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When left and right disagree: entropy and von Neumann algebras in quantum gravity with general AlAdS boundary conditions

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Abstract: Euclidean path integrals for UV-completions of d-dimensional bulk quantum gravity were recently studied in [1] by assuming that they satisfy axioms of finiteness, reality, continuity, reflection-positivity, and factorization. Sectors $\mathcal{H}_\mathcal{B}$ of the resulting Hilbert space were then defined for any $(d-2)$-dimensional surface $\mathcal{B}$, where $\mathcal{B}$ may be thought of as the boundary $\partial\Sigma$ of a bulk Cauchy surface in a corresponding Lorentzian description, and where $\mathcal{B}$ includes the specification of appropriate boundary conditions for bulk fields. Cases where $\mathcal{B}$ was the disjoint union $B \sqcup B$ of two identical $(d-2)$-dimensional surfaces B were studied in detail and, after the inclusion of finite-dimensional 'hidden sectors,' were shown to provide a Hilbert space interpretation of the associated Ryu-Takayanagi entropy. The analysis was performed by constructing type-I von Neumann algebras $\mathcal{A}_L^B, \mathcal{A}_R^B$ that act respectively at the left and right copy of B in $B \sqcup B$.

Below, we consider the case of general $\mathcal{B}$, and in particular for $\mathcal{B} = B_L \sqcup B_R$ with B_L, B_R distinct. For any B_R, we find that the von Neumann algebra at B_L acting on the off-diagonal Hilbert space sector $\mathcal{H}_{B_L \sqcup B_R}$ is a central projection of the corresponding type-I von Neumann algebra on the 'diagonal' Hilbert space $\mathcal{H}_{B_L \sqcup B_L}$. As a result, the von Neumann algebras $\mathcal{A}_L^{B_L}, \mathcal{A}_R^{B_L}$ defined in [1] using the diagonal Hilbert space $\mathcal{H}_{B_L \sqcup B_L}$ turn out to coincide precisely with the analogous algebras defined using the full Hilbert space of the theory (including all sectors $\mathcal{H}_\mathcal{B}$). A second implication is that, for any $\mathcal{H}_{B_L \sqcup B_R}$, including the same hidden sectors as in the diagonal case again provides a Hilbert space interpretation of the Ryu-Takayanagi entropy. We also show the above central projections to satisfy consistency conditions that lead to a universal central algebra relevant to all choices of B_L and B_R.

Keywords: AdS-CFT Correspondence, Models of Quantum Gravity

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Contents

1 Introduction

As emphasized in [1], a number of arguments regarding gravitational entropy that were originally motivated by the AdS/CFT correspondence have now been understood to follow directly from bulk physics. A primary example is the derivation [2, 3] of the Island Formula for the entropy of Hawking radiation transferred from an asymptotically-locally-AdS (AlAdS) gravitational system to a non-gravitational bath. This derivation simply combines the gravitational path integral arguments of [4–7] with the setting studied in [8, 9]. And as explained in [10], in this context the results may be safely interpreted in terms of standard von Neumann entropies without invoking holography at any intermediate step.

Another class of examples involves taking the semiclassical limit in which Hilbert space densities of states diverge. Using purely bulk methods, one can show that the algebra of observables is generated by a type-II von Neumann factor and its commutant. This observation then leads to an entropy on these algebras that agrees with the quantum-corrected RT formula up to an additive constant [11–14].

Motivated by such results, it was suggested in [1] that purely bulk arguments (i.e., without assuming the existence of a holographic dual field theory) should suffice to provide a Hilbert-space interpretation of an entropy defined by regions of an AlAdS boundary for which the semiclassical limit is given by the Ryu-Takayangi formula [15, 16] (or its covariant Hubeny-Rangamani-Takayanagi generalization [17]).

By assuming certain Euclidean-signature axioms, this was then shown to be the case in so-called 'diagonal' settings where the boundary $\partial\Sigma$ of a Cauchy surface Σ in a corresponding Lorentz-signature spacetime took the form $\partial\Sigma = B \sqcup B$, where B was a compact $(d-2)$-dimensional manifold (with $\partial B = \emptyset$), and on which appropriate boundary conditions were specified for bulk fields.

The argument of [1] was formulated in terms of a supposed path integral for a UV-complete finite-coupling[1] bulk asymptotically-locally-AdS$_d$ (AlAdS$_d$) theory. The previously-advertised axioms for such path integrals were called finiteness, reality,[2] reflection-positivity, continuity, and factorization. In the above diagonal setting, these properties sufficed to show the associated Hilbert space sectors $\mathcal{H}_{B\sqcup B}$ to be direct sums $\bigoplus_\mu \mathcal{H}^\mu_{B\sqcup B}$ of Hilbert spaces that factorize as $\mathcal{H}^\mu_{B\sqcup B} = \mathcal{H}^\mu_{B\sqcup B,L} \otimes \mathcal{H}^\mu_{B\sqcup B,R}$, where the left and right factors $\mathcal{H}^\mu_{B\sqcup B,L}$ and $\mathcal{H}^\mu_{B\sqcup B,R}$ for given μ are isomorphic but are associated with operator algebras that, in an appropriate sense, act at the respective left and right copy of B in $\mathcal{H}_{B\sqcup B}$. It was then further shown that the path integral defines a trace on such algebras that agrees with the standard sum-over-diagonal-matrix-elements Hilbert-space trace associated with the Hilbert space

$$\underset{\sim}{\mathcal{H}}_B = \bigoplus_\mu \mathcal{H}^\mu_{B\sqcup B,L} \otimes \mathbb{C}^{n_\mu} = \bigoplus_\mu \mathcal{H}^\mu_{B\sqcup B,R} \otimes \mathbb{C}^{n_\mu} \tag{1.1}$$

for appropriate integers n_μ. The corresponding entropies thus agree as well and, by first making use of an appropriate embedding of $\mathcal{H}_{B\sqcup B}$ in $\underset{\sim}{\mathcal{H}}_B \otimes \underset{\sim}{\mathcal{H}}_B$ and then connecting with the Lewkowycz-Maldacena argument [4] and its generalizations, one obtains a Hilbert space interpretation of the Ryu-Takayanagi entropy of either boundary B.

The reader should note that the Hilbert space $\underset{\sim}{\mathcal{H}}_B$ was not explicitly introduced in [1], though its use will simplify our discussion. Indeed, the tilde decoration at the bottom of $\underset{\sim}{\mathcal{H}}_B$ is intended to help to distinguish $\underset{\sim}{\mathcal{H}}_B$ from the various Hilbert spaces defined in [1] that were denoted by symbols with upper tildes. As shown by the 2nd equality in (1.1), it would be unnatural to assign either an L or an R label to $\underset{\sim}{\mathcal{H}}_B$. Furthermore, in the diagonal case we can lose nothing by using only a subscript B instead of $B \sqcup B$. This will turn out to be a good choice of notation as we will see below that the same space $\underset{\sim}{\mathcal{H}}_B$ arises in the analogous analysis of any off-diagonal sector $\mathcal{H}_{B\sqcup B'}$, where now $\mathcal{H}_{B\sqcup B'}$ is to be embedded in $\underset{\sim}{\mathcal{H}}_B \otimes \underset{\sim}{\mathcal{H}}_{B'}$.

The focus of [1] on diagonal sectors $\mathcal{H}_{B\sqcup B}$ had two primary drawbacks. The most obvious was of course that it provided the desired Hilbert space interpretation of RT entropy only when the two boundaries are identical. Directly generalizing the arguments of [1] to the case

[1]Gravitational path integrals associated with familiar classical actions diverge in the limit $G \to 0$ (or $S_0 \to \infty$ for JT gravity). As a result, the asymptotic expansion of such path integrals in powers of G will generally fail to satisfy the axioms of [1]; see [18] and especially section 5.4 of [1] for further discussion of this issue.

[2]The reality axiom implies that the theory to be invariant under a notion of time-reversal symmetry. We expect that this axiom is not in fact necessary.

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with $(d-2)$-dimensional boundary $B \sqcup B'$ turns out to be nontrivial due to the reliance of [1] on special properties of cylinders $C_\epsilon = B \times [0,\epsilon]$. The obstacle is that such cylinders are intrinsically diagonal in the sense that $\partial C_\epsilon = B \sqcup B$. We will thus seek other arguments below.

The second issue arises from the fact that the choice of Hilbert space plays a role in the construction of the desired operator algebras. In particular, while an algebra of simple operators can be defined directly using smooth boundary conditions in the path integral, the most useful mathematical structures turn out to be the left and right von Neumann algebras constructed by using a Hilbert space to define an appropriate notion of a completion. While this may seem like a mathematical technicality, it raises the interesting question of whether using the full quantum gravity Hilbert space might lead to von Neumann algebras (and thus to entropies) that differ from the ones constructed using only the diagonal sectors $\mathcal{H}_{B\sqcup B}$ as in [1].

We will address both of these shortfalls below. As further motivation for our study, it is useful to take inspiration from the AdS/CFT correspondence, in which the bulk path integral is simply equal to a path integral for the dual CFT. While it is not necessarily the most general allowed setting, this case certainly satisfies the axioms of [1]. Furthermore, in the AdS/CFT context, for any choice of B and B' we always have the so-called Harlow-factorization property[3] $\mathcal{H}_{B\sqcup B'} = \mathcal{H}_B \otimes \mathcal{H}_{B'}$ so that in particular, the left Hilbert space factor (and the associated type-I von Neumann factor) is manifestly independent of the choice of B'.[4]

One might then hope that a similar independence of B' follows more generally from the axioms of [1]. Below, it will be useful to rename B, B' as B_L, B_R and to refer to B_L, B_R as the left and right boundaries. In that notation, while we have already seen in the diagonal context that a given boundary B_L is associated with a *set* of left Hilbert space factors $\mathcal{H}_{B_L\sqcup B_L,L}$, one may nevertheless hope that the full set of such factors has no dependence on B_R.

As foreshadowed above, this will turn out to be nearly true in the sense that the off-diagonal Hilbert spaces $\mathcal{H}_{B_L\sqcup B_R}$ again decompose according to[5]

$$\mathcal{H}_{B_L\sqcup B_R} = \bigoplus_\mu \mathcal{H}^\mu_{B_L\sqcup B_R} \quad \text{with} \quad \mathcal{H}^\mu_{B_L\sqcup B_R} = \mathcal{H}^\mu_{B_L\sqcup B_R,L} \otimes \mathcal{H}^\mu_{B_L\sqcup B_R,R}, \tag{1.2}$$

such that every left Hilbert space factor $\mathcal{H}^\mu_{B_L\sqcup B_R,L}$ is in fact canonically isomorphic to a left factor $\mathcal{H}^\mu_{B_L\sqcup B_L,L}$ associated with the diagonal Hilbert space $\mathcal{H}_{B_L\sqcup B_L}$. However, for given B_R it may be that we find only a subset of the diagonal left Hilbert-space factors $\mathcal{H}^\mu_{B_L\sqcup B_L,L}$. Indeed, we will see that there is a natural sense in which the isomorphic factors $\mathcal{H}^\mu_{B_L\sqcup B_R,L}$ and $\mathcal{H}^\mu_{B_L\sqcup B_L,L}$ can be said to be associated with the same value of μ, so that for any μ appearing in (1.2) it is natural to write

$$\mathcal{H}^\mu_{B_L\sqcup B_R,L} = \mathcal{H}^\mu_{B_L\sqcup B_L,L} =: \mathcal{H}^\mu_{B_L}, \tag{1.3}$$

where this defines the symbol $\mathcal{H}^\mu_{B_L}$ used on the right-hand-side. Here we have refrained from adding an additional $_{,L}$ subscript on $\mathcal{H}^\mu_{B_L}$ since the identical Hilbert space arises from the

[3]The name comes from the emphasis on this property in [19].

[4]Indeed, since for AdS/CFT there is only a single value of μ and it has $n_\mu = 1$, for this case (1.1) gives $\utilde{\mathcal{H}}_B = \mathcal{H}_B$.

[5]Here we use notation chosen to mirror that of [1]. In this notation, we emphasize that an object labelled with $B_L \sqcup B_R$ may in fact depend on the partition of $\mathcal{B} = B_L \sqcup B_R$ into B_L and B_R, and thus that it is generally not determined entirely by $\mathcal{B}$ alone. Symbols in which an explicit $\sqcup$ does not appear will be free of this issue.

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corresponding construction when B_L is used as a right boundary (instead of a left boundary as above). In addition, for given μ the integer n_μ will be shown to be independent of the choice of boundary so that, in the notation of (1.3), replacing B in (1.1) by B_L, B_R we may write both

$$\underset{\sim}{\mathcal{H}}_{B_L} := \bigoplus_\mu \left(\mathcal{H}^\mu_{B_L} \otimes \mathbb{C}^{n_\mu}\right) \quad \text{and} \quad \underset{\sim}{\mathcal{H}}_{B_R} := \bigoplus_\mu \left(\mathcal{H}^\mu_{B_R} \otimes \mathbb{C}^{n_\mu}\right) \tag{1.4}$$

for the same integers n_μ.

The results of [1] then imply that the trace defined by the path integral on operators that act at any B_L or B_R coincides with the sum-over-diagonal-matrix-elements trace defined by the Hilbert spaces (1.4). A Hilbert space interpretation of the Ryu-Takayanagi entropy associated with either B_L or B_R of states in $\mathcal{H}_{B_L \sqcup B_R}$ then follows from an appropriate embedding of $\mathcal{H}_{B_L \sqcup B_R}$ in $\underset{\sim}{\mathcal{H}}_{B_L} \otimes \underset{\sim}{\mathcal{H}}_{B_R}$.

Finally, we will also verify the analogous statements for the above-mentioned von Neumann algebras, showing in particular that the algebras constructed in [1] using only the diagonal sectors $\mathcal{H}_{B \sqcup B}$ do in fact coincide with von Neumann algebras completed by using the topology defined by the entire quantum gravity Hilbert space. More specficially, we will see that the von Neumann algebra acting at B_L associated with an off-diagonal sector $\mathcal{H}_{B_L \sqcup B_R}$ is always a central projection of the corresponding algebra defined by the diagonal Hilbert space sector $\mathcal{H}_{B_L \sqcup B_L}$. Furthermore, these projections will be shown to satisfy compatibility conditions that allow us to assemble such central projections into a universal central algebra, independent of the choice of any particular B_L, from which the central algebra for each pair B_L,B_R can be recovered by acting with an appropriate projection. Such algebraic results are in fact more fundamental than the Hilbert-space results described above and will thus be addressed first in the work below.

This paper is organized as follows. We begin in section 2 by reviewing the construction of algebras and Hilbert spaces from Euclidean path integrals as described in [1]. This includes the definition of general off-diagonal Hilbert space sectors $\mathcal{H}_{B_L \sqcup B_R}$, as well as algebras $\hat{A}_L^{B_L \sqcup B_R}$, $\hat{A}_R^{B_L \sqcup B_R}$ of operators on $\mathcal{H}_{B_L \sqcup B_R}$ defined by attaching surfaces respectively to the left and right boundaries B_L, B_R. However, these algebras are not complete in any natural topology, and [1] defined von Neumann completions only in the diagonal context $B_L = B_R$. The new results begin in section 3, which shows that the off-diagonal left-algebra $\hat{A}_L^{B_L \sqcup B_R}$ is canonically identified with a quotient of the diagonal left-algebra $\hat{A}_L^{B_L \sqcup B_R}$, and similarly for the right-algebras. It then remains to study the completions that define the off-diagonal von Neumann algebras in section 4. After developing some useful technology, we demonstrate that the off-diagonal von Neumann algebras are again canonically identified with quotients of the diagonal von Neumann algebras. Section 5 then shows this identification to take the form of a central projection, discusses the relationship between the off-diagonal and diagonal Hilbert spaces, and organizes the discussion of centers in terms of a universal central algebra that is independent of the choices of boundaries. The fact that the left and right von Neumann algebras are commutants on $\mathcal{H}_{B_L \sqcup B_R}$ is also established in this section by making use of further supporting results from appendix A. With all of the above results in place, it is then straightforward to describe the Hilbert space interpretation of RT entropy in the off-diagonal context. This is done in section 6, after which further discussion and final comments are provided in section 7.

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2 Algebras and Hilbert spaces from gravitational path integrals

The results of [1] were established within an axiomatic framework for the Euclidean path integral in UV completions of quantum theories of gravity. The five axioms used in [1] are briefly summarized below, though we refer the reader to [1] for full details and additional discussion.

1. **Finiteness:** the boundary conditions for the path integral are assumed to form a space X^d of d-dimensional 'source manifolds' X^d. The path integral then defines a map $\zeta : X^d \to \mathbb{C}$ to the complex numbers; i.e., $\zeta(M)$ is well-defined and finite for every $M \in X^d$. Local restrictions on the sources may be imposed as needed to achieve this property. As an example, one may choose to require source manifolds to have non-negative scalar curvature.

2. **Reality:** let $\underline{X}^d$ denote formal finite linear combinations of source manifolds with coefficients in $\mathbb{C}$. We extend ζ to elements of $\underline{X}^d$ by linearity. For every $M \in \underline{X}^d$, we have both $M^* \in \underline{X}^d$ and $[\zeta(M)]^* = \zeta(M^*)$. This axiom is trivial if the original space X^d of source manifolds is taken to be real; i.e., if * is taken to act trivially on X^d. Furthermore, as noted in the introduction, this axiom also implies a time-reversal symmetry. We thus expect that it can be dropped without significant harm, though we leave such a study for future work.

3. **Reflection-Positivity:** $\zeta(M)$ is a non-negative real number for reflection-symmetric source manifolds M, i.e. $M \in \underline{X}^d$ can be written in the form $M = \sum_{I,J=1}^{n} \gamma_I^* \gamma_J M_{I,J}$ where $\gamma_I \in \mathbb{C}$, γ_I^* denotes the complex conjugate of γ_I, and where each $M_{I,J}$ can be sliced into two parts N_I^* and N_J, for some $n \in \mathbb{Z}^+$.

4. **Continuity:** suppose that the source manifold $M \in X^d$ contains a 'cylinder' C_ϵ of the form $B \times [0, \epsilon]$. Then ζ is continuous in the length ϵ of this cylinder for all $\epsilon > 0$.

5. **Factorization:** for closed boundary manifolds M_1, M_2 and their disjoint union $M_1 \sqcup M_2$, we have $\zeta(M_1 \sqcup M_2) = \zeta(M_1)\zeta(M_2)$.

The framework can also be applied to contexts like those in [20] and [21] where factorization fails, but where the path integral can be expressed as an integral ($\zeta = \int d\alpha\ \zeta_\alpha$) over path integrals ζ_α in which all of the above axioms hold. The results of [1] then clearly hold for each ζ_α, with corresponding implications for the full path integral ζ.

Another important ingredient in the discussion of [1] was the concept of a source-manifold N with boundary ∂N. An operation * (also used in axiom 3) was defined on $Y_{\mathcal{B}}^d$ by complex-conjugating the sources on N and simultaneously reflecting N about its boundary. This * was then used to define the notion of a *rimmed* source-manifold-with-boundary N, which is a was allowed to have a non-trivial boundary ∂N so long as some open set containing ∂N was a cylinder of the form $C_\epsilon = B \times [0, \epsilon]$ described above with $C_\epsilon = C_\epsilon^*$. We will discuss only boundaries B for which there exist cylinders satisfying this condition.

The space of such rimmed source-manifolds with boundary $\mathcal{B}$ was denoted $Y_{\mathcal{B}}^d$. We see that any $N_1, N_2 \in Y_{\mathcal{B}}^d$ can be naturally sewn together across $\mathcal{B}$ to define a smooth closed source-manifold $M_{N_1^* N_2}$. The space of formal finite linear combinations $\underline{Y}^d_{\mathcal{B}}$, equipped with a pre-inner product defined by the path integral of glued source manifolds $\langle N_1|N_2\rangle = \zeta(M_{N_1^* N_2})$, then forms a pre-inner product space[6] which we denote as $H_{\mathcal{B}}$. The Hilbert space $\mathcal{H}_{\mathcal{B}}$ is obtained by first taking the quotient by the space of null vectors $\mathcal{N}_{\mathcal{B}}$, and then taking the completion with respect to the norm. To reflect the fact that it is a dense subspace of $\mathcal{H}_{\mathcal{B}}$, we introduce the notation $\mathcal{D}_{\mathcal{B}} = H_{\mathcal{B}}/\mathcal{N}_{\mathcal{B}}$ for the pre-Hilbert space defined before taking the completion.

The analysis of [1] focused on the case when the boundary $\mathcal{B}$ is a disjoint union of two closed boundary manifolds $\mathcal{B} = B_1 \sqcup B_2$. The spaces $\underline{Y}^d_{B_1\sqcup B_1}$ and $\underline{Y}^d_{B_2\sqcup B_2}$ can then be promoted to algebras $A_L^{B_1}, A_R^{B_2}$ by defining products that simply glue together the two surfaces being multiplied. The product $a \cdot_L b$ on $\underline{Y}^d_{B_1\sqcup B_1}$ (which is used to define $A_L^{B_1}$) is defined by gluing the right boundary of a to the left boundary of b, while the product $c \cdot_R d$ on $\underline{Y}^d_{B_2\sqcup B_2}$ (which is used to define $A_R^{B_2}$) is defined by gluing the left boundary of c to the right boundary of d. Since left and right products are related by $a \cdot_L b = b \cdot_R a$, we may define $ab := a \cdot_L b = b \cdot_R a$. There is also a natural involution $\star$ defined by $a^\star := (a^t)^*$, where the transpose operation ^{t} simply interchanges the labels left and right on the boundaries of a. Thus a^t is the same source manifold as a but with the left boundary of a now called the right boundary of a^t, and vice versa.

If we interchange the two boundaries to instead use $\mathcal{B} = B_2 \sqcup B_1$, the same construction defines analogous algebras $A_L^{B_2}$ and $A_R^{B_1}$. The involution $\star$ then defines an anti-linear isomorphism between $A_L^{B_i}$ and $A_R^{B_i}$. Furthermore, a trace operation tr can be defined on these surface algebras using the path integral, $tr(a) := \zeta(M(a))$, where $M(a) \in \underline{X}^d$ denotes the source manifold obtained from gluing together the two copies of B in the boundary of $a \in \underline{Y}^d_{B\sqcup B}$.

Representations of the surface algebras $A_L^{B_1}$ and $A_R^{B_2}$ on the sector $\mathcal{H}_{B_1\sqcup B_2}$ were then constructed in two steps. The first step was to consider the natural actions of $A_L^{B_1}, A_R^{B_2}$ on the pre-inner product space $H_{B_1\sqcup B_2}$ by gluing the relevant surfaces along corresponding boundary components (B_1 or B_2). For example, $a \in A_L^{B_1}$ is represented by an operator $\hat{a}_L$ that acts on $|b\rangle \in H_{B_1\sqcup B_2}$ by gluing the right boundary of a to the left boundary of b so that $\hat{a}_L|b\rangle = |ab\rangle$. The next step used the trace inequality

$$tr(b^\star a a^\star b) \leq tr(a^\star a) tr(b^\star b). \tag{2.1}$$

derived in [1] for[7] a in either $A_L^{B_1}$ or $A_R^{B_2}$ and any $b \in H_{B_1\sqcup B_2}$. As shown in figure 1, the relation (2.1) is equivalent to the inequality

$$\langle b|\hat{a}_L \hat{a}_L^\dagger|b\rangle \leq tr(a^\star a)\langle b|b\rangle, \tag{2.2}$$

which immediately implies that each operator in the representation is bounded. It thus preserves the null space $\mathcal{N}_{B_1\sqcup B_2}$, and induces a (bounded) operator on $\mathcal{D}_{B_1\sqcup B_2}$. It follows that there is a unique bounded extension to the Hilbert space $\mathcal{H}_{B_1\sqcup B_2}$.

[6] This is the same $H_{\mathcal{B}}$ as in [1], where it was called a pre-Hilbert space.

[7] In fact, the inequality (2.1) was derived in [18] for any $a \in \underline{Y}^d_{B_1\sqcup B_2}$, $b \in \underline{Y}^d_{B_1\sqcup B_3}$. In that context, we can still define a corresponding operation $\star$ such that $a^\star \in \underline{Y}^d_{B_2\sqcup B_1}$, $b^\star \in \underline{Y}^d_{B_3\sqcup B_1}$, and concatenation of surfaces then defines $a^\star a \in A_L^{B_2}$ and $b^\star b, b^\star a a^\star b \in A_L^{B_3}$. This more general version will be useful in appendix B.

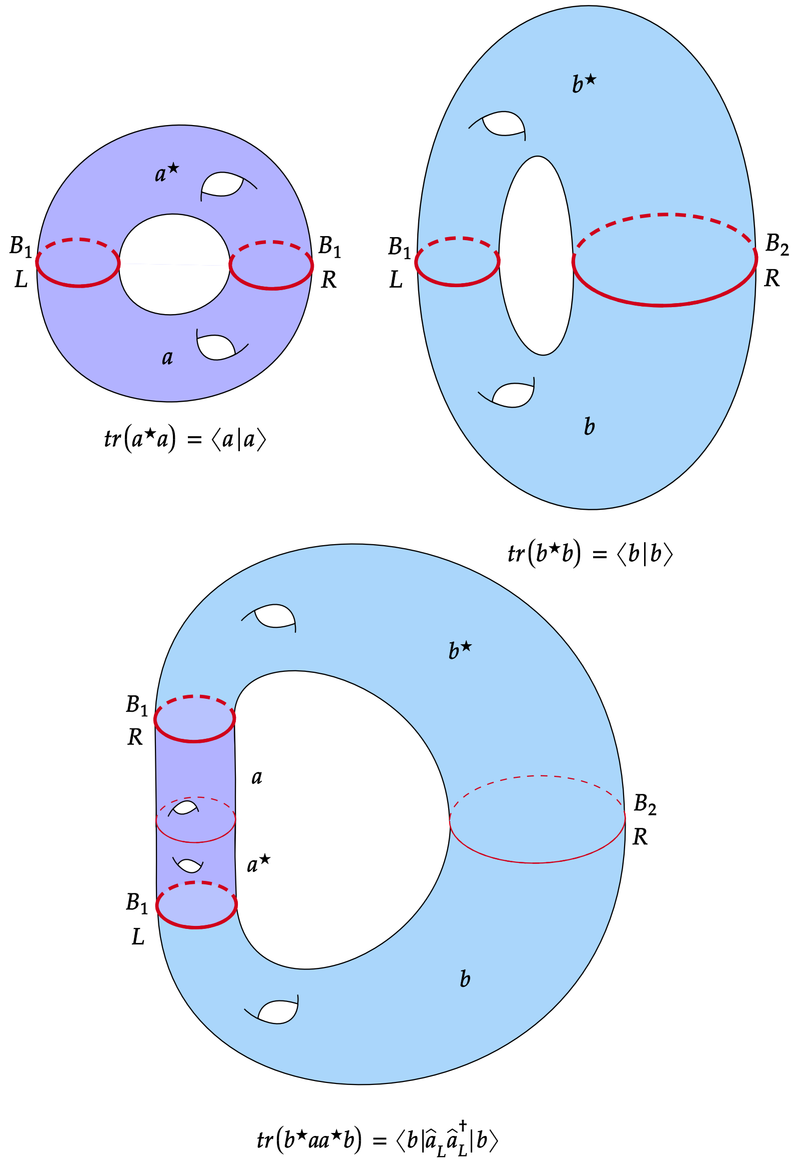

Figure 1. For surfaces $a \in Y^d_{B_L \sqcup B_L}$ and $b \in Y^d_{B_L \sqcup B_R}$, the traces of $a^\star a$ and $b^\star b$ coincide with $\langle a|a\rangle$ and $\langle b|b\rangle$ as shown in the upper panel. The left hand side $tr(b^\star aa^\star b)$ of the trace inequality (2.1) computes the inner product $\langle b|\hat{a}_L\hat{a}_L^\dagger|b\rangle$, as shown in the lower panel.

The left and right representations established on the sector $\mathcal{H}_{B_1 \sqcup B_2}$ are denoted by $\hat{A}_L^{B_1 \sqcup B_2}$ and $\hat{A}_R^{B_1 \sqcup B_2}$. The adjoint operation defined by $\mathcal{H}_{B_1 \sqcup B_2}$ then satisfies $(\hat{a}_L)^\dagger = \widehat{(a^\star)}_L$, and similarly for the right algebra. Finally, it is clear that operators in $\hat{A}_L^{B_1 \sqcup B_2}$ commute with those in $\hat{A}_R^{B_1 \sqcup B_2}$.

The remaining analysis of [1] was restricted to so-called diagonal sectors of the form $\mathcal{H}_{B_1 \sqcup B_1}$; i.e., with B_1 diffeomorphic to B_2. In that context, left and right von Neumann algebras $\mathcal{A}_L^{B_1}$ and $\mathcal{A}_R^{B_1}$ were constructed by taking completions of the representations $\hat{A}_L^{B_1 \sqcup B_1}$ and $\hat{A}_R^{B_1 \sqcup B_1}$ in the weak operator topology. The adjoint operation again acts as an involution on these von Neumann algebras.

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A key point was then that the above trace tr can be extended to positive elements of the von Neumann algebras $\mathcal{A}_L^{B_1}$ and $\mathcal{A}_R^{B_1}$ by taking it to be defined by

$$tr(a) := \lim_{\beta \downarrow 0} \langle C_\beta | a | C_\beta \rangle, \tag{2.3}$$

where C_β is an appropriate cylinder[8] of length β. The result is faithful, normal, and semifinite. In addition, it continues to satisfy the trace inequality (2.1), as well as related inequalities derived using larger numbers of boundaries. Together, these results require $tr(P)$ to be a non-negative integer for any projection P with finite trace. Each of the algebras $\mathcal{A}_L^{B_1}, \mathcal{A}_R^{B_1}$ must thus be a direct sum of type I factors. Furthermore, the Hilbert space $\mathcal{H}_{B_1 \sqcup B_1}$ is a direct sum of Hilbert spaces $\mathcal{H}^\mu_{B_1 \sqcup B_1}$ that factorize as $\mathcal{H}^\mu_{B_1 \sqcup B_1} = \mathcal{H}^\mu_{B_1 \sqcup B_1, L} \otimes \mathcal{H}^\mu_{B_1 \sqcup B_1, R}$ with $\mathcal{H}^\mu_{B_1 \sqcup B_1, L}$ canonically isomorphic to $\mathcal{H}^\mu_{B_1 \sqcup B_1, R}$ up to an overall phase. Finally, it was shown that such Hilbert-space factors $\mathcal{H}^\mu_{B_1 \sqcup B_1, L}$, $H^\mu_{B_1 \sqcup B_1, R}$ could be supplemented with finite dimensional Hilbert spaces $\mathbb{C}^{n_\mu}$ such that the trace $\widetilde{Tr}$ defined by summing diagonal matrix elements of operators on $\underset{\sim}{\mathcal{H}}_{B_1} := \bigoplus_\mu \mathcal{H}^\mu_{B_1 \sqcup B_1, L} \otimes \mathbb{C}^{n_\mu} = \bigoplus_\mu \mathcal{H}^\mu_{B_1 \sqcup B_1, R} \otimes \mathbb{C}^{n_\mu}$ coincides on $\mathcal{A}_L^{B_1}$ and $\mathcal{A}_R^{B_1}$ with the trace tr defined above. Since $\mathcal{H}_{B_1 \sqcup B_1} \subset \underset{\sim}{\mathcal{H}}_{B_1} \otimes \underset{\sim}{\mathcal{H}}_{B_1}$, this provided a Hilbert space interpretation of the entropy described by the gravitational replica trick. And by the argument of [4], this entropy is well approximated by the Ryu-Takayanagi entropy [15, 16] when the bulk theory admits an appropriate semiclassical limit.

3 Off-diagonal representations from the diagonal representation

The main goal of this paper is to generalize the above results to off-diagonal sectors $\mathcal{H}_{B_L \sqcup B_R}$ with $B_L \neq B_R$. We perform the first steps of that analysis in this section, focusing on the surface algebras $A_L^{B_L}$ and $A_R^{B_R}$ and their representations $\hat{A}_L^{B_L \sqcup B_R}$ and $\hat{A}_R^{B_L \sqcup B_R}$ on $\mathcal{H}_{B_L \sqcup B_R}$. In particular, we will show that any off-diagonal representation $\hat{A}_L^{B_L \sqcup B_R}$ can be identified with a quotient of $\hat{A}_L^{B_L \sqcup B_L}$.

The construction of these objects with $B_L \neq B_R$ was already given in [1] and was reviewed in section 2. We may thus proceed rapidly. The surface algebras $A_L^{B_1}$ and $A_R^{B_2}$ were in fact defined in section 2 using only properties that are intrinsic to the spaces of surfaces $Y^d_{B_1 \sqcup B_1}, Y^d_{B_2 \sqcup B_2}$, without mention of any Hilbert space sector. We thus need only set $B_1 = B_L$ and $B_2 = B_R$ to obtain surface algebras $A_L^{B_L}$ and $A_R^{B_R}$ which are identical to those used in the diagonal context.

We will show below that the representation of $A_L^{B_L}$ on any sector $\mathcal{H}_{B_L \sqcup B_R}$ is always a quotient of the representation on the diagonal sector $\mathcal{H}_{B_L \sqcup B_L}$, and similarly for the right surface algebra. This statement is equivalent to the claim that, if n lies in the diagonal null space $\mathfrak{N}_L^{B_L \sqcup B_L}$ of elements of $A_L^{B_L}$ that annihilate all states in the diagonal sector $\mathcal{H}_{B_L \sqcup B_L}$, then n must also annihilate all states in any non-diagonal sector $\mathcal{H}_{B_L \sqcup B_R}$.

To streamline our notation for boundaries, we now introduce the shorthand $LR = B_L \sqcup B_R$, $LL = B_L \sqcup B_L$, and $RR = B_R \sqcup B_R$. We will in particular write $\hat{A}_L^{LR} = \hat{A}_L^{B_L \sqcup B_R}$ and $\hat{A}_R^{LR} = \hat{A}_R^{B_L \sqcup B_R}$. As in the diagonal case, the adjoint operation defined by $\mathcal{H}_{B_L \sqcup B_R}$ satisfies

[8]Ref. [1] instead used so-called normalized cylinders $\tilde{C}_\beta$, but this is unnecessary since the appendix of [1] shows that the appropriate norm approaches 1 as $\beta \to 0$.

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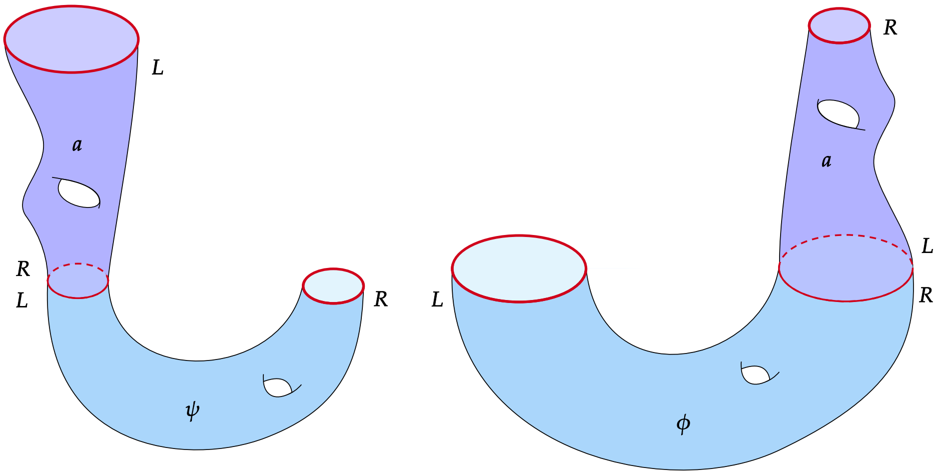

Figure 2. For surfaces $a \in Y^d_{LR}$, $\psi \in Y^d_{LL}$, $\phi \in Y^d_{RR}$, the left panel shows the action of $\hat{a}_L$ on $|\psi\rangle \in H_{RR}$ while the right panel shows the action of $\hat{a}_R$ on $|\phi\rangle \in H_{LL}$ as defined in equation (3.2). Both surfaces obtained belong to Y^d_{LR}.

$(\hat{a}_L)^\dagger = \widehat{(a^\star)}_L$, and similarly for the right algebra. It is also again clear that operators in $\hat{A}^{LR}_L$ commute with those in $\hat{A}^{LR}_R$.

The representation $\hat{A}^{LR}_L$ is a faithful representation of the quotient algebra $A^{B_L}_L/\mathfrak{N}^{LR}_L$, where $\mathfrak{N}^{LR}_L = \mathfrak{N}^{B_L \sqcup B_R}_L$ is the null space consisting of elements whose operator representations annihilate the entire sector $\mathcal{H}_{B_L \sqcup B_R}$. We now make the following claim:

Claim 3.1. *For any B_R, the null space $\mathfrak{N}^{B_L \sqcup B_R}_L$ contains the diagonal null space $\mathfrak{N}^{B_L \sqcup B_L}_L$:*

$$\mathfrak{N}^{B_L \sqcup B_R}_L \supseteq \mathfrak{N}^{B_L \sqcup B_L}_L. \tag{3.1}$$

Here, and throughout the rest of this work, we could of course also discuss the corresponding properties of algebras that act on the right boundary (which here would give $\mathfrak{N}^{B_L \sqcup B_R}_R \supseteq \mathfrak{N}^{B_R \sqcup B_R}_R$). For simplicity, we will generally refrain from doing so explicitly, though in all cases such properties clearly follow from analogous arguments.

It is useful to introduce an additional set of bounded operators before we prove claim 3.1. For any surface $a \in Y^d_{LR}$, we can define an operator $\hat{a}_R : H_{LL} \to H_{LR}$ via the usual gluing of surfaces; see figure 2. This is just the analogue of the action of the surface algebra $A^{B_L}_R$ on H_{LL}, but where the two boundaries of a are now allowed to be different (so that the set of such objects no longer forms a natural algebra, and so that $\hat{a}_R|\phi\rangle$ lives in a different Hilbert space sector than the original state $|\phi\rangle$). As usual, the inequality (2.2) implies that $\hat{a}_R$ maps zero-norm states to zero-norm states and so yields a well-defined operator on the image of H_{LL} in $\mathcal{H}_{LL}$. The action of the resulting operator is naturally written in the form

$$\hat{a}_R\,|\phi\rangle = |\phi a\rangle \quad \forall\,|\phi\rangle \in H_{LL}, \tag{3.2}$$

where ϕa denotes the surface formed by gluing the right boundary of $\phi \in Y^d_{LL}$ to the left boundary of $a \in Y^d_{LR}$. The subscript R on $\hat{a}_R$ denotes the fact that this operator acts on the right boundary of ϕ. We also define the analogous left operator $\hat{a}_L : H_{RR} \to H_{LR}$ for which

$$\hat{a}_L\,|\psi\rangle = |a\psi\rangle \quad \forall\,|\psi\rangle \in H_{RR}, \tag{3.3}$$

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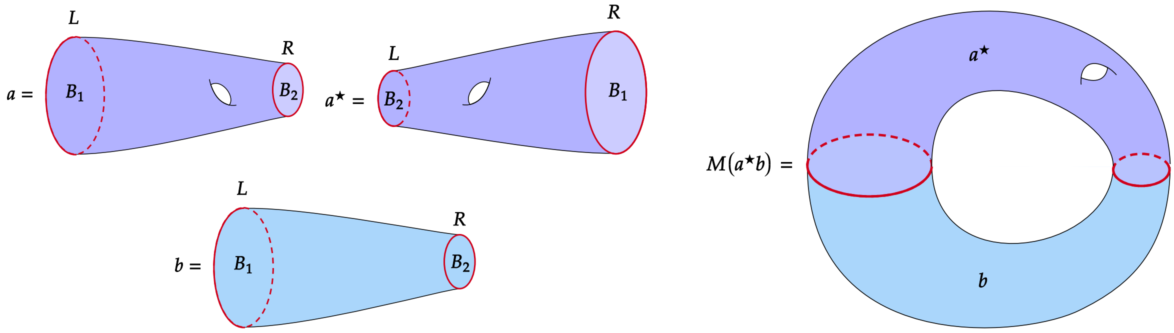

Figure 3. For surfaces $a, b \in Y^d_{B_1 \sqcup B_2}$, we can construct $a^\star$ and compute $tr(a^\star b)$ and $tr(ba^\star)$. Both are given by evaluating ζ on the closed surface shown at right, as is $\langle a|b\rangle$ This observation yields (3.6).

Now, as described in section 2, surfaces a define states $|a\rangle$ in the dense subspace $\mathcal{D}_{LR} \subset \mathcal{H}_{LR}$. It will be useful below to think about defining the operator $\hat{a}_R$ directly from such $|a\rangle$. To do so, we simply choose an arbitrary representative element $a \in H_{LR}$ of the equivalence class defined by $|a\rangle$. This a is a finite linear combination of surfaces in Y^d_{LR} to which we can extend our definition of $\hat{a}_R$ by linearity. We then need only observe that if two representatives a_1, a_2 differ by some zero-norm surface $N \in \mathcal{N}_{LR}$, then for any ϕ the norm of $|\phi N\rangle$ is given by

$$\langle \phi N|\phi N\rangle = \langle N|\hat{\phi}^\dagger_L \hat{\phi}_L|N\rangle \leq tr(\phi^\star \phi)\langle N|N\rangle = 0, \tag{3.4}$$

where the inequality as usual follows from (2.2). Thus the operators $\hat{a}_{1,R}$ and $\hat{a}_{2,R}$ are identical so that $\hat{a}_R$ is fully defined by the choice of the state $|a\rangle \in \mathcal{D}_{LR} \subset \mathcal{H}_{LR}$.

For later use, we denote the associated map on $\mathcal{D}_{LR}$ by Ψ_R, and we use Ψ_L for the corresponding left operator. In particular, we will use the notation

$$\Psi_R(|a\rangle) := \hat{a}_R, \quad \Psi_L(|a\rangle) := \hat{a}_L, \tag{3.5}$$

where we remind the reader that $\hat{a}_R$ maps H_{LL} to H_{LR} while $\hat{a}_L$ maps H_{RR} to H_{LR}.

To discuss the adjoints of $\hat{a}_R$ and $\hat{a}_L$, we first extend the definition of the $\star$ operation to off-diagonal surfaces:

Definition 1. *For any $a \in Y^d_{B_1 \sqcup B_2}$, we define $a^\star \in Y^d_{B_2 \sqcup B_1}$ to be the source manifold-with-boundary obtained from a by complex-conjugating all sources, relabeling the left boundary B_1 of a as the right boundary of $a^\star$, and similarly relabeling the right boundary B_2 of a as the left boundary of $a^\star$. As shown in figure 3, this definition then satisfies the relation*

$$\langle a|b\rangle = \zeta(M(a^\star b)) = tr(a^\star b) = tr(ba^\star) \quad \forall a, b \in H_{B_1 \sqcup B_2}, \tag{3.6}$$

where the first trace acts on $a^\star b \in \underline{Y}^d_{B_2 \sqcup B_2}$ and the second trace acts on $ba^\star \in \underline{Y}^d_{B_1 \sqcup B_1}$.

The adjoint $(\hat{a}_R)^\dagger : H_{LR} \to H_{LL}$ of $(\hat{a}_R)$ is associated with $a^\star$ in the sense that it satisfies

$$(\hat{a}_R)^\dagger |\phi\rangle = |\phi a^\star\rangle \quad \forall |\phi\rangle \in H_{LR}. \tag{3.7}$$

The result (3.7) can be verified by writing

$$\langle \phi|\psi a\rangle = tr(\phi^\star \psi a) = tr(a\phi^\star \psi) = tr\left((\phi a^\star)^\star \psi\right) = \langle \phi a^\star|\psi\rangle. \tag{3.8}$$

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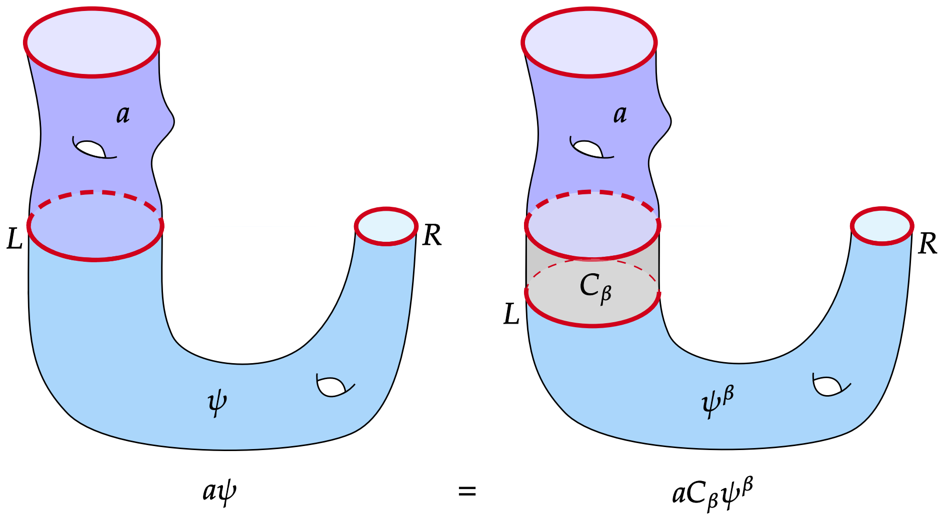

Figure 4. For rimmed surfaces $\psi \in Y^d_{LR}$ and $a \in Y^d_{LL}$, we define the action of $\hat{a}_L$ on $|\psi\rangle$ as $|a\psi\rangle$. As shown in the right panel, we can also separate out a cylinder C_β from the left rim of ψ and rewrite the surface $a\psi$ as $aC_\beta\psi^\beta$, where $\psi^\beta \in Y^d_{LR}$.

As usual, we can use the trace inequality (2.1) to show that $\hat{a}_R$ in fact defines a bounded operator that maps the Hilbert space $\mathcal{H}_{LL}$ to the Hilbert space $\mathcal{H}_{LR}$. In this case we consider again $|\phi\rangle \in H_{LL}$. After setting $b = \phi$, the right analogue of (2.2) yields

$$\langle\phi|\hat{a}_R^\dagger \hat{a}_R|\phi\rangle \leq \langle\phi|\phi\rangle\langle a|a\rangle. \tag{3.9}$$

Thus $\hat{a}_R$ annihilates null states in H_{LL} and defines a bounded operator on the dense subspace $\mathcal{D}_{LL} = H_{LL}/\mathcal{N}_{LL} \subset \mathcal{H}_{LL}$. A unique bounded extension to $\mathcal{H}_{LL}$ then follows.

As a result, the operation $\Psi_R : |a\rangle \mapsto \hat{a}_R$ defines a linear map from $\mathcal{D}_{LR} \subset \mathcal{H}_{LR}$ to the space $\mathcal{B}(\mathcal{H}_{LL}, \mathcal{H}_{LR})$ of bounded operators from $\mathcal{H}_{LL}$ to $\mathcal{H}_{LR}$. We will extend this map to the entire Hilbert space in section 4.2. In the diagonal context $B_L = B_R$, the map $\Psi_R : H_{LL} \to \hat{A}^{LL}_R$ coincides with the representation of the right surface algebra under the natural identification of H_{LL} with $A^{B_L}_R$.

Returning to claim 3.1, we need to show that any element a of the surface algebra $A^{B_L}_L$ that is represented by the zero operator on the diagonal Hilbert space $\mathcal{H}_{LL}$ is also represented by zero on any $\mathcal{H}_{LR}$. And since our operators are bounded, it in fact suffices to show that the representation $\hat{a}_L \in \hat{A}^{LR}_L$ of a on $\mathcal{H}_{LR}$ annihilates every state $|\psi\rangle$ in the dense subspace $\mathcal{D}_{LR}$.

Proof. Let us consider a rimmed surface $\psi \in Y^d_{B_L \sqcup B_R}$ and the corresponding state $|\psi\rangle \in \mathcal{D}_{LR}$. The rim at the left boundary requires there to be a neighborhood of the left boundary whose closure coincides with some cylinder C_β for some $\beta > 0$. As shown in figure 4 we may thus write the surface $a\psi$ as $aC_\beta\psi^\beta$ for some $\psi^\beta \in Y^d_{B_L \sqcup B_R}$. This observation yields the relations

$$\hat{a}_L\,|\psi\rangle = \hat{a}_L\,|C_\beta\psi^\beta\rangle = |aC_\beta\psi^\beta\rangle = \hat{\psi}^\beta_R\,|aC_\beta\rangle = \hat{\psi}^\beta_R \hat{a}^{LL}_L\,|C_\beta\rangle \quad \forall\,|\psi\rangle \in \mathcal{D}_{LR}, \tag{3.10}$$

where $\hat{a}^{LL}_L$ is the representation of a on the diagonal sector $\mathcal{H}_{LL}$. Thus $\hat{a}^{LL}_L = 0$ requires $\hat{a}_L\,|\psi\rangle = 0$. Since $\mathcal{D}_{LR}$ is the linear span of $|\psi\rangle$ with $\psi \in Y^d_{B_L \sqcup B_R}$, it follows that $\hat{a}_L$ annihilates $\mathcal{D}_{LR}$. This establishes $\mathfrak{N}^{LL}_L \subseteq \mathfrak{N}^{LR}_L$ as claimed above. $\square$

As a result, every left representation $\hat{A}_L^{LR}$ of the surface algebra $A_L^{B_L}$ is also a representation of the diagonal representation $\hat{A}_L^{LL}$. This is equivalent to the statement that there is a surjective $*$-homomorphism $\hat{\Phi}_{LR}$ from $\hat{A}_L^{LL}$ to any $\hat{A}_L^{LR}$. It follows that $\hat{A}_L^{LR}$ may be identified with the quotient of $\hat{A}_L^{LL}$ by an appropriate kernel, and thus that the diagonal representation contains all information about all representations $\hat{A}_L^{LR}$. We will show below that analogous statements continue to hold for the von Neumann algebra completions.

4 Completions and von Neumann algebras

We saw above that, for a given surface algebra $A_L^{B_L}$, the diagonal representation $\hat{A}_L^{LL}$ acts as universal 'covering algebra' for any representation $\hat{A}_L^{LR}$ in the sense that there is a $*$-homomorphsim $\hat{\Phi}_{LR}$ from $\hat{A}_L^{LL}$ to $\hat{A}_L^{LR}$. Furthermore, as reviewed in section 2, completing the diagonal representation $\hat{A}_L^{LL}$ yields an associated von Neumann algebra $\mathcal{A}_L^{B_L}$.

We will now generalize this construction to define an off-diagonal von Neumann algebra $\mathcal{A}_L^{LR} = \mathcal{A}_L^{B_L \sqcup B_R}$ by completing $\mathcal{A}_L^{LR}$ with respect to the weak operator topology on $\mathcal{B}(\mathcal{H}_{B_L \sqcup B_R})$. It is thus natural to ask if the results of section 3 extend to such completions. To make the notation more uniform between the diagonal and off-diagonal contexts, we will henceforth use the symbol $\mathcal{A}_L^{LL} = \mathcal{A}_L^{B_L \sqcup B_L}$ for the von Neumann algebra $\mathcal{A}_L^{B_L}$ defined using the weak operator topology on the diagonal Hilbert space $\mathcal{H}_{B_L \sqcup B_L}$.

The goal of this section is to show that this is indeed the case. The subtlety, however, is that the weak operator topology is defined by the Hilbert space on which the operators act. Thus, a priori, two completions might be very different even if we begin with isomorphic representations. It will turn out, however, that our axioms in fact require there to be a simple relation between $\mathcal{A}_L^{LR}$ and $\mathcal{A}_L^{LL}$.

As a result, we will be able to extend the surjective *-homomorphism $\hat{\Phi}_{LR} : \hat{A}_L^{LL} \to \hat{A}_L^{LR}$ to a surjective *-homomorphism $\Phi_{LR} : \mathcal{A}_L^{LL} \to \mathcal{A}_L^{LR}$ that is continuous with respect to the weak operator topology. Note that the extended Φ_{LR} is written without a hat decoration $\hat{}$. Extending $\hat{\Phi}_{LR}$ means in particular that, given any $a \in \mathcal{A}_L^{LL}$, we must construct an appropriate bounded operator $\Phi_{LR}(a)$ on the Hilbert space $\mathcal{H}_{LR}$. This first step will be accomplished in section 4.1, which also verifies that the extension defines a $*$-homomorphism. After an aside to introduce some useful technology in section 4.2, the desired continuity will then be established in section 4.3.

4.1 Extending our homomorphism to the von Neumann algebra $\mathcal{A}_L^{LL}$AL**LL

For each a in the von Neumann algebra $\mathcal{A}_L^{LL}$, we will first define the desired $\Phi_{LR}(a)$ as an operator on the dense domain $\mathcal{D}_{LR}$. Any state $|\psi\rangle \in \mathcal{D}_{LR}$ is a finite linear combination

$$\sum_{i=1}^{N} c_i |\psi_i\rangle \tag{4.1}$$

for states $|\psi_i\rangle$ defined by rimmed surfaces ψ_i with given right and left boundaries. Since each such surface will have a neighborhood of its left boundary that coincides (up to closure) with some cylinder C_ϵ for $\epsilon > 0$, by choosing $\epsilon > 0$ sufficiently small we can write $\psi_i = C_\epsilon \psi_i^\epsilon$ for all i; see again figure 4. Defining the state $|\psi^\epsilon\rangle = \sum_{i=1}^N c_i |\psi_i^\epsilon\rangle$, we may use the associated operator $\hat{\psi}_R^\epsilon := \Psi_R(|\psi^\epsilon\rangle)$ from $\mathcal{H}_{LL}$ to $\mathcal{H}_{LR}$ given by (3.2). We thus make the following definition for any $a \in \mathcal{A}_L^{LL}$:

$$\forall\, |\psi\rangle \in \mathcal{D}_{LR} \quad \text{we define} \quad \Phi_{LR}(a)\,|\psi\rangle := \hat{\psi}_R^\epsilon a|C_\epsilon\rangle \quad \text{for small enough } \epsilon. \tag{4.2}$$

It is manifest that $\Phi_{LR}(a) = \hat{\Phi}_{LR}(a)$ for $a \in \hat{A}_L^{LR}$, since then $\hat{\Phi}_{LR}(a)|\psi\rangle = |a\psi\rangle$ and (4.2) reduces to (3.2). Furthermore, for all a in the von Neumann algebra $\mathcal{A}_L^{LL}$, we will show below that the definition (4.2) is independent of ϵ for small enough ϵ. It is then also manifest that $\Phi_{LR}(ab) = \Phi_{LR}(a)\Phi_{LR}(b)$ and $\Phi_{LR}(\alpha a + \beta b) = \alpha\Phi_{LR}(a) + \beta\Phi_{LR}(b)$ for $\alpha, \beta \in \mathbb{C}$, so that Φ_{LR} is a homomorphism.

We now establish several claims regarding the definition (4.2).

Claim 4.1. *The definition (4.2) is independent of ϵ so long as C_ϵ is a cylinder common to all rimmed surfaces ψ_i appearing in (4.1).*

Proof. For $a \in \hat{A}_L^{LL}$, this claim follows from the observation that $\Phi_{LR}(a) = \hat{\Phi}_{LR}(a)$. Furthermore, a general a in the von Neumann algebra $\mathcal{A}_L^{LL}$ can be written as the limit of a net of operators $\{\hat{a}_\alpha\} \subset \hat{A}_L^{LL}$ that converges to a in the *strong* operator topology.[9] As a result, since Hilbert spaces are metrizable,[10] given any two common rims C_ϵ and C_δ there must be a *sequence* $\{\hat{a}_n\} \subset \hat{A}_L^{LL}$ such that the sequence $\{\hat{a}_n|C_\epsilon\rangle\}$ converges in norm to $a|C_\epsilon\rangle$ while the sequence $\{\hat{a}_n|C_\delta\rangle\}$ converges in norm to $a|C_\delta\rangle$. Since $\hat{\psi}_R^\epsilon$ and $\hat{\psi}_R^\delta$ are bounded, we similarly find the limits

$$\begin{aligned} &\hat{\psi}_R^\epsilon \hat{a}_n|C_\epsilon\rangle \to \hat{\psi}_R^\epsilon a|C_\epsilon\rangle \\ \text{and} \quad &\hat{\psi}_R^\delta \hat{a}_n|C_\delta\rangle \to \hat{\psi}_R^\delta a|C_\delta\rangle. \end{aligned} \tag{4.3}$$

But since $\hat{a}_n \in \hat{A}_L^{LL}$ we have $\hat{\psi}_R^\epsilon \hat{a}_n|C_\epsilon\rangle = \hat{\Phi}_{LR}(\hat{a}_n)|\psi\rangle = \hat{\psi}_R^\delta \hat{a}_n|C_\delta\rangle$ as noted in the opening sentence of this proof. Thus (4.3) implies $\hat{\psi}_R^\epsilon a|C_\epsilon\rangle = \hat{\psi}_R^\delta a|C_\delta\rangle$ as claimed.

□

Claim 4.2. *For any $a \in \mathcal{A}_L^{LL}$, our $\Phi_{LR}(a)$ is a bounded operator on $\mathcal{D}_{LR}$ (with operator norm no larger than the norm $||a||$ of a). It thus admits a unique bounded extension to the entire Hilbert space $\mathcal{H}_{LR}$.*

Proof. We begin by computing the norm of (4.2). Since $|\psi\rangle \in \mathcal{D}_{LR}$, we have

$$|\Phi_{LR}(a)\,|\psi\rangle\,|^2 = |\hat{\psi}_R^\epsilon a\,|C_\epsilon\rangle\,|^2 = \langle C_\epsilon|\, a^\dagger \hat{\psi}_R^{\epsilon,\dagger} \hat{\psi}_R^\epsilon a\,|C_\epsilon\rangle\,. \tag{4.4}$$

Recall now that $\hat{\psi}_R^\epsilon$ is a bounded operator from $\mathcal{H}_{LL}$ to $\mathcal{H}_{LR}$. As a result, $\hat{\psi}_R^{\epsilon,\dagger}\hat{\psi}_R^\epsilon$ is a bounded operator on $\mathcal{H}_{LL}$. Furthermore, since $|\psi\rangle$ is of the form (4.1), the operator

[9]For convex sets of bounded operators, the closure taken in the weak operator topology agrees with that taken in the strong operator topology; see e.g. theorem 5.1.12 in [22]. Here we regard the von Neumann $\mathcal{A}_L^{LL}$ as the closure of $\hat{A}_L^{LL}$ in the strong operator topology, which then requires that we include the limits of all strongly-convergent nets.

[10]Here we used an extended sequential property of metrizable spaces. Given two convergent nets in a metrizable space $\{|\psi_\alpha\rangle\} \to |\psi\rangle$ and $\{|\psi'_\alpha\rangle\} \to |\psi'\rangle$ with a common index set $\mathcal{J}$, there exist subsequences $\{|\psi_{\alpha_n}\rangle\} \to |\psi\rangle$ and $\{|\psi'_{\alpha_n}\rangle\} \to |\psi'\rangle$ with common indices $\alpha_n \in \mathcal{J}$ that converge to the same limit point. We presume the argument for this result' is standard, but reader's seeking an explict reference can consult the discussion around (3.42) in [1].

$\hat{\psi}_R^{\epsilon,\dagger}\hat{\psi}_R^\epsilon$ is a finite linear combination of the operators $\hat{\psi}_{i,R}^{\epsilon,\dagger}\hat{\psi}_{j,R}^\epsilon$, each of which lies in the right representation $\hat{A}_R^{LL}$ of the surface algebra $A_R^{B_L}$. Thus $\hat{\psi}_R^{\epsilon,\dagger}\hat{\psi}_R^\epsilon \in \hat{A}_R^{LL} \subset \mathcal{A}_R^{LL}$, so that its (unique) positive square root $|\hat{\psi}_R^\epsilon|$ also lies in the von Neumann algebra $\mathcal{A}_R^{LL}$. We may then use the fact that a and $a^\dagger$ are in the left von Neumann algebra $\mathcal{A}_L^{LL}$ to conclude that they commute with any operator in the right von Neumann algebra $\mathcal{A}_R^{LL}$, and in particular with $|\hat{\psi}_R^\epsilon|$. This observation allows us to write

$$\begin{aligned}\langle C_\epsilon| a^\dagger\hat{\psi}_R^{\epsilon,\dagger}\hat{\psi}_R^\epsilon a\,|C_\epsilon\rangle &= \langle C_\epsilon| a^\dagger|\hat{\psi}_R^\epsilon|^2 a\,|C_\epsilon\rangle = \langle C_\epsilon|\,|\hat{\psi}_R^\epsilon|\,a^\dagger a\,|\hat{\psi}_R^\epsilon|\,|C_\epsilon\rangle \\ &\leq ||a||^2\,\langle C_\epsilon|\,|\hat{\psi}_R^\epsilon|^2\,|C_\epsilon\rangle = ||a||^2\,\langle C_\epsilon|\,\hat{\psi}_R^{\epsilon,\dagger}\hat{\psi}_R^\epsilon\,|C_\epsilon\rangle = ||a||^2\,\langle\psi|\,\psi\rangle, \end{aligned} \tag{4.5}$$

where $||a||$ is the operator norm of a. Thus $\Phi_{LR}(a)$ is bounded as claimed and, in particular, its norm can be no larger than the norm of a. □

For each $a \in \mathcal{A}_L^{LL}$, we may now extend the domain of $\Phi_{LR}(a)$ to all of $\mathcal{H}_{LR}$ by continuity to define Φ_{LR} as a linear map from $\mathcal{A}_L^{LL}$ to the space $\mathcal{B}(\mathcal{H}_{LR})$ of bounded linear operators on $\mathcal{H}_{LR}$. Since bounded operators are determined by their matrix elements on the dense subspace $\mathcal{D}_{LR}$, this extension is again a homomorphism. One can also quickly use (4.2) (and the fact that $a \in \mathcal{A}_L^{LL}$ commutes with $\hat{\psi}_{1,R}^{\epsilon,\dagger}\hat{\psi}_{2,R}^\epsilon \in \mathcal{A}_R^{LL}$ for any $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{D}_{LR}$) to show that Φ_{LR} is a $*$-homomorphism, meaning that $\Phi_{LR}(a^\dagger) = (\Phi_{LR}(a))^\dagger$. We will also mention that we will find in section 4.3 that $\Phi_{LR}(a)$ is continuous with respect to the weak operator topology. Since we know that Φ_{LR} maps $\hat{A}_L^{LL} \subset \mathcal{A}_L^{LL}$ onto $\hat{A}_L^{LR}$, and since the $\mathcal{A}_L^{LR}$ is the closure of $\hat{A}_L^{LR}$ in the weak operator topology, it will then follow that $\Phi_{LR} : \mathcal{A}_L^{LL} \to A_L^{LR}$ is a surjective $*$-homomorphism. This is precisely the analogue of our result from section 3 at the level of the von Neumann algebras A_L^{LL}, A_L^{LR}.

4.2 Vectors in an off-diagonal sector as intertwining operators

Before proceeding to the proof of continuity in section 4.3, it will be useful to first derive some additional properties of the map Ψ_R. Recall that section 3 defined Ψ_R as a linear map from $\mathcal{D}_{LR}$ to $\mathcal{B}(\mathcal{H}_{LL}, \mathcal{H}_{LR})$. The results of section 4.1 will turn out to imply this Ψ_R to be continuous in the strong operator topology. In particular, we now establish the following claim:

Claim 4.3. *The map $\Psi_R : \mathcal{D}_{LR} \to \mathcal{B}(\mathcal{H}_{LL}, \mathcal{H}_{LR})$ is continuous w.r.t. the norm topology on $\mathcal{H}_{LR}$ and the strong operator topology on $\mathcal{B}(\mathcal{H}_{LL}, \mathcal{H}_{LR})$. It then follows that Ψ_R admits a unique continuous extension to the entire Hilbert space.*

Proof. Since $\mathcal{D}_{LR}$ is a metric space, the map Ψ_R is continuous if and only if it acts continuously on preserves all convergent sequences. Consider then an arbitrary sequence of vectors $\{|\psi_n\rangle\} \subset \mathcal{D}_{LR}$ that converges to some $|\psi\rangle \in \mathcal{H}_{LR}$. The first step is to show that the corresponding sequence of operators $\{\widehat{\psi_n}_R = \Psi_R(|\psi_n\rangle)\}$ is uniformly bounded. In particular, replacing $|a\rangle$ by $|\psi_n\rangle$ in (3.9) shows that $||\widehat{\psi_n}_R||$ is bounded by $\sqrt{\langle\psi_n|\psi_n\rangle}$. In addition, since the sequence $\{|\psi_n\rangle\}$ converges in $\mathcal{H}_{LR}$, we know the sequence of norms $\{\langle\psi_n|\psi_n\rangle\}$ is convergent and must be bounded. The sequence of operator norms $\{||\hat{\psi}_n||\}$ is thus bounded as well, so that the sequence $\{\hat{\psi}_n\}$ is uniformly bounded.

Next, for any $|x\rangle \in \mathcal{D}_{LL}$ we will show that the sequence of vectors $\{\widehat{\psi_n}_R\,|x\rangle\}$ converges in $\mathcal{H}_{LR}$. Let us first note that (3.2) implies

$$\widehat{\psi_n}_R\,|x\rangle = |x\psi_n\rangle = \hat{\Phi}_{LR}(\hat{x}_L)\,|\psi_n\rangle\,, \tag{4.6}$$

where $x \in H_{LL}$ is any representative of the vector $|x\rangle$, and where $\hat{x}_L := \Psi_L(|x\rangle) \in \hat{A}_L^{LL}$ is the bounded operator associated with the representation of $x \in A_L^{B_L}$ on the diagonal sector $\mathcal{H}_{LL}$. Since the sequence of vectors $|\psi_n\rangle$ converges to $|\psi\rangle$, and since $\hat{\Phi}_{LR}(\hat{x}_L)$ is bounded, (4.6) shows that the sequence of vectors $\widehat{\psi_n}_R\,|x\rangle$ also converges to $\hat{\Phi}_{LR}(\hat{x}_L)|\psi\rangle$.

Via a standard calculation, it then follows that, given any Cauchy sequence $\{|x_n\rangle\} \subset \mathcal{D}_{LR}$, the sequence $\widehat{\psi_n}_R|x_n\rangle$ is also Cauchy, and that it thus converges in $\mathcal{H}_{LR}$ (see e.g. theorem 6 in section 15.2 of [23]). This means that the sequence of operators $\widehat{\psi_n}_R$ converges in the strong operator topology to an operator $\hat{\psi}_R$. In particular, for any $|\psi\rangle \in \mathcal{H}_{LR}$, the limit yields a bounded operator $\hat{\psi}_R$ satisfying

$$\forall\,|x\rangle \in \mathcal{D}_{LL}, \quad \hat{\psi}_R\,|x\rangle = \hat{\Phi}_{LR}(\hat{x}_L)\,|\psi\rangle\,. \tag{4.7}$$

We may thus define $\Psi_R(|\psi\rangle) := \hat{\psi}_R$. $\square$

We will now show the extended Ψ_R to satisfy the following extension of (3.2).

Claim 4.4. *$\forall\,|\psi\rangle \in \mathcal{H}_{LR}$, the operator $\hat{\psi}_R = \Psi_R(|\psi\rangle) : \mathcal{H}_{LL} \to \mathcal{H}_{LR}$ is an intertwining operator between the von Neumann algebras $\mathcal{A}_L^{LL}$ and $\mathcal{A}_L^{LR}$. Specifically, we have*

$$\Phi_{LR}(a)\hat{\psi}_R = \hat{\psi}_R a \quad \forall a \in \mathcal{A}_L^{LL}. \tag{4.8}$$

Proof. Any $|\psi\rangle \in \mathcal{H}_{LR}$ is the Hilbert space limit of a sequence $\{|\psi_n\rangle\} \subset \mathcal{D}_{LR}$. Note that for any $\epsilon > 0$ we may construct the states $|\phi_n\rangle := |C_\epsilon\psi_n\rangle \in \mathcal{D}_{LR}$, and that $\widehat{\phi_n}^{\epsilon}_R = \widehat{\psi_n}_R$. For any $a \in \mathcal{A}_L^{LL}$ we may thus write

$$\Phi_{LR}(a)\widehat{\psi_n}_R|C_\epsilon\rangle = \Phi_{LR}(a)|C_\epsilon\psi_n\rangle = \widehat{\psi_n}_R a|C_\epsilon\rangle, \tag{4.9}$$

where the first equality is the definition of $\widehat{\psi_n}_R$ and the second is a direct application of (4.2). Recalling that $\widehat{\psi_n}_R = \Psi_R(|\psi_n\rangle)$, and that $\hat{\psi}_R = \Psi_R(|\psi\rangle)$, boundedness of $\Phi_{LR}(a)$ and the continuity property of Claim 4.3 allow us to take the limit $n \to \infty$ to find

$$\Phi_{LR}(a)\hat{\psi}_R|C_\epsilon\rangle = \hat{\psi}_R a|C_\epsilon\rangle. \tag{4.10}$$

To show that the above intertwining relation in fact holds when acting on arbitrary states in $\mathcal{H}_{LL}$, we simply consider any $\kappa \in \underline{Y}^d_{LL}$ and compute

$$\begin{aligned}
\hat{\psi}_R a|\kappa C_\epsilon\rangle = \hat{\psi}_R a\hat{\kappa}_L|C_\epsilon\rangle &= \Phi_{LR}(a\hat{\kappa}_L)\hat{\psi}_R|C_\epsilon\rangle \\
&= \Phi_{LR}(a)\Phi_{LR}(\hat{\kappa}_L)\hat{\psi}_R|C_\epsilon\rangle \\
&= \Phi_{LR}(a)\hat{\psi}_R\hat{\kappa}_L|C_\epsilon\rangle = \Phi_{LR}(a)\hat{\psi}_R|\kappa C_\epsilon\rangle.
\end{aligned} \tag{4.11}$$

Here the last equality on the first line uses (4.10) with the a of (4.10) replaced by $a\hat{\kappa}_L$. We then pass to the second line using the fact that Φ_{LR} is a homomorphism as shown at the end of section 4.1. The third line follows by applying (4.10) with the a of (4.10) replaced by $\hat{\kappa}_L$. Since the operators $\hat{\phi}_R, a, \Phi_{LR}(a)$ are bounded, and since every state in the dense subspace $\mathcal{D}_{LR} \subset \mathcal{H}_{LR}$ is of the form $|\kappa C_\epsilon\rangle$ for some κ, ϵ, the general result (4.8) follows by taking corresponding limits of (4.11). $\square$

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Before proceeding to the next section, we also wish to establish a further useful property of the extended map Ψ_R. This property is a partial extension of (3.6) associated with the fact that $\hat{\psi}_R^\dagger \hat{\psi}_R$ lies in $\mathcal{A}_R^{LL}$. In particular, we will find

$$tr(\hat{\psi}_R^\dagger \hat{\psi}_R) = \langle \psi | \psi \rangle \quad \forall |\psi\rangle \in \mathcal{H}_{LR}. \tag{4.12}$$

To see this, we will use the following two results:

Claim 4.5. *For cylinder operators $\widehat{C_\beta}_L \in \hat{A}_L^{LL}$, the limit $\lim_{\beta \downarrow 0} \Phi_{LR}(\widehat{C_\beta}_L)$ converges in the strong operator topology to the identity $\mathbb{1} \in B(\mathcal{H}_{LR})$.*

Proof. Since $\widehat{C_\beta}_L$ is the representation of the cylinder C_β on $\mathcal{H}_{LL}$, we know from section 3 that $\Phi_{LR}(\widehat{C_\beta}_L)$ is just the corresponding representation of C_β on the off-diagonal Hilbert space $\mathcal{H}_{LR}$. Furthermore, from claim (3.9) we have $||\Phi_{LR}(\widehat{C_\beta}_L)|| \leq ||\widehat{C_\beta}_L||$. And since it was shown in appendix A of [1] that $||\hat{C}_\beta|| \to 1$ as $\beta \downarrow 0$, the operators $\Phi_{LR}(\widehat{C_\beta}_L)$ are uniformly bounded at small β. We also observe that, since states in $\mathcal{H}_{LR}$ are determined by their inner products with states in $\mathcal{D}_{LR}$, for any $|x\rangle \in \mathcal{D}_{LR}$ the continuity axiom of [1] requires the vectors $\Phi_{LR}(\widehat{C_\beta}_L)\,|x\rangle = |C_\beta x\rangle$ to converge to $|x\rangle$ in the limit $\beta \downarrow 0$. Together, as in the proof of claim 4.3, these properties imply that $\Phi_{LR}(\widehat{C_\beta}_L)$ converges in the strong operator topology to $\mathbb{1}$ as $\beta \downarrow 0$; see e.g. theorem 15.2.6 of [23]. $\square$

Claim 4.6. *For any $|\psi\rangle \in \mathcal{H}_{B_1 \sqcup B_2}$, we have*

$$|\psi\rangle = \lim_{\beta \downarrow 0} \hat{\psi}_R |C_\beta\rangle \tag{4.13}$$

for C_β the cylinder of length β with boundary $B_1 \sqcup B_1$.

Proof. Since any such state $|\psi\rangle$ is fully determined by its inner products with states in $\mathcal{D}_{B_1 \sqcup B_2}$, is clear from the continuity axiom that (4.13) holds for $|\psi\rangle \in \mathcal{D}_{B_1 \sqcup B_2}$. For more general $|\psi\rangle$, we may consider a sequence of states $|\psi_n\rangle \in \mathcal{D}_{B_1 \sqcup B_2}$ that converge to $|\psi\rangle$. Continuity of Ψ_R and boundedness of each operator $\widehat{(C_\beta)}_L$ then gives

$$\begin{aligned}
\lim_{\beta \downarrow 0} \hat{\psi}_R |C_\beta\rangle &= \lim_{\beta \downarrow 0} \lim_{n \to \infty} \widehat{(\psi_n)}_R |C_\beta\rangle \\
&= \lim_{\beta \downarrow 0} \lim_{n \to \infty} \hat{\Phi}_{LR}\left(\widehat{(C_\beta)}_L\right) |\psi_n\rangle \\
&= \lim_{\beta \downarrow 0} \hat{\Phi}_{LR}\left(\widehat{(C_\beta)}_L\right) |\psi\rangle = |\psi\rangle,
\end{aligned} \tag{4.14}$$

where we pass from the first to the second line using (4.7), and where the final step uses claim 4.5. $\square$

Equation (4.12) then follows from (4.13) by applying definition (2.3) of the trace on positive elements of $\mathcal{A}_R^{LL}$.

4.3 Continuity of Φ_{LR} in the weak operator topology

The goal of this section is to show that our $*$-homomorphism Φ_{LR} is continuous with respect to the weak operator topology (imposed on both $\mathcal{B}(\mathcal{H}_{LL})$ and $\mathcal{B}(\mathcal{H}_{LR})$). In order to do so, we first establish the following intermediate claim.

Claim 4.7. *For any* $|\gamma\rangle \in \mathcal{H}_{LR}$ *and any* $a \in \mathcal{A}_L^{LL}$*, we have*

$$\langle\gamma|\Phi_{LR}(a)|\gamma\rangle = \langle\gamma_{LL}|a|\gamma_{LL}\rangle\,, \tag{4.15}$$

where we have defined $|\gamma_{LL}\rangle := \lim_{\beta\downarrow 0}|\hat{\gamma}_R||C_\beta\rangle \in \mathcal{H}_{LL}$ *and where* $|\hat{\gamma}_R|$ *is the positive square root of* $\hat{\gamma}_R^\dagger\hat{\gamma}_R$*.*

Proof. Let us first rewrite $\langle\gamma|\Phi_{LR}(a)|\gamma\rangle$ by inserting the identity operator in the form $\lim_{\beta\downarrow 0}\Phi_{LR}(\widehat{C_\beta}_L)$ established in claim 4.5. As noted in the proof of that claim, if $\widehat{C_\beta}_L$ is a cylinder operator on $\mathcal{H}_{LL}$, then $\hat{\Phi}_{LR}(\widehat{C_\beta}_L)$ is the corresponding cylinder operator on $\mathcal{H}_{LR}$. We thus have

$$\langle\gamma|\Phi_{LR}(a)|\gamma\rangle = \lim_{\beta\downarrow 0}\langle\gamma|\Phi_{LR}(a)\Phi_{LR}(\widehat{C_\beta}_L)|\gamma\rangle = \lim_{\beta\downarrow 0}\langle\gamma|\Phi(a)\hat{\gamma}_R|C_\beta\rangle = \lim_{\beta\downarrow 0}\langle\gamma|\hat{\gamma}_R a|C_\beta\rangle \tag{4.16}$$

where in the second equality we have used equation (4.7) with $|\psi\rangle = |\gamma\rangle$ and $|x\rangle = |C_\beta\rangle$, and where the third equality used (4.8) (again with $|\psi\rangle = |\gamma\rangle$).

We now insert another identity operator and perform similar manipulations to write

$$\hat{\gamma}_R^\dagger|\gamma\rangle = \lim_{\epsilon\downarrow 0}\hat{\gamma}_R^\dagger\Phi_{LR}(\widehat{C_\epsilon}_L)|\gamma\rangle = \lim_{\epsilon\downarrow 0}\hat{\gamma}_R^\dagger\hat{\gamma}_R|C_\epsilon\rangle = \lim_{\epsilon\downarrow 0}|\hat{\gamma}_R|^2|C_\epsilon\rangle\,. \tag{4.17}$$

Since $tr(|\hat{\gamma}_R|^2) = tr(\hat{\gamma}_R^\dagger\hat{\gamma}_R) = \langle\gamma|\gamma\rangle$ is finite, Corollary 1 of [1] then shows that the limit of $|\hat{\gamma}_R||C_\epsilon\rangle$ converges as $\epsilon\downarrow 0$ to a state $|\gamma_{LL}\rangle \in \mathcal{H}_{LL}$. Thus (4.17) yields

$$\hat{\gamma}_R^\dagger|\gamma\rangle = |\hat{\gamma}_R|\,|\gamma_{LL}\rangle \tag{4.18}$$

and we may evaluate (4.16) by writing

$$\lim_{\beta\downarrow 0}\langle\gamma|\hat{\gamma}_R a|C_\beta\rangle = \lim_{\beta\downarrow 0}\langle\gamma_{LL}|\,|\hat{\gamma}_R|a\,|C_\beta\rangle = \lim_{\beta\downarrow 0}\langle\gamma_{LL}|\,a|\hat{\gamma}_R|\,|C_\beta\rangle = \langle\gamma_{LL}|a|\gamma_{LL}\rangle\,, \tag{4.19}$$

where the second step follows by noting that $|\hat{\gamma}_R|$ lies in the right von Neumann algebra $\mathcal{A}_R^{LL}$ and so necessarily commutes with $a \in \mathcal{A}_L^{LL}$. Combining (4.19) with (4.16) then completes the desired proof. □

We have now acquired all of the tools we need to prove continuity of the map Φ_{LR} in the weak operator topology. Recall that convergence in the weak operator topology of the net of operators $\{\mathcal{O}_\alpha\}$ on a Hilbert space $\mathcal{H}$ is equivalent to convergence of the associated nets of matrix elements $\{\langle\phi_1|\mathcal{O}_\alpha|\phi_2\rangle\}$ for all $|\phi_1\rangle, |\phi_2\rangle \in \mathcal{H}$. In particular, let us consider any state $|\gamma\rangle \in \mathcal{H}_{LR}$ and the associated state $|\gamma_{LL}\rangle \in \mathcal{H}_{LL}$ defined as in Claim 4.7. If a net of operators $\{a_\alpha\} \subset \mathcal{A}_L^{LL}$ converges in the weak operator topology to $b \in \mathcal{A}_L^{LL}$, then the net of expectation values $\{\langle\gamma_{LL}|a_\alpha|\gamma_{LL}\rangle\}$ clearly converges to $\langle\gamma_{LL}|b|\gamma_{LL}\rangle$. Claim 4.7 then implies that the net of expectation values $\{\langle\gamma|\Phi_{LR}(a_\alpha)|\gamma\rangle\}$ also converges to $\{\langle\gamma|\Phi_{LR}(b)|\gamma\rangle\}$.

Using the standard construction of general matrix elements from expectation values, this is enough to establish that the net of matrix elements $\{\langle\phi_1|\Phi_{LR}(a_\alpha)|\phi_2\rangle\}$ also converges to $\langle\phi_1|\Phi_{LR}(b)|\phi_2\rangle$. In particular, given any two states $|\phi_1\rangle, |\phi_2\rangle$, we can define $|\alpha\rangle = |\phi_1\rangle + |\phi_2\rangle$ and $|\beta\rangle = |\phi_2\rangle + i\,|\phi_1\rangle$ to write (for any operator a)

$$\langle\phi_1|a|\phi_2\rangle = (\langle\alpha|a|\alpha\rangle + i\,\langle\beta|a|\beta\rangle)/2 - \langle\phi_1|a|\phi_1\rangle - \langle\phi_2|a|\phi_2\rangle\,. \tag{4.20}$$

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Convergence of expectation values of a net of operators thus implies convergence of all matrix elements. In particular, we see from the above that the net $\{\Phi_{LR}(a_\alpha)\}$ converges to $\Phi_{LR}(b)$ in the weak operator topology.

We thus see that Φ_{LR} is continuous in the weak operator topology. This fact can be used to provide an alternate proof that Φ_{LR} is a $*$-homomorphism directly from the corresponding properties of the map $\hat{\Phi}_{LR} : \hat{A}_L^{LL} \to \hat{A}_L^{LR}$. And since the von Neumann algebras $\mathcal{A}_L^{LL}, \mathcal{A}_L^{LR}$ are the weak-operator-topology closures of $\hat{A}_L^{LL}, \hat{A}_L^{LR}$, surjectivity of $\hat{\Phi}_{LR} : \hat{A}_L^{LL} \to \hat{A}_L^{LR}$ implies surjectivity of $\Phi_{LR} : \mathcal{A}_L^{LL} \to \mathcal{A}_L^{LR}$. As a result, the off-diagonal von Neumann algebra $\mathcal{A}_L^{LR}$ is isomorphic to the quotient $\mathcal{A}_L^{LL}/\ker(\Phi_{LR})$. This establishes analogues of the results of section 3 at the level of the corresponding von Neumann algebras.

Furthermore, since distinct sectors $\mathcal{H}_{\mathcal{B}}$ are orthogonal, the above observations achieve our primary goal. In particular, they show agreement between the von Neumann algebra $\mathcal{A}_L^{LL} = \mathcal{A}_L^{B_L \sqcup B_L}$ (defined from the surface algebra $A_L^{B_L}$ by the diagonal sector $\mathcal{H}_{LL} = \mathcal{H}_{B_L \sqcup B_L}$) and the von Neumann algebra defined by using the representation of same surface algebra $A_L^{B_L}$ on the larger Hilbert space

$$\mathbb{H}_{B_L} := \oplus_{B_R} \mathcal{H}_{B_L \sqcup B_R}. \tag{4.21}$$

Here the sum on the right is over all possible right boundaries B_R (including the empty set). This is the largest Hilbert space defined by the Euclidean path integral on which $A_L^{B_L}$ can naturally be said to act. In this sense the von Neumann algebra $\mathcal{A}_L^{LL} = \mathcal{A}_L^{B_L \sqcup B_L}$ defined by the diagonal sector $\mathcal{H}_{LL} = \mathcal{H}_{B_L \sqcup B_L}$ coincides with the von Neumann algebra acting on B_L defined by the full quantum gravity Hilbert space.

5 Off-diagonal central projections

In order to pave the way for a discussion of entropy in section 6, we devote this section to developing a better understanding of the off-diagonal central alebras and their relations to one another. We have already seen in section 4.3 that the off-diagonal von Neumann algebras $\mathcal{A}_L^{LR}$ are quotients $\mathcal{A}_L^{LL}/\ker(\Phi_{LR})$ of the digaonal von Neumann algebra $\mathcal{A}_L^{LL}$. Section 5.1 will make this more concrete by developing a better understanding of $\ker(\Phi_{LR})$. Section 5.2 then utilizes this understanding to show that our structure defines a universal central algebra, independent of any choice of boundaries, that can be said to contain all centers $\mathcal{Z}_{B_1 \sqcup B_2} \subset \mathcal{A}_L^{B_1 \sqcup B_2} \cap \mathcal{A}_R^{B_2 \sqcup B_2}$. This will in turn help to organize our discussion of entropy in section 6.

5.1 Off-diagonal algebras as central projection of diagonal algebras

The weak-operator-topology continuity of Φ_{LR} established in section 4.3 means that the inverse image of any weak-operator-topology-closed set is weak-operator-topology closed. This is in particular true of $\ker(\Phi_{LR}) = \Phi_{LR}^{-1}(0)$, since any single point is weak-operator-topology closed. Furthermore, as usual, the fact that Φ_{LR} is a homomorphsism implies that $\ker(\Phi_{LR})$ is a two-sided ideal. We may thus make use of theorem 6.8.8 of e.g. [22], which states that any weak-operator-topology-closed two-sided ideal in a von Neumann algebra $\mathcal{A}$ is of the form $\mathcal{P}\mathcal{A} = \mathcal{A}\mathcal{P}$ for some central projection $\mathcal{P} \in \mathcal{A}$. We denote the projection corresponding to

$\ker(\Phi_{LR}) \subset \mathcal{A}_L^{LL}$ by $P_{\ker(\Phi_{LR})}$, and we also define $P^{\perp}_{\ker(\Phi_{LR})} = \mathbb{1} - P_{\ker(\Phi_{LR})}$, so that we have

$$\begin{aligned} \ker(\Phi_{LR}) &= P_{\ker(\Phi_{LR})}\mathcal{A}_L^{LL} \quad \text{and} \\ \mathcal{A}_L^{LL}/\ker(\Phi) &\simeq P^{\perp}_{\ker(\Phi_{LR})}\mathcal{A}_L^{LL} = \mathcal{A}_L^{LL}P^{\perp}_{\ker(\Phi_{LR})} = \mathcal{A}_L^{LR}, \end{aligned} \tag{5.1}$$

where the final step used the fact that Φ_{LR} defines an isomorphism between $P^{\perp}_{\ker(\Phi_{LR})}\mathcal{A}_L^{LL}$ and $\mathcal{A}_L^{LR}$ in order to identify these algebras.

Now, the diagonal von Neumann algebra $\mathcal{A}_L^{LL}$ admits a central decomposition into a direct sum which, due to the trace inequality (2.1), is in fact a sum over a discrete set of type I factors indexed by $\mathcal{I}_{B_L}$ (see section 4.2 of [1]). Each factor $\mathcal{A}_{L,\mu}^{LL}$ (for $\mu \in \mathcal{I}_{B_L}$) is associated with a central projection P_μ that is minimal within the set of central projections (in the sense that there is no smaller non-trivial central projection) and which is orthogonal to p_ν when $\nu \neq \mu$. In particular, we have

$$\mathcal{A}_L^{LL} = \bigoplus_{\mu\in\mathcal{I}_{B_L}} \mathcal{A}_{L,\mu}^{LL} \quad \text{for} \quad \mathcal{A}_{L,\mu}^{LL} = P_\mu \mathcal{A}_L^{LL}. \tag{5.2}$$

Note that, since P_μ is a minimal central projection, we must have either $P^{\perp}_{\ker(\Phi_{LR})}P_\mu = 0$ or $P_{\ker(\Phi_{LR})}P_\mu = 0$ (else one of these would be a smaller central projection). The former case requires $\Phi_{LR}(P_\mu) = 0$, while in the latter case we have $P^{\perp}_{\ker(\Phi_{LR})}P_\mu = P_\mu \neq 0$ so that $\Phi_{LR}(P_\mu)$ is a non-trivial minimal central projection in $\mathcal{A}_L^{LR}$. We may thus write

$$P^{\perp}_{\ker(\Phi_{LR})} = \bigoplus_{\mu\in\mathcal{I}_{B_L}} \chi_{L,\mu}^{LR} P_\mu, \tag{5.3}$$

where $\chi_{L,\mu}^{LR} = 0$ when $\Phi_{LR}(P_\mu) = 0$ and $\chi_{L,\mu}^{LR} = 1$ when $\Phi_{LR}(P_\mu) \neq 0$. Furthermore, (5.3) immediately leads to a corresponding decomposition of the algebras (5.1).

As reviewed in section 2, the diagonal Hilbert space decomposes as a corresponding direct sum

$$\mathcal{H}_{B_L \sqcup B_L} = \bigoplus_{\mu\in\mathcal{I}_{B_L}} \mathcal{H}_\mu^{LL} \quad \text{with} \quad \mathcal{H}_\mu^{LL} = P_\mu \mathcal{H}_{LL}. \tag{5.4}$$

Furthermore, each $\mathcal{H}_\mu^{LL}$ admits a factorization

$$\mathcal{H}_\mu^{LL} = \mathcal{H}_{L,\mu}^{LL} \otimes \mathcal{H}_{R,\mu}^{LL}, \tag{5.5}$$

such that $\mathcal{A}_{L,\mu}^{LL}$ acts as $\mathcal{B}(\mathcal{H}_{L,\mu}^{LL}) \otimes \mathbb{1}_\mu^R$, where $\mathbb{1}_\mu^R$ is the identity on the right factor $\mathcal{H}_{R,\mu}^{LL}$. Since applying Φ_{LR} to (5.2) is equivalent to multiplying by $P^{\perp}_{\ker(\Phi_{LR})}$, the relation (5.3) yields a corresponding decomposition of the off-diagonal algebra

$$\mathcal{A}_L^{LR} = \bigoplus_{\mu\in\mathcal{I}_{B_L}} \chi_{L,\mu}^{LR}\Phi_{LR}(\mathcal{A}_{L,\mu}^{LL}) \quad \text{with} \quad \Phi_{LR}(\mathcal{A}_{L,\mu}^{LL}) = \Phi_{LR}(P_\mu)\mathcal{A}_L^{LR}, \tag{5.6}$$

and also for the off-diagonal Hilbert space

$$\mathcal{H}_{LR} = \bigoplus_{\mu\in\mathcal{I}_{B_L}} \chi_{L,\mu}^{LR}\Phi_{LR}(P_\mu)\mathcal{H}_{LR}. \tag{5.7}$$

In both (5.6) and (5.7) the factor of $\chi^{LR}_{L,\mu}$ is redundant (since e.g. $\chi^{LR}_{L,\mu}\Phi_{LR}(\mathcal{A}^{LL}_{L,\mu}) = \Phi_{LR}(\mathcal{A}^{LL}_{L,\mu}))$ but serves to emphasize which terms are non-zero.

We also emphasize that Φ_{LR} is an *isomorphism* when acting on any factor $\mathcal{A}^{LL}_{L,\mu}$ that it does not annihilate. Thus each $\mathcal{A}^{LR}_{L,\mu} := \Phi_{LR}(\mathcal{A}^{LL}_{L,\mu})$ is a type-I factor and the Hilbert space $\mathcal{H}^{LR}_{\mu} := \Phi_{LR}(P_\mu)\mathcal{H}_{LR}$ must again factorize as $\mathcal{H}^{LR}_{\mu} = \mathcal{H}^{LR}_{L,\mu} \otimes \mathcal{H}^{LR}_{R,\mu}$ for some $\mathcal{H}^{LR}_{L,\mu}$, $\mathcal{H}^{LR}_{R,\mu}$ such that $\mathcal{A}^{LR}_{L,\mu} := \Phi_{LR}(\mathcal{A}^{LL}_{L,\mu})$ acts as $\mathcal{B}(\mathcal{H}^{LR}_{L,\mu}) \otimes \mathbb{1}^{R}_{\mu}$. Up to the choice of an arbitrary overall phase, the isomorphism between $\mathcal{A}^{LL}_{L,\mu}$ and $\mathcal{A}^{LR}_{L,\mu}$ then also defines an isomorphism betwen the left Hilbert-space factors $\mathcal{H}^{LL}_{L,\mu}$ and $\mathcal{H}^{LR}_{L,\mu}$. We will thus henceforth write $\mathcal{H}^{LL}_{L,\mu} = \mathcal{H}^{LR}_{L,\mu}$

The algebra $\mathcal{B}(\mathcal{H}^{LR}_{R,\mu})$ of bounded operators on the right factor can be similarly associated with a type-I von Neumann factor given by a central projection of the right diagonal von Neumann algebra $\mathcal{A}^{RR}_{R} = \mathcal{A}^{B_R \sqcup B_R}_{R}$. To do so, we recall that a corresponding result for diagonal Hilbert space sectors $\mathcal{H}_{B\sqcup B}$ was derived in [1] by using the commutation theorem [24] for Hilbert semi-birigged spaces derived in to show that the left and right von Neumann algebras are commutants.[11] One may then check that the off-diagonal dense subspace $X = \mathcal{D}_{LR}$ equipped with representations $C = \hat{A}^{LR}_{L}$ and $D = \hat{A}^{LR}_{R}$ also satisfies the four axioms of Hilbert-semi-birigged spaces from [24], and where the notation X, C, D comes from that reference. One may also check that the so-called coupling condition from [24] is satisfied. See appendix A for further details.

As a result, theorem 1.3 of [24] implies that the von Neumann algebras $\mathcal{A}^{LR}_{L}$ and $\mathcal{A}^{LR}_{R}$ generated by A^{LR}_{L} and A^{LR}_{R} are commutants on $\mathcal{H}_{LR}$. This fact has two immediate implications. The first is that the central projections of $\mathcal{A}^{LR}_{L}$ are exactly the central projections of $\mathcal{A}^{LR}_{R}$. In particular, the minimal central projections of $\mathcal{A}^{LR}_{R}$ must again be $\Phi_{LR}(P_\mu)$, so that the right algebra admits a decomposition of the form (5.6):

$$\mathcal{A}^{LR}_{R} = \bigoplus_{\mu \in \mathcal{I}_{B_L}} \chi^{LR}_{L,\mu} \mathcal{A}^{LR}_{R,\mu}, \tag{5.8}$$

where for $\chi^{LR}_{L,\mu} = 1$ the corresponding $A^{LR}_{R,\mu} = \Phi_{LR}(P_\mu)\mathcal{A}^{LR}_{R}$ is a type-I factor.

The second implication is that (again for $\chi^{LR}_{L,\mu} = 1$) each $A^{LR}_{R,\mu}$ must act as $\mathbb{1}^{L}_{\mu} \otimes B(\mathcal{H}^{LR}_{R,\mu})$ on the Hilbert space $\mathcal{H}^{LR}_{\mu} = \mathcal{H}^{LR}_{L,\mu} \otimes \mathcal{H}^{LR}_{R,\mu}$. We have thus arrived at an off-diagonal analogue of the structure derived for diagonal sectors in section 4 of [1]. In particular, if we define $I_{B_L \sqcup B_R} \subset I_{B_L}$ as the set on which $\chi^{LR}_{L,\mu} = 1$, we may write

$$\mathcal{H}_{LR} = \bigoplus_{\mu \in \mathcal{I}_{B_L \sqcup B_R}} \mathcal{H}^{LR}_{\mu} = \bigoplus_{\mu \in \mathcal{I}_{B_L \sqcup B_R}} \mathcal{H}^{LR}_{L,\mu} \otimes \mathcal{H}^{LR}_{R,\mu}. \tag{5.9}$$

Now, as described above, we may use the isomorphism $\Phi_{LR} : \mathcal{A}^{LL}_{L,\mu} \to \mathcal{A}^{LR}_{L,\mu}$ for $\mu \in \mathcal{I}_{B_L \sqcup B_R}$ to write $\mathcal{H}^{LR}_{L,\mu} = \mathcal{H}^{LL}_{L,\mu}$ for such μ. But since all of the above discussion of the left von Neumann algebras $\mathcal{A}^{LL}_{L}$ can be repeated analogously for the right von Neumann algebras $\mathcal{A}^{RR}_{R}$, there must be another surjective $*$-homomorphism from $\mathcal{A}^{RR}_{R} \to \mathcal{A}^{LR}_{R}$. We will call this new homomorphism Φ^{R}_{LR} and, for clarity, we will sometimes also write Φ^{L}_{LR} for the previous left map Φ_{LR}. Furthermore, for $\mu \in \mathcal{I}_{B_L \sqcup B_R}$, the right homomorphism Φ^{R}_{LR} must define an isomorphism between $\mathcal{A}^{LR}_{R,\mu}$ and some type-I von Neumann factor $\mathcal{A}^{RR}_{R,\mu}$ in $\mathcal{A}^{RR}_{R}$.

[11]Though there was also a more direct proof in the diagonal case studied in [1].

We may then also write $\mathcal{H}^{LR}_{R,\mu} = \mathcal{H}^{RR}_{R,\mu}$, and thus

$$\mathcal{H}_{LR} = \bigoplus_{\mu\in\mathcal{I}_{B_L\sqcup B_R}} \mathcal{H}^{LR}_{L,\mu} \otimes \mathcal{H}^{LR}_{R,\mu} = \bigoplus_{\mu\in\mathcal{I}_{B_L\sqcup B_R}} \mathcal{H}^{LL}_{L,\mu} \otimes \mathcal{H}^{RR}_{R,\mu}; \tag{5.10}$$

i.e., the decomposition of the off-diagonal sectors involves the same left and right Hilbert spaces as the decomposition of the digaonal factors.

5.2 A universal central algebra for B

Our result (5.10) imposes a compatibility constraint on our von Neumann algebras. Since the left algebra $\mathcal{A}^{B_1\sqcup B_2}_L$ and the right algebra $\mathcal{A}^{B_1\sqcup B_2}_R$ associated with two arbitrary boundaries B_1 and B_2 must be commutants, they share a common center $\mathcal{Z}^{B_1\sqcup B_2}$. Furthermore, we saw that the center of such an off-diagonal algebra is identified with central projections of the centers of either diagonal algebra $\mathcal{A}^{B_1\sqcup B_1}_L$, $\mathcal{A}^{B_2\sqcup B_2}_R$. We thus have the following relations:

$$\mathcal{Z}^{B_1\sqcup B_2} \simeq P^{B_1\sqcup B_2}_L \mathcal{Z}^{B_1\sqcup B_1} \simeq P^{B_1\sqcup B_2}_R \mathcal{Z}^{B_2\sqcup B_2}. \tag{5.11}$$

Let us now define a universal Abelian algebra

$$\tilde{\mathcal{Z}}_{univ} = \bigoplus_B \mathcal{Z}^{B\sqcup B} \tag{5.12}$$

as the direct sum of all diagonal centers. We may again regard this as a von Neumann algebra by taking the elements to act on the direct sum Hilbert space $(\bigoplus_B \mathcal{H}_{B\sqcup B})$.

As it stands, the algebra (5.12) is a purely formal construction whose universality comes by fiat from our choice to sum over all source-manifolds B. However, the result (5.11) turns out to define an interesting equivalence relation $\sim$ on elements of $\tilde{\mathcal{Z}}^{univ}$ so that the quotient $\mathcal{Z}^{univ} = \tilde{\mathcal{Z}}^{univ}/\sim$ provides an algebraic encoding of the above compatibility condition. To see this, we begin by defining a relation $\sim$ that relates centrally-minimal projections in different centers:

Definition 2. *Consider projections $P_1 \in \mathcal{Z}^{B_1\sqcup B_1}$, $P_2 \in \mathcal{Z}^{B_2\sqcup B_2}$ that are each minimal in their respective central algebras. We will write $P_1 \sim P_2$ when there is some boundary B, a projection $P \in \mathcal{Z}^{B\sqcup B}$, and non-zero states $|\phi\rangle \in \mathcal{H}_{B_1\sqcup B}$, $|\psi\rangle \in \mathcal{H}_{B_2\sqcup B}$ that satisfy*

$$\begin{aligned} \Phi^L_{B_1B}(P_1)|\phi\rangle &= \Phi^R_{B_1B}(P)|\phi\rangle \neq 0 \quad \text{and} \\ \Phi^L_{B_2B}(P_2)|\psi\rangle &= \Phi^R_{B_2B}(P)|\psi\rangle \neq 0. \end{aligned} \tag{5.13}$$

Here $\Phi^L_{BB_2}, \Phi^L_{BB_1}$ are just Φ^L_{LR} for $B_L = B$ and $B_R = B_2, B_1$, and we will use analogous notation below.

This definition is manifestly symmetric, meaning that $P_1 \sim P_2$ is equivalent to $P_2 \sim P_1$. It also satisfies the reflexive property $P_1 \sim P_1$. This may be seen by setting $B = B_2 = B_1$, $P = P_2 = P_1$, and $|\psi\rangle = |\phi\rangle = P_1|C_\beta\rangle \in \mathcal{H}_{B_1\sqcup B_1}$ for some $\beta > 0$. Since it was shown in [1] that $P_1|C_\beta\rangle$ cannot vanish for any central projection $P_1 \in \mathcal{Z}^{B_1\sqcup B_1}$, to establish $P_1 \sim P_1$ in this context we need only recall that any such P_1 can be interpreted as a member of both $\mathcal{A}^{B_1\sqcup B_1}_L$ and $\mathcal{A}^{B_1\sqcup B_1}_R$ and that $\Phi^L_{B_1B_1}(P_1) = P_1 = \Phi^R_{B_1B_1}(P_1)$.

Showing that $\sim$ is an equivalence relation thus requires only that we establish transitivity, which means that $P_2 \sim P_3$ whenever $P_1 \sim P_2$ and $P_1 \sim P_3$. The conditions $P_1 \sim P_2$ and

$P_1 \sim P_3$ mean that there are boundaries B, B', projections $P \in \mathcal{Z}^{B\sqcup B}$, $P' \in \mathcal{Z}^{B'\sqcup B'}$, and non-zero states $\phi_{B_1B}, \phi_{B_2B}, \phi_{B_1B'}, \phi_{B_3B'}$ (which respectively lie in $\mathcal{H}_{B_1\sqcup B}$, $\mathcal{H}_{B_2\sqcup B}$, $\mathcal{H}_{B_1\sqcup B'}$, $\mathcal{H}_{B_3\sqcup B'}$) such that we have

$$\begin{aligned} \Phi^L_{B_1B}(P_1)|\phi_{B_1B}\rangle &= \Phi^R_{B_1B}(P)|\phi_{B_1B}\rangle \neq 0, \\ \Phi^L_{B_2B}(P_2)|\phi_{B_2B}\rangle &= \Phi^R_{B_2B}(P)|\phi_{B_2B}\rangle \neq 0, \\ \Phi^L_{B_1B'}(P_1)|\phi_{B_1B'}\rangle &= \Phi^R_{B_1B'}(P')|\phi_{B_1B'}\rangle \neq 0, \quad \text{and} \\ \Phi^L_{B_3B'}(P_3)|\phi_{B_3B'}\rangle &= \Phi^R_{B_3B'}(P')|\phi_{B_3B'}\rangle \neq 0. \end{aligned} \tag{5.14}$$

Given the existence of $|\phi_{B_3B'}\rangle$, we can show $P_1 \sim P_3$ by demonstrating the existence of a non-zero $|\phi_{B_2B'}\rangle \in \mathcal{H}_{B_2\sqcup B'}$ such that

$$\Phi^L_{B_2B'}(P_2)|\phi_{B_2B'}\rangle = \Phi^R_{B_2B'}(P')|\phi_{B_2B'}\rangle \neq 0. \tag{5.15}$$

We will build such a $|\phi_{B_2B'}\rangle$ by noting that the construction of our map $\Psi_R : \mathcal{H}_{LR} \to \mathcal{B}(\mathcal{H}_{LL}, \mathcal{H}_{LR})$ readily generalizes to define a map $\Psi_R^{B,B_1\to B_2} : \mathcal{H}_{B_1\sqcup B_2} \to \mathcal{B}(\mathcal{H}_{B\sqcup B_1}, \mathcal{H}_{B\sqcup B_2})$ involving arbitrary Hilbert spaces B, B_1, B_2. We will again use the simplified notation $\hat{a}_R := \Psi_R^{B,B_1\to B_2}(|a\rangle)$. As with the original Ψ_R, the full $\Psi_R^{B,B_1\to B_2}$ is first defined as a map from $\mathcal{D}_{B_1\sqcup B_2}$ to $\mathcal{B}(\mathcal{D}_{B\sqcup B_1}, \mathcal{D}_{B\sqcup B_2})$ where it is defined by sewing surfaces as in the definition of $\hat{a}_R$ in figure 2. As described in appendix B, in the same manner that it was demonstrated for the original Ψ_R, each $\hat{a}_R$ defined by the more general $\Psi_R^{B,B_1\to B_2}$ is a bounded operator and, moreover, the map $\Psi_R^{B,B_1\to B_2}$ itself can be shown to be continuous with respect to the Hilbert space topology on $\mathcal{H}_{B_1\sqcup B_2}$ and the strong operator topology on $B(\mathcal{H}_{B\sqcup B_1}, \mathcal{H}_{B\sqcup B_2})$. The full map $\Psi_R^{B,B_1\to B_2}$ is then given by the unique continuous extension to the full Hilbert space $\mathcal{H}_{B_1\sqcup B_2}$. Appendix B also derives two intertwining relations (B.2), (B.5), which we restate here for the convenience of the reader. The first of these is that for all $d \in \mathcal{A}_L^{B\sqcup B}$ we have

$$\Phi^L_{BB_2}(d)\hat{a}_R = \hat{a}_R\Phi^L_{BB_1}(d). \tag{5.16}$$

The second is the relation

$$(a^\star)_R := \hat{a}_R^\dagger, \tag{5.17}$$

where (as described in appendix B) the operation $\star$ has been extended from $\mathcal{D}_{B_1\sqcup B_2}$ to the entire Hilbert space by continuity.

There is also an analogous left map $\Psi_L^{B_2\to B_1,B} : \mathcal{H}_{B_1\sqcup B_2} \to \mathcal{B}(\mathcal{H}_{B_2\sqcup B}, \mathcal{H}_{B_1\sqcup B})$. Writing, $\hat{a}_L := \Psi_L^{B_2\to B_1,B}(|a\rangle)$, this map satisfies

$$\Phi^R_{B_1B}(d)\hat{a}_L = \hat{a}_L\Phi^R_{B_2B}(d) \tag{5.18}$$

for all $d \in \mathcal{A}_R^{B\sqcup B}$. It also satisfies

$$(a^\star)_L := \hat{a}_L^\dagger. \tag{5.19}$$

By including the projections $\Phi^L_{B_2B}(P_2), \Phi^R_{B_1B'}(P')$ in their definitions, we can take $|\phi_{B_2B}\rangle$ and $|\phi_{B_1B'}\rangle$ to satisfy

$$\Phi^L_{B_2B}(P_2)|\phi_{B_2B}\rangle = |\phi_{B_2B}\rangle \neq 0, \quad \Phi^R_{B_1B'}(P')|\phi_{B_1B'}\rangle = |\phi_{B_1B'}\rangle \neq 0. \tag{5.20}$$

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If we now choose any $b_R \in \mathcal{A}^R_{B_1B_1}, c_R \in \mathcal{A}^R_{BB}$, the above structures allow us to define a state

$$|\phi_{B_2B'}\rangle = \widehat{(\phi_{B_1B'})}_R\ \Phi^R_{B_2B_1}(b_R)\ \widehat{(\phi_{B_1B})}^\dagger_R \Phi^R_{B_2B}(c_R)\ |\phi_{B_2B}\rangle, \tag{5.21}$$

If all of the relevant states and operators above were defined directly by rimmed surfaces we could write

$$\begin{aligned} |\phi_{B_2B'}\rangle &= \widehat{(\phi_{B_1B'})}_R\ \Phi^R_{B_2B_1}(\hat{b}_R)\ \widehat{(\phi_{B_1B})}^\dagger_R \Phi^R_{B_2B}(\hat{c}_R)\ |\phi_{B_2B}\rangle \\ = |\phi_{B_2B}\ c\ \phi^\star_{B_1B}\ b\ \phi_{B_1B'}\rangle &= \widehat{(\phi_{B_2B})}_L\ \Phi^L_{BB'}(\hat{c}_L)\ \widehat{(\phi_{B_1B})}^\dagger_L \Phi^L_{B_1B'}(\hat{b}_L)\ |\phi_{B_1B'}\rangle, \end{aligned} \tag{5.22}$$

As described in appendix C, this observation generalizes to arbitrary states and operators as above when written in the form

$$\begin{aligned} |\phi_{B_2B'}\rangle &= \widehat{(\phi_{B_1B'})}_R\ \Phi^R_{B_2B_1}(b_R)\ \widehat{(\phi_{B_1B})}^\dagger_R \Phi^R_{B_2B}(c_R)\ |\phi_{B_2B}\rangle \\ &= \widehat{(\phi_{B_2B})}_L\ \Phi^L_{BB'}(c_L)\ \widehat{(\phi_{B_1B})}^\dagger_L \Phi^L_{B_1B'}(b_L)\ |\phi_{B_1B'}\rangle, \end{aligned} \tag{5.23}$$

where c_L, b_L are defined from b_R, c_R as described in appendix C.

We may now compute

$$\begin{aligned} \Phi^L_{B_2B'}(P_2)|\phi_{B_2B'}\rangle &= \Phi^L_{B_2B'}(P_2)\ \widehat{\phi_{B_1B'}}_R\ \Phi^R_{B_2B'}(b_R)\widehat{\phi_{B_1B}}^\dagger_R\ \Phi^R_{B_2B}(c_R)\ |\phi_{B_2B}\rangle \\ &= \widehat{\phi_{B_1B'}}_R\ \Phi^R_{B_2B'}(b_R)\ \widehat{\phi_{B_1B}}^\dagger_R\ \Phi^R_{B_2B}(c_R)\ \Phi^L_{B_2B}(P_2)|\phi_{B_2B}\rangle \\ &= |\phi_{B_2B'}\rangle, \end{aligned} \tag{5.24}$$

where we pass from the first to the 2nd line using intertwining relations of the form (5.16) and commutivity of left- and right-acting operators, and where the final step uses (5.20). We may also similarly write

$$\begin{aligned} \Phi^R_{B_2B'}(P')|\phi_{B_2B'}\rangle &= \Phi^R_{B_2B'}(P')\widehat{(\phi_{B_2B})}_L\ \Phi^L_{BB'}(c_L)\ \widehat{(\phi_{B_1B})}^\dagger_L \Phi^L_{B_1B'}(b_L)\ |\phi_{B_1B'}\rangle \\ &= \widehat{(\phi_{B_2B})}_L\ \Phi^L_{BB'}(c_L)\ \widehat{(\phi_{B_1B})}^\dagger_L \Phi^L_{B_1B'}(b_L)\ \Phi^R_{B_1B'}(P')|\phi_{B_1B'}\rangle \\ &= |\phi_{B_2B'}\rangle. \end{aligned} \tag{5.25}$$

This will establish transitivity so long as we also show that we can choose b_R, c_R such that $|\phi_{B_2B'}\rangle \neq 0$. We can do so by using the following two results (where the first will act as a Lemma that will be useful in proving the second.

Claim 5.1. *For non-zero $|\phi\rangle, |\kappa\rangle \in \mathcal{H}_{B_1 \sqcup B_2}$, the states $\hat{\phi}^\dagger_R|\phi\rangle \in \mathcal{H}_{B_1 \sqcup B_1}$ and $\hat{\kappa}_R|\kappa^\star\rangle \in \mathcal{H}_{B_2 \sqcup B_2}$ cannot vanish. Here $\hat{\phi}_R := \Psi^{B_1, B_1 \to B_2}_R(|\phi\rangle)$ and $\hat{\kappa}_R := \Psi^{B_2, B_1 \to B_2}_R(|\kappa\rangle)$.*

Proof. Let us first use (4.13) and boundedness of $\hat{\phi}_R$ (and thus of its adjoint) to rewrite the first state in the form

$$\hat{\phi}^\dagger_R|\phi\rangle = \lim_{\beta \downarrow 0} \hat{\phi}^\dagger_R \hat{\phi}_R |C_\beta\rangle. \tag{5.26}$$

The definition (2.3) of the trace tr then shows that the norm of our state is $tr\left((\hat{\phi}^\dagger_R\hat{\phi}_R)^2\right)$. But $\hat{\phi}^\dagger_R\hat{\phi}_R$ is non-zero since deleting $\hat{\phi}^\dagger_R$ from (5.26) and repeating the same argument gives $tr(\hat{\phi}^\dagger_R\hat{\phi}_R) = \langle\phi|\phi\rangle \neq 0$. Since $\hat{\phi}^\dagger_R\hat{\phi}_R$ is manifestly self-adjoint, the operator $(\hat{\phi}^\dagger_R\hat{\phi}_R)^2$ is again non-zero and faithfulness of tr as established in [1] means that $tr[(\hat{\phi}^\dagger_R\hat{\phi}_R)^2] \neq 0$. Thus our state has non-zero norm and cannot vanish. The analogous argument then also shows that $\hat{\kappa}_R|\kappa^\star\rangle \in \mathcal{H}_{B_2 \sqcup B_2}$ is non-zero. □

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Claim 5.2. *Given a centrally-minimal projection $P_1 \in \mathcal{A}_L^{B_1 \sqcup B_1}$ and non-zero states $|\phi\rangle \in \mathcal{H}_{B_1 \sqcup B_2}$ and $|\kappa\rangle \in \mathcal{H}_{B_1 \sqcup B_3}$ that satisfy*

$$\Phi^L_{B_1 B_2}(P_1)|\phi\rangle = |\phi\rangle \quad \Phi^L_{B_1 B_3}(P_1)|\kappa\rangle = |\kappa\rangle, \tag{5.27}$$

there is some a_R in the right von Neumann algebra $\mathcal{A}_R^{B_2 \sqcup B_1}$ for which the state $|\gamma\rangle := \hat{\kappa}_R a_R |\phi^\star\rangle \in \mathcal{H}_{B_2 \sqcup B_3}$ is non-zero, where as usual $\hat{\kappa}_R = \Psi_R^{B_2, B_1 \to B_3}(|\kappa\rangle)$

Proof. We will also write $\widehat{(\phi^\star)}_R := \Psi_R^{B_1, B_2 \to B_1}(|\phi^\star\rangle) = \left[\Psi_R^{B_1, B_1 \to B_2}(|\phi\rangle)\right]^\dagger =: \hat{\phi}_R^\dagger$, where the 2nd equality uses (B.5). As in the proof of 5.1 above, the operators $\hat{\kappa}_R^\dagger \hat{\kappa}_R$ and $\widehat{(\phi^\star)}_R \widehat{(\phi^\star)}_R^\dagger = \hat{\phi}_R^\dagger \hat{\phi}_R$ on $\mathcal{A}^L_{B_1 \sqcup B_1}$ are both non-zero.

Furthermore, for $P_1^\perp = \mathbb{1} - P_1$, we may write e.g.,

$$\lim_{\beta \downarrow 0} P_1^\perp \hat{\kappa}_R^\dagger \hat{\kappa}_R |C_\beta\rangle = P_1^\perp \lim_{\beta \downarrow 0} \hat{\kappa}_R^\dagger \hat{\kappa}_R |C_\beta\rangle = P_1^\perp \hat{\kappa}_R^\dagger |\kappa\rangle = \hat{\kappa}_R^\dagger \left[\Phi^L_{B_1 B_2}\left(P_1^\perp\right)\right] |\kappa\rangle = 0, \tag{5.28}$$

where the first equality uses boundedness of $P_1^\perp$, the 2nd follows as in the proof of 5.1, the third uses the intertwining relation (4.8), and the fourth follows from (5.27) since $\Phi^L_{B_1 B_2}\left(P_1^\perp\right) = \mathbb{1} - \Phi^L_{B_1 B_2}(P_1)$. Using the definition (2.3) of the trace tr on positive elements of the von Neumann algebra $\mathcal{A}_R^{B_1 \sqcup B_1}$, the above result requires $\left(\kappa_R^\dagger \hat{\kappa}_R (\mathbb{1} - P_1) \hat{\kappa}_R^\dagger \hat{\kappa}_R\right)$ to have vanishing trace.

Faithfulness of the trace now requires $\kappa_R^\dagger \hat{\kappa}_R (\mathbb{1} - P_1) \hat{\kappa}_R^\dagger \hat{\kappa}_R = 0$. In particular, for all $|\gamma\rangle \in \mathcal{H}_{B_1 \sqcup B_1}$ we have

$$\langle \gamma | \kappa_R^\dagger \hat{\kappa}_R (\mathbb{1} - P_1) \hat{\kappa}_R^\dagger \hat{\kappa}_R | \gamma \rangle = 0. \tag{5.29}$$

But (5.29) is the norm of the state $(\mathbb{1} - P_1)\hat{\kappa}_R^\dagger \hat{\kappa}_R |\gamma\rangle$, so this state must vanish for all γ and we have

$$P_1 \hat{\kappa}_R^\dagger \hat{\kappa}_R = \hat{\kappa}_R^\dagger \hat{\kappa}_R. \tag{5.30}$$

Since the same argument holds with κ replaced with ϕ, we must have

$$P_1 \hat{\phi}_R^\dagger \hat{\phi}_R = \hat{\phi}_R^\dagger \hat{\phi}_R, \quad \text{or, equivalently,} \quad P_1 \widehat{(\phi^\star)}_R \widehat{(\phi^\star)}_R^\dagger = \widehat{(\phi^\star)}_R \widehat{(\phi^\star)}_R^\dagger, \tag{5.31}$$

so that both $\hat{\kappa}_R^\dagger \hat{\kappa}_R$ and $\widehat{(\phi^\star)}_R \widehat{(\phi^\star)}_R^\dagger$ in fact lie in the same μ-sector $\mathcal{A}^{B_1 \sqcup B_1}_{R,\mu}$.

Due to (5.27) and the fact that $|\phi\rangle \neq 0$, we see that $\mathcal{A}^{B_1 \sqcup B_1}_{R,\mu}$ is also isomorphic to some off-diagonal factor $\mathcal{A}^{B_2 \sqcup B_1}_{R,\mu}$, and we may use $\Phi^R_{B_2 B_1}$ to denote this isomorphism. But let us also recall that $\mathcal{A}^{B_1 \sqcup B_1}_{R,\mu}$ is a type I von Neumann factor isomorphic to $\mathcal{B}(\mathcal{H}^\mu_{B_1 \sqcup B_1, R})$. Thus $\mathcal{A}^{B_2 \sqcup B_1}_{R,\mu}$ is also isomorphic to $\mathcal{B}(\mathcal{H}^\mu_{B_1 \sqcup B_1, R})$.

Furthermore, since traces on type-I factors are unique up to multiplication by constants, up to such constants we can use the identification of $\mathcal{A}^{B_2 \sqcup B_1}_{R,\mu}$ with $\mathcal{B}(\mathcal{H}^\mu_{B_1 \sqcup B_1, R})$ to evaluate traces on $\mathcal{A}^{B_2 \sqcup B_1}_{R,\mu}$. We can then use the trace on $\mathcal{B}(\mathcal{H}^\mu_{B_1 \sqcup B_1, R})$ to show that there is some $a_R \in \mathcal{A}^{B_2 \sqcup B_1}_{R,\mu}$ for which $tr(\widehat{(\phi^\star)}_R^\dagger a_R^\dagger \hat{\kappa}_R^\dagger \hat{\kappa}_R a_R \widehat{(\phi^\star)}_R)$ is non-zero. This is in particular true for the a_R obtained by applying the isomorphism $\Phi^R_{B_2 B_1}$ to the operator $|\overline{\phi^\star}\rangle\langle\overline{\kappa}| \in \mathcal{B}(\mathcal{H}^\mu_{B_1 \sqcup B_1, R})$ where $|\overline{\kappa}\rangle, |\overline{\phi^\star}\rangle \in H^\mu_{B_1 \sqcup B_1, R}$ are the largest-eigenvalue eigenvectors of $\hat{\kappa}_R^\dagger \hat{\kappa}_R$ and $\widehat{(\phi^\star)}_R \widehat{(\phi^\star)}_R^\dagger$. But by the same argument used in the proof of claim 5.1, this trace is $\langle \gamma | \gamma \rangle$ for $|\gamma\rangle := \hat{\kappa}_R a_R |\phi^\star\rangle \in \mathcal{H}_{B_2 \sqcup B_3}$. Thus $|\gamma\rangle$ is non-zero as claimed. $\square$

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To show that the $|\phi_{B_2B'}\rangle$ defined in (5.21) is non-zero, we simply apply Claim 5.2 twice. The first time uses $|\phi_1\rangle = |\phi^\star_{B_2B}\rangle$ and $|\kappa_1\rangle = |\phi^\star_{B_1B}\rangle$, while the second uses $|\kappa_2\rangle = |\phi_{B_1B'}\rangle$ and

$$|\phi_2\rangle = \widehat{\phi_{B_2B}}_R^\dagger \hat{c}_R^\dagger |\phi_{B_1B}\rangle. \tag{5.32}$$

The fact that $|\phi_2\rangle \neq 0$ follows from the first application of Claim 5.2, and the second application then gives $|\phi_{B_2B'}\rangle \neq 0$

This establishes the desired transitivity of $\sim$ and shows that it defines an equivalence relation on centrally-minimal projections. We can then extend the relation $\sim$ to more general elements of $Z^{B_1\sqcup B_1}$, $\mathcal{Z}^{B_2\sqcup B_2}$. We will write $z_1 \sim z_2$ when $z_1 = \sum_i c_i P_{1i}$ and $z_2 = \sum_i c_i P_{2i}$ for centrally-minimal projections P_{1i}, P_{2i} with $P_{1i} \sim P_{2i}$. It is then manifest that this extension is again an equivalence.

We may thus take the quotient of $\tilde{\mathcal{Z}}_{univ}$ by the relation $\sim$ to obtain a smaller algebra

$$\mathcal{Z}_{univ} = \tilde{\mathcal{Z}}_{univ}/\sim . \tag{5.33}$$

The algebra (5.33) is universal in the sense that any diagonal or non-diagonal center can be obtained as a projection of $\mathcal{Z}_{univ}$. This property allows us to then identify any center $\mathcal{Z}^{B_1\sqcup B_2}$ with a subalgebra of $\mathcal{Z}_{univ}$. In particular, this allows us to say that $z_{12} \in \mathcal{Z}^{B_1\sqcup B_2}$ and $z_{34} \in \mathcal{Z}^{B_3\sqcup B_4}$ are 'the same' if they map to the same element of $\mathcal{Z}_{univ}$.

6 Traces and entropy

The remaining properties (regarding entropy, hidden sectors, etc.) discussed in section 4 of [1] now follow by essentially the same arguments given there. However, we can use the universal structure identified in section 5.2 to provide a unified and clean discussion for any bipartite boundary $B_1 \sqcup B_2$. We briefly summarize such results below.

Recall that we have a $*$-homomorphism $\Phi_{BB'}$ from $\mathcal{A}_L^{B\sqcup B}$ to the bounded operators on $\mathcal{H}_{B\sqcup B'}$. Given any normalized state $|\psi\rangle \in \mathcal{H}_{B\sqcup B'}$, we can thus compute the expectation value $\langle\psi|\Phi_{BB'}(a)|\psi\rangle$ for any $a \in \mathcal{A}_L^{B\sqcup B}$. This is a positive linear functional on the type-I von Neumann algebra $\mathcal{A}_L^{B\sqcup B}$. Furthermore, our functional is normalized in the sense that

$$\langle\psi|\Phi_{BB'}(\mathbb{1})|\psi\rangle = \langle\psi|\mathbb{1}|\psi\rangle = 1. \tag{6.1}$$

As a result, given any trace tr on the set $\left(\mathcal{A}_L^{B\sqcup B}\right)^+$ of positive elements in $\mathcal{A}_L^{B\sqcup B}$, there is a density operator $\rho_\psi^{tr} \in \left(\mathcal{A}_L^{B\sqcup B}\right)^+$ with $tr(\rho_\psi^{tr}) = 1$ such that

$$tr\left(\left(\rho_\psi^{tr}\right)^{1/2} a \left(\rho_\psi^{tr}\right)^{1/2}\right) = \langle\psi|\Phi_{BB'}(a)|\psi\rangle \tag{6.2}$$

for all $a \in \left(\mathcal{A}_L^{B\sqcup B}\right)^+$. A brute-force argument for (6.2) can be found in the discussion around (4.63) in [1] where ρ_ψ^{tr} is constructed by tracing out the right Hilbert space factors from each term in the direct sum that defines $\mathbb{H}_B$. However, we suspect that the mathematical literature contains a more elegant derivation.

The notation ρ_ψ^{tr} explicitly indicates the dependence of this operator on the choice of the trace tr. Three traces on $\mathcal{A}_L^{B\sqcup B}$ were discussed in [1]. The first, called simply tr, is given by (2.3). The second trace, called Tr, was defined by choosing an arbitrary orthonormal basis $\{|i,\mu\rangle\}$ for each of the Hilbert spaces $\mathcal{H}^{B\sqcup B}_{\mu,L}$ for $\mu \in \mathcal{I}_{B\sqcup B}$ and writing

$$Tr(a) := \sum_{i,\mu} \langle i,\mu|a|i,\mu\rangle \tag{6.3}$$

for every $a \in \left(\mathcal{A}^{B\sqcup B}_L\right)^+$.

Now, on a given type-I factor $\mathcal{A}^{B\sqcup B}_{L,\mu}$, any two traces are proportional. In this context we must thus have

$$tr = n_\mu Tr. \tag{6.4}$$

It was shown in [1] that their axioms imply higher generalizations of the trace inequality (2.1) which require the n_μ to be positive integers. The third trace $\widetilde{Tr}$ from [1] can be defined by constructing the Hilbert spaces

$$\underset{\sim}{\mathcal{H}}_B := \bigoplus_\mu \mathcal{H}^{B\sqcup B}_{\mu,L} \otimes \mathbb{C}^{n_\mu}, \tag{6.5}$$

choosing an arbitrary orthonormal basis $\{|\tilde{i}\rangle\}$ of $\underset{\sim}{\mathcal{H}}^B$, and then writing

$$\widetilde{Tr}(a) := \sum_{\tilde{i}} \langle\tilde{i}|\,(a\otimes \mathbb{1})\,|\tilde{i}\rangle \tag{6.6}$$

for every $a \in \left(\mathcal{A}^{B\sqcup B}_L\right)^+$. Comparing (6.4) with (6.6) then shows that $tr = \widetilde{Tr}$, and thus that $\rho^{tr}_\psi = \rho^{\widetilde{Tr}}_\psi$. Following [1], to simplify the notation we will henceforth denote this density operator by $\tilde{\rho}_\psi$. The factors $\mathbb{C}^{n_\mu}$ in (6.5) were termed *hidden sectors* in [1], where their physical interpretation was discussed in more detail and illustrated with examples.[12]

An explicit formula for $\tilde{\rho}_\psi$ can be given by using an appropriate embedding of $\mathcal{H}_{B\sqcup B'}$ in $\underset{\sim}{\mathcal{H}}_B \otimes \underset{\sim}{\mathcal{H}}_{B'}$. To do so, simply choose a maximally entangled state $|\chi_\mu\rangle$ in $\mathbb{C}^{n_\mu}\otimes\mathbb{C}^{n_\mu}$ for each μ and write

$$|\tilde{\psi}\rangle = \sum_{P\in\mathcal{Z}^{MP}_{univ}} (\Phi_{BB'}(P)|\psi\rangle)\otimes|\chi_\mu\rangle, \tag{6.7}$$

where $\mathcal{Z}^{MP}_{univ}$ is the set of projections in $\mathcal{Z}_{univ}$ that are also minmal in $\mathcal{Z}_{univ}$. In (6.7), we have also defined $\Phi_{BB'}(P) := \Phi_{BB'}(P')$ when there is some $P' \sim P$ with $P' \in \mathcal{Z}^{B\sqcup B}$. When there is no such P', we set $\Phi_{BB'}(P) := 0$. Finally, we also regard μ as a function of P defined by the fact that the above P' here projects onto some $\mathcal{H}^\mu_{BB}$.

[12]The structure of (6.5) and the embedding (6.7) of $\mathcal{H}_{B\sqcup B'}$ in $\underset{\sim}{\mathcal{H}}_B \otimes \underset{\sim}{\mathcal{H}}_{B'}$ define a (sum of) quantum error correcting codes with two-sided recovery of the type described in [25]. This is not a coincidence, as the insertion of Hidden sectors can be taken to define the Hilbert space of a more fundamental theory in which $\mathcal{H}_{B\sqcup B'}$ is embedded. Some differences, however, are that the QEC structure described in [25] is expected to hold only approximately and to require some notion of cut-off in the bulk. In contrast, our $\mathcal{H}_{B\sqcup B'}$ is an exact construction. It is also often noted that the same mathematical structure also appears in discussions of so-called edge modes (see e.g. [26–28]). However, the physics of edge modes appears to be rather different. In particular, in contrast to edge modes, our $\mathbb{C}^{n_\mu}$ hidden sectors play no role in the factorization of the Hilbert space. Furthermore, there are interesting examples in which the hidden sectors are trivial.

The density operator (or density matrix) $\tilde{\rho}_\psi$ can then be constructed from $|\tilde{\psi}\rangle\langle\tilde{\psi}|$ by 'tracing out' the right factor $\underset{\sim}{\mathcal{H}}_{B'}$ in the above decomposition (using its natural trace $\widetilde{Tr}$). The result $\tilde{\rho}_\psi$ is a density matrix on $\underset{\sim}{\mathcal{H}}_B$ whose von Neumann entropy is given by $S(B) := -tr\,(\tilde{\rho}_\psi \ln \tilde{\rho}_\psi) = -\widetilde{Tr}\,(\tilde{\rho}_\psi \ln \tilde{\rho}_\psi)$. On the other hand, the connection of tr to our path integral means that, when $|\psi\rangle$ is defined by a finite linear combination of surfaces to which the Lewkowycz-Maldacena argument [4] can be applied in an appropriate semiclassical limit, our entropy $S(B)$ in that limit will conicide with that computed by the Ryu-Takayanagi formula [15, 16]. In this sense we obtain a Hilbert space interpretation of the Ryu-Takayangi formula for general states having the $(d-2)$-dimensional closed source-manifold B as any part of their boundary.

7 Discussion

The main primary goal of our work above was to generalize the analysis in [1] of UV-completions of asmptotically locally AdS Euclidean gravitational path integrals to Hilbert space sectors $\mathcal{H}_{B_L \sqcup B_R}$ defined by bipartite boundaries $B_L \sqcup B_R$ for general B_L, B_R. In analogy with the diagonal case $B_L = B_R$, the defining path integral provides a notion of entropy on both B_L and B_R. Furthermore, by the Lewkowycz-Maldacena argument [4], if the path integral admits an appropriate semi-classical limit then the entropy in that limit is given by the Ryu-Takayanagi formula with small corrections. Our main result is that there are Hilbert spaces $\underset{\sim}{\mathcal{H}}_{B_L}, \underset{\sim}{\mathcal{H}}_{B_L}$ such that $\mathcal{H}_{B_L \sqcup B_R}$ can be embedded in $\underset{\sim}{\mathcal{H}}_{B_L} \otimes \underset{\sim}{\mathcal{H}}_{B_R}$ so that the above entropy on B_L can be computed by tracing out $\underset{\sim}{\mathcal{H}}_{B_R}$ to define a density matrix $\tilde{\rho}$ and then calculating $-\widetilde{Tr}\,(\tilde{\rho}\,\ln \tilde{\rho})$ using the trace $\widetilde{Tr}$ defined by summing diagonal matrix elements of operators over an orthonormal basis of $\underset{\sim}{\mathcal{H}}_{B_L}$. In this sense we have provided a Hilbert space interpretation of Ryu-Takayanagi entropy without assuming the existence of a holographic dual CFT.

At the technical level, we showed that the non-diagonal sectors admit left and right von Neumann algebras, each of which can be obtained by appying central projections to the corresponding diagonal von Neumann algebras on $\mathcal{H}_{B_L \sqcup B_L}$ and $\mathcal{H}_{B_R \sqcup B_R}$. This property ultimately followed from our use of rimmed source-manifolds, which can always be well-approximated by the product of a cylinder ($B_L \times [0,1]$ and/or $B_R \times [0,1]$) with another manifold as illustrated for the source-manifold ψ in figure 4. An immediate implication of the above result is that both the left and right von Neumann algebras are type-I. We also showed the left and right von Neumann algebras on any non-diagonal sector to be commutants on $\mathcal{H}_{B_L \sqcup B_R}$. As in [1], we did not need to assume any of the various Hilbert spaces to be separable, though in realistic models one might expect that to be the case.

Natural directions for further research include providing a corresponding Lorentz-signature analysis and/or studying the effect of dropping the reality condition from the list of axioms. Such reality conditions imply that the theory is invariant under time-reversal which, based on analogy with quantum field theory, one does not expect to hold in general. Since the arguments both here and in [1] generally involved only the operation $\star$ (as opposed to separately using either the transpose operation ^{t} or the complex-conjugation operator *), we suspect that it will be straightforward to drop this axiom from the list of requirements. However, this remains to be checked in detail.

It is perhaps more important to emphasize that our work focused on bipartitions of the boundary, meaning that the boundary was written as a disjoint union of precisely *two* pieces $B_L \sqcup B_R$. It thus remains to study more complicated partitions of the boundary. In particular, for boundaries of the form $B_1 \sqcup B_2 \sqcup B_3$, we would not necessarily expect $\mathcal{H}_{B_1\sqcup B_2\sqcup B_3}$ to embed in a useful way into the tensor product $\underset{\sim}{\mathcal{H}}_{B_1} \otimes \underset{\sim}{\mathcal{H}}_{B_2} \otimes \underset{\sim}{\mathcal{H}}_{B_3}$ of our Hilbert spaces $\underset{\sim}{\mathcal{H}}_{B_1}$, $\underset{\sim}{\mathcal{H}}_{B_2}$, $\underset{\sim}{\mathcal{H}}_{B_3}$. It would be very interesting to understand if our definitions of these spaces can be modified in a way that makes the above property hold. Establishing that this is the case would bring us one step closer to showing that AdS/CFT-like results hold in a generic theory of gravity satisfying the axioms of [1].

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A The off-diagonal sector is a Hilbert semi-birigged space that satisfies the coupling condition

This appendix provides the details associated with the use of theorem 1.3 from [24] for Hilbert semi-birigged spaces as required for the arguments of section 5. In particular, it was stated in section 5 that, when equipped with representations $C := \hat{A}_L^{LR}$ and $D := \hat{A}_R^{LR}$, the off-diagonal dense subspace $X := \mathcal{D}_{LR}$ satisfies the four axioms of Hilbert-semi-birigged spaces found in [24], and that it also satisfies the so-called coupling condition from that reference. We explain these properties briefly below.

The axioms introduced in [24] for a Hilbert semi-birigged space require the existence of two sesquilinear forms on X that we call $(,)_C$ and $(,)_D$, and which take values in C and D as indicated. In our case, we take these to be given by the left and right gluing of any of their representative surfaces (or linear superpositions thereof) with one surface involuted according to

$$\begin{aligned} (|x\rangle, |y\rangle)_C &= [xy^\star]_L = \hat{x}_L \hat{y}_L^\dagger = \Psi_L(|x\rangle)\,\Psi_L(|y\rangle)^\dagger \quad \text{and} \\ (|x\rangle, |y\rangle)_D &= [x^\star y]_R = \hat{y}_R \hat{x}_R^\dagger = \Psi_R(|y\rangle)\,\Psi_R(|x\rangle)^\dagger\,. \end{aligned} \tag{A.1}$$

There are then 6 axioms to check:

1. C and D are faithfully represented on X.

2. $(x,y)_C z = x(y,z)_D$ for all $x,y,z \in X$.

3. If $c \in C$ is of the form $(x,y)_C$, then so is $c^\dagger$.

4. The linear span of the objects of the form $(x,y)_C\, z$ is dense in X.

5. $(x,y)_C = (y,x)_C$ and $(x,y)_D = (y,x)_D$ for all $x,y \in X$.

6. For any $x \in X$, both $(x,x)_C$ and $(x,x)_D$ act as non-negative operators on X.

Five of these axioms are essentially trivial to verify. Axiom 1 was established in section 3. Axioms 2 and 5 follow from short computations using our definitions (A.1). Axioms 3 and 6 are also manifest from (A.1).

This leaves only axiom 4. To verify this remaining axiom, we need only show[13] that there is no state $|\xi\rangle \in \mathcal{H}_{LR}$ that is orthogonal to all states of the form $(x,y)_C z$ for $x,y,z \in \mathcal{D}_{LR}$. To do so, we recall that $\mathcal{D}_{LR}$ is dense in $\mathcal{H}_{LR}$, so such a $|\xi\rangle$ would require there to be a sequence $\{|\xi_n\rangle \in \mathcal{D}_{LR}\}$ that converges in norm to ξ. Failure of axiom 4 would in particular require $|\xi\rangle$ to be orthogonal to all states of the form $(|\xi_n\rangle, |\xi_m\rangle)_C\, |\xi_k\rangle$, so that

$$0 = \langle\xi|\, (|\xi_n\rangle, |\xi_m\rangle)_C\, |\xi_k\rangle = \langle\xi|\Psi_L(|\xi_n\rangle)\, [\Psi_L(|\xi_m\rangle)]^\dagger\, |\xi_k\rangle \quad \text{for all } n,m,k, \tag{A.2}$$

where in the last step we have used (A.1) and the analogue of (3.7) for the left map Ψ_L. But we can use the weak-operator-topology continuity of the map Ψ_L (the left version of Claim 4.3) and of the adjoint map $\dagger$ to take the limits $n,m,k \to \infty$ (in any order) of the matrix elements (A.2) and obtain

$$\begin{aligned}
0 &= \langle\xi|\Psi_L(|\xi\rangle)\, [\Psi_L(|\xi\rangle)]^\dagger\, |\xi\rangle = \lim_{\beta\downarrow 0}\langle\xi|\Psi_L(|\xi\rangle)\widehat{C_{2\beta}}_R\, [\Psi_L(|\xi\rangle)]^\dagger\, |\xi\rangle \\
&= \lim_{\beta\downarrow 0}\langle\xi|\Phi^R_{LR}(\widehat{C_\beta}_R)\Psi_L(|\xi\rangle)\, [\Psi_L(|\xi\rangle)]^\dagger\, \Phi^R_{LR}(\widehat{C_\beta}_R)|\xi\rangle \\
&= \lim_{\beta\downarrow 0}\langle C_\beta|\, [\Psi_L(|\xi\rangle)]^\dagger\, \Psi_L(|\xi\rangle)\, [\Psi_L(|\xi\rangle)]^\dagger\, \Psi_L(|\xi\rangle)|C_\beta\rangle \\
&= tr(\{[\Psi_L(|\xi\rangle)]^\dagger\, [\Psi_L(|\xi\rangle)]\}^2),
\end{aligned} \tag{A.3}$$

where the 2nd equality follows by inserting the identity operator in the form $\lim_{\beta\downarrow 0}\widehat{C_{2\beta}}_R$, the 3rd uses the intertwining relation (4.10), the 4th uses (4.7) twice, and the final step then follows from the definition (2.3) of the trace on positive elements of the von Neumann algebra $\mathcal{A}^{LL}_L$.

Now, as in the proof of Claim (5.1), since $0 \neq \langle\xi|\xi\rangle = tr\left([\Psi_L(|\xi\rangle)]^\dagger]\Psi_L(|\xi\rangle)\right)$, the operator $[\Psi_L(|\xi\rangle)]^\dagger]\Psi_L(|\xi\rangle)$ cannot vanish. And since this operator is self-adjoint, its square must also be non-zero. Faithfulness of the trace (as derived in [1]) then requires that (A.3) be non-zero, contradicting (A.3). We thus see that states of the form $(x,y)_C z$ are dense in $\mathcal{D}_{LR}$, and thus that full set of axioms is satisfied.

In order to use theorem 1.3 from [24], we will also need to verify the so-called coupling condition. This condition states that if $m,n \in X$ and $x,y \in X$, and if

$$\langle m(x,z)_D, w\rangle = \langle z, n(y,w)_D\rangle \text{ for all } z,w \in X, \tag{A.4}$$

then for any fixed $z,w \in X$ there is a net $\{c_\alpha\}$ of elements of C such that

$$\begin{aligned}
c^\dagger_\alpha z &\to m(x,z)_D, \\
c_\alpha w &\to n(y,w)_D.
\end{aligned} \tag{A.5}$$

[13]This argument was inspired by the related proof of proposition I.9.2 in [29].

To see that the coupling condition holds in our context, let us use equation (3.6) to rewrite the two sides of equation (A.4) in the form:

$$\begin{aligned} tr\left((m(x,z)_D)^\star w\right) &= tr\left((x,z)_D^\star m^\star w\right) = tr((x^\star z)^\star m^\star w) \\ &= tr(z^\star x m^\star w) = \langle z|\hat{x}_L\hat{m}_L^\dagger|w\rangle \quad \text{and} \\ tr(z^\star n\,\langle y,w\rangle_D) &= tr(z^\star n y^\star w) = \langle z|\hat{n}_L\hat{y}_L^\dagger|w\rangle\,. \end{aligned} \tag{A.6}$$

Thus we have

$$\langle z|\hat{x}_L\hat{m}_L^\dagger|w\rangle = \langle z|\hat{n}_L\hat{y}_L^\dagger|w\rangle\,. \tag{A.7}$$

Since equation (A.7) holds for all $z, w \in \mathcal{D}_{LR}$, we must have $\hat{x}_L\hat{m}_L^\dagger = \hat{n}_L\hat{y}_L^\dagger$. In particular, both sides are elements of $C = \hat{A}_L^{LR}$. We are therefore free to take $\{c_\alpha\}$ to be the constant (α-independent) net $c_\alpha = \hat{x}_L\hat{m}_L^\dagger = \hat{n}_L\hat{y}_L^\dagger$, for which we trivially compute the limits

$$\begin{aligned} c_\alpha^\dagger z &\to \hat{m}_L\hat{x}_L^\dagger z, \\ c_\alpha w &\to \hat{n}_L\hat{y}_L^\dagger w. \end{aligned} \tag{A.8}$$

It is then straightforward to see the result (A.8) agrees with the condition (A.5) by writing

$$\begin{aligned} m(x,z)_D &= \hat{z}_R\hat{x}_R^\dagger m = |m x^\star z\rangle = \hat{m}_L\hat{x}_L^\dagger z \\ n(y,w)_D &= \hat{w}_R\hat{y}_R^\dagger n = |n y^\star w\rangle = \hat{n}_L\hat{y}_L^\dagger w, \end{aligned} \tag{A.9}$$

where the 3rd expression in each line has been written by choosing representatives of each object in $\underline{Y}^d_{B_L\sqcup B_R}$.

B A generalization of Ψ_R

As described in section 5.2, the construction of our map $\Psi_R : \mathcal{H}_{LR} \to \mathcal{B}(\mathcal{H}_{LL}, \mathcal{H}_{LR})$ generalizes readily to define a map $\Psi_R^{B,B_1\to B_2} : \mathcal{H}_{B_1\sqcup B_2} \to \mathcal{B}(\mathcal{H}_{B\sqcup B_1}, \mathcal{H}_{B\sqcup B_2})$ involving arbitrary Hilbert spaces B, B_1, B_2. The original map Ψ_R is then the special case of $\Psi_R^{B,B_1\to B_2}$ for which $B = B_1 = B_L, B_2 = B_R$. The more general map $\Psi_R^{B,B_1\to B_2}$ naturally satisfies properties analogous to those of the original Ψ_R. The purpose of this appendix is to state those properties explicitly and to describe the corresponding proofs. Our treatment below will be brief since we have already provided detailed arguments for Ψ_R in the main text.

We formalize the main result of this appendix as follows:

Claim B.1. *There is a map $\Psi_R^{B,B_1\to B_2} : \mathcal{H}_{B_1\sqcup B_2} \to \mathcal{B}(\mathcal{H}_{B\sqcup B_1}, \mathcal{H}_{B\sqcup B_2})$ that is continuous w.r.t. the norm topology on $\mathcal{H}_{B_1\sqcup B_2}$ and the strong operator topology on $\mathcal{B}(\mathcal{H}_{B\sqcup B_1}, \mathcal{H}_{B\sqcup B_2})$ which, if we define $\hat{a}_R := \Psi_R^{B,B_1\to B_2}(|a\rangle)$, for $|a\rangle \in \mathcal{D}_{B_1\sqcup B_2}$ and $|b\rangle \in \mathcal{D}_{B\sqcup B_1}$ satisfies*

$$\hat{a}_R|b\rangle = |ba\rangle \tag{B.1}$$

for any representatives $a \in H_{B_1\sqcup B_2}, b \in H_{B\sqcup B_1}$ of the equivalence classes defined by $|a\rangle, |b\rangle$. The map $\Psi_R^{B,B_1\to B_2}$ is then uniquely defined by (B.1) and the above continuity requirements. Furthermore, it satisfies the intertwining relation

$$\Phi^L_{BB_2}(d)\hat{a}_R = \hat{a}_R\Phi^L_{BB_1}(d) \tag{B.2}$$

for all $d \in \mathcal{A}_L^{B\sqcup B}$. Here $\Phi^L_{BB_2}, \Phi^L_{BB_1}$ are just Φ^L_{LR} for $B_L = B$ and $B_R = B_2, B_1$.

There is also a corresponding left-map $\Psi_L^{B_2\to B_1,B} : \mathcal{H}_{B_1\sqcup B_2} \to \mathcal{B}(\mathcal{H}_{B_2\sqcup B}, \mathcal{H}_{B_1\sqcup B})$.

Proof. The proof of this claim is directly analogous to our arguments for the corresponding properties of the original $\Psi_R : \mathcal{H}_{LR} \to \mathcal{B}(\mathcal{H}_{LL}, \mathcal{H}_{LR})$. The key point is that, because Ψ_R defines an operator in $\mathcal{B}(\mathcal{H}_{LL}, \mathcal{H}_{LR})$ that acts at the *right* boundary of $\mathcal{H}_{LL}$, the left boundary of this $\mathcal{H}_{LL}$ plays no role in the arugments and may be replaced with an arbitrary boundary B. In particular, using footnote 7 (with a replaced by $a^\star$) and the corresponding version of the trace inequality, for all $a \in H_{B_1 \sqcup B_2}, b \in H_{B \sqcup B_1}$ we have

$$\langle ba|ba\rangle \leq \langle a|a\rangle\langle b|b\rangle, \tag{B.3}$$

so that (B.1) requires that $\hat{a}_R = 0$ when $|a\rangle$ is a null state and, in addition, for all states $|a\rangle$ (B.1) implies that $\hat{a}_R$ annihilates all null states in $\mathcal{H}_{B \sqcup B_1}$. The condition (B.1) is thus well-defined for $|a\rangle \in \mathcal{D}_{B_1 \sqcup B_2}$ and $|b\rangle \in \mathcal{D}_{B_1 \sqcup B_2}$. Furthermore, the operator norm of $\Psi_R^{B, B_1 \to B_2}(|a\rangle)$ is bounded by $\langle a|a\rangle$ and so admits a unique continuous extension to all $|b\rangle \in \mathcal{H}_{B \sqcup B_1}$.

Continuity in the argument $|a\rangle$ of $\Psi_R^{B, B_1 \to B_2}$ then follows as in the proof of Claim 4.3, implying uniqueness of the corresponding extension to all $|a\rangle \in \mathcal{H}_{B_1 \sqcup B_2}$. The intertwining relation (B.2) can then be derived by noting that for $a \in H_{B_1 \sqcup B_2}, d \in H_{B \sqcup B}$, and $\phi \in H_{B \sqcup B_1}$ we have $\hat{d} \in \hat{A}_L^{B \sqcup B}$ and

$$\Phi^L_{BB_2}(\hat{d})\hat{a}_R|\phi\rangle = |d\psi a\rangle = \hat{a}_R \Phi^L_{BB_1}(\hat{d})|\phi\rangle. \tag{B.4}$$

Since $\Phi_{BB_2}(\hat{d})$, $\Phi_{BB_1}(\hat{d})$, and $\hat{a}_R$ are bounded operators, they are continuous. The left and right sides of (B.4) must thus in fact agree for all $|\phi\rangle \in \mathcal{H}_{B \sqcup B_1}$. Continuity of the maps $\Phi_{BB_1}, \Phi_{BB_2}, \Psi_R^{B, B_1 \to B_2}$ with respect their arguments then similarly requires the left and right sides of (B.4) to agree for all $a \in \mathcal{H}_{B_1 \sqcup B_2}, d \in \mathcal{H}_{B \sqcup B}$. This yields the desired intertwining relation (B.2). Note that this simple argument was not available when Claim 4.4 was originally proven in section 4.2 since continuity of Φ_{LR} had not yet been established. □

We will also use this appendix to state a small additional observation. Before doing so, let us first recall the operation $\star$ defined on surfaces in any $Y^d_{B_1 \sqcup B_2}$, where for $a \in Y^d_{B_1 \sqcup B_2}$ we have $a^\star \in Y^d_{B_2 \sqcup B_1}$. Note that $|a\rangle$ and $|a^\star\rangle$ have the same norm. Thus $\star$ is an anti-linear continuous map that extends to all states $|a\rangle \in \mathcal{H}_{B_1 \sqcup B_2}$. The output of this map will be denoted $|a^\star\rangle \in \mathcal{H}_{B_2 \sqcup B_1}$. This leads to the following result.

Claim B.2. *The maps* $\Psi_R^{B, B_1 \to B_2} : \mathcal{H}_{B_1 \sqcup B_2} \to \mathcal{B}(\mathcal{H}_{B \sqcup B_1}, \mathcal{H}_{B \sqcup B_2})$ *and* $\Psi_R^{B, B_2 \to B_1} : \mathcal{H}_{B_2 \sqcup B_1} \to \mathcal{B}(\mathcal{H}_{B \sqcup B_2}, \mathcal{H}_{B \sqcup B_1})$ *intertwine the above anti-linear map* $\star : \mathcal{H}_{B_1 \sqcup B_2} \to \mathcal{H}_{B_2 \sqcup B_1}$ *with the adjoint operation on* $B(\mathcal{H}_{B \sqcup B_1}, \mathcal{H}_{B \sqcup B_2})$*. In other words, for* $|a\rangle \in \mathcal{H}_{B_1 \sqcup B_2}$ *and defining* $\hat{a}_R := \Psi_R^{B, B_1 \to B_2}(|a\rangle)$, $\widehat{(a^\star)}_R := \Psi_R^{B, B_2 \to B_1}(|a^\star\rangle)$, *we have*

$$\widehat{(a^\star)}_R := \hat{a}_R^\dagger. \tag{B.5}$$

Proof. For $a \in \mathcal{D}_{B_1 B_2}$ this is just (3.7). Continuity of $\star$, $\Psi_R^{B, B_1 \to B_2}$, and $\Psi_R^{B, B_2 \to B_1}$ then imply the result for all $|a\rangle$. □

C Relating the left and right diagonal von Neumann algebras

Recall that our von Neumann algebras $\mathcal{A}_R^{B \sqcup B}$, $\mathcal{A}_L^{B \sqcup B}$ were defined by completing simpler algebras $\hat{A}_R^{B \sqcup B}$, $\hat{A}_L^{B \sqcup B}$ that were defined by concatenation of surfaces. The identification with surfaces then gives a natural map $\hat{\mathcal{L}}$ from $\hat{A}_R^{B\sqcup B}$ to $\hat{A}_L^{B\sqcup B}$. In particular, any $\hat{a}_R \in \hat{A}_R^{B\sqcup B}$ is defined to act via the right-gluing of some $a \in \underline{Y}^d_{B\sqcup B}$ in the sense that, for $|\phi\rangle \in \mathcal{D}_{B\sqcup B}$, we have

$$\hat{a}_R|\phi\rangle = |\phi a\rangle. \tag{C.1}$$

We may then define $\hat{\mathcal{L}}(\hat{a}_R) := \hat{a}_L$, where

$$\hat{a}_L|\kappa\rangle = |a\kappa\rangle \tag{C.2}$$

for $|\kappa\rangle \in \mathcal{D}_{B\sqcup B}$. For $|\phi\rangle, |\kappa\rangle \in \mathcal{D}_{B\sqcup B}$, we find the relation

$$\langle\phi^\star|\hat{a}_L|\kappa^\star\rangle = tr(\phi a \kappa^\star) = tr(\kappa^\star \phi a) = \langle\kappa|\hat{a}_R|\phi\rangle, \tag{C.3}$$

where we recall that the trace tr is defined on the entire space $\underline{Y}^d_{B\sqcup B}$ and not just on the manifestly positive such elements. We also find the relation

$$\hat{\kappa}_R\hat{a}_R|\phi\rangle = |\phi a \kappa\rangle = \hat{\phi}_L\hat{a}_L|\kappa\rangle. \tag{C.4}$$

We will use the relation (C.3) to extend $\hat{\mathcal{L}}$ to a map $\mathcal{L}$ defined on all of $\mathcal{A}_R^{B\sqcup B}$. In particular, for $a_R \in \mathcal{A}_R^{B\sqcup B}$ we define $a_L := \mathcal{L}(a_R)$ to be the operator in $\mathcal{A}_R^{B\sqcup B}$ whose matrix elements satisfy

$$\langle\phi|a_L|\kappa\rangle = \langle\kappa^\star|a_R|\phi^\star\rangle \tag{C.5}$$

for all $|\phi\rangle, |\kappa\rangle \in \mathcal{H}_{B\sqcup B}$.

To see that a_L is a bounded operator on $\mathcal{H}_{B\sqcup B}$, let us introduce an orthonormal basis $|i\rangle$ and write

$$\begin{aligned} |a_L|\kappa\rangle|^2 &= \sum_i |\langle i|a_L|\kappa\rangle|^2 \\ &= \sum_i |\kappa^\star|a_R|i^\star\rangle|^2 \\ &= |a_R|\kappa^\star\rangle|^2 \le ||a_R||^2\langle\kappa|\kappa\rangle. \end{aligned} \tag{C.6}$$

Furthermore, to see that $a_L \in \mathcal{A}_L^{B\sqcup B}$, let us write a_R as the weak-operator-topology limit of a net $\{\widehat{(a_\alpha)}_R\} \subset \hat{A}_R^{B\sqcup B}$. We then simply note that for $\widehat{(a_\alpha)}_L := \mathcal{L}\left(\widehat{(a_\alpha)}_R\right)$, the net $\{\widehat{(a_\alpha)}_L\} \subset \hat{A}_L^{B\sqcup B}$ converges in the weak operator topology to a_L.

We now wish to show the following:

Claim C.1. *For any boundary B, any $a_R \in \mathcal{A}_R^{B\sqcup B}$, and any states $|\phi\rangle, |\kappa\rangle \in \mathcal{H}_{B\sqcup B}$ the operator $a_L := \mathcal{L}(a_R)$ satisfies*

$$\hat{\kappa}_R a_R|\phi\rangle = |\phi a \kappa\rangle = \hat{\phi}_L a_L|\kappa\rangle. \tag{C.7}$$

Proof. To see this, we again consider a net $\{\widehat{(a_\alpha)}_R\} \subset \hat{A}_R^{B\sqcup B}$ that converges to a_R in the weak operator topology and the associated net $\{\widehat{(a_\alpha)}_L\} \subset \hat{A}_L^{B\sqcup B}$ that converges to a_L. Using relation (C.4) for each α and taking the inner product with a fixed state $|\gamma\rangle \in \mathcal{H}_{B\sqcup B}$ gives

$$\langle\gamma|\hat{\kappa}_R\widehat{(a_\alpha)}_R|\phi\rangle = \langle\gamma|\hat{\phi}_L\widehat{(a_\alpha)}_R|\kappa\rangle. \tag{C.8}$$

Taking limits then gives

$$\langle\gamma|\hat{\kappa}_R a_R|\phi\rangle = \langle\gamma|\phi a\kappa\rangle = \hat{\phi}_L a_L|\kappa\rangle \tag{C.9}$$

for all $|\gamma\rangle$, which is equivalent to (C.7). $\square$

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