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Hyperbolic limits of cantor set complements in the sphere

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Abstract

Let M be a hyperbolic 3-manifold with no rank two cusps admitting an embedding in $\mathbb{S}^3$. Then, if M admits an exhaustion by π_1-injective sub-manifolds there exists Cantor sets $C_n \subseteq \mathbb{S}^3$ such that $N_n = \mathbb{S}^3 \setminus C_n$ is hyperbolic and $N_n \to M$ geometrically.

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INTRODUCTION

In recent years, much work has been done in the study of infinite type hyperbolic manifolds, hyperbolizable manifolds with non-finitely generated fundamental group. For example, lot of work has gone into studying the mapping class group of infinite type surfaces, for example, [2, 3, 24]. Similarly, the first author has proven a hyperbolization result for a large class $\mathcal{M}^B$ of infinite type 3-manifolds, see [11]. The class $\mathcal{M}^B$ is characterized by the fact that each $M \in \mathcal{M}^B$ has an exhaustion $\{M_i\}_{i\in\mathbb{N}}$ in which each M_i is a compact, hyperbolizable 3-manifold with incompressible boundary and such that each $S \in \pi_0(\partial M_i)$ has genus at most $g = g(M)$. The class of hyperbolic 3-manifolds we will look at, denoted by $\mathcal{M}^{\mathbb{S}^3}$, are manifolds that need to admit exhaustions by π_1-injective sub-manifolds. Thus, we allow $M_i \subseteq M_{i+1}$ to have compressing disks in M_i, and we do not have any condition on the genus of the boundary components. However, we do need an embedding $\cup_{i\in\mathbb{N}} M_i \hookrightarrow \mathbb{S}^3$ and we will assume that $M \in \mathcal{M}^{\mathbb{S}^3}$ has no rank two cusps.

By work of Souto–Stover [31] and Cremaschi–Souto [13] and Cremaschi [10, 12] it is not hard to build hyperbolizable infinite type 3-manifolds that are homeomorphic to Cantor set complements in the 3-sphere $\mathbb{S}^3$. In particular, in [13], the manifold of Example 2 can be extended to be a Cantor set complement showing, for example, how one can have a hyperbolizable Cantor set complement in $\mathbb{S}^3$ whose fundamental group is not residually finite.

The flexibility to build hyperbolizable Cantor set complements in $\mathbb{S}^3$ is reminiscent of the fact that most knots in $\mathbb{S}^3$ are hyperbolic. For example, of the 1 701 936 knots with fewer than 16 crossings, all but 32 are hyperbolic, see [18]. Moreover, Purcell–Souto [25] showed that if $M \hookrightarrow \mathbb{S}^3$ is a one-ended hyperbolic 3-manifold of finite type without parabolics, then M is the geometric limit of hyperbolic knot complements. This shows how, under the geometric topology, hyperbolic knots are dense in the space of one ended-hyperbolic 3-manifolds admitting embeddings in $\mathbb{S}^3$.

The aim of the present work is to show a similar statement for hyperbolic 3-manifolds, not necessarily of finite type, admitting an embedding in $\mathbb{S}^3$. As approximating manifolds, we will use Cantor sets complements:

Theorem 2.6. *Let $M \cong \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold, not necessarily of finite type, without rank two cusps admitting an embedding $\iota : M \hookrightarrow \mathbb{S}^3$. Then, there exists a sequence of Cantor sets $C_i \subseteq \mathbb{S}^3$, $i \in \mathbb{N}$, such that:*

(i) *$N_i := \mathbb{S}^3 \setminus C_i$ is hyperbolic $N_i \cong \mathbb{H}^3/\Gamma_i$;*
(ii) *the N_i converge geometrically to M.*

As in [25], one can obtain hyperbolic Cantor set complements with small eigenvalues of the Laplacian, arbitrarily large isometrically embedded balls, arbitrarily many short geodesics or surfaces with arbitrarily small principal curvatures.

1 | BACKGROUND

1.1 | Notation and conventions

All appearing 3-manifolds are assumed to be aspherical and orientable. We use $\cong$ for homeomorphic. By $S \hookrightarrow M$, we denote an embedding of S into M while $S \looparrowright M$ denotes an immersion. By $\Sigma_{g,n}$ we denote an orientable surface of genus g with n boundary components. We say that a manifold is closed if it is compact and without boundary. By $\pi_0(M)$, we denote the set of connected components of M, and unless otherwise stated we use $I = [0, 1]$ to denote the closed unit interval.

Let M be an open manifold, by an *exhaustion* $\{M_i\}_{i\in\mathbb{N}}$ we mean a nested collection of compact sub-manifolds $M_i \subseteq \text{int}(M_{i+1})$ with $\cup_{i\in\mathbb{N}} M_i = M$. By *gaps* of an exhaustion $\{M_i\}_{i\in\mathbb{N}}$ we mean the connected components of $\overline{M_i \setminus M_{i-1}}$. We will use $\widehat{\mathbb{C}}$ to denote the Riemann sphere.

1.2 | Some 3-manifold topology

We now recall some facts and definitions about 3-manifold topology. For more details on the topology of 3-manifolds, some references are [16, 17, 19].

Let M be an orientable 3-manifold, then M is said to be *irreducible* if every embedded sphere $\mathbb{S}^2$ bounds a 3-ball $\mathbb{B}^3$. Given a connected properly immersed surface $S \looparrowright M$, we say it is π_1*-injective* if the induced map on the fundamental groups is injective. Furthermore, if $S \hookrightarrow M$ is embedded and π_1-injective we say that the surface S is *incompressible* in M. By the Loop Theorem [17, 19] if $S \hookrightarrow M$ is a two-sided surface that is not incompressible, we have that there is an embedded disk $D \subseteq M$ such that $\partial D = D \cap S$ and ∂D is non-trivial in $\pi_1(S)$. Such a disk is called a *compressing* disk.

An irreducible 3-manifold with boundary $(M, \partial M)$ is said to have *incompressible boundary* if every map of a disk: $(\mathbb{D}^2, \partial\mathbb{D}^2) \hookrightarrow (M, \partial M)$ is homotopic via maps of pairs into ∂M. Therefore, a manifold $(M, \partial M)$ has incompressible boundary if and only if each component S of ∂M is incompressible. We say that a 3-manifold M is *atoroidal* if any π_1-injective torus $T \subseteq M$ is homotopic into ∂M.

Definition 1.1. We say that a 3-manifold M is *hyperbolic*, or hyperbolizable, if $M \cong \mathbb{H}^3/\Gamma$ for $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{C})$ a discrete and torsion-free subgroup. The group Γ is called *Kleinian*.

In general, hyperbolic 3-manifolds that are not closed are open. We will make use of the following convention:

If we say that a compact 3-manifold is hyperbolic, we mean the interior and if M is a finite type hyperbolic 3-manifold, we use $\overline{M}$ to mean its compact manifold closure.

The above convention makes sense since by the Geometrization Theorem [20] and Tameness Theorem [1, 5], any hyperbolic 3-manifold M with finitely generated fundamental group is homeomorphic to the interior of a compact 3-manifold $\overline{M}$ such that $\overline{M}$ is irreducible, atoroidal and has infinite fundamental group.

Given a hyperbolic 3-manifold $M \cong \mathbb{H}^3/\Gamma$, the *convex core* $CC(M) \subseteq M$ is the smallest submanifold with convex boundary whose inclusion induces a homotopy equivalence to M. We say that $M \cong \mathbb{H}^3/\Gamma$ is *convex co-compact* if $CC(M)$ is a compact submanifold and we say that M is *geometrically finite* if $CC(M)$ has finite volume. Some reference for hyperbolic 3-manifolds are [4, 22, 23, 32].

We now prove a couple of topological Lemmas.

Lemma 1.2. *Let M be a compact 3-manifold with boundary and let $\mathcal{P} \subseteq \partial M$ be a collection of pairwise disjoint simple closed curves containing a pants decomposition of ∂M. Let γ_i, $1 \leqslant i \leqslant n$, be the components of $\mathcal{P}$ and assume that every γ_i is π_1-injective in M. For $0 < g_i < \infty$, $1 \leqslant i \leqslant n$, let M' be the 3-manifold obtained by attaching thickenings of $\Sigma_{g_i,1}$ to M by identifying regular neighbourhoods in M of γ_i and $\partial\Sigma_{g_i,1}$. Then, $\partial M'$ has incompressible boundary.*

Proof. Without loss of generality, we can assume that M has connected boundary. Let $(D, \partial D)$ be a compressing disk for $(M', \partial M')$. By an isotopy of D, we can assume that $D \pitchfork U$ for U a regular neighbourhood of $\mathcal{P}$ in ∂M.

If $U \cap D = \varnothing$, we have that $\partial D \subseteq \partial M \setminus \mathcal{P}$ hence D is either in M or in some $\Sigma_{g_i,1}$. If $D \subseteq M$, since $\mathcal{P}$ contains a pants decomposition, it means that ∂D is isotopic into $\mathcal{P}$ giving us a contradiction with the fact that each component of $\mathcal{P}$ π_1-injects in M. If D is contained in some $\Sigma_{g_i,1} \times I$ we have that $\partial D \subseteq \Sigma_{g_i,1} \times \partial I$ but $\Sigma_{g_i,1} \times \partial I$ has no compressing disks in the I-bundle.

Therefore, we have that $\mathcal{A} := D \cap U$ is a, non-empty, collection of essential arcs. Let $D' \subseteq D$ be an innermost disk with respect to the arc system $\mathcal{A} \subseteq D$. Then, $D' \cap U$ has only one component in $\partial D'$. Since $\mathcal{P}$ contains a pants decomposition, up to an isotopy of D', we obtain a disk in either M or $\Sigma_{g_i,1} \times I$ intersecting U in an essential arc α.

The disk D' cannot be contained in $\Sigma_{g_i,1} \times I$ because every compressing disk intersects $\partial\Sigma_{g_i,1} \times I$ in at least two components. If $D' \subseteq M$, then $\partial D'$ is decomposed into two arcs α, β with α an essential arc in U and β an essential arc in $\partial M \setminus U$. However, since $\mathcal{P}$ contains a pants decomposition and U is a thickening of $\mathcal{P}$, there cannot be such an essential β. $\square$

Our last preliminary topological lemma

Lemma 1.3. *Let M be a compact 3-manifold with non-empty boundary ∂M such that no component of ∂M is a torus. Given, $\iota : M \hookrightarrow \mathbb{S}^3$ with handle-body complement H, we can find a pants decomposition $\mathcal{P}$ of ∂M such that $\mathcal{P}$ is a disk-system for H and is π_1-injective in M.*

Proof. Let $\mathcal{D}$ be a disk system[†] for H such that no disk $D \in \mathcal{D}$ is separating in H. We now need to show that the loops $\iota^{-1}(\partial\mathcal{D})$ are essential in M. If not, by the loop Theorem if γ in $\iota^{-1}(\partial\mathcal{D})$ is not π_1-injective in M, then it bounds a disk D'. Let D be the disk of $\mathcal{D}$ corresponding to γ. Then $S = D \cup_\gamma D'$ is an embedded 2-sphere in $\mathbb{S}^3$, and so it is separating. However, since each D is non-separating in $H \subseteq \mathbb{S}^3$, we get a contradiction. $\square$

Remark 1.4. In the setup of Lemma 1.2 and 1.3, we can take a disk system so that no pair is separating in ∂H and so that the manifold M' of Lemma 1.2 has incompressible boundary and the JSJ decomposition of M' is given by the thickened surfaces we attach.

1.3 | Combination theorems

For the reader's convenience, we now recall some Theorems dealing with glueings of Kleinian groups, that is, hyperbolic 3-manifolds.

Theorem 1.5 [20], 4.97. *Let G_1, G_2 be Kleinian groups with fundamental domains D_1, D_2 in $\widehat{\mathbb{C}}$ such that: $\widehat{\mathbb{C}} \setminus D_2 \subseteq int(D_1)$ and $\widehat{\mathbb{C}} \setminus D_1 \subseteq int(D_2)$. Then, the group G generated by G_1 and G_2 is Kleinian and isomorphic to $G_1 * G_2$. Moreover, $D := D_1 \cap D_2$ is a fundamental domain for G on $\widehat{\mathbb{C}}$.*

Definition 1.6. Let $\Gamma \subseteq PSL_2(\mathbb{C})$ be a Kleinian group. Given a subgroup $H \subseteq \Gamma$, we say that $B \subseteq \widehat{\mathbb{C}}$ is *precisely invariant* under H if $H(B) \subseteq B$ and for all $\gamma \in \Gamma \setminus H$ we have that $\gamma(B) \cap B = \emptyset$.

Theorem 1.7 [20, 4.104]. *Let G_1, G_2 be a pair of Kleinian groups such that $G_1 \cap G_2 = H$, where H is a cyclic subgroup. Let D_j be fundamental domains for the actions of G_j on $\widehat{\mathbb{C}}$, $j = 1, 2$. Let B_1, B_2 be open disks in $\widehat{\mathbb{C}}$ such that $J := \overline{B}_1 \cap \overline{B}_2 = \partial B_1 = \partial B_2$ is a topological circle. Suppose the following.*

- *B_j is invariant under H in G_j, $j = 1, 2$.*
- *$D_j' := D_j \cap G_j(B_j) \subseteq B_j$, $j = 1, 2$.*
- *$D_1' \cap D_2$ and $D_1 \cap D_2'$ have non-empty interiors.*

Then, the subgroup $G \subseteq Isom(\mathbb{H}^3)$ generated by G_1, G_2 is Kleinian and isomorphic to $G_1 *_H G_2$. If G_1, G_2 are geometrically finite, then G is also geometrically finite. The quotient $\Omega(G)/G$ is naturally conformally equivalent to

$$\Omega(G_1 \setminus G_1(B_1))/G_1 \cup_L \Omega(G_2 \setminus G_2(B_2))/G_2$$

where the gluing is along $L = [J \cap \Omega(H)]/H$. Any parabolic element in G is either conjugate to G_1 or to G_2 or conjugate to an element commuting with a parabolic element of H.

[†] Such a disk system always exists and is even possible given a disk system $\mathcal{D}$ to surger it, by band sums, to get a new disk system $\mathcal{D}'$ that has no separating component.

Similarly:

Theorem 1.8 [20, 4.105]. *Let G_0 be Kleinian and H_1, H_2 a pair of cyclic subgroups. Let D_0 be a fundamental domain for the actions of G_0 on $\widehat{\mathbb{C}}$. Let B_1, B_2 be open disks in $\widehat{\mathbb{C}}$ and $A \in Isom(\mathbb{H}^3)$ be a Möbius transformation such that $AH_1A^{-1} = H_2$. This conjugation induces an isomorphism $\varphi : H_1 \to H_2$. Suppose the following.*

- *B_j is precisely invariant under H_j in G_0, $j = 1, 2$.*
- *$A(B_1) \cap B_2 = \emptyset$ and $A(\partial B_1) \cap \partial B_2 = J$ is a topological circle.*
- *$gB_1 \cap B_2 = \emptyset$ for all $g \in G_0$.*
- *$D_0 \cap (\widehat{\mathbb{C}} \setminus G_0(B_1 \cup B_2))$ has non-empty interior.*

Then, the subgroup $G \subseteq Isom(\mathbb{H}^3)$ generated by G_0, A is Kleinian and isomorphic to the HNN-extension $G_0 *_{\varphi : H_1 \to H_2}$ of G_0 via φ. If G_0 is geometrically finite, then G is also geometrically finite. The quotient $\Omega(G)/G$ is naturally conformally equivalent to

$$\sim /[\Omega(G_0) \setminus G_0(B_1 \cup B_2)]/G_0,$$

where the identification is such that $[J \cap \Omega(H_2)]/H_2$ is identified with $[A^{-1}(J) \cap \Omega(H_1)]/H_1$ via the projection of A. Any parabolic element in G is either conjugate to G_0 or conjugate to an element commuting with a parabolic element of H_j, $j = 1, 2$.

Remark 1.9 (Parabolic amalgamation). Let z be a parabolic fixed point for the action of a Kleinian group Γ corresponding to a 3-manifold M. By the Universal Horoball Theorem [22, 3.3.4], we can always find an embedded horoball H in $\Omega(\Gamma)$. Therefore, by using the universal horoball, it is easy to glue Kleinian groups Γ_1 and Γ_2 along a common parabolic group $\langle \alpha \rangle$.

2 | REDUCTION TO THE CONVEX CO-COMPACT CASE

We start by recalling a useful lemma about converging sequences of geometric limits.

Lemma 2.1. *If M is the geometric limit of $\{M_i\}_{i\in\mathbb{N}}$ and each M_i is the geometric limit of $\{N_i^n\}_{n\in\mathbb{N}}$, then M is the geometric limit of a sub-sequence $\{N_{a_n}^n\}_{n\in\mathbb{N}}$.*

Proof. Consider the diagram

$$\begin{tikzcd} (M_i, p_i) \arrow[r, "i"] & (M, p) \\ (N_i^n, q_i^n) \arrow[u, "n"] \arrow[ur, dotted] & \end{tikzcd}$$

By geometric convergence in i, we have that $\forall R > 0 : \exists i_R$ such that $\forall i \geqslant i_R$ we have embeddings

$$f_i : (B_R(p), p) \hookrightarrow (M_i, p_i) \qquad f_i \ \ (1 + \varepsilon_i)\text{-bilipschitz} \quad \varepsilon_i \to 0$$

and similar statements for (N_i^n, q_i^n) and (M_i, p_i).

For each i, we have that $f_i(B_R(p)) \subseteq B_{R+\varepsilon_i}(p_i)$ thus we have $(1+\zeta_{i,n})$-bilipschitz embeddings $g_i^n : B_{R+\varepsilon_i}(p_i) \to (N_i^n, q_i^n)$. Therefore, the embeddings

$$g_i^n \circ f_i : B_R(p) \to (N_i^n, q_i^n)$$

are $(1+\varepsilon_i)(1+\zeta_{i,n})$-bilipschitz. Thus, we can find a geometrically convergent sub-sequence. $\square$

We now reduce the general case to the convex co-compact case.

Definition 2.2. We say that a 3-manifold M is in $\mathcal{M}^{\mathbb{S}^3}$ if $M \hookrightarrow \mathbb{S}^3$ is hyperbolic without rank two cusps: $M \cong \mathbb{H}^3/\Gamma, \Gamma \leqslant PSL_2(\mathbb{C})$ and M is either of finite type, that is, $\pi_1(M)$ is finitely generated or $M = \cup_{i\in\mathbb{N}} M_i$ (as in the above sense) in which $\pi_1(M_i) \hookrightarrow \pi_1(M)$. The last condition is equivalent to, up to sub-sequence, $\pi_1(M_i) \hookrightarrow \pi_1(M_{i+1})$.

Lemma 2.3. *Let $M \cong \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold, not necessarily of finite type and with Γ not abelian, without rank two cusps, admitting an embedding $\iota : \overline{M} \hookrightarrow \mathbb{S}^3$. If M admits an exhaustion by π_1-injective compact sub-manifolds, then there is a sequence of finite type hyperbolic 3-manifolds with no parabolics (M_i, p_i) 3-manifolds with embeddings $f_i : (\overline{M}_i, p_i) \hookrightarrow \mathbb{S}^3$ such that $(M_i, p_i) \to (M, p)$ geometrically.*

Proof. Let subsets N_i be π_1-injective sub-manifolds giving us an exhaustion of M, and let $\Gamma_i \subseteq \Gamma$ be the corresponding Kleinian groups. Without loss of generality, we can assume that $N_i \not\cong N_{i+1}$ so that $\Gamma_i \neq \Gamma_{i+1}$. Then, $\Gamma_i \subsetneq \Gamma_{i+1}$ and $\cup_{i\in\mathbb{N}} \Gamma_i = \Gamma$. Then, we obtain the required sequence by

$$(M_i, p_i) := \left(\mathbb{H}^3 \big/ \Gamma_i, [0]\right).$$

Since the N_i are π_1-injective in M, they lift homeomorphically to the covers $\pi_i : M_i \to M$. By Tameness [1, 5], we have that $M_i \setminus N_i$ are product regions and so $M_i \cong \text{int}(N_i)$. Hence, the M_i also embed in $\mathbb{S}^3$, concluding the proof. $\square$

Proposition 2.4. *Let $M \cong \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold in $\mathcal{M}^{\mathbb{S}^3}$. Then, there is a sequence of convex co-compact hyperbolic 3-manifolds (M_i, p_i) with embeddings $f_i : (\overline{M}_i, p_i) \hookrightarrow \mathbb{S}^3$ such that $(M_i, p_i) \to (M, p)$ geometrically.*

Proof. We first deal with the case Γ is abelian, hence of finite type. Any such Kleinian group can be geometrically approximated by a classical Schottky group on two generators and we are done.

Let (M_i, p_i) be the sequence from Lemma 2.3. Since each M_i has no $\mathbb{Z}^2 \in \pi_1(M_i)$ by the Strong Density Theorem [30, 1.4], there is a collection of convex co-compact manifolds $N_n^i \in AH(M_i)$ converging strongly to M_i, moreover without loss of generality, by geometric convergence, we can assume that for all $n : N_n^i \cong M_i$. By Lemma 2.1, we have a sub-sequence $N_{n_i}^i$ that converges geometrically to M. Moreover, since each $N_{n_i}^i$ is homeomorphic to M_i, they admit embeddings $f_i : \overline{N}_{n_i}^i \to \mathbb{S}^3$. $\square$

Remark 2.5. The previous proposition is the only place in the paper in which we actually need the exhaustion and the fact that we have no rank two cusps.

2.1 | General proof assuming convex co-compact approximation

We now assume the following theorem, which we will prove in the next sections. The main step will be a gluing argument that is done in Section 3.

Theorem 4.2. *Let $M \cong \mathbb{H}^3/\Gamma$ be a convex co-compact hyperbolic 3-manifold admitting an embedding $\iota : \overline{M} \hookrightarrow \mathbb{S}^3$. Then, there exists a sequence of Cantor sets $C_i \subseteq \mathbb{S}^3$, $i \in \mathbb{N}$, such that:*

(i) *$N_i := \mathbb{S}^3 \setminus C_i$ is hyperbolic $N_i \cong \mathbb{H}^3/\Gamma_i$;*
(ii) *the N_i converge geometrically to M.*

and prove:

Theorem 2.6. *Let $M \cong \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold and let $M \in \mathcal{M}^{\mathbb{S}^3}$. Then, there exists a sequence of Cantor sets $C_i \subseteq \mathbb{S}^3$, $i \in \mathbb{N}$, such that:*

(i) *$N_i := \mathbb{S}^3 \setminus C_i$ is hyperbolic $N_i \cong \mathbb{H}^3/\Gamma_i$;*
(ii) *the N_i converge geometrically to M.*

Proof. By Proposition 2.4, we have a sequence of convex co-compact manifolds $\overline{M}_i \hookrightarrow \mathbb{S}^3$ that converge geometrically to M. By Theorem 4.2, each M_i is approximated by Cantor set complements; hence, by Lemma 2.1 M is approximated, geometrically, by Cantor set complements. □

3 | GLUING ARGUMENT

In this section, we will show how given $M \subseteq \mathbb{S}^3$ convex co-compact such that M^C, the complement of M in $\mathbb{S}^3$, is a collection of handlebodies H we can extend the metric of M to a new 3-manifold M' such that $M \subsetneq M' \subseteq \mathbb{S}^3$ and $H' :=(M')^C$ is a collection of handlebodies such that $H' \subsetneq H$. Moreover, each component of H contains at least two components of H', and for $h \in \pi_0(H)$ and $h' \in \pi_0(H')$ we have: $\operatorname{diam}(h') \leqslant \frac{1}{2}\operatorname{diam}(h)$. By iterating this argument, we will build our hyperbolic Cantor set complements. The aim of this section is to show our main gluing argument:

Proposition 3.5. *Let M be a convex co-compact hyperbolic manifold with the property that $\mathcal{P} \subseteq \partial M$ is a π_1-injective collection of pairwise disjoint simple closed curves. Let $m := |\mathcal{P}|$ and let $L \in [0, \infty)$. Then, there exists $\{g_i\}_{i=1}^m$ with $1 \leqslant g_i < \infty$ such that we can extend the hyperbolic metric of M to a convex co-compact manifold:*

$$M_L := M \cup_{\mathcal{P}} \coprod_{i=1}^{m} \Sigma_{g_i,1} \times I$$

with the property that:

(1) *in $\Sigma_{g_1,1} \times I$, the geodesic corresponding to $\mathcal{P}_i$ has a collar of width at least L;*
(2) *if $\mathcal{P}$ contains a pants decomposition, then M_L has incompressible boundary.*

Before showing Proposition 3.5, we show that given a compact convex co-compact manifold M embedding in $\mathbb{S}^3$, we can assume, up to geometric limit, that it has handle-body complement.

Lemma 3.1. *Let $\iota : M \hookrightarrow \mathbb{S}^3$ be a compact convex co-compact hyperbolic manifold. Then, by adding a collection of 1-handles H to M, we have an embedding $\iota' : M \cup_\partial H \hookrightarrow \mathbb{S}^3$, extending the metric, such that $\overline{\mathbb{S}^3 \setminus \iota'(M \cup_\partial H)}$ is a collection of handlebodies and $M \cup_\partial H$ is convex co-compact.*

Proof. If $\overline{\iota(M)^C}$ is a collection of handlebodies, there is nothing to do. Otherwise, let $N \subseteq \overline{\iota(M)^C}$ be a non-handlebody component. Let $C = \mathcal{H}_g \cup Q$ be a minimal genus Heegaard splitting of N, where $\mathcal{H}_g$ is a genus g handlebody, and Q is a collection of 2-handles. Attaching a 2-handle P to H_g is equivalent to attaching a 1-handle P' to $\overline{\iota(M)}$. Thus, we get that by attaching all 1-handles to $\iota(M)$ we can make N a handlebody component. Therefore, there is a collection of 1-handles H and an embedding $\iota' : M \cup H \hookrightarrow \mathbb{S}^3$ such that $\overline{\mathbb{S}^3 \setminus \iota'(M \cup_\partial H)}$ is a collection of handlebodies.

We now need to show that we can realize the above topological construction while extending the given hyperbolic metric on $M \cong \mathbb{H}^3/\Gamma$. This essentially follows from Ping-Pong Lemma (Theorem 1.5). There are two cases depending on whether the 1-handle P is attached to one or two boundary components of M. We will indicate by S_1 and S_2 these two boundary componentss3.4.

Assume $S_1 \neq S_2$. Let D_1 be a fundamental domain for the action of Γ on $\widehat{\mathbb{C}}$. Since Γ is convex co-compact $\Gamma.D_1$ has full measure and let $F_1 := \widehat{\mathbb{C}} \setminus D_1$. Pick two points x_1 and x_2 in $\mathrm{int}(D_1) \cap \widetilde{S}_1$ and $\mathrm{int}(D_1) \cap \widetilde{S}_2$ respectively, and let $h_\lambda \in \mathrm{Isom}^+(\mathbf{H}^3)$, $\lambda \in (0, \infty)$, be the loxodromic element with fixed points x_1 and x_2 and translation length λ. Let $D_2(\lambda)$ be the fundamental domain of $\langle h_\lambda \rangle$ and $F_2 := \widehat{\mathbb{C}} \setminus D_2$. Since as $\lambda \to \infty$:

$$D_2(\lambda) \xrightarrow{\text{Hausdorff}} \widehat{\mathbb{C}} \setminus \{x_1, x_2\} \qquad F_2(\lambda) \xrightarrow{\text{Hausdorff}} \{x_1, x_2\},$$

we get that there is $\lambda \in (0, \infty)$ such that

$$D_2(\lambda) \supset F_1 \qquad D_1 \supset F_2(\lambda)$$

Then, by Theorem 1.5, $\Gamma' := \langle \Gamma, h_\lambda \rangle$ is discrete, isomorphic to $\Gamma * h_\lambda$ and $\mathbb{H}^3/\Gamma'$ has the required topological type.

If $S_1 = S_2$, let D_1 and F_1 as before and pick $x \neq y$ to be points in $D_1 \cap \widetilde{S}_1$. Then, by the same reasoning as before, we can find h_λ such that $\Gamma' := \langle \Gamma, h_\lambda \rangle$ is discrete, isomorphic to $\Gamma * h_\lambda$ and $\mathbb{H}^3/\Gamma'$ has the required topological type. □

We now define:

Definition 3.2. Let N be a geometrically finite 3-manifold, we say that the convex core of N is homeomorphic to $\Sigma_{g,k,n} \times I$ if $CC(N)$ has n rank 1 cusps, k funnels and there is a type-preserving homeomorphism $f : N \xrightarrow{\cong} \Sigma_{g,k,n} \times I$.

The next Lemma constructs a handlebody piece that will be attached to M via cyclic amalgamation, Theorem 1.7, along a peripheral loxodromic γ. This particular construction produces a rank-1 cusp that we will have to deal with later. The loxodromic element γ and γ-invariant disk $B \subseteq \partial_\infty \mathbb{H}^3$ in the statement will be obtained from M by taking an incompressible curve in ∂M and lifting a collar around it.

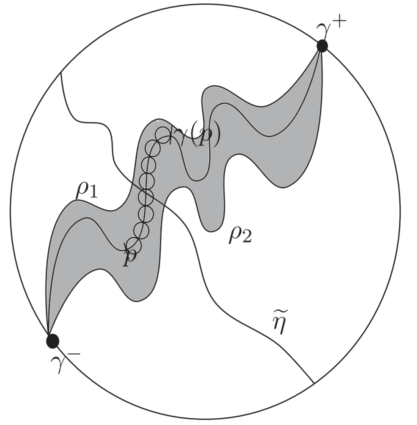

FIGURE 1 The disks Δ_i in the domain of discontinuity containing a lift of γ. The shaded region is the ball B'

Lemma 3.3. *Given $\gamma \in PSL_2(\mathbb{C})$ loxodromic element and a closed γ-invariant disk $B \subseteq \partial_\infty \mathbb{H}^3$, then there is a Schottky group extension of $\langle\gamma\rangle$, $\Gamma = \Gamma_B$, such that:*

(1) *the limit set of Γ is included in B;*
(2) *the convex core of $\mathbb{H}^3/\Gamma$ is homeomorphic to $\Sigma_{g,1,1} \times [0,1]$, where the boundary component of $\Sigma_{g,1,1}$ corresponds to γ and the puncture to a rank-1 cusp.*

Moreover, such group Γ can be taken so that γ has a collar larger than any given constant.

Proof. Take $B' \subseteq B$, a smaller region delimited by two γ-invariant smooth arcs ρ_1, ρ_2 joining the fixed points of γ. Furthermore, select a third γ-invariant smooth path $\rho \subseteq B'$ so that $A = B'/\langle\gamma\rangle$ is an annulus with boundary $\rho_1/\langle\gamma\rangle \cup \rho_2/\langle\gamma\rangle$ and π_1 representative embedded curve $\rho/\langle\gamma\rangle$.

We would have to find $F \subseteq B'$ so that F is a fundamental region of A and $\rho \cap F = \rho_F$ is connected. To do this, denote by $\gamma_\pm$ the fixed points of γ. Consider a closed path η in the annulus $(\partial_\infty \mathbb{H}^3 \setminus \{\gamma_\pm\})/\langle\gamma\rangle$, such that η intersects each one of $\rho/\langle\gamma\rangle, \rho_1/\langle\gamma\rangle, \rho_2/\langle\gamma\rangle$ exactly once (and the annulus $A = B'/\langle\gamma\rangle$ in a connected segment). Define F_0 as the lift in $\partial_\infty \mathbb{H}^3$ of the complement of η in $(\partial_\infty \mathbb{H}^3 \setminus \{\gamma_\pm\})/\langle\gamma\rangle$. This makes F_0 a disjoint union of connected components, and the closure of any of these components is a fundamental domain for $(\partial_\infty \mathbb{H}^3 \setminus \{\gamma_\pm\})/\langle\gamma\rangle$. Then one can verify that F can be obtained by $F := F_0 \cap B$. Cover ρ_F by closed disks $\{\Delta_i\}_{-N \leqslant i \leqslant N}$ in B', see Figure 1, such that:

(1) Δ_i, Δ_{i+1} are tangent for all $0 \leqslant i \leqslant 4N-2$, $\{p_{i+1}\} = \Delta_i \cap \Delta_{i+1}$;
(2) $\Delta_i \cap \Delta_j = \emptyset$ for $|i-j| \geqslant 2$;
(3) $\Delta_{4N-1} = \gamma(\Delta_0)$.

Iterate by powers of γ to obtain, $\{\Delta_i\}_{i\in\mathbb{Z}}$, a covering of ρ by disk in B such that:

(1) Δ_i, Δ_{i+1} are tangent for all $i \in \mathbb{Z}$, $\{p_{i+1}\} = \Delta_i \cap \Delta_{i+1}$;
(2) $\Delta_i \cap \Delta_j = \emptyset$ for $|i-j| \geqslant 2$;
(3) $\Delta_{i+4N} = \gamma(\Delta_i)$.

Select f_i a Mobius map that sends the triple $(\partial_\infty \mathbb{H}^3 \setminus (\mathring{\Delta}_i), p_i, p_{i+1})$ to the triple $(\Delta_{i+2}, p_{i+3}, p_{i+2})$, so that $\gamma^{-1} \circ f_i \circ \gamma = f_{i+4N}$ (make a priori such a selection). Furthermore, denote by $a_i = f_{4i}$, $b_i = f_{4i+1}$. Let Γ be the group generated by $a_0, b_0, \dots, a_{N-1}, b_{N-1}, \gamma$ (also generated by $\langle\{a_i, b_i\}_{i\in\mathbb{Z}}, \gamma\rangle$).

Then by modifying the proof of Theorem 1.5, we can prove that Γ is a Kleinian group freely generated by $a_0, b_0, \ldots, a_{N-1}, b_{N-1}, \gamma$. Indeed, for a_i, b_i take fundamental domains as the complement of the appropriate disks Δ_i, and take F_0 as the fundamental domain for γ. Taking any two of these fundamental domains (and denoting them by D_1, D_2), we have that $D_1 \supset (\mathbb{C} \setminus D_2), D_2 \supset (\mathbb{C} \setminus D_1)$ rather than $\mathrm{int}(D_1) \supset (\mathbb{C} \setminus D_2), \mathrm{int}(D_2) \supset (\mathbb{C} \setminus D_1)$, the latter as in Theorem 1.5. This is because of the tangencies we consider. Nevertheless, if D denotes the intersection of all fundamental domains and w is a nontrivial word generated by $a_0, b_0, \ldots, a_{N-1}, b_{N-1}, \gamma$, it follows that for any $z \in \mathrm{int}(D)$ we have $w(z) \notin D$. This implies that Γ is freely generated by $a_0, b_0, \ldots, a_{N-1}, b_{N-1}, \gamma$ and that D is a fundamental domain for Γ in $\partial_\infty \mathbb{H}^3 = \mathbb{S}^2$.

It remains to show that Γ is discrete. Take $x \in \mathbb{H}^3$ in the complement of all the half-spaces bounded by the functions Δ_i, intersected with a fundamental domain of γ bounded by F_0 (which is the complement of two topological half-spaces). Take into consideration that all half-spaces can be taken mutually disjoint. Then assume that there is a sequence $\{g_k\} \subseteq \Gamma$ so that $g_k(x) \to x$. For any $g_k \neq id$, we have that $g_k(x)$ belongs to one of the discarded half-spaces. Hence $g_k = id$ for k sufficiently large, and from which we know that Γ is a Kleinian group. And since D is a fundamental domain for Γ in $\partial_\infty \mathbb{H}^3 = \mathbb{S}^2$, it follows that the limit set of Γ is contained in B. This is because the complement of $\langle \gamma \rangle D$ is contained in B'.

Note that all the points of tangencies $\{p_i\}$ are identified with one another in the quotient by Γ, where the element $\alpha = \gamma[a_{N-1}, b_{N-1}], \ldots [a_0, b_0]$ fixes p_0. Moreover, α preserves the direction tangential to the disks meeting at p_0. In order to make α parabolic, we can make choices so that $D\alpha_{p_0}$ has norm 1 with respect to the standard $\mathbb{S}^2$ metric. Take the loxodromic element c_λ with real translation and fixed points p_2, p_3, so that the derivatives of c_λ at p_2, p_3 are λ^{-1}, λ, respectively. We can choose then $c_\lambda \circ a_0$ instead of a_0. The new choice $c_\lambda \circ a_0$ satisfies the same conditions as a_0 and introduces a factor λ twice while applying chain rule for $D\alpha_{p_0}$ (once for c_λ at p_3 and once for c_λ^{-1} at p_2). Then by taking the appropriate value for λ, we make α parabolic. We claim then that such Γ is geometrically finite with convex core homeomorphic to $\Sigma_{g,1,1} \times [0,1]$, where the boundary component of $\Sigma_{g,1,1}$ corresponds to γ and the puncture to the rank-1 cusp generated by α. Indeed, we can select a smooth metric in D/Γ so that p_0 is a hyperbolic cusp. By taking the Epstein envelope surface [14] of a sufficiently small multiple of the selected metric, we obtain a finite volume core with convex boundary. Then Γ is geometrically finite. Finally, the boundary of the core can be easily seen as $\Sigma_{2g,0,2}$, where γ is a separating curve that divides the quotient into components homeomorphic to $\Sigma_{g,1,1}$. From here we can see that the convex core of $\mathbb{H}^3/\Gamma_B$ is homeomorphic to $\Sigma_{g,1,1} \times [0,1]$, where γ corresponds to the boundary component of $\Sigma_{g,1,1}$ and the puncture to a rank-1 cusp.

As a final remark, observe that the collar around γ gets bigger as we take the region B and the disks Δ_i smaller. □

We now start the first step of our main gluing construction:

Lemma 3.4. *Let M be a convex co-compact hyperbolic manifold with the property that $\mathcal{P} \subseteq \partial M$ is a π_1-injective collection of disjoint non-homotopic curves. Let $n := |\mathcal{P}|$ and let $L \in [0, \infty)$. Then, there exists $\{g_i\}_{i=1}^n$ with $1 \leqslant g_i < \infty$ such that we can extend the hyperbolic metric of M to a geometrically finite manifold:*

$$M'_L := M \cup_{\mathcal{P}} \coprod_{i=1}^{n} \Sigma_{g_i,1,1} \times I$$

with the property that:

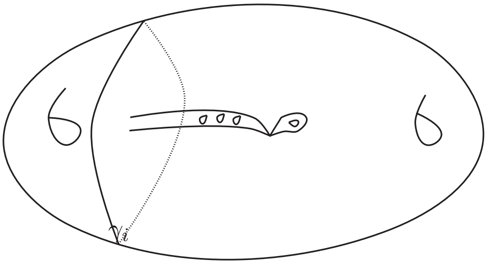

FIGURE 2 Partial stage in which we glued a punctured torus to a $\Sigma_{3,1,1} \times I$ along one γ_i. The rank 1 cusp C_i and accidental parabolic correspond to the node

(1) *$\Sigma_{g_1,1,1} \times I$ has a rank one cusp corresponding to a boundary component of $\Sigma_{g_1,1,1}$, and the other boundary is glued to a component $\mathcal{P}_i$ of $\mathcal{P}$;*
(2) *in $\Sigma_{g_1,1,1} \times I$, the geodesic corresponding to $\mathcal{P}_i$ has a collar of width at least L.*

Proof. Since each element $\mathcal{P}_i$ in $\mathcal{P}$ is π_1-injective, then it has a loxodromic element $\gamma_i \in \pi_1(M)$, and a γ_i-invariant disk B_i in the domain of discontinuity of M. By Lemma 3.3, there exist Schottky group extensions Γ_{B_i} with limit set in B_i and collars around γ_i as large as we desire. Then by Theorem 1.7, the manifold $M' = M'(L)$ with fundamental group generated by $\langle \pi_1(M), \Gamma_{B_1}, \dots \Gamma_{B_n} \rangle$ has the desired properties, provided that the groups $\{\Gamma_{B_i}\}$ from Lemma 3.3 have all collars bigger than L around the geodesics that each of them is extending. $\square$

We can now prove our main gluing step, where we will deal with the parabolics:

Proposition 3.5. *Let M be a convex co-compact hyperbolic manifold with the property that $\mathcal{P} \subseteq \partial M$ is a π_1-injective collection of pairwise disjoint simple closed curves. Let $m := |\mathcal{P}|$ and let $L \in [0, \infty)$. Then, there exists $\{g_i\}_{i=1}^m$ with $1 \leqslant g_i < \infty$ such that we can extend the hyperbolic metric of M to a convex co-compact manifold:*

$$M_L := M \cup_{\mathcal{P}} \coprod_{i=1}^{m} \Sigma_{g_i,1} \times I$$

with the property that:

(1) *in $\Sigma_{g_i,1} \times I$ the geodesic corresponding to $\mathcal{P}_i$ has a collar of width at least L;*
(2) *if $\mathcal{P}$ contains a pants decomposition, then M_L has incompressible boundary.*

Proof. Start with the manifold M'_L coming from Lemma 3.4 and let C_i be the rank 1 cusps corresponding to the $\Sigma_{g_i,1,1} \times I$ attached to γ_i. By applying Klein–Maskit combination (Theorem 1.7) to universal horoballs to each rank 1 cusps, we attach a $\Sigma_{1,1} \times I$ manifold. This gives us a new manifold:

$$M''_L := M \cup_{\mathcal{P}} \coprod_{i=1}^{m} \Sigma_{g_i+1,1} \times I$$

in which the $\Sigma_{g_i+1,1} \times I$ have an accidental parabolic δ_i corresponding to the remaining rank 1 cusp C_i coming from the Klein–Maskit combination, see Figure 2.

Note that if $\mathcal{P}$ contains a pants decomposition, then, by Lemma 1.2, the manifold M''_L has incompressible boundary.

For each rank-1 cusp, we can find invariant tangent disk at the corresponding fixed point, by cyclic amalgamation, see Remark 1.9, we can glue each rank-1 cusp onto itself to produce a geometrically finite manifold with rank-2 cusps. Each cusp has an embedded cylinder toward each of the two boundary components where it appears as an accidental parabolic.

Thus, we get manifolds:

$$\widehat{M}_L = M \cup_{\mathcal{P}} \coprod_{i=1}^{m} \left(\Sigma_{g_i+1,1} \times I \setminus \delta_i \times \{1/2\}\right)$$

still extending the metric on M.

By Thurston's Dehn Filling Theorem [4, 9 32], we have $N \in \mathbb{N}$ such that for all $n > N$ the manifolds $\widehat{M}^n_L$ obtained from $\widehat{M}_L$ by doing $\frac{1}{n}$-Dehn Filling on every rank two cusp, see [21], are convex co-compact. Moreover, by taking a larger N, if necessary, we can also assume that

$$\widehat{M}^n_L \cong M \cup_{\mathcal{P}} \coprod_{i=1}^{m} \left(\Sigma_{g_i+1,1} \times I\right),$$

where the homeomorphisms φ_n restrict to the identity on M and are induced by $\tau^n_{\gamma_i}$, the nth Dehn twist along δ_i, on $\Sigma_{g_i+1,1} \times I$. Hence, for all L and n, the manifolds $\widehat{M}^n_L$ are convex co-compact and have incompressible boundary by Lemma 1.2.

Finally, we have that

$$\widehat{M}^n_L \xrightarrow[n\to\infty]{geom} \widehat{M}_L.$$

Thus, by definition of geometric convergence, by taking n large enough and some $L' > L$, we can assume that in $M_L := \widehat{M}^n_{L'}$ all the geodesics corresponding to $\mathcal{P}$ have a collar of width at least L. Hence, the manifold M_L satisfies all the requirements of the proposition completing the proof. □

Corollary 3.6. *Let M be a convex co-compact hyperbolic manifold with the property that $\mathcal{P} \subseteq \partial M$ is a π_1-injective collection of pairwise disjoint simple closed curves. Let $m := |\mathcal{P}|$, $p \in CC(M)$, $R > 0$ and $n \in \mathbb{N}$ there exists $L = L(p, R, n)$ and*

$$f : N_R(CC(M)) \hookrightarrow M \cup_{\mathcal{P}} \coprod_{i=1}^{m} \left(\Sigma_{g_i+1,1} \times I\right)$$

such that f is $(1 + \frac{1}{n})$-bi-Lipschitz.

Proof. Pick $\{L_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ such that $L_n \nearrow \infty$. Build the manifolds $M_n := M_{L_n}$ as in Proposition 3.5. It is easy to see that for any $p \in CC(M)$, by property (1) of Proposition 3.5, the sequence

$$(M_n, p) \xrightarrow{geom} (M, p)$$

giving us the desired result. □

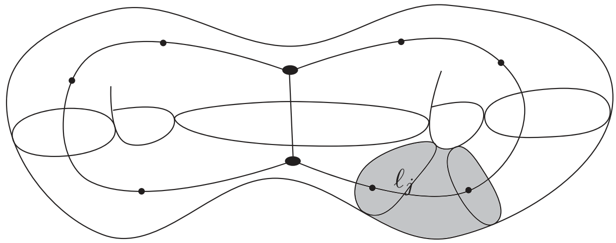

FIGURE 3 Nerve subdivision. The shaded ball B_j is the thickening of an ℓ_j section whose diameter is less than $\frac{1}{4}\operatorname{diam}(H)$

We can now prove our iteration step. One of the main takeaways is that we can choose our embeddings so that the diameter of the complement decays to 0 as we iterate the process, which is necessary to obtain a Cantor set complement.

Proposition 3.7. *Let $\iota : M \hookrightarrow \mathbb{S}^3$ be an embedding of a compact irreducible manifold whose complement is a collection of handlebodies H_i, $1 \leqslant i \leqslant n$. Then, by attaching a finite collection $\Sigma_{g_h^i,1} \times I$, $1 \leqslant h \leqslant n_i$ to a collection of disks D_i on ∂H_i, containing a disk system for H_i, we obtain a new embedding:*

$$\iota' : M \bigcup_{i=1}^{n} \cup_{h=1}^{n_i} \Sigma_{g_h^i,1} \times I \hookrightarrow \mathbb{S}^3$$

extending ι such that $\iota'(\cup_{h=1}^{n_i} \Sigma_{g_h,1}) \subseteq H_i$ and $\overline{H_i \setminus \iota'(\cup_{h=1}^{n_i} \Sigma_{g_h^i,1})}$ is a collection $J_1^i, \ldots, J_{m_i}^i$ of handlebodies with $m_i \geqslant 2$ and $\operatorname{diam}(J_{m_j}^i) \leqslant \frac{1}{2}\operatorname{diam}(H_1)$. Moreover, if int$(M) \cong \mathbb{H}^3/\Gamma$ is convex co-compact given $L > 0$, we can extend the hyperbolic metric to $M \bigcup_{i=1}^{n} \cup_{h=1}^{n_i} \Sigma_{g_h^i,1}$ so that each attaching region has a collar of width at least L.

Proof. Let Γ be the hyperbolic structure on M. It suffices to prove the statement for each handlebody component H_i, for the sake of notation, we will just refer to it as H. Let $\mathcal{D}$ be the disk system coming from Lemma 1.3.

Take a nerve on the handlebody H so that in each ball component of $\overline{H \setminus N_\varepsilon(\mathcal{D})}$ we have a trivalent vertex. By using copies of disks in $\mathcal{D}$, we subdivide the nerve into sections $\ell_1, \ldots, \ell_\kappa$ so that each ball component B_m, $1 \leqslant m \leqslant \kappa$, has diameter less than $\frac{1}{4}\operatorname{diam}(H)$, see Figure 3.

This gives us a collection of disks $\mathcal{D}' \subseteq H$ containing a pants decomposition of ∂H. Moreover, each component of $\mathcal{D}'$ π_1-injects in M. Then, by applying Corollary 3.6 to $(\mathcal{D}', \partial\mathcal{D}')$, we obtain a hyperbolic 3-manifold $M \cup_{h=1}^{n} \Sigma_{g_h,1}$ extending Γ.

We now construct the nested family of handlebodies obtained by successively attaching handles to the curves homotopic to $\partial\mathcal{D}'$. We do this so that each section from $\ell_1, \ldots, \ell_\kappa$ appear inside a handlebody. To each disk $D \in \pi_0(\mathcal{D}')$, we attach g_h 2-handles by drilling them from the adjacent 3-ball in an unknotted way so that they complement is a handlebody.

Each handlebody $J_1, \ldots, J_\kappa$ is a thickening of an element of $\ell_1, \ldots, \ell_\kappa$ with some handles attached or drilled in. Moreover, we can do it so that the resulting handle is still close to the corresponding element of $\ell_1, \ldots, \ell_\kappa$, and more importantly so that each complementary region's diameter is less than $\frac{1}{2}\operatorname{diam}(H)$. Since $\kappa \geqslant 3g(H) - 3 > 2$, we complete the proof. $\square$

Remark 3.8. Note that we can make the resulting manifold of Proposition 3.7 boundary incompressible by selecting a π_1-injective pants decomposition during the last iteration of the handle attaching.

4 | PROOF FOR CONVEX CO-COMPACT

Before proving the main result, we prove the following key Proposition:

Proposition 4.1. *Let (M, p) be a convex co-compact hyperbolic 3-manifold admitting an embedding $\iota : (\overline{M}, p) \hookrightarrow \mathbb{S}^3$ with complement given by a collection of handlebodies $\mathcal{H}$. Given $R > 0$, there exists a Cantor set $C_R \subseteq \mathcal{H}$ such that $\mathbb{S}^3 \setminus C_R$ is hyperbolizable and $B_R(p) \subseteq \mathbb{S}^3 \setminus C_R$ is $1 + e(R)$ bilipschitz to the R-ball around p in M. Moreover, $e(R) \to 0$ as $R \to \infty$.*

Proof. Pick $L > R$ and apply Proposition 3.7 to $\iota : M \hookrightarrow \mathbb{S}^3$ to obtain a new manifold N_1^L so that all new topology is at distance $L > R$ from $CC(M)$. We then reiterate this construction using the same L. We thus obtain a collection of convex co-compact hyperbolic 3-manifolds N_n^L admitting nested embedding $\overline{N}_n^L \subseteq \overline{N}_{n+1}^L$ whose complement in $\mathbb{S}^3$ is a collection of handlebodies H_n and whose direct limit N_∞^L is homeomorphic to the complement of a sub-set K of $\mathbb{S}^3$.

Claim 1: The set K is a Cantor set so that $N_\infty^L \cong \mathbb{S}^3 \setminus C_R$.

Proof of Claim:. To show that K is a Cantor set, we need to show that it is a compact, perfect, totally disconnected metric space. Let $C := \mathrm{diam}(H_1)$. By construction, it is easy to see that $K = \cap_{n\in\mathbb{N}} H_n$ where each H_n is a collection of handlebodies in which each component of H_n contains at least two components of H_{n+1}. Moreover, by Proposition 3.7, we have that for H a component of H_n: $\mathrm{diam}\, H \leqslant 2^{-n} C$ so that K is a collection of points. Since each component of H_n contains at least two components of H_{n+1} we see that K is also totally disconnected. Thus, being a closed sub-set of a compact metrizable space, it is compact and metrizable as well. The fact of it being perfect is also a straightforward consequence of the nesting construction. □

Claim 2: The $B_R(p) \subseteq \mathbb{S}^3 \setminus C_R$ is $1 + e(L)$ bi-lipschitz to the R-ball around p in M and $e(L) \to 0$ as $L \to \infty$.

Proof of Claim:. This follows from Proposition 3.5. □

If $R \to \infty$, so does L, and the last claim of the Proposition is proven. □

We now finish the proof of the main result:

Theorem 4.2. *Let $M \cong \mathbb{H}^3/\Gamma$ be a convex co-compact hyperbolic 3-manifold admitting an embedding $\iota : \overline{M} \hookrightarrow \mathbb{S}^3$. Then, there exists a sequence of Cantor sets $C_i \subseteq \mathbb{S}^3$, $i \in \mathbb{N}$, such that:*

(i) *$N_i := \mathbb{S}^3 \setminus C_i$ is hyperbolic $N_i \cong \mathbb{H}^3/\Gamma_i$;*
(ii) *the N_i converge geometrically to M.*

Proof. By Lemma 3.1, we can assume that we have $M_i \to M$ geometrically with embeddings $\iota_i : \overline{M}_i \to \mathbb{S}^3$ such that $\iota_i(\overline{M}_i)^C$ are handlebodies for every i. Then, by Lemma 2.1, it suffices to prove the Theorem for such an M_i.

Thus, let M be a convex co-compact hyperbolic 3-manifold with an embedding $\iota : \overline{M} \to \mathbb{S}^3$ that has for complement a collection of handlebodies $\mathcal{H} = \{H_1, \dots, H_n\}$.

Choose any strictly increasing sequence R_n. By applying Proposition 4.1 to (M, p, R_n), we obtain a sequence of Cantor set complements $(\mathbb{S}^3 \setminus C_n, p, R_n)$ that geometrically converge to M, concluding the proof. $\square$

Since, in particular, $\mathbb{H}^3 \hookrightarrow \mathbb{S}^3$ we have Cantor sets complements $N_n := \mathbb{S}^3 \setminus C_n$ and points $p \in \mathbb{H}^3$ and $p_n \in N_n$ such that

$$(N_n, p_n) \to (\mathbb{H}^3, p)$$

geometrically. Thus, the balls of radius $B_R(p) \subseteq \mathbb{H}^3$ can be $(1+\varepsilon_n)$-isometrically embedded in N_n. In particular, this means that for large enough n the set of points of distance, say, $\frac{R}{2}$ from p_n is simply connected and so $inj_{p_n}(N_n) \geqslant \frac{R}{2}$. Since R was arbitrary, we obtain:

Corollary 4.3. *For all $R > 0$, there exists a Cantor sets $C \subseteq \mathbb{S}^3$ such that $\mathbb{S}^3 \setminus C$ is hyperbolic and there is a point $p \in \mathbb{S}^3 \setminus C$ with injectivity radius at least R.*

However, we do not necessarily know what the shape of the corresponding Cantor set is. Moreover, as in [25], one can obtain hyperbolic Cantor set complements with small eigenvalues of the Laplacian, arbitrarily many short geodesics or surfaces with arbitrarily small principal curvatures.

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