AR-Classroom: Augmented Reality Technology for Learning 3D Spatial Transformations and Their Matrix Representation

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***Abstract*—Project AR-Classroom aims to enhance undergraduate students learning spatial transformations and their mathematical representations. Understanding closely allied spatial and mathematical concepts significantly contributes to STEM learning in fields of computer graphics, computer-aided design, computer vision, robotics, and many more. The technology and learning innovations of this research include novel AR features and their implications for learning. In AR-Classroom, a student can hold and manipulate a 3D physical model (a LEGO space shuttle as an example) while simultaneously interacting with AR visualization of 3D rotations. Two usability tests with 24 participants total have been conducted for AR-Classroom leading to promising results and recommendations for improvements. The project contributes to advancing our knowledge in (1) the role of interplay between physical and virtual manipulatives to engage students in embodied learning and (2) the features of AR to make difficult, invisible concepts visible for supporting an intuitive and formal understanding of spatial reasoning and mathematical formulation.**

***Index Terms*—Augmented Reality, Spatial Transformation, Mathematical Representation, Rotation Matrix, Embodied Learning**

I. Introduction

Spatial and mathematical thinking are closely allied. Understanding tightly coupled spatial transformations and mathematical concepts significantly contributes to STEM learning in fields of geometric modeling, computer graphics, computer-aided design (CAD), computer vision, robotics, video games, quantum mechanics, and more. For example, the phenomenon of gimbal lock when using Euler angles for spacecraft rotation rose in prominence in NASA's Project Apollo [1], inverse perspective transformation in the visual odometry process determined the position and orientation of Mars Exploration Rovers [2], and control of an object's orientation in a game can be very difficult and cause the "game developer's nightmare" [3]. The study of spatial transformations and linear algebra provides opportunities for students to learn important mathematical concepts, view mathematics as an interconnected discipline, and engage in higher-level reasoning activities using a variety of representations [4]. However, the difficulties students face when learning transformations and their mathematical representations are well documented, including: generalizing geometric ways of reasoning, understanding the symbolic notation of linear algebra [5], and switching to matrix representations of the transformations resulting in lost intuitive connection [6]. The most reliable results of research on improving linear algebra learning are 'negative' in nature and the previous recommendations are conjectures that are still open to questioning [7]. While there are mixed views on the impact of geometric visualization in learning linear algebra [7], [8], and less research with respect to learning, teaching, and assessment of haptic and proprioceptive sensation-based math learning [9], from literature [10]–[14] and our team's preliminary work [15], we learned about the unique potentials

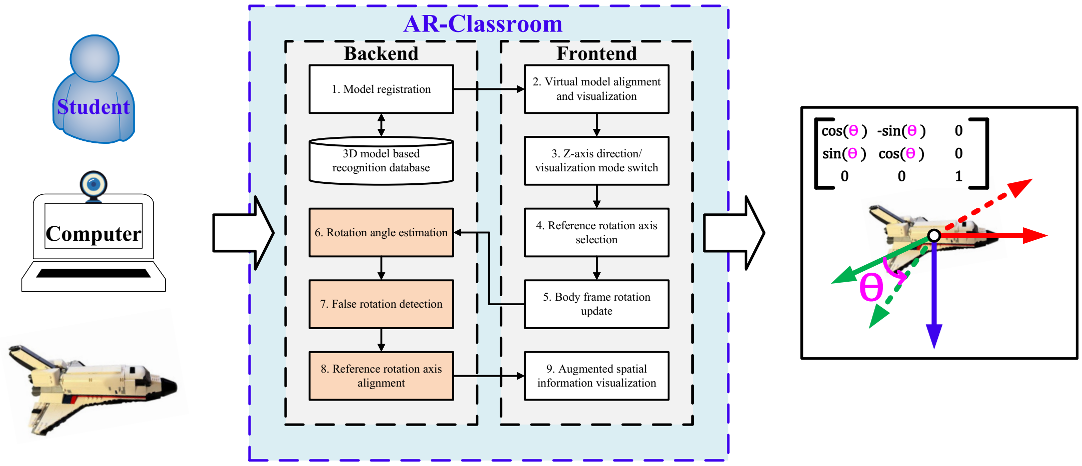


Fig. 1. System diagram of Workshop 2 in AR-Classroom.

of Augmented Reality (AR) and associated Artificial Intelligence (AI) in enhancing embodiment and visualization for learning. The goals of this research are to (1) systematically investigate how AR/AI-powered embodiment and visualization can enhance learning major mathematical representations of spatial transformations and (2) create an innovative learning environment - an app named AR-Classroom - for learning spatial transformations and their mathematical representations (rotation matrices) to contribute to Science, Technology, Engineering, and Mathematics (STEM) learning.

II. Related Work

Research shows that incorporating technologies into teaching and learning promotes students' understandings of mathematics [16], [17]. The use of dynamic math software with three-dimensional (3D) visualization for learning geometric transformations in the constructivist pedagogical style has proven to be effective, e.g., using Geometer's Sketchpad [18], [19] or GeoGebra [20], [21]. GeoGebra is free and open-source dynamic math software linking interactive geometry with algebra and is widely used in K-12 and college mathematics education [22]–[24]. GeoGebra has many possibilities to help students get an intuitive feeling and visualize adequate math process, make the connections between symbolic and visual representations [23], [25], and deepen students' knowledge and enthusiasm for the geometrical representations of linear algebra [22].

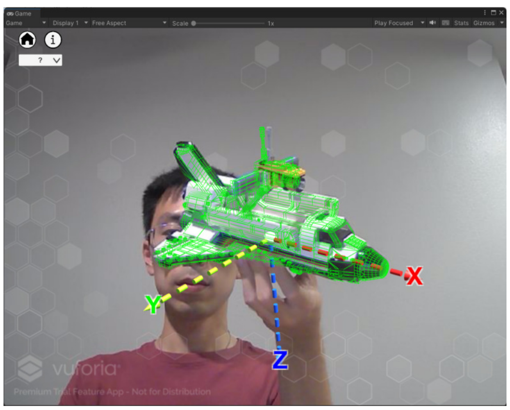
Augmented Reality (AR) superimposes digital images on a user's view of the real world through AR glasses or mobile devices, having significant potentials to benefit manufacturing, building construction, and part assembly, and to enhance geometric and spatial inference abilities of users [10]–[14], [26]. Existing research on AR-assisted geometry learning (including GeoGebra AR) has utilized virtual 3D models projected on physical surfaces (e.g., a desktop surface or a book page) but without registered virtual and physical models [10]–[12], [14], [27]–[30]. Based on the state-of-the-art, our AR-Classroom research investigates how AR can support the understanding of spatial transformations using the motion, mapping, and function conceptions [4], [31], [32], utilizing AR's inherent and unique power: integrating capabilities of (1) embodied learning and (2) visualization of $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ mapped images and superimposed synchronously changing math formulas.

III. AR-Classroom

AR-Classroom is an AR application, which provides an interactive learning environment for students. The app aims to run on AR device, such as an AR-enabled tablet, an AR headset, or even a computer with a camera (e.g., a laptop with a webcam, whose camera view is shown in Fig. 1). A student can hold and manipulate a 3D physical model (e.g., a LEGO space shuttle in Fig. 1) to learn spatial transformations by the help of the augmented mathematical representation (e.g., a rotation matrix multiplication displayed in the AR scene in Fig. 1) and visualization (e.g., a rotation angle arc and value θ with the directional arrow in Fig. 1) on the graphical user interface (GUI). The X-, Y-, and Z-axes of the models' coordinate systems are also displayed in the GUI. In the beginning of using the app, an instruction session is offered for helping students register the models in AR - aligning the virtual model with the physical model automatically. After the instruction session, Workshop 1 and 2 can start.

Workshop 1 & Workshop 2 in AR-Classroom: We designed two workshops in AR-Classroom: Workshop 1 and Workshop 2. Their major difference is how the spatial transformation is performed. In Workshop 1 the spatial transformation of the 3D virtual model is rotated by the UI component (a rotation angle slider), while in Workshop 2 the students manipulate the 3D physical model to perform the spatial transformation using their hands. By doing so, we can make Workshop 1 and Workshop 2 a continuous learning process, in which the students can learn the knowledge of spatial transformation in Workshop 1 and enhance the learning in Workshop 2, by rotating the virtual and physical models, respectively, while observing the changes in the matrix representation. Here we use both Workshop 1 and 2 to demonstrate AR-Classroom but with a focus on Workshop 2 (See Fig. 1) since hand motion for the physical model rotation introduces more challenges and opportunities associated with AR, for which we added three additional steps (See orange-colored boxes in Fig. 1) to handle the hand motion in AR-Classroom.

(a)



(b)

Fig. 2. (a) The guide view of model registration to assist the students for model registration - alignment between the virtual and physical models. (b) Virtual model visualization after model registration.

A. Model Registration

Model registration is a crucial step in AR to align our virtual content (e.g., the virtual model of the 3D LEGO space shuttle) with the physical object or the physical world. To do so, we use a model-based 3D object registration to find the relationship between the world coordinate (e.g., the coordinate system defined on the LEGO set) and the camera coordinate. Here we adopt the Model Target from Vuforia [33] to detect and localize the 3D physical model with the help of a trained 3D model-based recognition database of our 3D physical model. Fig. 2a shows the guide view provided by Model Target for model registration assistance. Note that we can switch to other model registration methods in our design. After the model registration, we can superimpose the virtual model and align it with the 3D physical model. We use the virtual model to assist in visualizing the spatial transformations. Fig. 2b shows that the virtual model and physical model are aligned.

B. Visualization of the Transformation

When a model starts to rotate after model registration, the visualization of the rotation can be seen on the AR screen. As an example, in Workshop 1, the following AR scene components are shown in Fig. 4a:

1) a physical model, called *pre-image* of the transformation, showing the state before rotation
2) a 3D reference frame coordinate with red, green, and blue solid arrows for X-, Y-, and Z-axis, attached to the physical model
3) a 3D wireframe model, called *image* of a transformation, showing the state after rotation
4) a 3D body frame coordinate with red, green, and blue dashed arrows for X-, Y-, and Z-axis, attached to the 3D wireframe model

Reference Frame vs. Body Frame: Since the spatial transformation is always described between the reference frame and the body frame, it is necessary to visualize the reference frame and the body frame. By convention, the reference frame is the coordinate system that defines the initial orientation of a rotating object, and the body frame is attached to the object to describe its orientation changes. We use the 3D physical model and the 3D wireframe model to represent the body frame and reference frame, respectively, and we interchange their roles according to the design of the two workshops. In Workshop 1, we assign the wireframe model to be the body frame. Then we can use the GUI component to control its transformation. The transformation in Workshop 2, on the other hand, is manipulated by the student's hand motion. Therefore, the 3D physical model is the body frame while the wireframe model is the reference frame. The 3D reference coordinate axes (solid arrow system) and the 3D body coordinate axes (dashed arrow system) are attached accordingly.

C. Z-axis Direction/Visualization Mode Switch

Different fields have different conventions to define the coordinate system. We provide a module to switch the coordinate definition according to the student's field of study by using a toggle on the GUI (See the red-circled box e in Fig. 3). For example, in aerospace and aircraft-related fields, the Z-axis is often pointing downward (See Fig. 4b). For mathematics, robotics, and architecture, the Z-axis, on the other hand, is pointing upward (See Fig. 4a). Our default setting follows the principle in aerospace and aircraft-related fields. The coordinate system is a right-handed system. We set the Z-axis to point downward, and the X-axis follows the heading direction.

Besides, we also provide a module to enable the component of the virtual model (e.g., item 2, 3, or 4 in Sec. III-B) to be visualized. It gives the students opportunities to choose the most beneficial visualization for their study. Fig. 4b, 4c, 4d shows the three visualization modes, and the mode switching can be done by clicking the visualization mode button (See the red-circled box d in Fig. 3) or pressing the "V" key on a computer.

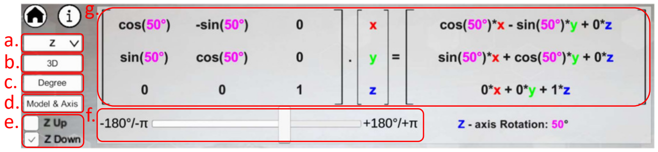


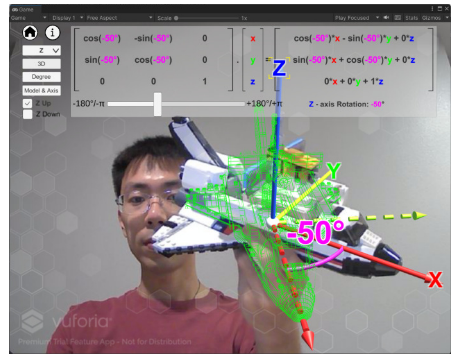
Fig. 3. A zoom-in view of our user interface components. (a) Dropdown list for rotation axis selection. (b) Button to switch rotation matrix representation between 2D space and 3D space. (c) Button to switch the unit of the rotation angle between degree and radian. (d) Button to switch the visualization mode. (e) Toggle group to select the Z-axis direction. (f) Slider to control the rotation angle (only in Workshop 1). (g) Rotation matrix display panel.

D. Reference Rotation Axis Selection

It is worth noting that we only focus on the rotation instead of the spatial transformation including the translation since the rotation is much more challenging and difficult in learning compared with the translation. Besides, adding additional translation information together on the GUI can raise the distraction for the students. Therefore, we are highly interested in teaching the rotation in AR-Classroom.

We emphasize on 1 degree of freedom (DoF) rotation instead of a complete 3 DoFs rotation. Namely, we want to rotate along one axis (X-, Y-, or Z-axis) every time. The rationale behind that is that 1 DoF rotation is fundamental since every rotation $\mathrm{R} \in \mathrm{SO}(3)$ can be decomposed as $\mathrm{R} = R_Z(\theta_z)R_Y(\theta_y)R_X(\theta_x)$, where $R_X(\theta_x)$, $R_Y(\theta_y)$, and $R_Z(\theta_z)$ denote the 1 DoF rotation with the angle, $\theta_x \in [-180°, 180°]$ along X-axis, $\theta_y \in [-180°, 180°]$ along Y-axis, and $\theta_z \in [-180°, 180°]$ along Z-axis, respectively. Besides, the clear correspondence between the rotation and the matrix representation will be obscure when the DoF is more than 1. Visualizing the rotation and the matrix representation with DoF more than 1 on GUI also can distract and hinder the students in learning. Therefore, we allow students to select any one axis to be the reference rotation axis using a dropdown list (See the red-circled box a in Fig. 3) before the rotation is performed. The selected reference axis is also highlighted in the AR scene (See the yellow box highlighted the "Z" text in Fig. 4b).

Meanwhile, in Workshop 1, the orientation of the physical model is set to be the reference frame after the reference rotation axis is selected. Because the physical model is held in hand, its orientation may change because of hand motion, thus the reference frame is set to be constantly updating to align with the physical model's orientation. The position of the reference frame is also dynamic by hand motion. The position of the reference frame is set to be co-centered with the physical model. In Workshop 2, the physical model coordinate is the body frame. Since we want to focus on learning rotation only, the reference frame (the wireframe model coordinate) is set to be co-centered with body frame (the physical model coordinate). In both Workshop 1 and 2, it is clearer for the students to see the rotation by constantly aligning the two frames in position (but not orientation) based on the physical model's center, eliminating the effect of translation.



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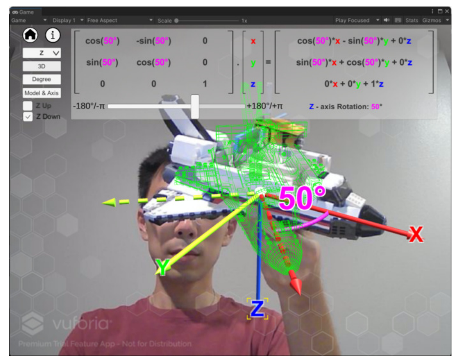

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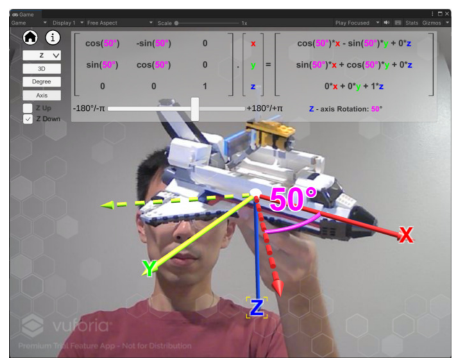

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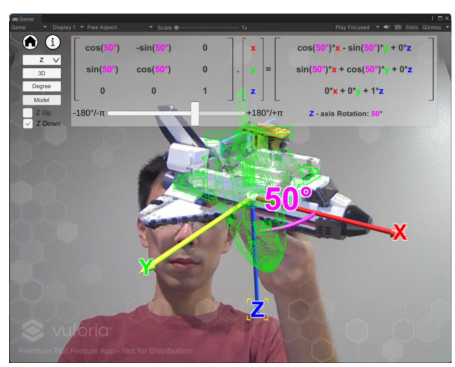
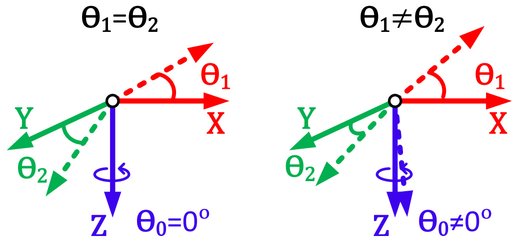


(d)

Fig. 4. Here we use Workshop 1 as an example to show the switch between visualization. (a) Virtual model with Z-axis pointed upwardly and all the three virtual items in Sec. III-B enabled. (b) Virtual model with Z-axis pointed downwardly and all three items enabled in Sec. III-B. (c) A visualization mode with the 3D wireframe model disabled. (d) Another visualization mode with the 3D body frame coordinate disabled.



(a) (b)

Fig. 5. (a) Illustration of general 1 DoF rotation along the Z-axis. (b) Illustration of 1 DoF rotation along the Z-axis manipulated by hand motion.

E. Body Frame Rotation Update

After the reference rotation axis is selected, we are ready to manipulate the rotation and update the body frame orientation. Note that the two workshops adopt different approaches: we use a slider on the GUI (See the red-circled box f in Fig. 3) to control the angle value $\theta \in [-180^\circ, 180^\circ]$ in Workshop 1 while the student rotates the 3D physical model in Workshop 2 by hand.

F. Hand Motion Challenges

When students try to rotate the LEGO model around a certain axis, undesired rotations will happen inevitably. These rotations are mainly caused by two reasons: the limitation of human hand motion and the misunderstanding of the rotation axis concept. There are two main challenges introduced by hand motion. First, the DoF of hand motion is more than 1. Second, it is difficult to ask the students to perform a 1 DoF rotation precisely by their hands. Both challenges can easily downgrade the students' learning experience. For example, the non-smooth rotation visualization can be caused by the inevitable hand-jittering. Therefore, it is necessary to tackle the challenges carefully with innovative ideas. Furthermore, when a student is asked to rotate the physical model around an axis, e.g., Z-axis, they may mistakenly rotate the model around another axis, e.g., X- or Y-axis.

The false rotation caused by hand-jittering should be automatically corrected by the app to provide a smooth user experience, but the rotation around a wrong axis should be conveyed to and corrected by the students, so that they can learn from the mistake. Our application distinguishes between these two types of false rotations using the tilt angle of the selected rotation axis. We set a threshold on the maximum allowed tilt angle and compensate for the undesired rotations within the threshold. When the tilt of the rotation axis exceeds the threshold, a warning will be shown to the students.

1) Rotation Angle Estimation: The rotation angle of a 1 DoF rotation along a selected reference rotation axis can be calculated by using an axis that is not selected. For example, if we rotate along the selected Z-axis, the rotation angle can be calculated by using the angle between the orientations before and after the rotation with either X-axis or Y-axis: θ_1 or θ_2, respectively, where the two angles $\theta_1 = \theta_2$ (See Fig. 5a). However, the reference rotation axis can move when the rotation is performed by the hand motion, which leads to the fact that θ_1 and θ_2 are not necessarily equal (See Fig. 5b). Using an example to illustrate this problem: if after a rotation around Z-axis with angle $\theta_1 = \theta_2$, another rotation is made, by hand-jittering, around the updated X-axis, then θ_1 will remain the same, but θ_2 will be changed. To solve this issue, we average θ_1 and θ_2 as the rotation angle around Z-axis. Given the angles θ_1 and θ_2 between the orientations before and after the rotation for the two non-rotation axes, i.e., X-axis and Y-axis, respectively, the hand-motion rotation angle can be calculated by

$$\theta = \frac{\theta_1 + \theta_2}{2}. \tag{1}$$

Note that (1) assumes that the rotation axis of the hand-motion rotation manipulated by the student is close to the selected reference rotation axis, and this assumption helps us design how we detect the hand motion which rotates along a different rotation axis in the next section Sec. III-F2.

2) False Rotation Detection: Recall that we assume that the rotation axis of the hand-motion rotation is close to the reference axis when we calculate the rotation angle in (1). However, this assumption can be easily violated, and create inconsistency between the rotation and the visualization since the hand motion has more DoFs. Plus, we cannot pose constraints on how the student performs the rotation, and the student can rotate along any direction. Both issues can impact the student's learning experience.

To handle these issues, we design a false rotation warning to check if the assumption is violated. Given the angle θ_0 between the body frame's selected axis and the reference frame's selected axis, we consider the false rotation happens when

$$\theta_0 > \epsilon, \tag{2}$$

where ϵ is determined by experiments ($\epsilon = 15^\circ$, but can be configured by the user). Fig. 6 shows an example of the false rotation. When the false rotation is detected, we pose a warning message on the GUI (See the red message "False Rotation !!!" on the top panel in Fig. 6b).

3) Reference Rotation Axis Alignment: The fact that the hand motion often does not rotate around a selected reference rotation axis can lead to the situation, in which there is no aligned axes between the body frame and the reference frame, which contradicts to our assumption of the rotation on a fixed axis. Besides, the dynamic body frame can cause the visualization distracting and confusion for the students. To clean up the visualization of rotations by showing the only rotation around the selected axis for enhanced learning, within the threshold ϵ, we re-align the selected axis of the reference frame to the corresponding axis of the body frame, to cancel the hand-jittering problem ($\theta_0 \neq 0^\circ$ problem). Note that the rotations of the other two axes are maintained and displayed using the angle arcs and values.

G. Switching Axes of Rotations

In both Workshop 1 and Workshop 2, students can switch the rotation axis at any time. Once the new rotation axis is selected from a dropdown menu in GUI (See the red-circled box a in Fig. 3), both the body frame and the reference frame are immediately aligned to the physical LEGO model and the tracking of the new rotation is restarted from 0 degree. In Workshop 2, the false rotation detection is also reset.

H. Augmented Spatial Information Visualization

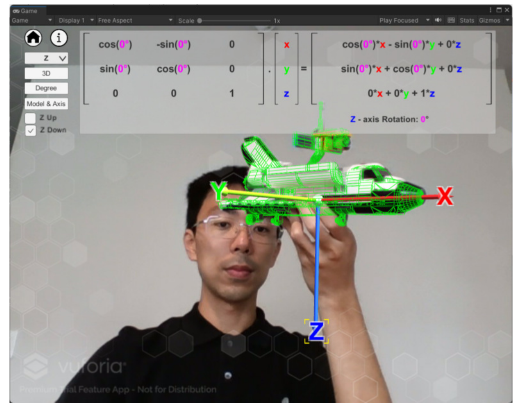
To assist the student in understanding the rotation, we display the rotation's mathematical representation - a rotation matrix, and its multiplication with a general column vector [x y z] representing any point on the geometry of the rotating model on the GUI. Given the rotation angle θ in (1), the rotation matrix displayed (See the red-circled box g in Fig. 3) is:

$$\begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}, \text{if rotating along X-axis,} \\ \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}, \text{if rotating along Y-axis,} \\ \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{if rotating along Z-axis.} \end{cases} \quad (3)$$

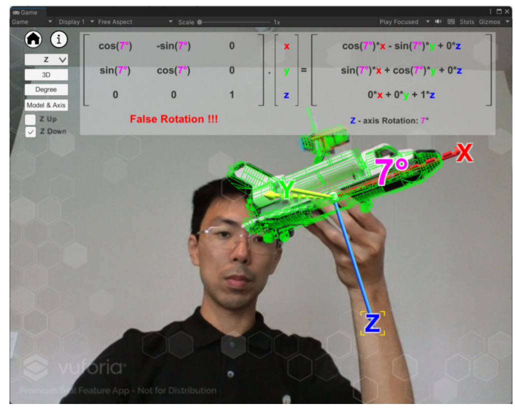
Each row of the rotation matrix is a unit vector and can be considered as a weight vector which is applied on the original 3D point to obtain its transformed position. Besides, we also visualize the rotation angle θ, and its rotating direction by a 3D arc with the arrow (See the pink θ and the pink arc in Fig. 1). For students learning 2D transformations and their matrices, switching between 3×3 and 2×2 matrices is enabled (See the red-circled box b in Fig. 3). The rotation angle value can be switched between degree and radian (See the red-circled box c in Fig. 3).

I. Switching Different Models and Separating Coordinate Systems

To expand AR-Classroom's accessibility to a wider user base, we made our app compatible with various LEGO models. By utilizing the latest Vuforia [33] Model Target Generator, we are able to train the Vuforia engine on multiple LEGO models and save the result into a database file. Once this database is imported into our app, a LEGO model in the database can be automatically recognized and tracked, so that the models can be freely switched in the AR scene while all the AR-Classroom functions remain the same. A demonstration of the app working with a LEGO race plane model is shown in Fig. 7b. However, during our development and experiments, we found that the tracking results provided by Vuforia are not consistent (See Fig. 7a). The inconsistent results often lead to misalignment in the model registration step, which can downgrade the students' learning experience if the alignment is not precise. The inconsistent tracking issue, in fact, is caused by the default coordinate system attached to the LEGO model by Vuforia, e.g., the X-axis is always parallel to the horizontal



(a)



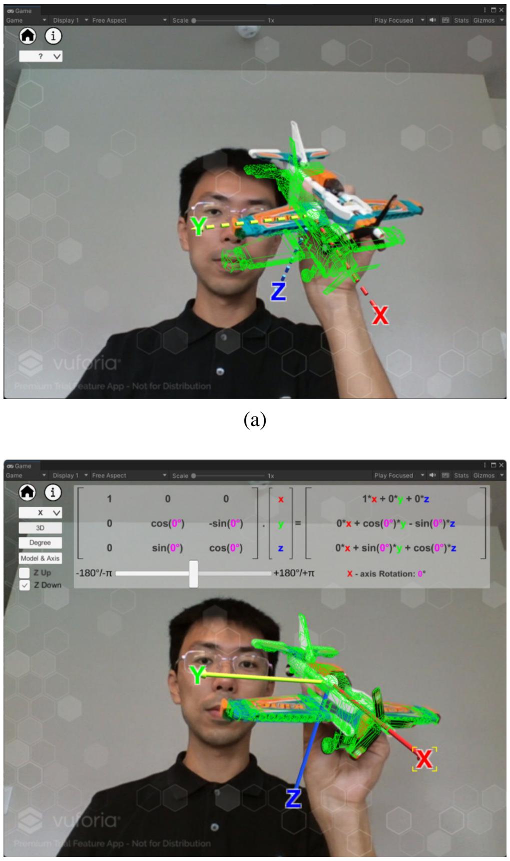
(b)

Fig. 6. False rotation detection example in Workshop 2 to handle high DoFs of hand motion challenges. (a) The Z-axis is selected as the rotation reference axis. It is initially pointing downward; (b) The user rotates along the Y-axis, and the orientation of the reference axis (Z-axis) is changed significantly. This is detected by the app and the warning message is displayed.

plane of the physical environment, but our models may have a different coordinate system, e.g., a LEGO race plane's X-axis may have a pitch angle of 15 degrees. To solve the issue, we measure the transformation between the coordinate system used by Vufoia and the coordinate system defined by us based on the actual geometry of the model. We correct the misalignment of the model registration by applying the transformation difference instead of merely relying on Vuforia tracking. The results of the correction are shown in Fig. 7.

IV. Experiments

Two user studies were conducted to investigate the AR classroom's usability [15]. First, a benchmark usability test was conducted to explore the AR Classroom's discoverability and usability under its starting conditions, then an updated usability study to examine user-app-interaction after improvements were made. The benchmark and updated usability test both have 12 participants and they followed similar procedures. Participants first completed a pre-test assessing their math abilities and confidence and watched a video on geometric transformations. Then, they were randomly assigned to either the Workshop 1 or 2 of the app. While interacting



(a)

(b)

Fig. 7. The model is switched from the previous LEGO Space Shuttle Adventure 31117, $39.99, to LEGO Technic Race Plane 42117, $9.99, which is more affordable for use in learning. (a) The virtual model alignment directly uses the transform provided by Vuforia [33]. In this case, both the position and orientation of the virtual model are inconsistent with the physical model. (b) After using the corrected transform, the virtual model is aligned with the physical model.

with the app, participants were asked to complete either virtual or physical model rotations around X-, Y-, and Z-axes, and observe the rotation matrix updates when rotation changes. Once done interacting with the app, participants completed a post-test assessing their math abilities and confidence, and provided feedback on their overall experience with the app (i.e., System Usability Scale, or SUS).

A thematic analysis was performed after each test to identify and code themes in user-app-interaction and compare findings from the benchmark and updated tests. Findings from the benchmark usability test were then used to formulate recommendations to enhance the AR Classroom's usability and functionality. After changes were made to the app such as adding additional instructions on model registration, restructuring and updating the instructions, and turning 'visualization type' into a button to make it easier to find, the updated usability test assessed usability again so that the impact of the modifications could be evaluated.

Both usability studies demonstrate substantial advancements in students' accuracy and confidence levels regarding the matrix representation of spatial transformations. The utilization of both Workshop 1 and 2 yielded an approximate 10% enhancement in math test scores, measured on a 100% scale, accompanied by a one-point increase in confidence levels, measured on a five-point scale. The updated usability study indicated that after changes were made to the app, the usability of both workshops in the app improved, users were better able to set up the LEGO space shuttle model, effectively utilize the in-app instructions, and easily access all of the app's functions. The details of the usability studies are described in "*AR-Classroom: Usability of AR educational technology for learning rotations using three-dimensional matrix algebra*" [15].

V. Conclusions and Future Work

To improve the learning experience in spatial transformations and their mathematical representations, we presented our new AR-Classroom, an AR-based application designed to combine the learning and the practice in real time. We focused on rotation only since rotation is less intuitive and more challenging to students in understanding compared with some other transformations such as translation and scale. To focus on learning rotation, the effect of model translation is eliminated by constantly aligning the two frames in position (but not orientation). The multiplication of the rotation matrix by the [x y z] coordinates is displayed in the AR scene and updated in real time when the models rotate. While in reality students cannot perform a rotation of the physical model strictly around only one axis by hand, which has 3 DoFs of rotation, it is desired to clearly visualize a rotation around only one axis at a time in the learning process. Such a clear visualization is achieved by aligning the selected rotation axis (e.g., Z-axis) between the reference frame and the body frame, and displaying the rotation angle between the two frames for each of the other non-selected axes (X and Y). Detection for human errors of rotating around wrong axes (false rotation) is enabled with a rotation angle estimation and a threshold setting. To help students understand different conventions in different disciplines, the Z-axis can be switched to point up (e.g., in robotics) or down (e.g., in aerospace engineering). The AR-Classroom's novel features and their implications for learning spatial transformations and their mathematical representations demonstrate technology and learning innovations of this research. Our two usability tests have shown promising results and recommendations for improvements of the innovations.

As part of our future work, we will conduct more user studies on how the AR's unique power of integrating embodied learning and visualization can enhance students learning spatial transformations and their mathematical representations. We will integrate other transformations such as translation into AR-Classroom for students to learn the challenging topic of composition of transformations. Other important mathematical representations of rotations such as Axis-Angles, Euler Angles, and Quaternions can be included into AR-Classroom. New features can be added to support students navigating in AR-Classroom more efficiently. We plan to add audio-visual AR instructions that demonstrate spatial transformations using a virtual model. Additionally, we intend to include AI-generated real-time audio guidance to enhance the learning experience by providing assistance to the students in terms of user instructions and mathematical concepts.

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References

[1] E. G. Hemingway and O. M. O'Reilly, "Perspectives on euler angle singularities, gimbal lock, and the orthogonality of applied forces and applied moments," *Multibody System Dynamics*, vol. 44, no. 1, p. 31–56, Sep 2018. [Online]. Available: https://doi.org/10.1007/s11044-018-9620-0

[2] Y. Cheng, M. Maimone, and L. Matthies, "Visual odometry on the mars exploration rovers," in *2005 IEEE International Conference on Systems, Man and Cybernetics*, vol. 1. IEEE, 2005, pp. 903–910.

[3] A. J. Hanson, "Visualizing quaternions," in *ACM SIGGRAPH 2005 Courses*, 2005, pp. 1–es.

[4] K. F. Hollebrands, "High school students' understandings of geometric transformations in the context of a technological environment," *The Journal of Mathematical Behavior*, vol. 22, no. 1, pp. 55–72, 2003.

[5] C. Andrews-Larson, M. Wawro, and M. Zandieh, "A hypothetical learning trajectory for conceptualizing matrices as linear transformations," *International Journal of Mathematical Education in Science and Technology*, vol. 48, no. 6, pp. 809–829, 2017.

[6] J. Dick and M. Childrey, "Enhancing understanding of transformation matrices," *The Mathematics Teacher*, vol. 105, no. 8, pp. 622–626, 2012.

[7] J.-L. Dorier and A. Sierpinska, "Research into the teaching and learning of linear algebra," *The teaching and learning of mathematics at university level: An ICMI study*, pp. 255–273, 2002.

[8] G. Harel, "Students' understanding of proofs: A historical analysis and implications for the teaching of geometry and linear algebra," *Linear Algebra and its applications*, vol. 302, pp. 601–613, 1999.

[9] M. Nathan, C. Williams-Pierce, C. Walkington, D. Abrahamson, E. Ottmar, H. Soto, and M. Alibali, "The future of embodied design for mathematical imagination and cognition," 2019.

[10] E. G. de Ravé, F. J. Jiménez-Hornero, A. B. Ariza-Villaverde, and J. Taguas-Ruiz, "Diedricar: a mobile augmented reality system designed for the ubiquitous descriptive geometry learning," *Multimedia Tools and Applications*, vol. 75, pp. 9641–9663, 2016.

[11] J. Martín-Gutiérrez, J. L. Saorín, M. Contero, M. Alcañiz, D. C. Pérez-López, and M. Ortega, "Design and validation of an augmented book for spatial abilities development in engineering students," *Computers & Graphics*, vol. 34, no. 1, pp. 77–91, 2010.

[12] A. Estapa and L. Nadolny, "The effect of an augmented reality enhanced mathematics lesson on student achievement and motivation," *Journal of STEM education*, vol. 16, no. 3, 2015.

[13] M.-B. Ibáñez and C. Delgado-Kloos, "Augmented reality for stem learning: A systematic review," *Computers & Education*, vol. 123, pp. 109–123, 2018.

[14] K. R. Bujak, I. Radu, R. Catrambone, B. MacIntyre, R. Zheng, and G. Golubski, "A psychological perspective on augmented reality in the mathematics classroom," *Computers & Education*, vol. 68, pp. 536–544, 2013.

[15] S. Aguilar, H. Burte, P. Yasskin, J. Liew, S. Yeh, C. Qian, D. Song, U. Monjoree, and W. Yan, "Ar-classroom: Usability of ar educational technology for learning rotations using three-dimensional matrix algebra," in *2023 IEEE Frontier in Education Conference*, 2023 (to appear).

[16] A. C. Cheung and R. E. Slavin, "The effectiveness of educational technology applications for enhancing mathematics achievement in k-12 classrooms: A meta-analysis," *Educational research review*, vol. 9, pp. 88–113, 2013.

[17] Q. Li and X. Ma, "A meta-analysis of the effects of computer technology on school students' mathematics learning," *Educational Psychology Review*, vol. 22, pp. 215–243, 2010.

[18] E. McClintock, Z. Jiang, and R. July, "Students' development of three-dimensional visualization in the geometer's sketchpad environment." 2002.

[19] K. A. Bakar, R. A. Tarmizi, A. F. M. Ayub, and A. S. M. Yunus, "Effect of utilizing geometer's sketchpad on performance and mathematical thinking of secondary mathematics learners: An initial exploration," *International journal of education and information technologies*, vol. 3, no. 1, pp. 20–27, 2009.

[20] M. Khalil, R. Farooq, E. ÇAKIROĞLU, U. Khalil, and D. Khan, "The development of mathematical achievement in analytic geometry of grade-12 students through geogebra activities," *Eurasia Journal of Mathematics Science and Technology Education*, vol. 14, no. 4, 2018.

[21] K. A. Bakar, A. F. M. Ayub, and R. A. Tarmizi, "Exploring the effectiveness of using geogebra and e-transformation in teaching and learning mathematics," in *Proc. of Intl. Conf. of Advanced Educational Technologies EDUTE*, vol. 2, 2010, pp. 19–23.

[22] V. Havelková, "Geogebra in teaching linear algebra," in *Proceedings Of The International Conference On E-Learning*, 2013, pp. 581–589.

[23] C. Aytekin and Y. Kiymaz, "Teaching linear algebra supported by geogebra visualization environment." *Acta Didactica Napocensia*, vol. 12, no. 2, pp. 75–96, 2019.

[24] S.-G. Lee, J.-E. Jang, and K.-W. Kim, "Visualization of linear algebra concepts with sage and geogebra," *Communications of Mathematical Education*, vol. 27, no. 1, pp. 1–17, 2013.

[25] L. Diković, "Applications geogebra into teaching some topics of mathematics at the college level," *Computer Science and Information Systems*, vol. 6, no. 2, pp. 191–203, 2009.

[26] H. Rao and W.-T. Fu, "Combining schematic and augmented reality representations in a remote spatial assistance system," in *IEEE ISMAR 2013 Workshop on Collaboration in Merging Realities*. Citeseer, 2013.

[27] M. Tomaschko and M. Hohenwarter, "Augmented reality in mathematics education: The case of geogebra ar," in *Augmented reality in educational settings*. Brill, 2019, pp. 325–346.

[28] T. H. Kramarenko, O. S. Pylypenko, and V. Zaselskyi, "Prospects of using the augmented reality application in stem-based mathematics teaching," 2019.

[29] N. Budinski and Z. Lavicza, "Teaching advanced mathematical concepts with origami and geogebra augmented reality," in *Proceedings of Bridges 2019: Mathematics, Art, Music, Architecture, Education, Culture*, 2019, pp. 387–390.

[30] F. Botana, Z. Kovács, Á. Martínez-Sevilla, and T. Recio, "Automatically augmented reality with geogebra," in *Augmented reality in educational settings*. Brill, 2019, pp. 347–368.

[31] L. D. Edwards, "The nature of mathematics as viewed from cognitive science," in *Third Conference of European Research in Mathematics Education, Bellaria, Italy*, 2003.

[32] H. B. Yanik and A. Flores, "Understanding rigid geometric transformations: Jeff's learning path for translation," *The Journal of Mathematical Behavior*, vol. 28, no. 1, pp. 41–57, 2009.

[33] Vuforia. (2019) Vuforia: Market-Leading Enterprise AR. [Online]. Available: https://www.ptc.com/en/products/vuforia