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General Section

Linear operators, the Hurwitz zeta function and Dirichlet L-functions

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ABSTRACT

At the 1900 International Congress of Mathematicians, Hilbert claimed that the Riemann zeta function is not the solution of any algebraic ordinary differential equation its region of analyticity [5]. In 2015, Van Gorder addresses the question of whether the Riemann zeta function satisfies a *non*-algebraic differential equation and constructs a differential equation of infinite order which zeta satisfies [7]. However, as he notes in the paper, this representation is formal and Van Gorder does not attempt to claim a region or type of convergence. In this paper, we show that Van Gorder's operator applied to the zeta function does not converge pointwise at any point in the complex plane. We also investigate the accuracy of truncations of Van Gorder's operator applied to the zeta function and show that a similar operator applied to zeta and other L-functions does converge.

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1. Introduction

In Hilbert's 1900 address at the International Congress of Mathematicians, he claimed that the Riemann zeta function is not the solution of any algebraic ordinary differential equation on its region of analyticity [5]. In [9], Van Gorder addresses the question of whether the Riemann zeta function satisfies a *non*-algebraic differential equation. As Van Gorder notes in the introduction of [9], it could be the case that $\zeta(z)$ satisfies a nonlinear differential equation or that it satisfies a linear differential equation of infinite order.[1] In [9], Van Gorder constructs a differential equation of infinite order that the Riemann zeta function satisfies [7]. However, as he notes in the paper, this representation is clearly formal and Van Gorder does not attempt to claim a region or type of convergence.[2]

In what follows we will examine the region of convergence for the differential equation in question. We will also extend the formal identity appearing in Van Gorder's work to see that the Hurwitz zeta function satisfies a similar differential equation.

In Section 2.1 we will begin with a brief overview of the differential operator introduced by Van Gorder. In Sections 2.2 and 2.3 we will extend Van Gorder's main results to show that the Hurwitz zeta function formally satisfies a similar infinite order differential equation to the one in [9]. These results subsume those of Van Gorder. In Section 2.4 we will address the issue of where such equations converge. We will show that, in fact, the differential equation under investigation in [9] diverges everywhere.

We will see through the course of Section 2 that the formal arguments given by Van Gorder rely on a non-global characterization of his operator T that only holds away from poles of the function on which it is being applied. If we define a new operator G in terms of this characterization globally, we can guarantee convergence. However, this new operator G is not a *differential* operator and furthermore does not converge to Van Gorder's operator T. We will investigate this new operator G in Section 3.

In Section 3.2 we will extend Van Gorder's argument to yield an operator equation involving G applied to Dirichlet L-functions. We will see that the inverse T^{-1} that Van Gorder presents in [9] actually yields G^{-1}. In Section 4 we make precise Van Gorder's claim that this inverse operator has a connection to the Bernoulli numbers, and in Section 4.1 we will use this connection to give identities for the Hurwitz zeta function and Dirichlet L-function and discuss the convergence of G^{-1}.

In Section 5, we examine the truncated version of the operator T. Though T does not converge when applied to ζ, it is possible that some truncation of T applied to ζ will provide a good approximation of Van Gorder's differential equation.

[1] In fact, in [4], Gauthier and Tarkhanov show that $\zeta(s)$ does satisfy an inhomogeneous linear differential equation. However, this equation is not algebraic.

[2] Though he does allude to some important things to be considered in Section 2 of [9].

2. Van Gorder's operator applied to the Hurwitz zeta function

2.1. Van Gorder's operator

The differential operator defined by Van Gorder in [9] is given by:

$$T = \sum_{n=0}^{\infty} L_n \tag{1}$$

where

$$L_n := p_n(s)\exp(nD)$$
$$p_n(s) := \begin{cases} 1 & \text{if } n = 0 \\ \frac{1}{(n+1)!}\prod_{j=0}^{n-1}(s+j) & \text{if } n > 0 \end{cases}$$
$$\exp(nD) := id + \sum_{k=1}^{\infty} \frac{n^k}{k!} D_s^k$$

for $D_s^k := \frac{\partial^k}{\partial s^k}$. For an overview of infinite order differential equations see Charmichael's [3] and for more recent applications involving infinite order differential equations with initial conditions see [2].

Van Gorder notes that $\exp(nD)$ acts as a shift operator for meromorphic functions in the sense that $\exp(nD)u(s) = u(s+n)$ *sufficiently far away from poles*. However, he does not attempt to answer the question of precisely what is "sufficiently far away from poles" but instead references Ritt's [6]. As we will see in Section 2.4.1, the operator that Ritt considers, $\exp(D)$, (though of infinite order) is simpler than Van Gorder's $T = \sum_{n=0}^{\infty} p_n(s)\exp(nD)$. Thus more work is necessary to address the convergence of T than is done by Ritt [6].

In [9], Van Gorder proves that

$$T[\zeta(s) - 1] = \frac{1}{s-1} \tag{2}$$

formally. The crux of the proof relies upon the characterization of $\exp(nD)$ as the "shift operator". In the following two sections, we prove that the Hurwitz zeta function satisfies a similar equation

$$T\left[\zeta(s,a) - \frac{1}{a^s}\right] = \frac{1}{(s-1)a^{s-1}}. \tag{3}$$

Our argument is akin to that of Van Gorder's.

It is important to note that in the proof of Corollary 4 we are assuming that $\exp(nD)u(s) = u(s+n)$ when claiming

$$L_n \left[\zeta(s,a) - \frac{1}{a^s} \right] = p_n(s) \left(\zeta(s+n,a) - \frac{1}{a^{s+n}} \right) \tag{4}$$

This assumption is also made at a similar place in [9]. However since this is only true "sufficiently far away from the poles" of u, this leads to the natural question of *where* (2) and (3) hold. We will begin to address this question by examining the convergence of the differential operator T in Section 2.4.

2.2. A useful identity for the Hurwitz zeta function

In order to show that ζ formally solves the differential equation (2), Van Gorder uses the following identity

$$\zeta(s) = \frac{s}{s-1} - \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!} (\zeta(s+n) - 1) \tag{5}$$

which can be found in [1] and [8]. Following the argument of Titchmarsh [8], we need to generalize the identity to the Hurwitz zeta function.

Lemma 1. *Let $\zeta(s,a)$ be the Hurwitz zeta function. Then, for $Re(s) > 2$ and $0 < a \leq 1$, we have that*

$$\zeta(s,a) - \frac{1}{(s-1)a^{s-1}} = \frac{1}{a^s} - \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!} \left(\zeta(s+n,a) - \frac{1}{a^{s+n}} \right) \tag{6}$$

Proof. Let $s \in \mathbb{C}$ satisfy $\mathrm{Re}(s) > 2$. We can then write the Hurwitz zeta as a series and it suffices to show that the series

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!} \left(\frac{1}{(k+a)^{s+n}} \right) \tag{7}$$

converges absolutely pointwise to $\frac{1}{(s-1)a^{s-1}} + \frac{1}{a^s} - \zeta(s,a)$, since this will mean that we can interchange the order of summation by Fubini's Theorem. To see why we get such result, observe that from geometric series we have that for an integer $k \geq 0$,

$$\begin{aligned} \left(\frac{k+a}{k+a-1} \right)^{s-1} &= \left(\frac{1}{1 - \frac{1}{k+a}} \right)^{s-1} \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=0}^{n-1}(s-1+j)}{n!} \frac{1}{(k+a)^n} \\ &= \sum_{n=0}^{\infty} \frac{\prod_{j=-1}^{n-2}(s+j)}{n!} \frac{1}{(k+a)^n} \end{aligned} \tag{8}$$

where the last series is absolutely convergent since by the triangle inequality and geometric series, we have

$$\sum_{n=0}^{\infty}\left|\frac{\prod_{j=-1}^{n-2}(s+j)}{n!}\frac{1}{(k+a)^n}\right| \leq \sum_{n=0}^{\infty}\frac{\prod_{j=-1}^{n-2}(|s|+j)}{n!}\frac{1}{(k+a)^n} = \left(\frac{k+a}{k+a-1}\right)^{|s|-1}$$

By absolute convergence, $\frac{1}{(k+a)^{s-1}(s-1)}\sum_{n=2}^{\infty}\frac{\prod_{j=-1}^{n-2}(s+j)}{n!}\frac{1}{(k+a)^n}$ is precisely the kth term of the left summation of (7). But this is the same as the expression $\frac{1}{(k+a)^{s-1}(s-1)} \times \left[\left(\frac{k+a}{k+a-1}\right)^{s-1} - 1 - \frac{s-1}{k+a}\right]$ and we have that

$$\begin{aligned}\frac{1}{(s-1)a^{s-1}} + \frac{1}{a^s} - \zeta(s,a) &= \frac{1}{(s-1)a^{s-1}} - \sum_{k=1}^{\infty}\frac{1}{(k+a)^s}\\ &= \frac{1}{s-1}\sum_{k=0}^{\infty}\frac{1}{(k+a)^{s-1}} - \sum_{k=1}^{\infty}\frac{1}{(k+a)^s} - \frac{1}{s-1}\sum_{k=1}^{\infty}\frac{1}{(k+a)^{s-1}}\end{aligned}$$

Since these three series converge absolutely, we can re-index the leftmost series and get that this is equal to an absolutely convergent series given by

$$\begin{aligned}&\sum_{k=1}^{\infty}\left(\frac{1}{(s-1)(k-1+a)^{s-1}} - \frac{1}{(s-1)(k+a)^{s-1}} - \frac{1}{(k+a)^s}\right)\\ &= \sum_{k=1}^{\infty}\frac{1}{s-1}\cdot\frac{1}{(k+a)^{s-1}}\left[\left(\frac{k+a}{k+a-1}\right)^{s-1} - 1 - \frac{s-1}{k+a}\right]\end{aligned}$$

Which is an absolutely convergent series and gives the desired result. □

In a more elegant way, we can express this identity in terms of the Γ function using the fact that $\frac{\Gamma(s+n)}{\Gamma(s)} = \prod_{j=0}^{n-1}(s+j)$ for $s \in \mathbb{C}, n \in \mathbb{N}$. Equation (1) then becomes:

$$\zeta(s,a) - \frac{1}{(s-1)a^{s-1}} = \frac{1}{a^s} - \sum_{n=1}^{\infty}\frac{\Gamma(s+n)}{(n+1)!\Gamma(s)}\left(\zeta(s+n,a) - \frac{1}{a^{s+n}}\right) \tag{9}$$

We now show that this identity holds for all $s \in \mathbb{C}$.

Lemma 2. *The right-hand side of equation (6) in Lemma 1 converges absolutely for all $s \in \mathbb{C}$.*

Proof. We first need to treat a delicate point. That is, equation (6) is well-defined when $s + n = 1$ for some integer $n \geq 0$. Namely, for such n, it makes sense to have the nth term of the sum be $\frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!}\left(\zeta(s+n,a) - \frac{1}{a^{s+n}}\right)$. Since $(s+n-1)\zeta(s+n,a)$ cancels the pole of the Hurwitz ζ at $s+n$ and $s+n-1$ appears as the last term in $\prod_{j=0}^{n-1}(s+j)$, the series is well-defined.

To show convergence, let $s \in \mathbb{C}$ and let $N > 0$ be an integer so that $\operatorname{Re}(s+N) > 1$. It suffices to show that

$$\sum_{n=N}^{\infty} \frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!} \left(\zeta(s+n,a) - \frac{1}{a^{s+n}}\right)$$

converges absolutely. First, we bound $\left|\zeta(s+n,a) - \frac{1}{a^{s+n}}\right|$. Since, $n \geq N$, we have that $\operatorname{Re}(s+n) \geq \operatorname{Re}(s+N) > 1$. By the triangle inequality and by the integral inequality for non-negative series, we can write

$$\begin{aligned}
\left|\zeta(s+n,a) - \frac{1}{a^{s+n}}\right| &= \left|\sum_{k=1}^{\infty} \frac{1}{(k+a)^{s+n}}\right| \\
&\leq \sum_{k=1}^{\infty} \left|\frac{1}{(k+a)^{s+n}}\right| \\
&= \sum_{k=1}^{\infty} \frac{1}{(k+a)^{\sigma+n}} \\
&\leq \frac{1}{(1+a)^{\sigma+n}} + \int_1^{\infty} \frac{1}{(x+a)^{\sigma+n}} dx \\
&= \frac{1}{(1+a)^{\sigma+n}} + \frac{1}{\sigma+n-1} \cdot \frac{1}{(1+a)^{\sigma+n}} \\
&= \frac{\sigma+n}{\sigma+n-1} \cdot \frac{1}{(1+a)^{\sigma+n}} \leq \frac{\sigma+n}{\sigma+n-1} \cdot \frac{1}{2^{\sigma+n}}
\end{aligned}$$

In addition, observe that $\left|\prod_{j=0}^{n-1}(s+j)\right| \leq \prod_{j=0}^{n-1}(|s|+j)$. We then have that

$$\sum_{n=N}^{\infty} \left|\frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!} \left(\zeta(s+n,a) - \frac{1}{a^{s+n}}\right)\right| \leq \sum_{n=N}^{\infty} \frac{\prod_{j=0}^{n-1}(|s|+j)}{(n+1)!} \cdot \frac{\sigma+n}{\sigma+n-1} \cdot \frac{1}{2^{\sigma+n}}$$

Now since $\sum_{n=N}^{\infty} \frac{\prod_{j=0}^{n-1}(|s|+j)}{(n+1)!} \cdot \frac{1}{2^{|s|+n}}$ converges and

$$\lim_{n\to\infty} \frac{2^{|s|+n}(\sigma+n)}{2^{\sigma+n}(\sigma+n-1)} = 2^{|s|-\sigma}$$

by the Limit Comparison test, $\sum_{n=N}^{\infty} \frac{\prod_{j=0}^{n-1}(|s|+j)}{(n+1)!} \cdot \frac{\sigma+n}{\sigma+n-1} \cdot \frac{1}{2^{\sigma+n}}$ must also converge. Thus, $\sum_{n=N}^{\infty} \frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!} \left(\zeta(s+n,a) - \frac{1}{a^{s+n}}\right)$ converges absolutely. $\square$

Since our series converges absolutely for all $s \in \mathbb{C}$, we must have that our identity in Lemma 1 actually holds for all $s \in \mathbb{C} \setminus \{1\}$. Thus, we have the following corollary.

Corollary 3. *For all $s \in \mathbb{C} \setminus \{1\}$, we have the following identity*

$$\zeta(s,a) = \frac{1}{(s-1)a^{s-1}} + \frac{1}{a^s} - \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!} \left(\zeta(s+n,a) - \frac{1}{a^{s+n}}\right) \quad \square$$

Following Van Gorder's use of equation (5) in [9], we will use equation (6) to show that (3) holds formally.

2.3. The Hurwitz zeta function formally satisfies a differential equation

Now we will show that the Hurwitz zeta function formally satisfies the differential equation (3). This result is a generalization of Theorem 3.1 from [9].

Corollary 4. *Let T be as defined above. Then $\zeta(s,a)$ formally satisfies the differential equation*

$$T\left[\zeta(s,a) - \frac{1}{a^s}\right] = \frac{1}{(s-1)a^{s-1}}$$

for $s \in \mathbb{C}$ satisfying $s + n \neq 1$ for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. Using equation (4)

$$L_n\left[\zeta(s,a) - \frac{1}{a^s}\right] = p_n(s)\left(\zeta(s+n,a) - \frac{1}{a^{s+n}}\right)$$

we have

$$\begin{aligned} T\left[\zeta(s,a) - \frac{1}{a^s}\right] &= \sum_{n=0}^{\infty} L_n\left[\zeta(s,a) - \frac{1}{a^s}\right] \\ &= \sum_{n=0}^{\infty} p_n(s)\left(\zeta(s+n,a) - \frac{1}{a^{s+n}}\right) \\ &= \zeta(s,a) - \frac{1}{a^s} + \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!}\left(\zeta(s+n,a) - \frac{1}{a^{s+n}}\right) \\ &= \frac{1}{(s-1)a^{s-1}} \quad \square \end{aligned}$$

2.4. Convergence

In this section we will show that T applied to the Hurwitz zeta function does not converge. However, for certain analytic functions f, we see in Section 2.4.2 that Tf does converge.

2.4.1. Convergence of T when applied to the Hurwitz zeta-function

As Van Gorder notes on page 781 of [9], "we must exercise some caution when working with infinite order differential equations if we are concerned with convergence of the operators near poles of the functions being operated upon." As a basis for this concern the author alludes to [6] where Ritt establishes formally that $(\exp(D) - z)\Gamma(z) = 0$. Ritt notes that Γ does not satisfy this differential equation on all of $\mathbb{C}$ but away from the infinitely many poles of Γ. Of course, in the case of the Riemann zeta function and Hurwitz zeta function, we only have one pole to be concerned about. As Van Gorder states, the operator $\exp(D)$ is only valid "outside of a neighborhood of the pole at $z = 1$." Our goal is to investigate *which* neighborhood and examine its effect on the convergence of the operator T as it is applied to zeta functions. In what follows we will consider T applied to the Hurwitz zeta function since (when $a = 1$) it also covers the case of the Riemann zeta function.

Recall that in the proof of Corollary 4, our use of equation (4) relies upon the characterization of exp as the shift operator $\exp(nD)[\zeta(s,a)] = \zeta(s+n,a)$ for $s \in \mathbb{C}$ and $0 < a \leq 1$ away from the poles of ζ. This characterization comes from the Taylor series expansion for ζ; thus, we must consider the radius of convergence of the Taylor series when considering when this characterization holds.

The critical observation is that this operator is, *formally*, the Taylor series about a point $s \in \mathbb{C}$ evaluated at $z \in \mathbb{C}$. Namely, the formal Taylor series is

$$\zeta(z,a) - \frac{1}{a^z} = \sum_{k=0}^{\infty} \frac{D_s^k\left(\zeta(s,a) - \frac{1}{a^s}\right)}{k!}(z-s)^k \tag{10}$$

which, at the point $s+n$ for $n \in \mathbb{Z}_{\geq 0}$, will formally satisfy

$$\zeta(s+n,a) - \frac{1}{a^{s+n}} = \sum_{k=0}^{\infty} \frac{D_s^k\left(\zeta(s,a) - \frac{1}{a^s}\right)}{k!}n^k = \exp(nD)[\zeta(s,a)] \tag{11}$$

Since $\zeta(z,a)$ has a pole at $z = 1$, these series converge pointwise for $|z-s| < |s-1|$ and $|(s+n) - s| = |n| < |s-1|$.

We claim that this operator applied to the function $\zeta(s,a) - \frac{1}{a^s}$ does not converge pointwise *anywhere*. More explicitly for any $s \in \mathbb{C}$, the sequence of partial sums of the series $T\left[\zeta(s,a) - \frac{1}{a^s}\right]$ is not a well-defined sequence of complex numbers. This comes from the fact that, to have a well-defined a complex-valued series, we need, first, a sequence of complex numbers $(z_n)_{n=0}^{\infty}$ so we can define the sequence of partial sums

$S_N = \sum_{n=0}^{N} z_n$ which is, again, a sequence of complex numbers. Then, if the sequence of partial sums converges to a complex number S, we write $\sum_{n=0}^{\infty} z_n = S$. What we will show now is that, for any $s \in \mathbb{C}$, the definition of the operator T evaluated at $\zeta(s,a) - \frac{1}{a^s}$ fails this first step by failing to make a sequence of complex numbers.

Proposition 5. *For any $s \in \mathbb{C}$, we can find some $N \geq 0$ so that the series*

$$\exp(ND)\left[\zeta(s,a) - \frac{1}{a^s}\right] = \sum_{k=0}^{\infty} \frac{D_s^k\left(\zeta(s,a) - \frac{1}{a^s}\right)}{k!} N^k$$

diverges.

Proof. Let $s \in \mathbb{C}$. If $s = 1$, then, the term at $k = 0$ of series (10) evaluated at $z = 1$ is undefined, so the series is not well-defined. In this case, $N = 0$ satisfies our claim.

To complete our proof, let $s \neq 1$. By Taylor's Theorem, there is a radius of convergence $r \geq 0$ so that the series (10) converges absolutely when evaluated at $z \in \mathbb{C}$ satisfying $|z - s| < r$. In addition, it must diverge when evaluated at $z \in \mathbb{C}$ satisfying $|z - s| > r$. Now, since $\zeta(z, a)$ has a pole at $z = 1$, we have that series (10) cannot converge when evaluated $z = 1$. Thus, we must have that $|s - 1| \geq r$.

Let $N > 0$ satisfy $|s + N - s| = N > |s - 1| > r$. Then, we have that the series (35) evaluated at $z = s + N$

$$\zeta(s+N,a) - \frac{1}{a^s} = \sum_{k=0}^{\infty} \frac{D_s^k\left(\zeta(s,a) - \frac{1}{a^s}\right)}{k!} N^k = \exp(ND)\left[\zeta(s,a) - \frac{1}{a^s}\right]$$

must be divergent. So we have found the desired $N \geq 0$. □

Lemma 6. *For $s \in \mathbb{Z}_{\geq 0}$ and $n > 0$, $p_n(s) \neq 0$. Specifically, $p_n(s) \geq \frac{1}{(s-1)!(n+1)}$ for $s \in \mathbb{Z}_{>0}$ and $n > 0$.*

Proof. For $s = 0$, notice that for $n > 1$, $p_n(0) = \frac{(n-1)!}{(n+1)!} = \frac{1}{n(n+1)} \neq 0$. Now, let $s \in \mathbb{Z}_{>0}$. Observe that, for all $n > 0$, we have $p_n(s) = \frac{(s+n-1)!}{(s-1)!(n+1)!} \geq \frac{(1+n-1)!}{(s-1)!(n+1)!} = \frac{1}{(s-1)!(n+1)} \neq 0$. □

Lemma 7. *For $s \in \mathbb{Z}_{<0}$ and $n \geq 1 - s$, $p_n(s) = 0$.*

Proof. Let $s \in \mathbb{Z}_{<0}$. For $n > 1$, $p_n(s) = \frac{1}{(n+1)!} \prod_{j=0}^{n-1} s + j$. Note that $1 - s \in \mathbb{Z}_{\geq 0}$ and so for $N := 1 - s$,

$$\prod_{j=0}^{N-1} s + j = (s + N - 1)(s + N - 2) \dots (s + 2)(s + 1) \cdot s$$

$$= (s + 1 - s - 1)(s + 1 - s - 2) \dots (s + 2)(s + 1) \cdot s = 0$$

Thus for each $n \geq N$, $p_n(s) = 0$. □

Theorem 8. *$T\left[\zeta(s,a) - \frac{1}{a^s}\right] = \sum_{n=0}^{\infty} p_n(s)\exp(nD)\left[\zeta(s,a) - \frac{1}{a^s}\right]$ diverges for all complex numbers $s \in \mathbb{C}$.*

Proof. From Proposition 5, since for all $s \in \mathbb{C}$ and all $n \geq 0$, we can find $N \geq 0$ so that $p_N(s)\exp(ND)[\zeta(s,a) - \frac{1}{a^s}]$ is divergent whenever $p_n(s) \neq 0$. Since p_n can be defined in terms of the Γ-function as in equation (9), $p_n(s)$ can only be equal to zero if $s \in \mathbb{Z}_{<0}$.

Assume that $s \in \mathbb{Z}_{<0}$. By Lemma 7, for $n \geq 1 - s$, $p_n(s) = 0$. By Proposition 5, there is some $N \geq 0$ so that $\exp(ND)\left[\zeta(s,a) - \frac{1}{a^s}\right]$ diverges. If $N < 1 - s$, then $p_N(s) \neq 0$ and $p_N(s)\exp(ND)\left[\zeta(s,a) - \frac{1}{a^s}\right]$ diverges as above.

If $s \in \mathbb{Z}_{<0}$ and $N \geq 1 - s$, then $p_N(s) = 0$. However, $0 \cdot \exp(ND)\left[\zeta(s,a) - \frac{1}{a^s}\right]$ is also not a complex number since $\exp(ND)\left[\zeta(s,a) - \frac{1}{a^s}\right]$ diverges.

Recall that $T = \sum_{n=0}^{\infty} p_n(s)\exp(nD)$. Then, for any $s \in \mathbb{C}$, we can find some $N \geq 0$ so that the Nth partial sum of $T[\zeta(s,a) - \frac{1}{a^s}]$,

$$\sum_{n=0}^{N} p_n(s)\exp(nD)\left[\zeta(s,a) - \frac{1}{a^s}\right]$$

is not a complex number. Thus, we cannot define the series $T[\zeta(s,a) - \frac{1}{a^s}]$ at any such points s. We conclude the series does not converge in $\mathbb{C}$. □

2.4.2. Convergence of T in a general setting

We now wish to discuss the convergence of T in a more general setting. To do so, we first look at T applied to the constant function with the goal of understanding the behavior of the series $\sum_{n=0}^{\infty} p_n(s)$ for $s \in \mathbb{C}$.

Lemma 9. *For $s \in \mathbb{Z}_{>0}$, the series $\sum_{n=0}^{\infty} p_n(s)$ diverges, and for $s \in \mathbb{Z}_{\leq 0}$, the series $\sum_{n=0}^{\infty} p_n(s)$ converges.*

Proof. Let $s \in \mathbb{Z}_{>0}$. From Lemma 6, for all $n > 0$, we have $p_n(s) \geq \frac{1}{(s-1)!(n+1)}$. Then, by series comparison, we have that, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{(s-1)!(n+1)}$ also diverges. By comparison with $\sum_{n=1}^{\infty} \frac{1}{(s-1)!(n+1)}$, the series $\sum_{n=1}^{\infty} p_n(s)$ diverges.

For $s = 0$, notice that for $n > 1$, $p_n(0) = \frac{(n-1)!}{(n+1)!} = \frac{1}{n(n+1)}$ and so $\sum_{n=0}^{\infty} p_n(0)$ converges. By the proof of Lemma 7, for $s \in \mathbb{Z}_{<0}$ and $N = 1-s$, we have $\prod_{j=0}^{N-1} s+j = 0$. Thus for each $n \geq N$, $p_n(s) = 0$ and so the series $\sum_{n=0}^{\infty} p_n(s)$ converges for $s \in \mathbb{Z}_{<0}$. □

It is more difficult to determine what happens outside of $\mathbb{Z}$.

Corollary 10. *For $s \in \mathbb{R}$ with $s > 1$, the series $\sum_{n=0}^{\infty} p_n(s)$ diverges.*

Proof. Let $s \in \mathbb{R}$ satisfy $s \geq 1$. First, observe that, for all integers $j \geq 0$, we also have that $s + j \geq 1 + j$. This means that $p_n(s) \geq p_n(1)$ and since, by Lemma 9, $\sum_{n=1}^{\infty} p_n(1)$ diverges we have that, by series comparison, $\sum_{n=1}^{\infty} p_n(s)$ also diverges. □

Corollary 11. *For $s \in \mathbb{C}$ with $Re(s) > 1$, the series $\sum_{n=0}^{\infty} p_n(s)$ does not converge absolutely.*

Proof. Let $s \in \mathbb{C}$ satisfy $\text{Re}(s) \geq 1$. For all integers $j \geq 0$, note that $|s + j| \geq 1 + j$. This means that $|p_n(s)| \geq |p_n(1)|$ and since, by Lemma 9, $\sum_{n=1}^{\infty} |p_n(1)|$ diverges we have that, by series comparison, $\sum_{n=1}^{\infty} |p_n(s)|$ also diverges. □

We can guarantee convergence when restricting to $L^1(\mathbb{R}_{\geq 0})$. First, we need to generalize $p_n(s)$. Writing it as $p(n, s) := p_n(s)$ and observing that, for all integers $n \geq 0$, we have that $p(n, s) = \frac{\Gamma(s+n)}{\Gamma(n+1)\Gamma(s)}$, we can make sense of $p(n, s)$ when n is any real number. We first define the set $U = \{s \in \mathbb{C} : s \neq n \text{ for all } n \in \mathbb{Z}_{<0}\}$. We, then, consider $p : \mathbb{R}_{\geq 0} \times U \to \mathbb{C}$ given by $p(x, s) = \frac{\Gamma(s+x)}{\Gamma(x+1)\Gamma(s)}$, which makes sense for all $x \in \mathbb{R}_{\geq 0}$ and all $s \in U$.

Proposition 12. *If $f : \mathbb{C} \to \mathbb{C}$ is analytic with radius of convergence equal to $+\infty$ and if we have that, for all $s \in U$, the function $p(\cdot, s)f(\cdot + s)$ is in $L^1(\mathbb{R}_{\geq 0})$, then $T[f](s)$ converges absolutely for all $s \in U$.*

Proof. Let $s \in U$. By assumption, $\int_0^{\infty} |p(x, s)f(x + s)|dx < \infty$. Then, by the integral test for series, $\sum_{n=0}^{\infty} |p_n(s)f(s + n)|$ must converge, as desired. □

This means that, in such cases, we can define a function $g : U \to \mathbb{C}$ given by $g(s) = T[f](s)$. Our next result seeks to give a sufficient condition for uniform convergence.

Proposition 13. *If $f : \mathbb{C} \to \mathbb{C}$ is analytic on all of $\mathbb{C}$ with radius of convergence equal to $+\infty$ and if there are constants $a, b \in \mathbb{R} \cup \{-\infty\}$ and $c \in \mathbb{R} \cup \{+\infty\}$ with $b < c$ so that, when we consider the set $U := \{z \in \mathbb{C} : Re(z) \in (a, \infty) \text{ and } Im(z) \in (b, c)\}$, we have that $\sum_{n=0}^{\infty} ||p_n(s)f(s + n)||_{C^{\infty}(U)}$ converges, then $T[f]$ converges uniformly to a continuous function $g : U \to \mathbb{C}$.*

Proof. Let $s \in \mathbb{C}$. Since f is analytic with radius of convergence equal to $+\infty$, for all integers $n > 0$, we have that $\sum_{k=0}^{\infty} \frac{f'(s)}{k!} n^k$ converges to $f(s + n)$ by Taylor's theorem. Then, we have that for each $s \in U$,

$$|p_n(s) \exp(nD)[f(s)]| = |p_n(s)f(n + s)| \leq ||p_n(s)f(n + s)||_{C^{\infty}(U)}$$

and

$$
\begin{aligned}
|T[f](s)| &= \left| \sum_{n=0}^{\infty} p_n(s) \exp(nD)[f](s) \right| \\
&\leq \sum_{n=0}^{\infty} \left\| p_n(s) \left(\limsup_{K \to \infty} \sum_{k=0}^{K} \frac{f'(s)}{k!} n^k \right) \right\|_{C^\infty(U)} = \sum_{n=0}^{\infty} ||p_n(s) f(s+n)||_{C^\infty(U)} < \infty
\end{aligned}
$$

By the Weierstrass M-test, we have that $\sum_{n=0}^{\infty} (p_n(\cdot) \exp(nD)[f(\cdot)])\big|_U$ converges uniformly to a continuous function $g : U \to \mathbb{C}$. □

We can have an analogous result when $U = \{z \in \mathbb{C} : \mathrm{Re}(z) \in [a, \infty) \text{ and } \mathrm{Im}(z) \in [c, b]\}$ as well as when $\mathrm{Im}(z)$ is in a half-open interval.

3. Generalizing Van Gorder's operator

The main reason the operator T is not well-defined when applied to ζ is because ζ is not analytic and so the radius of convergence of its Taylor series expansion is not $+\infty$. Specifically, the problem is that for all $s \in \mathbb{C}$, we can find some $N > 0$ for which $\exp(nD)[\zeta(s, a)]$ will not be convergent. However, when treating the operator $\exp(nD)$ as the shift operator, formally, we are able to show that (2) and (3) hold. With that in mind, we define an operator G, which agrees with T on analytic functions with radius of convergence equal to $+\infty$ but which can be applied to a wider range of functions.

Let $\mathcal{M}$ be the collection of meromorphic functions on $\mathbb{C}$ and $f \in \mathcal{M}$. Define $G : \mathcal{M} \to \mathcal{M}$ by

$$
G[f](s) = \sum_{n=0}^{\infty} p_n(s) f(s+n) \tag{12}
$$

For this operator to be well-defined, we do not require that f be differentiable, a significant gain from the definition of T. Note that G agrees with T on analytic functions. Thus G satisfies a version of Proposition 12 and Proposition 13. The assumption that f be analytic may be weakened in such versions.

When G is applied to $\zeta(s, a)$, we recover the identity (6) in Lemma 1 and we conclude that $G[\zeta(\cdot, a)]$ converges pointwise to a continuous function defined on $\mathbb{C} \setminus \{1\}$.

3.1. Using G to get an identity for the ζ-function

Recalling our discussion of T, observe that when we evaluate $\exp(nD)\left[\zeta(-m, a) - \frac{1}{a^{-m}}\right]$ for $m > 0$ and $n \geq 0$ integers, we have convergence of the Taylor series whenever $n \leq m$ because $|-m + n - (-m)| = n < |-m-1| = m+1$ and because $m+1$ is the radius of convergence of such series as we discussed in Section 2.4.1. In addition, from the proof of Lemma 7, $p_n(-m) = 0$ for $n > m$. In addition, by the fact that the pole of $\zeta(s, a)$ at $s = 1$ has residue 1, we have that

$$
\lim_{s \to -m} (s+m)\zeta(s+m+1, a) = 1
$$

and, thus,

$$\lim_{s\to -m} p_{m+1}(s)\zeta(s+m+1) = \frac{p_m(-m)}{m+2}$$

This gives us the following equality for $m > 0$ an integer, by (2), by our discussion above and by continuity.

$$\begin{aligned}
-\frac{a^{m+1}}{m+1} &= \sum_{n=0}^{m} p_n(-m)\left(\zeta(-m+n,a) - \frac{1}{a^{-m+n}}\right) + \frac{p_m(-m)}{m+2} \\
&= \frac{p_m(-m)}{m+2} + \sum_{n=0}^{m} p_n(-m)\left[\zeta(-m,a) - \frac{1}{a^{-m}} + \sum_{k=1}^{\infty}\frac{n^k}{k!}D_s^k\left[\zeta(-m,a) - \frac{1}{a^{-m}}\right]\right] \\
&= \frac{p_m(-m)}{m+2} + \sum_{n=0}^{m} p_n(-m)\left[\zeta(-m,a) - \frac{1}{a^{-m}}\right. \\
&\qquad \left. + \sum_{k=1}^{\infty}\frac{n^k}{k!}\left[\zeta^{(k)}(-m,a) - (\log(a))^k a^m\right]\right] \\
&= \frac{p_m(-m)}{m+2} + \left[\zeta(-m,a) - \frac{1}{a^{-m}}\right]\sum_{n=0}^{m} p_n(-m) \\
&\qquad + \sum_{k=1}^{\infty}\frac{1}{k!}\left[\zeta^{(k)}(-m,a) - (\log(a))^k a^m\right]\sum_{n=0}^{m} n^k p_n(-m)
\end{aligned}$$

The interchange between the finite sum and the series is justified by the absolute convergence of Taylor series within its radius of convergence. When we look at the zeta function, we get an interesting identity using the trivial zeros of the zeta function and the definition of $p_n(-m)$.

Theorem 14. *For $m > 0$ an integer, we have that*

$$\begin{aligned}
-\frac{1}{2m+1} =& \frac{1}{2m+1} - \sum_{n=0}^{2m}\frac{1}{(n+1)!}\prod_{j=0}^{n-1}(-2m+j) \\
&+ \sum_{k=1}^{\infty}\frac{1}{k!}\zeta^{(k)}(-2m)\sum_{n=1}^{2m}\frac{n^k}{(n+1)!}\prod_{j=0}^{n-1}(-2m+j)
\end{aligned}$$

and that, for $m \geq 0$,

$$\begin{aligned}
-\frac{1}{2m+2} =& -\frac{1}{2m+2} - \sum_{n=0}^{2m+1}\frac{1}{(n+1)!}\prod_{j=0}^{n-1}(-2m+j) \\
&+ \sum_{k=0}^{\infty}\frac{1}{k!}\zeta^{(k)}(-2m-1)\sum_{n=1}^{2m+1}\frac{n^k}{(n+1)!}\prod_{j=0}^{n-1}(-2m-1+j) \qquad \square
\end{aligned}$$

3.2. Applying G to Dirichlet L-functions

Corollary 3 can be reframed in terms of the operator G so that the equation corresponding to (3) does not *only* hold formally. Furthermore, Van Gorder's original result may also be extended to provide an operator equation involving Dirichlet L-functions.

Proposition 15. *Let G be the operator defined above. Then, for a Dirichlet character χ* mod k*, and for $s \in \mathbb{C} \setminus \{1\}$, we have that,*

$$G\left[k^s L(s,\chi) - \sum_{r=1}^{k} \chi(r)\frac{k^s}{r^s}\right] = \frac{k^{s-1}}{s-1}\sum_{r=1}^{k}\frac{\chi(r)}{r^{s-1}} \tag{13}$$

Proof. Let χ be a Dirichlet character. From the definition of G,

$$G\left[k^s L(s,\chi) - \sum_{r=1}^{k} \chi(r)\frac{k^s}{r^s}\right] = \sum_{n=0}^{\infty} p_n(s)\left(k^s L(s+n,\chi) - \sum_{r=1}^{k}\chi(r)\frac{k^{s+n}}{r^{s+n}}\right) \tag{14}$$

But we know that, for χ a character mod k,

$$L(s,\chi) = k^{-s}\sum_{r=1}^{k}\chi(r)\zeta(s, r/k) \tag{15}$$

Thus, (14) becomes,

$$\begin{aligned}
&\sum_{n=0}^{\infty} p_n(s)\left(k^s k^{-s}\sum_{r=1}^{k}\chi(r)\zeta(s+n,r/k) - \sum_{r=1}^{k}\chi(r)\frac{k^{s+n}}{r^{s+n}}\right)\\
&= \sum_{n=0}^{\infty} p_n(s)\left[\sum_{r=1}^{k}\left(\chi(r)\zeta(s+n,r/k) - \chi(r)\frac{k^{s+n}}{r^{s+n}}\right)\right]\\
&= \sum_{n=0}^{\infty}\sum_{r=1}^{k}\left[\chi(r)p_n(s)\left(\zeta(s+n,r/k) - \frac{k^{s+n}}{r^{s+n}}\right)\right]
\end{aligned}$$

Now, since $\sum_{n=0}^{\infty} p_n(s)\left(\zeta(s+n,r/k) - \frac{k^{s+n}}{r^{s+n}}\right)$ converges absolutely by Lemma 2, we can change the order of summation to get,

$$\sum_{r=1}^{k}\chi(r)\sum_{n=0}^{\infty}\left[p_n(s)\left(\zeta(s+n,r/k) - \frac{k^{s+n}}{r^{s+n}}\right)\right] \tag{16}$$

But, by Corollary 2, we have that the last summation equals $\frac{k^{s-1}}{(s-1)r^{s-1}}$ for $r = 1, .., k$ and (16) becomes

$$\sum_{r=1}^{k} \chi(r) \frac{k^{s-1}}{(s-1)r^{s-1}} = \frac{k^{s-1}}{s-1} \sum_{r=1}^{k} \frac{\chi(r)}{r^{s-1}}$$

as desired. □

This gives us an identity of Dirichlet L-functions. Namely, for χ a character mod k, we have

$$L(s,\chi) - \sum_{r=1}^{k} \frac{\chi(r)}{r^s} = \frac{1}{k(s-1)} \sum_{r=1}^{k} \frac{\chi(r)}{r^{s-1}} - \frac{1}{k^s} \sum_{n=1}^{\infty} \frac{\prod_{j=0}^{n-1}(s+j)}{(n+1)!} \left(k^{s+n} L(s+n,\chi) - \sum_{r=1}^{k} \chi(r) \frac{k^{s+n}}{r^{s+n}} \right)$$

3.3. Relating G^{-1} to the Hurwitz ζ and to Dirichlet L-functions

In [9], Van Gorder defines an inverse operator to T. We see in his proof of Theorem 4.1 that Van Gorder's definition of T^{-1} was the inverse operator for T he did not use the definition of $\exp(nD)$ but rather the characterization that it is a shift operator. Thus, we may use this construction to define an inverse operator to G, which we denote G^{-1}. Let f be a complex valued function. Then G^{-1} is given by

$$G^{-1}[f](s) = \sum_{n=1}^{\infty} q_n(s) f(s+n) \tag{17}$$

Where $q_0(s) = 1$ and $q_n(s) = -\sum_{k=0}^{n-1} q_k(s) p_{n-k}(s+k)$. The formal proof that this operator is the inverse of G is given by Van Gorder in the proof of Theorem 4.1 in [9]. However, this argument does not give reference to where this inverse converges. We will address this question in Section 4.1, but first we will make clear the relationship between G and the Bernoulli numbers which Van Gorder alluded to in [9].

4. Relationship to Bernoulli numbers

In his paper, Van Gorder alludes to a relationship between the $q_n(s)$ and Bernoulli numbers. We now provide a proof of such relationship.

Proposition 16. *For all integers $n \geq 0$ and all $s \in \mathbb{C}$, we have the identity*

$$q_n(s) = \frac{B_n}{n!} \prod_{j=0}^{n-1} (s+j)$$

Where B_n denotes the nth Bernoulli number.

Proof. We proceed with strong induction on n.

For $n = 0$, $q_0(s) = 1 = \frac{B_0}{0!}$. Suppose that we proved our claim for all $0 \leq k \leq n-1$ for some $n - 1 \geq 0$. We prove it for n. By the strong induction hypothesis and by the definition of $p_n(s)$, we have that, for all $s \in \mathbb{C}$

$$\begin{aligned} q_n(s) &= -\sum_{k=0}^{n-1} q_k(s)p_{n-k}(s+k) \\ &= -\sum_{k=0}^{n-1}\left[\left(\frac{B_k}{k!}\prod_{j=0}^{k-1}(s+j)\right)\left(\frac{1}{(n+1-k)!}\prod_{j=0}^{n-k-1}(s+k+j)\right)\right] \\ &= -\sum_{k=0}^{n-1}\left[\left(\frac{B_k}{k!(n+1-k)!}\prod_{j=0}^{k-1}(s+j)\right)\prod_{j=k}^{n-1}(s+j)\right] \\ &= -\sum_{k=0}^{n-1}\left[\frac{B_k}{k!(n+1-k)!}\prod_{j=0}^{n-1}(s+j)\right] \\ &= \left(\prod_{j=0}^{n-1}(s+j)\right)\left(-\sum_{k=0}^{n-1}\frac{B_k}{k!(n+1-k)!}\right) \\ &= \left(\prod_{j=0}^{n-1}(s+j)\right)\left(-\sum_{k=0}^{n-1}\binom{n+1}{k}\frac{B_k}{(n+1)!}\right) \end{aligned}$$

Since the Bernoulli numbers have the recursive formula $B_0 = 1$ and $(n+1)B_n = -\sum_{k=0}^{n-1}\binom{n+1}{k}B_k$ for $n > 0$, we conclude that

$$\begin{aligned} q_n(s) &= \frac{1}{(n+1)!}\left(\prod_{j=0}^{n-1}(s+j)\right)\left(-\sum_{k=0}^{n-1}\binom{n+1}{k}B_k\right) \\ &= \frac{(n+1)}{(n+1)!}B_n\prod_{j=0}^{n-1}(s+j) \\ &= \frac{B_n}{n!}\prod_{j=0}^{n-1}(s+j) \end{aligned}$$

This completes our induction. □

Proposition 16 gives a surprising connection between G and Bernoulli numbers. We now proceed to use G^{-1} to recover series representations of $\zeta(\cdot, a)$ and $L(\cdot, \chi)$.

4.1. Using G^{-1} to represent the Hurwitz ζ-function and Dirichlet L-functions

Using the fact that G^{-1} is an inverse operator to G, we have

$$\zeta(s,a)-\frac{1}{a^s}=G^{-1}\left[\frac{1}{(s-1)a^{s-1}}\right]=\sum_{n=0}^{\infty}\frac{q_n(s)}{(s+n-1)a^{s+n-1}}$$
$$=\sum_{n=0}^{\infty}\left(\frac{B_n}{n!}\cdot\frac{\prod_{j=0}^{n-1}(s+j)}{(s+n-1)a^{s+n-1}}\right) \tag{18}$$

and

$$k^sL(s,\chi)-\sum_{r=1}^{k}\chi(r)\frac{k^s}{r^s}=G^{-1}\left[\frac{k^{s-1}}{s-1}\sum_{r=1}^{k}\frac{\chi(r)}{r^{s-1}}\right]$$
$$=\sum_{n=0}^{\infty}\frac{B_n\prod_{j=0}^{n-1}(s+j)}{n!}\left(\frac{k^{s+n-1}}{s+n-1}\sum_{r=1}^{k}\frac{\chi(r)}{r^{s+n-1}}\right) \tag{19}$$

We now treat the convergence of these identities more rigorously. Notice that, for the Hurwitz zeta function, the coefficients in the sum are the same as the coefficients in the Euler-Maclaurin summation formula (which gives a convergent series for all $s\in\mathbb{C}\setminus\{1\}$). Thus we can conclude that (18) converges for all $s\in\mathbb{C}\setminus\{1\}$.

For Dirichlet L-functions, however, there is no clear way to apply the Euler-Maclaurin summation formula. One can, however, derive a series representation of Dirichlet L-functions from the Euler-Maclaurin summation formula for the Hurwitz zeta using the identity in (15), to get for χ a character mod k,

$$L(s,\chi)=k^{-s}\sum_{r=1}^{k}\chi(r)\left[\frac{k^s}{r^s}+\sum_{n=0}^{\infty}\left(\frac{B_n}{n!}\cdot\frac{k^{s+n-1}\prod_{j=0}^{n-1}(s+j)}{r^{s+n-1}(s+n-1)}\right)\right] \tag{20}$$

(which is a rearrangement of (19)). The question is whether such rearrangement gives us a convergent series of equal value.

Proposition 17. *The series in (19) and (20) both converge to the same value for all $s\in\mathbb{C}\setminus\{1\}$. And they both provide an analytic continuation of $L(s,\chi)$ to $\mathbb{C}\setminus\{1\}$.*

Proof. Let χ be a character mod k and fix an integer $N\geq 0$. Using the Euler-Maclaurin expression for the Hurwitz zeta, we get that for $s\in\mathbb{C}\setminus\{1\}$ with $\mathrm{Re}(s)>1$, $L(s,\chi)$ is equal to

$$T_N(s):=k^{-s}\sum_{r=1}^{k}\chi(r)\Big[\frac{k^s}{r^s}+\sum_{n=0}^{N}\left(\frac{B_n}{n!}\cdot\frac{k^{s+n-1}\prod_{j=0}^{n-1}(s+j)}{r^{s+n-1}(s+n-1)}\right) \tag{21}$$
$$-\frac{(-1)^N}{N!}\int_0^{\infty}\frac{\prod_{j=0}^{N-1}(s+j)}{(t+a)^{s+N}}\psi_N(t)dt\Big] \tag{22}$$

This is the same as

$$\sum_{r=1}^{k} \frac{\chi(r)}{r^s} + k^{-s} \left[\sum_{n=0}^{N} \sum_{r=1}^{k} \chi(r) \left(\frac{B_n}{n!} \cdot \frac{k^{s+n-1} \prod_{j=0}^{n-1}(s+j)}{r^{s+n-1}(s+n-1)} \right) - \frac{(-1)^N}{N!} \int_0^\infty \frac{\prod_{j=0}^{N-1}(s+j)}{(t+a)^{s+N}} \psi_N(t)dt \right] \tag{23}$$

For all $s \in \mathbb{C}$ with $\mathrm{Re}(s) + N > 1$, the integral is convergent. Thus, for $s \in \mathbb{C}$ with $\mathrm{Re}(s) + N > 1$, the right hand side of the above identity is convergent (the only problem being the term at $n = 0$ in the sum, where we have division by 0). It can also be shown that it is holomorphic in such region. This provides an analytic continuation of $L(s, \chi)$ to $\{s \in \mathbb{C} : \mathrm{Re}(s) > s - N; s \neq 1\}$. Furthermore, for all $m \leq N$, we have that the identity with N replaced by m agrees with the identity above for all $s \in \mathbb{C} \setminus \{1\}$ with $\mathrm{Re}(s) > 1 - m$.

Observe that both (21) and (23) agree for all integers $N \geq 0$ and that, letting $N \to \infty$, it is known that $\frac{(-1)^N}{N!} \int_0^\infty \frac{\prod_{j=0}^{N-1}(s+j)}{(t+a)^{s+N}} \psi_N(t)dt \to 0$.

We now show that the series given by $\sum_{n=0}^\infty \sum_{r=1}^k \chi(r) \left(\frac{B_n}{n!} \cdot \frac{k^{s+n-1} \prod_{j=0}^{n-1}(s+j)}{r^{s+n-1}(s+n-1)} \right)$ converges pointwise for all $s \in \mathbb{C} \setminus \{1\}$. To see this, let $s \in \mathbb{C} \setminus \{1\}$ and let $\epsilon > 0$. Choose an integer $N \geq 0$ so that $\mathrm{Re}(s) > 1 - N$ and $\left| \frac{(-1)^N}{N!} \int_0^\infty \frac{\prod_{j=0}^{N-1}(s+j)}{(t+a)^{s+N}} \psi_N(t)dt \right| < \epsilon/2$ for all $m \geq N$. We then have that by analytic continuation, $T_m(s) = T_N(s)$ for all $m \geq N$. Finally, using the triangle inequality, we have that for all $q \geq p \geq N$,

$$\left| \sum_{n=0}^{q} \sum_{r=1}^{k} \chi(r) \left(\frac{B_n}{n!} \cdot \frac{k^{s+n-1} \prod_{j=0}^{n-1}(s+j)}{r^{s+n-1}(s+n-1)} \right) - \sum_{n=0}^{p} \sum_{r=1}^{k} \chi(r) \left(\frac{B_n}{n!} \cdot \frac{k^{s+n-1} \prod_{j=0}^{n-1}(s+j)}{r^{s+n-1}(s+n-1)} \right) \right|$$
$$\leq |T_q(s) - T_p(s)| + 2(\epsilon/2) = \epsilon$$

Since ϵ was arbitrary, this proves that the sequence of partial sums is actually a Cauchy sequence and, since $\mathbb{C}$ is complete, this series must converge. Now, our point $s \in \mathbb{C} \setminus \{1\}$ was also arbitrary and, thus, we have proved the desired claim. □

5. Approximations

In Theorem 8, we establish that $T\left[\zeta(s,a) - \frac{1}{a^s}\right] = \sum_{n=0}^\infty p_n(s) \exp(nD) \left[\zeta(s,a) - \frac{1}{a^s}\right]$ diverges for $s \in \mathbb{C}$ so clearly it is not the case that $T\left[\zeta(s,a) - \frac{1}{a^s}\right]$ converges pointwise to $\frac{1}{(s-1)a^{s-1}}$ for $s \in \mathbb{C}$. However, it may be the case that truncating T may provide a good approximation even at values where the series does not converge. Of course, when considering the operator $T = \sum_{n=0}^\infty p_n(s)[id + \sum_{k=1}^\infty \frac{n^k}{k!} D_s^k]$, there are two sums that we may consider truncating: the sum over n and the sum over k. In what follows we will truncate in n.

Consider

$$T_N(s,a) := \sum_{n=0}^{N} p_n(s)\exp(nD)\left[\zeta(s,a) - \frac{1}{a^s}\right]$$

Note that though T does not converge when applied to the zeta function, T_N may converge when applied to $\zeta(s,a)$. Recall that from Proposition 5, for each s there is some N' so that $\exp(N'\cdot D)\left[\zeta(s,a) - \frac{1}{a^s}\right]$ diverges. However, from the Taylor series expansion,

$$\zeta(s+n,a) - \frac{1}{a^{s+n}} = \sum_{k=0}^{\infty} \frac{D_s^k\left(\zeta(s,a) - \frac{1}{a^s}\right)}{k!} n^k$$

converges pointwise for $|(s+n)-s| = |n| < |s-1|$ since $\zeta(z,a)$ has a pole at $z=1$. Thus we have

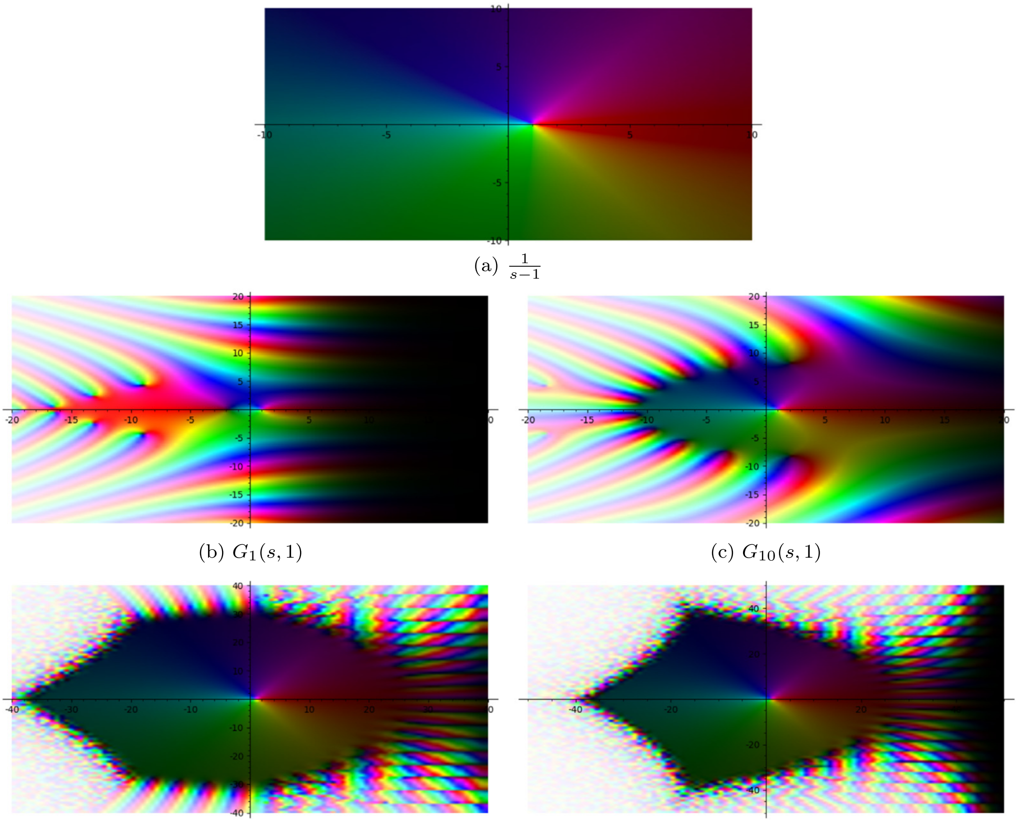
$$\begin{aligned} T_N(s,a) &:= \sum_{n=0}^{N} p_n(s)\exp(nD)\left[\zeta(s,a) - \frac{1}{a^s}\right] = \sum_{n=0}^{N} p_n(s)\left[\zeta(s+n,a) - \frac{1}{a^{s+n}}\right] \\ &=: G_N(s,a) \end{aligned}$$

for $N < |s-1|$. In other words, in the region of convergence (away from $s=1$), the truncation of T in n is equal to the truncation of the shift operator G.

In what follows we will use complex plots to examine whether $T_N(s,a)$ is good approximation to $\frac{1}{(s-1)a^{s-1}}$ for s in the region of convergence for T_N. Figs. 1 and 2 are obtained using "complex_plot" in Sage. This function takes a complex function of one variable, $f(z)$ and plots output of the function over the specified x_range and y_range. The magnitude of the output is indicated by the brightness (with zero being black and infinity being white) while the argument is represented by the hue. The hue of red is positive real, and increasing through orange, yellow, as the argument increases and the hue of green is positive imaginary. Note that, for simplicity, both figures only plot the specific case of the Riemann zeta function (when $a=1$).

Fig. 1 (a) is the complex plot of $\frac{1}{s-1}$ the right side of Van Gorder's equation (2). Subfigures (b), (c), (d) and (e) of approximations $G_N(s,1)$ of the left side of Van Gorder's equation (2) for $N = 1, 10, 50$ and 100. Note that the domain of the plots in subfigures (d) and (e) has been expanded.

We see in Fig. 1 that the first term of the expansion $G_N(s,1)$ is a good approximation of $\frac{1}{s-1}$ near the singularity $s=1$. (Note: This is not surprising since as $N\to\infty$, we have shown $G_N(s,1) \to G[\zeta(s)-1] = \frac{1}{s-1}$.) We also see from Fig. 1 that as N grows, the region on which $G_N(s,1)$ is a good approximation of $\frac{1}{s-1}$ also expands.



(a) $\frac{1}{s-1}$

(b) $G_1(s,1)$

(c) $G_{10}(s,1)$

(d) $G_{50}(s,1)$

(e) $G_{100}(s,1)$

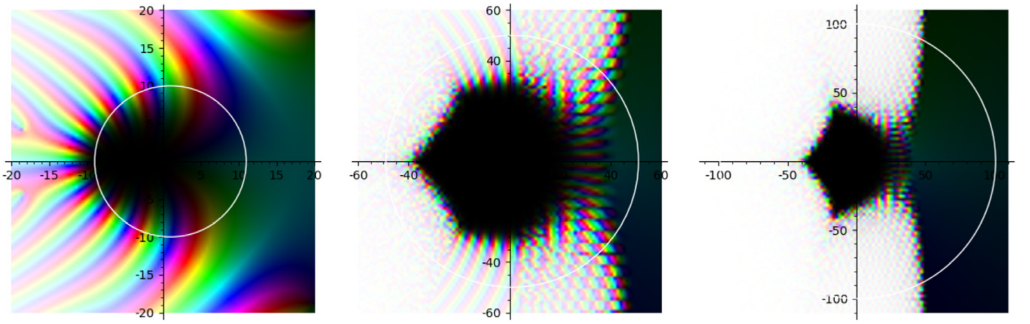
Fig. 1. Complex plots of $\frac{1}{s-1}$ and $G_N(s,1)$ for $N = 1, 10, 50$, and 100. Note that the domains for (a), (b) and (c) are $(-20, 20)$ on both the real and imaginary axes and for (d) and (e) is $(-40, 40)$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

As we observed at the beginning of this section, $T_N = G_N$ away from $s = 1$ so $T_N(s,a)$ may be a good approximation to $\frac{1}{(s-1)a^{s-1}}$ for s far enough from 1 (i.e. outside the circle $|s-1| = N$). One might wonder whether the expanding region on which $G_N(s,1)$ is a good approximation of $\frac{1}{s-1}$ will break into the region of convergence of $T_N(s,1)$.

Fig. 2 contains complex plots of $G_N(s,1) - \frac{1}{s-1}$ for $N = 10, 50$ and 100 along with the circle $|s-1| = N$ for $N = 10, 50$ and 100. The inclusion of this circle in plot is to be able to identify where $G_N(s,1) = T_N(s,1)$ (outside $|s-1| = N$).

Examining Fig. 2, we see that this "region of good approximation" is not expanding as quickly as the region of convergence for T_N (the radius of $|s-1| = N$). Thus, as N grows, $T_N(s,1)$ actually seems to be a less reasonable approximation for $\frac{1}{s-1}$.

Many natural questions remain about both the accuracy of the approximation T_N in this region and the accuracy of the approximation G_N in the disk and the rate of convergence of G.



(a) $N = 10$ (b) $N = 50$ (c) $N = 100$

Fig. 2. Complex plots of $G_N(s,1) - \frac{1}{s-1}$ for $N = 10, 50,$ and 100 as well as the circle $|s-1| = N$ for $N = 10, 50,$ and 100 (in white) which shows the region for which for T_N converges and equal G_N.

6. Conclusion

Using the operator G as opposed to T allows us to provide more than formal justification for the differential equation (2) as well as the corresponding generalizations to the Hurwitz zeta function and Dirichlet L-function. However, it is important to note that G is not a differential operator and so, in fact, this does not provide support for their being a non-algebraic differential equation which zeta satisfies.

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Corrigendum

Corrigendum to "Linear operators, the Hurwitz zeta function and Dirichlet L-functions" [J. Number Theory 217 (2020) 422–442]

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A B S T R A C T

We correct an error found in Section 3.4.1 of *Linear operators, the Hurwitz zeta function and Dirichlet L-functions* published in JNT 217 (2020) 422–442. The error is related to the convergence of the inverse operator G^{-1} defined in Section 3.3 and affects the statement and proof of Proposition 17. We provide a revised statement and proof.

In what follows we make a correction to our paper *Linear Operators, the Hurwitz Zeta Function and Dirichlet L-Functions* published in the Journal of Number Theory [1]. In this paper, we investigate an operator introduced by Van Gorder in [2]. Van Gorder attempts to provide differential equation of infinite order which the Riemann zeta function satisfies and we show that this equation does not converge pointwise. Ironically, our error occurs in Proposition 17 in which we state that the series in equations (18)

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and (19) converge. This error does not affect the main claims of our paper; however, we would like to fix our mistaken identities.

In [2], Van Gorder defines

$$T = \sum_{n=0}^{\infty} p_n(s) \exp(nD)$$

where $p_n(s) = \begin{cases} 1 & \text{if } n = 0 \\ \frac{1}{(n+1)!} \prod_{j=0}^{n-1} (s+j) & \text{if } n > 0 \end{cases}$ and $\exp(nD) = id + \sum_{k=1}^{\infty} \frac{n^k}{k!} D_s^k$ for $D_s^k := \frac{\partial^k}{\partial s^k}$. Van Gorder uses that fact that *exp* acts as a shift operator to show that the Riemann zeta function formally satisfies an operator equation "involving T." In [1], we show that this differential equation involving T (as it is defined) does not converge.

In [1] we use the operator defined by the shift operator, call it G, to give a convergent operator equation involving the Riemann zeta function. In Section 3 of [1], we define this shift operator as follows. Let $\mathcal{M}$ be the collection of meromorphic functions on $\mathbb{C}$ and $f \in \mathcal{M}$. Define $G : \mathcal{M} \to \mathcal{M}$ by

$$G[f](s) = \sum_{n=0}^{\infty} p_n(s) f(s+n) \tag{1}$$

For f a complex valued function, we then define G^{-1} by

$$G^{-1}[f](s) = \sum_{n=1}^{\infty} q_n(s) f(s+n) \tag{2}$$

where $q_0(s) = 1$ and $q_n(s) = -\sum_{k=0}^{n-1} q_k(s) p_{n-k}(s+k)$. The formal proof that this operator is the inverse of G is given by Theorem 4.1 in Van Gorder [2]. In Proposition 16 of [1], we show that for all integers $n \geq 0$ and all $s \in \mathbb{C}$,

$$q_n(s) = \frac{B_n}{n!} \prod_{j=0}^{n-1} (s+j)$$

where B_n denotes the n^{th} Bernoulli number. Up to this point we believe our paper has no errors.

The problem arises when we wrongfully attempted to justify the convergence of the application of G^{-1} to $\frac{1}{(s-1)a^{s-1}}$ using the Euler-Maclaurin expansion. Namely, in [1] we noticed that this characterization of $q_n(s)$ in terms of Bernoulli numbers formally gives

$$\zeta(s,a) - \frac{1}{a^s} = G^{-1}\left[\frac{1}{(s-1)a^{s-1}}\right] = \sum_{n=0}^{\infty} \frac{q_n(s)}{(s+n-1)a^{s+n-1}}$$

$$= \sum_{n=0}^{\infty} \left(\frac{B_n}{n!} \cdot \frac{\prod_{j=0}^{n-1}(s+j)}{(s+n-1)a^{s+n-1}} \right) \tag{3}$$

and hence

$$k^s L(s,\chi) - \sum_{r=1}^{k} \chi(r) \frac{k^s}{r^s} = G^{-1} \left[\frac{k^{s-1}}{s-1} \sum_{r=1}^{k} \frac{\chi(r)}{r^{s-1}} \right]$$
$$= \sum_{n=0}^{\infty} \frac{B_n \prod_{j=0}^{n-1}(s+j)}{n!} \left(\frac{k^{s+n-1}}{s+n-1} \sum_{r=1}^{k} \frac{\chi(r)}{r^{s+n-1}} \right) \tag{4}$$

where $\zeta(s,a)$ is the Hurwitz eta function and $L(s,\chi)$ is a Dirichlet L-function with χ a Dirichlet character modulo k. The second expression (4) was formally derived from (3) and the fact that $L(s,\chi) = k^{-s} \sum_{r=1}^{k} \chi(r)\zeta(s, r/k)$ for χ a Dirichlet character modulo k. (Note these are equations (18) and (19) respectively in [1].) Since the last series in (3) gives the series portion of the Euler-Maclaurin expansion, we wrongfully assumed that that finite series would converge as an infinite sum and it does not. Our argument that $\sum_{n=0}^{\infty} \frac{q_n(s)}{(s+n-1)a^{s+n-1}}$ and (4) converged depended on $\sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot \frac{\prod_{j=0}^{n-1}(s+j)}{(s+n-1)a^{s+n-1}}$ converging which we will now show it does not for $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

1. Divergence of (3) and (4)

The following is a proof that (18) and (19) in [1] diverge pointwise outside of the negative integers. We also provide an identity for the convergent series at the negative integers.

Proposition 1. *The sums*

$$\sum_{n=0}^{\infty} \frac{B_n}{n!} \cdot \frac{\prod_{j=0}^{n-1}(s+j)}{(s+n-1)a^{s+n-1}} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{B_n \prod_{j=0}^{n-1}(s+j)}{n!} \left(\frac{k^{s+n-1}}{s+n-1} \sum_{r=1}^{k} \frac{\chi(r)}{r^{s+n-1}} \right)$$

diverge for all $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and converge when $s = -M \in \mathbb{Z}_{\leq 0}$ giving

$$G^{-1}[f](-M) = \sum_{n=1}^{M} (-1)^n B_n \frac{M!}{n!(M-n)!} f(-M+n).$$

Proof. From Euler's formula and the fact that $\lim_{k\to\infty} \zeta(k) = 1$, $\left| \frac{B_{2k}}{(2k)!} \right| \sim \frac{2}{(2\pi)^{2k}}$. Note that the $2n+1$ Bernoulli numbers are 0. When c_n is the $2n^{th}$ coefficient of the corresponding series, $\left| \frac{c_{n+1}}{c_n} \right| \to \infty$ as $n \to \infty$ and the Ratio Test shows that each series diverges when $s \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

To see that these sums converge for $s \in \mathbb{Z}_{\leq 0}$ and achieve the G^{-1} identity, let $s = -M$ for some $M \in \mathbb{Z}_{\geq 0}$, and notice that for all $n > M$, $q_n(s) = q_n(-M) = \frac{B_n}{n!} \prod_{j=0}^{n-1}(-M+j) = 0$. $\square$

Proposition 17 of [1] also gives a proof of the meromorphic continuation of $L(s, \chi)$ for χ a Dirichlet character. The proof we give does not follow as given since it relied on the convergence of (3) and (4). Clearly, it is still possible to prove the meromorphic continuation on $L(s, \chi)$; however, the proof of Proposition 17 is no longer valid. One can still give a proof of the meromorphic continuations of $L(s, \chi)$ using the Euler-Maclaurin formula and this proof can be found in the version of the paper posted on the arXiv (1910.01192v3).

2. Acknowledgments

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