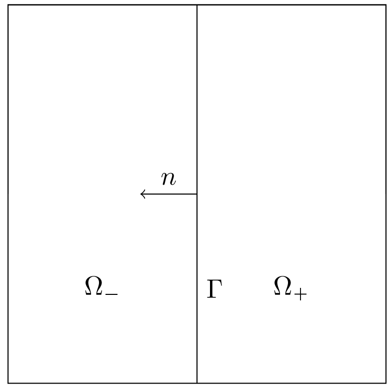

Figure 1: A model domain Ω with interface Γ

where $\partial_t^i f$ is the i-th time derivative in the sense of the Fréchet derivative in $\mathcal{X}$ (cf. [21]). For a function $f : [0,T] \to \mathcal{X}$, the Bochner norm is defined by

$$\|f\|_{L^r(0,T;\mathcal{X})} = \begin{cases} \left(\int_0^T \|f(s)\|_{\mathcal{X}}^r ds\right)^{1/r}, & 1 \leqslant r < \infty, \\ \operatorname{esssup}_{t\in(0,T)} \|f(t)\|_{\mathcal{X}}, & r = \infty. \end{cases}$$

$W^{k,r}(0,T;\mathcal{X})$ for a non-negative integer k and $1 \leqslant r \leqslant \infty$ is defined by the closure of $C^k([0,T];\mathcal{X})$ with the norm $\|f\|_{W^{k,r}(0,T;\mathcal{X})} = \sum_{i=0}^k \|\partial_t^i f\|_{L^r(0,T;\mathcal{X})}$. The semi-norm $\|f\|_{\dot{W}^{k,r}(0,T;\mathcal{X})}$ is defined by $\|f\|_{\dot{W}^{k,r}(0,T;\mathcal{X})} = \|\partial_t^k f\|_{L^r(0,T;\mathcal{X})}$.

For a normed space $\mathcal{X}$ with norm $\|\cdot\|_{\mathcal{X}}$ and functions $f_1, f_2 \in \mathcal{X}$, $\|f_1, f_2\|_{\mathcal{X}}$ will denote $\|f_1\|_X + \|f_2\|_{\mathcal{X}}$, and $\|f_1, f_2, f_3\|_{\mathcal{X}}$ is defined similarly.

2.1. Governing equations

In this subsection we introduce governing equations, a reformulation of the equations, and a variational formulation for finite element methods.

We assume that $\Omega_+, \Omega_- \subset \Omega$ are two disjoint subdomains with polygonal/polyhedral boundaries such that $\overline{\Omega_+} \cup \overline{\Omega_-} = \overline{\Omega}$, and let $\Gamma = \partial\Omega_+ \cap \partial\Omega_-$. For a function $v \in L^2(\Omega)$ such that $v|_{\Omega_j} \in H^1(\Omega_j)$ for $j = +, -$, we use $v|_{\Gamma_j}$ to denote the trace of v on Γ from $v|_{\Omega_j}$. Note that $v|_{\Gamma_+} \neq v|_{\Gamma_-}$ in general. Throughout this paper, the unit normal vector field n on Γ is the normal vector outward from Ω_+ (see Figure 1).

Suppose that u_i, $1 \leqslant i \leqslant N$ are real-valued functions on $[0,T] \times \Omega$. We use $u_i(t)$, $0 \leqslant t \leqslant T$, to denote a real-valued function $u_i(t,\cdot)$ defined on Ω. For given functions

$$f_i : \mathbb{R}^N \to \mathbb{R}, \quad g_i : [0,T] \times \partial\Omega \to \mathbb{R} \tag{1}$$

we consider the system of equations to find

$$(u_1, \cdots, u_N) : [0,T] \times \Omega \to \mathbb{R}^N$$

such that

$$\partial_t u_i(t) - \operatorname{div}(\kappa_i \nabla u_i(t)) = f_i(u_1(t), \cdots, u_N(t)) \qquad \text{in } \Omega, \tag{2a}$$

with interface condition

$$-(\kappa_i \nabla u_i(t)) \cdot n = K_i(u_i|_{\Gamma_+}(t) - u_i|_{\Gamma_-}(t)) \qquad \text{on } \Gamma, K_i > 0, \tag{3}$$

for all $0 < t \leqslant T$, $1 \leqslant i \leqslant N$ and with initial condition

$$(u_1(0), \cdots, u_N(0)). \tag{4}$$

To make (2) a well-posed system of partial differential equations, appropriate boundary conditions are necessary. A set of full Dirichlet boundary conditions

$$u_i(t) = g_i(t) \quad \text{ on } \partial\Omega \quad \forall 1 \leqslant i \leqslant N, 0 < t \leqslant T,$$

can be imposed to make (2) well-posed. For simplicity, we assume that $g_i = 0$ for $1 \leqslant i \leqslant N$, $0 < t \leqslant T$ in the rest of this paper but the discussions below can be extended to more general boundary conditions including $g_i \neq 0$ and Neumann or mixed boundary conditions on $\partial\Omega$ with appropriate modifications. Throughout this paper we assume that the functions $\{f_i\}_{k=1}^N$ satisfy a Lipschitz continuity assumption that as follows: For $v_i, w_i \in L^2(\Omega)$, $1 \leqslant i \leqslant N$,

$$\begin{aligned} &\|f_i(v_1(x), \cdots, v_N(x)) - f_i(w_1(x), \cdots, w_N(x))\| \\ &\qquad\qquad \leqslant L_i \left(\sum_{i=1}^N |v_i(x) - w_i(x)|^2 \right)^{\frac{1}{2}} \end{aligned} \tag{5}$$

for almost every $x \in \Omega$ with a constant $L_i > 0$ where $\| \cdot \|$ means the Euclidean norm in $\mathbb{R}^N$. We also assume that f_i, $1 \leqslant i \leqslant N$ satisfies $f_i(u_1, \cdots, u_N) \leqslant 0$ if $u_j = 0$ for all $1 \leqslant j \leqslant d$, $j \neq i$. This assumption is motivated from the physical modeling such that f_i is the increasing/decreasing rate of the i-th chemical by chemical reaction of other chemicals $\{u_j\}_{j,j\neq i}$. In particular,

$$f_i(0, \cdots, 0) = 0 \text{ for } 1 \leqslant i \leqslant N. \tag{6}$$

By introducing $\sigma_i = -\kappa_i \nabla u_i$, we have a system equivalent to (2) with unknowns $(\sigma_1(t), \cdots, \sigma_N(t))$, $(u_1(t), \cdots, u_N(t))$ such that

$$\kappa_i^{-1} \sigma_i(t) = -\nabla u_i(t) \qquad \text{in } \Omega, \tag{7a}$$

$$\partial_t u_i(t) + \operatorname{div} \sigma_i(t) = f_i(u_1(t), \cdots, u_N(t)) \qquad \text{in } \Omega \tag{7b}$$

with interface conditions

$$\sigma_i(t) \cdot n = K_i(u_i|_{\Gamma_+}(t) - u_i|_{\Gamma_-}(t)) \qquad \text{on } \Gamma \tag{8}$$

for all $0 \leqslant t \leqslant T$, $1 \leqslant i \leqslant N$. The boundary conditions

$$u_i(t) = 0 \qquad \text{on } \partial\Omega, \quad 1 \leqslant i \leqslant N, 0 < t \leqslant T \tag{9}$$

are imposed as before. For initial conditions, in addition to $(u_1(0), \cdots, u_N(0))$ in (4), we need $(\sigma_1(0), \cdots, \sigma_N(0))$ satisfying (7a), (8) for $t = 0$.

To derive a variational formulation of (7), let

$$\Sigma = \{\tau \in H(\mathrm{div}, \Omega) : \tau \cdot n|_\Gamma \in L^2(\Gamma)\}, \qquad V = L^2(\Omega),$$

where $H(\mathrm{div}, \Omega)$ is the subset of $L^2(\Omega; \mathbb{R}^d)$ such that the divergence of $\tau \in L^2(\Omega; \mathbb{R}^d)$ is well-defined as an element in $L^2(\Omega)$. Then, we define $\boldsymbol{\Sigma}$ and $\boldsymbol{V}$ by

$$\boldsymbol{\Sigma} = \Sigma_1 \times \cdots \times \Sigma_N, \quad \boldsymbol{V} = V_1 \times \cdots \times V_N$$

with $\Sigma_i = \Sigma$, $V_i = V$ for $1 \leqslant i \leqslant N$. Then, after the integration by parts of (7a) for $1 \leqslant i \leqslant N$, we can derive a system of variational equations from (7) and (8): Find $(\sigma_1, \cdots, \sigma_N) \in C^0([0, T]; \boldsymbol{\Sigma})$, $(u_1, \cdots, u_N) \in C^1([0, T]; \boldsymbol{V})$ such that

$$(\kappa_i^{-1} \sigma_i(t), \tau_i)_\Omega + \left\langle K_i^{-1} \sigma_i(t) \cdot n, \tau_i \cdot n \right\rangle_\Gamma - (u_i(t), \mathrm{div}\, \tau_i)_\Omega = 0, \tag{10a}$$

$$(\partial_t u_i(t), v_i)_\Omega + (\mathrm{div}\, \sigma_i(t), v_i)_\Omega - (f_i(u_1(t), \cdots, u_N(t)), v_i)_\Omega = 0 \tag{10b}$$

for all $0 \leqslant t \leqslant T$, $1 \leqslant i \leqslant N$ and for all $(\tau_1, \cdots, \tau_N) \in \boldsymbol{\Sigma}$, $(v_1, \cdots, v_N) \in \boldsymbol{V}$.

2.2. Finite element discretization

In this subsection we present discretization of (10) with finite element methods.

For an integer $l \geqslant 0$ and a set $D \subset \mathbb{R}^d$, $\mathcal{P}_l(D)$ is the space of polynomials defined on D of degree at most l. Similarly, $\mathcal{P}_l(D; \mathbb{R}^d)$ is the space of $\mathbb{R}^d$-valued polynomials of degree at most l. For given $l \geqslant 1$ let us define

$$\Sigma_h(T) = \mathcal{P}_{l-1}(T; \mathbb{R}^d) + \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} \mathcal{P}_{l-1}(T). \tag{11}$$

Suppose that $\Sigma_{h,i} \subset \Sigma_i$ is the Raviart–Thomas(–Nedelec) element ([22, 23, 18]) defined by

$$\Sigma_{h,i} = \{\tau \in \Sigma_i \,:\, \tau|_T \in \Sigma_h(T), \quad \forall T \in \mathcal{T}_h\}$$

and $V_{h,i}$ is defined by

$$V_{h,i} = \{v \in V \,:\, v|_T \in \mathcal{P}_{l-1}(T) \quad \forall T \in \mathcal{T}_h\}. \tag{12}$$

Then, it is well-known that the pair $(\Sigma_{h,i}, V_{h,i})$ satisfies

$$\mathrm{div}\, \Sigma_{h,i} = V_{h,i}, \qquad \inf_{v_i \in V_h} \sup_{\tau_i \in \Sigma_{h,i}} \frac{(v_i, \mathrm{div}\, \tau_i)_\Omega}{\|v_i\| \|\tau_i\|_{\mathrm{div}}} \geqslant C > 0 \tag{13}$$

with a uniform $C > 0$ independence of i and mesh sizes of $\mathcal{T}_h$ [18, p. 406].

2.3. Semidiscrete scheme and stability

In this subsection we define a semidiscrete scheme of (10) with $\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$ and discuss the stability of semidiscrete solutions. For simplicity define $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ by $(\sigma_1, \sigma_2, \cdots, \sigma_N)$ and $(u_1, \cdots, u_N)$, and semidiscrete solutions $\boldsymbol{\sigma}_h : [0,T] \to \boldsymbol{\Sigma}_h$, $\boldsymbol{u}_h : [0,T] \to \boldsymbol{V}_h$ are defined similarly.

For

$$\begin{aligned}\boldsymbol{\tau} &= (\tau_1, \cdots, \tau_N), \boldsymbol{\eta} = (\eta_1, \cdots, \eta_N) \in \boldsymbol{\Sigma},\\ \boldsymbol{v} &= (v_1, \cdots, v_N), \boldsymbol{w} = (w_1, \cdots, w_N) \in \boldsymbol{V},\end{aligned}$$

define three bilinear and one nonlinear forms

$$a(\boldsymbol{\tau}, \boldsymbol{\eta}) := \sum_{i=1}^{N} (\kappa_i^{-1} \tau_i, \eta_i)_\Omega + \sum_{i=1}^{N} \left\langle K_i^{-1} \tau_i \cdot n, \eta_i \cdot n \right\rangle_\Gamma, \tag{14}$$

$$b(\boldsymbol{\tau}, \boldsymbol{v}) := \sum_{i=1}^{N} (v_i, \operatorname{div} \tau_i)_\Omega, \tag{15}$$

$$c(\boldsymbol{v}, \boldsymbol{w}) := \sum_{i=1}^{N} (v_i, w_i)_\Omega, \tag{16}$$

$$d(\boldsymbol{v}, \boldsymbol{w}) := \sum_{i=1}^{N} (f_i(v_i, \cdots, v_N), w_i)_\Omega.$$

Then, the system (10) can be rewritten as

$$a(\boldsymbol{\sigma}(t), \boldsymbol{\tau}) - b(\boldsymbol{\tau}, \boldsymbol{u}(t)) = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \tag{17a}$$

$$b(\boldsymbol{\sigma}(t), \boldsymbol{v}) + c(\partial_t \boldsymbol{u}(t), \boldsymbol{v}) - d(\boldsymbol{u}(t), \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{17b}$$

A discrete-in-space and continuous-in-time semidiscrete scheme with finite element space $\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$, is to find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) : [0,T] \to \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$ such that

$$a(\boldsymbol{\sigma}_h(t), \boldsymbol{\tau}) - b(\boldsymbol{\tau}, \boldsymbol{u}_h(t)) = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \tag{18a}$$

$$b(\boldsymbol{\sigma}_h(t), \boldsymbol{v}) + c(\partial_t \boldsymbol{u}_h(t), \boldsymbol{v}) - d(\boldsymbol{u}_h(t), \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h \tag{18b}$$

for all $t \in [0,T]$. For stability analysis, let $\boldsymbol{\tau} = \boldsymbol{\sigma}_h(t)$, $\boldsymbol{v} = \boldsymbol{u}_h(t)$ and add the equations. Then,

$$\frac{1}{2} \frac{d}{dt} c(\boldsymbol{u}_h(t), \boldsymbol{u}_h(t)) + a(\boldsymbol{\sigma}_h(t), \boldsymbol{\sigma}_h(t)) = d(\boldsymbol{u}_h(t), \boldsymbol{u}_h(t)).$$

By the Lipschitz continuity assumption (5) and (6), we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}c(\boldsymbol{u}_h(t), \boldsymbol{u}_h(t)) + a(\boldsymbol{\sigma}_h(t), \boldsymbol{\sigma}_h(t)) \\
&\quad = d(\boldsymbol{u}_h(t), \boldsymbol{u}_h(t)) \\
&\quad = \sum_{i=1}^{N} (f_i(u_{h,1}(t), \cdots, u_{h,N}(t)), u_{h,i}(t))_\Omega \\
&\quad = \sum_{i=1}^{N} (f_i(u_{h,1}(t), \cdots, u_{h,N}(t)) - f_i(0, \cdots, 0), u_{h,i}(t))_\Omega \\
&\quad \leqslant L\|\boldsymbol{u}_h(t)\|^2_{L^2(\Omega)}
\end{aligned}$$

where $L = \max_{1\leqslant i\leqslant N}\{L_i\}$. Recalling that $c(\boldsymbol{u}_h(t), \boldsymbol{u}_h(t)) = \|\boldsymbol{u}_h(t)\|^2_{L^2(\Omega)}$, and $a(\boldsymbol{\sigma}_h(t), \boldsymbol{\sigma}_h(t)) \geqslant 0$, by Grönwall inequality,

$$\|\boldsymbol{u}_h(t)\|_{L^2(\Omega)} \leqslant e^{2Lt}\|\boldsymbol{u}_h(0)\|_{L^2(\Omega)}.$$

If $T > 0$ is fixed, then

$$\max_{0\leqslant t\leqslant T} \|\boldsymbol{u}_h(t)\|_{L^2(\Omega)} \leqslant e^{2LT}\|\boldsymbol{u}_h(0)\|_{L^2(\Omega)}. \tag{19}$$

To estimate $\|\boldsymbol{\sigma}_h(t)\|_{L^2(\Omega)}$, we take $\boldsymbol{\tau} = \boldsymbol{\sigma}_h(t)$ in the time derivative of (18a), $\boldsymbol{v} = \partial_t \boldsymbol{u}_h(t)$ in (18b), and add the two equations. Then, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}a(\boldsymbol{\sigma}_h(t), \boldsymbol{\sigma}_h(t)) + \|\partial_t \boldsymbol{u}_h(t)\|^2_{L^2(\Omega)} &= d(\boldsymbol{u}_h(t), \partial_t \boldsymbol{u}_h(t)) \\
&\leqslant L\|\boldsymbol{u}_h(t)\|_{L^2(\Omega)}\|\partial_t \boldsymbol{u}_h(t)\|_{L^2(\Omega)} \\
&\leqslant \frac{L^2}{4}\|\boldsymbol{u}_h(t)\|^2_{L^2(\Omega)} + \|\partial_t \boldsymbol{u}_h(t)\|^2_{L^2(\Omega)}.
\end{aligned}$$

By integrating in time,

$$a(\boldsymbol{\sigma}_h(t), \boldsymbol{\sigma}_h(t)) \leqslant a(\boldsymbol{\sigma}_h(0), \boldsymbol{\sigma}_h(0)) + \frac{L^2}{2}\int_0^t \|\boldsymbol{u}_h(s)\|^2_{L^2(\Omega)}\, ds.$$

Combining this with (19) gives

$$\begin{aligned}
a(\boldsymbol{\sigma}_h(t), \boldsymbol{\sigma}_h(t)) &\leqslant a(\boldsymbol{\sigma}_h(0), \boldsymbol{\sigma}_h(0)) + \frac{L^2 e^{4LT}}{2}\|\boldsymbol{u}_h(0)\|^2_{L^2(\Omega)}\int_0^t 1\, ds \\
&\leqslant a(\boldsymbol{\sigma}_h(0), \boldsymbol{\sigma}_h(0)) + \frac{tL^2 e^{4LT}}{2}\|\boldsymbol{u}_h(0)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore,

$$\max_{0\leqslant t\leqslant T} a(\boldsymbol{\sigma}_h(t), \boldsymbol{\sigma}_h(t))^{1/2} \leqslant a(\boldsymbol{\sigma}_h(0), \boldsymbol{\sigma}_h(0))^{1/2} + \frac{L\sqrt{T}e^{2LT}}{\sqrt{2}}\|\boldsymbol{u}_h(0)\|_{L^2(\Omega)}.$$

3. Fully discrete scheme and existence of solutions

In this section we present a fully discrete numerical scheme with the Crank–Nicolson method.

For fully discrete scheme, suppose that $(\boldsymbol{\sigma}_h^k, \boldsymbol{u}_h^k) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$, a numerical solution of the previous time step is given. The Crank–Nicolson scheme is to find $(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{u}_h^{k+1}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$ such that

$$\frac{1}{2}a(\boldsymbol{\sigma}_h^k + \boldsymbol{\sigma}_h^{k+1}, \boldsymbol{\tau}) - \frac{1}{2}b(\boldsymbol{\tau}, \boldsymbol{u}_h^k + \boldsymbol{u}_h^{k+1}) = 0, \tag{20a}$$

$$\begin{aligned} \frac{1}{2}b(\boldsymbol{\sigma}_h^k + \boldsymbol{\sigma}_h^{k+1}, \boldsymbol{v}) + \frac{1}{\Delta t}c\left(\boldsymbol{u}_h^{k+1} - \boldsymbol{u}_h^k, v\right) \\ -\frac{1}{2}(d(\boldsymbol{u}_h^k, \boldsymbol{v}) + d(\boldsymbol{u}_h^{k+1}, \boldsymbol{v})) = 0. \end{aligned} \tag{20b}$$

Since (20) is a nonlinear system, existence of $(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{u}_h^{k+1})$ is not guaranteed. We use a fixed point theorem to prove existence of $(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{u}_h^{k+1})$.

Theorem 3.1 (Existence and uniqueness of fully discrete solutions)**.** *Suppose that Δt is sufficiently small to satisfy*

$$L\Delta t < 2 \tag{21}$$

where $L > 0$ is the constant of Lipschitz continuity of $d(\cdot, \cdot)$ in (5)*. Then, there exists a unique $(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{u}_h^{k+1}) \in \boldsymbol{\Sigma_h} \times \boldsymbol{V}_h$ satisfying* (20)*.*

Proof. Recall the fully discrete scheme.

$$\begin{aligned} &\frac{1}{2}a(\boldsymbol{\sigma}_h^k + \boldsymbol{\sigma}_h^{k+1}, \boldsymbol{\tau}) - \frac{1}{2}b(\boldsymbol{\tau}, \boldsymbol{u}_h^k + \boldsymbol{u}_h^{k+1}) = 0, \\ &\frac{1}{2}b(\boldsymbol{\sigma}_h^k + \boldsymbol{\sigma}_h^{k+1}, \boldsymbol{v}) + c\left(\frac{\boldsymbol{u}_h^{k+1} - \boldsymbol{u}_h^k}{\Delta t}, \boldsymbol{v}\right) \\ &\quad - \frac{1}{2}(d(\boldsymbol{u}_h^k, \boldsymbol{v}) + d(\boldsymbol{u}_h^{k+1}, \boldsymbol{v})) = 0. \end{aligned}$$

Assuming that $\boldsymbol{\sigma}_h^k$, $\boldsymbol{u}_h^k$ are given, the system (20) is to find $(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{u}_h^{k+1})$ such that

$$\begin{aligned} &\Delta t(a(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{u}_h^{k+1}) - b(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{v})) + 2c(\boldsymbol{u}_h^{k+1}, \boldsymbol{v}) - \Delta t d(\boldsymbol{u}_h^{k+1}, \boldsymbol{v}) \\ &\quad = -\Delta t(a(\boldsymbol{\sigma}_h^k, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{u}_h^k - b(\boldsymbol{\sigma}_h^k, \boldsymbol{v})) + 2c(\boldsymbol{u}_h^k, \boldsymbol{v}) + \Delta t d(\boldsymbol{u}_h^k, \boldsymbol{v}) \\ &\quad =: G^k(\boldsymbol{\tau}, \boldsymbol{v}) \end{aligned} \tag{22}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$. For simplicity, let $\Phi_{\Delta t} : \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h \to \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$ be the map defined by

$$\begin{aligned} &\langle \Phi_{\Delta t}(\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\boldsymbol{\tau}, \boldsymbol{v}) \rangle_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} \\ &\quad = \Delta t(a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{u}_h) - b(\boldsymbol{\sigma}_h, \boldsymbol{v})) + 2c(\boldsymbol{u}_h, \boldsymbol{v}), \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h, \end{aligned} \tag{23}$$

$$(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h \mapsto \Phi_{\Delta t}(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h \tag{24}$$

where $\langle \cdot, \cdot \rangle_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h}$ is the standard $H(\mathrm{div}) \times L^2$ inner product on $\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$. Then, one can verify that $\Phi_{\Delta t}$ is well-posed by the Lax–Milgram lemma. From the definition of $\Phi_{\Delta t}$ in (23), we can rewrite (22) by

$$\left\langle \Phi_{\Delta t}(\boldsymbol{\sigma}_h^{k+1}, \boldsymbol{u}_h^{k+1}), (\boldsymbol{\tau}, \boldsymbol{v}) \right\rangle_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} - \Delta t d(\boldsymbol{u}_h^{k+1}, \boldsymbol{v}) = G^k(\boldsymbol{\tau}, \boldsymbol{v}).$$

Define $(\boldsymbol{\sigma}_{h,0}^{k+1}, \boldsymbol{u}_{h,0}^{k+1})$ by

$$\left\langle \Phi_{\Delta t}(\boldsymbol{\sigma}_{h,0}^{k+1}, \boldsymbol{u}_{h,0}^{k+1}), (\boldsymbol{\tau}, \boldsymbol{v}) \right\rangle_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} = G^k(\boldsymbol{\tau}, \boldsymbol{v}) \quad \forall (\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$$

and also define $\{(\boldsymbol{\sigma}_{h,m}^{k+1}, \boldsymbol{u}_{h,m}^{k+1})\}_{m=1}^{\infty}$ by

$$\left\langle \Phi_{\Delta t}(\boldsymbol{\sigma}_{h,m+1}^{k+1}, \boldsymbol{u}_{h,m+1}^{k+1}), (\boldsymbol{\tau}, \boldsymbol{v}) \right\rangle_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} - \Delta t d(\boldsymbol{u}_{h,m}^{k+1}, \boldsymbol{v}) = G^k(\boldsymbol{\tau}, \boldsymbol{v})$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$ and for $m \geqslant 0$. By taking difference of the above equations for m, $m+1$,

$$\begin{aligned}
&\left\langle \Phi_{\Delta t}(\boldsymbol{\sigma}_{h,m+1}^{k+1} - \boldsymbol{\sigma}_{h,m}^{k+1}, \boldsymbol{u}_{h,m+1}^{k+1} - \boldsymbol{u}_{h,m}^{k+1}), (\boldsymbol{\tau}, \boldsymbol{v}) \right\rangle_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} \\
&\quad = \Delta t (d(\boldsymbol{u}_{h,m}^{k+1}, \boldsymbol{v}) - d(\boldsymbol{u}_{h,m-1}^{k+1}, \boldsymbol{v}))
\end{aligned}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$. By Lipschitz continuity of the nonlinearity (5) of d,

$$|d(\boldsymbol{u}_{h,m}^{k+1}, \boldsymbol{v}) - d(\boldsymbol{u}_{h,m-1}^{k+1}, \boldsymbol{v})| \leqslant L \|\boldsymbol{u}_{h,m}^{k+1} - \boldsymbol{u}_{h,m-1}^{k+1}\|_{L^2(\Omega)} \|\boldsymbol{v}\|_{L^2(\Omega)}.$$

If Δt is small enough to satisfy $\Delta t L < 2$, then

$$\begin{aligned}
&\left\langle \Phi_{\Delta t}(\boldsymbol{\sigma}_{h,m+1}^{k+1} - \boldsymbol{\sigma}_{h,m}^{k+1}, \boldsymbol{u}_{h,m+1}^{k+1} - \boldsymbol{u}_{h,m}^{k+1}), (\boldsymbol{\sigma}_{h,m+1}^{k+1} - \boldsymbol{\sigma}_{h,m}^{k+1}, \boldsymbol{u}_{h,m+1}^{k+1} - \boldsymbol{u}_{h,m}^{k+1})) \right\rangle_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} \\
&\quad \leqslant \Delta t L \|\boldsymbol{u}_{h,m}^{k+1} - \boldsymbol{u}_{h,m-1}^{k+1}\|_{L^2(\Omega)} \|\boldsymbol{u}_{h,m+1}^{k+1} - \boldsymbol{u}_{h,m}^{k+1}\|_{L^2(\Omega)} \\
&\quad < 2 \|\boldsymbol{u}_{h,m}^{k+1} - \boldsymbol{u}_{h,m-1}^{k+1}\|_{L^2(\Omega)} \|\boldsymbol{u}_{h,m+1}^{k+1} - \boldsymbol{u}_{h,m}^{k+1}\|_{L^2(\Omega)}.
\end{aligned}$$

By the definition of $\Phi_{\Delta t}$,

$$\begin{aligned}
&\left\langle \Phi_{\Delta t}(\boldsymbol{\sigma}_{h,m+1}^{k+1} - \boldsymbol{\sigma}_{h,m}^{k+1}, \boldsymbol{u}_{h,m+1}^{k+1} - \boldsymbol{u}_{h,m}^{k+1}), (\boldsymbol{\sigma}_{h,m+1}^{k+1} - \boldsymbol{\sigma}_{h,m}^{k+1}, \boldsymbol{u}_{h,m+1}^{k+1} - \boldsymbol{u}_{h,m}^{k+1})) \right\rangle_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} \\
&\quad = \Delta t a(\boldsymbol{\sigma}_{h,m+1}^{k+1} - \boldsymbol{\sigma}_{h,m}^{k+1}, \boldsymbol{\sigma}_{h,m+1}^{k+1} - \boldsymbol{\sigma}_{h,m}^{k+1}) + 2 \|\boldsymbol{u}_{h,m+1}^{k+1} - \boldsymbol{u}_{h,m}^{k+1}\|_{L^2(\Omega)}^2.
\end{aligned}$$

The above inequality and equality imply that $\Phi_{\Delta t}$ is a contraction on $\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$ with the norm $\|(\boldsymbol{\tau}, \boldsymbol{v})\|_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} := (\Delta t a(\boldsymbol{\tau}, \boldsymbol{\tau}) + 2\|\boldsymbol{v}\|_{L^2(\Omega)}^2)^{1/2}$ if $\Delta t L < 2$. Therefore, there is a unique fixed point $(\boldsymbol{\sigma}_{h,\infty}^{k+1}, \boldsymbol{u}_{h,\infty}^{k+1}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$ such that

$$\left\| \left(\boldsymbol{\sigma}_{h,m}^{k+1} - \boldsymbol{\sigma}_{h,\infty}^{k+1}, \boldsymbol{u}_{h,m}^{k+1} - \boldsymbol{u}_{h,\infty}^{k+1} \right) \right\|_{\boldsymbol{\Sigma}_h \times \boldsymbol{V}_h} \to 0 \text{ as } m \to \infty.$$

By the Banach contraction principle, this fixed point is unique, so the proof is completed. $\square$

4. A priori error estimates

For $T > 0$ let $\Delta t = T/M$ for a natural number M and define $\{t_k\}_{n=0}^M$ by $t_k = k\Delta t$. For a variable $g : [0,T] \to X$ for a Hilbert space X, we will use g_h^k and g^k for the numerical solution of g at t_k and $g(t_k)$, respectively. The variable g can be $\boldsymbol{\sigma}$, $\boldsymbol{u}$ in the problem. For simplicity we will also use the definitions

$$\bar{\partial}_t g^{k+\frac{1}{2}} := \frac{1}{\Delta t}(g^{k+1} - g^k), \quad g^{k+\frac{1}{2}} := \frac{1}{2}(g^k + g^{k+1})$$

for any sequence $\{v^k\}_{k=0}^M$ of functions with upper index k.

Let $\Pi_h : H^1(\Omega;\mathbb{R}^d) \to \Sigma_h$ be the canonical interpolation operator of the Raviart–Thomas–Nedelec elements (see, e.g., [18]). If P_h is the L^2 projection to V_h, then (Π_h, P_h) satisfies the commuting diagram property

$$\operatorname{div}\Pi_h\tau = P_h \operatorname{div}\tau, \qquad \tau \in H^1(\Omega,\mathbb{R}^d). \tag{25}$$

On every facet F in $\mathcal{T}_h$ and a normal vector n_F on F,

$$\int_F (\tau - \Pi_h\tau)\cdot n_F q\, ds = 0 \quad \forall q \in \mathcal{P}_{l-1}(F). \tag{26}$$

By extending Π_h and P_h to the N-copies of $H^1(\Omega;\mathbb{R}^d)$ and $L^2(\Omega)$, we define

$$\boldsymbol{\Pi}_h : \underbrace{H^1(\Omega;\mathbb{R}^d) \times \cdots \times H^1(\Omega;\mathbb{R}^d)}_{N \text{ tuples}} \to \boldsymbol{\Sigma}_h, \quad \boldsymbol{P}_h : \boldsymbol{V} \to \boldsymbol{V}_h.$$

Let

$$e_{\boldsymbol{\sigma}}^k := \boldsymbol{\sigma}^k - \boldsymbol{\sigma}_h^k = (\sigma_1^k - \sigma_{1,h}^k, \cdots, \sigma_N^k - \sigma_{N,h}^k), \tag{27}$$
$$e_{\boldsymbol{u}} := \boldsymbol{u}^k - \boldsymbol{u}_h^k = (u_1^k - u_{1,h}^k, \cdots, u_N^k - u_{N,h}^k), \tag{28}$$

and define $e_{\boldsymbol{\sigma}}^{h,k}, e_{\boldsymbol{\sigma}}^{I,k}, e_{\boldsymbol{u}}^{h,k}, e_{\boldsymbol{u}}^{I,k}$ by

$$e_{\boldsymbol{\sigma}}^{h,k} := \boldsymbol{\Pi}_h\boldsymbol{\sigma}^k - \boldsymbol{\sigma}_h^k, \quad e_{\boldsymbol{u}}^{h,k} := \boldsymbol{P}_h\boldsymbol{u}^k - \boldsymbol{u}_h^k,$$
$$e_{\boldsymbol{\sigma}}^{I,k} := \boldsymbol{\Pi}_h\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^k, \quad e_{\boldsymbol{u}}^{I,k} := \boldsymbol{P}_h\boldsymbol{u}^k - \boldsymbol{u}^k.$$

By a standard approximation theory of interpolation operators, assuming that $\sigma_i^k \in H^r(\Omega;\mathbb{R}^d)$ and $u_i^k \in H^s(\Omega)$ with $r > 1/2$, $s \geqslant 0$,

$$\|\sigma_i^k - \Pi_h\sigma_i^k\|_{L^2(\Omega)} \leqslant Ch^m\|\sigma_i^k\|_{H^r(\Omega)} \qquad \frac{1}{2} < m \leqslant \max\{l,r\}, \tag{29}$$
$$\|u_i^k - P_h u_i^k\|_{L^2(\Omega)} \leqslant Ch^s\|u_i^k\|_{H^s(\Omega)} \qquad 0 \leqslant m \leqslant \max\{l,s\}. \tag{30}$$

As immediate extensions,

$$\|\boldsymbol{\sigma}_i^k - \boldsymbol{\Pi}_h\boldsymbol{\sigma}_i^k\|_{L^2(\Omega)} \leqslant Ch^m\|\boldsymbol{\sigma}^k\|_{H^r(\Omega)} \qquad \frac{1}{2} < m \leqslant \max\{l,r\}, \tag{31}$$
$$\|\boldsymbol{u}^k - \boldsymbol{P}_h\boldsymbol{u}^k\|_{L^2(\Omega)} \leqslant Ch^s\|\boldsymbol{u}^k\|_{H^s(\Omega)} \qquad 0 \leqslant m \leqslant \max\{l,s\}. \tag{32}$$

By the commuting diagram property (25) and the property $\operatorname{div} \Sigma_h = V_h$,

$$b(e_{\boldsymbol{\sigma}}^{I,k}, \boldsymbol{v}) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{33a}$$

$$b(\boldsymbol{\tau}, e_{\boldsymbol{u}}^{I,k}) = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h. \tag{33b}$$

Here we recall a discrete Grönwall inequality before we begin our proof of error estimates (cf. [24, 25]).

Lemma 4.1. *Let $\Delta t > 0$, $B, C > 0$ and $\{a_k\}_k$, $\{b_k\}_k$, $\{c_k\}_k$ be sequences of non-negative numbers satisfying*

$$a_k + \Delta t \sum_{i=0}^{k} b_i \leqslant B + C\Delta t \sum_{i=0}^{k} a_i + \sum_{i=0}^{k} c_i \tag{34}$$

for all $k \geqslant 0$. Then, if $C\Delta t < 1$,

$$a_k + \Delta t \sum_{i=0}^{k} b_i \leqslant e^{C(k+1)\Delta t} \left(B + \sum_{i=0}^{k} c_i \right). \tag{35}$$

Remark 4.2. *We remark that* (34) *and* (35) *are slightly different in [24]. In particular, the summation $\sum_{i=0}^{k} c_i$ is $\Delta t \sum_{i=0}^{k} c_i$ in [24] but we can show that* (34) *implies* (35) *with the same proof.*

Theorem 4.1. *Suppose that a pair $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N)$, $\boldsymbol{u} = (u_1, \cdots, u_N)$ is a solution of* (10)*. Suppose also that the assumption of Theorem 3.1 holds, and the sequence $\{(\boldsymbol{\sigma}_h^k, \boldsymbol{u}_h^k)\}_k$ is a solution of* (20) *for given numerical initial data $(\boldsymbol{\sigma}_h^0, \boldsymbol{u}_h^0) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$ satisfying $a(\boldsymbol{\sigma}_h^0, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{u}_h^0) = 0$. Recall the definitions of $e_{\boldsymbol{\sigma}}^{h,k}$ and $e_{\boldsymbol{u}}^{h,k}$ in* (27), (28)*. If $0 < \Delta t < 1/L$ for the L in Theorem 3.1, then*

$$\begin{aligned}
&\|e_{\boldsymbol{u}}^{h,k}\|_{L^2(\Omega)}^2 + \frac{\Delta t}{4} \sum_{m=0}^{k-1} a(e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1}, e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1}) \\
&+ a(e_{\boldsymbol{\sigma}}^{h,k}, e_{\boldsymbol{\sigma}}^{h,k}) + \frac{1}{2\Delta t} \sum_{m=0}^{k-1} \|e_{\boldsymbol{u}}^{m+1} - e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)}^2 \\
&\quad \leqslant \|e_{\boldsymbol{u}}^{h,0}\|_{L^2(\Omega)}^2 + a(e_{\boldsymbol{\sigma}}^{h,0}, e_{\boldsymbol{\sigma}}^{h,0}) \\
&\qquad + C\left(h^{2r}\|\boldsymbol{\sigma}, \boldsymbol{u}\|_{L^\infty(0,t_k;H^r(\Omega))}^2 + (\Delta t)^4 \|\partial_t^3 \boldsymbol{u}\|_{L^\infty(0,t_k;L^2(\Omega))}^2\right) \\
&\qquad + Ch^{2r}\|\partial_t \boldsymbol{\sigma}\|_{L^\infty(0,t_k;H^r(\Omega))}^2 + Ch^{2r} \sum \|\boldsymbol{u}\|_{L^\infty(0,t_k;H^r(\Omega))}^2
\end{aligned}$$

for $\frac{1}{2} < r \leqslant l$.

Proof. Note that solutions of (10) satisfy

$$\begin{aligned}
\frac{1}{2} a(\boldsymbol{\sigma}^k + \boldsymbol{\sigma}^{k+1}, \boldsymbol{\tau}) - \frac{1}{2} b(\boldsymbol{\tau}, \boldsymbol{u}^k + \boldsymbol{u}^{k+1}) = 0, \\
\frac{1}{2} b(\boldsymbol{\sigma}^k + \boldsymbol{\sigma}^{k+1}, \boldsymbol{v}) + \frac{1}{2} c(\partial_t \boldsymbol{u}^k + \partial_t \boldsymbol{u}^{k+1}, \boldsymbol{v}) - \frac{1}{2}(d(\boldsymbol{u}^k, \boldsymbol{v}) + d(\boldsymbol{u}^{k+1}, \boldsymbol{v})) = 0
\end{aligned}$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$, $k \geqslant 0$. The difference of the above equations and (20) gives

$$
\begin{aligned}
&\frac{1}{2} a(e_{\boldsymbol{\sigma}}^k + e_{\boldsymbol{\sigma}}^{k+1}, \boldsymbol{\tau}) - \frac{1}{2} b(\boldsymbol{\tau}, e_{\boldsymbol{u}}^k + e_{\boldsymbol{u}}^{k+1}) = 0, \\
&\frac{1}{2} b(e_{\boldsymbol{\sigma}}^k + e_{\boldsymbol{\sigma}}^{k+1}, \boldsymbol{v}) + c\left(\frac{1}{2}(\partial_t \boldsymbol{u}^k + \partial_t \boldsymbol{u}^{k+1}) - \frac{1}{\Delta t}(\boldsymbol{u}_h^{k+1} - \boldsymbol{u}_h^k), \boldsymbol{v}\right) \\
&\quad - \frac{1}{2}(d(\boldsymbol{u}^{k+1}, \boldsymbol{v}) - d(\boldsymbol{u}_h^{k+1}, \boldsymbol{v}) + d(\boldsymbol{u}^k, \boldsymbol{v}) - d(\boldsymbol{u}_h^k, \boldsymbol{v})) = 0
\end{aligned}
$$

for all $(\boldsymbol{\tau}, \boldsymbol{v}) \in \boldsymbol{\Sigma}_h \times \boldsymbol{V}_h$. Recalling that $e_{\boldsymbol{\sigma}}^k = e_{\boldsymbol{\sigma}}^{h,k} - e_{\boldsymbol{\sigma}}^{I,k}$, $e_{\boldsymbol{u}}^k = e_{\boldsymbol{u}}^{h,k} - e_{\boldsymbol{u}}^{I,k}$,

$$
\begin{aligned}
&\frac{1}{2} a(e_{\boldsymbol{\sigma}}^{h,k} + e_{\boldsymbol{\sigma}}^{h,k+1}, \boldsymbol{\tau}) - \frac{1}{2} b(\boldsymbol{\tau}, e_{\boldsymbol{u}}^{h,k} + e_{\boldsymbol{u}}^{h,k+1}) \\
&\quad = \frac{1}{2} a(e_{\boldsymbol{\sigma}}^{I,k} + e_{\boldsymbol{\sigma}}^{I,k+1}, \boldsymbol{\tau}) - \frac{1}{2} b(\boldsymbol{\tau}, e_{\boldsymbol{u}}^{I,k} + e_{\boldsymbol{u}}^{I,k+1}), \\
&\frac{1}{2} b(e_{\boldsymbol{\sigma}}^{h,k} + e_{\boldsymbol{\sigma}}^{h,k+1}, \boldsymbol{v}) + \frac{1}{\Delta t} c\left(e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}, \boldsymbol{v}\right) \\
&\quad = c\left(\frac{1}{\Delta t}(\boldsymbol{u}^{k+1} - \boldsymbol{u}^k) - \frac{1}{2}(\partial_t \boldsymbol{u}^k + \partial_t \boldsymbol{u}^{k+1}), \boldsymbol{v}\right) + \frac{1}{2} b\left(e_{\boldsymbol{\sigma}}^{I,k} + e_{\boldsymbol{\sigma}}^{I,k+1}, \boldsymbol{v}\right) \\
&\qquad - \frac{1}{2}\left(d(\boldsymbol{u}_h^{k+1}, \boldsymbol{v}) - d(\boldsymbol{u}^{k+1}, \boldsymbol{v}) + d(\boldsymbol{u}_h^k, \boldsymbol{v}) - d(\boldsymbol{u}^k, \boldsymbol{v})\right).
\end{aligned}
$$

By (33), we can get reduced error equations

$$
\begin{aligned}
&\frac{1}{2} a\left(e_{\boldsymbol{\sigma}}^{h,k} + e_{\boldsymbol{\sigma}}^{h,k+1}, \boldsymbol{\tau}\right) - \frac{1}{2} b\left(\boldsymbol{\tau}, e_{\boldsymbol{u}}^{h,k} + e_{\boldsymbol{u}}^{h,k+1}\right) = \frac{1}{2} a\left(e_{\boldsymbol{\sigma}}^{I,k} + e_{\boldsymbol{\sigma}}^{I,k+1}, \boldsymbol{\tau}\right), \\
&\frac{1}{2} b\left(e_{\boldsymbol{\sigma}}^{h,k} + e_{\boldsymbol{\sigma}}^{h,k+1}, \boldsymbol{v}\right) + \frac{1}{\Delta t} c\left(e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}, \boldsymbol{v}\right) \\
&= c\left(\frac{1}{\Delta t}(\boldsymbol{u}^{k+1} - \boldsymbol{u}^k) - \frac{1}{2}(\partial_t \boldsymbol{u}^k + \partial_t \boldsymbol{u}^{k+1}), \boldsymbol{v}\right) \\
&\quad - \frac{1}{2}\left(d(\boldsymbol{u}_h^{k+1}, \boldsymbol{v}) - d(\boldsymbol{u}^{k+1}, \boldsymbol{v}) + d(\boldsymbol{u}_h^k, \boldsymbol{v}) - d(\boldsymbol{u}^k, \boldsymbol{v})\right).
\end{aligned}
$$

Take $\boldsymbol{\tau} = e_{\boldsymbol{\sigma}}^{h,k+1} + e_{\boldsymbol{\sigma}}^{h,k}$, $\boldsymbol{v} = e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}$ and add the equations and get

$$
\begin{aligned}
&\frac{1}{2} a\left(e_{\boldsymbol{\sigma}}^{h,k+1} + e_{\boldsymbol{\sigma}}^{h,k}, e_{\boldsymbol{\sigma}}^{h,k+1} + e_{\boldsymbol{\sigma}}^{h,k}\right) + \frac{1}{\Delta t}\left(\|e_{\boldsymbol{u}}^{h,k+1}\|_{L^2(\Omega)}^2 - \|e_{\boldsymbol{u}}^{h,k}\|_{L^2(\Omega)}^2\right) \\
&\quad = \frac{1}{2} a\left(e_{\boldsymbol{\sigma}}^{I,k} + e_{\boldsymbol{\sigma}}^{I,k+1}, e_{\boldsymbol{\sigma}}^{h,k} + e_{\boldsymbol{\sigma}}^{h,k+1}\right) \\
&\qquad + c\left(\frac{1}{\Delta t}(\boldsymbol{u}^{k+1} - \boldsymbol{u}^k) - \frac{1}{2}(\partial_t \boldsymbol{u}^k + \partial_t \boldsymbol{u}^{k+1}), e_{\boldsymbol{u}}^{h,k} + e_{\boldsymbol{u}}^{h,k+1}\right) \\
&\qquad - \frac{1}{2}\left(d(\boldsymbol{u}_h^{k+1}, e_{\boldsymbol{u}}^{h,k} + e_{\boldsymbol{u}}^{h,k+1}) - d(\boldsymbol{u}^{k+1}, e_{\boldsymbol{u}}^{h,k} + e_{\boldsymbol{u}}^{h,k+1}) + d(\boldsymbol{u}_h^k, e_{\boldsymbol{u}}^{h,k} + e_{\boldsymbol{u}}^{h,k+1})\right) \\
&\qquad + \frac{1}{2} d(\boldsymbol{u}^k, e_{\boldsymbol{u}}^{h,k} + e_{\boldsymbol{u}}^{h,k+1}).
\end{aligned}
$$

By multiplying Δt and by a simple algebraic computation,

$$\|e_{\boldsymbol{u}}^{h,k+1}\|_{L^2(\Omega)}^2 + \frac{\Delta t}{2} a\left(e_{\boldsymbol{\sigma}}^{h,k+1} + e_{\boldsymbol{\sigma}}^{h,k}, e_{\boldsymbol{\sigma}}^{h,k+1} + e_{\boldsymbol{\sigma}}^{h,k}\right) = \|e_{\boldsymbol{u}}^{h,k}\|_{L^2(\Omega)}^2 + \sum_{j=1}^{6} I_j^k \quad (36)$$

where

$$\begin{aligned}
I_1^k &:= \frac{\Delta t}{2} a(e_{\boldsymbol{\sigma}}^{I,k} + e_{\boldsymbol{\sigma}}^{I,k+1}, e_{\boldsymbol{\sigma}}^{h,k} + e_{\boldsymbol{\sigma}}^{h,k+1}), \\
I_2^k &:= c\left(\boldsymbol{u}^{k+1} - \boldsymbol{u}^k - \frac{\Delta t}{2}\left(\partial_t \boldsymbol{u}^k + \partial_t \boldsymbol{u}^{k+1}\right), e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}\right), \\
I_3^k &:= \frac{\Delta t}{2}\left(d\left(\boldsymbol{u}_h^{k+1}, e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}\right) - d\left(\boldsymbol{P}_h \boldsymbol{u}^{k+1}, e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}\right)\right), && (37) \\
I_4^k &:= \frac{\Delta t}{2}\left(d\left(\boldsymbol{u}_h^{k}, e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}\right) - d\left(\boldsymbol{P}_h \boldsymbol{u}^{k}, e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}\right)\right), && (38) \\
I_5^k &:= \frac{\Delta t}{2}\left(d\left(\boldsymbol{P}_h \boldsymbol{u}^{k+1}, e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}\right) - d\left(\boldsymbol{u}^{k+1}, e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}\right)\right), \\
I_6^k &:= \frac{\Delta t}{2}\left(d\left(\boldsymbol{P}_h \boldsymbol{u}^{k}, e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}\right) - d\left(\boldsymbol{u}^{k}, e_{\boldsymbol{u}}^{h,k+1} + e_{\boldsymbol{u}}^{h,k}\right)\right).
\end{aligned}$$

If we take the summation of (36) over k, then we can obtain

$$\|e_{\boldsymbol{u}}^{h,k}\|_{L^2(\Omega)}^2 + \frac{\Delta t}{2}\sum_{m=0}^{k-1} a\left(e_{\boldsymbol{\sigma}}^{h,m+1} + e_{\boldsymbol{\sigma}}^{h,m}, e_{\boldsymbol{\sigma}}^{h,m+1} + e_{\boldsymbol{\sigma}}^{h,m}\right) = \|e_{\boldsymbol{u}}^{h,0}\|_{L^2(\Omega)}^2 + \sum_{m=0}^{k-1}\sum_{j=1}^{6} I_j^m. \quad (39)$$

By the Lipschitz continuity assumption (5) and the triangle inequality,

$$\begin{aligned}
|I_3^m| &\leqslant 2L\Delta t \|e_{\boldsymbol{u}}^{h,m+1}\|_{L^2(\Omega)}\left(\|e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)} + \|e_{\boldsymbol{u}}^{h,m+1}\|_{L^2(\Omega)}\right), && (40) \\
|I_4^m| &\leqslant 2L\Delta t \|e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)}\left(\|e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)} + \|e_{\boldsymbol{u}}^{h,m+1}\|_{L^2(\Omega)}\right), && (41)
\end{aligned}$$

so

$$|I_3^m + I_4^m| \leqslant 4\Delta t L\left(\|e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)}^2 + \|e_{\boldsymbol{u}}^{h,m+1}\|_{L^2(\Omega)}^2\right). \quad (42)$$

By (5), (32), the triangle inequality, and Young's inequality,

$$\begin{aligned}
|I_5^m + I_6^m| &\leqslant \Delta t h^r C\left(\|\boldsymbol{u}^m\|_{H^r(\Omega)} + \|\boldsymbol{u}^{m+1}\|_{H^r(\Omega)}\right)\left(\|e_{\boldsymbol{u}}^{h,m} + e_{\boldsymbol{u}}^{h,m+1}\|_{L^2(\Omega)}\right) \\
&\leqslant C\Delta t h^{2r}\|\boldsymbol{u}\|_{L^\infty(t_m,t_{m+1};H^r(\Omega))}^2 && (43) \\
&\quad + \frac{\Delta t}{4}\left(\|e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)}^2 + \|e_{\boldsymbol{u}}^{h,m+1}\|_{L^2(\Omega)}^2\right).
\end{aligned}$$

Note that

$$\sum_{i=1}^{N} \left\langle K_i^{-1} e_{\sigma_i}^{I,k} \cdot n, \tau_i \cdot n \right\rangle = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h$$

by (26). Then, (31), the Cauchy–Schwarz and Young's inequalities give

$$\begin{aligned} |I_1^m| &= \frac{\Delta t}{2} |a(e_{\boldsymbol{\sigma}}^{I,m} + e_{\boldsymbol{\sigma}}^{I,m+1}, e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1})| \\ &\leqslant \frac{\Delta t}{2} \| e_{\boldsymbol{\sigma}}^{I,m} + e_{\boldsymbol{\sigma}}^{I,m+1} \|_{L^2(\Omega)} \| e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1} \|_{L^2(\Omega)} \\ &\leqslant C \Delta t h^{2r} \| \boldsymbol{\sigma} \|^2_{L^\infty(t_m, t_{m+1}; H^r(\Omega))} \\ &\quad + \frac{\Delta t}{4} a(e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1}, e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1}). \end{aligned} \tag{44}$$

Lastly, we can estimate I_2^m by Cauchy–Schwarz and Young's inequalities,

$$\begin{aligned} |I_2^m| &\leqslant C(\Delta t)^3 \| \partial_t^3 \boldsymbol{u} \|_{L^\infty(t_m, t_{m+1}; L^2(\Omega))} \| e_{\boldsymbol{u}}^{h,m} + e_{\boldsymbol{u}}^{h,m+1} \|_{L^2(\Omega)} \\ &\leqslant C(\Delta t)^5 \| \partial_t^3 \boldsymbol{u} \|^2_{L^\infty(t_m, t_{m+1}; L^2(\Omega))} \\ &\quad + \frac{\Delta t}{4} \left(\| e_{\boldsymbol{u}}^{h,m} \|^2_{L^2(\Omega)} + \| e_{\boldsymbol{u}}^{h,m+1} \|^2_{L^2(\Omega)} \right). \end{aligned} \tag{45}$$

Applying (42), (43), (44), (45) to (39), we get

$$\begin{aligned} &\| e_{\boldsymbol{u}}^{h,k} \|^2_{L^2(\Omega)} + \frac{\Delta t}{4} \sum_{m=0}^{k-1} a(e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1}, e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1}) \\ &\quad \leqslant \| e_{\boldsymbol{u}}^{h,0} \|^2_{L^2(\Omega)} + \Delta t \left(4L^2 + \frac{1}{2} \right) \sum_{m=0}^{k-1} \left(\| e_{\boldsymbol{u}}^{h,m} \|^2_{L^2(\Omega)} + \| e_{\boldsymbol{u}}^{h,m+1} \|^2_{L^2(\Omega)} \right) \\ &\qquad + C \Delta t \sum_{m=0}^{k-1} \left(h^{2r} \| \boldsymbol{\sigma}, \boldsymbol{u} \|^2_{L^\infty(t_m, t_{m+1}; H^r(\Omega))} + (\Delta t)^4 \| \partial_t^3 \boldsymbol{u} \|^2_{L^\infty(t_m, t_{m+1}; L^2(\Omega))} \right). \end{aligned} \tag{46}$$

Recall that $a(\boldsymbol{\sigma}_h^0, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, \boldsymbol{u}_h^0) = 0$ as a condition of numerical initial data. Combining this with the fully discrete scheme, we can get

$$a(e_{\boldsymbol{\sigma}}^k, \boldsymbol{\tau}) - b(\boldsymbol{\tau}, e_{\boldsymbol{u}}^k) = 0, \quad \forall k \geqslant 0.$$

The difference of k and $(k+1)$ time step of the above error equations is

$$\frac{1}{2} a(e_{\boldsymbol{\sigma}}^{k+1} - e_{\boldsymbol{\sigma}}^k, \boldsymbol{\tau}) - \frac{1}{2} b(\boldsymbol{\tau}, e_{\boldsymbol{u}}^{k+1} - e_{\boldsymbol{u}}^k) = 0,$$

so we get another set of error equations

$$
\begin{aligned}
&\frac{1}{2}a(e_{\boldsymbol{\sigma}}^{h,k+1} - e_{\boldsymbol{\sigma}}^{h,k}, \boldsymbol{\tau}) - \frac{1}{2}b(\boldsymbol{\tau}, e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}) \\
&\quad = \frac{1}{2}a(e_{\boldsymbol{\sigma}}^{I,k+1} - e_{\boldsymbol{\sigma}}^{I,k}, \boldsymbol{\tau}) + \frac{1}{2}b(\boldsymbol{\tau}, e_{\boldsymbol{u}}^{I,k+1} - e_{\boldsymbol{u}}^{I,k}), \\
&\frac{1}{2}b(e_{\boldsymbol{\sigma}}^{h,k} + e_{\boldsymbol{\sigma}}^{h,k+1}, \boldsymbol{v}) + \frac{1}{\Delta t}c\left(e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}, \boldsymbol{v}\right) \\
&\quad = c\left(\frac{1}{\Delta t}(\boldsymbol{u}^{k+1} - \boldsymbol{u}^{k}) - \frac{1}{2}(\partial_t \boldsymbol{u}^k + \partial_t \boldsymbol{u}^{k+1}), \boldsymbol{v}\right) - \frac{1}{2}b\left(e_{\boldsymbol{\sigma}}^{I,k} + e_{\boldsymbol{\sigma}}^{I,k+1}, \boldsymbol{v}\right) \\
&\qquad - \frac{1}{2}\left(d(\boldsymbol{u}_h^{k+1}, \boldsymbol{v}) - d(\boldsymbol{u}^{k+1}, \boldsymbol{v}) + d(\boldsymbol{u}_h^{k}, \boldsymbol{v}) - d(\boldsymbol{u}^{k}, \boldsymbol{v})\right).
\end{aligned}
$$

Again by (33), we get reduced error equations

$$
\begin{aligned}
&\frac{1}{2}a(e_{\boldsymbol{\sigma}}^{h,k+1} - e_{\boldsymbol{\sigma}}^{h,k}, \boldsymbol{\tau}) - \frac{1}{2}b(\boldsymbol{\tau}, e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}) = \frac{1}{2}a(e_{\boldsymbol{\sigma}}^{I,k+1} - e_{\boldsymbol{\sigma}}^{I,k}, \boldsymbol{\tau}), \\
&\frac{1}{2}b(e_{\boldsymbol{\sigma}}^{h,k} + e_{\boldsymbol{\sigma}}^{h,k+1}, \boldsymbol{v}) + \frac{1}{\Delta t}c\left(e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}, \boldsymbol{v}\right) \\
&\quad = c\left(\frac{1}{\Delta t}(\boldsymbol{u}^{k+1} - \boldsymbol{u}^{k}) - \frac{1}{2}(\partial_t \boldsymbol{u}^k + \partial_t \boldsymbol{u}^{k+1}), \boldsymbol{v}\right) \\
&\qquad - \frac{1}{2}\left(d(\boldsymbol{u}_h^{k+1}, \boldsymbol{v}) - d(\boldsymbol{u}^{k+1}, \boldsymbol{v}) + d(\boldsymbol{u}_h^{k}, \boldsymbol{v}) - d(\boldsymbol{u}^{k}, \boldsymbol{v})\right).
\end{aligned}
$$

By taking $\boldsymbol{\tau} = 2(e_{\boldsymbol{\sigma}}^{h,k+1} + e_{\boldsymbol{\sigma}}^{h,k})$, $\boldsymbol{v} = 2(e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k})$, and adding these two equations,

$$
\begin{aligned}
&a(e_{\boldsymbol{\sigma}}^{h,k+1}, e_{\boldsymbol{\sigma}}^{h,k+1}) - a(e_{\boldsymbol{\sigma}}^{h,k}, e_{\boldsymbol{\sigma}}^{h,k}) + \frac{2}{\Delta t}c(e_{\boldsymbol{u}}^{k+1} - e_{\boldsymbol{u}}^{h,k}, e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}) \\
&\quad = a(e_{\boldsymbol{\sigma}}^{I,k+1} - e_{\boldsymbol{\sigma}}^{I,k}, e_{\boldsymbol{\sigma}}^{h,k+1} + e_{\boldsymbol{\sigma}}^{h,k}) \\
&\qquad + 2c\left(\frac{1}{\Delta t}(\boldsymbol{u}^{k+1} - \boldsymbol{u}^{k}) - \frac{1}{2}(\partial_t \boldsymbol{u}^k + \partial_t \boldsymbol{u}^{k+1}), e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}\right) \\
&\qquad - (d(\boldsymbol{u}_h^{k+1}, e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}) - d(\boldsymbol{u}^{k+1}, e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k})) \\
&\qquad - (d(\boldsymbol{u}_h^{k}, e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k}) - d(\boldsymbol{u}^{k}, e_{\boldsymbol{u}}^{h,k+1} - e_{\boldsymbol{u}}^{h,k})) \\
&\quad =: J_1^k + J_2^k + J_3^k + J_4^k.
\end{aligned}
$$

Taking the summation of the above equation over k, we can get

$$
\begin{aligned}
a(e_{\boldsymbol{\sigma}}^{h,k}, e_{\boldsymbol{\sigma}}^{h,k}) &+ \frac{2}{\Delta t}\sum_{m=0}^{k-1} c(e_{\boldsymbol{u}}^{m+1} - e_{\boldsymbol{u}}^{h,m}, e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}) \\
&= a(e_{\boldsymbol{\sigma}}^{h,0}, e_{\boldsymbol{\sigma}}^{h,0}) + \sum_{m=0}^{k-1}\left(J_1^m + J_2^m + J_3^m + J_4^m\right). \qquad (47)
\end{aligned}
$$

By an argument similar to (44), we estimate J_1^m with Young's inequality by

$$\begin{aligned}
|J_1^m| &\leqslant \|e_{\boldsymbol{\sigma}}^{I,m+1} - e_{\boldsymbol{\sigma}}^{I,m}\|_{L^2(\Omega)} \|e_{\boldsymbol{\sigma}}^{h,m+1} + e_{\boldsymbol{\sigma}}^{h,m}\|_{L^2(\Omega)}, \\
&\leqslant C\Delta t h^r \|\partial_t \boldsymbol{\sigma}\|_{L^\infty(t_m,t_{m+1};H^r(\Omega))} \|e_{\boldsymbol{\sigma}}^{h,m+1} + e_{\boldsymbol{\sigma}}^{h,m}\|_{L^2(\Omega)} \qquad (48) \\
&\leqslant C\Delta t h^{2r} \|\partial_t \boldsymbol{\sigma}\|^2_{L^\infty(t_m,t_{m+1};H^r(\Omega))} + \frac{\Delta t}{4}\left(a(e_{\boldsymbol{\sigma}}^{h,m+1}, e_{\boldsymbol{\sigma}}^{h,m+1}) + a(e_{\boldsymbol{\sigma}}^{h,m}, e_{\boldsymbol{\sigma}}^{h,m})\right).
\end{aligned}$$

For J_2^m,

$$\begin{aligned}
|J_2^m| &\leqslant 2\left\|\frac{1}{\Delta t}(\boldsymbol{u}^{m+1} - \boldsymbol{u}^m) - \frac{1}{2}(\partial_t \boldsymbol{u}^m + \partial_t \boldsymbol{u}^{m+1})\right\|_{L^2(\Omega)} \|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)}, \\
&\leqslant 2\Delta t \left\|\frac{1}{\Delta t}(\boldsymbol{u}^{m+1} - \boldsymbol{u}^m) - \frac{1}{2}(\partial_t \boldsymbol{u}^m + \partial_t \boldsymbol{u}^{m+1})\right\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2\Delta t}\|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}\|^2_{L^2(\Omega)} \\
&\leqslant 2(\Delta t)^{-1} \left\|(\boldsymbol{u}^{m+1} - \boldsymbol{u}^m) - \frac{\Delta t}{2}(\partial_t \boldsymbol{u}^m + \partial_t \boldsymbol{u}^{m+1})\right\|^2_{L^2(\Omega)} \\
&\quad + \frac{1}{2\Delta t}\|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}\|^2_{L^2(\Omega)} \\
&\leqslant C(\Delta t)^5 \left\|\partial_t^3 \boldsymbol{u}\right\|^2_{L^\infty(t_m,t_{m+1};L^2(\Omega))} + \frac{1}{2\Delta t}\|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}\|^2_{L^2(\Omega)}. \qquad (49)
\end{aligned}$$

By (5), the Cauchy–Schwarz inequality, Young's inequality, and (30),

$$\begin{aligned}
|J_3^m| &\leqslant L\|\boldsymbol{u}_h^{m+1} - \boldsymbol{u}^{m+1}\|_{L^2(\Omega)} \|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)} \\
&\leqslant L\|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{I,m+1}\|_{L^2(\Omega)} \|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)} \\
&\leqslant \Delta t L^2 \left(\|e_{\boldsymbol{u}}^{h,m+1}\|^2_{L^2(\Omega)} + \|e_{\boldsymbol{u}}^{I,m+1}\|^2_{L^2(\Omega)}\right) + \frac{1}{2\Delta t}\|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}\|^2_{L^2(\Omega)} \\
&\leqslant \Delta t L^2 \|e_{\boldsymbol{u}}^{h,m+1}\|^2_{L^2(\Omega)} + \frac{1}{2\Delta t}\|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}\|^2_{L^2(\Omega)} \\
&\quad + C\Delta t h^{2r}\|\boldsymbol{u}\|^2_{L^\infty(t_m,t_{m+1};H^r(\Omega))}. \qquad (50)
\end{aligned}$$

A completely same argument gives

$$\begin{aligned}
|J_4^m| &\leqslant \Delta t L^2 \|e_{\boldsymbol{u}}^{h,m}\|^2_{L^2(\Omega)} + \frac{1}{2\Delta t}\|e_{\boldsymbol{u}}^{h,m+1} - e_{\boldsymbol{u}}^{h,m}\|^2_{L^2(\Omega)} \\
&\quad + C\Delta t h^{2r}\|\boldsymbol{u}\|^2_{L^\infty(t_m,t_{m+1};H^r(\Omega))}. \qquad (51)
\end{aligned}$$

By combining the estimates of $J_1^m, J_2^m, J_3^m, J_4^m$ in (48), (49), (50), (51), we have

$$
\begin{aligned}
&a(e_{\boldsymbol{\sigma}}^{h,k}, e_{\boldsymbol{\sigma}}^{h,k}) + \frac{1}{4\Delta t}\sum_{m=0}^{k-1}\|e_{\boldsymbol{u}}^{m+1} - e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)}^2 \\
&\leqslant a(e_{\boldsymbol{\sigma}}^{h,0}, e_{\boldsymbol{\sigma}}^{h,0}) + C\Delta t h^{2r}\sum_{m=0}^{k-1}\|\partial_t \boldsymbol{\sigma}\|_{L^\infty(t_m,t_{m+1};H^r(\Omega))}^2 \\
&\quad + \sum_{m=0}^{k-1}\frac{\Delta t}{4}\sum_{m=0}^{k-1}\left(a(e_{\boldsymbol{\sigma}}^{h,m+1}, e_{\boldsymbol{\sigma}}^{h,m+1}) + a(e_{\boldsymbol{\sigma}}^{h,m}, e_{\boldsymbol{\sigma}}^{h,m})\right) \\
&\quad + C(\Delta t)^5\sum_{m=0}^{k-1}\left\|\partial_t^3 \boldsymbol{u}\right\|_{L^\infty(t_m,t_{m+1};L^2(\Omega))}^2 \\
&\quad + \Delta t L^2\sum_{m=0}^{k-1}\left(\|e_{\boldsymbol{u}}^{h,m+1}\|_{L^2(\Omega)}^2 + \|e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)}^2\right) \\
&\quad + C\Delta t h^{2r}\sum_{m=0}^{k-1}\|\boldsymbol{u}\|_{L^\infty(t_m,t_{m+1};H^r(\Omega))}^2.
\end{aligned}
\tag{52}
$$

The sum of (46) and (52) gives

$$
\begin{aligned}
&\|e_{\boldsymbol{u}}^{h,k}\|_{L^2(\Omega)}^2 + \frac{\Delta t}{4}\sum_{m=0}^{k-1} a(e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1}, e_{\boldsymbol{\sigma}}^{h,m} + e_{\boldsymbol{\sigma}}^{h,m+1}) \\
&+ a(e_{\boldsymbol{\sigma}}^{h,k}, e_{\boldsymbol{\sigma}}^{h,k}) + \frac{1}{2\Delta t}\sum_{m=0}^{k-1}\|e_{\boldsymbol{u}}^{m+1} - e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)}^2 \\
&\quad \leqslant \|e_{\boldsymbol{u}}^{h,0}\|_{L^2(\Omega)}^2 + a(e_{\boldsymbol{\sigma}}^{h,0}, e_{\boldsymbol{\sigma}}^{h,0}) + \Delta t\left(5L^2 + \frac{1}{2}\right)\sum_{m=0}^{k-1}\left(\|e_{\boldsymbol{u}}^{h,m}\|_{L^2(\Omega)}^2 + \|e_{\boldsymbol{u}}^{h,m+1}\|_{L^2(\Omega)}^2\right) \\
&\qquad + C\Delta t\sum_{m=0}^{k-1}\left(h^{2r}\|\boldsymbol{\sigma}, \boldsymbol{u}\|_{L^\infty(t_m,t_{m+1};H^r(\Omega))}^2 + (\Delta t)^4\|\partial_t^3 \boldsymbol{u}\|_{L^\infty(t_m,t_{m+1};L^2(\Omega))}^2\right) \\
&\qquad + C\Delta t h^{2r}\sum_{m=0}^{k-1}\|\partial_t \boldsymbol{\sigma}\|_{L^\infty(t_m,t_{m+1};H^r(\Omega))}^2 \\
&\qquad + \frac{\Delta t}{4}\sum_{m=0}^{k-1}\left(a(e_{\boldsymbol{\sigma}}^{h,m+1}, e_{\boldsymbol{\sigma}}^{h,m+1}) + a(e_{\boldsymbol{\sigma}}^{h,m}, e_{\boldsymbol{\sigma}}^{h,m})\right) \\
&\qquad + C\Delta t h^{2r}\sum_{m=0}^{k-1}\|\boldsymbol{u}\|_{L^\infty(t_m,t_{m+1};H^r(\Omega))}^2.
\end{aligned}
$$

We remark that $\Delta t \sum_{m=0}^{k-1}\|g\|_{L^\infty(t_m,t_{m+1};\mathcal{X})} \leqslant k\Delta t\|g\|_{L^\infty(0,t_k;\mathcal{X})}$ for $g \in L^\infty(0,T;\mathcal{X})$ with a normed space $\mathcal{X}$, and $k\Delta t = T$ at the final time step $k = M$. Thus, this Δt in $\Delta t \sum_{m=0}^{k-1}\|g\|_{L^\infty(t_m,t_{m+1};\mathcal{X})}$ does not give an additional order of convergence. Finally, the conclusion follows if we apply the discrete Grönwall inequality in Lemma 4.1 to the above inequality. □

As an immediate consequence, we can prove the a priori error estimates.

Theorem 4.2. *Suppose that all assumptions in Theorem 4.1 hold, and the numerical initial data* $(\boldsymbol{\sigma}_h^0, \boldsymbol{u}_h^0)$ *satisfy*

$$\|\boldsymbol{u}(0) - \boldsymbol{u}_h^0\|_{L^2(\Omega)} + a(\boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_h^0, \boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_h^0)^{1/2} \leqslant Ch^r(\|\boldsymbol{u}(0)\|_{H^r(\Omega)} + \|\boldsymbol{\sigma}(0)\|_{H^r(\Omega)})$$

for $\frac{1}{2} < r \leqslant l$*. Then,*

$$\begin{aligned}
&\|\boldsymbol{u}(t_k) - \boldsymbol{u}_h^k\|_{L^2(\Omega)}^2 + \|\boldsymbol{\sigma}(t_k) - \boldsymbol{\sigma}_h^k\|_{L^2(\Omega)}^2 \\
&\quad \leqslant Ch^r(\|\boldsymbol{u}(0)\|_{H^r(\Omega)} + \|\boldsymbol{\sigma}(0)\|_{H^r(\Omega)}) \\
&\qquad + C\left(h^{2r}\|\boldsymbol{\sigma}, \boldsymbol{u}\|_{L^\infty(0,t_k;H^r(\Omega))}^2 + (\Delta t)^4\|\partial_t^3 \boldsymbol{u}\|_{L^\infty(0,t_k;L^2(\Omega))}^2\right) \\
&\qquad + Ch^{2r}\|\partial_t \boldsymbol{\sigma}\|_{L^\infty(0,t_k;H^r(\Omega))}^2 + Ch^{2r}\sum \|\boldsymbol{u}\|_{L^\infty(0,t_k;H^r(\Omega))}^2
\end{aligned}$$

for $\frac{1}{2} < r \leqslant l$ *and* $1 \leqslant k \leqslant M$*.*

Proof. The proof follows immediately by Theorem 4.1, (31), (32), and the triangle inequality. □

5. Numerical experiments

In this section we present numerical experiment results to illustrate that our theoretical error estimates are valid. All numerical experiments are carried out with FEniCS 2019.1.0 (see [26]).

For numerical experiments we set $\Omega = [0,1] \times [0,1]$, $\Gamma = \{1/2\} \times [0,1]$, $\Omega_= = [0,1/2] \times [0,1]$, $\Omega_+ = [1/2,1] \times [0,1]$. We use an unstructured mesh such that Γ is in the union of the edges of the mesh. In numerical experiments for convergence rates of errors, we refine meshes by subdividing each triangle to four congruent subtriangles (see Figure 2).

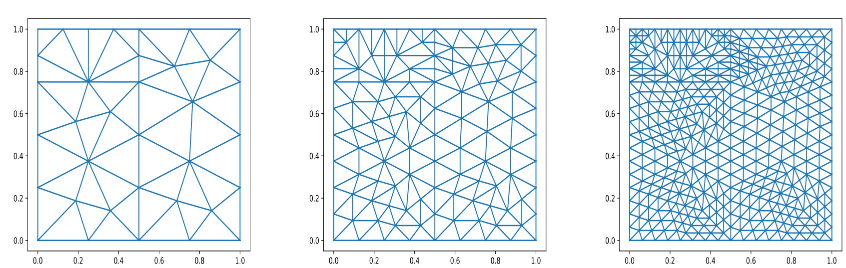

Figure 2: Initial unstructured mesh and its two nested refinements

In our experiments, we used the lowest and the second lowest Raviart–Thomas elements, denoted by RT_0 and RT_1 for V_h. The finite element spaces with piecewise constant and discontinuous piecewise linear polynomials are denoted by DG_0 and DG_1, and these spaces are used for V_h. The stable mixed finite element pairs are (RT_0, DG_0) and (RT_1, DG_1).

In our error analysis, the expected convergence rates of all errors are the first and second orders, respectively. We impose Dirichlet boundary conditions on the top and bottom boundary components of Ω for $i = 1, 2$, and impose Neumann boundary conditions on the left and right boundary components of Ω for $i = 1, 2$.

For manufactured solutions we define

$$u_i = \begin{cases} u_{i,+}, & \text{in } \Omega_+, \\ u_{i,-}, & \text{in } \Omega_-, \end{cases}$$

for $i = 1, 2$ with appropriate functions $u_{i,\pm}$ which will be given below. First, let

$$\phi(x, t) = 1 + (\cos t) \left(x - \frac{1}{2} \right)^2 ,$$

and define

$$\begin{aligned}
\tilde{u}_{1,-}(x, y) &= \sin \frac{\pi x}{3} + \left(x - \frac{1}{2} \right)^2 y(1 - y), \\
\tilde{u}_{1,+}(x, y) &= \sin \frac{\pi x}{3} + 1 + \left(x - \frac{1}{2} \right)^2 \sin(\pi y), \\
\tilde{u}_{2,-}(x, y) &= \cos \frac{\pi x}{3} + 2 \left(x - \frac{1}{2} \right)^2 y(1 - y), \\
\tilde{u}_{2,+}(x, y) &= \cos \frac{\pi x}{3} - 1 + 2 \left(x - \frac{1}{2} \right)^2 \sin(\pi y).
\end{aligned}$$

Then, $u_{i,\pm}$, $i = 1, 2$ are defined by

$$u_{i,-} = \phi(x, t) u_{i,-}, \quad u_{i,+} = \phi(x, t) u_{i,+}.$$

For nonlinearities we take $f_1(u_1, u_2) = u_1^2 u_2^3$ and $f_2(u_1, u_2) = u_1^3 u_2^3$. Then, $\sigma_{i,\pm}$, $f_{i,\pm}$, $i = 1, 2$ are also defined by

$$\begin{aligned}
\sigma_{i,\pm} &= -\nabla u_{i,\pm}, \\
f_{1,\pm} &= \operatorname{div} \sigma_{1,\pm} + u_{1,\pm}^2 u_{2,\pm}^3, \\
f_{2,\pm} &= \operatorname{div} \sigma_{2,\pm} + u_{1,\pm}^3 u_{2,\pm}^3.
\end{aligned}$$

We remark that these nonlinearities are not Lipschitz continuous with uniform Lipschitz constants in general. However, if u_1 and u_2 are functions in $L^\infty(0, T; L^\infty(\Omega))$, then the Lipschitz continuity assumption (5) is satisfied for $0 \leqslant t \leqslant T$. Since we use manufactured solutions which are in $L^\infty(0, T; L^\infty(\Omega))$ in our numerical experiments, our theoretical error estimates are still valid in our numerical experiments.

In Table 1 and Table 2 we present convergence of errors for $\Delta t = h$ and for (RT_0, DG_0), (RT_1, DG_1) pairs. The results show that optimal convergence rates, which we expected in theoretical analysis, are obtained in all cases.

$h_{\max}$	$\|u_1 - u_{1,h}\|_{L^2(\Omega)}$ error	rate	$\|u_2 - u_{2,h}\|_{L^2(\Omega)}$ error	rate	$\|\sigma_1 - \sigma_{1,h}\|_{L^2(\Omega)}$ error	rate	$\|\sigma_2 - \sigma_{2,h}\|_{L^2(\Omega)}$ error	rate
0.3082	5.4351e-02	–	5.5267e-02	–	5.6489e-02	–	3.6849e-02	–
0.1875	2.7398e-02	0.99	2.7816e-02	0.99	2.8086e-02	1.01	1.8283e-02	1.01
0.0938	1.3721e-02	1.00	1.3926e-02	1.00	1.4041e-02	1.00	9.1296e-03	1.00
0.0469	6.8628e-03	1.00	6.9657e-03	1.00	7.0187e-03	1.00	4.5619e-03	1.00
0.0234	3.4315e-03	1.00	3.4831e-03	1.00	3.5084e-03	1.00	2.2802e-03	1.00

Table 1: Convergence results with $\Delta t = h$, the Crank–Nicolson method, and (RT_0, DG_0).

$h_{\max}$	$\|u_1 - u_{1,h}\|_{L^2(\Omega)}$ error	rate	$\|u_2 - u_{2,h}\|_{L^2(\Omega)}$ error	rate	$\|\sigma_1 - \sigma_{1,h}\|_{L^2(\Omega)}$ error	rate	$\|\sigma_2 - \sigma_{2,h}\|_{L^2(\Omega)}$ error	rate
0.3082	1.7406e-03	–	1.6257e-03	–	1.5422e-03	–	1.5721e-03	–
0.1875	4.3491e-04	2.00	4.0582e-04	2.00	3.8080e-04	2.02	3.8980e-04	2.01
0.0938	1.0904e-04	2.00	1.0181e-04	2.00	9.5514e-05	2.00	9.7276e-05	2.00
0.0469	2.7314e-05	2.00	2.5486e-05	2.00	2.3894e-05	2.00	2.4287e-05	2.00
0.0234	6.8363e-06	2.00	6.3889e-06	2.00	5.9716e-06	2.00	6.0672e-06	2.00

Table 2: Convergence results with $\Delta t = h$, the Crank–Nicolson method, and (RT_1, DG_1).

6. Conclusion

In this paper we develop mixed finite element methods for nonlinear reaction-diffusion equations with Robin-type interface conditions on membrane structures in the domain. We proved well-posedness of fully discrete scheme with the Crank–Nicolson method and the a priori error estimates of solutions with a sufficiently small time-step size assumption. In some numerical results, we observed that the errors of solutions converge as expected by our theoretical analysis. In our future research, we will study positivity-preserving numerical methods for the problems.

Statements and Declarations

Funding Xinran Jin and Jeonghun J. Lee gratefully acknowledge support from the National Science Foundation (DMS-2110781).

References

[1] O. Kedem, A. Katchalsky, A Physical Interpretation of the Phenomenological Coefficients of Membrane Permeability , Journal of General Physiology 45 (1) (1961) 143–179. arXiv:https://rupress.org/jgp/article-pdf/45/1/143/1242549/143.pdf, doi:10.1085/jgp.45.1.143. URL https://doi.org/10.1085/jgp.45.1.143

[2] F. Kleinhans, Membrane permeability modeling: Kedem-katchalsky vs a two-parameter formalism, Cryobiology 37 (4) (1998) 271–289. doi:10.1006/cryo.1998.2135.
URL https://doi.org/10.1137/19M1292618

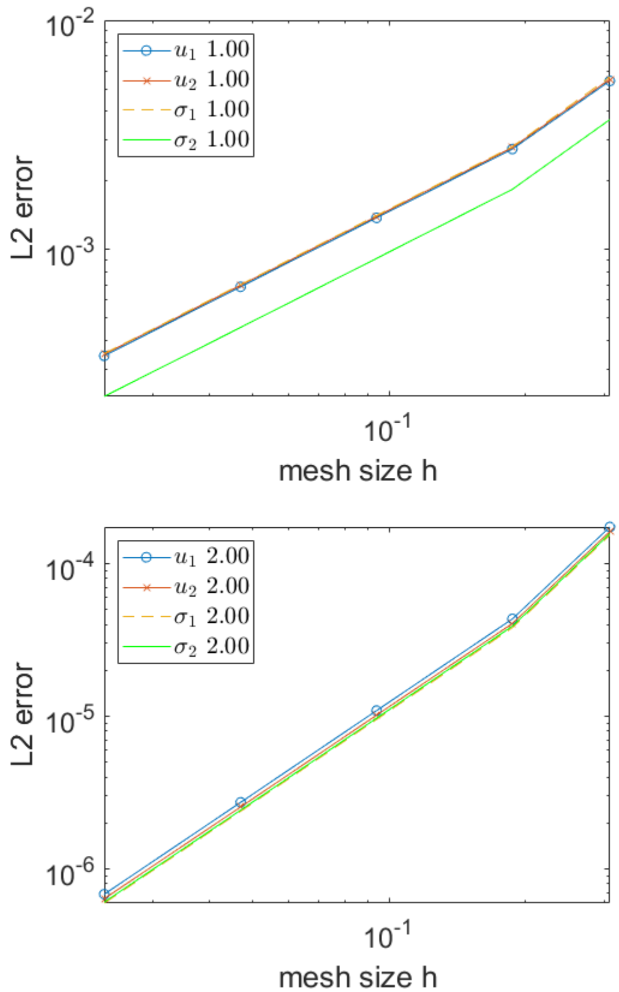

Figure 3: Graphs for asymptotic convergence rates of errors for Table 1 and Table 2

[3] A. Cangiani, E. H. Georgoulis, Y. A. Sabawi, Convergence of an adaptive discontinuous Galerkin method for elliptic interface problems, J. Comput. Appl. Math. 367 (2020) 112397, 15. `doi:10.1016/j.cam.2019.112397`. URL `https://doi.org/10.1016/j.cam.2019.112397`

[4] A. Cangiani, E. H. Georgoulis, M. Jensen, Discontinuous Galerkin methods for mass transfer through semipermeable membranes, SIAM J. Numer. Anal. 51 (5) (2013) 2911–2934. `doi:10.1137/120890429`. URL `https://doi.org/10.1137/120890429`

[5] A. Cangiani, E. H. Georgoulis, M. Jensen, Discontinuous Galerkin methods

for fast reactive mass transfer through semi-permeable membranes, Appl. Numer. Math. 104 (2016) 3–14. doi:10.1016/j.apnum.2014.06.007.
URL https://doi.org/10.1016/j.apnum.2014.06.007

[6] Z. Cai, X. Ye, S. Zhang, Discontinuous Galerkin finite element methods for interface problems: a priori and a posteriori error estimations, SIAM J. Numer. Anal. 49 (5) (2011) 1761–1787. doi:10.1137/100805133.
URL https://doi.org/10.1137/100805133

[7] G. Ciavolella, B. Perthame, Existence of a global weak solution for a reaction-diffusion problem with membrane conditions, J. Evol. Equ. 21 (2) (2021) 1513–1540. doi:10.1007/s00028-020-00633-7.
URL https://doi.org/10.1007/s00028-020-00633-7

[8] J. Li, L. Su, X. Wang, Y. Wang, Bulk-surface coupling: derivation of two models, J. Differential Equations 289 (2021) 1–34. doi:10.1016/j.jde.2021.04.011.
URL https://doi.org/10.1016/j.jde.2021.04.011

[9] M. Bathory, M. Bulíček, O. Souček, Existence and qualitative theory for nonlinear elliptic systems with a nonlinear interface condition used in electrochemistry, Z. Angew. Math. Phys. 71 (3) (2020) Paper No. 74, 24. doi:10.1007/s00033-020-01293-w.
URL https://doi.org/10.1007/s00033-020-01293-w

[10] D. Bothe, M. Pierre, Quasi-steady-state approximation for a reaction-diffusion system with fast intermediate, J. Math. Anal. Appl. 368 (1) (2010) 120–132. doi:10.1016/j.jmaa.2010.02.044.
URL https://doi.org/10.1016/j.jmaa.2010.02.044

[11] F. Calabrò, P. Zunino, Analysis of parabolic problems on partitioned domains with nonlinear conditions at the interface. Application to mass transfer through semi-permeable membranes, Math. Models Methods Appl. Sci. 16 (4) (2006) 479–501. doi:10.1142/S0218202506001236.
URL https://doi.org/10.1142/S0218202506001236

[12] A. Quarteroni, A. Veneziani, P. Zunino, Mathematical and numerical modeling of solute dynamics in blood flow and arterial walls, SIAM J. Numer. Anal. 39 (5) (2001/02) 1488–1511. doi:10.1137/S0036142900369714.
URL https://doi.org/10.1137/S0036142900369714

[13] M. Brera, J. W. Jerome, Y. Mori, R. Sacco, A conservative and monotone mixed-hybridized finite element approximation of transport problems in heterogeneous domains, Comput. Methods Appl. Mech. Engrg. 199 (41-44) (2010) 2709–2770. doi:10.1016/j.cma.2010.05.016.
URL https://doi.org/10.1016/j.cma.2010.05.016

[14] M. A. J. Chaplain, C. Giverso, T. Lorenzi, L. Preziosi, Derivation and application of effective interface conditions for continuum mechanical models of cell invasion through thin membranes, SIAM J. Appl. Math. 79 (5)

(2019) 2011–2031. doi:10.1137/19M124263X.
URL https://doi.org/10.1137/19M124263X

[15] Z. Chen, J. Zou, Finite element methods and their convergence for elliptic and parabolic interface problems, Numer. Math. 79 (2) (1998) 175–202. doi:10.1007/s002110050336.
URL https://doi.org/10.1007/s002110050336

[16] J. Li, J. M. Melenk, B. Wohlmuth, J. Zou, Optimal a priori estimates for higher order finite elements for elliptic interface problems, Appl. Numer. Math. 60 (1-2) (2010) 19–37. doi:10.1016/j.apnum.2009.08.005.
URL https://doi.org/10.1016/j.apnum.2009.08.005

[17] M. Plum, C. Wieners, Optimal a priori estimates for interface problems, Numer. Math. 95 (4) (2003) 735–759. doi:10.1007/s002110200395.
URL https://doi.org/10.1007/s002110200395

[18] D. Boffi, F. Brezzi, M. Fortin, Mixed finite element methods and applications, Vol. 44 of Springer Series in Computational Mathematics, Springer, Heidelberg, 2013. doi:10.1007/978-3-642-36519-5.
URL https://doi.org/10.1007/978-3-642-36519-5

[19] A. Budiša, W. M. Boon, X. Hu, Mixed-dimensional auxiliary space preconditioners, SIAM J. Sci. Comput. 42 (5) (2020) A3367–A3396. doi:10.1137/19M1292618.
URL https://doi.org/10.1137/19M1292618

[20] L. C. Evans, Partial differential equations, Vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 1998.

[21] K. Yosida, Functional Analysis, 6th Edition, Springer Classics in Mathematics, Springer-Verlag, 1980.

[22] J.-C. Nédélec, Mixed finite elements in $\mathbf{R}^3$, Numer. Math. 35 (3) (1980) 315–341. doi:10.1007/BF01396415.
URL http://dx.doi.org/10.1007/BF01396415

[23] P.-A. Raviart, J. M. Thomas, A mixed finite element method for 2nd order elliptic problems, in: Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Springer, Berlin, 1977, pp. 292–315. Lecture Notes in Math., Vol. 606.

[24] J. G. Heywood, R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization, SIAM J. Numer. Anal. 19 (2) (1982) 275–311. doi:10.1137/0719018.
URL https://doi.org/10.1137/0719018

[25] B. Rivière, Discontinuous Galerkin methods for solving elliptic and parabolic equations, Vol. 35 of Frontiers in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2008.

[26] A. Logg, K.-A. Mardal, G. N. Wells (Eds.), Automated solution of differential equations by the finite element method, Vol. 84 of Lecture Notes in Computational Science and Engineering, Springer, Heidelberg, 2012, the FEniCS book. doi:10.1007/978-3-642-23099-8.
URL http://dx.doi.org/10.1007/978-3-642-23099-8