

Strategyproof Mechanisms for Group-Fair Obnoxious Facility Location Problems

Jiaqian Li¹, Minming Li^{1*}, Hau Chan²

¹Department of Computer Science, City University of Hong Kong, HKSAR China

²School of Computing, University of Nebraska-Lincoln, USA

jiaqian.li@my.cityu.edu.hk, minming.li@cityu.edu.hk, hchan3@unl.edu

Abstract

We study the group-fair obnoxious facility location problems from the mechanism design perspective where agents belong to different groups and have private location preferences on the undesirable locations of the facility. Our main goal is to design strategyproof mechanisms that elicit the true location preferences from the agents and determine a facility location that approximately optimizes several group-fair objectives. We first consider the maximum total and average group cost (group-fair) objectives. For these objectives, we propose deterministic mechanisms that achieve 3-approximation ratios and provide matching lower bounds. We then provide the characterization of 2-candidate strategyproof randomized mechanisms. Leveraging the characterization, we design randomized mechanisms with improved approximation ratios of 2 for both objectives. We also provide randomized lower bounds of $5/4$ for both objectives. Moreover, we investigate intergroup and intragroup fairness (IIF) objectives, addressing fairness between groups and within each group. We present a mechanism that achieves a 4-approximation for the IIF objectives and provide tight lower bounds.

Introduction

In recent decades, facility location problems have received considerable attention in operations research, computer science, and economic communities due to their applicability to address real-world challenges and theoretical interests (see, e.g., (Hakimi 1964; Church and ReVelle 1976; Jain et al. 2003)). In a typical facility location problem, a social planner is tasked with identifying a suitable geographical location for a facility to best serve the targeted agent population according to a given optimization objective, defined based on the distances of agents' locations or preferences to the facility location (Hakimi 1964). These problems can be used to address locating a public school to provide education to students living in the surrounding area, locating a public park to provide recreational space for local residents, and locating a hospital to provide health services to citizens in the communities.

In artificial intelligence and economics, the studies of facility location problems have largely focused on the mech-

anism design perspective where the facility location preferences of the agents are private (Chan et al. 2021). The primary goal of these studies aims to design (*strategyproof*) mechanisms that elicit agent location truthfully and determine a facility location to (approximately) optimize certain objectives (Moulin 1980; Procaccia and Tennenholz 2013).

While the majority of the related mechanism design studies consider facilities that are pleasant for the agents, a few studies (Cheng, Yu, and Zhang 2013; Ibara and Nagamochi 2012; Ye, Mei, and Zhang 2015) consider locating obnoxious facilities which are undesirable for agents. Indeed, facilities such as nuclear power reactors, landfill sites, and chemical plants are often unpleasant and do not provide service directly to the agents. For instance, landfill sites are unpleasant for the nearby agents because of the potential exposure to odors/gases/chemicals (e.g., lead, ammonia, and hydrogen sulfide) and other health risks (e.g., cancers and developmental disabilities). Similarly, exposure to radiation from nuclear power plants can cause cancer and leukemia. As a result, agents have preferences on the undesirable locations of the obnoxious facility. Therefore, existing studies design mechanisms to locate the facility *far away* from the agents' undesirable location preferences.

Group Fairness and Obnoxious Facilities. In real-world situations, it is well-documented that groups of agents (e.g., environmental and community activists) fight to advocate for the location of an undesirable facility (e.g., a coal ash landfill, nuclear plant, or petrochemical plant) through public and legal channels. Moreover, when locating obnoxious facilities, unfairness can unexpectedly occur for groups of agents. For instance, for any obnoxious facility location, we can create a problem instance that is good for one group of agents and bad for another group of agents who dislike locations near the obnoxious facility. Therefore, obnoxious facility location problems should carefully consider groups of agents and the potential unfairness incurred by these groups.

While recent studies propose to enforce fairness in facility location by designing strategyproof mechanisms subject to group fairness objectives (Zhou, Li, and Chan 2022) or criteria/constraints (Aziz et al. 2022a,b, 2023), no previous studies consider group fairness for obnoxious facility location problems. Therefore, following the work of Zhou, Li, and Chan (2022), our focus is to consider natural group-fair

*Corresponding author

Copyright © 2024, Association for the Advancement of Artificial Intelligence (www.aaai.org). All rights reserved.

Objective	Deterministic	Randomized
$mtgc$	UB: 3 (Thm. 1) LB: 3 (Thm. 2)	UB: 2 (Thm. 4) LB: $\frac{5}{4}$ (Thm. 5)
$magc$	UB: 3 (Thm. 7) LB: 3 (Thm. 8)	UB: 2 (Thm. 9) LB: $\frac{5}{4}$ (Thm. 10)
Mechanism	UB (IIF₁ & IIF₂)	LB
BGMV	4 (Thm. 11)	4 (Thm. 12)

Table 1: Summary of the approximation ratios and lower bounds for group-fair objectives (Section) and IIF objectives (Section). UB and LB stand for the upper bound and lower bound, respectively.

objectives that qualify the unfairness of groups of agents and design strategyproof mechanisms that approximately optimize the objectives in obnoxious facility location problems.

Our Contributions

We investigate the group-fair obnoxious facility location problems where agents are located on a closed interval. We design strategyproof mechanisms which (approximately) optimize the maximum total and average group cost (group-fair) objectives, along with the intergroup and intragroup fairness (IIF) objectives. Our results are summarized in Table 1. More specifically:

- We introduce deterministic mechanisms (LHGV and BGMV) for minimizing the maximum total group cost ($mtgc$) or the maximum average group cost ($magc$). These mechanisms achieve 3-approximation ratios. We also provide tight lower bounds for these objectives.
- We provide a characterization of any (group) strategyproof 2-candidate randomized mechanisms. We define monotonicity and establish the equivalence between strategyproofness and group strategyproofness within the context of a 2-candidate mechanism. Building on this insight, we propose optimal randomized 2-candidate mechanisms (PBPM and NPBPM) for both group-fair objectives, effectively improving the approximation ratio to 2.
- Lastly, we investigate two objectives that capture intergroup and intragroup fairness (IIF), considering group fairness within and among groups. We show that the BGMV mechanism achieves a 4-approximation for the IIF objectives. Additionally, we establish a tight lower bound that applies to all mechanisms for these objectives.

We note that all the designed mechanisms are group strategyproof. Due to the page limit, we only included some proof sketches.

Related Work

We discuss related work on facility location problems that consider fairness or obnoxious facilities from the mechanism design perspective.

Fairness in Facility Location Problems A growing body of research is exploring fairness from the mechanism design perspective. Procaccia and Tennenholtz (2013) initiate the study on approximate mechanism design without money. They examine the fairness objective of minimizing the maximum cost among agents for facility location problems. More recently, various envy-related concepts are investigated, including minimax envy (Cai, Filos-Ratsikas, and Tang 2016) and envy ratio (Ding et al. 2020). These concepts aim to minimize the maximum normalized cost difference between any two agents and the maximum ratios of utility for any pair of agents. A very recent study (Zhou, Li, and Chan 2022) incorporates group fairness into the facility location problems while adopting the approximate mechanism design approach. Beyond the objective-centric mechanism design, Aziz et al. (2022a,b, 2023) investigate mechanism design for the proportional fairness notions (Individual Fair Share and Unanimous Fair Share) for both standard and obnoxious facility location problems.

Obnoxious Facility Location Problems In certain situations, agents may express displease towards the presence of a facility, preferring greater distance from it. These scenarios align with obnoxious facility location problems, where the objective shifts to maximizing the sum of distances from agents to the facility (Church and Garfinkel 1978). From the mechanism design perspective, the first work is by Cheng, Yu, and Zhang (2013), following Procaccia and Tennenholtz’s seminal work (Procaccia and Tennenholtz 2013). In their context, the distance between agents and the facility is defined as the utility, and the objective becomes maximizing social welfare. Later on, Ibara and Nagamochi (2012) provide characterizations of deterministic 2-candidate strategyproof mechanisms for the obnoxious facility game and Ye, Mei, and Zhang (2015) study some other objectives (e.g., the sum of squares of agents’ utility). A recent survey (Chan et al. 2021) offers a comprehensive overview of recent progress from the mechanism design perspective.

Preliminaries

In this section, we provide necessary notations, objectives, and definitions for the group-fair obnoxious facility location problems and the corresponding mechanism design settings.

Notations

Consider a set of n agents denoted as $N = \{1, 2, \dots, n\}$, located along the interval $I = [0, 1]$. Each agent $i \in N$ belongs to a specific group based on their affiliation (e.g., different groups of activists). Let $G = \{G_1, G_2, \dots, G_m\}$ represent the set of mutually exclusive groups of agents, with $|G_j|$ indicating the number of agents in group G_j .

The profile of each agent i is denoted as $r_i = (x_i, g_i)$, where $x_i \in I$ is the location of agent i , and $g_i \in \{1, 2, \dots, m\}$ corresponds to the group affiliation of agent i . The full profile $\mathbf{r} = \{r_1, r_2, \dots, r_n\}$ represents the location and group information for all agents, and $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ is the associated location profile. We refer the set $\mathbf{x}_g = \{x_i : i \in G_g\}$ to be the location profiles of agents within group G_g .

A deterministic mechanism is defined as a function f that maps a profile \mathbf{r} to a facility location $y \in I$. Alternatively, a randomized mechanism is a function f that assigns a profile \mathbf{r} to a probability distribution Y over I , where the facility location y is a potential realization of Y . The distance between two points $a, b \in I$ is calculated as $d(a, b) = |a - b|$. Given a deterministic (or randomized) mechanism f and a profile \mathbf{r} , the utility of an agent $i \in N$ is represented as $u(f(\mathbf{r}), x_i) = d(f(\mathbf{r}), x_i)$ (or the expected distance $\mathbb{E}_{y \sim Y}[d(y, x_i)]$). The cost for agent i is subsequently defined as $c(f(\mathbf{r}), x_i) = 1 - u(f(\mathbf{r}), x_i)$. In the context of this obnoxious facility location problem, each agent aims to increase their distance from the facility to enhance their utility and reduce their cost.

As each agent is aware of the mechanism, they may be incentivized to falsely report their location in an attempt to manipulate the outcome to benefit themselves. The challenge is to design mechanisms that promote truthful reporting while (approximately) optimizing a specific cost objective.

In our setting, we assume an agent can only misreport their location. The group information is known publicly.

Definition 1 (Strategyproofness). A mechanism f is strategyproof (SP) if no agent can gain an advantage by misreporting their location, regardless of the locations reported by other agents. Formally, given any profile $\mathbf{r} = \{r_1, r_2, \dots, r_n\}$, let \mathbf{r}_{-i} denote the profile excluding agent i . Then, for every $i \in N$, let $r'_i = (x'_i, g_i)$ be another profile with arbitrary $x'_i \in I$, it holds that $c(f(\mathbf{r}), x_i) \leq c(f(r'_i, \mathbf{r}_{-i}), x_i)$.

Definition 2 (Group Strategyproofness). A mechanism f is group strategyproof (GSP) if, for any group of agents, at least one agent cannot benefit from false reporting. Formally, given any profile $\mathbf{r} = \{r_1, r_2, \dots, r_n\}$, for any $S \subseteq N$ and for any $\mathbf{x}'_S \in I^S$, let \mathbf{r}_{-S} be the profile excluding the set of agents S , and $\mathbf{r}'_S = (x'_i, g_i)_{i \in S}$. Then, at least one agent $i \in S$ must satisfy $c(f(\mathbf{r}), x_i) \leq c(f(\mathbf{r}'_S, \mathbf{r}_{-S}), x_i)$.

Given fixed group sizes $|G_1|, \dots, |G_m|$, a deterministic mechanism $f : (I, [m])^n \rightarrow I$ is classified as a p -candidate mechanism if there is a set $C_f \in I^p$, such that for every \mathbf{r} , $f(\mathbf{r})$ is an element of C_f . Likewise, a randomized mechanism $f : (I, [m])^n \rightarrow I$ is a p -candidate mechanism if there is a set $C_f \in I^p$, such that for every \mathbf{r} , $f(\mathbf{r})$ is a distribution over C_f , i.e., the outcome of $f(\mathbf{r})$ is always within the set C_f .

Optimization Objectives

We adopt two group-fair objectives proposed by Zhou, Li, and Chan (2022). The first objective considers total group cost, which is the cumulative cost of all members of a group. Our goal is to prevent any group from bearing a high total cost. Hence, the first group-fair objective is to minimize the maximum total group cost ($mtgc$). Formally, the maximum total group cost ($mtgc$) of a facility location y with respect to the profile \mathbf{r} is defined as:

$$mtgc(y, \mathbf{r}) = \max_{1 \leq j \leq m} \left\{ \sum_{i \in G_j} c(y, x_i) \right\};$$

Type	Mechanism	Approx. Ratio	LB
Deterministic	MV LHGV	$\Omega(m)$ 3	3
Randomized	PBPM	2	$\frac{5}{4}$

Table 2: Summary of the approximation ratios and lower bounds for deterministic and randomized mechanisms ($mtgc$).

the maximum total group cost of a distribution Y with respect to \mathbf{r} is $mtgc(Y, \mathbf{r}) = \mathbb{E}_{y \sim Y}[mtgc(y, \mathbf{r})]$.

The second group-fair objective aims to minimize the maximum average group cost ($magc$). This objective aims to alleviate the impact of group size and normalize each total group cost by their group size. It is defined as follows:

$$magc(y, \mathbf{r}) = \max_{1 \leq j \leq m} \left\{ \frac{\sum_{i \in G_j} c(y, x_i)}{|G_j|} \right\};$$

the maximum average group cost of a distribution Y is $magc(Y, \mathbf{r}) = \mathbb{E}_{y \sim Y}[magc(y, \mathbf{r})]$.

Let $y^* \in \operatorname{argmin}_y mtgc(y, \mathbf{r})$ denote the optimal location for minimizing the objective $mtgc$. We measure the performance of a mechanism f by comparing the objective achieved by f and the optimal solution. Specifically, f is an α -approximation if for any profile \mathbf{r} , it holds that $mtgc(f(\mathbf{r}), \mathbf{r}) \leq \alpha \cdot mtgc(y^*, \mathbf{r})$. The approximation performance of a mechanism for $magc$ can be defined similarly.

Group-Fair Cost Objectives

In this section, we develop both deterministic and randomized strategyproof mechanisms that approximately optimize the group-fair objectives defined in Section . Additionally, we provide lower bounds applicable to all strategyproof mechanisms.

The Maximum Total Group Cost

In this subsection, we focus on minimizing the maximum total group cost ($mtgc$). Table 2 summarizes our results.

We begin by examining the mechanism proposed by Cheng, Yu, and Zhang (2013), which investigates the classical obnoxious facility location problems. This mechanism aims to maximize social welfare (the sum of each agent's utility) without considering any group-fair objective.

Mechanism 1 (Majority Vote (MV)). Given a location profile \mathbf{x} on I , let n_1 be the number of agents located on $[0, \frac{1}{2}]$ and n_2 be the number of agents on $(\frac{1}{2}, 1]$. If $n_1 \leq n_2$, the mechanism returns the left endpoint 0; otherwise, it returns the right endpoint 1.

Since Mechanism 1 does not leverage the group information, the strategyproofness still holds in our model. However, for the $mtgc$ objective, it may perform poorly, as demonstrated in the following example.

Example 1. There are $(2m - 3)$ agents from m groups. For $1 \leq j \leq m - 1$, group G_j only contains one agent located at 0, while all other $(m - 2)$ agents belong to group G_m and are located at 1. In this case, the optimal solution places the facility at $\frac{1}{m-1}$ with a total group cost of $\frac{m-2}{m-1}$ in each group. However, Mechanism 1 outputs 1 with a total group cost of $(m - 2)$ in group G_m , which implies an $(m - 1)$ -approximation for the objective $mtgc$.

We propose the following deterministic mechanism, which incorporates the group information.

Mechanism 2. (Largest Half-Group Vote (LHGV)) For each group G_j , let $n_{j,1}$ and $n_{j,2}$ be the number of agents on $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$, respectively. We further denote the larger one of $n_{j,1}$ or $n_{j,2}$ by n_j . Let $g \in \operatorname{argmax}_{1 \leq j \leq m} n_j$, which represents the group containing the most agents on half of the interval among all groups (break ties by choosing the smallest index). The mechanism places the facility y at 1 if $n_{g,1} > n_{g,2}$. Otherwise, it places the facility at 0.

Notice that Mechanism 2 first selects a group and then performs Mechanism 1 on this subset of agents. We summarize this as a two-step mechanism design model which will be frequently used later:

1. Select a group G_g .
2. Design a mechanism for x_g . Recall that x_g is the location profile of agents in G_g .

The following proposition and theorem show that Mechanism 2 is GSP and has a good constant approximation ratio.

Proposition 1. Mechanism 2 is group strategyproof.

Theorem 1. Mechanism 2 is a 3-approximation for the objective $mtgc$.

Proof Sketch. For any profile r , let ALG and OPT denote the maximum total group cost given by LHGV and the optimal solution, respectively. Recall that in Mechanism 2, G_g represents the group with the largest half-group.

Without loss of generality, we assume that LHGV places the facility at 1. When $y^* \in (\frac{1}{2}, 1]$, y^* is closer to 1, we prove that a slight movement from y^* to 1 will not make a big difference between OPT and ALG as the cost of an agent can be reduced by at most $|1 - y^*|$. When $y^* \in [0, \frac{1}{2}]$, we prove that $OPT \geq \frac{1}{2}n_{g,1}$ and $ALG \leq \frac{3}{2}n_{g,1}$. \square

Next, we provide a tight lower bound to complement our upper bound result.

Theorem 2. Any deterministic strategyproof mechanism has an approximation ratio of at least 3 for the objective $mtgc$.

Before the proof, we introduce a lemma based on Ibara and Nagamochi (2012)'s result.

Lemma 1. Any deterministic strategyproof mechanism is a 2-candidate mechanism.

Ibara and Nagamochi prove Lemma 1 for the classical obnoxious facility problem (single-group case), and the idea can be extended to the multiple-group case, which is used to prove two later theorems (Theorem 12 and 13). Based on Lemma 1, we provide the following proof for the theorem.

Proof of Theorem 2. In the proof, we always consider the profiles with two agents from a single group. Hence, we use the location profile to represent the agent profile since the agent's group identity is always 1.

For the profile $(0, 0)$ (two agents located at 0), the maximum total group cost is 0 when the facility is placed at 1. Hence, the mechanism must place the facility at 1 to ensure a finite approximation ratio. Similarly, for the profile $(1, 1)$, the mechanism must place the facility at 0. Therefore, the candidate set $C_f = \{0, 1\}$ and there is no space to include other candidates due to Lemma 1.

Next, we show any deterministic mechanism with $C_f = \{0, 1\}$ has an approximation ratio of at least 3. We start with the profile $(0, 1)$. Without loss of generality, we assume that the output is 0. Then, for the profile $(0, \frac{1}{2} + \epsilon)$ with sufficiently small $\epsilon > 0$, the facility needs to stay at 0 with a total group cost of $\frac{3}{2} - \epsilon$. Otherwise, the agent at $\frac{1}{2} + \epsilon$ can deviate to 1 to obtain a lower cost. However, the optimal solution places the facility at 1 with a total group cost of $\frac{1}{2} + \epsilon$. This leads to an approximation ratio of 3 as ϵ goes to 0. \square

Next, we study whether randomized mechanisms can help improve the approximation ratio. We define the following monotonicity property.

Definition 3 (2-candidate Monotonicity). Let $C_f = \{c_1, c_2\} \in I^2$ be the candidate set of a randomized mechanism f . Without loss of generality, we assume $c_1 < c_2$. Let α be the probability of placing the facility at c_1 , and then the probability to place the facility at c_2 is $1 - \alpha$. f is monotone if

- α is a function with respect to $\mathbf{n} = (n_{j,1}, n_{j,2}, n_{j,3})_{j=1}^m$, where $n_{j,1}, n_{j,2}, n_{j,3}$ are numbers of agents in G_j located in $[0, \frac{c_1+c_2}{2}]$, at $\frac{c_1+c_2}{2}$, and in $(\frac{c_1+c_2}{2}, 1]$, respectively.
- Let \mathbf{n}_{-j} be the input except group G_j 's information. We have that

$$\alpha(\mathbf{n}) \leq \alpha((n_{j,1} - 1, n_{j,2}, n_{j,3} + 1), \mathbf{n}_{-j}) \quad (1)$$

and

$$\begin{aligned} & \alpha((n_{j,1}, n_{j,2} + 1, n_{j,3} - 1), \mathbf{n}_{-j}) \\ & \leq \alpha(\mathbf{n}) \leq \alpha((n_{j,1} - 1, n_{j,2} + 1, n_{j,3}), \mathbf{n}_{-j}) \end{aligned} \quad (2)$$

hold.

Inequality (1) relates to the case where an agent misreports either from $[0, \frac{c_1+c_2}{2}]$ to $(\frac{c_1+c_2}{2}, 1]$ or from $(\frac{c_1+c_2}{2}, 1]$ to $[0, \frac{c_1+c_2}{2}]$. Inequality (2) relates to the case where an agent misreports from either the left or right half-interval to the midpoint at $\frac{c_1+c_2}{2}$.

Theorem 3. For any randomized 2-candidate mechanism f with $C_f = \{c_1, c_2\}$, the following statements are equivalent: (1) f is strategyproof. (2) f is group strategyproof. (3) f is monotone.

Proof Sketch. We prove the equivalence using Fig. 1.

From SP to Monotonicity. By SP, the outcome does not change when agents within $[0, \frac{c_1+c_2}{2}]$ or $[\frac{c_1+c_2}{2}, 1]$ misreport to any location within their respective intervals. Hence,

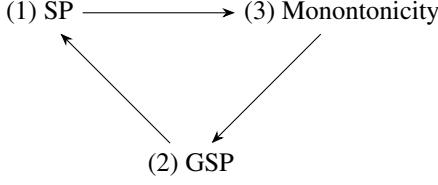


Figure 1: Proving the equivalence of statements about 2-candidate mechanisms.

we can prove that α depends solely on \mathbf{n} . Then, consider any profile \mathbf{r} , we prove the validity of Inequality (1) and (2) using strategyproofness for the case that an agent misreports across the midpoint $\frac{c_1+c_2}{2}$.

From Monotonicity to GSP. Let S be the set of agents that misreport their locations. If every agent in set S is located within the interval $[0, \frac{c_1+c_2}{2}]$, or alternatively, if they all reside within the interval $(\frac{c_1+c_2}{2}, 1]$, we inductively use Inequality (1) and (2) to prove they will not benefit from deviating. Otherwise, let i and j be two agents in S such that $x_i \leq \frac{c_1+c_2}{2}$ and $x_j \geq \frac{c_1+c_2}{2}$. We can prove either agent i or j will not benefit from deviating.

From GSP to SP. Consider a single individual as a group of size 1, the agent cannot benefit by misreporting their location due to group strategyproofness. This implies strategyproofness. \square

To facilitate our mechanism design, we define the simplified monotonicity property using a simple tie-breaking rule for agents at $\frac{c_1+c_2}{2}$ inspired by the above characterization.

Definition 4 (Simplified Monotonicity). *f* is simplified monotone if

- α , the probability of placing the facility at c_1 , is a function with respect to $\mathbf{n} = (n_{j,1}, n_{j,2})_{j=1}^m$, where $n_{j,1}, n_{j,2}$ are numbers of agents in G_j and located in $[0, \frac{c_1+c_2}{2}]$ and in $(\frac{c_1+c_2}{2}, 1]$, respectively.
- Let \mathbf{n}_{-j} be the input excluding group G_j 's information. $\alpha(\mathbf{n}) \leq \alpha((n_{j,1} - 1, n_{j,2} + 1), \mathbf{n}_{-j})$ holds true.

Corollary 1. Any simplified monotone mechanism is GSP.

Simplified monotonicity is a stronger concept than monotonicity, although the inequality constraint becomes simpler. Consider the “toy” mechanism that takes a single agent's location, denoted by x , as input. If $x \in [0, \frac{1}{2}]$ or $x \in (\frac{1}{2}, 1]$, the facility is placed at 1 or 0 with probability 1, respectively. Otherwise, the facility is placed at 0 or 1 with equal probability. While this mechanism adheres to monotonicity, it does not satisfy simplified monotonicity.

Theorem 3 and Corollary 1 indicate how to better utilize the group information. We propose the following 2-candidate mechanism.

Mechanism 3 (Probabilistic Balanced Placement Mechanism (PBPM)). *Let $C_f = \{0, 1\}$. Given an input profile \mathbf{r} , the mechanism counts the numbers of agents in $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ for each group G_j , denoted by a count tuple $\mathbf{n} = (n_{j,1}, n_{j,2})_{j=1}^m$.*

Let α solve

$$\begin{aligned} \alpha + (1 - \alpha) \frac{\max_{j=1}^m \{\frac{1}{2}n_{j,1} + n_{j,2}\}}{\max_{j=1}^m \{\frac{1}{2}n_{j,1}\}} \\ = (1 - \alpha) + \alpha \frac{\max_{j=1}^m \{\frac{1}{2}n_{j,2} + n_{j,1}\}}{\max_{j=1}^m \{\frac{1}{2}n_{j,2}\}}. \end{aligned} \quad (3)$$

Specially, set $\alpha = 1$ when $\max_{j=1}^m \{n_{j,1}\} = 0$, and set $\alpha = 0$ when $\max_{j=1}^m \{n_{j,2}\} = 0$. The mechanism places the facility at 0 (resp. 1) with probability α (resp. $1 - \alpha$).

Proposition 2. Mechanism 3 is group strategyproof.

The idea behind Mechanism 3 is based on minimizing the worst ratio among profiles where the optimal solution places the facility at one endpoint. Specifically, when the optimal facility location is at 0, placing the facility at 1 results in a ratio of at most $\frac{\max_{j=1}^m \{\frac{1}{2}n_{j,1} + n_{j,2}\}}{\max_{j=1}^m \{\frac{1}{2}n_{j,1}\}}$. Conversely, if the optimal facility location is at 1, placing the facility at 0 incurs a ratio of at most $\frac{\max_{j=1}^m \{\frac{1}{2}n_{j,2} + n_{j,1}\}}{\max_{j=1}^m \{\frac{1}{2}n_{j,2}\}}$. Mechanism 3 identifies the best value of α that achieves a balance in both cases.

The next theorem shows that Mechanism 3 improves the approximation ratio of the earlier mechanism even for profiles where the optimal location is not at 0 and 1.

Theorem 4. Mechanism 3 is a 2-approximation for the objective mtgc .

Proof Sketch. For any profile \mathbf{r} , let ALG and OPT denote the expected maximum total group cost given by PBPM and the optimal solution, respectively. Let c_0 and c_1 be the maximum total group cost when the facility is placed at 0 or 1, respectively. By definition, $ALG = \alpha \cdot c_0 + (1 - \alpha) \cdot c_1$. Without loss of generality, we assume that $y^* \in [0, \frac{1}{2}]$.

We can prove that $c_0 \leq 2 \cdot OPT$. Thus, we can further assume that $c_1 \geq 2 \cdot OPT$. Otherwise, the approximation ratio is at most 2. We notice that α, c_0 and c_1 are determined by a constant number of groups. We aim to construct a succinct profile \mathbf{r}' with the property that $c'_0 \geq c_0, c'_1 \geq c_1$ and $\alpha' \leq \alpha$. Then, the approximation ratio of \mathbf{r}' is greater than or equal to that of \mathbf{r} as $ALG \leq ALG' \leq OPT'$.

The construction of \mathbf{r}' depends on the range of y^* . For example, when $y^* \in [\frac{1}{4}, \frac{1}{2}]$, the construction is as follows:

- In profile \mathbf{r} , there is a group that achieves the highest total group cost at location 1. Within this group, we relocate agents on $[2y^* - \frac{1}{2}, \frac{1}{2}]$ to the position $\frac{1}{2}$ and agents on $(\frac{1}{2}, 1]$ to 1. We denote this group by G_g . We introduce group 1 in \mathbf{r}' , which contains $n'_{1,1}$ agents located at $\frac{1}{2}$ and $n'_{1,2}$ agents located at 1, where $n'_{1,2}$ is the same as the number of agents in G_g on $(\frac{1}{2}, 1]$, and $n'_{1,1}$ is determined by the equation:

$$\begin{aligned} & n'_{1,1} \cdot (1 - \underbrace{|\frac{1}{2} - y^*|}_{\text{cost at } y^* \text{ incurred by agents at } \frac{1}{2} \text{ in the new group 1}}) \\ &= \underbrace{\sum_{j \in G_g \wedge x_j \leq \frac{1}{2}} (1 - |y^* - x_j|)}_{\text{cost at } y^* \text{ incurred by agents in } G_g \text{ on } [0, \frac{1}{2}] \text{ after relocation}}. \end{aligned}$$

Type	Mechanism	Approx. Ratio	LB
Deterministic	MV	4	3
	LHGV	4	
	BGMV	3	
Randomized	NPBPM	2	$\frac{5}{4}$

Table 3: Summary of the approximation ratios and lower bounds for deterministic and randomized mechanisms (*magc*).

- Group 2 in profile \mathbf{r}' consists of $\frac{OPT}{1-y^*}$ agents located at 0.
- In profile \mathbf{r} , there exists a group G_g that achieves $\max_{j=1}^m \{n_{j,1} + \frac{1}{2}n_{j,2}\}$. We introduce group 3 in profile \mathbf{r}' , where the number of agents in the intervals $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ match the number of agents in G_g within the intervals $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$, respectively. Moreover, all agents in the newly created group are located at 0 or 1.

We notice that \mathbf{r}' only contains agents located at 0, $\frac{1}{2}$ and 1. We can prove that for such succinct profile \mathbf{r}' , the approximation ratio is at most 2. \square

The following theorem establishes a lower bound for randomized strategyproof mechanisms in general.

Theorem 5. *Any randomized strategyproof mechanism has an approximation ratio of at least $\frac{5}{4}$ for the objective *mtgc*.*

Proof Sketch. Let Y and Y' be the output distributions for profiles $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{1}{3}, 1)$ (two agents, single group), respectively. We use Y to find a constraint on Y' . We notice that $\mathbb{E}_Y(|y - \frac{1}{3}|) + \mathbb{E}_Y(|y - \frac{2}{3}|) \leq 1$ holds for the profile $(\frac{1}{3}, \frac{2}{3})$. Without loss of generality, we assume $\mathbb{E}_Y(|y - \frac{2}{3}|) \leq \frac{1}{2}$. It follows that $\mathbb{E}_{Y'}(|y' - \frac{2}{3}|) \leq \frac{1}{2}$ for the profile $(\frac{1}{3}, 1)$. Meanwhile, the optimal location for the profile $(\frac{1}{3}, 1)$ is 0. We prove the lower bound using this small misalignment. \square

The Maximum Average Group Cost

In this subsection, we focus on minimizing the maximum average group cost (*magc*). Table 3 summarizes our findings.

Some mechanisms for the *mtgc* objective already achieve a constant approximation ratio for the *magc* objective.

Theorem 6. *Mechanism 1 (MV) and 2 (LHGV) have an approximation ratio of 4 for the objective *magc*.*

Using the two-step model (as discussed in Section), we present the following deterministic mechanism.

Mechanism 4 (Biased Group Majority Vote (BGMV)). *For each group G_j , let $n_{j,1}$ and $n_{j,2}$ be the number of agents on $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$, respectively. We further denote the larger one of $\frac{n_{j,1}}{|G_j|}$ or $\frac{n_{j,2}}{|G_j|}$ by p_j , which is the larger proportion of the number of agents on the half interval to the number of agents on the whole interval. Let $g \in \arg\max_{1 \leq j \leq m} p_j$, the most biased group in proportion (break ties by choosing*

the smallest index). The mechanism places the facility y at 1 if $n_{g,1} > n_{g,2}$. Otherwise it places the facility at 0.

We notice that compared with Mechanism 2, we change the first step – the group selection step in Mechanism 4. We will show that such a change can improve the approximation ratio from 4 to 3 in Theorem 7.

Proposition 3. *Mechanism 4 is group strategyproof.*

Theorem 7. *Mechanism 4 is a 3-approximation for the objective *magc*.*

Using the same profiles constructed in Theorem 2, we have the following lower bound.

Theorem 8. *Any deterministic strategyproof mechanism for the objective *magc* has an approximation ratio of at least 3.*

Next, we study whether randomized mechanisms can enhance the approximation ratio. We propose the following randomized mechanism based on simplified monotonicity.

Mechanism 5 (Normalized Probabilistic Balanced Placement Mechanism, NPBPM). *Let $C_f = \{0, 1\}$. Given an input profile \mathbf{r} , the mechanism counts the numbers of agents in $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ for each group G_j , denoted by $(n_{j,1}, n_{j,2})_{j=1}^m$. Let $p_{j,1}$ (resp. $p_{j,2}$) be $\frac{n_{j,1}}{n_{j,1} + n_{j,2}}$ (resp. $\frac{n_{j,2}}{n_{j,1} + n_{j,2}}$) for $j = 1, 2, \dots, m$.*

Let α solve

$$\alpha + (1 - \alpha) \frac{\max_{j=1}^m \{\frac{1}{2}p_{j,1} + p_{j,2}\}}{\max_{j=1}^m \{\frac{1}{2}p_{j,1}\}} = (1 - \alpha) + \alpha \frac{\max_{j=1}^m \{\frac{1}{2}p_{j,2} + p_{j,1}\}}{\max_{j=1}^m \{\frac{1}{2}p_{j,2}\}}. \quad (4)$$

Specially, set $\alpha = 1$ when $\max_{j=1}^m \{p_{j,1}\} = 0$, and set $\alpha = 0$ when $\max_{j=1}^m \{p_{j,2}\} = 0$. The mechanism puts the facility at 0 (resp. 1) with probability α (resp. $1 - \alpha$).

Using a similar method as that employed to prove the strategyproofness of Mechanism 3, we can show that Mechanism 5 is GSP.

Proposition 4. *Mechanism 5 is group strategyproof.*

Next, we explore the relationship between PBPM and NPBPM, showing that the approximation ratio of NPBPM is at most equivalent to that of PBPM.

Theorem 9. *Mechanism 5 is a 2-approximation for the objective *magc*.*

We begin the proof by focusing on the “duplication” property. This property is defined as duplicating a group, denoted by G_g , l ($l \geq 0$) times, if we generate l additional agents at the same position for each agent that belongs to G_g .

Lemma 2. *Duplicating any group an arbitrary number of times does not affect:*

- The value of the *magc* objective at any point $y \in I$.
- The optimal solution y^* and $\text{magc}(y^*, \mathbf{r})$.
- The output or approximation ratio of Mechanism 5.

Lemma 3. *PBPM and NPBPM have the same output and approximation ratio if all groups have an equal number of agents.*

Now we prove Theorem 9 based on the lemmas above.

Proof of Theorem 9. Consider \mathbf{r} as any feasible profile. We duplicate each group G_j exactly $\prod_{k \neq j} |G_k|$ times, resulting in a new profile, \mathbf{r}' . In this new profile \mathbf{r}' , each group contains an equal number of agents, and it maintains the same approximation ratio as \mathbf{r} due to Lemma 2.

PBPM and NPBPM yield the same approximation ratio on \mathbf{r}' (Lemma 3), which is at most 2 due to Theorem 4. Consequently, the approximation ratio for the input profile \mathbf{r} is also at most 2. \square

The proof of Theorem 5 uses profiles consisting only of agents from a single group. In such scenarios, the approximation ratios of $mtgc$ and $magc$ are identical. Hence, the $magc$ objective maintains the lower bound result.

Theorem 10. *Any randomized strategyproof mechanism has an approximation ratio of at least $\frac{5}{4}$ for the objective $magc$.*

Intergroup and Intragroup Fairness

In this section, we investigate Intergroup and Intragroup Fairness (IIF) proposed by Zhou, Li, and Chan (2022), which captures not only the fairness between groups but also fairness within groups.

To facilitate our discussion, given profile \mathbf{r} , group G_g , and facility location y , let $avgc(\mathbf{r}, g, y)$, $maxc(\mathbf{r}, g, y)$ and $minc(\mathbf{r}, g, y)$ be the average cost, maximum cost and minimum cost among agents in group G_g , respectively. We define below our group-fair IIF objectives which measure both intergroup and intragroup fairness:

$$\begin{aligned} IIF_1(y, \mathbf{r}) &= \max_{1 \leq j \leq m} \{avgc(\mathbf{r}, j, y)\} \\ &\quad + \max_{1 \leq j \leq m} \{maxc(\mathbf{r}, j, y) - minc(\mathbf{r}, j, y)\}, \\ IIF_2(y, \mathbf{r}) &= \max_{1 \leq j \leq m} \{avgc(\mathbf{r}, j, y) \\ &\quad + maxc(\mathbf{r}, j, y) - minc(\mathbf{r}, j, y)\}. \end{aligned}$$

Our goal is to minimize IIF_1 or IIF_2 . Here, the $avgc(\mathbf{r}, j, y)$ term is to measure the intergroup fairness, and the $(maxc(\mathbf{r}, j, y) - minc(\mathbf{r}, j, y))$ term is to measure the intragroup fairness. Surprisingly, a mechanism we introduced in Section 4 performs well for the two objectives.

Theorem 11. *Mechanism 4 is a 4-approximation for minimizing both IIF_1 and IIF_2 .*

Proof Sketch. Let $lm(\mathbf{x})$ (resp. $rm(\mathbf{x})$) be the leftmost (resp. rightmost) location of a location profile \mathbf{x} . Let y be the solution given by BGMV. We discuss two cases based on the optimal location y^* to build the facility.

Suppose y^* is outside of all groups. In this case, we can prove that y^* is at either 0 or 1. Comparing y with y^* , the intergroup fairness term of each group ($maxc(\mathbf{r}, j, y) - minc(\mathbf{r}, j, y)$) remains unchanged. Recall that BGMV achieves a 3-approximation ratio for minimizing $magc$, we have $\max_{1 \leq j \leq m} \{avgc(\mathbf{r}, j, y)\} \leq 3 \max_{1 \leq j \leq m} \{avgc(\mathbf{r}, j, y^*)\}$. We can combine these two facts to show the approximation ratio for both IIF_1 and IIF_2 is 4.

Otherwise, there is a group g , such that $lm(\mathbf{x}_g) \leq y^* \leq rm(\mathbf{x}_g)$. We can prove $avgc(\mathbf{r}, g, y^*) + maxc(\mathbf{r}, g, y^*) - minc(\mathbf{r}, g, y^*) \geq \frac{1}{2}$, and hence $IIF_1(y, \mathbf{r}) \geq IIF_2(y, \mathbf{r}) \geq \frac{1}{2}$. We notice that $IIF_2(y, \mathbf{r}) \leq IIF_1(y, \mathbf{r}) \leq 2$ always holds. Therefore, the approximation ratio is at most 4. \square

Next, we establish lower bounds for the IIF objectives.

Theorem 12. *Any deterministic strategyproof mechanism has an approximation ratio of at least 4 for minimizing either IIF_1 or IIF_2 .*

Discussion and Future Work

In this section, we discuss interesting future directions.

Alternative Group-Fair Objectives

One could alternatively consider the minimum total group utility as a measure of fairness amongst groups. This would translate into group-fair objectives aiming to maximize either the **minimum total** or **average group utility** ($mtgu$ or $magu$). However, when more than one group is present, achieving a finite approximation ratio becomes impossible.

Theorem 13. *Any deterministic strategyproof mechanism cannot achieve a finite approximation ratio for maximizing the minimum total or average group utility.*

Following a similar approach to that of (Fong et al. 2018), we define the cost of an agent as one minus their utility, which then shifts the focus towards minimizing the maximum total or average group cost. However, the randomized strategyproof mechanism design for both $mtgu$ and $magu$ still remains open.

Beyond 2-Candidate Randomized Mechanisms

This work focuses on 2-candidate randomized mechanisms. By confining the mechanism’s output to a 2-candidate set, PBPM and NPBPM deliver the optimal approximation ratio for the $mtgc$ and $magc$ objectives, respectively.

Theorem 14. *Any randomized 2-candidate strategyproof mechanism has an approximation ratio of at least 2 for minimizing either $mtgc$ or $magc$.*

The domain of mechanisms involving more candidates remains largely unexplored. The existence of a strategyproof mechanism involving more than two candidates with a constant approximation ratio is an interesting direction.

Conclusion

We study group-fair obnoxious facility location problems. We developed several (group) strategyproof mechanisms that minimize costs among different groups measured by the $mtgc$ and $magc$ objectives. For the objectives, the designed deterministic and 2-candidate randomized mechanisms obtain approximation ratios of 3 and 2, respectively. We also designed mechanisms for two objectives capturing both intergroup and intragroup fairness that obtain 4-approximation ratios. Finally, we provide lower bounds for these objectives to complement our results.

Acknowledgements

Minming Li is supported by the Research Grants Council of the Hong Kong Special Administrative Region, China [Project No. CityU 11213620]. Hau Chan is supported by the National Institute of General Medical Sciences of the National Institutes of Health [P20GM130461], the Rural Drug Addiction Research Center at the University of Nebraska-Lincoln, and the National Science Foundation under grant IIS:RI #2302999. The content is solely the responsibility of the authors and does not necessarily represent the official views of the funding agencies. We would like to thank the anonymous reviewers for their valuable feedback.

References

Aziz, H.; Lam, A.; Lee, B.; and Walsh, T. 2022a. Strategyproof and Proportionally Fair Facility Location. In *Web and Internet Economics: 18th International Conference, WINE 2022, Troy, NY, USA, December 12–15, 2022, Proceedings*, volume 13778, 351.

Aziz, H.; Lam, A.; Li, B.; Ramezani, F.; and Walsh, T. 2023. Proportional Fairness in Obnoxious Facility Location. In *Proceedings of the 2023 International Conference on Autonomous Agents and Multiagent Systems*, 2646–2648.

Aziz, H.; Lam, A.; Suzuki, M.; and Walsh, T. 2022b. Random Rank: The One and Only Strategyproof and Proportionally Fair Randomized Facility Location Mechanism. *Advances in Neural Information Processing Systems*, 35: 29076–29087.

Cai, Q.; Filos-Ratsikas, A.; and Tang, P. 2016. Facility location with minimax envy. In *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence*, 137–143.

Chan, H.; Filos-Ratsikas, A.; Li, B.; Li, M.; and Wang, C. 2021. Mechanism Design for Facility Location Problems: A Survey. In *30th International Joint Conference on Artificial Intelligence*, 4356–4365. International Joint Conferences on Artificial Intelligence Organization.

Cheng, Y.; Yu, W.; and Zhang, G. 2013. Strategy-proof approximation mechanisms for an obnoxious facility game on networks. *Theoretical Computer Science*, 497: 154–163.

Church, R. L.; and Garfinkel, R. S. 1978. Locating an obnoxious facility on a network. *Transportation science*, 12(2): 107–118.

Church, R. L.; and ReVelle, C. S. 1976. Theoretical and computational links between the p-median, location set-covering, and the maximal covering location problem. *Geographical Analysis*, 8(4): 406–415.

Ding, Y.; Liu, W.; Chen, X.; Fang, Q.; and Nong, Q. 2020. Facility location game with envy ratio. *Computers & Industrial Engineering*, 148: 106710.

Fong, K. C.; Li, M.; Lu, P.; Todo, T.; and Yokoo, M. 2018. Facility Location Games with Fractional Preferences. In *32nd AAAI Conference on Artificial Intelligence (AAAI-18)*, 1039–1046. AAAI Press.

Hakimi, S. L. 1964. Optimum locations of switching centers and the absolute centers and medians of a graph. *Operations research*, 12(3): 450–459.

Ibara, K.; and Nagamochi, H. 2012. Characterizing mechanisms in obnoxious facility game. In *Combinatorial Optimization and Applications: 6th International Conference, COCOA 2012, Banff, AB, Canada, August 5–9, 2012, Proceedings* 6, 301–311. Springer.

Jain, K.; Mahdian, M.; Markakis, E.; Saberi, A.; and Vazirani, V. V. 2003. Greedy facility location algorithms analyzed using dual fitting with factor-revealing LP. *Journal of the ACM (JACM)*, 50(6): 795–824.

Moulin, H. 1980. On strategy-proofness and single peakedness. *Public Choice*, 35(4): 437–455.

Procaccia, A. D.; and Tennenholtz, M. 2013. Approximate mechanism design without money. *ACM Transactions on Economics and Computation (TEAC)*, 1(4): 1–26.

Ye, D.; Mei, L.; and Zhang, Y. 2015. Strategy-proof mechanism for obnoxious facility location on a line. In *Computing and Combinatorics: 21st International Conference, COCOON 2015, Beijing, China, August 4–6, 2015, Proceedings*, 45–56. Springer.

Zhou, H.; Li, M.; and Chan, H. 2022. Strategyproof Mechanisms for Group-Fair Facility Location Problems. In *31st International Joint Conference on Artificial Intelligence and the 25th European Conference on Artificial Intelligence (IJCAI-ECAI 2022)*, 613–619. International Joint Conferences on Artificial Intelligence.