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A HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR THE COUPLED NAVIER–STOKES/BIOT PROBLEM

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Abstract. In this paper we present a hybridizable discontinuous Galerkin method for the time-dependent Navier–Stokes equations coupled to the quasi-static poroelasticity equations via interface conditions. We determine a bound on the data that guarantees stability and well-posedness of the fully discrete problem and prove a priori error estimates. A numerical example confirms our analysis.

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1. Introduction

In this paper we consider a system of partial differential equations such that the governing equations of two different physical models on two disjoint subdomains are coupled across an interface. The two models are the time-dependent Navier–Stokes equations of incompressible fluids and the quasi-static poroelasticity (or Biot) equations [6–8]. The interface conditions coupling the two governing equations are derived by fundamental physical laws and experimental data. This fluid and poroelastic structure interaction problem, which we refer to here as the coupled Navier–Stokes/Biot problem, has applications in engineering fields such as hydrogeology, petroleum engineering, and biomechanics.

To the best of our knowledge, the coupled Stokes/Biot model with general interface conditions was first proposed in [50]. Soon after, Badia et al. [3] studied conforming finite element methods for the spatial discretization of the coupled Navier–Stokes/Biot problem and monolithic and domain decomposition (partitioned) algorithms to solve the fully discrete problem. A mathematical proof of existence and uniqueness of weak solutions to the fully dynamic coupled Navier–Stokes and Biot problem, under a small data assumption, was given in [17]. In this paper we consider the time-dependent Navier–Stokes equations coupled to the quasi-static Biot equations. Well-posedness of this model is still an open question.

Various finite element methods have been studied for the coupled Stokes/Biot and Navier–Stokes/Biot problems. A Lagrange multiplier method for the coupled stationary Stokes and quasi-static poroelasticity equations

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was studied in [2], which was extended to a nonlinear model with non-Newtonian fluids in [1]. A conforming/mixed finite element method was studied for the coupled stationary Stokes and quasi-static poroelasticity equations in [9, 45] using the total pressure formulation [34, 38]. Other formulations of this system of equations have also been studied. These include the velocity-pressure (for Stokes) and stress-displacement-velocity-pressure (for poroelasticity) formulation [37] and the stress-velocity-pressure (for Stokes) and stress-displacement-velocity-pressure (for poroelasticity) formulation [16]. A conforming finite element method using Nitsche's technique for the Stokes/Biot problem is studied in [29]. They consider the velocity-pressure formulation of the Stokes equations and a formulation using displacement, total pressure, and fluid content as primary variables for the poroelasticity equations. For the coupled stationary Navier–Stokes and quasi-static poroelasticity equations, an augmented mixed method using a pseudo-stress formulation of the Navier–Stokes equations and a stress-displacement-velocity-pressure formulation of the poroelasticity model is studied in [36]. They prove existence and uniqueness of a continuous weak and a semidiscrete continuous-in-time formulation of these equations. A conforming finite element method with stabilization for the time-dependent Stokes equations coupled to the dynamic poroelasticity equations was studied in [18]. Furthermore, many partitioned time discretization schemes for efficient time discretization of the (Navier–)Stokes/Biot model have been studied, see, for example, [5, 14, 15, 30, 39].

In our previous work [21], we presented a locking-free hybridizable discontinuous Galerkin (HDG) [26] method for the coupled stationary Stokes equations and quasi-static poroelasticity equations. This HDG method was constructed such that: (i) the discrete velocities and displacement are divergence-conforming; (ii) the compressibility equations are satisfied pointwise on the elements; and (iii) mass is conserved pointwise on the elements for the semi-discrete problem in the absence of source/sink terms. In this paper we expand on our work in [21] and propose and analyze an HDG method for the coupled time-dependent Navier–Stokes and quasi-static Biot equations that inherits the three aforementioned properties of the HDG method for the coupled Stokes/Biot problem. For this, we couple the exactly divergence-free HDG method for the time-dependent Navier–Stokes equations of [41] to the locking-free HDG method for the Biot equations of [20].

We consider Backward Euler time-stepping for the time discretization, lagging the convective velocity in the nonlinear term of the Navier–Stokes equations. To prove stability and well-posedness of our discretization, the convective velocity across the interface must be small enough (this observation was also made in [28] for the stationary Navier–Stokes/Darcy problem). We show this by assuming the data is small and extending the stability and well-posedness analysis of [25] for a discontinuous Galerkin discretization of the Navier–Stokes/Darcy problem to our HDG discretization of the Navier–Stokes/Biot problem. With well-posedness established we proceed with an a priori error analysis. Here it is interesting to remark that all error bounds are independent of the fluid pressure analogous to pressure-robust estimates found elsewhere in the literature for divergence-conforming discretizations of incompressible flows (see, for example, [27, 32, 35, 43, 49, 51] and the review paper [31]).

The remainder of this paper is organized as follows. In section 2, we present the coupled Navier–Stokes/Biot problem. Notation, definitions, useful preliminary results, and the HDG method are introduced in section 3. The HDG method is shown to be stable and well-posed in section 4 and a priori error estimates are proven in section 5. A numerical result is presented in section 6 and we end this paper with concluding remarks in section 7.

2. The coupled Navier–Stokes/Biot problem

Let Ω be a bounded connected open subset of $\mathbb{R}^d$, $d = 2, 3$, with polygonal/polyhedral boundaries, and let Ω^f and Ω^b be two disjoint open connected subsets, both with polygonal/polyhedral boundaries, such that $\overline{\Omega} = \overline{\Omega^f} \cup \overline{\Omega^b}$. Furthermore, let $J = [0, T]$ denote the time interval of interest.

Let $\sigma^j := 2\mu^j \varepsilon(u^j) - p^j \mathbb{I}$ ($j = f, b$) and $\varepsilon(u) := \left(\nabla u + (\nabla u)^T\right)/2$. The Navier–Stokes equations in $\Omega^f \times J$ are given by

$$\partial_t u^f + \nabla \cdot (u^f \otimes u^f) - \nabla \cdot \sigma^f = f^f, \qquad \nabla \cdot u^f = 0, \tag{1}$$

where u^f is the velocity in Ω^f and p^f is the pressure in Ω^f. Furthermore, $\mu^f > 0$ is the fluid viscosity and f^f is a given body force term. Biot's equations in $\Omega^b \times J$ are given by

$$\begin{aligned} -\nabla \cdot u^b + \lambda^{-1}(\alpha p^p - p^b) = 0, \quad -\nabla \cdot \sigma^b = f^b, \quad \mu^f \kappa^{-1} z + \nabla p^p = 0, \\ c_0 \partial_t p^p + \alpha\lambda^{-1}(\alpha \partial_t p^p - \partial_t p^b) + \nabla \cdot z = g^b, \end{aligned} \tag{2}$$

where u^b is the displacement, p^b, which is defined as

$$p^b := \alpha p^p - \lambda \nabla \cdot u^b, \tag{3}$$

is the total pressure, p^p is the pore pressure, and z is the Darcy velocity. Furthermore, μ^b and λ are the Lamé constants, $\kappa > 0$ is the permeability constant, $\alpha \in (0, 1)$ is the Biot–Willis constant, and $c_0 \geq 0$ is the specific storage coefficient. The body force in Ω^b is denoted by f^b while g^b is a source/sink term.

Various equations are prescribed on the interface $\Gamma_I = \overline{\partial\Omega}^f \cap \overline{\partial\Omega}^b$ that couple the Navier–Stokes and Biot equations. First, mass conservation across the interface is prescribed by

$$u^f \cdot n = (\partial_t u^b + z) \cdot n \text{ on } \Gamma_I \times J. \tag{4a}$$

Here we use the convention that n^j, $j = f, b$ is the unit outward normal to Ω^j and that on Γ_I, $n = n^f = -n^b$. Next, the balance of stresses is prescribed by

$$\sigma^f n = \sigma^b n, \qquad -(\sigma^f n) \cdot n = p^p \text{ on } \Gamma_I \times J. \tag{4b}$$

The Beavers–Joseph–Saffmann condition [4, 46] prescribes slip with friction and is given by

$$-2\mu^f \left(\varepsilon(u^f)n\right)^t = \gamma \mu^f \kappa^{-1/2} (u^f - \partial_t u^b)^t \text{ on } \Gamma_I \times J, \tag{4c}$$

where $\gamma > 0$ is an experimentally determined dimensionless constant and where $(w)^t := w - (w \cdot n)n$.

The boundary of the domain is partitioned as follows. On each subdomain we define $\Gamma^j = \partial\Omega \cap \partial\Omega^j$, $j = f, b$. We then partition both Γ^f and Γ^b into Dirichlet Γ_D^j and Neumann Γ_N^j parts such that $\Gamma^j = \Gamma_D^j \cup \Gamma_N^j$. Note that $\Gamma_D^j \cap \Gamma_N^j = \emptyset$ and we will assume that $|\Gamma_D^j|, |\Gamma_N^j| > 0$. It will also be useful to define $\Gamma_{IN}^f := \Gamma_I \cup \Gamma_N^f$. A second partitioning of Γ^b is defined as $\Gamma^b = \Gamma_P^b \cup \Gamma_F^b$ with $\Gamma_P^b \cap \Gamma_F^b = \emptyset$ and $|\Gamma_P^b| > 0$. See fig. 1 for an example domain configuration depicting the notation. We now impose the following boundary conditions:

$$\begin{aligned} u^j &= 0 \quad \text{on } \Gamma_D^j \times J, \quad j = f, b, \quad & \sigma^j n &= 0 \quad \text{on } \Gamma_N^j \times J, \quad j = f, b, \\ p^p &= 0 \quad \text{on } \Gamma_P^b \times J, & z \cdot n &= 0 \quad \text{on } \Gamma_F^b \times J. \end{aligned} \tag{5}$$

Initial conditions are given by

$$u^f(x, 0) = u_0^f(x) \text{ in } \Omega^f, \qquad p^p(x, 0) = p_0^p(x) \text{ in } \Omega^b. \tag{6}$$

Initial conditions for u^b, z, and p^b are determined by u_0^f and p_0^p assuming that $u_0^f \in \left[H^s(\Omega^f)\right]^d$, $s > 3/2$ and $p_0^p \in H^1(\Omega^b)$. Indeed, first note that $z_0(x) = -(\kappa/\mu^f)\nabla p_0^p(x)$ by the third equation in eq. (2). A weak formulation of $-\nabla \cdot \sigma_0^b = f^b(x, 0)$ in eq. (2), with $\sigma_0^b = 2\mu^b \varepsilon(u_0^b) + \lambda \nabla \cdot u_0^b \mathbb{I} - \alpha p_0^p \mathbb{I}$, gives

$$\int_{\Omega^b} (2\mu^b \varepsilon(u_0^b) + \lambda \nabla \cdot u_0^b \mathbb{I}) : \varepsilon(v^b) \,\mathrm{d}x = \int_{\Omega^b} (\alpha p_0^p \nabla \cdot v^b + f^b(x, 0) \cdot v^b) \,\mathrm{d}x + \int_{\Gamma_I} \sigma_0^f n \cdot v^b \,\mathrm{d}s \quad \forall v^b \in \left[H_D^1(\Omega^b)\right]^d, \tag{7}$$

where $\left[H_D^1(\Omega^b)\right]^d := \{v^b \in \left[H^1(\Omega^b)\right]^d : v^b|_{\Gamma_D^b} = 0\}$ and the interface integral is by the first equality in eq. (4b) and the second boundary condition in eq. (5), i.e., $\sigma_0^b n = 0$ on Γ_N^b. Decomposing $\sigma_0^f n$ into its normal and

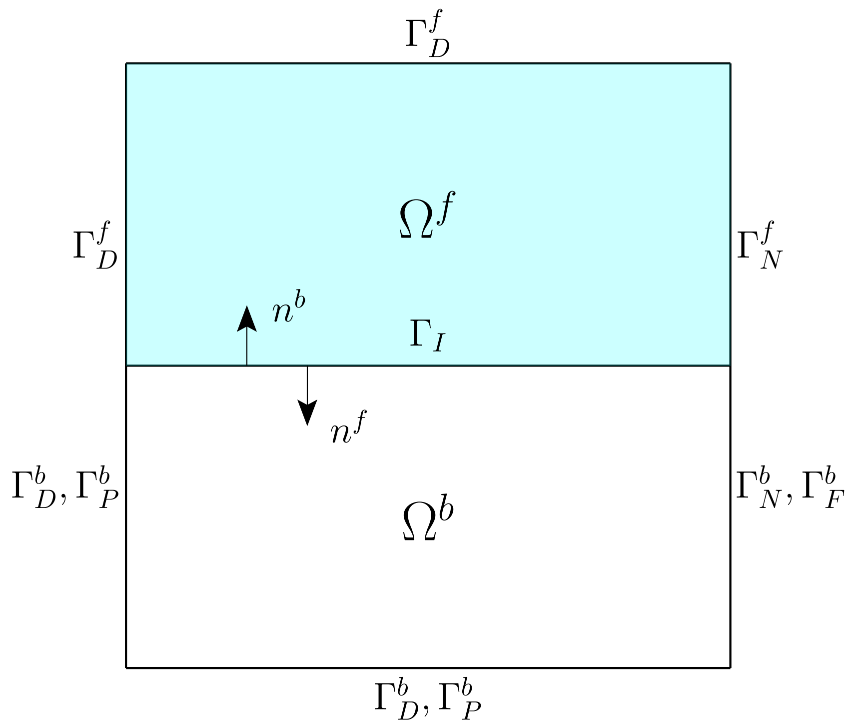

FIGURE 1. An example domain configuration depicting the domain and boundary notation. Note that $\Gamma_{IN} = \Gamma_I \cup \Gamma_N^f$, $\Gamma^f = \Gamma_D^f \cup \Gamma_N^f$, and $\Gamma^b = \Gamma_D^b \cup \Gamma_N^b = \Gamma_P^b \cup \Gamma_F^b$.

tangential components, using the second equality in eq. (4b), and the orthogonality of normal and tangential vectors, eq. (7) can be rewritten as

$$\int_{\Omega^b} (2\mu^b \varepsilon(u_0^b) + \lambda \nabla \cdot u_0^b \mathbb{I}) : \varepsilon(v^b) \, \mathrm{d}x$$
$$= \int_{\Omega^b} (\alpha p_0^p \nabla \cdot v^b + f^b(x,0) \cdot v^b) \, \mathrm{d}x + \int_{\Gamma_I} (-p_0^p v^b \cdot n + (2\mu^f \varepsilon(u_0^f) n)^t \cdot (v^b)^t) \, \mathrm{d}s \quad \forall v^b \in \left[H_D^1(\Omega^b)\right]^d. \quad (8)$$

The integral on Γ_I is well-defined so that, by Korn's inequality and the Lax–Milgram lemma, eq. (8) has a unique solution $u_0^b \in \left[H_D^1(\Omega^b)\right]^d$. Finally, using eq. (3), $p_0^b(x) = \alpha p_0^p(x) - \lambda \nabla \cdot u_0^b(x)$.

In what follows, we write $u|_{\Omega^j} = u^j$, $p|_{\Omega^j} = p^j$, $f|_{\Omega^j} = f^j$, and $\mu|_{\Omega^j} = \mu^j$ for $j = f, b$ so that u, p, f, and μ are defined on the whole domain Ω.

3. Notation, the HDG method, and preliminary results

3.1. Mesh and time partitioning

We discretize the domains Ω^j, $j = f, b$, by shape-regular triangulations which we denote by $\mathcal{T}^j$. We will assume that the triangulations consist of simplices, denoted by K, that match at the interface and that the triangulations are free of hanging nodes. The set of all simplices is denoted by $\mathcal{T} := \mathcal{T}^f \cup \mathcal{T}^b$. The boundary of an element K is denoted by ∂K and we define $\partial \mathcal{T}^j := \{\partial K \, : \, K \in \mathcal{T}^j\}$ and $\partial \mathcal{T} := \{\partial K \, : \, K \in \mathcal{T}\}$. We also consider various sets of facets. The sets of all facets in $\overline{\Omega}^j$ and Ω^j are denoted by $\mathcal{F}^j$ and $\mathcal{F}_{int}^j$, respectively.

The sets of facets on the Dirichlet Γ_D^j and Neumann Γ_N^j boundaries are denoted by, respectively, $\mathcal{F}_D^j$ and $\mathcal{F}_N^j$, while on Γ_F^b and Γ_P^b we denote the sets of facets by, respectively, $\mathcal{F}_F^b$ and $\mathcal{F}_P^b$. The set of facets on the interface is denoted by $\mathcal{F}_I$ and the set of all facets is denoted by $\mathcal{F}$. The union of facets in $\mathcal{F}^j$ is denoted by Γ_0^j. Furthermore, we denote by $\mathcal{F}(K)$ the set of all facets of K. The diameter of an element K is denoted by h_K and we define $h := \max_{K\in\mathcal{T}} h_K$. On the boundary of an element we denote by n_K the unit outward normal vector, however, we will drop the subscript K where no confusion can occur.

The time interval J is partitioned as follows: $0 = t_0 < t_1 < \cdots < t_N = T$. For simplicity, we assume a fixed time step, i.e., $\Delta t = T/N = t_n - t_{n-1}$ for $n = 1, \dots, N$. A function f evaluated at time level n will be denoted by $f^n := f(t_n)$. We further introduce the difference $\delta f^{n+1} := f^{n+1} - f^n$, the first order time derivative $d_t f^{n+1} := (f^{n+1} - f^n)/\Delta t$ for $n = 0, \cdots, N-1$, and the second order time derivative $d_{tt} f^{b,n+1} = (f^{b,n+1} - 2f^{b,n} + f^{b,n-1})/(\Delta t)^2$ for $n = 1, \cdots, N-1$.

3.2. **Function spaces and norms**

Various function spaces will be used throughout this paper. First, the usual Sobolev spaces are denoted by $W^{k,p}(D)$ for $k \geq 0$ and $1 \leq p \leq \infty$ on a Lipschitz domain $D \subset \mathbb{R}^d$. The norm on $W^{k,p}(D)$ is denoted by $\|\cdot\|_{p,k,D}$. As usual, $H^k(D) = W^{k,2}(D)$ with norm $\|\cdot\|_{k,D} = \|\cdot\|_{2,k,D}$ and $L^p(D) = W^{0,p}(D)$ with norm $\|\cdot\|_{p,0,D}$. If $k = 0$ and $p = 2$ we define $L^2(D) = W^{0,2}(D)$ with norm $\|\cdot\|_D = \|\cdot\|_{2,0,D}$. The $L^p(S)$ norm on a surface $S \subset \mathbb{R}^{d-1}$ is defined similarly.

Let X be a Banach space with norm $\|\cdot\|_X$, then $W^{k,p}(J;X)$ denotes a Bochner space with norm $\|f\|^p_{W^{k,p}(J;X)} := \int_0^T \sum_{i=0}^k \|\partial_t^i f(t)\|_X^p \,\mathrm{d}t$ for $1 \leq p < \infty$. If $k = 0$, then $L^p(J;X) = W^{0,p}(J;X)$. The norm on $L^\infty(J;X)$, i.e., the Bochner space for $k = 0$ and $p = \infty$, is defined as $\|f\|_{L^\infty(J;X)} := \operatorname{ess\,sup}_{t\in J} \|f(t)\|_X$. Furthermore, we denote by $\ell^p(J;X)$ the space equipped with the norm $\|f\|_{\ell^\infty(J;X)} := \max_{1\leq i\leq N} \|f^i\|_X$ for $p = \infty$ and $\|f\|^p_{\ell^p(J;X)} := \Delta t \sum_{i=1}^N \|f^i\|_X^p$ for $1 \leq p < \infty$.

Let us now define the following function spaces (for $j = f, b$):

$$V^j := \{v \in [H^2(\Omega^j)]^d \;:\; v|_{\Gamma_D^j} = 0\}, \quad Q^j := H^1(\Omega^j), \quad Q^{b0} := \{q \in H^2(\Omega^b) \;:\; q|_{\Gamma_P^b} = 0\},$$

and denote by $\bar{V}^j$ the trace space of V^j restricted to Γ_0^j, $\bar{Q}^j$ is the trace space of Q^j restricted to Γ_0^j, and $\bar{Q}^{b0}$ is the trace space of Q^{b0} restricted to Γ_0^b. For a compact notation, we define $\boldsymbol{V}^j := V^j \times \bar{V}^j$, $\boldsymbol{Q}^j := Q^j \times \bar{Q}^j$, and $\boldsymbol{Q}^{b0} := Q^{b0} \times \bar{Q}^{b0}$. Furthermore, we define

$$Z := \{v \in [H^1(\Omega^b)]^d \;:\; v \cdot n|_{\Gamma_F^b} = 0\}.$$

To define the HDG method we require the following element and facet function space pairs on each domain Ω^j, $j = f, b$:

$$\begin{aligned} V_h^j &:= \{v_h \in [L^2(\Omega^j)]^d : \; v_h \in [P_k(K)]^d, \; \forall\, K \in \mathcal{T}^j\}, \\ \bar{V}_h^j &:= \{\bar{v}_h \in [L^2(\Gamma_0^j)]^d : \; \bar{v}_h \in [P_k(F)]^d \;\; \forall\, F \in \mathcal{F}^j, \; \bar{v}_h = 0 \text{ on } \Gamma_D^j\}, \end{aligned}$$

and

$$Q_h^j := \{q_h \in L^2(\Omega^j) : \; q_h \in P_{k-1}(K), \; \forall\, K \in \mathcal{T}^j\}, \quad \bar{Q}_h^j := \{\bar{q}_h \in L^2(\Gamma_0^j) : \; \bar{q}_h \in P_k(F) \;\forall\, F \in \mathcal{F}^j\},$$

where $P_r(D)$ denotes the set of polynomials of total degree at most $r \geq 0$ defined on D. We will furthermore require:

$$V_h := \{v_h \in [L^2(\Omega)]^d : \; v_h \in [P_k(K)]^d \,, \; \forall K \in \mathcal{T}\} \,, \qquad Q_h := \{q_h \in L^2(\Omega) : \; q_h \in P_{k-1}(K), \; \forall \; K \in \mathcal{T}\} \,,$$
$$\bar{Q}_h^{b0} := \{\bar{q}_h \in \bar{Q}_h^b : \bar{q}_h = 0 \text{ on } \Gamma_P^b\} \,.$$

For $(u_h, p_h) \in V_h \times Q_h$, we will write $u_h|_{\Omega^j} = u_h^j \in V_h^j$ and $p_h|_{\Omega^j} = p_h^j \in Q_h^j$ for $j = f, b$. We group element and facet unknowns together as follows:

$$\begin{aligned}
\boldsymbol{v}_h &= (v_h, \bar{v}_h^f, \bar{v}_h^b) \in \boldsymbol{V}_h := V_h \times \bar{V}_h^f \times \bar{V}_h^b, & \boldsymbol{v}_h^j &= (v_h^j, \bar{v}_h^j) \in \boldsymbol{V}_h^j := V_h^j \times \bar{V}_h^j, \\
\boldsymbol{q}_h &= (q_h, \bar{q}_h^f, \bar{q}_h^b) \in \boldsymbol{Q}_h := Q_h \times \bar{Q}_h^f \times \bar{Q}_h^b, & \boldsymbol{q}_h^j &= (q_h^j, \bar{q}_h^j) \in \boldsymbol{Q}_h^j := Q_h^j \times \bar{Q}_h^j, \\
\boldsymbol{q}_h^p &= (q_h^p, \bar{q}_h^p) \in \boldsymbol{Q}_h^{b0} := Q_h^b \times \bar{Q}_h^{b0},
\end{aligned}$$

where $j = f, b$, and $(\boldsymbol{v}_h, \boldsymbol{q}_h, w_h, \boldsymbol{q}_h^p) \in \boldsymbol{X}_h := \boldsymbol{V}_h \times \boldsymbol{Q}_h \times V_h^b \times \boldsymbol{Q}_h^{b0}$. We will also require the following two subspaces of $\boldsymbol{V}_h^j$ and $\boldsymbol{V}_h$, respectively:

$$\widetilde{\boldsymbol{V}}_h^j := \{\boldsymbol{v}_h \in \boldsymbol{V}_h^j \;:\; \bar{v}_h|_{\Gamma_I} = 0\} \,, \qquad \widehat{\boldsymbol{V}}_h := \{\boldsymbol{v}_h \in \boldsymbol{V}_h \;:\; \bar{v}_h^f \cdot n = \bar{v}_h^b \cdot n \text{ on } \Gamma_I\} \,. \tag{9}$$

Extended function spaces are defined as (for $j = f, b$):

$$\begin{aligned}
&V^j(h) := V_h^j + V^j, && \boldsymbol{V}^j(h) := \boldsymbol{V}_h^j + \boldsymbol{V}^j, && Z(h) := V_h^b + Z, \\
&\boldsymbol{Q}^j(h) := \boldsymbol{Q}_h^j + \boldsymbol{Q}^j, && \boldsymbol{Q}^{b0}(h) := \boldsymbol{Q}_h^{b0} + \boldsymbol{Q}^{b0},
\end{aligned}$$

and

$$V^{f,\mathrm{div}}(h) := \{v \in V^f(h) \cap H(\mathrm{div}; \Omega^f) \;:\; \nabla \cdot v = 0 \text{ for } x \in K, \; \forall K \in \mathcal{T}^f\} \,.$$

We will work with the following norms:

$$\begin{aligned}
|||\boldsymbol{v}^j|||_{v,j}^2 &:= \sum_{K \in \mathcal{T}^j} \left(\|\varepsilon(v^j)\|_K^2 + h_K^{-1} \|v^j - \bar{v}^j\|_{\partial K}^2\right) && \forall \boldsymbol{v}^j \in \boldsymbol{V}^j(h), && j = f, b, \\
|||\boldsymbol{v}^j|||_{v',j}^2 &:= |||\boldsymbol{v}^j|||_{v,j}^2 + \sum_{K \in \mathcal{T}^j} h_K^2 |v^j|_{2,K}^2 && \forall \boldsymbol{v}^j \in \boldsymbol{V}^j(h), && j = f, b, \\
|||\boldsymbol{q}|||_{q,j}^2 &:= \|q\|_{\Omega^j}^2 + \sum_{K \in \mathcal{T}^j} h_K \|\bar{q}^j\|_{\partial K}^2 && \forall \boldsymbol{q} \in \boldsymbol{Q}^j(h), && j = f, b, \\
|||\boldsymbol{v}_h|||_v^2 &:= |||\boldsymbol{v}_h^f|||_{v,f}^2 + |||\boldsymbol{v}_h^b|||_{v,b}^2 + \|(\bar{v}_h^f - \bar{v}_h^b)^t\|_{\Gamma_I}^2 && \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\
\|v_h\|_{1,h,\Omega^j} &:= |||(v_h, \{\!\{v_h\}\!\})|||_{v,j} && \forall v_h \in V_h^j, \\
|||\boldsymbol{q}_h|||_q^2 &:= |||\boldsymbol{q}_h^f|||_{q,f}^2 + |||\boldsymbol{q}_h^b|||_{q,b}^2 && \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h, \\
\|q_h\|_{1,h,\Omega^b}^2 &:= \sum_{K \in \mathcal{T}^b} \|\nabla q_h\|_K^2 + \sum_{F \in \mathcal{F}_{int}^b \cup \mathcal{F}_P^b} h_F^{-1} \|[\![q_h]\!]\|_F^2 && \forall q_h \in Q_h^b, \\
|||\boldsymbol{q}_h|||_{1,h,b}^2 &:= \sum_{K \in \mathcal{T}^b} \left(\|\nabla q_h\|_K^2 + h_K^{-1} \|q_h - \bar{q}_h\|_{\partial K}^2\right) && \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h^{b0},
\end{aligned}$$

where the average operator $\{\!\{\cdot\}\!\}$ across an interior facet $F = \partial K^+ \cap \partial K^- \in \mathcal{F}_{int}^j$ is defined as $\{\!\{v_h\}\!\} := (v_h|_{K^+} + v_h|_{K^-})$. On $F \in \mathcal{F}_N^j \cup \mathcal{F}_I$ we define $\{\!\{v_h\}\!\} := v_h$ and on $F \in \mathcal{F}_D^j$ we define $\{\!\{v_h\}\!\} := 0$. Furthermore, the jump operator $[\![\cdot]\!]$ is defined as $[\![v]\!]_h := v_h|_{K^+} - v_h|_{K^-}$ across an interior facet and as $[\![v]\!]_h := v_h|_K$ on a boundary facet.

It is useful to remark that the discrete bilinear form of the diffusion term in the Navier–Stokes equation is continuous on $\boldsymbol{V}^j(h)$ in the norm $|\!|\!|\cdot|\!|\!|_{v',j}$ (see eq. (11b)). On $\boldsymbol{V}_h^j$ the norm $|\!|\!|\cdot|\!|\!|_{v',j}$ is equivalent to $|\!|\!|\cdot|\!|\!|_{v,j}$, i.e., there exists a constant $c_e > 0$ independent of h such that $|\!|\!|\boldsymbol{v}|\!|\!|_{v,j} \le |\!|\!|\boldsymbol{v}|\!|\!|_{v',j} \le c_e|\!|\!|\boldsymbol{v}|\!|\!|_{v,j}$ for all $\boldsymbol{v} \in \boldsymbol{V}_h^j$ (see [52, eq. (5.5)]).

In appendix A we prove that there exist constants $c_p, c_{si,r} > 0$, independent of h, such that

$$\|v_h\|_{\Omega^f} \le c_p\|v_h\|_{1,h,\Omega^f} \le c_p|\!|\!|\boldsymbol{v}_h|\!|\!|_{v,f} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^f, \tag{10a}$$

$$\|v_h\|_{\Omega^b} \le c_p\|v_h\|_{1,h,\Omega^b} \le c_p|\!|\!|\boldsymbol{v}_h|\!|\!|_{v,b} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^b, \tag{10b}$$

$$\|v_h^f\|_{r,0,\Gamma_{IN}^f} \le c_{si,r}\|v_h\|_{1,h,\Omega^f} \le c_{si,r}|\!|\!|\boldsymbol{v}_h|\!|\!|_{v,f} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^f, \tag{10c}$$

$$\|q_h\|_{\Omega^b} \le c_{pp}\,\|q_h\|_{1,h,\Omega^b} \le c_{pp}|\!|\!|\boldsymbol{q}_h|\!|\!|_{1,h,b} \qquad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h^{b0}, \tag{10d}$$

where, in eq. (10c), $\|v_h^f\|_{r,0,\Gamma_{IN}^f}$ is a trace norm with $1 \le r < \infty$ if $d = 2$ and $1 \le r \le 4$ if $d = 3$.

Consider two scalar functions w and z. We will denote by $(w, z)_D$ the integral of wz over a domain $D \subset \mathbb{R}^d$ and by $\langle w, z\rangle_D$ the integral of wz over a domain $D \subset \mathbb{R}^{d-1}$. We furthermore introduce the notation

$$(w,z)_{\Omega^j} := \sum_{K\in\mathcal{T}^j}(w,z)_K, \qquad (w,z)_{\Omega} := \sum_{K\in\mathcal{T}}(w,z)_K, \qquad \langle w,z\rangle_{\Gamma_I} := \sum_{F\in\mathcal{F}_I}\langle w,z\rangle_F,$$

$$\langle w,z\rangle_{\partial\mathcal{T}^j} := \sum_{K\in\mathcal{T}^j}\langle w,z\rangle_{\partial K}, \qquad \langle w,z\rangle_{\partial\mathcal{T}} := \sum_{K\in\mathcal{T}}\langle w,z\rangle_{\partial K}, \qquad \langle w,z\rangle_{\Gamma_{IN}^f} := \sum_{F\in\mathcal{F}_I\cup\mathcal{F}_N^f}\langle w,z\rangle_F.$$

If w and z are vector functions, then $(w, z)_D := \sum_{i=1}^d (w_i, z_i)_D$ and $\langle w, z\rangle_D := \sum_{i=1}^d \langle w_i, z_i\rangle_D$. Similarly, if w and z are matrix functions, then $(w, z)_D := \sum_{i,j=1}^d (w_{ij}, z_{ij})_D$ and $\langle w, z\rangle_D := \sum_{i,j=1}^d \langle w_{ij}, z_{ij}\rangle_D$.

3.3. **Forms and their properties**

For $\boldsymbol{u}^j, \boldsymbol{v}^j \in \boldsymbol{V}^j(h)$, we define

$$\begin{aligned} a_h^j(\boldsymbol{u}^j,\boldsymbol{v}^j) :=& (2\mu^j\varepsilon(u),\varepsilon(v))_{\Omega^j} + \langle 2\beta^j\mu^j h_K^{-1}(u-\bar{u}^j), v-\bar{v}^j\rangle_{\partial\mathcal{T}^j} \\ & - \langle 2\mu^j\varepsilon(u)n^j, v-\bar{v}^j\rangle_{\partial\mathcal{T}^j} - \langle 2\mu^j\varepsilon(v)n^j, u-\bar{u}^j\rangle_{\partial\mathcal{T}^j}, \\ a_h(\boldsymbol{u},\boldsymbol{v}) :=& a_h^f(\boldsymbol{u}^f,\boldsymbol{v}^f) + a_h^b(\boldsymbol{u}^b,\boldsymbol{v}^b), \end{aligned}$$

where $\beta^j > 0$ is a penalty parameter. It was shown in [24, Lemmas 2 and 3] and [40, Lemmas 4.2 and 4.3] that there exist constants $\beta_0 > 0$, $c_{ae}^j > 0$, and $c_{ab}^j > 0$, independent of h and Δt, such that

$$a_h^j(\boldsymbol{v}_h^j,\boldsymbol{v}_h^j) \ge c_{ae}^j\mu^j|\!|\!|\boldsymbol{v}_h^j|\!|\!|_{v,j}^2 \qquad \forall \boldsymbol{v}_h^j \in \boldsymbol{V}_h^j,\ \beta > \beta_0, \tag{11a}$$

$$|a_h^j(\boldsymbol{u}^j,\boldsymbol{v}^j)| \le c_{ac}^j\mu^j|\!|\!|\boldsymbol{u}^j|\!|\!|_{v',j}|\!|\!|\boldsymbol{v}^j|\!|\!|_{v',j} \qquad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}^j(h). \tag{11b}$$

For $\boldsymbol{v}^j \in \boldsymbol{V}^j(h)$ and $\boldsymbol{q}^j \in \boldsymbol{Q}^j(h)$ we define

$$b_h^j(\boldsymbol{v}^j,\boldsymbol{q}^j) := -(q,\nabla\cdot v)_{\Omega^j} + \langle \bar{q}^j, (v-\bar{v}^j)\cdot n^j\rangle_{\partial\mathcal{T}^j}, \qquad b_h(\boldsymbol{v},\boldsymbol{q}) := b_h^f(\boldsymbol{v}^f,\boldsymbol{q}^f) + b_h^b(\boldsymbol{v}^b,\boldsymbol{q}^b).$$

The form $b_h^b(\cdot,\cdot)$ is also defined on $(Z(h)\times\{0\})\times\boldsymbol{Q}^{b0}(h)$. We have the following inf-sup conditions (see [21, eqs (17)-(19)] and [42]):

$$\inf_{\boldsymbol{0}\neq\boldsymbol{q}_h\in\boldsymbol{Q}_h^j}\ \sup_{\boldsymbol{0}\neq\boldsymbol{v}_h\in\widetilde{\boldsymbol{V}}_h^j}\frac{b_h^j(\boldsymbol{v}_h,\boldsymbol{q}_h)}{|||\boldsymbol{v}_h|||_{v,j}|||\boldsymbol{q}_h|||_{q,j}}\geq c_{bj},\quad j=f,b, \tag{12a}$$

$$\inf_{\boldsymbol{0}\neq\boldsymbol{q}_h\in\boldsymbol{Q}_h}\ \sup_{\boldsymbol{0}\neq\boldsymbol{v}_h\in\widehat{\boldsymbol{V}}_h}\frac{b_h(\boldsymbol{v}_h,\boldsymbol{q}_h)}{|||\boldsymbol{v}_h|||_{v}|||\boldsymbol{q}_h|||_{q}}\geq c_{b}, \tag{12b}$$

$$\inf_{\boldsymbol{0}\neq\boldsymbol{q}_h^p\in\boldsymbol{Q}_h^{b0}}\ \sup_{0\neq w_h\in V_h^b}\frac{b_h^b((w_h,0),\boldsymbol{q}_h^p)}{\|w_h\|_{\Omega^b}\,|||\boldsymbol{q}_h^p|||_{q,b}}\geq c_{bp}, \tag{12c}$$

where c_{bf}, c_{bb}, c_b, and c_{bp} are positive constants independent of h. On $(\boldsymbol{Q}^{b0}(h)\times\boldsymbol{Q}^b(h))\times\boldsymbol{Q}^b(h)$, we define

$$c_h((p,r),q):=(\lambda^{-1}(\alpha p-r),q)_{\Omega^b},$$

while on the interface we define

$$a_h^I((\bar{u}^f,\bar{u}^b),(\bar{v}^f,\bar{v}^b)):=\langle\gamma\mu^f\kappa^{-1/2}(\bar{u}^f-\bar{u}^b)^t,(\bar{v}^f-\bar{v}^b)^t\rangle_{\Gamma_I},\qquad b_h^I((\bar{v}^f,\bar{v}^b),\bar{q}):=\langle\bar{q},(\bar{v}^f-\bar{v}^b)\cdot n^f\rangle_{\Gamma_I},$$

where a_h^I is defined on $((\bar{V}^f+\bar{V}_h^f)\times(\bar{V}^b+\bar{V}_h^b))\times((\bar{V}^f+\bar{V}_h^f)\times(\bar{V}^b+\bar{V}_h^b))$ and b_h^I is defined on $((\bar{V}^f+\bar{V}_h^f)\times(\bar{V}^b+\bar{V}_h^b))\times(\bar{Q}^{b0}+\bar{Q}_h^{b0})$.

The forms $a_h(\cdot,\cdot)$, $a_h^I(\cdot,\cdot)$, $b_h(\cdot,\cdot)$, $b_h^I(\cdot,\cdot)$, and $c_h(\cdot,\cdot)$ discussed above are identical to those considered for the coupled Stokes and Biot problem in [21]. In addition to these forms, we now also require the following discrete convection term for $w\in V^{f,div}(h)$ and $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}^f(h)$:

$$t_h(w;\boldsymbol{u},\boldsymbol{v}):=-(u\otimes w,\nabla v)_{\Omega^f}+\tfrac{1}{2}\langle w\cdot n^f\,(u+\bar{u}),v-\bar{v}\rangle_{\partial\mathcal{T}^f}+\tfrac{1}{2}\langle|w\cdot n^f|\,(u-\bar{u}),v-\bar{v}\rangle_{\partial\mathcal{T}^f}+\langle(w\cdot n^f)\bar{u},\bar{v}\rangle_{\Gamma_{IN}^f}.$$

We have the following properties for t_h.

Proposition 3.1. *For all $w_1,w_2\in V^{f,div}(h)$, and $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{V}^f(h)$, there exists a $c_w>0$ such that*

$$|t_h(w_1;\boldsymbol{u},\boldsymbol{v})-t_h(w_2;\boldsymbol{u},\boldsymbol{v})|\leq c_w\,\|w_1-w_2\|_{1,h,\Omega^f}\,|||\boldsymbol{u}|||_{v,f}|||\boldsymbol{v}|||_{v,f}. \tag{13}$$

Proof. Using that $\frac{1}{2}(w\cdot n^f+|w\cdot n^f|)u+\frac{1}{2}(w\cdot n^f-|w\cdot n^f|)\bar{u}=w\cdot n^f\bar{u}+S_w(u-\bar{u})$, where $S_w=\max(w\cdot n^f,0)$, we can write t_h after integration by parts as

$$t_h(w;\boldsymbol{u},\boldsymbol{v})=(\nabla u,v\otimes w)_{\Omega^f}-\langle((u-\bar{u})\otimes w)n^f,v\rangle_{\partial\mathcal{T}^f}+\langle S_w(u-\bar{u}),v-\bar{v}\rangle_{\partial\mathcal{T}^f}.$$

The remainder of the proof is given by [19, Proposition 3.4]. □

Proposition 3.2. *For $w\in V^{f,div}(h)$ and $\boldsymbol{v}\in\boldsymbol{V}(h)$ it holds that*

$$t_h(w;\boldsymbol{v},\boldsymbol{v})=\tfrac{1}{2}\langle|w\cdot n^f|,|v-\bar{v}|^2\rangle_{\partial\mathcal{T}^f}+\tfrac{1}{2}\langle w\cdot n^f,|\bar{v}|^2\rangle_{\Gamma_{IN}^f}. \tag{14}$$

Proof. Note that

$$t_h(w;\boldsymbol{v},\boldsymbol{v})=-(v\otimes w,\nabla v)_{\Omega^f}+\tfrac{1}{2}\langle w\cdot n^f\,(v+\bar{v}),v-\bar{v}\rangle_{\partial\mathcal{T}^f}+\tfrac{1}{2}\langle|w\cdot n^f|,|v-\bar{v}|^2\rangle_{\partial\mathcal{T}^f}+\langle w\cdot n^f,|\bar{v}|^2\rangle_{\Gamma_{IN}^f}.$$

Note that $(v+\bar{v})\cdot(v-\bar{v}) = |v|^2 - |\bar{v}|^2$. Furthermore, $-(v\otimes w, \nabla v)_{\Omega^f} = -\langle \frac{1}{2} w\cdot n^f, |v|^2\rangle_{\partial\mathcal{T}^f}$ since $-v\otimes w : \nabla v = -\frac{1}{2}\nabla\cdot(|v|^2 w)$. Therefore, also using that $\langle \frac{1}{2}w\cdot n^f, |\bar{v}|^2\rangle_{\partial\mathcal{T}^f} = \langle \frac{1}{2}w\cdot n^f, |\bar{v}|^2\rangle_{\Gamma_I} + \langle \frac{1}{2}w\cdot n^f, |\bar{v}|^2\rangle_{\Gamma_N^f}$, the result follows. □

Proposition 3.3. *Let $w\in V^{f,div}(h)$ and $\|w\cdot n\|_{\Gamma_{IN}^f} \le \frac{1}{2}\mu^f c_{ae}^f (c_{pq}^2 + c_{si,4}^2)^{-1}$. Then, for $\beta > \beta_0$,*

$$t_h(w;\boldsymbol{v}_h,\boldsymbol{v}_h) + a_h^f(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge \tfrac{1}{2}c_{ae}^f\mu^f |||\boldsymbol{v}_h^f|||_{v,f}^2 \quad \forall \boldsymbol{v}_h\in \boldsymbol{V}_h^f. \tag{15}$$

Proof. By eq. (14), we find that

$$t_h(w;\boldsymbol{v}_h,\boldsymbol{v}_h) \ge -\tfrac{1}{2}\langle |w\cdot n^f|, |\bar{v}_h|^2\rangle_{\Gamma_{IN}^f}.$$

By a scaling identity there exists a constant $c_{pq}>0$ independent of h such that $\|\bar{v}\|_{4,0,\partial K} \le c_{pq}h^{(1-d)/4}\|\bar{v}\|_{\partial K}$ for $\bar{v}\in\{q\in L^2(\partial K) : q|_F \in P_k(F),\ \forall F\in\mathcal{F}(K)\}$. By the identical steps as used in the proof of [23, Lemma 6] (see also [28, Lemma 2]) it then follows that

$$t_h(w;\boldsymbol{v}_h,\boldsymbol{v}_h) \ge -(c_{pq}^2 + c_{si,4}^2)\,\|w\cdot n\|_{\Gamma_{IN}^f}\,|||\boldsymbol{v}_h|||_{v,f}^2,$$

where $c_{si,4}$ is the constant from eq. (10c) with $r=4$. We find, using eq. (11a),

$$t_h(w;\boldsymbol{v}_h,\boldsymbol{v}_h) + a_h^f(\boldsymbol{v}_h,\boldsymbol{v}_h) \ge (c_{ae}^f\mu^f - (c_{pq}^2 + c_{si,4}^2)\,\|w\cdot n\|_{\Gamma_{IN}^f})\,|||\boldsymbol{v}_h^f|||_{v,f}^2.$$

The result follows by the assumption on $\|w\cdot n\|_{\Gamma_{IN}^f}$. □

Remark 3.4. In Proposition 3.3, we assume a smallness condition on $\|w\cdot n\|_{\Gamma_{IN}^f}$. If Γ_N^f represents an outflow boundary (on which $w\cdot n>0$), then the smallness condition only needs to hold on $\|w\cdot n\|_{\Gamma_I}$. Numerically, however, it cannot be guaranteed that $w\cdot n>0$ on Γ_N^f.

3.4. **The HDG method**

The semi-discrete HDG method for the coupled Navier–Stokes/Biot problem eqs. (1), (2), (4) and (5) is given by: For $t\in J$, find $(\boldsymbol{u}_h(t),\boldsymbol{p}_h(t),z_h(t),\boldsymbol{p}_h^p(t))\in\boldsymbol{X}_h$ such that for all $(\boldsymbol{v}_h,\boldsymbol{q}_h,w_h,\boldsymbol{q}_h^p)\in\boldsymbol{X}_h$,

$$(\partial_t u_h, v_h)_{\Omega^f} + t_h(u_h^f;\boldsymbol{u}_h^f,\boldsymbol{v}_h^f) + a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h,\boldsymbol{p}_h) + a_h^I((\bar{u}_h^f,\partial_t\bar{u}_h^b),(\bar{v}_h^f,\bar{v}_h^b)) + b_h^I((\bar{v}_h^f,\bar{v}_h^b),\bar{p}_h^p) = (f,v_h)_\Omega, \tag{16a}$$

$$b_h(\boldsymbol{u}_h,\boldsymbol{q}_h) + c_h((p_h^p,p_h^b),q_h^b) = 0, \tag{16b}$$

$$(c_0\partial_t p_h^p, q_h^p)_{\Omega^b} + c_h((\partial_t p_h^p,\partial_t p_h^b),\alpha q_h^p) - b_h^b((z_h,0),\boldsymbol{q}_h^p) - b_h^I((\bar{u}_h^f,\partial_t\bar{u}_h^b),\bar{q}_h^p) = (g^b,q_h^p)_{\Omega^b}, \tag{16c}$$

$$(\mu^f\kappa^{-1}z_h,w_h)_{\Omega^b} + b_h^b((w_h,0),\boldsymbol{p}_h^p) = 0. \tag{16d}$$

Using Backward Euler time-stepping, with lagging of the convective velocity, the fully-discrete HDG method is given by: For $n=0,1,\cdots,N-1$, find $(\boldsymbol{u}_h^{n+1},\boldsymbol{p}_h^{n+1},z_h^{n+1},\boldsymbol{p}_h^{p,n+1})\in\boldsymbol{X}_h$ such that for all $(\boldsymbol{v}_h,\boldsymbol{q}_h,w_h,\boldsymbol{q}_h^p)\in\boldsymbol{X}_h$,

$$
\begin{aligned}
&(d_t u_h^{n+1}, v_h)_{\Omega^f} + t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n+1}, \boldsymbol{v}_h^f) + a_h(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \boldsymbol{p}_h^{n+1}) + a_h^I((\bar{u}_h^{f,n+1}, d_t\bar{u}_h^{b,n+1}), (\bar{v}_h^f, \bar{v}_h^b)) \qquad (17a)\\
&\qquad + b_h^I((\bar{v}_h^f, \bar{v}_h^b), \bar{p}_h^{p,n+1}) = (f^{n+1}, v_h)_\Omega,\\
&b_h(\boldsymbol{u}_h^{n+1}, \boldsymbol{q}_h) + c_h((p_h^{p,n+1}, p_h^{b,n+1}), q_h^b) = 0, \qquad (17b)\\
&(c_0 d_t p_h^{p,n+1}, q_h^p)_{\Omega^b} + c_h((d_t p_h^{p,n+1}, d_t p_h^{b,n+1}), \alpha q_h^p) - b_h^b((z_h^{n+1}, 0), \boldsymbol{q}_h^p) - b_h^I((\bar{u}_h^{f,n+1}, d_t\bar{u}_h^{b,n+1}), \bar{q}_h^p) \qquad (17c)\\
&\qquad = (g^{b,n+1}, q_h^p)_{\Omega^b},\\
&(\mu^f \kappa^{-1} z_h^{n+1}, w_h)_{\Omega^b} + b_h^b((w_h, 0), \boldsymbol{p}_h^{p,n+1}) = 0. \qquad (17d)
\end{aligned}
$$

Note that despite the coupled Navier–Stokes/Biot problem being nonlinear, the fully-discrete HDG method eq. (17) is linear at each time step due to lagging of the convective velocity.

Remark 3.5. The HDG method eq. (17) is an extension of the HDG method previously presented in [21] for the coupled Stokes/Biot model. For this HDG method, it was proven in [21, Lemma 1] that the discrete velocities and displacement are divergence-conforming. Furthermore, it was shown that the compressibility equations are satisfied pointwise on the elements and that, for the semi-discrete problem eq. (16) and in the absence of source/sink terms, that mass is conserved pointwise on the elements. These properties are inherited by eq. (17). The proof is identical to that of [21, Lemma 1] and therefore not included here.

4. Stability and well-posedness

Before showing well-posedness of the discretization we introduce some definitions and inequalities. First, we define the discrete function spaces

$$
\begin{aligned}
\boldsymbol{V}_h^{div} &:= \{\boldsymbol{v}_h \in \boldsymbol{V}_h^f \,:\, b_h^f(\boldsymbol{v}_h, \boldsymbol{q}_h) = 0 \;\forall \boldsymbol{q}_h \in \boldsymbol{Q}_h^f\},\\
\boldsymbol{B}_h^f &:= \{\boldsymbol{v}_h \in \boldsymbol{V}_h^{div} \,:\, |||\boldsymbol{v}_h|||_{v,f} \le \min\big(\frac{\mu^f c_{ae}^f}{2c_{si,2}(c_{pq}^2 + c_{si,4}^2)}, \frac{c_{ae}^f \mu^f}{4c_w}\big)\},
\end{aligned}
$$

where we note that if $\boldsymbol{u}_h^f \in \boldsymbol{B}_h^f$, then by eq. (10c),

$$
\|u_h^f \cdot n\|_{\Gamma_{IN}^f} \le c_{si,2} |||\boldsymbol{u}_h^f|||_{v,f} \le \frac{\mu^f c_{ae}^f}{2(c_{pq}^2 + c_{si,4}^2)}. \qquad (18)
$$

For $0 \le j \le N-1$, we define

$$
X^j := \|d_t u_h^{f,j+1}\|_{\Omega^f}^2 + \tfrac{1}{2} a_h^b(d_t\boldsymbol{u}_h^{b,j+1}, d_t\boldsymbol{u}_h^{b,j+1}) + \lambda^{-1} \|\alpha d_t p_h^{p,j+1} - d_t p_h^{b,j+1}\|_{\Omega^b}^2 + c_0 \|d_t p_h^{p,j+1}\|_{\Omega^b}^2. \qquad (19)
$$

Furthermore,

$$
\begin{aligned}
F^0 :=& 3\|f^{f,1}\|_{\Omega^f}^2 + \frac{3c_p^2}{c_{ae}^b \mu^b} \|d_t f^{b,1}\|_{\Omega^b}^2 + \frac{3c_{td}^2 \mu^f}{\kappa} \Delta t \|d_t g^{b,1}\|_{\Omega^b}^2, \qquad (20a)\\
F^m :=& \frac{4c_p^2}{c_{ae}^f \mu^f} \Delta t \sum_{k=1}^m \|d_t f^{f,k+1}\|_{\Omega^f}^2 + \frac{2c_{td}^2 \mu^f}{\kappa} \Delta t \sum_{k=1}^m \|d_t g^{b,k+1}\|_{\Omega^b}^2 \qquad (20b)\\
&+ 2c_p^2 (c_{ac}^b \mu^b)^{-1} \big(\max_{1\le k\le m} \|d_t f^{b,k+1}\|_{\Omega^b} + \|d_t f^{b,2}\|_{\Omega^b} + \Delta t \sum_{k=2}^m \|d_{tt} f^{b,k+1}\|_{\Omega^b}\big)^2, \qquad 1 \le m \le N-1,
\end{aligned}
$$

where $\sum_{k=2}^{m} \|d_{tt} f^{b,k+1}\|_{\Omega^b}$ in eq. (20b) is zero if $m = 1$. It will also be useful to define

$$G^0 := F^0 \quad \text{and} \quad G^m := \tfrac{145}{24} F^0 + F^m \text{ for } 1 \leq m \leq N-1. \tag{21}$$

Finally, let us define

$$H := 2\|f^f\|_{\ell^1(J;\Omega^f)} + c_{td}(\mu^f/\kappa)^{1/2}\|g^b\|_{\ell^2(J;\Omega^b)} + 2c_p(c_{ae}^b\mu^b)^{-1/2}\|d_t f^b\|_{\ell^1(J;\Omega^b)} + 2c_p(c_{ae}^b\mu^b)^{-1/2}\|f^b\|_{\ell^\infty(J;\Omega^b)}. \tag{22}$$

In this section we will prove well-posedness of the HDG method eq. (17) assuming the data satisfies

$$\max\big(H^2, \|f^f\|_{\ell^\infty(J;\Omega^f)} H + \tfrac{1}{2}c_{td}^2\mu^f\kappa^{-1}\|g^b\|^2_{\ell^\infty(J;\Omega^b)} + c_p(c_{ae}^b\mu^b)^{-1/2}\|f^b\|_{\ell^\infty(J;\Omega^b)}(2G^{N-1})^{1/2} + H(2G^{N-1})^{1/2}\big) \leq \tfrac{1}{2}c_{ae}^f\mu^f\Big[\min\Big(\frac{\mu^f c_{ae}^f}{2c_{si,2}(c_{pq}^2 + c_{si,4}^2)}, \frac{\mu^f c_{ae}^f}{4c_w}\Big)\Big]^2. \tag{23}$$

In our proofs we will use the following result from [22, Lemma 4.2]: if $\boldsymbol{p}_h^{p,n}$ and z_h^n are part of the solution to eq. (17) for $n \geq 1$, then there exists a constant c_{pd}, independent of h, such that

$$\|p_h^{p,n}\|_{\Omega^b} \leq c_{pp}\|p_h^{p,n}\|_{1,h,\Omega^b} \leq c_{pp}|||\boldsymbol{p}_h^{p,n}|||_{1,h,b} \leq c_{td}\mu^f\kappa^{-1}\|z_h^n\|_{\Omega^b}, \tag{24}$$

where $c_{td} = c_{pp}c_{pd}$. We remark that the final inequality in eq. (24) follows by using modified local BDM degrees-of-freedom that incorporate HDG facet unknowns and eq. (17d).

The main goal of this section is to prove well-posedness under the assumptions that $f^{b,0} = 0$, $g^{b,0} = 0$, $\boldsymbol{u}_h^{f,0} = 0$ and $\boldsymbol{p}_h^{p,0} = 0$. These assumptions are necessary to prove well-posedness for the first time-step.

Theorem 4.1 (Well-posedness)**.** *Assume that eq. (23) holds, and that $f^{b,0} = 0$ and $g^{b,0} = 0$. Then, starting with $\boldsymbol{u}_h^{f,0} = 0$ and $\boldsymbol{p}_h^{p,0} = 0$, a unique solution to eq. (17) exists. We furthermore have the following uniform bounds (in n and h) for $1 \leq n \leq N$:*

$$\tfrac{1}{2}c_{ae}^f\mu^f|||\boldsymbol{u}_h^{f,n}|||^2_{v,f} + \gamma\mu^f\kappa^{-1/2}\|(\bar{u}_h^{f,n} - d_t\bar{u}_h^{b,n})^t\|^2_{\Gamma_I} + \tfrac{1}{2}\mu^f\kappa^{-1}\|z_h^n\|^2_{\Omega^b} \leq \tfrac{1}{2}c_{ae}^f\mu^f\Big[\min\Big(\frac{\mu^f c_{ae}^f}{2c_{si,2}(c_{pq}^2 + c_{si,4}^2)}, \frac{\mu^f c_{ae}^f}{4c_w}\Big)\Big]^2, \tag{25a}$$

$$c_{ae}^b\mu^b|||\boldsymbol{u}_h^{b,n}|||^2_{v,b} + \lambda^{-1}\|\alpha p_h^{p,n} - p_h^{b,n}\|^2_{\Omega^b} + c_0\|p_h^{p,n}\|^2_{\Omega^b} \leq \tfrac{1}{2}c_{ae}^f\mu^f\Big[\min\Big(\frac{\mu^f c_{ae}^f}{2c_{si,2}(c_{pq}^2 + c_{si,4}^2)}, \frac{\mu^f c_{ae}^f}{4c_w}\Big)\Big]^2. \tag{25b}$$

The pressures are bounded as:

$$|||\boldsymbol{p}_h^{p,n}|||^2_{1,h,b} \leq \tfrac{1}{2}c_{pd}^2\mu^f(c_{ae}^f)^2\kappa^{-1}\min\Big(\frac{1}{c_{si,2}(c_{pq}^2 + c_{si,4}^2)}, \frac{1}{2c_w}\Big), \tag{26a}$$

$$\begin{aligned} c_b|||\boldsymbol{p}_h^{n+1}|||_q \leq\, & c_p\|f^f\|_{\ell^\infty(J;L^2(\Omega^f))} + c_p\|f^b\|_{\ell^\infty(J;L^2(\Omega^b))} + c_p(G^{N-1})^{1/2} \\ & + \tfrac{1}{2}\mu^f c_{ae}^f c_w \min\Big(\frac{1}{c_{si,2}(c_{pq}^2 + c_{si,4}^2)}, \frac{1}{2c_w}\Big) \\ & + (c_{ac}^f\mu^f + c_{ac}^b(c_{ae}^f/c_{ae}^b)^{1/2}(\mu^f/\mu^b)^{1/2} + (\tfrac{1}{2}c_{ae}^f\gamma\kappa^{-1/2})^{1/2}\mu^f)\times \\ & \Big[\min\Big(\frac{\mu^f c_{ae}^f}{2c_{si,2}(c_{pq}^2 + c_{si,4}^2)}, \frac{c_{ae}^f\mu^f}{4c_w}\Big)\Big]^{1/2}. \end{aligned} \tag{26b}$$

The proof of Theorem 4.1 is by induction and follows at the end of this section after first proving some intermediate results. The following lemma shows uniqueness of the discrete solution at t_{n+1} assuming that $\boldsymbol{u}_h^{f,n} \in \boldsymbol{B}_h^f$ for some $1 \le n \le N-1$.

Lemma 4.2 (Uniqueness)**.** *Let $u_h^{f,0} = 0$ and $p_h^{p,0} = 0$. Assume that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n, z_h^n, \boldsymbol{p}_h^{p,n}) \in \boldsymbol{X}_h$ is the solution to eq. (17) for some $1 \le n \le N-1$. If $\boldsymbol{u}_h^{f,n} \in \boldsymbol{B}_h^f$, then a unique solution $(\boldsymbol{u}_h^{n+1}, \boldsymbol{p}_h^{n+1}, z_h^{n+1}, \boldsymbol{p}_h^{p,n+1}) \in \boldsymbol{X}_h$ to eq. (17) exists.*

Proof. See appendix B.1. □

In the following lemma, which extends [22, Lemma 4.5] for Navier–Stokes/Darcy to Navier–Stokes/Biot, and the following corollary, we obtain bounds on X^n (see eq. (19) for the definition of X^n). These bounds will be used to prove the pressure bound eq. (26b) and to show that if $\boldsymbol{u}_h^{f,k} \in \boldsymbol{B}_h^f$ for all $0 \le k \le n$ with $0 \le n \le N-1$, then $\boldsymbol{u}_h^{f,n+1} \in \boldsymbol{B}_h^f$ (see Lemma 4.5 and Remark 4.6).

Lemma 4.3. *Assume $f^{b,0} = 0$, $g^{b,0} = 0$, $\boldsymbol{u}_h^{f,0} = 0$, and $\boldsymbol{p}_h^{p,0} = 0$. If $(\boldsymbol{u}_h^k, \boldsymbol{p}_h^k, z_h^k, \boldsymbol{p}_h^{p,k}) \in \boldsymbol{X}_h$ is a solution to eq. (17) for $1 \le k \le n$, then*

$$X^0 \le F^0, \tag{27a}$$

$$c_{ae}^f \mu^f \Delta t |||d_t \boldsymbol{u}_h^{f,1}|||_{v,f}^2 \le \tfrac{1}{12} F^0. \tag{27b}$$

Furthermore, if $\boldsymbol{u}_h^{f,k} \in \boldsymbol{B}_h^f$ for all $0 \le k \le n$, with $1 \le n \le N-1$, then

$$X^n \le 6X^0 + \tfrac{1}{2} c_{ae}^f \mu^f \Delta t |||d_t \boldsymbol{u}_h^{f,1}|||_{v,f}^2 + F^n. \tag{28}$$

Proof. See appendix B.2. □

An immediate consequence of eqs. (27) and (28) is the following result.

Corollary 4.4. *Under the assumptions of Lemma 4.3,*

$$X^n \le G^n \qquad \forall 0 \le n \le N-1, \tag{29}$$

where G^n is defined in eq. (21).

In the following lemma we obtain results that are used to prove eqs. (25a), (25b) and (26a).

Lemma 4.5. *For $1 \le i \le N$, let*

$$A_i^2 := \tfrac{1}{2} \left\| u_h^{f,i} \right\|_{\Omega^f}^2 + \tfrac{1}{2} a_h^b(\boldsymbol{u}_h^{b,i}, \boldsymbol{u}_h^{b,i}) + \tfrac{1}{2} \lambda^{-1} \left\| \alpha p_h^{p,i} - p_h^{b,i} \right\|_{\Omega^b}^2 + \tfrac{1}{2} c_0 \left\| p_h^{p,i} \right\|_{\Omega^b}^2, \tag{30a}$$

$$B_i^2 := \tfrac{1}{4} \Delta t c_{ae}^f \mu^f |||\boldsymbol{u}_h^{f,i}|||_{v,f}^2 + \tfrac{1}{2} \Delta t \gamma \mu^f \kappa^{-1/2} \left\| (\bar{u}_h^{f,i} - d_t \bar{u}_h^{b,i})^t \right\|_{\Gamma_I}^2 + \tfrac{1}{2} \Delta t \mu^f \kappa^{-1} \left\| z_h^i \right\|_{\Omega^b}^2. \tag{30b}$$

Assume $f^{b,0} = 0$, $g^{b,0} = 0$, $\boldsymbol{u}_h^{f,0} = 0$, and $\boldsymbol{p}_h^{p,0} = 0$. Let $1 \le n \le N-1$ and assume that $(\boldsymbol{u}_h^k, \boldsymbol{p}_h^k, z_h^k, \boldsymbol{p}_h^{p,k}) \in \boldsymbol{X}_h$ is a solution to eq. (17) for all $0 \le k \le n$ such that $\boldsymbol{u}_h^{f,k} \in \boldsymbol{B}_h^f$. Then

$$\Big(A_{n+1}^2 + \sum_{i=1}^{n+1} B_i^2\Big)^{1/2} \le \tfrac{1}{\sqrt{2}} H, \tag{31}$$

and

$$\tfrac{1}{2} c_{ae}^f \mu^f |||\boldsymbol{u}_h^{f,n+1}|||_{v,f}^2 + \gamma \mu^f \kappa^{-1/2} \left\| (\bar{u}_h^{f,n+1} - d_t \bar{u}_h^{b,n+1})^t \right\|_{\Gamma_I}^2 + \tfrac{1}{2} \mu^f \kappa^{-1} \left\| z_h^{n+1} \right\|_{\Omega^b}^2$$
$$\le \left\| f^{f,n+1} \right\|_{\Omega^f} H + \tfrac{1}{2} c_{td}^2 \mu^f \kappa^{-1} \left\| g^{b,n+1} \right\|_{\Omega^b}^2 + c_p (c_{ae}^b \mu^b)^{-1/2} \left\| f^{b,n+1} \right\|_{\Omega^b} (2G^n)^{1/2} + H (2G^n)^{1/2}. \tag{32}$$

Proof. See appendix B.3. □

Remark 4.6. A consequence of eq. (32) and eq. (23) is that $\boldsymbol{u}_h^{f,n+1} \in \boldsymbol{B}_h^f$. This result will be used in the proof of Theorem 4.1 to prove uniqueness.

We end this section with proving Theorem 4.1.

Proof of Theorem 4.1. Equation (25a) follows directly from eq. (32) and eq. (23). Furthermore, eq. (25a) implies $\boldsymbol{u}_h^{f,n} \in \boldsymbol{B}_h^f$ for $1 \le n \le N$ so that existence and uniqueness follow from Lemma 4.2. The bound eq. (25b) follows from eq. (31), eq. (11a), and eq. (23). The bound eq. (26a) is a direct consequence of eq. (24) and eq. (25a). Finally, we consider the pressure bound eq. (26b). From eq. (12b) and eq. (17) we obtain:

$$c_b|||\boldsymbol{p}_h^{n+1}|||_q \le \sup_{\boldsymbol{0}\neq\boldsymbol{v}_h\in\widehat{\boldsymbol{V}}_h} \frac{|b_h(\boldsymbol{v}_h, \boldsymbol{p}_h^{n+1})|}{|||\boldsymbol{v}_h|||_v} \le \sup_{\boldsymbol{0}\neq\boldsymbol{v}_h\in\widehat{\boldsymbol{V}}_h} \frac{B}{|||\boldsymbol{v}_h|||_v} \tag{33}$$

where

$$B = |(f^{n+1}, v_h)_\Omega| + |(d_t u_h^{n+1}, v_h)_{\Omega^f}| + |t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n+1}, \boldsymbol{v}_h^f)| + |a_h(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)| \\ + |a_h^I((\bar{u}_h^{f,n+1}, d_t\bar{u}_h^{b,n+1}), (\bar{v}_h^f, \bar{v}_h^b))| + |b_h^I((\bar{v}_h^f, \bar{v}_h^b), \bar{p}_h^{p,n+1})|.$$

First, note that $b_h^I((\bar{v}_h^f, \bar{v}_h^b), \bar{p}_h^{p,n+1}) = 0$ by the definition of $\widehat{\boldsymbol{V}}_h$. Using the Cauchy–Schwarz inequality, eq. (10a), eq. (10b), eq. (13), eq. (11b):

$$\begin{aligned}
|(f^{n+1}, v_h)_\Omega| &\le c_p \, \|f^{f,n+1}\|_{\Omega^f} \, |||\boldsymbol{v}_h^f|||_{v,f} + c_p \, \|f^{b,n+1}\|_{\Omega^b} \, |||\boldsymbol{v}_h^b|||_{v,b}, \\
|(d_t u_h^{n+1}, v_h)_{\Omega^f}| &\le c_p \, \|d_t u_h^{n+1}\|_{\Omega^f} \, |||\boldsymbol{v}_h^f|||_{v,f}, \\
|t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n+1}, \boldsymbol{v}_h^f)| &\le c_w |||\boldsymbol{u}_h^{f,n}|||_{v,f} |||\boldsymbol{u}_h^{f,n+1}|||_{v,f} |||\boldsymbol{v}_h^f|||_{v,f}, \\
|a_h(\boldsymbol{u}_h^{n+1}, \boldsymbol{v}_h)| &\le c_{ac}^f \mu^f |||\boldsymbol{u}_h^{f,n+1}|||_{v,f} |||\boldsymbol{v}_h^f|||_{v,f} + c_{ac}^b \mu^b |||\boldsymbol{u}_h^{b,n+1}|||_{v,b} |||\boldsymbol{v}_h^b|||_{v,b}, \\
|a_h^I((\bar{u}_h^{f,n+1}, d_t\bar{u}_h^{b,n+1}), (\bar{v}_h^f, \bar{v}_h^b))| &\le \gamma \mu^f \kappa^{-1/2} \, \|(\bar{u}_h^{f,n+1} - d_t\bar{u}_h^{b,n+1})^t\|_{\Gamma_I} \, |||\boldsymbol{v}_h|||_v.
\end{aligned}$$

Combined now with eq. (33), Corollary 4.4, eq. (25a), and eq. (25b) we obtain the result eq. (26b). □

5. Error analysis

For the error analysis we first define interpolation operators to decompose the errors. For scalar functions, we denote by Π_Q^j, $\bar{\Pi}_Q^j$, and $\bar{\Pi}_{Q^0}^b$ the L^2-projection operators onto Q_h^j, $\bar{Q}_h^j$, and $\bar{Q}_h^{b0}$, respectively. For vector valued functions, we define $\Pi_V^j : H(\text{div}, \Omega^j) \cap [L^r(\Omega^j)]^d \to V_h^j$, for $r > 2$ and $j = f, b$, to be the interpolation operator to the Brezzi–Douglas–Marini (BDM) finite element spaces [12, Section III.3] and $\bar{\Pi}_V^j : [L^2(\mathcal{F}^j)]^d \to \bar{V}_h^j$ to be the L^2-projection onto $\bar{V}_h^j$. It is known that

$$(q_h, \nabla\cdot\Pi_V^j u^j)_K = (q_h, \nabla\cdot u^j)_K \quad \forall q_h \in P_{k-1}(K), K \in \mathcal{T}^j, \quad \langle \bar{q}_h, n\cdot\Pi_V^j u^j\rangle_F = \langle \bar{q}_h, n\cdot u^j\rangle_F \quad \forall \bar{q}_h \in P_k(F), F \in \mathcal{F}^j,$$

and

$$\begin{aligned}
&\text{if } u^j|_K \in [H^{k+1}(K)]^d: && \|u^j - \Pi_V^j u^j\|_{m,K} \le C h_K^{l-m} \, \|u^j\|_{l,K}, \quad m = 0,1,2, \quad \max\{1,m\} \le l \le k+1, \\
&\text{if } u^j|_K \in [W_\infty^1(K)]^d: && \|u^j - \Pi_V^j u^j\|_{L^\infty(K)} \le C h_K |u|_{W_\infty^1(K)}.
\end{aligned}$$

For time stepping index n, we introduce the following notation for the errors:

$$u^{j,n}-u_h^{j,n}=(u^{j,n}-\Pi_V^j u^{j,n})-(u_h^{j,n}-\Pi_V^j u^{j,n})=:e_{u^j}^{I,n}-e_{u^j}^{h,n}, \tag{34a}$$

$$\bar{u}^{j,n}-\bar{u}_h^{j,n}=(\bar{u}^{j,n}-\bar{\Pi}_V^j \bar{u}^{j,n})-(\bar{u}_h^{j,n}-\bar{\Pi}_V^j \bar{u}^{j,n})=:\bar{e}_{u^j}^{I,n}-\bar{e}_{u^j}^{h,n}, \tag{34b}$$

$$z^{n}-z_h^{n}=(z^{n}-\Pi_V^b z^{n})-(z_h^{n}-\Pi_V^b z^{n})=:e_{z}^{I,n}-e_{z}^{h,n}, \tag{34c}$$

$$p^{j,n}-p_h^{j,n}=(p^{j,n}-\Pi_Q^j p^{j,n})-(p_h^{j,n}-\Pi_Q^j p^{j,n})=:e_{p^j}^{I,n}-e_{p^j}^{h,n}, \tag{34d}$$

$$\bar{p}^{j,n}-\bar{p}_h^{j,n}=(\bar{p}^{j,n}-\bar{\Pi}_Q^j \bar{p}^{j,n})-(\bar{p}_h^{j,n}-\bar{\Pi}_Q^j p^{j,n})=:\bar{e}_{p^j}^{I,n}-\bar{e}_{p^j}^{h,n}, \tag{34e}$$

$$p^{p,n}-p_h^{p,n}=(p^{p,n}-\Pi_Q^b p^{p,n})-(p_h^{p,n}-\Pi_Q^b p^{p,n})=:e_{p^p}^{I,n}-e_{p^p}^{h,n}, \tag{34f}$$

$$\bar{p}^{p,n}-\bar{p}_h^{p,n}=(\bar{p}^{p,n}-\bar{\Pi}_{Q^0}^b \bar{p}^{p,n})-(\bar{p}_h^{p,n}-\bar{\Pi}_{Q^0}^b \bar{p}^{p,n})=:\bar{e}_{p^p}^{I,n}-\bar{e}_{p^p}^{h,n}, \tag{34g}$$

and define $e_u^{\omega,n}$ and $e_p^{\omega,n}$ such that $e_u^{\omega,n}|_{\Omega^j}=e_{u^j}^{\omega,n}$ and $e_p^{\omega,n}|_{\Omega^j}=e_{p^j}^{\omega,n}$, for $\omega=I,h$.

The following lemma now determines the error equations.

Lemma 5.1 (Error equations)**.** *Suppose that $\{(\boldsymbol{u}_h^n,\boldsymbol{p}_h^n,z_h^n,\boldsymbol{p}_h^{p,n})\}_n$ are the solutions to eq. (17). Furthermore, assume that (u,p,z,p^p) is the solution to eqs. (1), (2) and (4) to (6) on the time interval $J=(0,T]$ with $u_0^f(x)=0$ and $p_0^p(x)=0$. Define $\boldsymbol{u}:=(u,u|_{\Gamma_0^f},u|_{\Gamma_0^b})$, $\boldsymbol{p}:=(p,p|_{\Gamma_0^f},p|_{\Gamma_0^b})$, and $\boldsymbol{p}^p:=(p^p,p^p|_{\Gamma_0^b})$. Then, for $n\geq 0$ and for all $(\boldsymbol{v}_h,\boldsymbol{q}_h,w_h,\boldsymbol{q}_h^p)\in\boldsymbol{X}_h$ we have:*

$$\begin{aligned}
&(d_t e_{u^f}^{h,n+1},v_h^f)_{\Omega^f}+t_h(u_h^{f,n};\boldsymbol{e}_{u^f}^{h,n+1},\boldsymbol{v}_h^f)+a_h(\boldsymbol{e}_u^{h,n+1},\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,\boldsymbol{e}_p^{h,n+1})\\
&\quad+a_h^I((\bar{e}_{u^f}^{h,n+1},d_t\bar{e}_{u^b}^{h,n+1}),(\bar{v}_h^f,\bar{v}_h^b))+b_h^I((\bar{v}_h^f,\bar{v}_h^b),\bar{e}_{p^p}^{h,n+1})\\
&\quad=(\partial_t u^{f,n+1}-(\Delta t)^{-1}(\Pi_V^f u^{f,n+1}-\Pi_V^f u^{f,n}),v_h^f)_{\Omega^f}+a_h(\boldsymbol{e}_u^{I,n+1},\boldsymbol{v}_h)\\
&\qquad+a_h^I((0,\partial_t\bar{u}^{b,n+1}-d_t\bar{u}^{b,n+1}),(\bar{v}_h^f,\bar{v}_h^b))+[t_h(u^{f,n+1};\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)-t_h(u^{f,n};\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)]\\
&\qquad+t_h(u^{f,n};\boldsymbol{e}_{u^f}^{I,n+1},\boldsymbol{v}_h^f)+[t_h(u^{f,n};\boldsymbol{\Pi}_V^f\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)-t_h(u_h^{f,n};\boldsymbol{\Pi}_V^f\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)]\,,
\end{aligned} \tag{35a}$$

$$b_h^f(\boldsymbol{e}_{u^f}^{h,n+1},\boldsymbol{q}_h^f)+b_h^b(d_t\boldsymbol{e}_{u^b}^{h,n+1},\boldsymbol{q}_h^b)+c_h((d_te_{p^p}^{h,n+1},d_te_{p^b}^{h,n+1}),q_h^b)=0, \tag{35b}$$

$$\begin{aligned}
&(c_0d_te_{p^p}^{h,n+1},q_h^p)_{\Omega^b}+c_h((d_te_{p^p}^{h,n+1},d_te_{p^b}^{h,n+1}),\alpha q_h^p)-b_h^b((e_z^{h,n+1},0),\boldsymbol{q}_h^p)-b_h^I((\bar{e}_{u^f}^{h,n+1},d_t\bar{e}_{u^b}^{h,n+1}),\bar{q}_h^p)\\
&\quad=(c_0(\partial_tp^{p,n+1}-d_tp^{p,n+1}),q_h^p)_{\Omega^b}+c_h((\partial_tp^{p,n+1}-d_tp^{p,n+1},\partial_tp^{b,n+1}-d_tp^{b,n+1}),\alpha q_h^p)\\
&\qquad-b_h^I((0,\partial_t\bar{u}^{b,n+1}-d_t\bar{u}^{b,n+1}),\bar{q}_h^p),
\end{aligned} \tag{35c}$$

$$(\mu^f\kappa^{-1}e_z^{h,n+1},w_h)_{\Omega^b}+b_h^b((w_h,0),\boldsymbol{e}_{p^p}^{h,n+1})=(\mu^f\kappa^{-1}e_z^{I,n+1},w_h)_{\Omega^b}. \tag{35d}$$

Proof. It can easily be shown, by standard arguments, that the semi-discrete HDG method eq. (16) is consistent. Therefore, substituting the exact solution at time level $t=t_{n+1}$ into eq. (16) and subtracting eq. (17) from the result, we obtain:

$$\begin{aligned}
&(\partial_tu^{f,n+1}-d_tu_h^{f,n+1},v_h^f)_{\Omega^f}+t_h(u^{f,n+1};\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)-t_h(u_h^{f,n};\boldsymbol{u}_h^{f,n+1},\boldsymbol{v}_h^f)\\
&\quad+a_h(\boldsymbol{u}^{n+1}-\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,\boldsymbol{p}^{n+1}-\boldsymbol{p}_h^{n+1})\\
&\quad+a_h^I((\bar{u}^{f,n+1}-\bar{u}_h^{f,n+1},\partial_t\bar{u}^{b,n+1}-d_t\bar{u}_h^{b,n+1}),(\bar{v}_h^f,\bar{v}_h^b))+b_h^I((\bar{v}_h^f,\bar{v}_h^b),\bar{p}^{p,n+1}-\bar{p}_h^{p,n+1})=0,
\end{aligned} \tag{36a}$$

$$b_h(\boldsymbol{u}^{n+1}-\boldsymbol{u}_h^{n+1},\boldsymbol{q}_h)+c_h((p^{p,n+1}-p_h^{p,n+1},p^{b,n+1}-p_h^{b,n+1}),q_h^b)=0, \tag{36b}$$

$$\begin{aligned}
&(c_0(\partial_tp^{p,n+1}-d_tp_h^{p,n+1}),q_h^p)_{\Omega^b}+c_h((\partial_tp^{p,n+1}-d_tp_h^{p,n+1},\partial_tp^{b,n+1}-d_tp_h^{b,n+1}),\alpha q_h^p)\\
&\quad-b_h^b((z^{n+1}-z_h^{n+1},0),\boldsymbol{q}_h^p)-b_h^I((\bar{u}^{f,n+1}-\bar{u}_h^{f,n+1},\partial_t\bar{u}^{b,n+1}-d_t\bar{u}_h^{b,n+1}),\bar{q}_h^p)=0,
\end{aligned} \tag{36c}$$

$$\mu^f\kappa^{-1}(z^{n+1}-z_h^{n+1},w_h)_{\Omega^b}+b_h^b((w_h,0),\boldsymbol{p}^{p,n+1}-\boldsymbol{p}_h^{p,n+1})=0. \tag{36d}$$

Noting that $\partial_t u^{f,n+1} - d_t u_h^{f,n+1} = -d_t e_{u^f}^{h,n+1} + [\partial_t u^{f,n+1} - (\Delta t)^{-1}(\Pi_V^f u^{f,n+1} - \Pi_V^f u^{f,n})]$, and similar for the other time derivative terms, and using the error decomposition eq. (34), we can write eq. (36) as

$$
\begin{aligned}
&(d_t e_{u^f}^{h,n+1}, v_h^f)_{\Omega^f} - t_h(u^{f,n+1}; \boldsymbol{u}^{f,n+1}, \boldsymbol{v}_h^f) + t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n+1}, \boldsymbol{v}_h^f) \\
&\quad + a_h(\boldsymbol{e}_u^{h,n+1}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \boldsymbol{e}_p^{h,n+1}) + a_h^I((\bar{e}_{u^f}^{h,n+1}, d_t \bar{e}_{u^b}^{h,n+1}), (\bar{v}_h^f, \bar{v}_h^b)) + b_h^I((\bar{v}_h^f, \bar{v}_h^b), \bar{e}_{p^p}^{h,n+1}) \\
&\quad = (\partial_t u^{f,n+1} - (\Delta t)^{-1}(\Pi_V^f u^{f,n+1} - \Pi_V^f u^{f,n}), v_h^f)_{\Omega^f} + a_h(\boldsymbol{e}_u^{I,n+1}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \boldsymbol{e}_p^{I,n+1}) \\
&\qquad + a_h^I((\bar{e}_{u^f}^{I,n+1}, \partial_t \bar{u}^{b,n+1} - (\Delta t)^{-1}(\bar{\Pi}_V^b \bar{u}^{b,n+1} - \bar{\Pi}_V^b \bar{u}^{b,n})), (\bar{v}_h^f, \bar{v}_h^b)) + b_h^I((\bar{v}_h^f, \bar{v}_h^b), \bar{e}_{p^p}^{I,n+1}),
\end{aligned} \tag{37a}
$$

$$
b_h(\boldsymbol{e}_u^{h,n+1}, \boldsymbol{q}_h) + c_h((e_{p^p}^{h,n+1}, e_{p^b}^{h,n+1}), q_h^b) = b_h(\boldsymbol{e}_u^{I,n+1}, \boldsymbol{q}_h) + c_h((e_{p^p}^{I,n+1}, e_{p^b}^{I,n+1}), q_h^b), \tag{37b}
$$

$$
\begin{aligned}
&(c_0 d_t e_{p^p}^{h,n+1}, q_h^p)_{\Omega^b} + c_h((d_t e_{p^p}^{h,n+1}, d_t e_{p^b}^{h,n+1}), \alpha q_h^p) - b_h^b((e_z^{h,n+1}, 0), \boldsymbol{q}_h^p) \\
&\quad - b_h^I((\bar{e}_{u^f}^{h,n+1}, d_t \bar{e}_{u^b}^{h,n+1}), \bar{q}_h^p) \\
&\quad = (c_0 \left[\partial_t p^{p,n+1} - (\Delta t)^{-1}(\Pi_Q^b p^{p,n+1} - \Pi_Q^b p^{p,n})\right], q_h^p)_{\Omega^b} \\
&\qquad + c_h((\partial_t p^{p,n+1} - (\Delta t)^{-1}(\Pi_Q^b p^{p,n+1} - \Pi_Q^b p^{p,n})), \partial_t p^{b,n+1} - (\Delta t)^{-1}(\Pi_Q^b p^{b,n+1} - \Pi_Q^b p^{b,n})), \alpha q_h^p) \\
&\qquad - b_h^b((e_z^{I,n+1}, 0), \boldsymbol{q}_h^p) - b_h^I((\bar{e}_{u^f}^{I,n+1}, \partial_t \bar{u}^{b,n+1} - (\Delta t)^{-1}(\bar{\Pi}_V^b \bar{u}^{b,n+1} - \bar{\Pi}_V^b \bar{u}^{b,n})), \bar{q}_h^p),
\end{aligned} \tag{37c}
$$

$$
(\mu^f \kappa^{-1} e_z^{h,n+1}, w_h)_{\Omega^b} + b_h^b((w_h, 0), \boldsymbol{e}_{p^p}^{h,n+1}) = (\mu^f \kappa^{-1} e_z^{I,n+1}, w_h)_{\Omega^b} + b_h^b((w_h, 0), \boldsymbol{e}_{p^p}^{I,n+1}). \tag{37d}
$$

By properties of Π_V^j and $\bar{\Pi}_V^j$, $j = f, b$, we have that $b_h(\boldsymbol{e}_u^{I,n+1}, \boldsymbol{q}_h) = 0$ for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$ and $b_h^b((e_z^{I,n+1}, 0), \boldsymbol{q}_h^p) = 0$ for all $\boldsymbol{q}_h^p \in \boldsymbol{Q}_h^{b0}$. Similarly, $c_h((e_{p^p}^{I,n+1}, e_{p^b}^{I,n+1}), q_h^b) = 0$ for all $q_h^b \in Q_h^b$ because Π_Q^b is the L^2-projection onto Q_h^b, and $b_h(\boldsymbol{v}_h, \boldsymbol{e}_p^{I,n+1}) = 0$ for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$, $b_h^b((w_h, 0), \boldsymbol{e}_{p^p}^{I,n+1}) = 0$ for all $w_h \in V_h^b$, and $b_h^I((\bar{v}_h^f, \bar{v}_h^b), \bar{e}_{p^p}^{I,n+1}) = 0$ for all $(\bar{v}_h^f, \bar{v}_h^b) \in \bar{V}_h^f \times \bar{V}_h^b$ because Π_Q^j, $\bar{\Pi}_Q^j$, $j = f, b$, and $\bar{\Pi}_{Q^0}^b$ are L^2-projections. Furthermore, since $\bar{\Pi}_V^j$, $\bar{\Pi}_Q^j$, $j = f, b$, $\bar{\Pi}_{Q^0}^b$ are L^2-projections,

$$
\begin{aligned}
a_h^I((\bar{e}_{u^f}^{I,n+1}, (\Delta t)^{-1}(\bar{\Pi}_V^b \bar{u}^{b,n+1} - \bar{\Pi}_V^b \bar{u}^{b,n}) - \partial_t \bar{u}^{b,n+1}), (\bar{v}_h^f, \bar{v}_h^b)) &= a_h^I((0, d_t \bar{u}^{b,n+1} - \partial_t \bar{u}^{b,n+1}), (\bar{v}_h^f, \bar{v}_h^b)), \\
b_h^I((\bar{e}_{u^f}^{I,n+1}, (\Delta t)^{-1}(\bar{\Pi}_V^b \bar{u}^{b,n+1} - \bar{\Pi}_V^b \bar{u}^{b,n}) - \partial_t \bar{u}^{b,n+1}), \bar{q}_h^p) &= b_h^I((0, d_t \bar{u}^{b,n+1} - \partial_t \bar{u}^{b,n+1}), \bar{q}_h^p), \\
(c_0((\Delta t)^{-1}(\Pi_Q^b p^{p,n+1} - \Pi_Q^b p^{p,n}) - \partial_t p^{p,n+1}), q_h^p)_{\Omega^b} &= (c_0(d_t p^{p,n+1} - \partial_t p^{p,n+1}), q_h^p)_{\Omega^b}, \\
c_h(((\Delta t)^{-1}(\Pi_Q^b p^{p,n+1} - \Pi_Q^b p^{p,n}) - \partial_t p^{p,n+1}, (\Delta t)^{-1}(\Pi_Q^b p^{b,n+1} &- \Pi_Q^b p^{b,n}) - \partial_t p^{b,n+1}), \alpha q_h^p) \\
&= c_h((d_t p^{p,n+1} - \partial_t p^{p,n+1}, d_t p^{b,n+1} - \partial_t p^{b,n+1}), \alpha q_h^p).
\end{aligned}
$$

Therefore, we can write eq. (37) as

$$
\begin{aligned}
&(d_t e_{u^f}^{h,n+1}, v_h^f)_{\Omega^f} - t_h(u^{f,n+1}; \boldsymbol{u}^{f,n+1}, \boldsymbol{v}_h^f) + t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n+1}, \boldsymbol{v}_h^f) \\
&\quad + a_h(\boldsymbol{e}_u^{h,n+1}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, \boldsymbol{e}_p^{h,n+1}) + a_h^I((\bar{e}_{u^f}^{h,n+1}, d_t \bar{e}_{u^b}^{h,n+1}), (\bar{v}_h^f, \bar{v}_h^b)) + b_h^I((\bar{v}_h^f, \bar{v}_h^b), \bar{e}_{p^p}^{h,n+1}) \\
&\quad = (\partial_t u^{f,n+1} - (\Delta t)^{-1}(\Pi_V^f u^{f,n+1} - \Pi_V^f u^{f,n}), v_h^f)_{\Omega^f} + a_h(\boldsymbol{e}_u^{I,n+1}, \boldsymbol{v}_h) \\
&\qquad + a_h^I((0, \partial_t \bar{u}^{b,n+1} - d_t \bar{u}^{b,n+1})), (\bar{v}_h^f, \bar{v}_h^b)),
\end{aligned} \tag{38a}
$$

$$
b_h(\boldsymbol{e}_u^{h,n+1}, \boldsymbol{q}_h) + c_h((e_{p^p}^{h,n+1}, e_{p^b}^{h,n+1}), q_h^b) = 0, \tag{38b}
$$

$$
\begin{aligned}
&(c_0 d_t e_{p^p}^{h,n+1}, q_h^p)_{\Omega^b} + c_h((d_t e_{p^p}^{h,n+1}, d_t e_{p^b}^{h,n+1}), \alpha q_h^p) - b_h^b((e_z^{h,n+1}, 0), \boldsymbol{q}_h^p) - b_h^I((\bar{e}_{u^f}^{h,n+1}, d_t \bar{e}_{u^b}^{h,n+1}), \bar{q}_h^p) \\
&\quad = (c_0(\partial_t p^{p,n+1} - d_t p^{p,n+1}), q_h^p)_{\Omega^b} + c_h((\partial_t p^{p,n+1} - d_t p^{p,n+1}, \partial_t p^{b,n+1} - d_t p^{b,n+1}), \alpha q_h^p) \\
&\qquad - b_h^I((0, \partial_t \bar{u}^{b,n+1} - d_t \bar{u}^{b,n+1}), \bar{q}_h^p),
\end{aligned} \tag{38c}
$$

$$
(\mu^f \kappa^{-1} e_z^{h,n+1}, w_h)_{\Omega^b} + b_h^b((w_h, 0), \boldsymbol{e}_{p^p}^{h,n+1}) = (\mu^f \kappa^{-1} e_z^{I,n+1}, w_h)_{\Omega^b}. \tag{38d}
$$

To simplify this further, note that

$$
\begin{aligned}
-t_h(u^{f,n+1};\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)+t_h(u_h^{f,n};\boldsymbol{u}_h^{f,n+1},\boldsymbol{v}_h^f)=&-t_h(u^{f,n+1};\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)+t_h(u^{f,n};\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)\\
&-t_h(u^{f,n};\boldsymbol{e}_{u^f}^{I,n+1},\boldsymbol{v}_h^f)-t_h(u^{f,n};\boldsymbol{\Pi}_V^f\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)\\
&+t_h(u_h^{f,n};\boldsymbol{\Pi}_V^f\boldsymbol{u}^{f,n+1},\boldsymbol{v}_h^f)+t_h(u_h^{f,n};\boldsymbol{e}_{u^f}^{h,n+1},\boldsymbol{v}_h^f).
\end{aligned}
\tag{39}
$$

Furthermore, splitting eq. (38b) into its terms on Ω^f and Ω^b and applying the discrete time derivative on Ω^b, we can write eq. (38b) as

$$
b_h^f(\boldsymbol{e}_{u^f}^{h,n+1},\boldsymbol{q}_h^f)+b_h^b(d_t\boldsymbol{e}_{u^b}^{h,n+1},\boldsymbol{q}_h^b)+c_h((d_te_{p^p}^{h,n+1},d_te_{p^b}^{h,n+1}),q_h^b)=0. \tag{40}
$$

Combining eqs. (38a), (38c), (38d), (39) and (40) we find eq. (35). □

The following auxiliary result will be useful to prove the error estimate in Theorem 5.3.

Lemma 5.2. *Let $\boldsymbol{p}_h^{p,n}$, z_h^n, p^p, and z^n be as defined in Lemma 5.1. There exists a $C>0$, independent of h, Δt, and n, such that for $n\ge 0$:*

$$
\|e_{p^p}^{h,n+1}\|_{\Omega^b}\le C\mu^f\kappa^{-1}(\|e_z^{h,n+1}\|_{\Omega^b}+\|e_z^{I,n+1}\|_{\Omega^b}), \tag{41a}
$$
$$
\|\bar{e}_{p^p}^{h,n+1}\|_{\Gamma_I}\le C\mu^f\kappa^{-1}(\|e_z^{h,n+1}\|_{\Omega^b}+\|e_z^{I,n+1}\|_{\Omega^b}). \tag{41b}
$$

Proof. In [21, eq. (42)] we proved that $\|e_{p^p}^{h,n+1}\|_{1,h,\Omega^b}\le C\mu^f\kappa^{-1}\|z-z_h\|_{\Omega^b}$. Equation (41a) now follows by eq. (10d) and a triangle inequality. The proof of eq. (41b) is given by [21, Lemma 4]. □

The main result of this section is the following theorem.

Theorem 5.3. *Suppose that $\{(\boldsymbol{u}_h^n,\boldsymbol{p}_h^n,z_h^n,\boldsymbol{p}_h^{p,n})\}_n$ are the solutions to eq. (17), that the assumptions of Theorem 4.1 hold and that (u,p,z,p^p) is the solution to eqs. (1), (2) and (4) to (6) on the time interval $J=(0,T]$ with $f^b(x,0)=0$, $g^b(x,0)=0$, $u_0^f(x)=0$, and $p_0^p(x)=0$. Furthermore, let $\boldsymbol{u}:=(u,u|_{\Gamma_0^f},u|_{\Gamma_0^b})$, $\boldsymbol{p}:=(p,p|_{\Gamma_0^f},p|_{\Gamma_0^b})$, and $\boldsymbol{p}^p:=(p^p,p^p|_{\Gamma_0^b})$. If*

$$
\begin{aligned}
&u^f\in H^1(J,[H^k(\Omega^f)]^d)\cap H^2(J,[L^2(\Omega^f)]^d)\cap\ell^\infty(J,[W^{1,3}(\Omega^f)\cap H^{k+1}(\Omega^f)]^d),\\
&u^b\in H^2(J,[H^1(\Omega^b)]^d)\cap W^{2,1}(J,[H^{k+1}(\Omega^b)]^d),\\
&\ z\in\ell^\infty(J,H^k(\Omega^b)),\qquad p^p,p^b\in H^2(J,L^2(\Omega^b)),
\end{aligned}
\tag{42}
$$

then,

$$
\begin{aligned}
&\|e_{u^f}^{h,m}\|_{\Omega^f}^2+a_h^b(\boldsymbol{e}_{u^b}^{h,m},\boldsymbol{e}_{u^b}^{h,m})+\lambda^{-1}\|\alpha e_{p^p}^{h,m}-e_{p^b}^{h,m}\|_{\Omega^b}^2+c_0\|e_{p^p}^{h,m}\|_{\Omega^b}^2\\
&\quad+\Delta t\sum_{i=1}^m[\mu^f|||\boldsymbol{e}_{u^f}^{h,i}|||_{v,f}^2+\gamma\mu^f\kappa^{-1/2}\|\alpha\bar{e}_{u^f}^{h,i}-d_t\bar{e}_{u^b}^{h,i}\|_{\Gamma_I}^2+\mu^f\kappa^{-1}\|e_z^{h,i}\|_{\Omega^b}^2]\le C_G\left[h^{2k}+(\Delta t)^2\right],
\end{aligned}
\tag{43}
$$

with C_G a constant resulting from a discrete Grönwall inequality that depends on T and the norms of u^f, u^b, z, p^p, and p^b in eq. (42). Moreover,

$$
|||\boldsymbol{e}_{p^b}^{h,m}|||_{q,b}\le C(\mu^b)^{1/2}a_h^b(\boldsymbol{e}_{u^b}^{h,m},\boldsymbol{e}_{u^b}^{h,m})^{1/2}+C\mu^bh^k\|u^{b,m}\|_{k+1,\Omega^b}. \tag{44}
$$

Proof. Set $\boldsymbol{v}_h^f = \boldsymbol{e}_{u^f}^{h,n+1}$, $\boldsymbol{v}_h^b = d_t\boldsymbol{e}_{u^b}^{h,n+1}$, $\boldsymbol{q}_h^f = -\boldsymbol{e}_{p^f}^{h,n+1}$, $\boldsymbol{q}_h^b = -\boldsymbol{e}_{p^b}^{h,n+1}$, $w_h = e_z^{h,n+1}$, $\boldsymbol{q}_h^p = \boldsymbol{e}_{p^p}^{h,n+1}$ in eq. (35), sum the equations, use the algebraic inequality $a(a-b) \geq (a^2-b^2)/2$ for the time-derivative terms, and use eq. (15), to obtain:

$$
\begin{aligned}
&\tfrac{1}{2\Delta t}(\|e_{u^f}^{h,n+1}\|_{\Omega^f}^2 - \|e_{u^f}^{h,n}\|_{\Omega^f}^2) + \tfrac{1}{2}c_{ae}^f\mu^f|||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f}^2 + \tfrac{1}{2\Delta t}(a_h^b(\boldsymbol{e}_{u^b}^{h,n+1},\boldsymbol{e}_{u^b}^{h,n+1}) - a_h^b(\boldsymbol{e}_{u^b}^{h,n},\boldsymbol{e}_{u^b}^{h,n}))\\
&\qquad + \gamma\mu^f\kappa^{-1/2}\|(\bar{e}_{u^f}^{h,n+1} - d_t\bar{e}_{u^b}^{h,n+1})^t\|_{\Gamma_I}^2 + \tfrac{\lambda^{-1}}{2\Delta t}(\|\alpha e_{p^p}^{h,n+1} - e_{p^b}^{h,n+1}\|_{\Omega^b}^2 - \|\alpha e_{p^p}^{h,n} - e_{p^b}^{h,n}\|_{\Omega^b}^2)\\
&\qquad + \tfrac{c_0}{2\Delta t}(\|e_{p^p}^{h,n+1}\|_{\Omega^b}^2 - \|e_{p^p}^{h,n}\|_{\Omega^b}^2) + \mu^f\kappa^{-1}\|e_z^{h,n+1}\|_{\Omega^b}^2\\
&\leq I_1^n + I_2^n + I_3^n + I_4^n,
\end{aligned}
\tag{45}
$$

where $e_{u^f}^n = u^{f,n} - u_h^{f,n}$ and where

$$
\begin{aligned}
I_1^n &:= I_{1a}^n + I_{1b}^n + I_{1c}^n + I_{1d}^n + I_{1e}^n + I_{1f}^n\\
&:= (\partial_t u^{f,n+1} - (\Delta t)^{-1}(\Pi_V^f u^{f,n+1} - \Pi_V^f u^{f,n}), e_{u^f}^{h,n+1})_{\Omega^f}\\
&\quad + [t_h(u^{f,n+1};\boldsymbol{u}^{f,n+1},\boldsymbol{e}_{u^f}^{h,n+1}) - t_h(u^{f,n};\boldsymbol{u}^{f,n+1},\boldsymbol{e}_{u^f}^{h,n+1})]\\
&\quad + t_h(u^{f,n};\boldsymbol{e}_{u^f}^{I,n+1},\boldsymbol{e}_{u^f}^{h,n+1}) + [t_h(u^{f,n};\boldsymbol{\Pi}_V^f\boldsymbol{u}^{f,n+1},\boldsymbol{e}_{u^f}^{h,n+1}) - t_h(u_h^{f,n};\boldsymbol{\Pi}_V^f\boldsymbol{u}^{f,n+1},\boldsymbol{e}_{u^f}^{h,n+1})]\\
&\quad + a_h^f(\boldsymbol{e}_{u^f}^{I,n+1},\boldsymbol{e}_{u^f}^{h,n+1}) + a_h^b(\boldsymbol{e}_{u^b}^{I,n+1}, d_t\boldsymbol{e}_{u^b}^{h,n+1}),\\
I_2^n &:= a_h^I((0,\partial_t\bar{u}^{b,n+1} - d_t\bar{u}^{b,n+1}),(\bar{e}_{u^f}^{h,n+1}, d_t\bar{e}_{u^b}^{h,n+1})),\\
I_3^n &:= I_{3a}^n + I_{3b}^n + I_{3c}^n\\
&:= (c_0(\partial_t p^{p,n+1} - d_t p^{p,n+1}), e_{p^p}^{h,n+1})_{\Omega^b}\\
&\qquad + c_h((\partial_t p^{p,n+1} - d_t p^{p,n+1}, \partial_t p^{b,n+1} - d_t p^{b,n+1}), \alpha e_{p^p}^{h,n+1})\\
&\qquad - b_h^I((0,\partial_t\bar{u}^{b,n+1} - d_t\bar{u}^{b,n+1}), \bar{e}_{p^p}^{h,n+1}),\\
I_4^n &:= (\mu^f\kappa^{-1}e_z^{I,n+1}, e_z^{h,n+1})_{\Omega^b}.
\end{aligned}
$$

By definition of I_{1a}^n, the Cauchy–Schwarz inequality, eq. (10a), the triangle inequality, approximation properties of the BDM interpolation operator (Π_V^f), Taylor's theorem, and Young's inequality,

$$
\begin{aligned}
|I_{1a}^n| &\leq C\,\|-d_t e_{u^f}^{I,n+1} + d_t u^{f,n+1} - \partial_t u^{f,n+1}\|_{\Omega^f}\,|||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f}\\
&\leq 2\psi|||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f}^2 + \tfrac{C}{\psi}h^{2k}(\Delta t)^{-1}\|\partial_t u^f\|_{L^2(t_n,t_{n+1};H^k(\Omega^f))}^2 + \tfrac{C}{\psi}\Delta t\,\|\partial_{tt}u^f\|_{L^2(t_n,t_{n+1};L^2(\Omega^f))}^2,
\end{aligned}
\tag{46}
$$

where $\psi > 0$ will be chosen later. Next, by eq. (13) and Young's inequality,

$$
\begin{aligned}
|I_{1b}^n| &\leq C\,\|u^{f,n+1} - u^{f,n}\|_{1,h,\Omega^f}\,\|u^{f,n+1}\|_{1,\Omega^f}\,|||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f}\\
&\leq C(\Delta t)^{1/2}\,\|\partial_t u^f\|_{L^2(t_n,t_{n+1};H^1(\Omega^f))}\,\|u^{f,n+1}\|_{1,\Omega^f}\,|||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f}\\
&\leq \psi|||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f}^2 + \frac{C}{\psi}\Delta t\,\|\partial_t u^f\|_{L^2(t_n,t_{n+1};H^1(\Omega^f))}^2\,\|u^{f,n+1}\|_{1,\Omega^f}^2.
\end{aligned}
\tag{47}
$$

By eq. (13), approximation properties of the BDM interpolation operator (Π_V^f) and facet L^2-projection ($\bar{\Pi}_V^f$), and Young's inequality,

$$\begin{aligned} |I_{1c}^n| &\leq C \left\| u^{f,n} \right\|_{1,h,\Omega^f} |||\boldsymbol{e}_{u^f}^{I,n+1}|||_{v,f} |||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f} \\ &\leq Ch^k \left\| u^{f,n} \right\|_{1,\Omega^f} \left\| u^{f,n+1} \right\|_{k+1,\Omega^f} |||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f} \\ &\leq \psi |||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f}^2 + \frac{C}{\psi} h^{2k} \left\| u^{f,n} \right\|_{1,\Omega^f}^2 \left\| u^{f,n+1} \right\|_{k+1,\Omega^f}^2 . \end{aligned} \tag{48}$$

For I_{1d}^n we have for $\psi > 0$ (see [22, Appendix C]):

$$|I_{1d}^n| \leq 2\psi |||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f}^2 + \frac{C}{\psi} h^{2k} \left\| u^{f,n+1} \right\|_{k+1,\Omega^f}^2 \left\| u^{f,n} \right\|_{k+1,\Omega^f}^2 + \frac{C}{\psi} \left\| e_{u^f}^{h,n} \right\|_{\Omega^f}^2 \left\| u^{f,n+1} \right\|_{W_3^1(\Omega^f)}^2 . \tag{49}$$

For I_{1e}^n, using eq. (11b), Young's inequalities, and interpolation properties, we find

$$|I_{1e}^n| \leq C\mu^f h^{2k} \left\| u^{f,n+1} \right\|_{k+1,\Omega^f}^2 + \tfrac{1}{8} c_{ae}^f \mu^f |||\boldsymbol{e}_{u^f}^{h,n+1}|||_{v,f}^2 . \tag{50}$$

We postpone estimating I_{1f}^n until later and proceed with estimating I_2^n. By the Cauchy–Schwarz inequality, Young's inequality, and the trace inequality,

$$\begin{aligned} |I_2^n| &\leq \gamma\mu^f \kappa^{-1/2} \left\| d_t \bar{u}^{b,n+1} - \partial_t \bar{u}^{b,n+1} \right\|_{\Gamma_I} \left\| \bar{e}_{u^f}^{h,n+1} - d_t \bar{e}_{u^b}^{h,n+1} \right\|_{\Gamma_I} \\ &\leq \tfrac{1}{2}\gamma\mu^f \kappa^{-1/2} \left\| \bar{e}_{u^f}^{h,n+1} - d_t \bar{e}_{u^b}^{h,n+1} \right\|_{\Gamma_I}^2 + \tfrac{1}{2}\gamma\mu^f \kappa^{-1/2} \left\| d_t \bar{u}^{b,n+1} - \partial_t \bar{u}^{b,n+1} \right\|_{\Gamma_I}^2 \\ &\leq \tfrac{1}{2}\gamma\mu^f \kappa^{-1/2} \left\| \bar{e}_{u^f}^{h,n+1} - d_t \bar{e}_{u^b}^{h,n+1} \right\|_{\Gamma_I}^2 + C\mu^f \kappa^{-1/2} \Delta t \left\| \partial_{tt} \bar{u}^b \right\|_{L^2(t_n,t_{n+1};L^2(\Gamma_I))}^2 \\ &\leq \tfrac{1}{2}\gamma\mu^f \kappa^{-1/2} \left\| \bar{e}_{u^f}^{h,n+1} - d_t \bar{e}_{u^b}^{h,n+1} \right\|_{\Gamma_I}^2 + C\mu^f \kappa^{-1/2} \Delta t \left\| \partial_{tt} u^b \right\|_{L^2(t_n,t_{n+1};H^1(\Omega^b))}^2 . \end{aligned} \tag{51}$$

By the Cauchy–Schwarz inequality, Lemma 5.2, Young's inequality, and an argument similar to the estimate in eq. (46),

$$\begin{aligned} |I_{3a}^n| &\leq c_0 \left\| d_t p^{p,n+1} - \partial_t p^{p,n+1} \right\|_{\Omega^b} \left\| e_{p^p}^{h,n+1} \right\|_{\Omega^b} \\ &\leq c_0 \mu^f \kappa^{-1} C \left\| d_t p^{p,n+1} - \partial_t p^{p,n+1} \right\|_{\Omega^b} (\left\| e_z^{h,n+1} \right\|_{\Omega^b} + \left\| e_z^{I,n+1} \right\|_{\Omega^b}) \\ &\leq \tfrac{1}{5}\mu^f \kappa^{-1} \left\| e_z^{h,n+1} \right\|_{\Omega^b}^2 + C\mu^f \kappa^{-1} h^{2k} \left\| z^{n+1} \right\|_{H^k(\Omega^b)}^2 + Cc_0^2 \mu^f \kappa^{-1} \Delta t \left\| \partial_{tt} p^p \right\|_{L^2(t_n,t_{n+1};L^2(\Omega^b))}^2 . \end{aligned} \tag{52}$$

Likewise, and using that $0 < \alpha \leq 1$, we find for I_{3b}^n,

$$\begin{aligned} |I_{3b}^n| &\leq \lambda^{-1} \left\| \alpha(d_t p^{p,n+1} - \partial_t p^{p,n+1}) - (d_t p^{b,n+1} - \partial_t p^{b,n+1}) \right\|_{\Omega^b} \left\| e_{p^p}^{h,n+1} \right\|_{\Omega^b} \\ &\leq \tfrac{1}{5}\mu^f \kappa^{-1} \left\| e_z^{h,n+1} \right\|_{\Omega^b}^2 + C\mu^f \kappa^{-1} h^{2k} \left\| z^{n+1} \right\|_{H^k(\Omega^b)}^2 \\ &\quad + C\lambda^{-2} \mu^f \kappa^{-1} \Delta t (\left\| \partial_{tt} p^p \right\|_{L^2(t_n,t_{n+1};L^2(\Omega^b))}^2 + \left\| \partial_{tt} p^b \right\|_{L^2(t_n,t_{n+1};L^2(\Omega^b))}^2). \end{aligned} \tag{53}$$

For I_{3c}^n, by eq. (41b) and an argument similar to the estimate eq. (51),

$$|I_{3c}^n| \leq \tfrac{1}{5}\mu^f \kappa^{-1} \left\| e_z^{h,n+1} \right\|_{\Omega^b}^2 + C\mu^f \kappa^{-1} h^{2k} \left\| z^{n+1} \right\|_{H^k(\Omega^b)}^2 + C\mu^f \kappa^{-1} \Delta t \left\| \partial_{tt} u^b \right\|_{L^2(t_n,t_{n+1};H^1(\Omega^b))}^2 . \tag{54}$$

For I_4^n, using the Cauchy–Schwarz inequality, Young's inequality, and the approximation properties of the BDM interpolant, we find

$$|I_4^n| \leq \tfrac{1}{5}\mu^f \kappa^{-1} \left\| e_z^{h,n+1} \right\|_{\Omega^b}^2 + C\mu^f \kappa^{-1} h^{2k} \left\| z^{n+1} \right\|_{H^k(\Omega^b)}^2 . \tag{55}$$

Combining the estimates in eqs. (46) to (55) with eq. (45), choosing $\psi = c^f_{ae}\mu^f/24$, summing for $n = 0$ to $n = m-1$, multiplying both sides of the inequality by Δt, and taking into account the vanishing initial data, we find:

$$
\begin{aligned}
&\tfrac{1}{2}\left\|e^{h,m}_{u^f}\right\|^2_{\Omega^f} + \tfrac{\Delta t}{8} c^f_{ae}\mu^f \sum_{i=1}^{m} \|\!|\boldsymbol{e}^{h,i}_{u^f}|\!\|^2_{v,f} + \tfrac{1}{2} a^b_h(\boldsymbol{e}^{h,m}_{u^b}, \boldsymbol{e}^{h,m}_{u^b}) + \tfrac{\Delta t}{2}\gamma\mu^f\kappa^{-1/2}\sum_{i=1}^{m}\left\|(\bar{e}^{h,i}_{u^f} - d_t\bar{e}^{h,i}_{u^b})^t\right\|^2_{\Gamma_I} \\
&\quad + \tfrac{\lambda^{-1}}{2}\left\|\alpha e^{h,m}_{p^p} - e^{h,m}_{p^b}\right\|^2_{\Omega^b} + \tfrac{c_0}{2}\left\|e^{h,m}_{p^p}\right\|^2_{\Omega^b} + \tfrac{\Delta t}{5}\mu^f\kappa^{-1}\sum_{i=1}^{m}\left\|e^{h,i}_{z}\right\|^2_{\Omega^b} \\
&\le C(\mu^f)^{-1}h^{2k}\left\|\partial_t u^f\right\|^2_{L^2(0,t_m;H^k(\Omega^f))} + C(\mu^f)^{-1}(\Delta t)^2\left\|\partial_{tt}u^f\right\|^2_{L^2(0,t_m;L^2(\Omega^f))} \\
&\quad + C(\mu^f)^{-1}(\Delta t)^2\sum_{i=1}^{m}\left\|\partial_t u^f\right\|^2_{L^2(t_{i-1},t_i;H^1(\Omega^f))}\left\|u^{f,i}\right\|^2_{1,\Omega^f} \\
&\quad + C(\mu^f)^{-1}\Delta t h^{2k}\sum_{i=1}^{m}((\mu^f)^2 + \left\|u^{f,i-1}\right\|^2_{1,\Omega^f} + \left\|u^{f,i-1}\right\|^2_{k+1,\Omega^f})\left\|u^{f,i}\right\|^2_{k+1,\Omega^f} + \Delta t\sum_{i=1}^{m} I^{i-1}_{1f} \\
&\quad + C(\mu^f)^{-1}\Delta t\sum_{i=1}^{m}\left\|e^{h,i-1}_{u^f}\right\|^2_{\Omega^f}\left\|u^{f,i}\right\|^2_{W^1_3(\Omega^f)} + C\mu^f\kappa^{-1/2}(\Delta t)^2\left\|\partial_{tt}u^b\right\|^2_{L^2(0,t_m;H^1(\Omega^b))} \\
&\quad + C\mu^f\kappa^{-1}\Delta t h^{2k}\sum_{i=1}^{m}\left\|z^i\right\|^2_{H^k(\Omega^b)} + Cc_0^2\mu^f\kappa^{-1}(\Delta t)^2\left\|\partial_{tt}p^p\right\|^2_{L^2(0,t_m;L^2(\Omega^b))} \\
&\quad + C\lambda^{-2}\mu^f\kappa^{-1}(\Delta t)^2(\left\|\partial_{tt}p^p\right\|^2_{L^2(0,t_m;L^2(\Omega^b))} + \left\|\partial_{tt}p^b\right\|^2_{L^2(0,t_m;L^2(\Omega^b))}) \\
&\quad + C\mu^f\kappa^{-1}(\Delta t)^2\left\|\partial_{tt}u^b\right\|^2_{L^2(0,t_m;H^1(\Omega^b))}.
\end{aligned}
\tag{56}
$$

Let us now consider the term I_{1f}. Using summation-by-parts and that the initial data vanishes,

$$
\Delta t\sum_{i=1}^{m} I^{i-1}_{1f} = a^b_h(\boldsymbol{e}^{I,m}_{u^b}, \boldsymbol{e}^{h,m}_{u^b}) + \sum_{i=2}^{m} a^b_h(\boldsymbol{e}^{I,i-1}_{u^b} - \boldsymbol{e}^{I,i}_{u^b}, \boldsymbol{e}^{h,i-1}_{u^b}).
$$

Note also that, by the Cauchy–Schwarz and Young's inequalities, eq. (11b), and approximation properties of the BDM interpolant,

$$
\begin{aligned}
a^b_h(\boldsymbol{e}^{I,i-1}_{u^b} - \boldsymbol{e}^{I,i}_{u^b}, \boldsymbol{e}^{h,i-1}_{u^b}) &\le a^b_h(\boldsymbol{e}^{I,i-1}_{u^b} - \boldsymbol{e}^{I,i}_{u^b}, \boldsymbol{e}^{I,i-1}_{u^b} - \boldsymbol{e}^{I,i}_{u^b})^{1/2} a^b_h(\boldsymbol{e}^{h,i-1}_{u^b}, \boldsymbol{e}^{h,i-1}_{u^b})^{1/2} \\
&\le (\Delta t)^{-1} a^b_h(\boldsymbol{e}^{I,i-1}_{u^b} - \boldsymbol{e}^{I,i}_{u^b}, \boldsymbol{e}^{I,i-1}_{u^b} - \boldsymbol{e}^{I,i}_{u^b}) + \tfrac{\Delta t}{4} a^b_h(\boldsymbol{e}^{h,i-1}_{u^b}, \boldsymbol{e}^{h,i-1}_{u^b}) \\
&\le (\Delta t)^{-1} c^b_{ac}\mu^b \|\!|\boldsymbol{e}^{I,i-1}_{u^b} - \boldsymbol{e}^{I,i}_{u^b}|\!\|^2_{v',j} + \tfrac{\Delta t}{4} a^b_h(\boldsymbol{e}^{h,i-1}_{u^b}, \boldsymbol{e}^{h,i-1}_{u^b}) \\
&\le (\Delta t)^{-1} C\mu^b h^{2k}\left\|u^{b,i-1} - u^{b,i}\right\|^2_{k+1,\Omega^b} + \tfrac{\Delta t}{4} a^b_h(\boldsymbol{e}^{h,i-1}_{u^b}, \boldsymbol{e}^{h,i-1}_{u^b}) \\
&\le C\mu^b h^{2k}\left\|\partial_t u^b\right\|^2_{L^2(t_{i-1},t_i;H^{k+1}(\Omega^b))} + \tfrac{\Delta t}{4} a^b_h(\boldsymbol{e}^{h,i-1}_{u^b}, \boldsymbol{e}^{h,i-1}_{u^b}),
\end{aligned}
$$

and, similarly,

$$
a^b_h(\boldsymbol{e}^{I,m}_{u^b}, \boldsymbol{e}^{h,m}_{u^b}) \le C\mu^b h^{2k}\left\|u^{b,m}\right\|^2_{k+1,\Omega^b} + \tfrac{1}{4} a^b_h(\boldsymbol{e}^{h,m}_{u^b}, \boldsymbol{e}^{h,m}_{u^b}).
$$

The above inequalities together with eq. (56) result in:

$$
\begin{aligned}
&\tfrac{1}{2}\left\|e^{h,m}_{u^f}\right\|^2_{\Omega^f} + \tfrac{\Delta t}{8} c^f_{ae}\mu^f \sum_{i=1}^{m} \vert\vert\vert \boldsymbol{e}^{h,i}_{u^f} \vert\vert\vert^2_{v,f} + \tfrac{1}{4} a^b_h(\boldsymbol{e}^{h,m}_{u^b}, \boldsymbol{e}^{h,m}_{u^b}) + \tfrac{\Delta t}{2}\gamma\mu^f\kappa^{-1/2} \sum_{i=1}^{m} \left\|(\bar{e}^{h,i}_{u^f} - d_t \bar{e}^{h,i}_{u^b})^t\right\|^2_{\Gamma_I} \\
&\quad + \tfrac{\lambda^{-1}}{2}\left\|\alpha e^{h,m}_{p^p} - e^{h,m}_{p^b}\right\|^2_{\Omega^b} + \tfrac{c_0}{2}\left\|e^{h,m}_{p^p}\right\|^2_{\Omega^b} + \tfrac{\Delta t}{5}\mu^f\kappa^{-1}\sum_{i=1}^{m}\left\|e^{h,i}_{z}\right\|^2_{\Omega^b} \\
&\le C(\mu^f)^{-1}h^{2k}\left\|\partial_t u^f\right\|^2_{L^2(0,t_m;H^k(\Omega^f))} + C(\mu^f)^{-1}(\Delta t)^2\left\|\partial_{tt}u^f\right\|^2_{L^2(0,t_m;L^2(\Omega^f))} \\
&\quad + CT(\mu^f)^{-1}(\Delta t)^2\left\|\partial_t u^f\right\|^2_{L^2(0,t_m;H^1(\Omega^f))}\left\|u^f\right\|^2_{\ell^\infty(0,t_m;H^1(\Omega^f))} \\
&\quad + CT(\mu^f)^{-1}h^{2k}((\mu^f)^2 + \left\|u^f\right\|^2_{\ell^\infty(0,t_m;H^1(\Omega^f))} + \left\|u^f\right\|^2_{\ell^\infty(0,t_m;H^{k+1}(\Omega^f))})\left\|u^f\right\|^2_{\ell^\infty(0,t_m;H^{k+1}(\Omega^f))} \\
&\quad + C\mu^b h^{2k}\left\|\partial_t u^b\right\|^2_{L^2(0,t_m;H^{k+1}(\Omega^b))} + C\mu^b h^{2k}\left\|u^b\right\|^2_{\ell^\infty(0,t_m;H^{k+1}(\Omega^b))} \\
&\quad + C(\mu^f)^{-1}\Delta t\sum_{i=1}^{m}\left\|e^{h,i-1}_{u^f}\right\|^2_{\Omega^f}\left\|u^{f,i}\right\|^2_{W^1_3(\Omega^f)} + C\mu^f\kappa^{-1/2}(\Delta t)^2\left\|\partial_{tt}u^b\right\|^2_{L^2(0,t_m;H^1(\Omega^b))} \\
&\quad + \tfrac{\Delta t}{4}\sum_{i=2}^{m} a^b_h(\boldsymbol{e}^{h,i-1}_{u^b}, \boldsymbol{e}^{h,i-1}_{u^b}) + C\mu^f\kappa^{-1}Th^{2k}\left\|z\right\|^2_{\ell^\infty(0,t_m;H^k(\Omega^b))} \\
&\quad + Cc_0^2\mu^f\kappa^{-1}(\Delta t)^2\left\|\partial_{tt}p^p\right\|^2_{L^2(0,t_m;L^2(\Omega^b))} + C\mu^f\kappa^{-1}(\Delta t)^2\left\|\partial_{tt}u^b\right\|^2_{L^2(0,t_m;H^1(\Omega^b))} \\
&\quad + C\lambda^{-2}\mu^f\kappa^{-1}(\Delta t)^2(\left\|\partial_{tt}p^p\right\|^2_{L^2(0,t_m;L^2(\Omega^b))} + \left\|\partial_{tt}p^b\right\|^2_{L^2(0,t_m;L^2(\Omega^b))}).
\end{aligned}
$$

Equation (43) follows by a discrete Grönwall inequality (see, e.g., [33, Lemma 28]).

We next prove eq. (44). For this, let us first note that by eq. (12a) there exists a $\tilde{\boldsymbol{v}}^b_h \in \tilde{\boldsymbol{V}}^b_h$, with $\tilde{\boldsymbol{V}}^b_h$ defined in eq. (9), such that $b^b_h(\tilde{\boldsymbol{v}}^b_h, \boldsymbol{e}^{h,n+1}_{p^b}) = \vert\vert\vert \boldsymbol{e}^{h,n+1}_{p^b} \vert\vert\vert^2_{q,b}$ and $\vert\vert\vert \tilde{\boldsymbol{v}}^b_h \vert\vert\vert_{v,b} \le C \vert\vert\vert \boldsymbol{e}^{h,n+1}_{p^b} \vert\vert\vert_{q,b}$. Take $\boldsymbol{v}^f_h = \boldsymbol{0}$ and $\boldsymbol{v}^b_h = \tilde{\boldsymbol{v}}^b_h$ in eq. (35a) which then reduces to $a^b_h(\boldsymbol{e}^{h,n+1}_{u^b}, \tilde{\boldsymbol{v}}^b_h) + \vert\vert\vert \boldsymbol{e}^{h,n+1}_{p^b} \vert\vert\vert^2_{q,b} = a^b_h(\boldsymbol{e}^{I,n+1}_{u^b}, \tilde{\boldsymbol{v}}^b_h)$. By eq. (11b) we obtain:

$$
\vert\vert\vert \boldsymbol{e}^{h,n+1}_{p^b} \vert\vert\vert^2_{q,b} \le |a^b_h(\boldsymbol{e}^{h,n+1}_{u^b}, \tilde{\boldsymbol{v}}^b_h)| + |a^b_h(\boldsymbol{e}^{I,n+1}_{u^b}, \tilde{\boldsymbol{v}}^b_h)| \le C\mu^b\left(\vert\vert\vert \boldsymbol{e}^{h,n+1}_{u^b} \vert\vert\vert_{v,b} + \vert\vert\vert \boldsymbol{e}^{I,n+1}_{u^b} \vert\vert\vert_{v',b}\right)\vert\vert\vert \boldsymbol{e}^{h,n+1}_{p^b} \vert\vert\vert_{q,b},
$$

so that eq. (44) follows by using eq. (11a) and approximation properties of the interpolant. □

An immediate consequence of Theorem 5.3, the triangle inequality, and approximation properties of the different interpolants is the following corollary.

Corollary 5.4. *Suppose that all the assumptions in Theorem 5.3 hold. Then:*

$$
\begin{aligned}
&\left\|u^{f,m} - u^{f,m}_h\right\|^2_{\Omega^f} + \mu^b \vert\vert\vert \boldsymbol{u}^{b,m} - \boldsymbol{u}^{b,m}_h \vert\vert\vert^2_{v,b} + \lambda^{-1}\left\|\alpha(p^{p,m} - p^{p,m}_h) - (p^{b,m} - p^{b,m}_h)\right\|^2_{\Omega^b} \\
&+ c_0\left\|p^{p,m} - p^{p,m}_h\right\|^2_{\Omega^b} + \vert\vert\vert \boldsymbol{p}^{b,m} - \boldsymbol{p}^{b,m}_h \vert\vert\vert^2_{q,b} \\
&+ \Delta t\sum_{i=1}^{m}\left[\mu^f \vert\vert\vert \boldsymbol{u}^{f,i} - \boldsymbol{u}^{f,i}_h \vert\vert\vert^2_{v,f} + \gamma\mu^f\kappa^{-1/2}\left\|\alpha(\bar{u}^{f,i} - \bar{u}^{f,i}_h) - d_t(\bar{u}^{b,i} - \bar{u}^{b,i}_h)\right\|^2_{\Gamma_I} + \mu^f\kappa^{-1}\left\|z^i - z^i_h\right\|^2_{\Omega^b}\right] \\
&\le C'_G\left[h^{2k} + (\Delta t)^2\right],
\end{aligned}
$$

with C'_G depending on C_G (see Theorem 5.3), the norms of the exact solutions, the constants of the approximation properties of the interpolation operators Π^j_V, $\bar{\Pi}^j_V$, Π^j_Q, $\bar{\Pi}^j_Q$, and the different model parameters.

6. Numerical example

In this final section, we present a numerical example to confirm our analysis. For this, we consider the time-dependent manufactured solution of [21, Section 6.2]. We consider the domain $\Omega := (0,1)^2$ with $\Omega^f := (0,1)\times(0.5,1)$ and $\Omega^b := (0,1)\times(0,0.5)$. The boundaries of the domain are defined as:

$$\Gamma_D^f := \{x \in \Gamma^f \ : \ x_1 = 0 \text{ or } x_2 = 1\}, \qquad \Gamma_N^f := \{x \in \Gamma^f \ : \ x_1 = 1\},$$
$$\Gamma_P^b = \Gamma_D^b := \{x \in \Gamma^b \ : \ x_0 = 0 \text{ or } x_2 = 0\}, \qquad \Gamma_F^b = \Gamma_N^b := \{x \in \Gamma^b \ : \ x_1 = 1\}.$$

We consider the Navier–Stokes/Biot problem eqs. (1) and (2) with boundary conditions

$$\begin{aligned} u^f &= U^f \quad \text{on } \Gamma_D^f \times J, & u^b &= U^b \quad \text{on } \Gamma_D^b \times J, & p^p &= P^p \quad \text{on } \Gamma_P^b \times J,\\ \sigma^f n &= S^f \quad \text{on } \Gamma_N^f \times J, & \sigma^b n &= S^b \quad \text{on } \Gamma_N^b \times J, & z\cdot n &= Z^d \quad \text{on } \Gamma_F^b \times J, \end{aligned}$$

and interface conditions

$$\begin{aligned} u^f\cdot n &= (\partial_t u^b + z)\cdot n + M^u && \text{on } \Gamma_I \times J,\\ \sigma^f n &= \sigma^b n + M^s && \text{on } \Gamma_I \times J,\\ (\sigma^f n)\cdot n &= p^p + M^p && \text{on } \Gamma_I \times J,\\ -2\mu^f\left(\varepsilon(u^f)n\right)^t &= \gamma\mu^f\kappa^{-1/2}(u^f - \partial_t u^b)^t + M^e && \text{on } \Gamma_I \times J. \end{aligned}$$

The functions M^u, M^s, M^p, and M^e in the aforementioned modified interface conditions, as well as the boundary data, U^f, U^b, P^p, S^f, S^b, and Z^d, body forces f^f and f^b, source/sink term g^b, and the initial conditions are chosen such that the exact solution is given by

$$u^f = \begin{bmatrix} \pi x_1\cos(\pi(x_1x_2 - t)) + 1\\ -\pi x_2\cos(\pi(x_1x_2 - t)) + 2x_1\end{bmatrix}, \qquad u^b = \begin{bmatrix}\sin(10\pi t)\cos(4(x_1 - t))\cos(3x_2)\\ \sin(10\pi t)\sin(5x_1)\cos(2(x_2 - t))\end{bmatrix},$$
$$p^f = \sin(3x_1)\cos(4(x_2 - t)), \qquad p^p = \sin(3(x_1x_2 - t)).$$

The model parameters are chosen as follows: $\mu^f = 10^{-2}$, $\mu^b = 10^{-3}$, $\alpha = 0.2$, $\lambda = 10^2$, $\kappa = 10^{-2}$, $c_0 = 10^{-2}$, and $\gamma = 0.3$, while the HDG penalty parameters are chosen as $\beta^f = \beta^b = 8k^2$, where k is the polynomial degree. We consider the time interval $J = [0, 0.01]$ and implement the HDG method in the Netgen/NGSolve finite element library [47,48].

We first consider the spatial rates of convergence. For this, we compute the solution using $k = 1$ and $k = 2$ and list the errors measured in the L^2-norm and rates of convergence of the unknowns in Ω^f in table 1 and in Ω^b in table 1. We use a time step of $\Delta t = \frac{1}{10}h^{k+2}$. From both tables we observe that $\|u_h^f - u^f\|_{\Omega^f}$, $\|u_h^b - u^b\|_{\Omega^b}$, and $\|z_h - z\|_{\Omega^b}$ are $\mathcal{O}(h^{k+1})$ while $\|p_h^f - p^f\|_{\Omega^f}$, $\|p_h^b - p^b\|_{\Omega^b}$, and $\|p_h^p - p^p\|_{\Omega^b}$ are $\mathcal{O}(h^k)$.

We next consider the temporal rates of convergence. The errors, measured in the L^2-norm, and temporal rates of convergence for the unknowns in Ω^f are given in table 3 and in Ω^b are given in table 4. To compute these results we choose $k = 4$ and compute on the solution on a mesh consisting of 37548 cells. We observe that the error for all unknowns is $\mathcal{O}(\Delta t)$.

7. Conclusions

In this paper we introduced and analyzed an HDG discretization for the time-dependent Navier–Stokes equations coupled to the Biot equations. Appealing properties of the discretization include that the velocities and displacement are divergence-conforming, that the compressibility equations are satisfied pointwise on the elements, and that mass is conserved pointwise for the semi-discrete problem when source and sink terms are

Cells	$\|u_h^f - u^f\|_{\Omega^f}$	r	$\|p_h^f - p^f\|_{\Omega^f}$	r	$\|\nabla \cdot u_h^f\|_{\Omega^f}$
			$k=1$		
8	2.4e-01	-	6.0e-01	-	6.3e-16
28	5.8e-02	2.0	1.2e-01	2.4	5.3e-16
152	1.5e-02	2.0	4.7e-02	1.3	9.6e-16
576	3.2e-03	2.2	2.2e-02	1.1	8.5e-16
2348	6.8e-04	2.2	1.1e-02	1.1	8.4e-16
			$k=2$		
8	5.8e-02	-	1.1e-01	-	1.1e-15
28	9.6e-03	2.6	2.6e-02	2.0	1.2e-15
152	1.5e-03	2.7	4.7e-03	2.5	1.5e-15
576	1.6e-04	3.2	9.5e-04	2.3	1.3e-15
2348	1.5e-05	3.4	2.2e-04	2.1	1.2e-15

TABLE 1. Errors and spatial rates of convergence r for the solution in Ω^f for the test case described in section 6.

Cells	$\|u_h^b - u^b\|_{\Omega^b}$	r	$\|p_h^b - p^b\|_{\Omega^b}$	r	$\|z_h - z\|_{\Omega^b}$	r	$\|p_h^p - p^p\|_{\Omega^b}$	r
				$k=1$				
8	6.9e-02	-	3.2e+01	-	1.6e-01	-	1.4e-01	-
28	1.8e-02	2.0	1.6e+01	0.9	4.0e-02	2.0	6.8e-02	1.0
152	2.8e-03	2.6	6.6e+00	1.3	6.6e-03	2.6	2.9e-02	1.2
576	7.2e-04	2.0	3.4e+00	1.0	1.9e-03	1.8	1.5e-02	1.0
2348	1.7e-04	2.1	1.7e+00	1.0	5.2e-04	1.9	7.3e-03	1.0
				$k=2$				
8	9.7e-03	-	6.8e+00	-	4.5e-02	-	2.0e-02	-
28	2.2e-03	2.1	2.3e+00	1.5	4.9e-03	3.2	5.3e-03	1.9
152	1.9e-04	3.6	4.4e-01	2.4	4.7e-04	3.4	1.2e-03	2.2
576	2.2e-05	3.1	1.0e-01	2.1	1.0e-04	2.2	2.8e-04	2.1
2348	2.3e-06	3.2	2.6e-02	2.0	1.7e-05	2.7	6.6e-05	2.1

TABLE 2. Errors and spatial rates of convergence r for the solution in Ω^b for the test case described in section 6.

Δt	$\|u_h^f - u^f\|_{\Omega^f}$	r	$\|p_h^f - p^f\|_{\Omega^f}$	r	$\|\nabla \cdot u_h^f\|_{\Omega^f}$
$T/8$	6.0e-02	-	8.2e-02	-	6.1e-12
$T/16$	3.6e-02	0.7	5.3e-02	0.6	5.9e-12
$T/32$	2.1e-02	0.8	3.2e-02	0.7	5.7e-12
$T/64$	1.1e-02	0.9	1.8e-02	0.8	4.3e-12
$T/128$	6.0e-03	0.9	9.8e-03	0.9	3.5e-12

TABLE 3. Errors and temporal rates of convergence r for the solution in Ω^f for the test case described in section 6.

Δt	$\|u_h^b - u^b\|_{\Omega^b}$	r	$\|p_h^b - p^b\|_{\Omega^b}$	r	$\|z_h - z\|_{\Omega^b}$	r	$\|p_h^p - p^p\|_{\Omega^b}$	r
$T/8$	5.5e-04	-	9.5e-06	-	1.1e-02	-	1.4e-03	-
$T/16$	3.0e-04	0.9	4.5e-06	1.1	5.7e-03	1.0	7.1e-04	1.0
$T/32$	1.6e-04	0.9	2.2e-06	1.0	2.9e-03	1.0	3.6e-04	1.0
$T/64$	8.7e-05	0.9	1.1e-06	1.0	1.4e-03	1.0	1.8e-04	1.0
$T/128$	4.6e-05	0.9	5.6e-07	1.0	7.2e-04	1.0	8.9e-05	1.0

TABLE 4. Errors and temporal rates of convergence r for the solution in Ω^b for the test case described in section 6.

ignored. We proved stability and well-posedness under a small data assumption and presented an a priori error analysis of the method.

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Appendix A. Proofs of the inequalities in eq. (10)

Before proving eq. (10) we present a few useful results. First, we have the following discrete Poincaré and Korn's inequalities (see [10, 11] and [44, eqs. (3.4) and (5.4)]):

$$\|v_h\|_{\Omega^j} \leq C\,(\sum_{K\in\mathcal{T}^j} \|\nabla v_h\|_K^2 + \sum_{F\in\mathcal{F}^j_{int}\cup\mathcal{F}^j_D} h_F^{-1}\,\|[\![v_h]\!]\|_F^2) \qquad \forall v_h\in V_h^j,\ j=f,b, \tag{57a}$$

$$\sum_{K\in\mathcal{T}^j} \|\nabla v_h\|_K^2 \leq C_{Korn}\,(\sum_{K\in\mathcal{T}^j} \|\varepsilon(v_h)\|_K^2 + \sum_{F\in\mathcal{F}^j_{int}\cup\mathcal{F}^j_D} h_F^{-1}\,\|[\![v_h]\!]\|_F^2) \qquad \forall v_h\in V_h^j,\ j=f,b. \tag{57b}$$

Next, we note that

$$\begin{aligned}\sum_{F\in\mathcal{F}^j_{int}\cup\mathcal{F}^j_D} h_F^{-1}\,\|[\![v_h]\!]\|_F^2 &= \sum_{F\in\mathcal{F}^j_{int}} h_F^{-1}\,\|(v_h^+ - \{\!\{v_h\}\!\}) - (v_h^- - \{\!\{v_h\}\!\})\|_F^2 + \sum_{F\in\mathcal{F}^j_D} h_F^{-1}\,\|v_h\|_F^2 \\ &\leq C\,(\sum_{F\in\mathcal{F}^j_{int}} (h_{K^+}^{-1}\,\|v_h^+ - \{\!\{v_h\}\!\}\|_F^2 + h_{K^-}^{-1}\,\|v_h^- - \{\!\{v_h\}\!\}\|_F^2) + \sum_{F\in\mathcal{F}^j_D} (h_K^{-1}\,\|v_h\|_F^2)) \\ &= C\sum_{K\in\mathcal{T}^j} h_K^{-1}\,\|v_h - \{\!\{v_h\}\!\}\|_{\partial K}^2\,,\end{aligned} \tag{58}$$

where we assumed shape regularity of the mesh. This result is used to show the second inequality in the following equivalence result:

$$c_1\|v_h\|_{1,h,\Omega^j}^2 \leq \sum_{K\in\mathcal{T}^j} \|\varepsilon(v_h)\|_K^2 + \sum_{F\in\mathcal{F}^j_{int}\cup\mathcal{F}^j_D} h_F^{-1}\,\|[\![v_h]\!]\|_F^2 \leq c_2\|v_h\|_{1,h,\Omega^j}^2\,. \tag{59}$$

The first inequality in (59) was shown in [21, Appendix A]. Finally, let us note that we have the following inequalities between different norms:

$$c_1\|v_h\|_{1,h,\Omega^j}^2 \leq \sum_{K\in\mathcal{T}^j} \|\varepsilon(v_h)\|_K^2 + \sum_{F\in\mathcal{F}^j_{int}\cup\mathcal{F}^j_D} h_F^{-1}\,\|[\![v_h]\!]\|_F^2 \leq C|\!|\!|\boldsymbol{v}_h|\!|\!|_{v,j}^2. \tag{60}$$

The first inequality is by eq. (59) while the second inequality follows by identical steps as in eq. (58) but with $\{\!\{v_h\}\!\}$ replaced by $\bar{v}_h$.

We now prove eq. (10). The first inequalities in eqs. (10a) and (10b) are a consequence of eqs. (57a), (57b) and (59). The second inequalities in eqs. (10a) and (10b) are a consequence of eq. (60). The proof of eq. (10d) is similar to eqs. (10a) and (10b). Finally, [13, Theorem 4.4] implies that for $1\leq r<\infty$ when $d=2$ and $1\leq r\leq 4$ when $d=3$

$$\|v_h^f\|_{r,0,\Gamma^f_{IN}}^2 \leq C\,(\|v_h^f\|_{1,0,\Omega^f}^2 + \sum_{K\in\mathcal{T}^f} \|\nabla v_h\|_K^2 + \sum_{F\in\mathcal{F}^f_{int}} h_F^{-1}\,\|[\![v_h^f]\!]\|_F^2)\,.$$

By Hölder's inequality, eqs. (57a), (57b) and (59) we obtain the first inequality of eq. (10c). The second inequality is a consequence of eq. (60).

Appendix B. Proofs of lemma's in section 4

B.1. **Proof of Lemma 4.2**

We first show uniqueness. Assume that both $(\boldsymbol{u}_h^{n+1}, \boldsymbol{p}_h^{n+1}, z_h^{n+1}, \boldsymbol{p}_h^{p,n+1})$ and $(\widehat{\boldsymbol{u}}_h^{n+1}, \widehat{\boldsymbol{p}}_h^{n+1}, \widehat{z}_h^{n+1}, \widehat{\boldsymbol{p}}_h^{p,n+1})$ are solutions to the fully discrete system eq. (17). Let us define their difference by $(\boldsymbol{x}_h^{n+1}, \boldsymbol{r}_h^{n+1}, y_h^{n+1}, \boldsymbol{r}_h^{p,n+1}) =$

$(\boldsymbol{u}_h^{n+1} - \widehat{\boldsymbol{u}}_h^{n+1}, \boldsymbol{p}_h^{n+1} - \widehat{\boldsymbol{p}}_h^{n+1}, z_h^{n+1} - \widehat{z}_h^{n+1}, \boldsymbol{p}_h^{p,n+1} - \widehat{\boldsymbol{p}}_h^{p,n+1})$. We need to show that $(\boldsymbol{x}_h^{n+1}, \boldsymbol{r}_h^{n+1}, y_h^{n+1}, \boldsymbol{r}_h^{p,n+1}) = (\boldsymbol{0}, \boldsymbol{0}, 0, \boldsymbol{0})$. Let us first note that $(\boldsymbol{x}_h^{n+1}, \boldsymbol{r}_h^{n+1}, y_h^{n+1}, \boldsymbol{r}_h^{p,n+1})$ satisfies

$$\frac{1}{\Delta t}(x_h^{n+1}, v_h^f)_{\Omega^f} + t_h(u_h^{f,n}; \boldsymbol{x}_h^{f,n+1}, \boldsymbol{v}_h^f) + a_h^f(\boldsymbol{x}_h^{f,n+1}, \boldsymbol{v}_h^f) + a_h^b(\boldsymbol{x}_h^{b,n+1}, \boldsymbol{v}_h^b) + b_h^f(\boldsymbol{v}_h^f, \boldsymbol{r}_h^{f,n+1}) \\ + b_h^b(\boldsymbol{v}_h^b, \boldsymbol{r}_h^{b,n+1}) + a_h^I((\bar{x}_h^{f,n+1}, \frac{1}{\Delta t}\bar{x}_h^{b,n+1}), (\bar{v}_h^f, \bar{v}_h^b)) + b_h^I((\bar{v}_h^f, \bar{v}_h^b), \bar{r}_h^{p,n+1}) = 0, \tag{61a}$$

$$b_h^f(\boldsymbol{x}_h^{f,n+1}, \boldsymbol{q}_h^f) + b_h^b(\boldsymbol{x}_h^{b,n+1}, \boldsymbol{q}_h^b) + c_h((r_h^{p,n+1}, r_h^{b,n+1}), q_h^b) = 0, \tag{61b}$$

$$\frac{1}{\Delta t}(c_0 r_h^{p,n+1}, q_h^p)_{\Omega^b} + \frac{1}{\Delta t}c_h((r_h^{p,n+1}, r_h^{b,n+1}), \alpha q_h^p) - b_h^b((y_h^{n+1}, 0), \boldsymbol{q}_h^p) - b_h^I((\bar{x}_h^{f,n+1}, \frac{1}{\Delta t}\bar{x}_h^{b,n+1}), \bar{q}_h^p) = 0, \tag{61c}$$

$$(\mu^f \kappa^{-1} y_h^{n+1}, w_h)_{\Omega^b} + b_h^b((w_h, 0), \boldsymbol{r}_h^{p,n+1}) = 0, \tag{61d}$$

for all $(\boldsymbol{v}_h, \boldsymbol{q}_h, w_h, \boldsymbol{q}_h^p) \in \boldsymbol{X}_h$. Add the above equations, choose $\boldsymbol{v}_h^f = \boldsymbol{x}_h^{f,n+1}$, $\boldsymbol{v}_h^b = \frac{1}{\Delta t}\boldsymbol{x}_h^{b,n+1}$, $\boldsymbol{q}_h^f = -\boldsymbol{r}_h^{f,n+1}$, $\boldsymbol{q}_h^b = -\frac{1}{\Delta t}\boldsymbol{r}_h^{b,n+1}$, $w_h = y_h^{n+1}$, and $\boldsymbol{q}_h^p = \boldsymbol{r}^{p,n+1}$. Since $\boldsymbol{u}_h^{f,n} \in \boldsymbol{B}_h^f$ we find, using Proposition 3.3 and eq. (11a), that

$$\tfrac{1}{\Delta t}\|x_h^{f,n+1}\|_{\Omega^f}^2 + \tfrac{1}{2}c_{ae}^f \mu^f |||\boldsymbol{x}_h^{f,n+1}|||_{v,f}^2 + \tfrac{1}{\Delta t}c_{ae}^b \mu^b |||\boldsymbol{x}_h^{b,n+1}|||_{v,b}^2 + \gamma\mu^f\kappa^{-1/2}\|(\bar{x}_h^{f,n+1} - \tfrac{1}{\Delta t}\bar{x}_h^{b,n+1})^t\|_{\Gamma_I}^2 \\ + \tfrac{1}{\Delta t}\lambda^{-1}\|\alpha r_h^{p,n+1} - r_h^{b,n+1}\|_{\Omega^b}^2 + \tfrac{c_0}{\Delta t}\|r_h^{p,n+1}\|_{\Omega^b}^2 + \mu^f\kappa^{-1}\|y_h^{n+1}\|_{\Omega^b}^2 \le 0,$$

so that $\boldsymbol{x}_h^{n+1} = \boldsymbol{0}$ and $y_h^{n+1} = 0$. We are left to show that $\boldsymbol{r}_h^{n+1}$ and $\boldsymbol{r}_h^{p,n+1}$ are zero. To show that $\boldsymbol{r}_h^{n+1} = \boldsymbol{0}$, substitute $\boldsymbol{x}_h^{n+1} = \boldsymbol{0}$ into eq. (61a) and choose $\bar{v}_h^f = \bar{v}_h^b = 0$ on Γ_I to find $b_h^j(\boldsymbol{v}_h^j, \boldsymbol{r}_h^{j,n+1}) = 0\ \forall \boldsymbol{v}_h^j \in \widetilde{V}_h^j$, $j = f, b$. The result follows by the inf-sup condition eq. (12a). Similarly, to show that $\boldsymbol{r}_h^{p,n+1} = \boldsymbol{0}$, substitute $y_h^{n+1} = 0$ into eq. (61d) to find $b_h^b((w_h, 0), \boldsymbol{r}_h^{p,n+1}) = 0\ \forall w_h \in V_h^b$. The result follows by the inf-sup condition eq. (12c). Finally, existence of a solution is a consequence of uniqueness since eq. (17) is a linear and finite dimensional problem.

B.2. **Proof of Lemma 4.3**

Step 1: proof of eq. (27a). Let $n = 0$ in eq. (17). Use that the initial conditions are zero and choose $\boldsymbol{v}_h^f = \boldsymbol{u}_h^{f,1}$, $\boldsymbol{v}_h^b = \frac{1}{\Delta t}\boldsymbol{u}_h^{b,1}$, $\boldsymbol{q}_h^f = -\boldsymbol{p}_h^{f,1}$, $\boldsymbol{q}_h^p = \boldsymbol{p}_h^{p,1}$, and $w_h = z_h^1$. Furthermore, choose $\boldsymbol{q}_h^b = -\frac{1}{\Delta t}\boldsymbol{p}_h^{b,1}$ and note that $b_h^b(\boldsymbol{u}_h^{b,1}, -\frac{1}{\Delta t}\boldsymbol{p}_h^{b,1}) = -b_h^b(\frac{1}{\Delta t}\boldsymbol{u}_h^{b,1}, \boldsymbol{p}_h^{b,1})$ and $c_h((p_h^{p,1}, p_h^{b,1}), -\frac{1}{\Delta t}p_h^{b,1}) = -c_h((\frac{1}{\Delta t}p_h^{p,1}, \frac{1}{\Delta t}p_h^{b,1}), p_h^{b,1})$. Equation (17) becomes:

$$\tfrac{1}{\Delta t}\|u_h^{f,1}\|_{\Omega^f}^2 + a_h^f(\boldsymbol{u}_h^{f,1}, \boldsymbol{u}_h^{f,1}) + a_h^b(\boldsymbol{u}_h^{b,1}, \tfrac{1}{\Delta t}\boldsymbol{u}_h^{b,1}) + \gamma\mu^f\kappa^{-1/2}\|(\bar{u}_h^{f,1} - \tfrac{1}{\Delta t}\bar{u}_h^{b,1})^t\|_{\Gamma_I}^2 + \tfrac{\lambda^{-1}}{\Delta t}\|\alpha p_h^{p,1} - p_h^{b,1}\|_{\Omega^b}^2 \\ + \tfrac{c_0}{\Delta t}\|p_h^{p,1}\|_{\Omega^b}^2 + \mu^f\kappa^{-1}\|z_h^1\|_{\Omega^b}^2 = (f^{f,1}, u_h^{f,1})_{\Omega^f} + (f^{b,1}, \tfrac{1}{\Delta t}u_h^{b,1})_{\Omega^b} + (g^{b,1}, p_h^{p,1})_{\Omega^b}.$$

Coercivity of a_h^f (see eq. (11a)), the Cauchy–Schwarz inequality, and using eq. (24) so that $\|p_h^{p,1}\|_{\Omega^b} \le c_{td}\mu^f\kappa^{-1}\|z_h^1\|_{\Omega^b}$, we obtain

$$\tfrac{1}{\Delta t}\|u_h^{f,1}\|_{\Omega^f}^2 + c_{ae}^f\mu^f|||\boldsymbol{u}_h^{f,1}|||_{v,f}^2 + \tfrac{1}{\Delta t}a_h^b(\boldsymbol{u}_h^{b,1}, \boldsymbol{u}_h^{b,1}) + \gamma\mu^f\kappa^{-1/2}\|(\bar{u}_h^{f,1} - \tfrac{1}{\Delta t}\bar{u}_h^{b,1})^t\|_{\Gamma_I}^2 + \tfrac{\lambda^{-1}}{\Delta t}\|\alpha p_h^{p,1} - p_h^{b,1}\|_{\Omega^b}^2 \\ + \tfrac{c_0}{\Delta t}\|p_h^{p,1}\|_{\Omega^b}^2 + \mu^f\kappa^{-1}\|z_h^1\|_{\Omega^b}^2 \le \|f^{f,1}\|_{\Omega^f}\|u_h^{f,1}\|_{\Omega^f} + \tfrac{1}{\Delta t}\|f^{b,1}\|_{\Omega^b}\|u_h^{b,1}\|_{\Omega^b} + c_{td}\mu^f\kappa^{-1}\|g^{b,1}\|_{\Omega^b}\|z_h^1\|_{\Omega^b}. \tag{62}$$

Using eq. (10b) and eq. (11a) so that

$$\|u_h^{b,1}\|_{\Omega^b} \le c_p|||\boldsymbol{u}_h^{b,1}|||_{v,b} \le c_p(c_{ae}^b\mu^b)^{-1/2}a_h^b(\boldsymbol{u}_h^{b,1}, \boldsymbol{u}_h^{b,1})^{1/2}, \tag{63}$$

and noting the nonnegativity of the second and the fourth terms of eq. (62), we further have

$$\begin{aligned}&\tfrac{1}{\Delta t}\left\|u_h^{f,1}\right\|_{\Omega^f}^2+\tfrac{1}{\Delta t}a_h^b(\boldsymbol{u}_h^{b,1},\boldsymbol{u}_h^{b,1})+\tfrac{\lambda^{-1}}{\Delta t}\left\|\alpha p_h^{p,1}-p_h^{b,1}\right\|_{\Omega^b}^2+\tfrac{c_0}{\Delta t}\left\|p_h^{p,1}\right\|_{\Omega^b}^2+\mu^f\kappa^{-1}\left\|z_h^1\right\|_{\Omega^b}^2\\&\quad\le\left\|f^{f,1}\right\|_{\Omega^f}\left\|u_h^{f,1}\right\|_{\Omega^f}+\tfrac{1}{\Delta t}c_p(c_{ae}^b\mu^b)^{-1/2}\left\|f^{b,1}\right\|_{\Omega^b}a_h^b(\boldsymbol{u}_h^{b,1},\boldsymbol{u}_h^{b,1})^{1/2}+c_{td}\mu^f\kappa^{-1}\left\|g^{b,1}\right\|_{\Omega^b}\left\|z_h^1\right\|_{\Omega^b}.\end{aligned}\tag{64}$$

Let us define

$$\mathcal{Z}^2:=\left\|u_h^{f,1}\right\|_{\Omega^f}^2+a_h^b(\boldsymbol{u}_h^{b,1},\boldsymbol{u}_h^{b,1})+\lambda^{-1}\left\|\alpha p_h^{p,1}-p_h^{b,1}\right\|_{\Omega^b}^2+c_0\left\|p_h^{p,1}\right\|_{\Omega^b}^2+\Delta t\mu^f\kappa^{-1}\left\|z_h^1\right\|_{\Omega^b}^2,$$

and write eq. (64) as:

$$\tfrac{1}{\Delta t}\mathcal{Z}^2\le\big(\left\|f^{f,1}\right\|_{\Omega^f}+\tfrac{1}{\Delta t}c_p(c_{ae}^b\mu^b)^{-1/2}\left\|f^{b,1}\right\|_{\Omega^b}+c_{td}\tfrac{1}{(\Delta t)^{1/2}}(\mu^f\kappa^{-1})^{1/2}\left\|g^{b,1}\right\|_{\Omega^b}\big)\mathcal{Z}.$$

This immediately implies, using that $(X^0)^{1/2}\le\frac{1}{\Delta t}\mathcal{Z}$ and that $\frac{1}{\Delta t}\left\|f^{b,1}\right\|_{\Omega^b}=\left\|d_tf^{b,1}\right\|_{\Omega^b}$ and $\frac{1}{(\Delta t)^{1/2}}\left\|g^{b,1}\right\|_{\Omega^b}=(\Delta t)^{1/2}\left\|d_tg^{b,1}\right\|_{\Omega^b}$ because $f^{b,0}=0$ and $g^{b,0}=0$,

$$(X^0)^{1/2}\le\left\|f^{f,1}\right\|_{\Omega^f}+c_p(c_{ae}^b\mu^b)^{-1/2}\left\|d_tf^{b,1}\right\|_{\Omega^b}+c_{td}(\Delta t)^{1/2}(\mu^f\kappa^{-1})^{1/2}\left\|d_tg^{b,1}\right\|_{\Omega^b},$$

so that eq. (27a) follows after squaring.

Step 2: proof of eq. (27b). We return to eq. (62). Applying Young's inequality $ab\le a^2/(2\psi)+\psi b^2/2$ to each term on the right hand side, using eq. (63) for the second term, choosing $\psi=2/\Delta t$, $\psi=2/\Delta t$, and $\psi=2\mu^f\kappa^{-1}$ for the first, second, and third terms, respectively, and dividing both sides by Δt, we obtain, using that $f^{b,0}=0$ and $g^{b,0}=0$,

$$c_{ae}^f\mu^f\Delta t\|\!|d_t\boldsymbol{u}_h^{f,1}|\!\|_{v,f}^2\le\tfrac14\left\|f^{f,1}\right\|_{\Omega^f}^2+\tfrac14c_p^2(c_{ae}^b\mu^b)^{-1}\left\|d_tf^{b,1}\right\|_{\Omega^b}^2+\tfrac14c_{td}^2\mu^f\kappa^{-1}\Delta t\left\|d_tg^{b,1}\right\|_{\Omega^b}^2.$$

This proves eq. (27b).

Step 3: proof of eq. (28). Let $1\le n\le N-1$. Subtract eq. (17) for the solution at time-level t_n from eq. (17) for the solution at time-level t_{n+1}, choose $\boldsymbol{v}_h^f=\delta\boldsymbol{u}_h^{f,n+1}$, $\boldsymbol{v}_h^b=\frac{1}{\Delta t}(\delta\boldsymbol{u}_h^{b,n+1}-\delta\boldsymbol{u}_h^{b,n})$, $\boldsymbol{q}_h^f=-\delta\boldsymbol{p}_h^{f,n+1}$, $\boldsymbol{q}_h^b=-\frac{1}{\Delta t}\delta\boldsymbol{p}_h^{b,n+1}$, $w_h=\delta z_h^{n+1}$, $\boldsymbol{q}_h^p=\delta\boldsymbol{p}_h^{p,n+1}$ and add the resulting equations:

$$\begin{aligned}&\tfrac{1}{\Delta t}(\delta u_h^{f,n+1}-\delta u_h^{f,n},\delta u_h^{f,n+1})_{\Omega^f}+t_h(u_h^{f,n};\boldsymbol{u}_h^{f,n+1},\delta\boldsymbol{u}_h^{f,n+1})-t_h(u_h^{f,n-1};\boldsymbol{u}_h^{f,n},\delta\boldsymbol{u}_h^{f,n+1})\\&\qquad+a_h^f(\delta\boldsymbol{u}_h^{f,n+1},\delta\boldsymbol{u}_h^{f,n+1})+a_h^b(\delta\boldsymbol{u}_h^{b,n+1},\tfrac{1}{\Delta t}(\delta\boldsymbol{u}_h^{b,n+1}-\delta\boldsymbol{u}_h^{b,n}))\\&\qquad+b_h^b(\tfrac{1}{\Delta t}(\delta\boldsymbol{u}_h^{b,n+1}-\delta\boldsymbol{u}_h^{b,n}),\delta\boldsymbol{p}_h^{b,n+1})-b_h^b(\delta\boldsymbol{u}_h^{b,n+1},\tfrac{1}{\Delta t}\delta\boldsymbol{p}_h^{b,n+1})\\&\qquad+a_h^I((\delta\bar{u}_h^{f,n+1},\tfrac{1}{\Delta t}(\delta\bar{u}_h^{b,n+1}-\delta\bar{u}_h^{b,n})),(\delta\bar{u}_h^{f,n+1},\tfrac{1}{\Delta t}(\delta\bar{u}_h^{b,n+1}-\delta\bar{u}_h^{b,n})))\\&-c_h((\delta p_h^{p,n+1},\delta p_h^{b,n+1}),\tfrac{1}{\Delta t}\delta p_h^{b,n+1})+\tfrac{1}{\Delta t}c_h((\delta p_h^{p,n+1}-\delta p_h^{p,n},\delta p_h^{b,n+1}-\delta p_h^{b,n}),\alpha\delta p_h^{p,n+1})\\&+\tfrac{c_0}{\Delta t}(\delta p_h^{p,n+1}-\delta p_h^{p,n},\delta p_h^{p,n+1})_{\Omega^b}+(\mu^f\kappa^{-1}\delta z_h^{n+1},\delta z_h^{n+1})_{\Omega^b}\\&=(\delta f^{f,n+1},\delta u_h^{f,n+1})_{\Omega^f}+(\delta f^{b,n+1},\tfrac{1}{\Delta t}(\delta u_h^{b,n+1}-\delta u_h^{b,n}))_{\Omega^b}+(\delta g^{b,n+1},\delta p_h^{p,n+1})_{\Omega^b}.\end{aligned}\tag{65}$$

We simplify next the sum of the 6$^{\text{th}}$, 7$^{\text{th}}$, 9$^{\text{th}}$, and 10$^{\text{th}}$ terms on the left hand side of eq. (65) as follows. First,

$$\begin{aligned}&[b_h^b(\tfrac{1}{\Delta t}(\delta\boldsymbol{u}_h^{b,n+1}-\delta\boldsymbol{u}_h^{b,n}),\delta\boldsymbol{p}_h^{b,n+1})-b_h^b(\delta\boldsymbol{u}_h^{b,n+1},\tfrac{1}{\Delta t}\delta\boldsymbol{p}_h^{b,n+1})]-c_h((\delta p_h^{p,n+1},\delta p_h^{b,n+1}),\tfrac{1}{\Delta t}\delta p_h^{b,n+1})\\&\qquad\qquad+\tfrac{1}{\Delta t}c_h((\delta p_h^{p,n+1}-\delta p_h^{p,n},\delta p_h^{b,n+1}-\delta p_h^{b,n}),\alpha\delta p_h^{p,n+1})=:I_1+I_2+I_3.\end{aligned}$$

Note that

$$I_1 = b_h^b(\tfrac{1}{\Delta t}\delta\boldsymbol{u}_h^{b,n+1}, \delta\boldsymbol{p}_h^{b,n+1}) - b_h^b(\tfrac{1}{\Delta t}\delta\boldsymbol{u}_h^{b,n}, \delta\boldsymbol{p}_h^{b,n+1}) - b_h^b(\delta\boldsymbol{u}_h^{b,n+1}, \tfrac{1}{\Delta t}\delta\boldsymbol{p}_h^{b,n+1}) = - b_h^b(\tfrac{1}{\Delta t}\delta\boldsymbol{u}_h^{b,n}, \delta\boldsymbol{p}_h^{b,n+1}) \\ = c_h((\delta p_h^{p,n}, \delta p_h^{b,n}), \tfrac{1}{\Delta t}\delta p_h^{b,n+1}),$$

where the last equality follows by subtracting eq. (17b) at time-level t_n from eq. (17b) at time-level t_{n+1} and choosing $\boldsymbol{q}_h^f = \boldsymbol{0}$, $\boldsymbol{q}_h^b = \frac{1}{\Delta t}\delta\boldsymbol{p}_h^{b,n+1}$. We immediately find that

$$I_1 + I_2 + I_3 = \tfrac{1}{\Delta t}c_h((\delta p_h^{p,n+1} - \delta p_h^{p,n}, \delta p_h^{b,n+1} - \delta p_h^{b,n}), \alpha\delta p_h^{p,n+1} - \delta p_h^{b,n+1}).$$

Noting also that $t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n+1}, \delta\boldsymbol{u}_h^{f,n+1}) = t_h(u_h^{f,n}; \delta\boldsymbol{u}_h^{f,n+1}, \delta\boldsymbol{u}_h^{f,n+1}) + t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n}, \delta\boldsymbol{u}_h^{f,n+1})$, we write eq. (65) as:

$$\begin{aligned}
&\tfrac{1}{\Delta t}(\delta u_h^{f,n+1} - \delta u_h^{f,n}, \delta u_h^{f,n+1})_{\Omega^f} + t_h(u_h^{f,n}; \delta\boldsymbol{u}_h^{f,n+1}, \delta\boldsymbol{u}_h^{f,n+1}) + a_h^f(\delta\boldsymbol{u}_h^{f,n+1}, \delta\boldsymbol{u}_h^{f,n+1}) \\
&\quad + a_h^b(\delta\boldsymbol{u}_h^{b,n+1}, \tfrac{1}{\Delta t}(\delta\boldsymbol{u}_h^{b,n+1} - \delta\boldsymbol{u}_h^{b,n})) \\
&\quad + a_h^I((\delta\bar{u}_h^{f,n+1}, \tfrac{1}{\Delta t}(\delta\bar{u}_h^{b,n+1} - \delta\bar{u}_h^{b,n})), (\delta\bar{u}_h^{f,n+1}, \tfrac{1}{\Delta t}(\delta\bar{u}_h^{b,n+1} - \delta\bar{u}_h^{b,n}))) \\
&\quad + \tfrac{1}{\Delta t}c_h((\delta p_h^{p,n+1} - \delta p_h^{p,n}, \delta p_h^{b,n+1} - \delta p_h^{b,n}), \alpha\delta p_h^{p,n+1} - \delta p_h^{b,n+1}) \\
&\quad + \tfrac{c_0}{\Delta t}(\delta p_h^{p,n+1} - \delta p_h^{p,n}, \delta p_h^{p,n+1})_{\Omega^b} + (\mu^f\kappa^{-1}\delta z_h^{n+1}, \delta z_h^{n+1})_{\Omega^b} \\
&= (\delta f^{f,n+1}, \delta u_h^{f,n+1})_{\Omega^f} + (\delta f^{b,n+1}, \tfrac{1}{\Delta t}(\delta u_h^{b,n+1} - \delta u_h^{b,n}))_{\Omega^b} + (\delta g^{b,n+1}, \delta p_h^{p,n+1})_{\Omega^b} \\
&\quad - [t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n}, \delta\boldsymbol{u}_h^{f,n+1}) - t_h(u_h^{f,n-1}; \boldsymbol{u}_h^{f,n}, \delta\boldsymbol{u}_h^{f,n+1})]\,.
\end{aligned} \tag{66}$$

For the left hand side of eq. (66), since we assumed that $\boldsymbol{u}_h^{f,n} \in \boldsymbol{B}_h^f$, we have by eq. (15),

$$t_h(u_h^{f,n}; \delta\boldsymbol{u}_h^{f,n+1}, \delta\boldsymbol{u}_h^{f,n+1}) + a_h^f(\delta\boldsymbol{u}_h^{f,n+1}, \delta\boldsymbol{u}_h^{f,n+1}) \ge \tfrac{1}{2}c_{ae}^f\mu^f |||\delta\boldsymbol{u}_h^{f,n+1}|||_{v,f}^2.$$

For the right hand side of eq. (66) (see also [22, Proof of Lemma 4.5]):

$$\begin{aligned}
|t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n}, \delta\boldsymbol{u}_h^{f,n+1}) - t_h(u_h^{f,n-1}; \boldsymbol{u}_h^{f,n}, \delta\boldsymbol{u}_h^{f,n+1})| &\le c_w \left\|u_h^{f,n} - u_h^{f,n-1}\right\|_{1,h,\Omega^f} |||\boldsymbol{u}_h^{f,n}|||_{v,f} |||\delta\boldsymbol{u}_h^{f,n+1}|||_{v,f} \\
&= c_w \left\|\delta u_h^{f,n}\right\|_{1,h,\Omega^f} |||\boldsymbol{u}_h^{f,n}|||_{v,f} |||\delta\boldsymbol{u}_h^{f,n+1}|||_{v,f} \\
&\le c_w |||\delta\boldsymbol{u}_h^{f,n}|||_{v,f} |||\boldsymbol{u}_h^{f,n}|||_{v,f} |||\delta\boldsymbol{u}_h^{f,n+1}|||_{v,f}.
\end{aligned}$$

Applying also the Cauchy–Schwarz inequality to the terms on the right hand side of eq. (66) and using $\|\delta u_h^{f,n+1}\|_{\Omega^f} \le c_p |||\delta\boldsymbol{u}_h^{f,n+1}|||_{v,f}$ (by eq. (10a)), and $\|\delta p_h^{p,n+1}\|_{\Omega^b} \le c_{td}\mu^f\kappa^{-1}\left\|\delta z_h^{n+1}\right\|_{\Omega^b}$ (by a simple modification of the proof to eq. (24)), we obtain

$$\begin{aligned}
&\tfrac{1}{\Delta t}(\delta u_h^{f,n+1} - \delta u_h^{f,n}, \delta u_h^{f,n+1})_{\Omega^f} + \tfrac{1}{\Delta t}a_h^b(\delta\boldsymbol{u}_h^{b,n+1}, \delta\boldsymbol{u}_h^{b,n+1} - \delta\boldsymbol{u}_h^{b,n}) \\
&\quad + \tfrac{1}{\Delta t}c_h((\delta p_h^{p,n+1} - \delta p_h^{p,n}, \delta p_h^{b,n+1} - \delta p_h^{b,n}), \alpha\delta p_h^{p,n+1} - \delta p_h^{b,n+1}) \\
&\quad + \tfrac{c_0}{\Delta t}(\delta p_h^{p,n+1} - \delta p_h^{p,n}, \delta p_h^{p,n+1})_{\Omega^b} + \tfrac{1}{2}c_{ae}^f\mu^f |||\delta\boldsymbol{u}_h^{f,n+1}|||_{v,f}^2 \\
&\quad + \mu^f\kappa^{-1}\left\|\delta z_h^{n+1}\right\|_{\Omega^b}^2 + \gamma\mu^f\kappa^{-1/2}\left\|(\delta\bar{u}_h^{f,n+1} - \tfrac{1}{\Delta t}(\delta\bar{u}_h^{b,n+1} - \delta\bar{u}_h^{b,n})^t\right\|_{\Gamma_I}^2 \\
&\le c_p \left\|\delta f^{f,n+1}\right\|_{\Omega^f} |||\delta\boldsymbol{u}_h^{f,n+1}|||_{v,f} + \tfrac{1}{\Delta t}(\delta f^{b,n+1}, \delta u_h^{b,n+1} - \delta u_h^{b,n})_{\Omega^b} \\
&\quad + c_{td}\mu^f\kappa^{-1}\left\|\delta g^{b,n+1}\right\|_{\Omega^b}\left\|\delta z_h^{n+1}\right\|_{\Omega^b} + c_w |||\delta\boldsymbol{u}_h^{f,n}|||_{v,f} |||\boldsymbol{u}_h^{f,n}|||_{v,f} |||\delta\boldsymbol{u}_h^{f,n+1}|||_{v,f}.
\end{aligned} \tag{67}$$

Apply Young's inequality $ab \le a^2/(2\psi) + \psi b^2/2$ to the first, third, and fourth terms on the right hand side of eq. (67), choose $\psi = \frac{1}{2}c_{ae}^f\mu^f$, $\psi = \mu^f\kappa^{-1}$, and $\psi = 1$ for the first, third, and fourth terms, respectively, and note that since we assumed that $\boldsymbol{u}_h^{f,n} \in \boldsymbol{B}_h^f$ that $\frac{1}{2}c_{ae}^f\mu^f - c_w|||\boldsymbol{u}_h^{f,n}|||_{v,f} \ge \frac{1}{4}c_{ae}^f\mu^f$. We find from eq. (67):

$$
\begin{aligned}
&\tfrac{1}{\Delta t}(\delta u_h^{f,n+1} - \delta u_h^{f,n}, \delta u_h^{f,n+1})_{\Omega^f} + \tfrac{1}{\Delta t}a_h^b(\delta \boldsymbol{u}_h^{b,n+1}, \delta \boldsymbol{u}_h^{b,n+1} - \delta \boldsymbol{u}_h^{b,n}) \\
&\quad + \tfrac{1}{\Delta t}c_h((\delta p_h^{p,n+1} - \delta p_h^{p,n}, \delta p_h^{b,n+1} - \delta p_h^{b,n}), \alpha\delta p_h^{p,n+1} - \delta p_h^{b,n+1}) \\
&\quad + \tfrac{c_0}{\Delta t}(\delta p_h^{p,n+1} - \delta p_h^{p,n}, \delta p_h^{p,n+1})_{\Omega^b} + \tfrac{1}{8}c_{ae}^f\mu^f|||\delta \boldsymbol{u}_h^{f,n+1}|||_{v,f}^2 \\
&\quad + \tfrac{1}{2}\mu^f\kappa^{-1}\left\|\delta z_h^{n+1}\right\|_{\Omega^b}^2 + \gamma\mu^f\kappa^{-1/2}\left\|(\delta\bar{u}_h^{f,n+1} - \tfrac{1}{\Delta t}(\delta\bar{u}_h^{b,n+1} - \delta\bar{u}_h^{b,n})^t\right\|_{\Gamma_I}^2 \\
&\le \frac{c_p^2}{c_{ae}^f\mu^f}\left\|\delta f^{f,n+1}\right\|_{\Omega^f}^2 + \frac{c_{td}^2\mu^f}{2\kappa}\left\|\delta g^{b,n+1}\right\|_{\Omega^b}^2 + \tfrac{1}{\Delta t}(\delta f^{b,n+1}, \delta u_h^{b,n+1} - \delta u_h^{b,n})_{\Omega^b} + \frac{c_w}{2}|||\boldsymbol{u}_h^{f,n}|||_{v,f}|||\delta\boldsymbol{u}_h^{f,n}|||_{v,f}^2.
\end{aligned}
\tag{68}
$$

For the last term on the right hand side of eq. (68) use $|||\boldsymbol{u}_h^{f,n}||| \le c_{ae}^f\mu^f/(4c_w)$ (assumption $\boldsymbol{u}_h^{f,n} \in \boldsymbol{B}_h^f$). Furthermore, by $2a(a-b) = a^2 - b^2 + (a-b)^2$, coercivity of a_h^b (see eq. (11a)), and eq. (10b), we find:

$$
\begin{aligned}
2a_h^b(\delta\boldsymbol{u}_h^{b,n+1}, \delta\boldsymbol{u}_h^{b,n+1} - \delta\boldsymbol{u}_h^{b,n}) =& a_h^b(\delta\boldsymbol{u}_h^{b,n+1}, \delta\boldsymbol{u}_h^{b,n+1}) - a_h^b(\delta\boldsymbol{u}_h^{b,n}, \delta\boldsymbol{u}_h^{b,n}) \\
&+ a_h^b(\delta\boldsymbol{u}_h^{b,n+1} - \delta\boldsymbol{u}_h^{b,n}, \delta\boldsymbol{u}_h^{b,n+1} - \delta\boldsymbol{u}_h^{b,n}) \\
\ge& a_h^b(\delta\boldsymbol{u}_h^{b,n+1}, \delta\boldsymbol{u}_h^{b,n+1}) - a_h^b(\delta\boldsymbol{u}_h^{b,n}, \delta\boldsymbol{u}_h^{b,n}) + c_{ae}^b\mu^b c_p^{-1}\left\|\delta u_h^{b,n+1} - \delta u_h^{b,n}\right\|_{\Omega^b}^2.
\end{aligned}
$$

Also using $2a(a-b) \ge a^2 - b^2$, we obtain after multiplying eq. (68) by $2/\Delta t$, replacing n by k, using that $d_t f^{k+1} = \Delta t^{-1}\delta f^{k+1}$, and summing for $k=1$ to $k=n$:

$$
\begin{aligned}
&\left\|d_t u_h^{f,n+1}\right\|_{\Omega^f}^2 + a_h^b(d_t\boldsymbol{u}_h^{b,n+1}, d_t\boldsymbol{u}_h^{b,n+1}) + c_{ae}^b\mu^b c_p^{-1}\sum_{k=1}^n\left\|d_t u_h^{b,k+1} - d_t u_h^{b,k}\right\|_{\Omega^b}^2 \\
&+ \lambda^{-1}\left\|\alpha d_t p_h^{p,n+1} - d_t p_h^{b,n+1}\right\|_{\Omega^b}^2 + c_0\left\|d_t p_h^{p,n+1}\right\|_{\Omega^b}^2 + \tfrac{1}{4}c_{ae}^f\mu^f\Delta t|||d_t\boldsymbol{u}_h^{f,n+1}|||_{v,f}^2 \\
&+ \mu^f\kappa^{-1}\Delta t\sum_{k=1}^n\left\|d_t z_h^{k+1}\right\|_{\Omega^b}^2 + 2\gamma\mu^f\kappa^{-1/2}\Delta t\sum_{k=1}^n\left\|d_t\bar{u}_h^{f,k+1} - \tfrac{1}{\Delta t}(d_t\bar{u}_h^{b,k+1} - d_t\bar{u}_h^{b,k})^t\right\|_{\Gamma_I}^2 \\
\le& \left\|d_t u_h^{f,1}\right\|_{\Omega^f}^2 + a_h^b(d_t\boldsymbol{u}_h^{b,1}, d_t\boldsymbol{u}_h^{b,1}) + \lambda^{-1}\left\|\alpha d_t p_h^{p,1} - d_t p_h^{b,1}\right\|_{\Omega^b}^2 + c_0\left\|d_t p_h^{p,1}\right\|_{\Omega^b}^2 \\
&+ \tfrac{1}{4}c_{ae}^f\mu^f\Delta t|||d_t\boldsymbol{u}_h^{f,1}|||_{v,f}^2 + \frac{2c_p^2}{c_{ae}^f\mu^f}\Delta t\sum_{k=1}^n\left\|d_t f^{f,k+1}\right\|_{\Omega^f}^2 \\
&+ \frac{c_{td}^2\mu^f}{\kappa}\Delta t\sum_{k=1}^n\left\|d_t g^{b,k+1}\right\|_{\Omega^b}^2 + \tfrac{2}{(\Delta t)^2}\sum_{k=1}^n(\delta f^{b,k+1}, \delta u_h^{b,k+1} - \delta u_h^{b,k})_{\Omega^b}.
\end{aligned}
\tag{69}
$$

Apply summation-by-parts to the last term on the right hand side of eq. (69):

$$
\begin{aligned}
&\tfrac{1}{(\Delta t)^2}\sum_{k=1}^n(\delta f^{b,k+1}, \delta u_h^{b,k+1} - \delta u_h^{b,k})_{\Omega^b} \\
&= (d_t f^{b,n+1}, d_t u_h^{b,n+1})_{\Omega^b} - (d_t f^{b,2}, d_t u_h^{b,1})_{\Omega^b} - \Delta t\sum_{k=2}^n(d_{tt}f^{b,k+1}, d_t u_h^{b,k})_{\Omega^b} \\
&\le \left\|d_t f^{b,n+1}\right\|_{\Omega^b}\left\|d_t u_h^{b,n+1}\right\|_{\Omega^b} + \left\|d_t f^{b,2}\right\|_{\Omega^b}\left\|d_t u_h^{b,1}\right\|_{\Omega^b} + \Delta t\sum_{k=2}^n\left\|d_{tt}f^{b,k+1}\right\|_{\Omega^b}\left\|d_t u_h^{b,k}\right\|_{\Omega^b},
\end{aligned}
\tag{70}
$$

where the summation term on the right hand side is zero if $n = 1$. Note that by eq. (10b) and eq. (11a) we have

$$\|d_t u_h^{b,k}\|_{\Omega^b}^2 \leq c_p^2 |||d_t \boldsymbol{u}_h^{b,k}|||_{v,b} \leq c_p^2 (c_{ae}^b \mu^b)^{-1} a_h^b(d_t \boldsymbol{u}_h^{b,k}, d_t \boldsymbol{u}_h^{b,k}). \tag{71}$$

Combining eqs. (69) to (71), and using the definition of X^n (see eq. (19)), we obtain

$$\begin{aligned} X^n \leq 3X^0 + \tfrac{1}{4} c_{ae}^f \mu^f \Delta t |||d_t \boldsymbol{u}_h^{f,1}|||_{v,f}^2 + \frac{2c_p^2}{c_{ae}^f \mu^f} \Delta t \sum_{k=1}^{n} \|d_t f^{f,k+1}\|_{\Omega^f}^2 + \frac{c_{td}^2 \mu^f}{\kappa} \Delta t \sum_{k=1}^{n} \|d_t g^{b,k+1}\|_{\Omega^b}^2 \\ + c_p (c_{ac}^b \mu^b)^{-1/2} (\|d_t f^{b,n+1}\|_{\Omega^b} (2X^n)^{1/2} + \|d_t f^{b,2}\|_{\Omega^b} (2X^0)^{1/2} + \Delta t \sum_{k=2}^{n} \|d_{tt} f^{b,k+1}\|_{\Omega^b} (2X^{k-1})^{1/2}). \end{aligned}$$

Assume $\max_{1 \leq k \leq n} X^k = X^n$. Then,

$$\begin{aligned} X^n \leq 3X^0 + \tfrac{1}{4} c_{ae}^f \mu^f \Delta t |||d_t \boldsymbol{u}_h^{f,1}|||_{v,f}^2 + \frac{2c_p^2}{c_{ae}^f \mu^f} \Delta t \sum_{k=1}^{n} \|d_t f^{f,k+1}\|_{\Omega^f}^2 + \frac{c_{td}^2 \mu^f}{\kappa} \Delta t \sum_{k=1}^{n} \|d_t g^{b,k+1}\|_{\Omega^b}^2 \\ + \sqrt{2} c_p (c_{ac}^b \mu^b)^{-1/2} (\|d_t f^{b,n+1}\|_{\Omega^b} + \|d_t f^{b,2}\|_{\Omega^b} + \Delta t \sum_{k=2}^{n} \|d_{tt} f^{b,k+1}\|_{\Omega^b}) (X^n)^{1/2}. \end{aligned} \tag{72}$$

For nonnegative A and B the following holds:

$$X^n \leq A + B(X^n)^{1/2} \Leftrightarrow ((X^n)^{1/2} - \tfrac{1}{2}B)^2 \leq A + \tfrac{1}{4}B^2 \Rightarrow X^n \leq [\tfrac{1}{2}B + (A + \tfrac{1}{4}B^2)^{1/2}]^2 \leq 2A + B^2.$$

Applying this to eq. (72) and using the definition of F^n (see eq. (20b)),

$$X^n \leq 6X^0 + \tfrac{1}{2} c_{ae}^f \mu^f \Delta t |||d_t \boldsymbol{u}_h^{f,1}|||_{v,f}^2 + F^n. \tag{73}$$

Note that if the assumption $\max_{1 \leq k \leq n} X^k = X^n$ does not hold, then there exists $0 \leq m < n$ such that $\max_{1 \leq k \leq n} X^k = X^m$. In this case, eq. (73) holds with n replaced by m and we find:

$$X^n < X^m \leq 6X^0 + \tfrac{1}{2} c_{ae}^f \mu^f \Delta t |||d_t \boldsymbol{u}_h^{f,1}|||_{v,f}^2 + F^m \leq 6X^0 + \tfrac{1}{2} c_{ae}^f \mu^f \Delta t |||d_t \boldsymbol{u}_h^{f,1}|||_{v,f}^2 + F^n.$$

This completes the proof of eq. (28).

B.3. **Proof of Lemma 4.5**

Before proving Lemma 4.5 we first prove the following minor modification of [20, Lemma 2].

Lemma B.1. *Let $\{A_i\}_i, \{B_i\}_i$, $\{E_i\}_i$, $\{\overline{E}_i\}_i$, $\{\widetilde{E}_i\}_i$, and $\{D_i\}_i$ be nonnegative sequences. Suppose these sequences satisfy*

$$A_n^2 + \sum_{i=1}^{n} B_i^2 \leq \sum_{i=1}^{n} E_i A_i + \sum_{i=1}^{n-1} \overline{E}_i A_i + \widetilde{E}_n A_n + \sum_{i=1}^{n} D_i, \tag{74}$$

for all $n \geq 1$. Then for any $n \geq 1$,

$$(A_n^2 + \sum_{i=1}^{n} B_i^2)^{1/2} \leq \sum_{i=1}^{n} E_i + \sum_{i=1}^{n-1} \overline{E}_i + \max_{1 \leq i \leq n} \widetilde{E}_i + (\sum_{i=1}^{n} D_i)^{1/2} \tag{75}$$

with $C > 0$ independent of n.

Proof. Suppose that

$$A_n^2 + \sum_{i=1}^{n} B_i^2 = \max_{1\le\ell\le n} \{A_\ell^2 + \sum_{i=1}^{\ell} B_i^2\}\,. \tag{76}$$

If $A_n^2 + \sum_{i=1}^n B_i^2 \le \sum_{i=1}^n D_i$, then eq. (75) naturally holds. If $A_n^2 + \sum_{i=1}^n B_i^2 > \sum_{i=1}^n D_i$, then eq. (74) and eq. (76) imply

$$\begin{aligned} A_n^2 + \sum_{i=1}^{n} B_i^2 &\le (\sum_{i=1}^{n} E_i + \sum_{i=1}^{n-1} \overline{E}_i + \widetilde{E}_n) \max_{1\le i\le n} A_i + (\sum_{i=1}^{n} D_i)^{1/2} (\sum_{i=1}^{n} D_i)^{1/2} \\ &\le (\sum_{i=1}^{n} E_i + \sum_{i=1}^{n-1} \overline{E}_i + \widetilde{E}_n + (\sum_{i=1}^{n} D_i)^{1/2})\,(A_n^2 + \sum_{i=1}^{n} B_i^2)^{1/2}\,. \end{aligned}$$

Equation (75) follows from dividing this by $(A_n^2 + \sum_{i=1}^n B_i^2)^{1/2}$.

If eq. (76) is not true, then there exists $1 \le n_0 < n$ such that

$$A_{n_0}^2 + \sum_{i=1}^{n_0} B_i^2 = \max_{1\le\ell\le n} \{A_\ell^2 + \sum_{i=0}^{\ell} B_i^2\}\,.$$

By the same argument as above, we have

$$(A_{n_0}^2 + \sum_{i=1}^{n_0} B_i^2)^{1/2} \le \sum_{i=1}^{n_0} E_i + \sum_{i=0}^{n_0-1} \overline{E}_i + \max_{1\le i\le n_0} \widetilde{E}_i + (\sum_{i=1}^{n_0} D_i)^{1/2}\,. \tag{77}$$

Then, eq. (75) follows by eq. (77), $n_0 < n$, and the nonnegativities of E_i, $\overline{E}_i$, $\widetilde{E}_i$, D_i. □

We now proceed with the proof of Lemma 4.5.

Proof of Lemma 4.5. **Step 1: proof of eq. (31).** Split eq. (17b) as

$$b_h^b(d_t\boldsymbol{u}_h^{b,n+1}, \boldsymbol{q}_h^b) + c_h((d_t p_h^{p,n+1}, d_t p_h^{b,n+1}), q_h^b) = 0, \tag{78a}$$

$$b_h^f(\boldsymbol{u}_h^{f,n+1}, \boldsymbol{q}_h^f) = 0, \tag{78b}$$

where we applied d_t to the terms in the Biot region to obtain eq. (78a). Now, choose $\boldsymbol{v}_h^f = \boldsymbol{u}_h^{f,n+1}$, $\boldsymbol{v}_h^b = d_t\boldsymbol{u}_h^{b,n+1}$, $\boldsymbol{q}_h^f = -\boldsymbol{p}_h^{f,n+1}$, $\boldsymbol{q}_h^b = -\boldsymbol{p}_h^{b,n+1}$, $w_h = z_h^{n+1}$, $\boldsymbol{q}_h^p = \boldsymbol{p}_h^{p,n+1}$ in eqs. (17) and (78) and add the resulting equations to find:

$$\begin{aligned} &(d_t u_h^{f,n+1}, u_h^{f,n+1})_{\Omega^f} + t_h(u_h^{f,n}; \boldsymbol{u}_h^{f,n+1}, \boldsymbol{u}_h^{f,n+1}) + a_h^f(\boldsymbol{u}_h^{f,n+1}, \boldsymbol{u}_h^{f,n+1}) + a_h^b(\boldsymbol{u}_h^{b,n+1}, d_t\boldsymbol{u}_h^{b,n+1}) \\ &\quad + \gamma\mu^f\kappa^{-1/2} \|(\bar{u}_h^{f,n+1} - d_t\bar{u}_h^{b,n+1})^t\|_{\Gamma_I}^2 + c_h((d_t p_h^{p,n+1}, d_t p_h^{b,n+1}), \alpha p_h^{p,n+1} - p_h^{b,n+1}) \\ &\quad + (c_0 d_t p_h^{p,n+1}, p_h^{p,n+1})_{\Omega^b} + \mu^f\kappa^{-1} \|z_h^{n+1}\|_{\Omega^b}^2 \\ &= (f^{f,n+1}, u_h^{f,n+1})_{\Omega^f} + (f^{b,n+1}, d_t u_h^{b,n+1})_{\Omega^b} + (g^{b,n+1}, p_h^{p,n+1})_{\Omega^b}. \end{aligned} \tag{79}$$

Using the algebraic inequality $a(a-b) \ge (a^2-b^2)/2$ for the discrete time-derivative terms, eq. (15) (which holds by the assumption that $\boldsymbol{u}_h^{f,n} \in \boldsymbol{B}_h^f$), the Cauchy–Schwarz inequality applied to the first and third term on the right hand side of eq. (79), and eq. (24), we find, using the definitions of A_i and B_i:

$$A_{n+1}^2 - A_n^2 + 2B_{n+1}^2 \leq \sqrt{2}\Delta t \left\| f^{f,n+1} \right\|_{\Omega^f} A_{n+1} + c_{td}(2\Delta t \mu^f \kappa^{-1})^{1/2} \left\| g^{b,n+1} \right\|_{\Omega^b} B_{n+1} + \Delta t (f^{b,n+1}, d_t u_h^{b,n+1})_{\Omega^b}. \quad (80)$$

Let us pause to note that, by summation-by-parts, using that $u_h^{b,0} = 0$, the Cauchy–Schwarz inequality, eq. (10b), the coercivity result eq. (11a), and the definition of A_i, we have

$$\begin{aligned}
\Delta t \sum_{i=0}^{n} (f^{b,i+1}, d_t u_h^{b,i+1})_{\Omega^b} =& (f^{b,n+1}, u_h^{b,n+1})_{\Omega^b} - \Delta t \sum_{i=1}^{n} (d_t f^{b,i+1}, u_h^{b,i})_{\Omega^b} \\
\leq& c_p \left\| f^{b,n+1} \right\|_{\Omega^b} |||\boldsymbol{u}_h^{b,n+1}|||_{v,b} + c_p \Delta t \sum_{i=1}^{n} \left\| d_t f^{b,i+1} \right\|_{\Omega^b} |||\boldsymbol{u}_h^{b,i}|||_{v,b} \\
\leq& c_p \left\| f^{b,n+1} \right\|_{\Omega^b} (c_{ae}^b \mu^b)^{-1/2} \sqrt{2} A_{n+1} \\
&+ c_p \Delta t \sum_{i=1}^{n} \left\| d_t f^{b,i+1} \right\|_{\Omega^b} (c_{ae}^b \mu^b)^{-1/2} \sqrt{2} A_i.
\end{aligned}$$

Therefore, replacing n by i in eq. (80) and summing for i from 0 to n, and Young's inequality,

$$\begin{aligned}
A_{n+1}^2 + \sum_{i=0}^{n} B_{i+1}^2 \leq& A_0^2 + \sqrt{2} \sum_{i=0}^{n} \Delta t \left\| f^{f,i+1} \right\|_{\Omega^f} A_{i+1} + \Delta t \sum_{i=1}^{n} \left\| d_t f^{b,i+1} \right\|_{\Omega^b} c_p (c_{ae}^b \mu^b)^{-1/2} \sqrt{2} A_i \\
&+ \left\| f^{b,n+1} \right\|_{\Omega^b} c_p (c_{ae}^b \mu^b)^{-1/2} \sqrt{2} A_{n+1} + \tfrac{1}{2} \sum_{i=0}^{n} c_{td}^2 \Delta t \mu^f \kappa^{-1} \left\| g^{b,i+1} \right\|_{\Omega^b}^2.
\end{aligned} \quad (81)$$

Defining

$$\begin{aligned}
E_i &= \sqrt{2}\Delta t \left\| f^{f,i} \right\|_{\Omega^f}, & \bar{E}_i &= \Delta t \left\| d_t f^{b,i+1} \right\|_{\Omega^b} c_p (c_{ae}^b \mu^b)^{-1/2} \sqrt{2}, \\
\tilde{E}_i &= \left\| f^{b,i} \right\|_{\Omega^b} c_p (c_{ae}^b \mu^b)^{-1/2} \sqrt{2}, & D_i &= \tfrac{1}{2} c_{td}^2 \Delta t \mu^f \kappa^{-1} \left\| g^{b,i} \right\|_{\Omega^b}^2,
\end{aligned}$$

and noting that $A_0 = 0$ by our assumption on the initial conditions, we can write eq. (81), for all $n \geq 1$, as

$$A_n^2 + \sum_{i=1}^{n} B_i^2 \leq \sum_{i=1}^{n} E_i A_i + \sum_{i=1}^{n-1} \bar{E}_i A_i + \tilde{E}_n A_n + \sum_{i=1}^{n} D_i.$$

Therefore, by Lemma B.1, for any $n \geq 1$,

$$\begin{aligned}
\Big(A_n^2 + \sum_{i=1}^{n} B_i^2\Big)^{1/2} \leq& \sqrt{2}\Delta t \sum_{i=1}^{n} \left\| f^{f,i} \right\|_{\Omega^f} + c_{td} (\tfrac{1}{2}\mu^f/\kappa)^{1/2} \Big(\Delta t \sum_{i=1}^{n} \left\| g^{b,i} \right\|_{\Omega^b}^2\Big)^{1/2} \\
&+ \sqrt{2} c_p (c_{ae}^b \mu^b)^{-1/2} \Delta t \sum_{i=1}^{n-1} \left\| d_t f^{b,i+1} \right\|_{\Omega^b} + \sqrt{2} c_p (c_{ae}^b \mu^b)^{-1/2} \max_{1 \leq i \leq n} \left\| f^{b,i} \right\|_{\Omega^b}.
\end{aligned}$$

By definition of A_i and B_i (see eq. (30)) and H (see eq. (22)) we conclude eq. (31) (since we consider $1 \leq n \leq N$).

Step 2: proof of eq. (32). Let us start with eq. (79). For $1 \le n \le N$:

$$
\begin{aligned}
&(d_t u_h^{f,n}, u_h^{f,n})_{\Omega^f} + t_h(u_h^{f,n-1}; \boldsymbol{u}_h^{f,n}, \boldsymbol{u}_h^{f,n}) + a_h^f(\boldsymbol{u}_h^{f,n}, \boldsymbol{u}_h^{f,n}) + a_h^b(\boldsymbol{u}_h^{b,n}, d_t \boldsymbol{u}_h^{b,n}) \\
&\quad + \gamma \mu^f \kappa^{-1/2} \left\| (\bar{u}_h^{f,n} - d_t \bar{u}_h^{b,n})^t \right\|_{\Gamma_I}^2 + c_h((d_t p_h^{p,n}, d_t p_h^{b,n}), \alpha p_h^{p,n} - p_h^{b,n}) \\
&\quad + (c_0 d_t p_h^{p,n}, p_h^{p,n})_{\Omega^b} + \mu^f \kappa^{-1} \left\| z_h^n \right\|_{\Omega^b}^2 = (f^{f,n}, u_h^{f,n})_{\Omega^f} + (f^{b,n}, d_t u_h^{b,n})_{\Omega^b} + (g^{b,n}, p_h^{p,n})_{\Omega^b}.
\end{aligned}
$$

By the assumption that $\boldsymbol{u}_h^{f,n-1} \in \boldsymbol{B}_h^f$ we know that

$$
t_h(u_h^{f,n-1}; \boldsymbol{u}_h^{f,n}, \boldsymbol{u}_h^{f,n}) + a_h^f(\boldsymbol{u}_h^{f,n}, \boldsymbol{u}_h^{f,n}) \ge \tfrac{1}{2} c_{ae}^f \mu^f \|\!|\boldsymbol{u}_h^{f,n}|\!\|_{v,f}^2,
$$

and so,

$$
\begin{aligned}
&\tfrac{1}{2} c_{ae}^f \mu^f \|\!|\boldsymbol{u}_h^{f,n}|\!\|_{v,f}^2 + \gamma \mu^f \kappa^{-1/2} \left\| (\bar{u}_h^{f,n} - d_t \bar{u}_h^{b,n})^t \right\|_{\Gamma_I}^2 + \mu^f \kappa^{-1} \left\| z_h^n \right\|_{\Omega^b}^2 \\
&\le |(f^{f,n}, u_h^{f,n})_{\Omega^f} + (f^{b,n}, d_t u_h^{b,n})_{\Omega^b} + (g^{b,n}, p_h^{p,n})_{\Omega^b} \\
&\quad - (d_t u_h^{f,n}, u_h^{f,n})_{\Omega^f} - a_h^b(\boldsymbol{u}_h^{b,n}, d_t \boldsymbol{u}_h^{b,n}) - c_h((d_t p_h^{p,n}, d_t p_h^{b,n}), \alpha p_h^{p,n} - p_h^{b,n}) - (c_0 d_t p_h^{p,n}, p_h^{p,n})_{\Omega^b}|.
\end{aligned}
\tag{82}
$$

The Cauchy–Schwarz and Young's inequalities, eq. (10b), eq. (11a), eq. (24), and the definitions of X^n eq. (19) and A_n eq. (30a) yields

$$
\begin{aligned}
|(f^{f,n}, u_h^{f,n})_{\Omega^f}| &\le \left\| f^{f,n} \right\|_{\Omega^f} \left\| u_h^{f,n} \right\|_{\Omega^f} \le \left\| f^{f,n} \right\|_{\Omega^f} \sqrt{2} A_n, && \text{(83a)} \\
|(g^{b,n}, p_h^{p,n})_{\Omega^b}| &\le \left\| g^{b,n} \right\|_{\Omega^b} c_{td} \mu^f \kappa^{-1} \left\| z_h^n \right\|_{\Omega^b} \le \tfrac{1}{2} c_{td}^2 \mu^f \kappa^{-1} \left\| g^{b,n} \right\|_{\Omega^b}^2 + \tfrac{1}{2} \mu^f \kappa^{-1} \left\| z_h^n \right\|_{\Omega^b}^2, && \text{(83b)} \\
|(f^{b,n}, d_t u_h^{b,n})_{\Omega^b}| &\le \left\| f^{b,n} \right\|_{\Omega^b} \left\| d_t u_h^{b,n} \right\|_{\Omega^b} \le c_p \left\| f^{b,n} \right\|_{\Omega^b} \|\!|d_t \boldsymbol{u}_h^{b,n}|\!\|_{v,b} && \text{(83c)} \\
&\le c_p (c_{ae}^b \mu^b)^{-1/2} \left\| f^{b,n} \right\|_{\Omega^b} a_h^b \left(d_t \boldsymbol{u}_h^{b,n}, d_t \boldsymbol{u}_h^{b,n} \right)^{1/2} \\
&\le c_p (c_{ae}^b \mu^b)^{-1/2} \left\| f^{b,n} \right\|_{\Omega^b} (2X^{n-1})^{1/2} \\
|(d_t u_h^{f,n}, u_h^{f,n})_{\Omega^f}| &\le \left\| d_t u_h^{f,n} \right\|_{\Omega^f} \left\| u_h^{f,n} \right\|_{\Omega^f}, && \text{(83d)} \\
|a_h^b(\boldsymbol{u}_h^{b,n}, d_t \boldsymbol{u}_h^{b,n})| &\le a_h^b(\boldsymbol{u}_h^{b,n}, \boldsymbol{u}_h^{b,n})^{1/2} a_h^b(d_t \boldsymbol{u}_h^{b,n}, d_t \boldsymbol{u}_h^{b,n})^{1/2}, && \text{(83e)} \\
|c_h((d_t p_h^{p,n}, d_t p_h^{b,n}), \alpha p_h^{p,n} - p_h^{b,n})| &\le \lambda^{-1/2} \left\| d_t(\alpha p_h^{p,n} - p_h^{b,n}) \right\|_{\Omega^b} \lambda^{-1/2} \left\| \alpha p_h^{p,n} - p_h^{b,n} \right\|_{\Omega^b}, && \text{(83f)} \\
|(c_0 d_t p_h^{p,n}, p_h^{p,n})_{\Omega^b}| &\le c_0^{1/2} \left\| p_h^{p,n} \right\|_{\Omega^b} c_0^{1/2} \left\| d_t p_h^{p,n} \right\|_{\Omega^b}. && \text{(83g)}
\end{aligned}
$$

Furthermore, combining eqs. (83d) to (83g) and using the definitions of A_n and X^n, we find

$$
\begin{aligned}
&|(d_t u_h^{f,n}, u_h^{f,n})_{\Omega^f}| + |a_h^b(\boldsymbol{u}_h^{b,n}, d_t \boldsymbol{u}_h^{b,n})| + |c_h((d_t p_h^{p,n}, d_t p_h^{b,n}), \alpha p_h^{p,n} - p_h^{b,n})| + |(c_0 d_t p_h^{p,n}, p_h^{p,n})_{\Omega^b}| \\
&\qquad \le 2A_n (X^{n-1})^{1/2}.
\end{aligned}
$$

Combining the above inequality, eqs. (83a) to (83c) with eq. (82), and using eq. (31) and eq. (29),

$$
\begin{aligned}
&\tfrac{1}{2} c_{ae}^f \mu^f \|\!|\boldsymbol{u}_h^{f,n}|\!\|_{v,f}^2 + \gamma \mu^f \kappa^{-1/2} \left\| (\bar{u}_h^{f,n} - d_t \bar{u}_h^{b,n})^t \right\|_{\Gamma_I}^2 + \tfrac{1}{2} \mu^f \kappa^{-1} \left\| z_h^n \right\|_{\Omega^b}^2 \\
&\le \left\| f^{f,n} \right\|_{\Omega^f} \sqrt{2} A_n + \tfrac{1}{2} c_{td}^2 \mu^f \kappa^{-1} \left\| g^{b,n} \right\|_{\Omega^b}^2 + c_p (c_{ae}^b \mu^b)^{-1/2} \left\| f^{b,n} \right\|_{\Omega^b} (2X^{n-1})^{1/2} + 2A_n (X^{n-1})^{1/2} \\
&\le \left\| f^{f,n} \right\|_{\Omega^f} H + \tfrac{1}{2} c_{td}^2 \mu^f \kappa^{-1} \left\| g^{b,n} \right\|_{\Omega^b}^2 + c_p (c_{ae}^b \mu^b)^{-1/2} \left\| f^{b,n} \right\|_{\Omega^b} (2G^{n-1})^{1/2} + H (2G^{n-1})^{1/2}.
\end{aligned}
$$

We therefore conclude eq. (32). □