

RESEARCH ARTICLE

Hölder regularity of the Boltzmann equation past an obstacle

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RS-2023-00212304**Abstract**

Regularity and singularity of the solutions according to the shape of domains is a challenging research theme in the Boltzmann theory. In this paper, we prove an Hölder regularity in $C_{x,v}^{0,\frac{1}{2}-}$ for the Boltzmann equation of the hard-sphere molecule, which undergoes the elastic reflection in the intermolecular collision and the contact with the boundary of a convex obstacle. In particular, this Hölder regularity result is a stark contrast to the case of other physical boundary conditions (such as the diffuse reflection boundary condition and in-flow boundary condition), for which the solutions of the Boltzmann equation develop discontinuity in a codimension 1 subset (Kim [Comm. Math. Phys. 308 (2011)]), and therefore the best possible regularity is BV, which has been proved by Guo et al. [Arch. Rational Mech. Anal. 220 (2016)].

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1 | INTRODUCTION

The Boltzmann equation is one of the fundamental mathematical equations for rarefied collisional gases. It describes the motion of binary collisional gas as a partial differential equation of distribution function $F(t, x, v)$. Taking no external forces into account, the distribution function $F(t, x, v)$ solves

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \quad F(0, x, v) = F_0(x, v), \quad (1.1)$$

where nonlinear quadratic term $Q(F, F)$ means collision operator which has form of (we abbreviate arguments t, x for convenience)

$$Q(F_1, F_2) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - u, \omega) [F_1(u') F_2(v') - F_1(u) F_2(v)] d\omega du, \quad (1.2)$$

with the post collision velocities u' and v' which can be written as $u' = u - ((v - u) \cdot \omega)\omega$ and $v' = v + ((v - u) \cdot \omega)\omega$, respectively, for $\omega \in \mathbb{S}^2$. For collision kernel $B(v - u, \omega) = |v - u|^\kappa q_0(|\frac{v-u}{|v-u|} \cdot \omega|)$, we work on a hard sphere model $B(v - u, \omega) = |(v - u) \cdot \omega|$, or $\kappa = 1$ with $q_0(|\cdot|) = |\cdot|$.

Due to its own importance and applications in the statistical physics, there have been extensive researches on the Boltzmann equation ([4, 6]). Guo established the energy method for the Boltzmann equation to construct global *smooth* solutions and their asymptotical stability near Maxwellians (e.g. [16]). In [5], Desvillettes and Villani proved the asymptotic stability of the global Maxwellian under the assumptions of high order apriori bound and Gaussian lower bound of the *smooth* solutions. More recently, Imbert and Silvestre [18] derived conditional C^∞ -*smoothness* of the non-cutoff Boltzmann equation, using the theory of integro-differential equations. Forementioned works, especially [6, 16, 18], deal with idealized periodic domains or whole space, in which the solutions can remain *smooth* if initially so.

However, a physical boundary is present in many applications, which changes global properties (such as *smoothness*) of solutions in general. In [13], Guo et al. proved that in convex domains, the first-order derivatives are continuous away from a grazing set, where a particle hits the boundary with tangential velocity, for the first time. In [3], Chen and Kim proved some higher regularity up to $C^{1,1-}$ away from the boundary for the steady problem of diffuse the boundary. When the domain is non-convex, Kim proved the formation and propagation of discontinuity along the characteristics emanating from the grazing set of non-convex boundary of the diffuse/inflow/bounce-back reflection in [19]. As such characteristics form a codimension 1 subset in the phase space, the optimal regularity is BV. Indeed in [14], they proved this optimal BV regularity for the diffuse and inflow boundary. We also refer very recent result [25] which shows that higher order regularity is hard to obtain in general even for free transport equation in the presence of physical boundary conditions.

The regularity question of the Boltzmann equation with the specular reflection boundary condition (a particle hits the boundary and bounces back like a billiard) in the non-convex domains has been a challenging open problem. Even the global well-posedness of such problem has been an outstanding open problem since an announcement of [27] without full justification in 1977. Only very recently, the question is settled affirmatively in [17, 23]. In particular, in [23], Kim-Lee constructed the first unique global-in-time solution and proved asymptotic stability of the Boltzmann equation near equilibria in smooth convex domains, which completely settled the classical long-standing (40 years) open question of the kinetic community in the affirmative. As the key of

the proof, they establish a novel $L^2 - L^\infty$ estimate using triple iterations of the Duhamel representation along the billiard trajectory in smooth convex domains. In [22, 24], they further extended the result of [23] to the cylindrical non-convex domains and general 3D toroidal domains, which are the first results for non-convex domains.

In this paper, we are studying Hölder regularity of the Boltzmann equation in some non-convex domains with the specular reflection boundary condition in a full 3D setting. For the reader's convenience, we state an informal statement of Theorem 2.9:

Theorem 1.1 (Informal statement). *A local-in-time solution of the Boltzmann equation outside a general convex domain satisfying the specular reflection boundary condition is $C_{x,v}^{0, \frac{1}{2}-}$ if the initial datum is $C_{x,v}^{0,1-}$.*

The major difficulty of the problem is that the result of the velocity lemma of [13], which is a powerful tool of the regularity estimate in convex domains, is no longer true for non-convex domains. Therefore, we have to develop a *completely new* technique in the paper. Moreover, this Hölder regularity result is a stark contrast to the result of other physical boundary conditions [14, 19], for which the formation and propagation of discontinuity happens in general.

Beyond the theoretical importance in general, the quantitative regularity estimate has several significant applications. Based on the regularity estimate results [13], later Cao-Kim-Lee studied Vlasov-Poisson-Boltzmann systems. When particles are charged (e.g., plasma), they interact through not only collisions but also the Lorentz force generated by charged particles. Master equations for this situation are Vlasov-Boltzmann equation coupled with Maxwell system or simpler Poisson equation. Mathematical theory of boundary problems of such coupled system is not developed satisfactorily mainly due to the intrinsic singularity of solutions. In [1], Cao-Kim-Lee construct the first unique global-in-time solution of Vlasov-Poisson-Boltzmann system with the diffuse reflection boundary condition in smooth convex domains and proved its convergence toward equilibria when the initial data is small enough. In [2] the authors construct a unique local-in-time solution of Vlasov-Poisson-Boltzmann system with a generalized diffuse reflection boundary condition in smooth convex domains without the size restriction.

Lastly, we note that the $L^2 - L^\infty$ method established in [17, 23] has been widely used to various problems, such as global well-posedness with large amplitude external potential ([20]), rotational setting [21], and large amplitude problem with smooth convex domain [10]. We also refer recent global well-posedness results with large amplitude initial data for various setting [8, 9, 11, 26].

First let us define a convex ball $\mathcal{O}$ and the domain Ω .

Definition 1.2. Throughout this paper, we assume that the domain $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{O}}$ is an exterior domain of $\mathcal{O}$, which is a uniformly convex bounded open subset in $\mathbb{R}^3$: $\mathcal{O} = \{x \in \mathbb{R}^3 : \xi(x) > 0\}$ and there exists $\theta_\Omega > 0$ such that

$$\zeta \cdot (-\nabla^2 \xi(x)) \zeta := \sum_{i,j=1}^3 (-\partial_{ij} \xi(x)) \zeta_i \zeta_j \geq \theta_\Omega |\zeta|^2 \text{ for all } \zeta \in \mathbb{R}^3. \quad (1.3)$$

In other words, $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$ with ξ of (1.3). We define unit normal vector of the exterior domain Ω as

$$n(x) = \frac{\nabla \xi(x)}{|\nabla \xi(x)|}.$$

Moreover, we impose the specular reflection boundary condition on the boundary $\partial\Omega$,

$$F(t, x, v) = F(t, x, R_x v) \text{ for } x \in \partial\Omega, \text{ where } R_x v = v - 2(n(x) \cdot v)n(x). \quad (1.4)$$

In the classical Boltzmann theory, an equilibrium state of the problem (1.1) and (1.4) is well-known as a global Maxwellian

$$\mu(v) = e^{-\frac{|v|^2}{2}}.$$

2 | MAIN THEOREM AND SCHEME OF PROOF

We rewrite the Boltzmann equation (1.1) and the specular reflection boundary condition (1.4) using $F(t, x, v) = \sqrt{\mu}f(t, x, v)$ to obtain

$$\partial_t f + v \cdot \nabla_x f = \Gamma(f, f) := \Gamma_{\text{gain}}(f, f) - \nu(f)f, \quad f|_{t=0} = f_0, \quad f(t, x, v)|_{\partial\Omega} = f(t, x, R_x v), \quad (2.1)$$

where

$$\Gamma_{\text{gain}}(f_1, f_2) := \frac{1}{\sqrt{\mu}} Q_{\text{gain}}(\sqrt{\mu}f_1, \sqrt{\mu}f_2), \quad \nu(f) := \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v-u) \cdot \omega| \sqrt{\mu(u)} f(t, x, u) d\omega du. \quad (2.2)$$

We define a backward exit time and position as

$$t_b(x, v) := \sup\{s \geq 0 : x - \tau v \in \Omega \text{ for all } \tau \in (0, s)\}, \quad t^1(x, v) := t - t_b(x, v), \\ x_b(x, v) := x - t_b(x, v)v.$$

Then the trajectory of (2.1) satisfies that for $(t, x, v) \in [0, \infty) \times \Omega \times \mathbb{R}^3$,

$$\frac{d}{ds}(X(s; t, x, v), V(s; t, x, v)) = (V(s; t, x, v), 0), \quad (X(s; t, x, v), V(s; t, x, v))|_{s=t} = (x, v). \quad (2.3)$$

While the explicit formula is given as

$$(X(s; t, x, v), V(s; t, x, v)) \\ = \begin{cases} (x - (t-s)v, v) & \text{for } s \in (t - t_b(x, v), t] \\ (x_b(x, v) - (t - t_b(x, v) - s)R_{x_b(x, v)}v, R_{x_b(x, v)}v) & \text{for } s \in [0, t - t_b(x, v)], \end{cases} \quad (2.4)$$

where $R_{x_b(x, v)} = I - 2n(x_b) \otimes n(x_b)$ is reflection operator. From (2.1), we obtain the Duhamel's formula

$$f(t, x, v) = e^{-\int_0^t \nu(f)(\tau, X(\tau; t, x, v), V(\tau; t, x, v)) d\tau} f(0, X(0; t, x, v), V(0; t, x, v)) \\ + \int_0^t e^{-\int_s^t \nu(f)(\tau, X(\tau; t, x, v), V(\tau; t, x, v)) d\tau} \Gamma_{\text{gain}}(f, f)(s, X(s; t, x, v), V(s; t, x, v)) ds. \quad (2.5)$$

Local in time existence of the mild solution (2.5) is given as the following.

Lemma 2.1 (Local existence). Assume initial data f_0 satisfies $\|w_0(v)f_0\|_\infty := \|e^{\vartheta_0|v|^2}f_0\|_\infty < \infty$ for $0 < \vartheta_0 < \frac{1}{4}$ and initial compatibility condition (1.4). Then there exists $T^* > 0$ and a local in time unique solution $f(t, x, v)$ of (2.5) for $0 \leq t \leq T^*$. Moreover, the solution satisfies

$$\sup_{0 \leq s \leq T^*} \|w(v)f(s)\|_\infty := \sup_{0 \leq s \leq T^*} \|e^{\vartheta|v|^2}f(s)\|_\infty \lesssim \|e^{\vartheta_0|v|^2}f_0\|_\infty,$$

for some $0 < \vartheta < \vartheta_0$.

Proof. We refer [13]. □

Now, we define *shifted* position and velocity which will be *crucially* used when we perform Hölder regularity estimates.

Definition 2.2 (Shift). We define shifted position $\tilde{x} = \tilde{x}(x, \bar{x}, v)$ and velocity $\tilde{v} = \tilde{v}(v, \bar{v}, \zeta)$, respectively.

(i) For fixed $x, \bar{x}, v$, let us assume

$$x - \bar{x} \neq 0 \text{ is neither parallel nor anti-parallel to } v \neq 0, \text{ that is, } (x - \bar{x}) \cdot v \neq \pm |x - \bar{x}| |v|. \quad (2.6)$$

In this case, we define shifted $\tilde{x}$ as

$$\tilde{x} = \tilde{x}(x, \bar{x}, v) := \bar{x} + ((x - \bar{x}) \cdot \hat{v})\hat{v}. \quad (2.7)$$

(ii) For fixed $v, \bar{v}, \zeta$, let us assume

$$\begin{aligned} v + \zeta \neq 0 \text{ is neither parallel nor anti-parallel to } \bar{v} + \zeta \neq 0, \\ \text{that is, } (v + \zeta) \cdot (\bar{v} + \zeta) \neq \pm |v + \zeta| |\bar{v} + \zeta|. \end{aligned} \quad (2.8)$$

In this case, we define shifted $\tilde{v}$ as

$$\tilde{v} + \zeta = \tilde{v}(v, \bar{v}, \zeta) + \zeta := |v + \zeta| \widehat{(\bar{v} + \zeta)}. \quad (2.9)$$

Note that we have used the following notation,

$$\hat{v} := \begin{cases} \frac{v}{|v|}, & v \neq 0, \\ 0, & v = 0. \end{cases}$$

Next, we parametrize position and velocity using above shifted position and velocity. Before we introduce parametrizations, we first define cross section and argument.

Definition 2.3.

(i) Let us assume (2.6). We define

$$S_{(x, \bar{x}, v)} := x + \text{span}\{x - \bar{x}, v\} = x + \text{span}\{x - \bar{x}, v\} \quad (2.10)$$

where $\tilde{x} = \tilde{x}(x, \bar{x}, v)$ is defined in (2.7).

(ii) Let us assume (2.8). We define

$$S_{(x,v,\bar{v},\zeta)} := x + \text{span}\{v + \zeta, \bar{v} + \zeta\} = x + \text{span}\{v + \zeta, \bar{v} + \zeta\}, \quad (2.11)$$

where $\bar{v} = \bar{v}(v, \bar{v}, \zeta)$ is defined in (2.9). In the plane $S_{(x,v,\bar{v},\zeta)}$, we define $\arg : S_{(x,v,\bar{v},\zeta)} \setminus \{0\} \mapsto [0, 2\pi)$ by

$$\arg(w) = \cos^{-1} \left(\widehat{w \cdot (\bar{v} + \zeta)} \right), \quad \forall w \in S_{(x,v,\bar{v},\zeta)} \setminus \{0\}, \quad (2.12)$$

so that $\arg(\bar{v} + \zeta) = 0$.

Note that we are mainly interested in when cross section $\partial\Omega \cap S_{(x,\bar{x},v)}$ and $\partial\Omega \cap S_{(x,v,\bar{v},\zeta)}$ have meaningful geometry. So we assume

$$\partial\Omega \cap S_{(x,\bar{x},v)} \text{ is closed curve, that is, } \partial\Omega \cap S_{(x,\bar{x},v)} \neq \emptyset \text{ is neither an empty set nor a single point,} \quad (2.13)$$

for x perturbation case and

$$\partial\Omega \cap S_{(x,v,\bar{v},\zeta)} \text{ is closed curve, that is, } \partial\Omega \cap S_{(x,v,\bar{v},\zeta)} \neq \emptyset \text{ is neither an empty set nor a single point,} \quad (2.14)$$

for v perturbation case.

Definition 2.4 (Parametrizations). Recall $\tilde{x}(x, \bar{x}, v)$ in Definition 2.2 under assumption (2.6). Also we recall $\bar{v}(v, \bar{v}, \zeta)$ in Definition 2.2 under assumption (2.8).

(i) We define parametrizations,

$$\begin{aligned} \mathbf{x}(\tau) &= \mathbf{x}(\tau; x, \bar{x}, v) = (1 - \tau)\tilde{x}(x, \bar{x}, v) + \tau x, \quad \dot{\mathbf{x}} := \dot{\mathbf{x}}(\tau) = x - \tilde{x}, \\ \mathbf{v}(\tau) &= \mathbf{v}(\tau; x, v, \bar{v}, \zeta) = |v + \zeta| R_{(v,\bar{v},\zeta)} \begin{bmatrix} \cos \Theta(\tau) \\ \sin \Theta(\tau) \\ 0 \end{bmatrix}, \quad \Theta(\tau) := \tau\theta, \quad \dot{\Theta}(\tau) = \dot{\Theta} := \theta, \end{aligned} \quad (2.15)$$

where $\theta = \cos^{-1}(\widehat{v + \zeta} \cdot \widehat{\bar{v} + \zeta}) \in [0, 2\pi)$ is the angle between $v + \zeta$ and $\bar{v} + \zeta$, and

$$R_{(v,\bar{v},\zeta)} := \begin{bmatrix} \widehat{\bar{v} + \zeta} & \widehat{v + \zeta} & (\bar{v} + \zeta) \times (v + \zeta) \end{bmatrix} \begin{bmatrix} 1 & \cos \theta & 0 \\ 0 & \sin \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1},$$

so that

$$\begin{aligned} \mathbf{x}(0) &= \tilde{x}, \mathbf{x}(1) = x, \Theta(0) = \arg(\bar{v} + \zeta) = \arg(\bar{v} + \zeta) = 0 \text{ by (2.12), and } \Theta(1) = \arg(v + \zeta) := \theta, \\ \mathbf{v}(0) &= \bar{v} + \zeta, \mathbf{v}(1) = v + \zeta. \end{aligned}$$

(ii) We further assume (2.13) and (2.14), respectively. By convexity (1.3) of Ω , we can define $\tau_{\pm}(x, \bar{x}, v)$ and $\tau_{\pm}(x, v, \bar{v}, \zeta)$ as

$$\begin{aligned} v \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_{\pm}), v)) &= 0, \quad \tau_{-} < \tau_{+}, \quad \tau_{\pm} = \tau_{\pm}(x, \bar{x}, v), \\ \mathbf{v}(\tau_{\pm}) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau_{\pm}))) &= 0, \quad \tau_{-} < \tau_{+}, \quad \tau_{\pm} = \tau_{\pm}(x, v, \bar{v}, \zeta), \end{aligned} \quad (2.16)$$

where $x_{\mathbf{b}}(\mathbf{x}(\tau), v)$ and $x_{\mathbf{b}}(x, \mathbf{v}(\tau))$ are well defined for $\tau_{-} \leq \tau \leq \tau_{+}$, respectively. Here we excuse our abuse of notations, different $\tau_{\pm}$'s are only distinguished by their arguments, for the sake of briefness in the later use of them.

Remark 2.5. By Definition 2.2 and 2.4, we can check the following crucial properties:

$$\dot{\mathbf{x}} \cdot v = 0, \quad \dot{\mathbf{v}}(\tau) \cdot \mathbf{v}(\tau) = 0, \quad |\dot{\mathbf{v}}(\tau)| = \theta |\mathbf{v}(\tau)|, \quad \ddot{\mathbf{v}}(\tau) = -\theta^2 \mathbf{v}(\tau), \quad \text{and} \quad \ddot{\mathbf{v}}(\tau) \cdot \mathbf{v}(\tau) = -\theta^2 |\mathbf{v}(\tau)|^2, \quad (2.17)$$

and

$$|\mathbf{v}(\tau)| = |v + \zeta|, \quad |\dot{\mathbf{v}}(\tau)| = \theta |v + \zeta|, \quad \text{and} \quad |\ddot{\mathbf{v}}(\tau)| = \theta^2 |v + \zeta|.$$

Next definition precisely describes the kernels of integral operators which come from $\Gamma_{\text{gain}}(f, f)$.

Definition 2.6. For $c > 0$, we define

$$k_c(v, v + \zeta) := \frac{1}{|\zeta|} e^{-c|\zeta|^2 - c \frac{||v|^2 - |v + \zeta|^2|^2}{|\zeta|^2}}, \quad (2.18)$$

and

$$\mathbf{k}_c(v, \bar{v}, \zeta) := k_c(v, v + \zeta) + k_c(\bar{v}, \bar{v} + \zeta). \quad (2.19)$$

Note that the following equalities are obvious.

$$k_c(v, v + \zeta) = k_c(R_x v, R_x v + R_x \zeta), \quad \mathbf{k}_c(v, \bar{v}, \zeta) = \mathbf{k}_c(R_x v, R_x \bar{v}, R_x \zeta), \quad x \in \partial\Omega.$$

The following two definitions are the most important quantities of this paper.

Definition 2.7 (Specular Singularity). Let $x, \bar{x} \in \Omega$, $v, \bar{v}, \zeta \in \mathbb{R}^3$. We assume (2.6) and (2.13) for $\mathbf{x}(\tau)$ case, and assume (2.8) and (2.14) for $\mathbf{v}(\tau)$ case, respectively. Recall the notations $\mathbf{x}(\tau)$ and $\mathbf{v}(\tau)$ in (2.15). Suppose $x_{\mathbf{b}}(\mathbf{x}(\tau), v)$ and $x_{\mathbf{b}}(x, \mathbf{v}(\tau))$ are well-defined on $\partial\Omega$, respectively. We define a reciprocal of the specular singularity

$$\mathfrak{S}_{sp}(\tau; x, \bar{x}, v) := \frac{-\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot v}{\left| \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \right|}, \quad (2.20)$$

$$\mathfrak{S}_{vel}(\tau; x, v, \bar{v}, \zeta) := \frac{-\nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \mathbf{v}(\tau)}{t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \left| \frac{\dot{\mathbf{v}}(\tau)}{|\dot{\mathbf{v}}(\tau)|} \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \right|}. \quad (2.21)$$

Definition 2.8 (Seminorms). Let $x, \bar{x} \in \Omega$ and $v, \bar{v}, \zeta \in \mathbb{R}^3$. For $\varpi > 0$, we define

$$\mathfrak{S}_{vel}^{2\beta}(s) := \sup_{\substack{x \in \Omega, \\ 0 < |v - \bar{v}| \leq 1}} e^{-\varpi \langle v \rangle^2 s} \int_{\zeta} \mathbf{k}_c(v, \bar{v}, \zeta) \frac{|f(s, x, v + \zeta) - f(s, x, \bar{v} + \zeta)|}{|v - \bar{v}|^{2\beta}} d\zeta,$$

$$\mathfrak{S}_{sp}^{2\beta}(s) := \sup_{\substack{v \in \mathbb{R}^3, \\ 0 < |x - \bar{x}| \leq 1}} e^{-\varpi \langle v \rangle^2 s} \int_{\zeta} k_c(v, v + \zeta) \frac{|f(s, x, v + \zeta) - f(s, \bar{x}, v + \zeta)|}{|x - \bar{x}|^{2\beta}} d\zeta,$$

where $\langle v \rangle := \sqrt{1 + |v|^2}$.

Theorem 2.9 (Main theorem). Suppose the domain is given as in Definition 1.2 and (1.3). Assume f_0 satisfies compatibility condition (1.4), $\|e^{\vartheta_0 |v|^2} f_0\|_{\infty} < \infty$ for $0 < \vartheta_0 < \frac{1}{4}$, and

$$\sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \sup_{\substack{x \in \Omega \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} < \infty, \quad \beta < \frac{1}{2}.$$

Then there exists $0 < T \ll 1$ such that we have a unique solution $f(t, x, v)$ of (2.1) for $0 \leq t \leq T$ with $\sup_{0 \leq t \leq T} \|e^{\vartheta |v|^2} f(t)\|_{\infty} \lesssim \|e^{\vartheta_0 |v|^2} f_0\|_{\infty}$ for some $0 \leq \vartheta < \vartheta_0$. Moreover, $f(t, x, v)$ is Hölder continuous in the following sense:

$$\begin{aligned} & \sup_{0 \leq t \leq T} \sup_{\substack{(x, v) \in \Omega \times \mathbb{R}^3 \\ 0 < |(x, v) - (\bar{x}, \bar{v})| \leq 1}} \left| \langle v \rangle^{-2\beta} e^{-\varpi \langle v \rangle^2 t} \frac{|f(t, x, v) - f(t, \bar{x}, \bar{v})|}{|(x, v) - (\bar{x}, \bar{v})|^{\beta}} \right| \\ & \lesssim_{\beta} \|e^{\vartheta_0 |v|^2} f_0\|_{\infty} \left[\sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \sup_{\substack{x \in \Omega \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} \right] \\ & + \mathcal{P}_2(\|e^{\vartheta_0 |v|^2} f_0\|_{\infty}), \end{aligned} \quad (2.22)$$

for some $\varpi \gg_{f_0, \beta} 1$, where $\mathcal{P}_2(s) = |s| + |s|^2$.

Now we explain main ideas of the proof.

2.1 | A roadmap to the main theorem

To the sake of simplicity we drop out $v(f)f$ from (2.1) in this exhibition of key ideas:

$$\partial_t f + v \cdot \nabla_x f = \Gamma_{\text{gain}}(f, f), \quad f|_{t=0} = f_0, \quad f(t, x, v)|_{\partial\Omega} = f(t, x, R_x v). \quad (2.23)$$

From Hamiltonian (2.3) and (2.4), f of (2.23) is given by

$$f(t, x, v) = f(0, X(0; t, x, v), V(0; t, x, v)) + \int_0^t \Gamma_{\text{gain}}(f, f)(s, X(s; t, x, v), V(s; t, x, v)) ds. \quad (2.24)$$

Step 1 ($C_{x,v}^{0,\frac{1}{2}}$ trajectory and nonlocal to local iteration) In order to estimate the Hölder semi-norm of f we consider the difference quotients directly. Note that $V(s; t, x, v)$ has jump discontinuity at $s = t - t_b(x, v)$ which has to be handled carefully by applying geometric splitting of $V(s)$ along the trajectory and the specular reflection BC. Then using a version of Carleman's representation and a priori L^∞ -bound of f , we can derive that for some θ and β ,

$$\begin{aligned} \frac{|f(t, x, v + \zeta) - f(t, \bar{x}, \bar{v} + \zeta)|}{|(x, v) - (\bar{x}, \bar{v})|^\beta} &\lesssim \int_0^t \frac{|X(s) - \bar{X}(s)|^\theta}{|(x, v) - (\bar{x}, \bar{v})|^\beta} J_x[f(s)](V(s); X(s), \bar{X}(s)) ds \\ &+ \int_0^t \frac{|V(s) - \bar{V}(s)|^\theta}{|(x, v) - (\bar{x}, \bar{v})|^\beta} J_v[f(s)](X(s); V(s), \bar{V}(s)) + \text{good terms}, \end{aligned} \quad (2.25)$$

where **J-seminorms** are defined as

$$\begin{aligned} J_x[f(s)](V(s); X(s), \bar{X}(s)) &:= \int_{|u| \leq 1} \frac{|f(s, X(s), u + V(s)) - f(s, \bar{X}(s), u + V(s))|}{|X(s) - \bar{X}(s)|^\theta} du, \\ J_v[f(s)](X(s); V(s), \bar{V}(s)) &:= \int_{|u| \leq 1} \frac{|f(s, X(s), u + V(s)) - f(s, X(s), u + \bar{V}(s))|}{|V(s) - \bar{V}(s)|^\theta} du. \end{aligned} \quad (2.26)$$

Here, $X(s) = X(s; t, x, v + \zeta)$, $V(s) = V(s; t, x, v + \zeta)$, $\bar{X}(s) = \bar{X}(s; t, \bar{x}, \bar{v} + \zeta)$, and $\bar{V}(s) = \bar{V}(s; t, \bar{x}, \bar{v} + \zeta)$. Our first key observation is that $(x, v) \mapsto X(s; t, x, v)$ is $C_{x,v}^{0,\frac{1}{2}}$ and $(x, v) \mapsto V(s; t, x, v)$ with $s \neq t - t_b(x, v)$ belongs to $C_{x,v}^{0,\frac{1}{2}}$! (For example, if we consider 2D circle $x^2 + (y - 1)^2 = 1$, then near grazing regime (let $x = (1, 0)$, $\bar{x} = (1, \varepsilon)$, $v = (1, 0)$), we have $|x_b(x, v) - x_b(\bar{x}, v)| = \sqrt{2\varepsilon - \varepsilon^2} \simeq \sqrt{\varepsilon}$, for $\varepsilon \ll 1$.) Thereby a natural choice of θ is

$$\theta = 2\beta.$$

A schematic bound (2.25) exhibits the nonlocal effect of the Boltzmann equation. Our key strategy is a **nonlocal-to-local iteration**: namely we first estimate the J -seminorm and then use (2.25) to bound the Hölder seminorm. Indeed the J -seminorms (2.26) can be bounded in a closed form through the Duhamel form of (2.24) as

$$\begin{aligned} J_x[f(s)](v; x, \bar{x}) &\lesssim \text{good terms} + \int_0^t \int_{|\zeta| \leq 1} \frac{|X(s) - \bar{X}(s)|^{2\beta}}{|x - \bar{x}|^{2\beta}} d\zeta ds \times \sup_{s,v,x \neq \bar{x}} J_x[f(s)](v; x, \bar{x}) \\ &+ \int_0^t \int_{|\zeta| \leq 1} \frac{|V(s) - \bar{V}(s)|^{2\beta}}{|x - \bar{x}|^{2\beta}} d\zeta ds \times \sup_{s,x,v \neq \bar{v}} J_v[f(s)](x; v, \bar{v}), \end{aligned} \quad (2.27)$$

and a similar expression for $J_v[f(s)](x; v, \bar{v})$. Main ingredients of (2.27) are difference quotients of $X(s)$ and $V(s)$. In fact, the difference quotients would be very singular $\sim \frac{1}{|(v+\zeta) \cdot \nabla \xi(x_b)|}$ when the trajectories hit $\partial\Omega$ near grazing region. However, measuring this singular behavior is very tricky, because of the curvature of the cross section $\partial\Omega \cap \text{span}\{x - \bar{x}\}$.

Step 2 (Shift method and ODE of Specular Singularity) This step is the most technical step in this paper. To get precise estimate of the difference quotients, we introduce two very important ideas : **Specular Singularity** and **Shift method**.

First, let us consider some simple linear parametrizations of position $x(\tau)$ and velocity $v(\tau)$. If all the trajectories from $(x(\tau), v)$ with $0 \leq \tau \leq 1$ hit $\partial\Omega$, then difference quotient of trajectory will

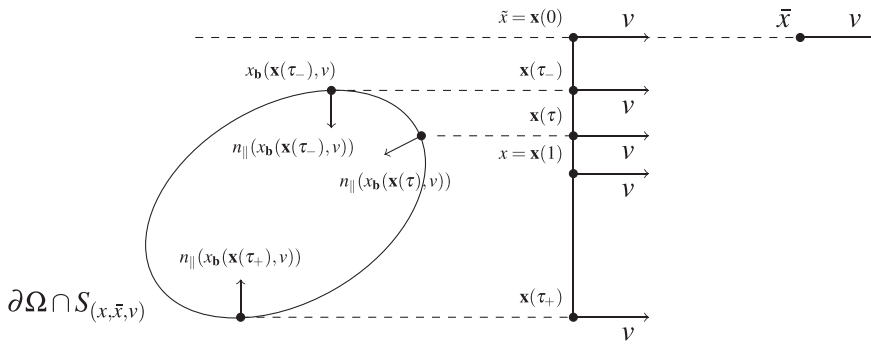


FIGURE 1 $\tilde{x}$, $x(\tau)$, and trajectories on projected plane $S_{(x, \tilde{x}, v)} := x + \text{span}\{x - \tilde{x}, v\}$.

look like

$$\frac{|X(s) - \tilde{X}(s)|}{|x - \tilde{x}|} \sim \int_0^1 |\nabla X(s; t, x(\tau), v)| d\tau \sim \int_0^1 \frac{1}{|v \cdot \nabla \xi(x_b(x(\tau), v))|} d\tau, \quad (2.28)$$

since $\nabla_x X(s) \sim \frac{1}{|v \cdot \nabla \xi(x_b(x(\tau), v))|}$. (We get similar result for $v - \bar{v}$ case with $|v \cdot \nabla \xi(x_b(x, v(\tau)))|$.) Evidently, the success of estimating $J_x[f(s)]$ and $J_v[f(s)]$ relies on an efficient estimate of the integration with respect to ζ of the different quotients in (2.28). Unfortunately, however, estimate (2.28) is not efficient: it does not consider the angle between $\dot{x}(\tau)$ (or $\dot{v}(\tau)$) and $\nabla \xi$ which is not uniform in general and hence causes trouble, that is, it fails to reveal geometric property of Ω very well.

Meanwhile, to clarify the effect of convexity of Ω to the difference quotients, we introduce **shifted** position and velocity. We shift position and velocity to define $\tilde{x}$ and $\tilde{v}$ as in (2.7) and (2.9), respectively. Now, we use new parametrizations $\mathbf{x}(\tau)$ and $\mathbf{v}(\tau)$ in (2.15) with shifted $\tilde{x}$ and $\tilde{v}$ to redefine Specular Singularity (See (2.20) and (2.21) for their exact forms.)

$$\begin{aligned} \mathfrak{S}_{sp}(\tau; x, \tilde{x}, v) &:= \left| \frac{\nabla \xi(x_b(\mathbf{x}(\tau), v)) \cdot v}{\dot{\mathbf{x}}(\tau) \cdot \nabla \xi(x_b(\mathbf{x}(\tau), v))} \right|, \\ \mathfrak{S}_{vel}(\tau; x, v, \tilde{v}, \zeta) &:= \left| \frac{\nabla \xi(x_b(x, \mathbf{v}(\tau))) \cdot \mathbf{v}(\tau)}{\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))} \right|. \end{aligned} \quad (2.29)$$

These crucial quantities measure how singular region does the trajectory hits by its numerators. Moreover, the denominator takes into account the angle between $\dot{\mathbf{x}}(\tau)$ (also for $\dot{\mathbf{v}}(\tau)$) and $\nabla \xi$ and hence clarifies the effect of non-convexity in ODE method which will be explained very soon. Note that with the new parametrizations, we also have the following orthogonal geometric properties

$$\dot{\mathbf{x}} \perp v \quad \text{and} \quad \mathbf{v}(\tau) \perp \dot{\mathbf{v}}(\tau) \quad (2.30)$$

which are used *crucially* throughout this paper. See Figures 1 and 2. Now, with (2.29), (2.28) newly look like

$$\frac{|X(s) - \tilde{X}(s)|}{|x - \tilde{x}|} \sim \int_0^1 |\nabla X(s; t, x(\tau), v)| d\tau \sim \int_0^1 \frac{1}{\mathfrak{S}_{sp}(\tau; x, \tilde{x}, v)} d\tau. \quad (2.31)$$

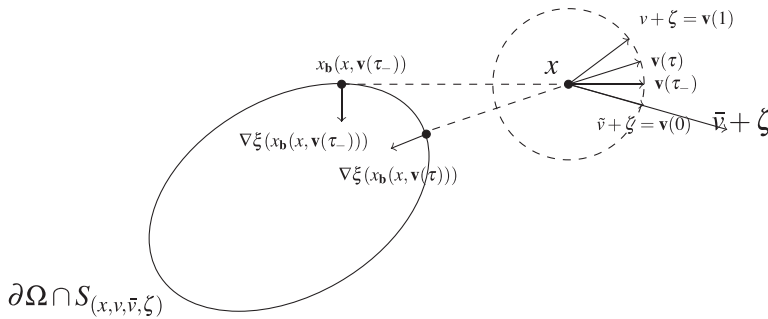


FIGURE 2 $\bar{v}$, $v(\tau)$, and trajectories on projected plane $S_{(x,v,\bar{v},\zeta)} := x + \text{span}\{v + \zeta, \bar{v} + \zeta\}$.

To perform estimate for (2.31), our next step is **ODE method for $\mathfrak{S}_{sp,vel}$** . Using convexity (1.3) and (2.30) *crucially*, the Specular Singularity $\mathfrak{S}_{sp}$ solves ODE look like

$$\frac{d}{d\tau} \mathfrak{S}_{sp}^2(\tau) \gtrsim_{\Omega} \frac{1}{|\dot{x} \cdot \nabla \xi(x_b(x(\tau), v))|} \mathfrak{S}_{sp}^2(\tau).$$

ODE for $\mathfrak{S}_{vel}$ is also similar. (See (5.16) and (5.18) in Lemma 5.4 for exact ODEs.) We note that the denominators of (2.29) are very important in its definition when we derive above type of ODEs. Its qualitative aspect was already explained. In the technical aspect, $\frac{d}{d\tau} \mathfrak{S}_{sp,vel}^2$ gain positive lower bound via convexity (see (5.20) and (5.22)) only when they contain the denominators. Therefore, we obtain difference quotients estimates via integration of Specular Singularity, (in general we will use $v + \zeta$ instead of v for notational convenience in velocity integration)

$$\frac{|X(s) - \bar{X}(s)|}{|x - \bar{x}|} \lesssim \frac{1}{|(v + \zeta) \cdot \nabla \xi(x_b(x, v + \zeta))|} + \frac{1}{|(v + \zeta) \cdot \nabla \xi(x_b(\bar{x}, v + \zeta))|}. \quad (2.32)$$

(See (5.23) and (5.24) of Proposition 5.6 for exact estimates.) We also note that $\bar{x}$ depends on ζ surely, but $(v + \zeta) \cdot \nabla \xi(x_b(\bar{x}, v)) = (v + \zeta) \cdot \nabla \xi(x_b(\bar{x}, v))$ by definition of $\bar{x}$ in (2.7). Thus **we can fix positions on x or $\bar{x}$** , independent to ζ , for each terms in (2.32). This is a huge advantage in terms of (2.27), because we should integrate (2.32) with respect to $\zeta \in \mathbb{R}^3$ again. If the position moves depending on velocity ζ , it looks nearly impossible to perform the integration!

Now performing sharp integral estimate (Lemma 6.5)

$$\int_{\zeta} \frac{1}{|(v + \zeta) \cdot \nabla \xi(x_b(x, v + \zeta))|^{2\beta}} d\zeta \lesssim 1, \quad \beta < \frac{1}{2},$$

we obtain uniform bound of $J_x[f]$ and $J_v[f]$ in (2.26). Exact definitions and estimate of the **J-seminorms** are given in Definition 2.8 and (6.32) in Proposition 6.7, respectively.

Lastly, from uniform estimates of J-seminorm and $C_{x,v}^{0, \frac{1}{2}-}$ estimate of trajectory (Lemma 7.1), we obtain $C_{x,v}^{0, \frac{1}{2}-}$ regularity of f via nonlocal-to-local estimate using (2.25).

3 | PRELIMINARIES

In this section, we state lemmas for some estimates of $\Gamma_{\text{gain}}(f, f)$.

Lemma 3.1 (Carleman estimate). Recall (1.2) and (2.2). We have the following expressions:

$$\Gamma_{\text{gain}}(g_1, g_2)(v) = C \int_{\mathbb{R}^3} du \int_{u \cdot z=0} dz g_1(v+z) g_2(v+u) q_0^* \left(\frac{|u|}{|u+z|} \right) \frac{|u+z|^{\kappa-1}}{|u|} e^{-\frac{|u+v+z|^2}{4}}, \quad (3.1)$$

$$\Gamma_{\text{gain}}(g_1, g_2)(v) = C \int_{\mathbb{R}^3} dz \int_{u \cdot z=0} du g_1(v+z) g_2(v+u) q_0^* \left(\frac{|u|}{|u+z|} \right) \frac{|u+z|^{\kappa-1}}{|z|} e^{-\frac{|u+v+z|^2}{4}}, \quad (3.2)$$

where $q_0^*(|\cos \theta|) := \frac{1}{|\cos \theta|} q_0(|\cos \theta|)$. For the hard sphere case ($\kappa = 1$), we have $q_0^*(|\cos \theta|) := \frac{1}{|\cos \theta|} |\cos \theta| = 1$.

Proof. Using $u' = u + ((v-u) \cdot \omega)\omega$ and $v' = v - ((v-u) \cdot \omega)\omega$, we write

$$\begin{aligned} \Gamma_{\text{gain}}(g_1, g_2) &= \int_{u \in \mathbb{R}^3} \int_{\omega \in \mathbb{S}^2} |v-u|^\kappa q_0 \left(\frac{|(v-u) \cdot \omega|}{|v-u|} \right) \sqrt{\mu}(u) \\ &\quad \times g_1(u + ((v-u) \cdot \omega)\omega) g_2(v - ((v-u) \cdot \omega)\omega) d\omega du. \end{aligned}$$

We change $u \rightarrow u+v$ to get

$$\begin{aligned} \Gamma_{\text{gain}}(g_1, g_2) &= \int_{u \in \mathbb{R}^3} \int_{\omega \in \mathbb{S}^2} |u|^\kappa q_0 \left(\frac{|u \cdot \omega|}{|u|} \right) \sqrt{\mu}(u+v) \\ &\quad \times g_1(u+v - (u \cdot \omega)\omega) g_2(v + (u \cdot \omega)\omega) d\omega du. \end{aligned}$$

Let us decompose and define $u_{\parallel} := (u \cdot \omega)\omega$ and $u_{\perp} := u - u_{\parallel}$. Then,

$$\Gamma_{\text{gain}}(g_1, g_2) = \int_{u \in \mathbb{R}^3} \int_{\omega \in \mathbb{S}^2} |u_{\parallel} + u_{\perp}|^\kappa q_0 \left(\frac{|u_{\parallel}|}{|u_{\parallel} + u_{\perp}|} \right) \times e^{-\frac{1}{4}|u_{\parallel} + u_{\perp} + v|^2} g_1(u_{\perp} + v) g_2(v + u_{\parallel}) d\omega du.$$

Now, for (3.1), for fixed $\omega \in \mathbb{S}^2$, we have $du = 2du_{\perp} d|u_{\parallel}|$, where $u_{\parallel} = |u_{\parallel}|\omega$. Therefore (also see [15]),

$$dud\omega = \frac{2}{|u_{\parallel}|^2} du_{\perp} |u_{\parallel}|^2 d|u_{\parallel}| d\omega = \frac{2}{|u_{\parallel}|^2} du_{\perp} du_{\parallel}.$$

Now we just rewrite $u_{\parallel} \rightarrow u$ and $u_{\perp} \rightarrow w$ to obtain

$$\begin{aligned} \Gamma_{\text{gain}}(g_1, g_2) &= \int_{u \in \mathbb{R}^3} \int_{u \cdot w=0} |u+w|^\kappa q_0 \left(\frac{|u|}{|u+w|} \right) e^{-\frac{1}{4}|u+w+v|^2} g_1(v+w) g_2(v+u) \frac{2}{|u|^2} dw du \\ &= 2 \int_{u \in \mathbb{R}^3} \int_{u \cdot w=0} \frac{|u+w|^{\kappa-1}}{|u|} \frac{|u+w|}{|u|} q_0 \left(\frac{|u|}{|u+w|} \right) e^{-\frac{1}{4}|u+w+v|^2} g_1(v+w) g_2 \\ &\quad \times (v+u) dw du. \end{aligned}$$

We define $q_0^*(\cos \theta) := \frac{1}{|\cos \theta|} q_0(\cos \theta)$ and this finishes the proof for (3.1).

For (3.2), for fixed $\omega \in \mathbb{S}^2$, we choose $\bar{\omega} = \bar{\omega}(\omega) \in \mathbb{S}^2$ so that $\bar{\omega} \perp \omega$ and a map $\omega \mapsto \bar{\omega}(\omega)$ is measure preserving bijective map via rotation from $\mathbb{S}^2$ to $\mathbb{S}^2$. Then we have

$$dud\omega = 2|u_{\perp}|d|u_{\parallel}||d|u_{\perp}||d\theta d\omega,$$

where θ is angle of polar coordinate with $u_{\parallel}$ axis. Now consider the rotation $\omega \rightarrow \bar{\omega}$ which makes $\theta \rightarrow \bar{\theta}$, where $\bar{\theta}$ is angle of polar coordinate with $u_{\perp}$. Therefore (also see [15]),

$$dud\omega = 2|u_{\perp}|d|u_{\parallel}||d|u_{\perp}||d\bar{\theta}d\bar{\omega} = 2\frac{1}{|u_{\perp}|}(|u_{\perp}|^2d|u_{\perp}||d\bar{\omega})\frac{1}{|u_{\parallel}|}(|u_{\parallel}|d|u_{\parallel}||d\bar{\theta}) = \frac{2}{|u_{\parallel}||u_{\perp}|}du_{\parallel}du_{\perp},$$

where $u_{\perp} \in \mathbb{R}^3$ and $u_{\parallel} \perp u_{\perp}$. Finally rewriting $u_{\parallel} \mapsto u$ and $u_{\perp} \mapsto w$, we get

$$\begin{aligned} \Gamma_{\text{gain}}(g_1, g_2) &= 2 \int_{w \in \mathbb{R}^3} \int_{u \cdot w = 0} |u + w|^{\kappa} q_0 \left(\frac{|u|}{|u + w|} \right) e^{-\frac{1}{4}|u+w+v|^2} \\ &\quad \times g_1(v + w) g_2(v + u) \frac{1}{|w||u|} dudw, \end{aligned}$$

and this gives (3.2) with the definition of q_0^* . $\square$

Lemma 3.2 (Nonlinear estimates). *Let $w(v) = e^{\vartheta|v|^2}$ for $0 < \vartheta < \frac{1}{4}$ and $x, \bar{x} \in \Omega$, $v, \bar{v}, \zeta \in \mathbb{R}^3$. For $|(x, v) - (\bar{x}, \bar{v})| \leq 1$, we have the following estimates*

$$\frac{|\Gamma_{\text{gain}}(f, f)(x, v) - \Gamma_{\text{gain}}(f, f)(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} \lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} \mathbf{k}_c(v, \bar{v}, u) \frac{|f(x, v + u) - f(x, \bar{v} + u)|}{|v - \bar{v}|^{2\beta}} du + \|wf\|_{\infty}^2 \min\{\langle v \rangle^{-1}, \langle \bar{v} \rangle^{-1}\}, \quad (3.3)$$

$$\frac{|\Gamma_{\text{gain}}(f, f)(x, v) - \Gamma_{\text{gain}}(f, f)(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} \lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} k_c(v, v + u) \frac{|f(x, v + u) - f(\bar{x}, v + u)|}{|x - \bar{x}|^{2\beta}} du, \quad (3.4)$$

for some $c > 0$, where k_c and $\mathbf{k}_c$ are defined in (2.18) and (2.19) of Definition 2.6.

Proof. Applying (3.1) to $\Gamma_{\text{gain}}(f, f)(x, v)$, and (3.2) to $\Gamma_{\text{gain}}(f, f)(x, \bar{v})$, we derive

$$\begin{aligned} \Gamma_{\text{gain}}(f, f)(v) - \Gamma_{\text{gain}}(f, f)(\bar{v}) &= C \int_{\mathbb{R}^3} du \int_{u \cdot z = 0} dz \frac{1}{|u|} \sqrt{\mu(u + v + z)} f(v + z) f(v + u) \\ &\quad - C \int_{\mathbb{R}^3} dz \int_{u \cdot z = 0} du \frac{1}{|z|} \sqrt{\mu(u + \bar{v} + z)} f(\bar{v} + u) f(\bar{v} + z) \\ &= C \int_{u \in \mathbb{R}^3} \int_{u \cdot z = 0} \frac{1}{|u|} \sqrt{\mu(u + v + z)} f(v + z) (f(v + u) - f(v + u - (v - \bar{v}))) dz du \\ &\quad + C \int_{z \in \mathbb{R}^3} \int_{u \cdot z = 0} \frac{1}{|z|} \sqrt{\mu(u + \bar{v} + z)} f(\bar{v} + u) (f(v + z) - f(v + z - (v - \bar{v}))) du dz \end{aligned}$$

$$\begin{aligned}
& + C \left\{ \int_{\mathbb{R}^3} \int_{u \cdot z=0} \frac{\sqrt{\mu(u+v+z)}}{|u|} f(v+z) f(\bar{v}+u) dz du \right. \\
& \left. - \int_{\mathbb{R}^3} \int_{u \cdot z=0} \frac{\sqrt{\mu(u+\bar{v}+z)}}{|z|} f(v+z) f(\bar{v}+u) du dz \right\}. \quad (3.5)
\end{aligned}$$

For the first term in RHS of (3.5), from the standard estimates (e.g. (3.11) and (3.52) in [12]), we derive, for some $c_{\vartheta'} > 0$,

$$\begin{aligned}
(3.5)_1 & \lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} \int_{u \cdot z=0} \frac{1}{|u|} e^{-\frac{|u+v+z|^2}{4}} e^{-\vartheta'|v+z|^2} |f(v+u) - f(v+u-(v-\bar{v}))| dz du \\
& \lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} \frac{1}{|u|} e^{-c_{\vartheta'}|u|^2 - c_{\vartheta'} \frac{||v|^2 - |v+u|^2|^2}{|u|^2}} |f(v+u) - f(v+u-(v-\bar{v}))| du.
\end{aligned}$$

Similarly, the second term in RHS of (3.5) is bounded by

$$(3.5)_2 \lesssim \|wf\|_{\infty} \int_{\mathbb{R}^3} \frac{1}{|u|} e^{-c_{\vartheta'}|u|^2 - c_{\vartheta'} \frac{||\bar{v}|^2 - |\bar{v}+u|^2|^2}{|u|^2}} |f(v+u) - f(v+u-(v-\bar{v}))| du.$$

We only need to bound last line of (3.5) by the last term of (3.3). Following the proof of Lemma 3.1, we re-express the second integral of (3.5) into $u \in \mathbb{R}^3$ integration (and an integral over $z \in \{z \cdot u = 0\}$). Then we bound the last line of (3.5) by

$$|v - \bar{v}|^{2\beta} \|wf\|_{\infty}^2 \times \underbrace{\int_u \int_{u \cdot z=0} \frac{1}{w(v+z)w(\bar{v}+u)|u|} \frac{|\sqrt{\mu(u+v+z)} - \sqrt{\mu(u+\bar{v}+z)}|}{|v - \bar{v}|^{2\beta}} dz du}_{(3.6)_*}. \quad (3.6)$$

Note that

$$\left| \sqrt{\mu(u+\bar{v}+v-\bar{v}+z)} - \sqrt{\mu(u+\bar{v}+z)} \right| = \left| \int_0^1 (v-\bar{v}) \cdot \nabla \sqrt{\mu(u+\bar{v}+(v-\bar{v})s+z)} ds \right| \lesssim |v - \bar{v}|.$$

Hence

$$(3.6)_* \lesssim |v - \bar{v}|^{1-2\beta} \int_0^1 ds \int_{\mathbb{R}^3} du |u|^{-1} e^{-\vartheta'|\bar{v}+u|^2} \int_{u \cdot z=0} dz e^{-\vartheta'|v+z|^2} \lesssim \langle \bar{v} \rangle^{-1} |v - \bar{v}|^{1-2\beta}.$$

Hence we conclude that the last line of (3.5) is bounded above by $\langle \bar{v} \rangle^{-1} |v - \bar{v}| \|wf\|_{\infty}^2$. Now changing v and $\bar{v}$ and then following the same argument we also get the upper bound $\langle v \rangle^{-1} |v - \bar{v}| \|wf\|_{\infty}^2$. The proof of (3.4) is similar but simpler. $\square$

Similar estimate for $\nu(f)$ is simpler.

Lemma 3.3. Let $w(v) = e^{\vartheta|v|^2}$ for $0 < \vartheta < \frac{1}{4}$, and $x, \bar{x} \in \Omega$, $v, \bar{v} \in \mathbb{R}^3$. For some positive $c > 0$, we obtain

$$\nu(f)(t, x, v) - \nu(f)(t, x, \bar{v}) \leq C |v - \bar{v}| \|f(t)\|_{\infty}, \quad (3.7)$$

and

$$\nu(f)(t, x, v) - \nu(f)(t, \bar{x}, v) \leq C\langle v \rangle \int_{\mathbb{R}^3} k_c(0, u) |f(t, x, u) - f(t, \bar{x}, u)| du. \quad (3.8)$$

Proof. For (3.7),

$$\begin{aligned} \nu(f)(t, x, v) - \nu(f)(t, x, \bar{v}) &= C \int_{\mathbb{R}_u^3} |v - u| \sqrt{\mu(u)} f(t, x, u) du - C \int_{\mathbb{R}_u^3} |\bar{v} - u| \sqrt{\mu(u)} f(t, x, u) du \\ &\leq C|v - \bar{v}| \|f(t)\|_\infty. \end{aligned}$$

For (3.8),

$$\begin{aligned} \nu(f)(t, x, v) - \nu(f)(t, \bar{x}, v) &\leq C \int_{\mathbb{R}_u^3} |v - u| \sqrt{\mu(u)} |f(t, x, u) - f(t, \bar{x}, u)| du \\ &\leq C\langle v \rangle \int_{\mathbb{R}^3} k_c(0, u) |f(t, x, u) - f(t, \bar{x}, u)| du. \end{aligned}$$

for some generic small $c > 0$. □

Lemma 3.4 (Uniform negativity). *For $|\varpi s| < c$,*

$$e^{-\varpi(1+|v|^2)s} e^{\varpi(1+|v+\zeta|^2)s} k_c(v, v+\zeta) \leq k_{\frac{c}{2}}(v, v+\zeta). \quad (3.9)$$

When $|v - \bar{v}| \leq 1$ and $\varpi s < (\sqrt{20} - 4)\frac{c}{2}$,

$$e^{-\varpi(1+|\bar{v}|^2)s + \varpi(1+|\bar{v}+\zeta|^2)s} k_c(v, v+\zeta) \lesssim k_{\frac{c}{2}}(v, v+\zeta). \quad (3.10)$$

Moreover, from (3.9) and (3.10),

$$e^{-\varpi(1+|v|^2)s + \varpi(1+|v+\zeta|^2)s} \mathbf{k}_c(v, \bar{v}, \zeta) \lesssim \mathbf{k}_{\frac{c}{2}}(v, \bar{v}, \zeta), e^{-\varpi(1+|\bar{v}|^2)s + \varpi(1+|\bar{v}+\zeta|^2)s} \mathbf{k}_c(v, \bar{v}, \zeta) \lesssim \mathbf{k}_{\frac{c}{2}}(v, \bar{v}, \zeta), \quad (3.11)$$

hold when $|v - \bar{v}| \leq 1$ and $\varpi s < (\sqrt{20} - 4)\frac{c}{2}$.

Proof. For $0 \leq \theta < 2c$, we have

$$-\theta|v|^2 + \theta|v + \zeta|^2 - c|\zeta|^2 - c \frac{1}{|\zeta|^2} ||v|^2 - |v + \zeta|^2|^2 < 0,$$

by uniform negativity of quadratic form (refer Lemma 3 of [17]). (3.9) is obtained by replacing θ into ϖs . When $|v - \bar{v}| \leq 1$, for $\theta < (\sqrt{20} - 4)c$,

$$\begin{aligned} &-\theta|\bar{v}|^2 + \theta|\bar{v} + \zeta|^2 - c|\zeta|^2 - c \frac{1}{|\zeta|^2} ||v|^2 - |v + \zeta|^2|^2 < 2\theta, \\ &-\theta|\bar{v}|^2 + \theta|\bar{v} + \zeta|^2 - c|\zeta|^2 - c \frac{1}{|\zeta|^2} ||v|^2 - |v + \zeta|^2|^2 \\ &= 2\theta(\bar{v} \cdot \zeta) + \theta|\zeta|^2 - c|\zeta|^2 - c \left(|\zeta|^2 + 4(v \cdot \zeta) + 4 \frac{|v \cdot \zeta|^2}{|\zeta|^2} \right) \end{aligned}$$

$$\begin{aligned}
&\leq 2\theta(v \cdot \zeta) + 2\theta|\zeta| + \theta|\zeta|^2 - c|\zeta|^2 - c\left(|\zeta|^2 + 4(v \cdot \zeta) + 4\frac{|v \cdot \zeta|^2}{|\zeta|^2}\right) \\
&\leq 2\theta(v \cdot \zeta) + 2\theta(1 + |\zeta|^2) + \theta|\zeta|^2 - c|\zeta|^2 - c\left(|\zeta|^2 + 4(v \cdot \zeta) + 4\frac{|v \cdot \zeta|^2}{|\zeta|^2}\right). \quad (3.12)
\end{aligned}$$

By choosing $\theta < \min\{(\sqrt{20} - 4)c, \frac{2}{3}c\} = (\sqrt{20} - 4)c$ and considering quadratic form of ζ and $\frac{v \cdot \zeta}{|\zeta|}$,

$$(3.12) - 2\theta \leq -(2c - 3\theta)|\zeta|^2 + (2\theta - 4c)(v \cdot \zeta) - 4c\frac{|v \cdot \zeta|^2}{|\zeta|^2} < 0,$$

because discriminant of RHS satisfies

$$4(\theta + 4c)^2 - 80c^2 < 0,$$

when $\theta < (\sqrt{20} - 4)c$. (3.10) is obtained by replacing θ into ϖs . $\square$

Lemma 3.5 (Specular reflection of $\Gamma(f, f)$ and $\nu(f)$). *If f satisfies the specular reflection boundary condition (1.4), then $\Gamma(f, f)$ also satisfies the specular reflection boundary condition. In particular,*

$$\Gamma_{\text{gain}}(f, f)(t, x, v) = \Gamma_{\text{gain}}(f, f)(t, x, R_x v) \text{ and } \Gamma_{\text{loss}}(f, f)(t, x, v) = \Gamma_{\text{loss}}(f, f)(t, x, R_x v),$$

where $R_x = I - 2n(x) \otimes n(x)$ is specular reflection operator on $x \in \partial\Omega$. Moreover, $\nu(f)$ also satisfy (1.4).

Proof. Since R_x is orthonormal we have $(R_x v - R_x u) \cdot R_x \sigma = (v - u) \cdot \sigma$. Also using specular reflection condition,

$$\begin{aligned}
\Gamma_{\text{gain}}(f, f)(t, x, v) &= \iint |(v - u) \cdot \sigma| \sqrt{\mu}(u) \\
&\quad \times f(t, x, u + ((v - u) \cdot \sigma)\sigma) f(t, x, v - ((v - u) \cdot \sigma)\sigma) d\sigma du \\
&= \iint |(R_x v - R_x u) \cdot R_x \sigma| \sqrt{\mu}(R_x u) \\
&\quad \times f(t, x, R_x u + (R_x(v - u) \cdot R_x \sigma)R_x \sigma) f(t, x, R_x v - (R_x(v - u) \cdot R_x \sigma)R_x \sigma) d\sigma du \\
&= \iint |(R_x v - R_x u) \cdot R_x \sigma| \sqrt{\mu}(R_x u) f(t, x, R_x u + (R_x(v - u) \cdot R_x \sigma)R_x \sigma) \\
&\quad \times f(t, x, R_x v - (R_x(v - u) \cdot R_x \sigma)R_x \sigma) d(R_x \sigma) d(R_x u) = \iint |(R_x v - u) \cdot \sigma| \sqrt{\mu}(u) \\
&\quad \times f(t, x, u + ((R_x v - u) \cdot \sigma)\sigma) f(t, x, R_x v - ((R_x v - u) \cdot \sigma)\sigma) d\sigma du = \Gamma_{\text{gain}}(f, f)(t, x, R_x v).
\end{aligned}$$

Similarly, for $\Gamma_{\text{loss}}(f, f)$,

$$\begin{aligned}
\Gamma_{\text{loss}}(f, f)(t, x, v) &= f(t, x, v) \iint |(v - u) \cdot \sigma| \sqrt{\mu}(u) f(t, x, u) d\sigma du \\
&= f(t, x, R_x v) \iint |(R_x v - R_x u) \cdot R_x \sigma| \sqrt{\mu}(u) f(t, x, R_x u) d\sigma du
\end{aligned}$$

$$\begin{aligned}
&= f(t, x, R_x v) \iint |(R_x v - R_x u) \cdot R_x \sigma| \sqrt{\mu}(u) f(t, x, R_x u) dR_x \sigma dR_x u \\
&= f(t, x, R_x v) \Gamma_{\text{loss}}(f, f)(t, x, R_x v).
\end{aligned}$$

□

4 | GEOMETRIC LEMMAS

Definition 4.1. Let S be a plane in $\mathbb{R}^3$ and Ω is given as in Definition 1.2. Assume

$$\partial\Omega \cap S \text{ is closed curve, that is, } \partial\Omega \cap S \text{ is neither an empty set nor a single point.} \quad (4.1)$$

We define projected normal vector,

$$\begin{aligned}
n_{\parallel}(x) &:= \text{Proj}_S n(x) = \text{projection of } n(x) \text{ on } S \\
&= (I - \hat{q} \otimes \hat{q})n(x),
\end{aligned} \quad (4.2)$$

where $x \in \partial\Omega$ and $\hat{q}$ is a unit vector orthogonal to the plane S . We also parametrize the curve $\partial\Omega \cap S$ as regularized curve $\mathbf{r}_S : [0, L_S] \rightarrow \partial\Omega \cap S$ (regularized means $|\mathbf{r}'_S(s)| = 1$), where $L_S > 0$ is the length of $\partial\Omega \cap S$. Note that we do not specify S in the definition $n_{\parallel}(x)$, because it can be understood properly in the context.

Lemma 4.2 (Uniform comparability of $n_{\parallel}$). *Let us consider a plane S which satisfies (4.1) with a domain Ω as in Definition 1.2. Then $|n_{\parallel}(x)|$ in (4.2) is uniformly comparable for all $x \in \partial\Omega \cap S$, that is, there exist uniformly positive constants c and C , which only depend on Ω , such that*

$$c < \frac{|n_{\parallel}(x)|}{|n_{\parallel}(y)|} \leq C, \quad \forall x, y \in \partial\Omega \cap S \text{ and } \forall S \text{ which satisfies (4.1),} \quad (4.3)$$

and

$$c < \frac{|\mathbf{r}''_S(x)|}{|\mathbf{r}''_S(y)|} \leq C, \quad \forall x, y \in \partial\Omega \cap S \text{ and } \forall S \text{ which satisfies (4.1).} \quad (4.4)$$

(For example, it is obvious that

$$|\mathbf{r}''_S(x)| = |\mathbf{r}''_S(y)| \text{ and } |n_{\parallel}(x)| = |n_{\parallel}(y)| \text{ for all } x, y \in \partial\Omega \cap S$$

for any S that satisfies (4.1), if $\mathcal{O}$ is a sphere.)

Proof. We parametrize the curve $\partial\Omega \cap S$ as regularized curve ($|\mathbf{r}'_S(s)| = 1$), $\mathbf{r}_S : [0, L_S] \rightarrow \partial\Omega \cap S$, where $L_S > 0$ is the length of $\partial\Omega \cap S$.

Considering normal curvature $n(x) \cdot \mathbf{r}''_S(s)$, we have

$$|n_{\parallel}(x)| = \left| n(x) \cdot \frac{\mathbf{r}''_S(s)}{|\mathbf{r}''_S(s)|} \right| = |n(x) \cdot \mathbf{r}''_S| \times \frac{1}{|\mathbf{r}''_S(s)|}, \quad x = \mathbf{r}_S(s) \in \partial\Omega \cap S. \quad (4.5)$$

Step 1 First, let us recall some standard definitions and properties of differential geometry ([7]).

We can consider Gauss map $N : \partial\Omega \mapsto \mathbb{S}^2$, $N(x) = n(x) = \frac{\nabla \xi(x)}{|\nabla \xi(x)|}$. For Gauss map N , we define

differential $dN_x : T_x(\partial\Omega) \mapsto T_x(\partial\Omega)$ as

$$dN_x(\alpha'(0)) = \frac{d}{dt}N(\alpha(t)) \Big|_{t=0},$$

where α is a curve on $\partial\Omega$ such that $|\alpha'(0)| = 1$ and $\alpha(0) = x \in \partial\Omega$. For self-adjoint linear map dN_x , there exists an orthonormal basis $\hat{x}_1, \hat{x}_2 \in T_x(\partial\Omega)$ and $k_1, k_2 \in \mathbb{R}$ such that

$$dN_x(\hat{x}_1) = -k_1\hat{x}_1, \quad dN_x(\hat{x}_2) = -k_2\hat{x}_2, \quad k_1 \geq k_2.$$

$\hat{x}_1$ and $\hat{x}_2$ are called principal directions at $x \in \partial\Omega$ and k_1 (resp, k_2) is maximum (resp, minimum) normal curvature. (See page 140, 144 of [7] for above definitions and properties.)

Now, let us fix $x \in \partial\Omega$ and tangential plane $T_x(\partial\Omega)$. We consider a local parametrization $\Phi : B(0, \varepsilon) \rightarrow B(x, \varepsilon)$

$$\Phi(y) = x + y_1\hat{x}_1 + y_2\hat{x}_2 + y_3n(x), \quad y = (y_1, y_2, y_3), \quad (4.6)$$

so that

$$\nabla\Phi = \begin{pmatrix} \hat{x}_1 & \hat{x}_2 & n(x) \end{pmatrix}, \quad \text{an orthonormal matrix.}$$

We can locally define height function $h(y)$ in $B(0, \varepsilon)$

$$\xi(\Phi(y)) = h(y_1, y_2) - y_3,$$

so that

$$\{x : \xi(x) = 0\} = \Phi\{y : h(y_1, y_2) = y_3\} \subset \partial\Omega, \quad \text{locally.}$$

If we choose $\zeta \in T_x(\partial\Omega)$, there exists $w = \begin{pmatrix} w_1 \\ w_2 \\ 0 \end{pmatrix}$ such that $\zeta = \nabla\Phi w$ and

$$\nabla^2(h(y_1, y_2) - y_3) = \nabla(\nabla\xi^T \nabla\Phi) = \nabla\Phi^T \nabla^2\xi \nabla\Phi. \quad (4.7)$$

From (4.7), we obtain

$$\zeta \cdot \nabla^2\xi(x)\zeta = w_h \cdot \nabla^2h(0, 0)w_h, \quad (4.8)$$

where $w_h = (w_1, w_2)$ and $\nabla_h^2 h$ is upper left 2×2 matrix of $\nabla^2 h$. (4.8) is quadratic form of height function based on $T_x(\partial\Omega)$ when $|\zeta| = |w| = 1$. It is well-known that (see page 173 of [7] for example) the quadratic form of a height function (RHS of (4.7)) is normal curvature at $x \in \partial\Omega$ in the direction $\zeta \in \mathbb{S}^2$, that is,

$$|w_h \cdot \nabla^2 h(0, 0)w_h| = |n(x) \cdot \alpha''(0)|$$

for any (parametrized) curve α such that $\alpha(0) = x \in \partial\Omega$ and $\alpha'(0) = \zeta$ with $|\alpha'(0)| = |\zeta| = 1$. Hence, combining with (1.3) and (4.8), we obtain

$$\theta_\Omega \leq |\zeta \cdot \nabla^2\xi(x)\zeta| = |n(x) \cdot \alpha''(0)| \lesssim_\Omega \|\xi\|_{C^2}, \quad \text{where } \alpha(0) = x, \alpha'(0) = \zeta, |\alpha'(0)| = |\zeta| = 1. \quad (4.9)$$

From (4.9), for any point $x \in \partial\Omega$ and any direction $\zeta \in T_x(\partial\Omega)$, corresponding normal curvature is uniformly bounded from below and above.

Step 2 Now, to consider all possible planes S that satisfy (4.1), we first parametrize such planes. We choose a fixed point $p \in \Omega$ and a unit vector $\ell \in \mathbb{S}^2$. Then, let us use $S_{p,\ell}$ to denote the plane

$$S_{p,\ell} := \{x \in \mathbb{R}^3 : (x - p) \cdot \ell = 0\},$$

the plane which is perpendicular to ℓ and passes $p \in \Omega$. Moreover, by uniform convexity, for fixed $p \in \Omega$ and $\ell \in \mathbb{S}^2$, there exists $a_{p,\ell} < b_{p,\ell}$ such that plane $S_{p+r\ell,\ell}$ also satisfies (4.1) for all $a_{p,\ell} < r < b_{p,\ell}$, that is, plane perpendicular to ℓ and passes $p + r\ell$. Since ℓ is unit,

$$|b_{p,\ell} - a_{p,\ell}| \leq \max_{x,y \in \partial\Omega} |x - y| \lesssim_{\Omega} 1 \quad \forall \ell \in \mathbb{S}^2. \quad (4.10)$$

For fixed p, ℓ , we split

$$(a_{p,\ell}, b_{p,\ell}) = (a_{p,\ell}, a_{p,\ell} + \varepsilon_{p,\ell}] \cup [a_{p,\ell} + \varepsilon_{p,\ell}, b_{p,\ell} - \varepsilon_{p,\ell}] \cup [b_{p,\ell} - \varepsilon_{p,\ell}, b_{p,\ell}).$$

There exists $\varepsilon_{p,\ell} \ll 1$ such that for $r \in (a_{p,\ell}, a_{p,\ell} + \varepsilon_{p,\ell}]$, we can parametrize

$$\{\partial\Omega \cap S_{p+r\ell,\ell} : r \in [a_{p,\ell}, a_{p,\ell} + \varepsilon_{p,\ell}]\}$$

as a graph $\eta_{p,\ell}$ over tangent plane $S_{p+a_{p,\ell}\ell,\ell}$. Similar as local parametrization of Step 1 (rotation and translation similar as (4.6)), we consider local height function $\eta_{p,\ell}$ such that $\vec{x} \mapsto (\vec{x}, \eta_{p,\ell}(\vec{x}))$ with $\vec{x} = (x, y)^T$ is an orthogonal parametrization near $S_{p+a_{p,\ell}\ell,\ell} \cap \partial\Omega$. It is obvious that $\eta_{p,\ell}(0) = \nabla\eta_{p,\ell}(0) = 0$ and its principal directions are (1,0) and (0,1) (for maximum and minimum curvature, respectively). By the Taylor expansion,

$$\eta_{p,\ell}(\vec{x}) = \frac{1}{2} \vec{x} \cdot \nabla^2 \eta_{p,\ell}|_{(0,0)} \vec{x} + O_{\Omega}(|\vec{x}|^3) = C_{p,\ell,1} x^2 + C_{p,\ell,2} y^2 + O_{\Omega}(|\vec{x}|^3),$$

for some $C_{p,\ell,1} > C_{p,\ell,2} > 0$. Here, note that two principal directions are eigenvectors of $\nabla^2 \eta_{p,\ell}$ in fact. Moreover, $\nabla^2 \eta_{p,\ell}$ is self-adjoint and positive definite by (1.3) and (4.8), so both two eigenvalues have positive sign. Thus, we obtain the RHS above. In other words, $\partial\Omega \cap S_{p+r\ell,\ell}$ is a small perturbation of ellipse for any $r \in (a_{p,\ell}, a_{p,\ell} + \varepsilon_{p,\ell}]$, that is,

$$C_{p,\ell,1} x^2 + C_{p,\ell,2} y^2 + O_{\Omega}(|\vec{x}|^3) = r - a_{p,\ell}. \quad (4.11)$$

We can choose sufficiently small $\varepsilon_{p,\ell} \ll 1$ and note that $|\vec{x}| \rightarrow 0$ as $\varepsilon_{p,\ell} \rightarrow 0$.

The curve and corresponding curvature that satisfies (4.11) is sufficiently smooth and therefore, the ratio

$$\frac{\min_s \left| \mathbf{r}''_{S_{p+r\ell}}(s) \right|}{\max_s \left| \mathbf{r}''_{S_{p+r\ell}}(s) \right|}$$

is continuous in $r \in (a_{p,\ell}, a_{p,\ell} + \varepsilon_{p,\ell}]$. Moreover,

$$1 \geq \frac{\min_s |\mathbf{r}_{S_{p+r\ell}}''(s)|}{\max_s |\mathbf{r}_{S_{p+r\ell}}''(s)|} \rightarrow \left(\frac{C_{p,\ell,2}}{C_{p,\ell,1}} \right)^{\frac{3}{2}} \text{ as } r \rightarrow a_{p,\ell},$$

because the maximum and minimum curvatures of the ellipse $C_{p,\ell,1}x^2 + C_{p,\ell,2}y^2 = r - a_{p,\ell}$ are given by $\frac{C_{p,\ell,1}}{\sqrt{(r-a_{p,\ell})C_{p,\ell,2}}}$ and $\frac{C_{p,\ell,2}}{\sqrt{(r-a_{p,\ell})C_{p,\ell,1}}}$, respectively. (Note that it is easy to check the maximum and minimum curvatures of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given by $\frac{a}{b^2}$ and $\frac{b}{a^2}$, respectively, when $a \geq b$.) Now by continuity argument, we can choose $\varepsilon_{p,\ell} \ll 1$ such that

$$\frac{1}{2} \left(\frac{C_{p,\ell,2}}{C_{p,\ell,1}} \right)^{\frac{3}{2}} \leq \frac{\min_s |\mathbf{r}_{S_{p+r\ell}}''(s)|}{\max_s |\mathbf{r}_{S_{p+r\ell}}''(s)|} \leq 1, \quad \forall r \in (a_{p,\ell}, a_{p,\ell} + \varepsilon_{p,\ell}]. \quad (4.12)$$

Similarly, we repeat above process near $\partial\Omega \cap S_{p+b_{p,\ell}\ell}$ using height function, say $\tilde{\eta}_{p,\ell}(\vec{x})$ to get

$$\tilde{\eta}_{p,\ell}(\vec{x}) = \frac{1}{2} \vec{x} \cdot \nabla^2 \tilde{\eta}_{p,\ell}|_{(0,0)} \vec{x} + O_\Omega(|\vec{x}|^3) = C'_{p,\ell,1}x^2 + C'_{p,\ell,2}y^2 + O_\Omega(|\vec{x}|^3),$$

for some $C'_{p,\ell,1} > C'_{p,\ell,2} > 0$. By same argument, we can choose $\varepsilon_{p,\ell} \ll 1$ (even smaller if necessary)

$$\frac{1}{2} \left(\frac{C'_{p,\ell,2}}{C'_{p,\ell,1}} \right)^{\frac{3}{2}} \leq \frac{\min_s |\mathbf{r}_{S_{p+r\ell}}''(s)|}{\max_s |\mathbf{r}_{S_{p+r\ell}}''(s)|} \leq 1, \quad \forall r \in [b_{p,\ell} - \varepsilon_{p,\ell}, b_{p,\ell}]. \quad (4.13)$$

For $r \in [a_{p,\ell} + \varepsilon_{p,\ell}, b_{p,\ell} - \varepsilon_{p,\ell}]$, we obtain

$$0 < c_{p,\ell,r} \leq \frac{\min_s |\mathbf{r}_{S_{p+r\ell}}''(s)|}{\max_s |\mathbf{r}_{S_{p+r\ell}}''(s)|} \leq 1,$$

where strict positivity of $c_{p,\ell,r} > 0$ comes from (4.9) and $\max_s |\mathbf{r}_{S_{p+r\ell}}''(s)| < \infty$. Moreover, this is continuous function in $r \in [a_{p,\ell} + \varepsilon_{p,\ell}, b_{p,\ell} - \varepsilon_{p,\ell}]$. Combining with compactness of $[a_{p,\ell} + \varepsilon_{p,\ell}, b_{p,\ell} - \varepsilon_{p,\ell}]$ (by (4.10)), we derive some r -independent $D_{p,\ell} > 0$ such that

$$D_{p,\ell} \lesssim_\Omega \frac{\min_s |\mathbf{r}_{S_{p+r\ell}}''(s)|}{\max_s |\mathbf{r}_{S_{p+r\ell}}''(s)|} \leq 1, \quad r \in [a_{p,\ell} + \varepsilon_{p,\ell}, b_{p,\ell} - \varepsilon_{p,\ell}]. \quad (4.14)$$

Combining (4.12), (4.13), and (4.14), we have $C_{p,\ell} > 0$ such that

$$C_{p,\ell} \lesssim \frac{\min_s \left| \mathbf{r}_{S_{p+r\ell}}''(s) \right|}{\max_s \left| \mathbf{r}_{S_{p+r\ell}}''(s) \right|} \leq 1 \quad \text{for all } r \in (a_{p,\ell}, b_{p,\ell}).$$

Now, we consider all possible $\ell \in \mathbb{S}^2$ and can use similar continuity argument as (4.14). From compactness of $\mathbb{S}^2$, we can drop ℓ independence from lower bound and obtain

$$C'_p \leq \frac{\min_s \left| \mathbf{r}_{S_{p+r\ell}}''(s) \right|}{\max_s \left| \mathbf{r}_{S_{p+r\ell}}''(s) \right|} \leq 1, \quad (4.15)$$

which gives (4.4). Also, combining (4.15), (4.5) and (4.9), we obtain (4.3). $\square$

Lemma 4.3. Suppose the domain Ω is given as in Definition 1.2 and (1.3).

- (i) Let two distinct points $x, \bar{x} \in \Omega$, and a nonzero velocity $v \neq 0$ are given. We assume (2.6) and (2.13), and recall $\mathbf{x}(\tau)$ in Definition 2.4. There exists $\tau_0(x, \bar{x}, v) \in (\tau_-(x, \bar{x}, v), \tau_+(x, \bar{x}, v))$ such that

$$\begin{cases} \dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) > 0 & \text{for } \tau_-(x, \bar{x}, v) \leq \tau < \tau_0(x, \bar{x}, v), \\ \dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) = 0 & \text{for } \tau = \tau_0(x, \bar{x}, v), \\ \dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) < 0 & \text{for } \tau_0(x, \bar{x}, v) < \tau \leq \tau_+(x, \bar{x}, v), \end{cases} \quad (4.16)$$

where $\tau_{\pm}(x, \bar{x}, v)$ is defined in (2.16). Moreover, there exists $C_{\Omega} \gtrsim 1$ such that

$$|\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))| \leq C_{\Omega} |\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))|, \quad \tau_- \leq \tau \leq \tau_0, \quad (4.17)$$

$$|\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))| \leq C_{\Omega} |\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_+), v))|, \quad \tau_0 \leq \tau \leq \tau_+, \quad (4.18)$$

and

$$|v \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))| \leq C_{\Omega} |v \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_0), v))|, \quad \tau_- \leq \tau \leq \tau_+. \quad (4.19)$$

- (ii) Let $x \in \Omega$ and $v, \bar{v}, \zeta \in \mathbb{R}^3$ are given. We assume (2.8) and (2.14), and recall $\mathbf{v}(\tau)$ in Definition 2.4. There exists $\tau_0(x, v, \bar{v}, \zeta) \in (\tau_-(x, v, \bar{v}, \zeta), \tau_+(x, v, \bar{v}, \zeta))$ such that

$$\begin{cases} \dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) < 0 & \text{for } \tau_-(x, v, \bar{v}, \zeta) \leq \tau < \tau_0(x, v, \bar{v}, \zeta), \\ \dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) = 0 & \text{for } \tau = \tau_0(x, v, \bar{v}, \zeta), \\ \dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) > 0 & \text{for } \tau_0(x, v, \bar{v}, \zeta) < \tau \leq \tau_+(x, v, \bar{v}, \zeta). \end{cases} \quad (4.20)$$

Here, $\tau_{\pm}(x, v, \bar{v}, \zeta)$ is defined in Definition 2.4. Moreover, there exists $C_{\Omega} \gtrsim 1$ such that

$$|\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))| \leq C_{\Omega} |\dot{\mathbf{v}}(\tau_-) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau_-)))|, \quad \tau_- \leq \tau \leq \tau_0, \quad (4.21)$$

$$|\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))| \leq C_{\Omega} |\dot{\mathbf{v}}(\tau_+) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau_+)))|, \quad \tau_0 \leq \tau \leq \tau_+, \quad (4.22)$$

and

$$|\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))| \leq C_{\Omega} |\mathbf{v}(\tau_0) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau_0)))|, \quad \tau_- \leq \tau \leq \tau_+. \quad (4.23)$$

Proof.

(i) First we prove (4.16). From (5.11), (5.13), and (1.3), we derive that

$$\begin{aligned} & \frac{d}{d\tau} \left[(\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))) e^{\int_{\tau_-}^{\tau} \frac{(v \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(s), v)) \cdot \dot{\mathbf{x}})}{\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(s), v)) \cdot v} ds} \right] \\ &= -(-\dot{\mathbf{x}} \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot \dot{\mathbf{x}}) e^{\int_{\tau_-}^{\tau} \frac{(v \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(s), v)) \cdot \dot{\mathbf{x}})}{\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(s), v)) \cdot v} ds} < 0. \end{aligned} \quad (4.24)$$

Since $x_{\mathbf{b}}(\mathbf{x}(\tau), v)$ is well-defined for $\tau_- \leq \tau \leq \tau_+$ on $\partial\Omega$, we have

$$\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v)) > 0 \quad \text{and} \quad \dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_+), v)) < 0. \quad (4.25)$$

Since $e^{\int_{\tau_-}^{\tau} \frac{(v \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(s), v)) \cdot \dot{\mathbf{x}})}{\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(s), v)) \cdot v} ds} > 0$ always, combining with (4.24) and (4.25), there exists a unique $\tau_0(x, \bar{x}, v) \in (\tau_-(x, \bar{x}, v), \tau_+(x, \bar{x}, v))$ such that

$$(\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_0), v))) e^{\int_{\tau_-}^{\tau_0} \frac{(v \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(s), v)) \cdot \dot{\mathbf{x}})}{\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(s), v)) \cdot v} ds} = 0 \Leftrightarrow \dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_0), v)) = 0.$$

This proves (4.16). To prove (4.17), first note that $\dot{\mathbf{x}}$ and $\widehat{n}_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_{\pm}), v))$ are parallel or antiparallel to each other because $\dot{\mathbf{x}}$ and $\widehat{n}_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_{\pm}), v))$ belong to the plan $S_{(x, \bar{x}, v)}$, and $\dot{\mathbf{x}} \perp v$ and $\widehat{n}_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_{\pm}), v)) \perp v$ by (2.30) and (2.16), respectively, that is,

$$|\dot{\mathbf{x}}| = |\dot{\mathbf{x}} \cdot \widehat{n}_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_{\pm}), v))| = \max_{\tau_- \leq \tau \leq \tau_+} |\dot{\mathbf{x}} \cdot \widehat{n}_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau), v))|. \quad (4.26)$$

Now, since $1 \lesssim_{\Omega} |\nabla \xi| \lesssim_{\Omega} 1$, using (4.26) and Lemma 4.2,

$$\begin{aligned} |\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))| &\lesssim |\dot{\mathbf{x}} \cdot n(x_{\mathbf{b}}(\mathbf{x}(\tau), v))| \\ &\lesssim |\dot{\mathbf{x}} \cdot n_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau), v))| = |n_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau), v))| |\dot{\mathbf{x}} \cdot \widehat{n}_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau), v))| \\ &\leq C_{\Omega} |n_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))| |\dot{\mathbf{x}} \cdot \widehat{n}_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))| \\ &= C_{\Omega} |\dot{\mathbf{x}} \cdot n_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))| = C_{\Omega} |\dot{\mathbf{x}} \cdot n(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))| \\ &\lesssim C_{\Omega} |\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))|. \end{aligned} \quad (4.27)$$

Estimate (4.18) is obtained similarly.

Now, let us prove (4.19). v and $n_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_0), v))$ are parallel or antiparallel to each other because $\dot{\mathbf{x}} \perp v$ (by (2.17)) and (4.16), that is,

$$|v \cdot n_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_0), v))| = |v| |n_{\parallel}(x_{\mathbf{b}}(\mathbf{x}(\tau_0), v))|. \quad (4.28)$$

Therefore, using (4.28) and Lemma 4.2,

$$\begin{aligned} |v \cdot \nabla \xi(x_b(\mathbf{x}(\tau_0), v))| &\gtrsim |v \cdot n_{\parallel}(x_b(\mathbf{x}(\tau_0), v))| = |v| |n_{\parallel}(x_b(\mathbf{x}(\tau_0), v))| \\ &\gtrsim_{\Omega} |v \cdot \widehat{n}_{\parallel}(x_b(\mathbf{x}(\tau), v))| |n_{\parallel}(x_b(\mathbf{x}(\tau), v))| \\ &= |v \cdot n_{\parallel}(x_b(\mathbf{x}(\tau), v))| \gtrsim |v \cdot \nabla \xi(x_b(\mathbf{x}(\tau), v))|. \end{aligned}$$

(ii) To prove (4.20), we derive that

$$\begin{aligned} &\frac{d}{d\tau} \left[(\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))) e^{\int_{\tau_-}^{\tau} \frac{-t_b(x, \mathbf{v}(s))(-\mathbf{v}(s) \cdot \nabla^2 \xi(x_b(x, \mathbf{v}(s))) \cdot \dot{\mathbf{v}}(s))}{\nabla \xi(x_b(x, \mathbf{v}(s))) \cdot \mathbf{v}(s)} ds} \right] \\ &= [t_b(x, \mathbf{v}(\tau))(-\dot{\mathbf{v}}(\tau) \cdot \nabla^2 \xi(x_b(x, \mathbf{v}(\tau))) \cdot \dot{\mathbf{v}}(\tau)) + \dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))] \\ &\quad \times e^{\int_{\tau_-}^{\tau} \frac{-t_b(x, \mathbf{v}(s))(-\mathbf{v}(s) \cdot \nabla^2 \xi(x_b(x, \mathbf{v}(s))) \cdot \dot{\mathbf{v}}(s))}{\nabla \xi(x_b(x, \mathbf{v}(s))) \cdot \mathbf{v}(s)} ds} > 0, \end{aligned}$$

where we have used the fact: $\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_b(x, v(\tau))) = -\theta^2 \mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau))) \geq 0$, by (2.17) and $\mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau))) \leq 0$. Similar as (4.25), we have

$$\dot{\mathbf{v}}(\tau_-) \cdot \nabla \xi(x_b(\mathbf{x}(\tau_-), v)) < 0 \quad \text{and} \quad \dot{\mathbf{v}}(\tau_+) \cdot \nabla \xi(x_b(\mathbf{x}(\tau_+), v)) > 0.$$

Now, (4.20) is obtained similar as proof of (4.16), since $t_b(x, \mathbf{v}(\tau)) > 0$ always. To prove (4.21), note that $\dot{\mathbf{v}}(\tau_{\pm})$ and $\widehat{n}_{\parallel}(x_b(x, \mathbf{v}(\tau_{\pm})))$ are parallel or antiparallel to each other because $\dot{\mathbf{v}}(\tau_{\pm})$ and $\widehat{n}_{\parallel}(x_b(x, \mathbf{v}(\tau_{\pm})))$ belong to the plan $S_{(x, v, \bar{v}, \xi)}$, and $\dot{\mathbf{v}}(\tau) \perp \mathbf{v}(\tau)$ and $\widehat{n}_{\parallel}(x_b(x, \mathbf{v}(\tau_{\pm}))) \perp \mathbf{v}(\tau_{\pm})$ by (2.17) and (2.16), respectively, that is,

$$|\dot{\mathbf{v}}(\tau_{\pm})| = |\dot{\mathbf{v}}(\tau_{\pm}) \cdot \widehat{n}_{\parallel}(x_b(x, \mathbf{v}(\tau_{\pm})))| = \max_{\tau_- \leq \tau \leq \tau_+} |\dot{\mathbf{v}}(\tau) \cdot \widehat{n}_{\parallel}(x_b(x, \mathbf{v}(\tau)))|. \quad (4.29)$$

Using (4.29) and Lemma 4.2,

$$\begin{aligned} |\dot{\mathbf{v}} \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))| &\lesssim |\dot{\mathbf{v}} \cdot n(x_b(x, \mathbf{v}(\tau)))| \\ &= |\dot{\mathbf{v}} \cdot n_{\parallel}(x_b(x, \mathbf{v}(\tau)))| = |n_{\parallel}(x_b(x, \mathbf{v}(\tau)))| |\dot{\mathbf{v}} \cdot \widehat{n}_{\parallel}(x_b(x, \mathbf{v}(\tau)))| \\ &\leq C_{\Omega} |n_{\parallel}(x_b(x, \mathbf{v}(\tau_-)))| |\dot{\mathbf{v}} \cdot \widehat{n}_{\parallel}(x_b(x, \mathbf{v}(\tau_-)))| \\ &\leq C_{\Omega} |\dot{\mathbf{v}} \cdot n(x_b(x, \mathbf{v}(\tau_-)))| \leq C_{\Omega} |\dot{\mathbf{v}} \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau_-)))|. \end{aligned}$$

Estimate (4.22) is obtained similarly.

Now, let us prove (4.23). $\mathbf{v}(\tau_0)$ and $n_{\parallel}(x_b(x, \mathbf{v}(\tau_0)))$ are parallel to each other because $\mathbf{v}(\tau_0)$ and $n_{\parallel}(x_b(x, \mathbf{v}(\tau_0)))$ belong to $S_{(x, v, \bar{v}, \xi)}$, and $\dot{\mathbf{v}}(\tau) \perp \mathbf{v}(\tau)$ (by (2.17)) and (4.20). Therefore,

$$|\mathbf{v}(\tau_0) \cdot n_{\parallel}(x_b(x, \mathbf{v}(\tau_0)))| = |\mathbf{v}(\tau_0)| |n_{\parallel}(x_b(x, \mathbf{v}(\tau_0)))|. \quad (4.30)$$

Therefore, using (4.30) and Lemma 4.2,

$$\begin{aligned} |\mathbf{v}(\tau_0) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau_0)))| &\gtrsim |\mathbf{v}(\tau_0) \cdot n_{\parallel}(x_b(x, \mathbf{v}(\tau_0)))| = |\mathbf{v}(\tau_0)| |n_{\parallel}(x_b(x, \mathbf{v}(\tau_0)))| \\ &\gtrsim_{\Omega} |\mathbf{v}(\tau) \cdot \widehat{n}_{\parallel}(x_b(x, \mathbf{v}(\tau)))| |n_{\parallel}(x_b(x, \mathbf{v}(\tau)))| \\ &= |\mathbf{v}(\tau) \cdot n_{\parallel}(x_b(x, \mathbf{v}(\tau)))| \gtrsim |\mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))|. \quad \square \end{aligned}$$

Lemma 4.4. Recall $\mathbf{x}(\tau) = \mathbf{x}(\tau; x, \bar{x}, v)$, $\mathbf{v}(\tau) = \mathbf{v}(\tau; x, v, \bar{v}, \zeta)$ as in Definition 2.4, and $\tau_0(x, \bar{x}, v)$ and $\tau_0(x, v, \bar{v}, \zeta)$ at (4.16) and (4.20), respectively, in Lemma 4.3.

(i) Recall all assumptions in (i) of Lemma 4.3. When $\tau_- \leq \tau_* \leq \tau_0$, we get

$$\max_{\tau_- \leq s \leq \tau_*} t_{\mathbf{b}}(\mathbf{x}(s), v) = t_{\mathbf{b}}(\mathbf{x}(\tau_-), v), \quad \min_{\tau_- \leq s \leq \tau_*} t_{\mathbf{b}}(\mathbf{x}(s), v) = t_{\mathbf{b}}(\mathbf{x}(\tau_*), v). \quad (4.31)$$

By symmetry, when $\tau_0 \leq \tau_* \leq \tau_+$,

$$\max_{\tau_* \leq s \leq \tau_+} t_{\mathbf{b}}(\mathbf{x}(s), v) = t_{\mathbf{b}}(\mathbf{x}(\tau_+), v), \quad \min_{\tau_* \leq s \leq \tau_+} t_{\mathbf{b}}(\mathbf{x}(s), v) = t_{\mathbf{b}}(\mathbf{x}(\tau_*), v).$$

(ii) Recall all assumptions in (ii) of Lemma 4.3. When $\tau_- \leq \tau_* \leq \tau_0$, we get

$$\max_{\tau_- \leq s \leq \tau_*} t_{\mathbf{b}}(x, \mathbf{v}(s)) = t_{\mathbf{b}}(x, \mathbf{v}(\tau_-)), \quad \min_{\tau_- \leq s \leq \tau_*} t_{\mathbf{b}}(x, \mathbf{v}(s)) = t_{\mathbf{b}}(x, \mathbf{v}(\tau_*)). \quad (4.32)$$

By symmetry, when $\tau_0 \leq \tau_* \leq \tau_+$,

$$\max_{\tau_* \leq s \leq \tau_+} t_{\mathbf{b}}(x, \mathbf{v}(s)) = t_{\mathbf{b}}(x, \mathbf{v}(\tau_+)), \quad \min_{\tau_* \leq s \leq \tau_+} t_{\mathbf{b}}(x, \mathbf{v}(s)) = t_{\mathbf{b}}(x, \mathbf{v}(\tau_*)).$$

also holds. Moreover, for all $\tau_- \leq \tau \leq \tau_+$,

$$\max_{\tau_- \leq \tau \leq \tau_+} (|\mathbf{v}(\tau)| t_{\mathbf{b}}(x, \mathbf{v}(\tau))) \lesssim_{\Omega} 1 + \min_{\tau_- \leq \tau \leq \tau_+} (|\mathbf{v}(\tau)| t_{\mathbf{b}}(x, \mathbf{v}(\tau))). \quad (4.33)$$

Proof.

(i) Let $\tau_- \leq \tau_* \leq \tau_0$. From (5.15) and (4.16),

$$\frac{d}{d\tau} t_{\mathbf{b}}(\mathbf{x}(\tau), v) = \frac{\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot \dot{\mathbf{x}}}{\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot v} < 0$$

since $\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot v < 0$. $\tau_0 \leq \tau_* \leq \tau_+$ case is similar. We also note that $\frac{d}{d\tau} t_{\mathbf{b}}(\mathbf{x}(\tau), v) = 0$ only at τ_0 by (4.16).

(ii) We can prove similarly as (i) using (5.15) and (4.20), since

$$\frac{d}{d\tau} t_{\mathbf{b}}(x, \mathbf{v}(\tau)) = -t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \frac{\nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \dot{\mathbf{v}}(\tau)}{\nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \mathbf{v}(\tau)} < 0.$$

To prove (4.33), let us consider $|\mathbf{v}(\tau)| t_{\mathbf{b}}(x, \mathbf{v}(\tau))$. If $\text{dist}(x, \partial\Omega) \leq 1$, then we have $\max_{\tau} |\mathbf{v}(\tau)| t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \lesssim_{\Omega} 1$ also. If $\text{dist}(x, \partial\Omega) \geq 1$, we have

$$\begin{aligned} \max_{\tau} (|\mathbf{v}(\tau)| t_{\mathbf{b}}(x, \mathbf{v}(\tau))) &\leq \text{dist}(x, \partial\Omega) + \max_{p, q \in \partial\Omega} |p - q| \\ &\leq \text{dist}(x, \partial\Omega)(1 + C_{\Omega}) \leq (1 + C_{\Omega}) \min_{\tau} (|\mathbf{v}(\tau)| t_{\mathbf{b}}(x, \mathbf{v}(\tau))). \quad \square \end{aligned}$$

Lemma 4.5.

(i) Assume (2.6) and (2.13), and recall $\tau_0(x, \bar{x}, v)$ from (4.16). Then,

$$\frac{\tau_0(x, \bar{x}, v) - \tau_-(x, \bar{x}, v)}{\tau_+(x, \bar{x}, v) - \tau_-(x, \bar{x}, v)} \gtrsim_{\Omega} 1. \quad (4.34)$$

(ii) Assume (2.8) and (2.14), and recall $\tau_0(x, v, \bar{v}, \zeta)$ from (4.20). Then,

$$\frac{\tau_0(x, v, \bar{v}, \zeta) - \tau_-(x, v, \bar{v}, \zeta)}{\tau_+(x, v, \bar{v}, \zeta) - \tau_-(x, v, \bar{v}, \zeta)} \gtrsim_{\Omega} 1. \quad (4.35)$$

Proof. For fixed S , let

$$R_S := \left(\min_{x \in \partial\Omega \cap S} \kappa_S(x) \right)^{-1}, \quad r_S := \left(\max_{x \in \partial\Omega \cap S} \kappa_S(x) \right)^{-1}. \quad (4.36)$$

Now, let us consider a circles C_1 with radius r . At any point $y \in \partial\Omega \cap S$, we can place C_1 inside of $\partial\Omega \cap S$ while C_1 is tangential at $y \in \partial\Omega \cap S$ by (4.36). Similarly, consider another circles C_2 with radius R and for any point $y \in \partial\Omega \cap S$, we can place C_2 outside of $\partial\Omega \cap S$ while C_2 is tangential at $y \in \partial\Omega \cap S$.

(i) If we choose $y = x_b(\mathbf{x}(\tau_0), v)$, it is obvious that

$$r_S \leq |\dot{\mathbf{x}}|(\tau_0 - \tau_-), \quad |\dot{\mathbf{x}}|(\tau_+ - \tau_0) \leq 2R_S.$$

Therefore,

$$|\dot{\mathbf{x}}(\tau_0 - \tau_-)| \geq r_S \geq \frac{r_S}{2R_S} |\dot{\mathbf{x}}(\tau_+ - \tau_0)|$$

holds and we obtain (4.34) using (4.4).

(ii) If $\sin(\theta(\tau_0 - \tau_-)) \geq \frac{1}{2}$, we have

$$\theta(\tau_0 - \tau_-) \geq \frac{1}{2}$$

and (4.35) holds since $\theta(\tau_+ - \tau_-) \leq \pi$.

If $0 \leq \sin(\theta(\tau_0 - \tau_-)) \leq \frac{1}{2}$, let us choose $y = x_b(x, \mathbf{v}(\tau_0))$ and circle C_1 to get

$$r_S \leq (|x_b(x, \mathbf{v}(\tau_0)) - x| + r) \sin(\theta(\tau_0 - \tau_-))$$

because C_1 is inside of $\partial\Omega \cap S$. From $0 \leq \sin(\theta(\tau_0 - \tau_-)) \leq \frac{1}{2}$,

$$\frac{r_S}{2} \leq |x_b(x, \mathbf{v}(\tau_0)) - x| \sin(\theta(\tau_0 - \tau_-)).$$

Also, considering outer circle C_2 tangential at $y = x_b(x, \mathbf{v}(\tau_0))$,

$$|x_b(x, \mathbf{v}(\tau_0)) - x|(\theta(\tau_+ - \tau_0)) \lesssim |x_b(x, \mathbf{v}(\tau_0)) - x| \sin(\theta(\tau_+ - \tau_0)) \leq \max_{x, y \in \partial\Omega \cap S} |x - y| \leq 2R_S,$$

where $A \lesssim B$ means $A \leq CB$ with some generic constant $C > 0$, since $(\theta(\tau_+ - \tau_0)) \leq \frac{\pi}{2}$. From above two inequalities,

$$|x_b(x, \mathbf{v}(\tau_0)) - x|(\theta(\tau_0 - \tau_-)) \gtrsim \frac{r_S}{2} \gtrsim \frac{r_S}{4R_S} |x_b(x, \mathbf{v}(\tau_0)) - x|(\theta(\tau_+ - \tau_0))$$

holds and we obtain (4.35) using (4.4). □

5 | SPECULAR SINGULARITY

5.1 | From fraction to specular singularity

Lemma 5.1. Suppose the domain is given as in Definition 1.2 and (1.3).

- (i) Let $x, \bar{x} \in \Omega$, $v \in \mathbb{R}^3$ and assume (2.6), (2.13). Recall shifted position $\tilde{x} = \tilde{x}(x, \bar{x}, v)$ defined in (2.7) of Definition 2.2. For $|x - \bar{x}| \leq 1$,

$$\frac{|V(s; t, x, v) - V(s; t, \bar{x}, v)|}{|x - \bar{x}|} \mathbf{1}_{\left\{ \begin{array}{l} s \leq \min\{t^1(x, v), t^1(\bar{x}, v)\} \\ s > \max\{t^1(x, v), t^1(\bar{x}, v)\} \end{array} \right\}} \leq |v| + |v|^2 \int_0^1 \frac{1}{\mathfrak{S}_{sp}(\tau; x, \bar{x}, v)} d\tau, \quad (5.1)$$

$$\begin{aligned} & \frac{|X(s; t, x, v) - X(s; t, \bar{x}, v)|}{|x - \bar{x}|} \mathbf{1}_{\left\{ \begin{array}{l} s \leq \min\{t^1(x, v), t^1(\bar{x}, v)\} \\ s > \max\{t^1(x, v), t^1(\bar{x}, v)\} \end{array} \right\}} \leq 1 + |v|(t - s) + |v|^2(t - s) \\ & \times \int_0^1 \frac{1}{\mathfrak{S}_{sp}(\tau; x, \bar{x}, v)} d\tau. \end{aligned} \quad (5.2)$$

- (ii) Let $x \in \Omega$, $v, \bar{v}, \zeta \in \mathbb{R}^3$ and assume (2.8), (2.14). Recall shifted velocity $\bar{v} = \bar{v}(v, \bar{v}, \zeta)$ defined in (2.9) of Definition 2.2. For $|v - \bar{v}| \leq 1$,

$$\begin{aligned} & \frac{|V(s; t, x, v + \zeta) - V(s; t, x, \bar{v} + \zeta)|}{|v - \bar{v}|} \mathbf{1}_{\left\{ \begin{array}{l} s \leq \min\{t^1(x, v + \zeta), t^1(x, \bar{v} + \zeta)\} \\ s > \max\{t^1(x, v + \zeta), t^1(x, \bar{v} + \zeta)\} \end{array} \right\}} \\ & \lesssim 1 + |v + \zeta|(t - s) + |v + \zeta|^2 \int_0^1 \frac{1}{\mathfrak{S}_{vel}(\tau; x, v, \bar{v}, \zeta)} d\tau, \end{aligned} \quad (5.3)$$

$$\begin{aligned} & \frac{|X(s; t, x, v + \zeta) - X(s; t, x, \bar{v} + \zeta)|}{|v - \bar{v}|} \mathbf{1}_{\left\{ \begin{array}{l} s \leq \min\{t^1(x, v + \zeta), t^1(x, \bar{v} + \zeta)\} \\ s > \max\{t^1(x, v + \zeta), t^1(x, \bar{v} + \zeta)\} \end{array} \right\}} \\ & \lesssim (t - s) + |v + \zeta|(t - s)^2 + |v + \zeta|^2(t - s) \int_0^1 \frac{1}{\mathfrak{S}_{vel}(\tau; x, v, \bar{v}, \zeta)} d\tau. \end{aligned} \quad (5.4)$$

Remark 5.2. In the regularity estimate, we are only interested in the case that $s \in [0, t]$. Therefore, if either $t^1(x, v) = -\infty$ or $t^1(\bar{x}, v) = -\infty$, which means either one of the trajectory from (t, x, v) or $(t, \bar{x}, v)$ missed the boundary $\partial\Omega$, then $\mathbf{1}_{\left\{ \begin{array}{l} s \leq \min\{t^1(x, v), t^1(\bar{x}, v)\} \\ s > \max\{t^1(x, v), t^1(\bar{x}, v)\} \end{array} \right\}} \equiv 0$. Note that it is easy to get the same conclusion for $\mathbf{1}_{\left\{ \begin{array}{l} s \leq \min\{t^1(x, v + \zeta), t^1(x, \bar{v} + \zeta)\} \\ s > \max\{t^1(x, v + \zeta), t^1(x, \bar{v} + \zeta)\} \end{array} \right\}}$. The other nontrivial case, only one trajectory hits the boundary, will be discussed in the next lemma.

Proof. If $s > \max\{t^1(x, v), t^1(\bar{x}, v)\}$ or $s > \max\{t^1(x, v + \zeta), t^1(x, \bar{v} + \zeta)\}$, it is trivial. We consider $s \leq \min\{t^1(x, v + \zeta), t^1(x, \bar{v} + \zeta)\}$ and $s \leq \min\{t^1(x, v), t^1(\bar{x}, v)\}$ cases only.

Step 1 From (5.15), for $s \leq t - t_b(x, v)$,

$$\begin{aligned} \nabla_x V(s; t, x, v) &= \nabla_x [v - 2(n(x_b) \cdot v)n(x_b)] \\ &= -2(v \cdot n(x_b)) \frac{1}{|\nabla \xi(x_b)|} (I - n(x_b) \otimes n(x_b)) \nabla^2 \xi(x_b) \nabla_x x_b \end{aligned}$$

$$\begin{aligned}
& -\frac{2}{|\nabla \xi(x_b)|} (n(x_b) \otimes v)(I - n(x_b) \otimes n(x_b)) \nabla^2 \xi(x_b) \nabla_x x_b \\
& = -\frac{2}{|\nabla \xi(x_b)|} ((v \cdot n(x_b)) R_{x_b} + n(x_b) \otimes v) \nabla^2 \xi(x_b) \left(I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} \right) \quad (5.5)
\end{aligned}$$

and

$$\begin{aligned}
\nabla_x X(s; t, x, v) &= \nabla_x [x_b - (t - t_b - s)(v - 2(n(x_b) \cdot v)n(x_b))] \\
&= I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} + (v - 2(v \cdot n(x_b))n(x_b)) \otimes \nabla_x t_b - (t - t_b - s) \nabla_x V(s) \\
&= R_{x_b} - (t - t_b - s) \nabla_x V(s; t, x, v). \quad (5.6)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\nabla_v V(s; t, x, v) &= \nabla_v [v - 2(n(x_b) \cdot v)n(x_b)] \\
&= R_{x_b} + \frac{2t_b}{|\nabla \xi(x_b)|} ((v \cdot n(x_b)) R_{x_b} + n(x_b) \otimes v) \nabla^2 \xi(x_b) \left(I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} \right) \quad (5.7)
\end{aligned}$$

and

$$\begin{aligned}
\nabla_v X(s; t, x, v) &= \nabla_v [x_b - (t - t_b - s)(v - 2(n(x_b) \cdot v)n(x_b))] \\
&= -t_b \left(I - \frac{v \otimes n(x_b)}{v \cdot n(x_b)} \right) + (v - 2(v \cdot n(x_b))n(x_b)) \otimes \nabla_v t_b - (t - t_b - s) \nabla_v V(s) \\
&= -t_b R_{x_b} - (t - t_b - s) \nabla_v V(s; t, x, v). \quad (5.8)
\end{aligned}$$

Step 2 First, we consider (5.1). From (5.5),

$$\begin{aligned}
|V(s; t, x, v) - V(s; t, \tilde{x}, v)| &= \left| \int_0^1 \nabla_x V(s; t, \mathbf{x}(\tau), v) \dot{\mathbf{x}} d\tau \right| \\
&\lesssim |x - \tilde{x}| |v| \int_0^1 \left(1 + |v| \frac{\left| \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi(x_b(\mathbf{x}(\tau), v)) \right|}{|v \cdot \nabla \xi(x_b(\mathbf{x}(\tau), v))|} \right) d\tau.
\end{aligned}$$

Using (2.20), we derive (5.1).

For (5.2), we use (5.6) and then similarly as above,

$$\begin{aligned}
|X(s; t, x, v) - X(s; t, \tilde{x}, v)| &= \left| \int_0^1 \nabla_x X(s; t, \mathbf{x}(\tau), v) \dot{\mathbf{x}} d\tau \right| \\
&\lesssim |x - \tilde{x}| + (t - s) |x - \tilde{x}| |v| \int_0^1 \left(1 + |v| \frac{\left| \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi(x_b(\mathbf{x}(\tau), v)) \right|}{|v \cdot \nabla \xi(x_b(\mathbf{x}(\tau), v))|} \right) d\tau
\end{aligned}$$

holds.

For v -directional parametrization, using (5.7),

$$\begin{aligned} |V(s; t, x, v + \zeta) - V(s; t, x, \tilde{v} + \zeta)| &= \left| \int_0^1 \nabla_v V(s; t, x, v(\tau)) \dot{\mathbf{v}} d\tau \right| \\ &\lesssim |v - \tilde{v}| |v + \zeta| \int_0^1 \left((t - s) + |v + \zeta| \frac{t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \left| \frac{\dot{\mathbf{v}}(\tau)}{|\dot{\mathbf{v}}(\tau)|} \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \right|}{|\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))|} \right) d\tau + |v - \tilde{v}| \end{aligned}$$

which yields (5.3). (5.4) is also gained similarly using (5.8). $\square$

Lemma 5.3. Suppose the domain is given as in Definition 1.2 and (1.3).

- (i) Let $x, \tilde{x} \in \Omega$, $v \in \mathbb{R}^3$ and assume (2.6), (2.13). Recall shifted position $\tilde{x} = \tilde{x}(x, \tilde{x}, v)$ defined in (2.7) of Definition 2.2. If $t_{\mathbf{b}}(\tilde{x}, v), t_{\mathbf{b}}(x, v) < \infty$,

$$(t^1(x, v) - s) \mathbf{1}_{t^1(\tilde{x}, v) < s \leq t^1(x, v)} \leq |x - \tilde{x}| \int_0^1 \frac{1}{\mathfrak{S}_{sp}(\tau; x, \tilde{x}, v)} d\tau. \quad (5.9)$$

- (ii) Let $x \in \Omega$, $v, \tilde{v}, \zeta \in \mathbb{R}^3$ and assume (2.8), (2.14). Recall shifted velocity $\tilde{v} = \tilde{v}(v, \tilde{v}, \zeta)$ defined in (2.9) of Definition 2.2. If $t_{\mathbf{b}}(x, \tilde{v} + \zeta), t_{\mathbf{b}}(x, v + \zeta) < \infty$,

$$(t^1(x, v + \zeta) - s) \mathbf{1}_{t^1(x, \tilde{v} + \zeta) < s \leq t^1(x, v + \zeta)} \lesssim |v - \tilde{v}| \int_0^1 \frac{1}{\mathfrak{S}_{vel}(\tau; x, v, \tilde{v}, \zeta)} d\tau. \quad (5.10)$$

Proof. For (5.9)

$$\begin{aligned} (t^1(x, v) - s) \mathbf{1}_{t^1(\tilde{x}, v) < s \leq t^1(x, v)} &\leq |t_{\mathbf{b}}(x, v) - t_{\mathbf{b}}(\tilde{x}, v)| = \left| \int_0^1 \nabla_x t_{\mathbf{b}}(\mathbf{x}(\tau), v) \dot{\mathbf{x}} d\tau \right| \\ &\leq \int_0^1 \left| \frac{\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))}{v \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))} \right| d\tau \leq |\dot{\mathbf{x}}| \int_0^1 \frac{1}{\mathfrak{S}_{sp}(\tau; x, \tilde{x}, v)} d\tau. \end{aligned}$$

Similarly, for (5.10)

$$\begin{aligned} (t^1(x, v + \zeta) - s) \mathbf{1}_{t^1(x, \tilde{v} + \zeta) < s \leq t^1(x, v + \zeta)} &\leq |t_{\mathbf{b}}(x, v + \zeta) - t_{\mathbf{b}}(x, \tilde{v} + \zeta)| = \left| \int_0^1 \nabla_v t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \dot{\mathbf{v}}(\tau) d\tau \right| \\ &\leq \int_0^1 \left| t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \frac{\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))}{\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))} \right| d\tau \leq |\dot{\mathbf{v}}(\tau)| \int_0^1 \frac{1}{\mathfrak{S}_{vel}(\tau; x, v, \tilde{v}, \zeta)} d\tau. \quad \square \end{aligned}$$

5.2 | Averaging specular singularity

We start with the ODEs for the specular singularities. Let $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$ for a C^2 -function $\xi : \mathbb{R}^3 \rightarrow \mathbb{R}$. Let two arbitrary maps $\tau \mapsto \mathbf{x}(\tau) \in \Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$ and $\tau \mapsto \mathbf{v}(\tau) \in \mathbb{R}^3$ are differentiable and $\dot{\mathbf{x}} \equiv 0$. As long as $x_{\mathbf{b}}(\mathbf{x}(\tau), v)$ is well-defined on $\partial\Omega$, we compute

$$\begin{aligned}
& \frac{d}{d\tau}(\dot{\mathbf{x}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))) \\
&= \frac{-1}{-\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot v} \{(\dot{\mathbf{x}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)))(-v \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))\dot{\mathbf{x}}(\tau)) \\
&+ (-\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot v)(-\dot{\mathbf{x}}(\tau) \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))\dot{\mathbf{x}}(\tau))\}, \tag{5.11}
\end{aligned}$$

$$\begin{aligned}
& \frac{d}{d\tau}(-\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot v) \\
&= \frac{1}{-\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot v} \{(\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)))(-v \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))v) \\
&+ (-\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) \cdot v)(-\dot{\mathbf{x}} \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))v)\}. \tag{5.12}
\end{aligned}$$

As long as $x_{\mathbf{b}}(x, \mathbf{v}(\tau))$ is well-defined on $\partial\Omega$, we obtain

$$\begin{aligned}
& \frac{d}{d\tau}(\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))) = \ddot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) + \frac{t_{\mathbf{b}}(x, \mathbf{v}(\tau))}{-\nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \mathbf{v}(\tau)} \\
&\times \{(\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))))(-\mathbf{v}(\tau) \cdot \nabla^2 \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))\dot{\mathbf{v}}(\tau)) \\
&+ (-\nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \mathbf{v}(\tau))(-\dot{\mathbf{v}}(\tau) \cdot \nabla^2 \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))\dot{\mathbf{v}}(\tau))\}, \tag{5.13}
\end{aligned}$$

$$\begin{aligned}
& \frac{d}{d\tau}(-\nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \mathbf{v}(\tau)) = -\nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \dot{\mathbf{v}}(\tau) + \frac{-t_{\mathbf{b}}(x, \mathbf{v}(\tau))}{-\nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \mathbf{v}(\tau)} \\
&\times \{(\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))))(-\mathbf{v}(\tau) \cdot \nabla^2 \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))\mathbf{v}(\tau)) \\
&+ (-\nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \mathbf{v}(\tau))(-\dot{\mathbf{v}}(\tau) \cdot \nabla^2 \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))\mathbf{v}(\tau))\}. \tag{5.14}
\end{aligned}$$

These are the direct outcome of the basic computation ([1, 13])

$$\begin{aligned}
& \nabla_x t_{\mathbf{b}} = \frac{\nabla \xi(x_{\mathbf{b}})}{\nabla \xi(x_{\mathbf{b}}) \cdot v}, \quad \nabla_v t_{\mathbf{b}} = -t_{\mathbf{b}} \nabla_x t_{\mathbf{b}}, \\
& \nabla_x x_{\mathbf{b}} = I - \frac{v \otimes n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})}, \quad \nabla_v x_{\mathbf{b}} = -t_{\mathbf{b}} \nabla_x x_{\mathbf{b}}, \\
& \nabla_x x_{\mathbf{b}} = I - \frac{v \otimes n(x_{\mathbf{b}})}{v \cdot n(x_{\mathbf{b}})}, \quad \nabla_v x_{\mathbf{b}} = -t_{\mathbf{b}} \nabla_x x_{\mathbf{b}}, \\
& \nabla_x n(x_{\mathbf{b}}) = \frac{1}{|\nabla \xi(x_{\mathbf{b}})|} (I - n(x_{\mathbf{b}}) \otimes n(x_{\mathbf{b}})) \nabla^2 \xi(x_{\mathbf{b}}). \tag{5.15}
\end{aligned}$$

The next differential inequalities are crucially used to prove Proposition 6.7.

Lemma 5.4 (ODE for Specular Singularity). *Suppose the domain is given as in Definitions 1.2 and (1.3).*

(i) Recall $\tilde{x} = \tilde{x}(x, \tilde{x}, v)$ in Definition (2.2) under assumption (2.6) and (2.13).

We also recall $\mathfrak{S}_{sp}(\tau; x, \tilde{x}, v)$ in (2.20) and $\mathbf{x}(\tau)$ in (2.15). For $\tau \in (\tau_-(x, \tilde{x}, v), \tau_+(x, \tilde{x}, v))$,

$$\frac{d\mathfrak{S}_{sp}(\tau; x, \tilde{x}, v)}{d\tau} \geq \frac{1}{\mathfrak{S}_{sp}(\tau; x, \tilde{x}, v)} \frac{\theta_\Omega |\dot{\mathbf{x}}|^2}{|\dot{\mathbf{x}} \cdot \nabla \xi(\mathbf{x}_b(\mathbf{x}(\tau), v))|} (|v|^2 + \mathfrak{S}_{sp}^2(\tau; x, \tilde{x}, v)). \quad (5.16)$$

(ii) We recall $\tilde{v} = \tilde{v}(v, \tilde{v}, \zeta)$ in Definition (2.2) under assumption (2.8) and (2.14). Recall $\mathfrak{S}_{vel}(\tau; x, v, \tilde{v}, \zeta)$ in (2.21) and $\mathbf{v}(\tau)$ in (2.15). Define

$$\tilde{\mathfrak{S}}_{vel}(\tau; x, v, \tilde{v}, \zeta) := \left| \frac{\mathbf{v}(\tau) \cdot \nabla \xi(\mathbf{x}_b(x, \mathbf{v}(\tau)))}{\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(\mathbf{x}_b(x, \mathbf{v}(\tau)))} \right| = \frac{t_b(x, \mathbf{v}(\tau))}{|\dot{\mathbf{v}}(\tau)|} \mathfrak{S}_{vel}(\tau; x, v, \tilde{v}, \zeta). \quad (5.17)$$

For $\tau \in (\tau_-(x, v, \tilde{v}, \zeta), \tau_+(x, v, \tilde{v}, \zeta))$,

$$\begin{aligned} & \frac{d\tilde{\mathfrak{S}}_{vel}(\tau; x, v, \tilde{v}, \zeta)}{d\tau} \\ & \geq 1 + \frac{\theta_\Omega |\mathbf{v}(\tau)|^2 t_b(x, \mathbf{v}(\tau))}{\tilde{\mathfrak{S}}_{vel}(\tau; x, v, \tilde{v}, \zeta) |\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(\mathbf{x}_b(x, \mathbf{v}(\tau)))|} \left(1 + \tilde{\mathfrak{S}}_{vel}^2(\tau; x, v, \tilde{v}, \zeta) \right). \end{aligned} \quad (5.18)$$

Remark 5.5. Actually for the differential inequalities (5.16) and (5.18) we do not need the whole setting of Definition 2.2, but only arbitrary $x, \tilde{x}, v, \tilde{v}, \zeta$ satisfy (2.17).

Proof. Step 1. First we prove (5.16). Recall $\tau_0(x, \tilde{x}, v)$ in (4.16) and let us consider the case $\tau \in [\tau_-(x, \tilde{x}, v), \tau_0(x, \tilde{x}, v)]$. Simply let us write $\mathfrak{S}_{sp}(\tau) = \mathfrak{S}_{sp}(\tau; x, \tilde{x}, v)$ here. Using (5.15), (5.11), and (5.12),

$$\frac{d}{d\tau} \mathfrak{S}_{sp}(\tau) = \frac{1}{\mathfrak{S}_{sp}(\tau)} \frac{|\dot{\mathbf{x}}|^2}{\dot{\mathbf{x}} \cdot \nabla \xi(\mathbf{x}_b(\mathbf{x}(\tau), v))} (5.19)_*, \quad (5.19)$$

where

$$\begin{aligned} (5.19)_* &= \frac{1}{\left| \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi \right|^2} \left[\left(\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi \right) v - (\nabla \xi \cdot v) \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \right] \\ & \quad \cdot (-\nabla^2 \xi) \cdot \left[\left(\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi \right) v - (\nabla \xi \cdot v) \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \right] \\ & \geq \frac{\theta_\Omega}{\left| \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi \right|^2} \left| \left(\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi \right) v - (\nabla \xi \cdot v) \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \right|^2. \end{aligned} \quad (5.20)$$

Here, we have used the convexity (1.3) to derive the lower bound estimate in (5.20). Here, we abbreviated $\nabla \xi = \nabla \xi(\mathbf{x}_b(\mathbf{x}(\tau), v))$ and $\nabla^2 \xi = \nabla^2 \xi(\mathbf{x}_b(\mathbf{x}(\tau), v))$ for notational simplicity. Now, using the decomposition

$$v = \left(v \cdot \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \right) \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} + \left(I - \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \otimes \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \right) v,$$

and $\dot{\mathbf{x}} \cdot v = 0$ from (2.17),

$$\begin{aligned}
 (5.20) &= \theta_{\Omega} \left| \left(I - \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \otimes \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \right) \mathbf{v} \right|^2 + \frac{\theta_{\Omega}}{\left| \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi \right|^2} \left| \nabla \xi \cdot \mathbf{v} - \left(\nabla \xi \cdot \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \right) \left(\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \mathbf{v} \right) \right|^2 \\
 &= \theta_{\Omega} |\mathbf{v}|^2 + \frac{\theta_{\Omega}}{\left| \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} \cdot \nabla \xi \right|^2} |\nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), \mathbf{v})) \cdot \mathbf{v}|^2 = \theta_{\Omega} |\mathbf{v}|^2 + \theta_{\Omega} \mathfrak{S}_{sp}^2(\tau),
 \end{aligned}$$

since $\dot{\mathbf{x}} \cdot \mathbf{v} = 0$ from (2.17).

From the above equality, combining with (5.19) and (5.20), we conclude (5.16). Proof for $\tau \in [\tau_0(x, \tilde{x}, v), \tau_+(x, \tilde{x}, v)]$ is same.

Step 2. Next we prove (5.18). Recall $\tau_0(x, v, \bar{v}, \zeta)$ in (4.20) and let us consider $\tau \in [\tau_-(x, v, \bar{v}, \zeta), \tau_0(x, v, \bar{v}, \zeta)]$. Simply let us write $\mathfrak{S}_{vel}(\tau) = \mathfrak{S}_{vel}(\tau; x, v, \bar{v}, \zeta)$ here. Using (5.13), and (5.14),

$$\begin{aligned}
 &\frac{d}{d\tau} \left(\frac{\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))}{\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))} \right) \\
 &= \frac{1}{\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))} [\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \\
 &\quad + \frac{t_{\mathbf{b}}(x, \mathbf{v}(\tau))}{\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))} \{ (\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))) \mathbf{v}(\tau) \cdot \nabla^2 \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \mathbf{v}(\tau) \\
 &\quad - (\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))) \dot{\mathbf{v}}(\tau) \cdot \nabla^2 \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \mathbf{v}(\tau) \}] \\
 &\quad - \frac{\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))}{|\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))|^2} [\ddot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \\
 &\quad + \frac{t_{\mathbf{b}}(x, \mathbf{v}(\tau))}{\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))} \{ (\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))) \mathbf{v}(\tau) \cdot \nabla^2 \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \dot{\mathbf{v}}(\tau) \\
 &\quad - (\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))) \dot{\mathbf{v}}(\tau) \cdot \nabla^2 \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \dot{\mathbf{v}}(\tau) \}]. \tag{5.21}
 \end{aligned}$$

Since $\dot{\mathbf{v}}(\tau) = -\theta^2 \mathbf{v}(\tau)$ from (2.17), $-\frac{\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))}{|\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))|^2} \dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) > 0$ and then we use (5.21) to obtain

$$\begin{aligned}
 \frac{d}{d\tau} \mathfrak{S}_{vel}(\tau; x, v, \bar{v}, \zeta) &\geq 1 + t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \frac{1}{AB^2} (B\mathbf{v}(\tau) - A\dot{\mathbf{v}}(\tau)) \cdot \nabla^2 \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) (B\mathbf{v}(\tau) - A\dot{\mathbf{v}}(\tau)) \\
 &\geq 1 - t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \frac{\theta_{\Omega}}{AB^2} |(\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))) \mathbf{v}(\tau) - (\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))) \dot{\mathbf{v}}(\tau)|^2 \\
 &= 1 - \frac{\theta_{\Omega} t_{\mathbf{b}}(x, \mathbf{v}(\tau))}{(\mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))))} \\
 &\quad \times \left\{ \frac{\left| \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \cdot \left(I - \frac{\dot{\mathbf{v}}(\tau)}{|\dot{\mathbf{v}}(\tau)|} \otimes \frac{\dot{\mathbf{v}}(\tau)}{|\dot{\mathbf{v}}(\tau)|} \right) \mathbf{v}(\tau) \right|^2}{\left| \frac{\dot{\mathbf{v}}(\tau)}{|\dot{\mathbf{v}}(\tau)|} \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) \right|^2} + \left| \left(I - \frac{\dot{\mathbf{v}}(\tau)}{|\dot{\mathbf{v}}(\tau)|} \otimes \frac{\dot{\mathbf{v}}(\tau)}{|\dot{\mathbf{v}}(\tau)|} \right) \mathbf{v}(\tau) \right|^2 \right\} \\
 &= 1 + t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \frac{\theta_{\Omega} |\mathbf{v}(\tau)|^2}{\mathfrak{S}_{vel}(\tau; x, v, \bar{v}, \zeta) |\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))|} \left(1 + \mathfrak{S}_{vel}^2(\tau; x, v, \bar{v}, \zeta) \right), \tag{5.22}
 \end{aligned}$$

where we used (1.3), $\dot{\mathbf{v}}(\tau) \cdot \mathbf{v}(\tau) = 0$ by (2.17), and notation,

$$A := \mathbf{v}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) < 0, \quad B := \dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau))) < 0.$$

Sign of B comes from (4.20). Proof for $\tau \in [\tau_0, \tau_+)$ is same. $\square$

Proposition 5.6. Suppose the domain is given as in Definition 1.2 and (1.3).

(i) Assume $t_{\mathbf{b}}(x(\tau_*), v) < \infty$ and $\tau_* \in [\tau_-(x, \bar{x}, v), \tau_+(x, \bar{x}, v)]$. Then we have

$$\int_{\tau_-(x, \bar{x}, v)}^{\tau_*} \frac{d\tau}{\mathfrak{S}_{sp}(\tau; x, \bar{x}, v)} \lesssim C_{\Omega} \frac{\tau_* - \tau_-(x, \bar{x}, v)}{|v \cdot \nabla \xi(x_{\mathbf{b}}(x(\tau_*), v))|}. \quad (5.23)$$

(ii) Assume $t_{\mathbf{b}}(x, v(\tau)) < \infty$ and $\tau_* \in [\tau_-(x, v, \bar{v}, \zeta), \tau_+(x, v, \bar{v}, \zeta)]$. Then we have

$$\begin{aligned} \int_{\tau_-(x, v, \bar{v}, \zeta)}^{\tau_*} \frac{d\tau}{\mathfrak{S}_{vel}(\tau; x, v, \bar{v}, \zeta)} &\lesssim C_{\Omega} \frac{(\tau_* - \tau_-)}{|\mathbf{v}(\tau_*) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau_*)))|} \frac{1}{|\mathbf{v}(\tau_*)|} \\ &\times \left(1 + \min_{\tau} (|\mathbf{v}(\tau)| t_{\mathbf{b}}(x, \mathbf{v}(\tau)))\right). \end{aligned} \quad (5.24)$$

(Remind that $|\mathbf{v}(\tau)| = |v + \zeta|$ for all τ .)

Proof. Step 1. We first prove (5.23) when $\tau \in [\tau_-, \tau_0]$, where $\tau_-(x, \bar{x}, v)$ and $\tau_0 = \tau_0(x, \bar{x}, v)$ are defined in (2.16) and (4.16). From the ODE (5.16),

$$\begin{aligned} \frac{d}{d\tau} G_{sp}(\tau; x, \bar{x}, v) &\geq \frac{2\theta_{\Omega} |\dot{\mathbf{x}}|^2}{|\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))|} G_{sp}(\tau; x, \bar{x}, v), \text{ where } G_{sp}(\tau; x, \bar{x}, v) \\ &:= |v|^2 + \mathfrak{S}_{sp}^2(\tau; x, \bar{x}, v). \end{aligned} \quad (5.25)$$

From $\mathfrak{S}_{sp}(\tau_-) = 0$, we derive an upper bound of $G_{sp}(\tau; x, \bar{x}, v)$ by applying the Gronwall's inequality to (5.25). Then we derive that, in terms of $\mathfrak{S}_{sp}(\tau; x, \bar{x}, v)$,

$$\begin{aligned} \frac{1}{\mathfrak{S}_{sp}(\tau; x, \bar{x}, v)} &\leq \frac{1}{|v|} \left(e^{\int_{\tau_-}^{\tau} \frac{2\theta_{\Omega} |\dot{\mathbf{x}}|^2}{|\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(s), v))|} ds} - 1 \right)^{-\frac{1}{2}} \leq \frac{1}{|v|} \left[\int_{\tau_-}^{\tau} \frac{2\theta_{\Omega} |\dot{\mathbf{x}}|^2}{\max_{\tau_- \leq s \leq \tau} |\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(s), v))|} \right]^{-\frac{1}{2}} \\ &\lesssim_{\Omega} \frac{1}{|v| |\dot{\mathbf{x}}|} \sqrt{\frac{|\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_-(x, \bar{x}, v)), v))|}{(\tau - \tau_-(x, \bar{x}, v))}}. \end{aligned}$$

Here, we have used the fact $0 \leq |\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))| \leq C_{\Omega} |\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))|$ for $\tau \in [\tau_-, \tau_0]$ by (4.17). Hence, for $\tau_* \leq \tau_0$,

$$\begin{aligned} \int_{\tau_-}^{\tau_*} \frac{1}{\mathfrak{S}_{sp}(s; x, \bar{x}, v)} ds &\leq C_{\Omega} \frac{\sqrt{|\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))|}}{|v| |\dot{\mathbf{x}}|} \sqrt{\tau_* - \tau_-} \\ &\leq \frac{C_{\Omega} |\hat{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))|}{|v| \sqrt{|\dot{\mathbf{x}}|}} \sqrt{\tau_* - \tau_-(x, \bar{x}, v)}. \end{aligned} \quad (5.26)$$

Next we claim that

$$\frac{1}{|v|\sqrt{|\dot{\mathbf{x}}|}} \leq \frac{C_\Omega \|\nabla \xi\|_{L^\infty(\partial\Omega)} \sqrt{\tau_* - \tau_-(x, \bar{x}, v)}}{|v \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_*), v))|} \text{ for any } \tau_* \in [\tau_-(x, \bar{x}, v), \tau_0(x, \bar{x}, v)]. \quad (5.27)$$

Combining (5.27) and (5.26), we can prove (5.23).

The proof of (5.27) comes from (5.12):

$$\begin{aligned} \frac{d}{d\tau} (v \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)))^2 &= 2v \cdot \nabla^2 \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v)) [(v \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))) \dot{\mathbf{x}} - (\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))) v] \\ &\leq C_\Omega |v|^2 |\dot{\mathbf{x}} \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_-), v))| \leq C_\Omega \|\nabla \xi\|_{L^\infty(\partial\Omega)} |v|^2 |\dot{\mathbf{x}}|. \end{aligned}$$

We integrate the above inequality from τ_- to τ_* and use $v \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau), v))|_{\tau=\tau_-} = 0$ from (2.16). Then we can prove the claim (5.27).

Next, let us consider the case $\tau_* \in [\tau_0, \tau_+]$. Following the same argument to prove (5.23), using (4.18) instead of (4.17), we can derive that for $\tau_* \in [\tau_-, \tau_0]$

$$\int_{\tau_*}^{\tau_+} \frac{d\tau}{\mathfrak{S}_{sp}(\tau; x, \bar{x}, v)} \lesssim C_\Omega \frac{\tau_+ - \tau_*}{|v \cdot \nabla \xi(x_{\mathbf{b}}(\mathbf{x}(\tau_*), v))|}. \quad (5.28)$$

Now we split $\int_{\tau_-}^{\tau_*} = \int_{\tau_-}^{\tau_0} + \int_{\tau_0}^{\tau_+} - \int_{\tau_*}^{\tau_+} \leq |\int_{\tau_-}^{\tau_0}| + |\int_{\tau_0}^{\tau_+}| + |\int_{\tau_*}^{\tau_+}|$ and apply (5.23) with $\tau_* \in [\tau_-, \tau_0]$ and (5.28) to derive that

$$\int_{\tau_-}^{\tau_*} \frac{d\tau}{\mathfrak{S}_{sp}(\tau; x, \bar{x}, v)} \lesssim_\Omega \frac{(\tau_0 - \tau_-)}{|v \cdot \nabla \xi(x_{\mathbf{b}}(x(\tau_0), v))|} + \frac{(\tau_+ - \tau_0)}{|v \cdot \nabla \xi(x_{\mathbf{b}}(x(\tau_0), v))|} + \frac{(\tau_+ - \tau_*)}{|v \cdot \nabla \xi(x_{\mathbf{b}}(x(\tau_*), v))|}.$$

Note that from (4.19), we have $\frac{1}{|v \cdot \nabla \xi(x_{\mathbf{b}}(x(\tau_0), v))|} \leq \frac{C_\Omega}{|v \cdot \nabla \xi(x_{\mathbf{b}}(x(\tau_*), v))|}$. On the other hand, from (4.34), we easily obtain $\tau_+ - \tau_0 \lesssim \tau_0 - \tau_- \lesssim \tau_* - \tau_-$ and $\tau_+ - \tau_* \lesssim \tau_+ - \tau_0 \lesssim \tau_* - \tau_-$. Hence all three terms above (on the RHS) are bounded by

$$\frac{C(\tau_* - \tau_-)}{|v \cdot \nabla \xi(x_{\mathbf{b}}(x(\tau_*), v))|}.$$

Therefore we prove (5.23) for $\tau_* \in [\tau_0, \tau_+]$.

Step 2. We prove (5.24) first when $\tau \in [\tau_-, \tau_0]$, where $\tau_-(x, v, \bar{v}, \zeta)$ and $\tau_0 = \tau_0(x, v, \bar{v}, \zeta)$ are defined in (2.16) and (4.20). From the differential inequality (5.18),

$$\frac{d}{d\tau} G_{vel}(\tau; x, v, \bar{v}, \zeta) \geq t_{\mathbf{b}}(x, \mathbf{v}(\tau)) \frac{2\theta_\Omega |\mathbf{v}(\tau)|^2}{|\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau)))|} G_{vel}(\tau; x, v, \bar{v}, \zeta), \quad (5.29)$$

where $G_{vel}(\tau; x, v, \bar{v}, \zeta) := 1 + \tilde{\mathfrak{S}}_{vel}^2(\tau; x, v, \bar{v}, \zeta)$. We apply the Gronwall's inequality to (5.29) using $\tilde{\mathfrak{S}}_{vel}(\tau_-) = 0$. Then in terms of $\tilde{\mathfrak{S}}_{vel}(\tau; x, v, \bar{v}, \zeta)$, we have that for $\tau \in [\tau_-, \tau_0]$,

$$\begin{aligned} \frac{1}{\tilde{\mathfrak{S}}_{vel}(\tau; x, v, \bar{v}, \zeta)} &\leq \left(e^{\int_{\tau_-}^{\tau} \frac{2\theta_\Omega |\mathbf{v}(s)|^2 t_{\mathbf{b}}(x, \mathbf{v}(s))}{|\dot{\mathbf{v}}(s) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(s)))|} ds} - 1 \right)^{-\frac{1}{2}} \leq \left[\int_{\tau_-}^{\tau} \frac{2\theta_\Omega |\mathbf{v}(\tau)|^2 \min_{\tau_- \leq s \leq \tau} t_{\mathbf{b}}(x, \mathbf{v}(s))}{\max_{\tau_- \leq s \leq \tau} |\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(s)))|} \right]^{-\frac{1}{2}} \\ &\lesssim_\Omega \frac{\sqrt{|\dot{\mathbf{v}}(\tau_-) \cdot \nabla \xi(x_{\mathbf{b}}(x, \mathbf{v}(\tau_-)))|}}{|\mathbf{v}(\tau)| \sqrt{t_{\mathbf{b}}(x, \mathbf{v}(\tau))} \sqrt{\tau - \tau_-}}, \end{aligned}$$

where we have used (4.31) and (4.21). Hence, from definition (5.17), we have that for $\tau_* \leq \tau_0$

$$\int_{\tau_-}^{\tau_*} \frac{d\tau}{\mathfrak{E}_{vel}(\tau; x, v, \bar{v}, \zeta)} \lesssim_{\Omega} \sqrt{t_b(x, \mathbf{v}(\tau_-))} \frac{\sqrt{|\dot{\mathbf{v}}(\tau_-) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau_-)))|}}{|\dot{\mathbf{v}}(\tau_*)| |\mathbf{v}(\tau_*)|} \sqrt{\tau_* - \tau_-}. \quad (5.30)$$

Next we claim that

$$\begin{aligned} |\mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))| &\lesssim \sqrt{\theta |\widehat{\mathbf{v}}(\tau_-) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau_-)))| |\mathbf{v}(\tau)|^2} \\ &\times \sqrt{\left(|\widehat{\mathbf{v}}(\tau_-) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau_-)))| + t_b(x, \mathbf{v}(\tau_-)) |\mathbf{v}(\tau)| \right) \sqrt{\tau - \tau_-}} \\ &\lesssim_{\Omega} \sqrt{|\dot{\mathbf{v}}(\tau)| |\mathbf{v}(\tau)| (1 + t_b(x, \mathbf{v}(\tau_-)) |\mathbf{v}(\tau)|)} \sqrt{\tau - \tau_-}. \end{aligned} \quad (5.31)$$

From (5.30), (5.31), and (4.33), we can easily get (5.24) for $\tau_* \in [\tau_-, \tau_0]$. For the case $\tau \in [\tau_0, \tau_+]$, we follow the same argument of the last part in **Step 1**, using (4.23) and (4.35). This finishes the proof.

Now we only need to prove the claim (5.31). From (5.14), (4.21) and (4.32),

$$\begin{aligned} \frac{d}{d\tau} (\mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))) &= \dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau))) \\ &- t_b(x, \mathbf{v}(\tau)) \mathbf{v}(\tau) \cdot \nabla^2 \xi(x_b(x, \mathbf{v}(\tau))) \left(I - \frac{\mathbf{v}(\tau) \otimes \nabla \xi(x_b(x, \mathbf{v}(\tau)))}{\mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))} \right) \dot{\mathbf{v}}(\tau). \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{2} \frac{d}{d\tau} (\mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau))))^2 \\ &= (\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))) (\mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))) \\ &\quad - t_b(x, \mathbf{v}(\tau)) (\mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))) \mathbf{v}(\tau) \cdot \nabla^2 \xi(x_b(x, \mathbf{v}(\tau))) \dot{\mathbf{v}}(\tau) \\ &\quad + t_b(x, \mathbf{v}(\tau)) (\dot{\mathbf{v}}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))) \mathbf{v}(\tau) \cdot \nabla^2 \xi(x_b(x, \mathbf{v}(\tau))) \mathbf{v}(\tau) \\ &\leq C_{\Omega} \theta |\widehat{\mathbf{v}}(\tau_-) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau_-)))| |\mathbf{v}(\tau)|^2 \\ &\quad \times \left(|\widehat{\mathbf{v}}(\tau_-) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau_-)))| + t_b(x, \mathbf{v}(\tau_-)) |\mathbf{v}(\tau)| \right), \end{aligned}$$

where we perform the following estimate for the second term,

$$\begin{aligned} &|t_b(x, \mathbf{v}(\tau)) (\mathbf{v}(\tau) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau)))) \mathbf{v}(\tau) \cdot \nabla^2 \xi(x_b(x, \mathbf{v}(\tau))) \dot{\mathbf{v}}(\tau)| \\ &\leq C_{\Omega} \theta t_b(x, \mathbf{v}(\tau_-)) |\mathbf{v}(\tau)|^3 |\widehat{\mathbf{v}}(\tau_-) \cdot n_{\parallel}(x_b(x, \mathbf{v}(\tau)))| \\ &\leq C_{\Omega} \theta t_b(x, \mathbf{v}(\tau_-)) |\mathbf{v}(\tau)|^3 |\widehat{\mathbf{v}}(\tau_-) \cdot \widehat{n}_{\parallel}(x_b(x, \mathbf{v}(\tau_-)))| |n_{\parallel}(x_b(x, \mathbf{v}(\tau)))| \\ &\leq C_{\Omega} \theta t_b(x, \mathbf{v}(\tau_-)) |\mathbf{v}(\tau)|^3 |\widehat{\mathbf{v}}(\tau_-) \cdot \nabla \xi(x_b(x, \mathbf{v}(\tau_-)))|. \end{aligned}$$

Here, we have used the facts $|\dot{\mathbf{v}}(\tau)| = \theta |\mathbf{v}(\tau)|$ and $\theta = \cos^{-1}(\widehat{v + \zeta} \cdot \widehat{\bar{v} + \zeta})$ in (2.17) and (4.3) in Lemma 4.2. □

6 | $\mathfrak{H}_{sp,vel}^{2\beta}$ ESTIMATES

6.1 | Difference estimates

When (2.6) with $v + \zeta$ and (2.8) hold, we split

$$f(s, X(s; t, x, v + \zeta), V(s; t, x, v + \zeta)) - f(s, X(s; t, \bar{x}, \bar{v} + \zeta), V(s; t, \bar{x}, \bar{v} + \zeta)) \quad (6.1)$$

$$\leq f(s, X(s; t, x, v + \zeta), V(s; t, x, v + \zeta)) - f(s, X(s; t, \bar{x}, v + \zeta), V(s; t, \bar{x}, v + \zeta)) \quad (6.2)$$

$$+ f(s, X(s; t, \bar{x}, v + \zeta), V(s; t, \bar{x}, v + \zeta)) - f(s, X(s; t, \bar{x}, v + \zeta), V(s; t, \bar{x}, \bar{v} + \zeta)) \quad (6.3)$$

$$+ f(s, X(s; t, \bar{x}, v + \zeta), V(s; t, \bar{x}, v + \zeta)) - f(s, X(s; t, \bar{x}, \bar{v} + \zeta), V(s; t, \bar{x}, \bar{v} + \zeta)) \quad (6.4)$$

$$+ f(s, X(s; t, \bar{x}, \bar{v} + \zeta), V(s; t, \bar{x}, \bar{v} + \zeta)) - f(s, X(s; t, \bar{x}, \bar{v} + \zeta), V(s; t, \bar{x}, \bar{v} + \zeta)), \quad (6.5)$$

where $\tilde{x} = \tilde{x}(x, \bar{x}, v + \zeta)$ and $\tilde{v} = \tilde{v}(v, \bar{v}, \zeta)$ are defined in Definition (2.2). We estimate each (6.2)–(6.5).

Lemma 6.1. *Let $f : \bar{\Omega} \times \mathbb{R}^3 \rightarrow \mathbb{R}_+ \cup \{0\}$ be a function which satisfies specular reflection (1.4), where Ω is a domain as in Definition 1.2. Let $w(v) = e^{\vartheta|v|^2}$ for some $0 < \vartheta$.*

If (2.6) holds with $v + \zeta$, then (6.2) and (6.3) enjoy the following estimates.

$$\begin{aligned} \frac{(6.2)}{|x - \bar{x}|^\gamma} &\lesssim [1 + |v + \zeta|(t - s) + (|v + \zeta| + |v + \zeta|^2(t - s))\mathcal{T}_{sp}(x, \bar{x}, v + \zeta)]^\gamma \\ &\times \left[\frac{e^{\varpi(v+\zeta)^2s}}{\langle v + \zeta \rangle^{s_1}} \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} e^{-\varpi(v)^2s} \langle v \rangle^{s_1} \frac{|f(s, x, v) - f(s, \bar{x}, v)|}{|x - \bar{x}|^\gamma} + \frac{\|wf(s)\|_\infty}{w(v + \zeta)} \right] \\ &+ [1 + |v + \zeta| + (|v + \zeta| + |v + \zeta|^2)\mathcal{T}_{sp}(x, \bar{x}, v + \zeta)]^\gamma \\ &\times \left[\frac{e^{\varpi(v+\zeta)^2s}}{\langle v + \zeta \rangle^{s_2}} \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} e^{-\varpi(v)^2s} \langle v \rangle^{s_2} \frac{|f(s, x, v) - f(s, x, \bar{v})|}{|v - \bar{v}|^\gamma} + \frac{\|wf(s)\|_\infty}{w(v + \zeta)} \right], \quad (6.6) \end{aligned}$$

$$\frac{(6.3)}{|x - \bar{x}|^\gamma} \lesssim \frac{e^{\varpi(v+\zeta)^2s}}{\langle v + \zeta \rangle^{s_1}} \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \left(e^{-\varpi(v)^2s} \langle v \rangle^{s_1} \frac{|f(s, x, v) - f(s, \bar{x}, v)|}{|x - \bar{x}|^\gamma} \right) + \frac{\|wf(s)\|_\infty}{w(v + \zeta)}. \quad (6.7)$$

Similarly, if (2.8) holds, then (6.4) and (6.5) enjoy the following estimates.

$$\begin{aligned} \frac{(6.4)}{|v - \bar{v}|^\gamma} &\lesssim [(t - s) + |v + \zeta|(t - s)^2 + (|v + \zeta| + |v + \zeta|^2(t - s))\mathcal{T}_{vel}(\bar{x}, v, \bar{v}, \zeta; t, s)]^\gamma \\ &\times \left[\frac{e^{\varpi(v+\zeta)^2s}}{\langle v + \zeta \rangle^{s_1}} \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} e^{-\varpi(v)^2s} \langle v \rangle^{s_1} \frac{|f(s, x, v) - f(s, \bar{x}, v)|}{|x - \bar{x}|^\gamma} + \frac{\|wf(s)\|_\infty}{w(v + \zeta)} \right] \end{aligned}$$

$$+ [1 + |v + \zeta|(t - s) + |v + \zeta|^2 \mathcal{T}_{vel}(\bar{x}, v, \bar{v}, \zeta; t, s)]^\gamma \\ \times \left[\frac{e^{\varpi(v+\zeta)^2 s}}{\langle v + \zeta \rangle^{s_2}} \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} e^{-\varpi(v)^2 s} \langle v \rangle^{s_2} \frac{|f(s, x, v) - f(s, x, \bar{v})|}{|v - \bar{v}|^\gamma} + \frac{\|wf(s)\|_\infty}{w(v + \zeta)} \right], \quad (6.8)$$

$$\frac{(6.5)}{|v - \bar{v}|^\gamma} \lesssim \left[\frac{e^{\varpi(v+\zeta)^2 s}}{\langle v + \zeta \rangle^{s_1}} \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \left(e^{-\varpi(v)^2 s} \langle v \rangle^{s_1} \frac{|f(s, x, v) - f(s, \bar{x}, v)|}{|x - \bar{x}|^\gamma} \right) + \frac{\|wf(s)\|_\infty}{w(v + \zeta)} \right] (t - s)^\gamma \\ + \left[\frac{e^{\varpi(v+\zeta)^2 s}}{\langle v + \zeta \rangle^{s_2}} \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \left(e^{-\varpi(v)^2 s} \langle v \rangle^{s_2} \frac{|f(s, x, v) - f(s, x, \bar{v})|}{|v - \bar{v}|^\gamma} \right) + \frac{\|wf(s)\|_\infty}{w(v + \zeta)} \right]. \quad (6.9)$$

Here $\mathcal{T}_{sp}$ and $\mathcal{T}_{vel}$ are defined as

$$\mathcal{T}_{sp}(x, \bar{x}, v + \zeta) := \left[\int_0^1 \frac{1}{\mathfrak{S}_{sp}(\tau; x, \bar{x}, v + \zeta)} d\tau \mathbf{1}_{\{t_b(x(\tau), v + \zeta) < \infty, 0 \leq \tau \leq 1\}} \right. \\ + \int_{\tau_-}^1 \frac{1}{\mathfrak{S}_{sp}(\tau; x, \bar{x}, v + \zeta)} d\tau \mathbf{1}_{\{t_b(x(\tau), v + \zeta) < \infty, \tau_- \leq \tau \leq 1\}} \\ \left. + \int_0^{\tau_-} \frac{1}{\mathfrak{S}_{sp}(\tau; x, \bar{x}, v + \zeta)} d\tau \mathbf{1}_{\{t_b(x(\tau), v + \zeta) < \infty, 0 \leq \tau \leq \tau_-\}} \right] \mathbf{1}_{\zeta \in P^1_{(x, \bar{x}, v + \zeta)}}, \quad (6.10)$$

where

$$P^1_{(x, \bar{x}, v + \zeta)} := \{\zeta \in \mathbb{R}^3 : (2.6) \text{ and } (2.13) \text{ with } S_{(x, \bar{x}, v + \zeta)} \text{ hold}\},$$

and

$$\mathcal{T}_{vel}(\bar{x}, v, \bar{v}, \zeta; t, s) := \left[\int_0^1 \frac{1}{\mathfrak{S}_{vel}(\tau; \bar{x}, v, \bar{v}, \zeta)} d\tau \mathbf{1}_{\left\{ \begin{array}{l} t_b(\bar{x}, v(\tau)) < \infty, 0 \leq \tau \leq 1 \\ \min_{0 \leq \tau \leq 1} t_b(\bar{x}, v(\tau)) \leq t - s \end{array} \right\}} \right. \\ + \int_{\tau_-}^1 \frac{1}{\mathfrak{S}_{vel}(\tau; \bar{x}, v, \bar{v}, \zeta)} d\tau \mathbf{1}_{\left\{ \begin{array}{l} t_b(\bar{x}, v(\tau)) < \infty, \tau_- \leq \tau \leq 1 \\ \min_{\tau_- \leq \tau \leq 1} t_b(\bar{x}, v(\tau)) \leq t - s \end{array} \right\}} \\ \left. + \int_0^{\tau_-} \frac{1}{\mathfrak{S}_{vel}(\tau; \bar{x}, v, \bar{v}, \zeta)} d\tau \mathbf{1}_{\left\{ \begin{array}{l} t_b(\bar{x}, v(\tau)) < \infty, 0 \leq \tau \leq \tau_- \\ \min_{0 \leq \tau \leq \tau_-} t_b(\bar{x}, v(\tau)) \leq t - s \end{array} \right\}} \right] \mathbf{1}_{\zeta \in P^2_{(\bar{x}, v, \bar{v}, \zeta)}}, \quad (6.11)$$

where

$$P^2_{(\bar{x}, v, \bar{v}, \zeta)} := \{\zeta \in \mathbb{R}^3 : (2.8) \text{ and } (2.14) \text{ with } S_{(\bar{x}, v, \bar{v}, \zeta)} \text{ hold}\}.$$

Here, we used notation, $f_a^b := \frac{1}{b-a} \int_a^b$.

Remark 6.2. In the definition $\mathcal{T}_{vel}$ (6.11), we put extra condition of the following type

$$\min_{0 \leq \tau \leq 1} t_b(x, \mathbf{v}(\tau)) \leq t - s,$$

which is missing in the definition of $\mathcal{T}_{sp}$ in (6.10). In fact, we can put similar condition in the definition of $\mathcal{T}_{sp}$ also. However, above condition is not important for $\mathcal{T}_{sp}$ case as we see in (5.23): unlike to (5.24), it does not contain any $t_b(x, \mathbf{v}(\tau))$ -related terms.

Proof. In this proof, we drop ζ in (6.2)–(6.5) for notational convenience.

Step 0 (Trivial dynamics) Assume (2.13) with $S_{(x, \bar{x}, v)}$ ($S_{(x, \bar{x}, v + \zeta)}$ in fact) and (2.14) with $S_{(\bar{x}, v, \bar{v}, 0)}$ ($S_{(\bar{x}, v, \bar{v}, \zeta)}$ in fact) do not hold. (The definition of $S_{(x, \bar{x}, v)}$ and $S_{(\bar{x}, v, \bar{v}, 0)}$ are given in (2.10), (2.11).) Then, all the backward in time trajectories do not hit $\partial\Omega$, hence (6.6)–(6.9) hold obviously, using the following trivial trajectory estimates,

$$\begin{aligned} \frac{|X(s; t, x, v) - X(s; t, \bar{x}, v)|}{|x - \bar{x}|} &= 1, & \frac{|X(s; t, \bar{x}, v) - X(s; t, \bar{x}, v)|}{|\bar{x} - \bar{x}|} &= 1, \\ \frac{|V(s; t, x, v) - V(s; t, \bar{x}, v)|}{|x - \bar{x}|} &= 0, & \frac{|V(s; t, \bar{x}, v) - V(s; t, \bar{x}, v)|}{|\bar{x} - \bar{x}|} &= 0, \\ \frac{|X(s; t, \bar{x}, v) - X(s; t, \bar{x}, \bar{v})|}{|v - \bar{v}|} &= (t - s), & \frac{|X(s; t, \bar{x}, \bar{v}) - X(s; t, \bar{x}, \bar{v})|}{|\bar{v} - \bar{v}|} &= (t - s), \\ \frac{|V(s; t, \bar{x}, v) - V(s; t, \bar{x}, \bar{v})|}{|v - \bar{v}|} &= 1, & \frac{|V(s; t, \bar{x}, \bar{v}) - V(s; t, \bar{x}, \bar{v})|}{|\bar{v} - \bar{v}|} &= 1. \end{aligned}$$

We omit the details.

Now let us consider nontrivial cases. In the following **Step 1** and **Step 2**, we assume (2.13) with $S_{(x, \bar{x}, v)}$ ($S_{(x, \bar{x}, v + \zeta)}$ in fact) and (2.14) with $S_{(\bar{x}, v, \bar{v}, 0)}$ ($S_{(\bar{x}, v, \bar{v}, \zeta)}$ in fact), in addition to (2.6) and (2.8).

Step 1 (Nonsingular parts) Let us treat (6.3) first. Since $\bar{x} - \bar{x}$ is parallel to v ($v + \zeta$ in fact), we do not see any specular singularity in (6.3).

$$\begin{aligned} (6.3) &\leq \frac{|f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v)) - f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v))|}{|X(s; t, \bar{x}, v) - X(s; t, \bar{x}, v)|^\gamma} \\ &\quad \times |X(s; t, \bar{x}, v) - X(s; t, \bar{x}, v)|^\gamma (\mathbf{1}_{s > \max\{t^1(\bar{x}, v), t^1(\bar{x}, v)\}} + \mathbf{1}_{s \leq \min\{t^1(\bar{x}, v), t^1(\bar{x}, v)\}}) \\ &\quad + |f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v)) - f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v))| \\ &\quad \times \mathbf{1}_{\min\{t^1(\bar{x}, v), t^1(\bar{x}, v)\} < s \leq \max\{t^1(\bar{x}, v), t^1(\bar{x}, v)\}} \\ &\leq \frac{|f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v)) - f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v))|}{|X(s; t, \bar{x}, v) - X(s; t, \bar{x}, v)|^\gamma} |\bar{x} - \bar{x}|^\gamma \\ &\quad \times (\mathbf{1}_{s > \max\{t^1(\bar{x}, v), t^1(\bar{x}, v)\}} + \mathbf{1}_{s \leq \min\{t^1(\bar{x}, v), t^1(\bar{x}, v)\}}) \\ &\quad + (|f(s, X(s; t, \bar{x}, v), v) - f(s, x_b(\bar{x}, v), v)| \\ &\quad + |f(s, x_b(\bar{x}, v), R_{x_b(\bar{x}, v)} v) - f(s, X(s; t, \bar{x}, v), R_{x_b(\bar{x}, v)} v)|) \times \mathbf{1}_{t^1(\bar{x}, v) < s \leq t^1(\bar{x}, v)} \\ &\quad + (|f(s, X(s; t, \bar{x}, v), R_{x_b(\bar{x}, v)} v) - f(s, x_b(\bar{x}, v), R_{x_b(\bar{x}, v)} v)| \\ &\quad + |f(s, x_b(\bar{x}, v), v) - f(s, X(s; t, \bar{x}, v), v)|) \times \mathbf{1}_{t^1(\bar{x}, v) < s \leq t^1(\bar{x}, v)} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{|f(s, X(s; t, \tilde{x}, v), V(s; t, \tilde{x}, v)) - f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v))|}{|X(s; t, \tilde{x}, v) - X(s; t, \bar{x}, v)|^\gamma} |\tilde{x} - \bar{x}|^\gamma \\
&\quad \times (\mathbf{1}_{s > \max\{t^1(\tilde{x}, v), t^1(\bar{x}, v)\}} + \mathbf{1}_{s \leq \min\{t^1(\tilde{x}, v), t^1(\bar{x}, v)\}}) \\
&\quad + \left(\frac{|f(s, X(s; t, \tilde{x}, v), v) - f(s, x_b(\tilde{x}, v), v)|}{|X(s; t, \tilde{x}, v) - x_b(\tilde{x}, v)|^\gamma} \right. \\
&\quad \left. + \frac{|f(s, x_b(\tilde{x}, v), R_{x_b(\tilde{x}, v)} v) - f(s, X(s; t, \bar{x}, v), R_{x_b(\bar{x}, v)} v)|}{|x_b(\tilde{x}, v) - X(s; t, \bar{x}, v)|^\gamma} \right) |\tilde{x} - \bar{x}|^\gamma \mathbf{1}_{t^1(\tilde{x}, v) < s \leq t^1(\bar{x}, v)} \\
&\quad + \left(\frac{|f(s, X(s; t, \tilde{x}, v), R_{x_b(\tilde{x}, v)} v) - f(s, x_b(\tilde{x}, v), R_{x_b(\tilde{x}, v)} v)|}{|X(s; t, \tilde{x}, v) - x_b(\tilde{x}, v)|^\gamma} \right. \\
&\quad \left. + \frac{|f(s, x_b(\tilde{x}, v), v) - f(s, X(s; t, \bar{x}, v), v)|}{|x_b(\tilde{x}, v) - X(s; t, \bar{x}, v)|^\gamma} \right) |\tilde{x} - \bar{x}|^\gamma \mathbf{1}_{t^1(\tilde{x}, v) < s \leq t^1(\bar{x}, v)}, \tag{6.12}
\end{aligned}$$

where we used the facts that $(\tilde{x} - \bar{x}) \parallel v$ ($v + \zeta$ in fact) and $|\tilde{x} - \bar{x}| \leq |x - \bar{x}|$ by (2.7). When a denominator is larger than 1, the second term of the RHS in (6.7) controls LHS. Therefore, (6.12) gives (6.7).

Let us treat (6.5). Since $\bar{v}$ and $\tilde{v}$ are parallel to each other ($\bar{v} + \zeta$ and $\tilde{v} + \zeta$ in fact), we do not see any specular singularity in (6.5), neither.

$$\begin{aligned}
(6.5) &\leq \left(\frac{|f(s, X(s; t, \bar{x}, \bar{v}), V(s; t, \bar{x}, \bar{v})) - f(s, X(s; t, \tilde{x}, \bar{v}), V(s; t, \tilde{x}, \bar{v}))|}{|V(s; t, \bar{x}, \bar{v}) - V(s; t, \tilde{x}, \bar{v})|^\gamma} |\bar{v} - \tilde{v}|^\gamma \right. \\
&\quad \left. + \frac{|f(s, X(s; t, \bar{x}, \bar{v}), V(s; t, \bar{x}, \bar{v})) - f(s, X(s; t, \tilde{x}, \bar{v}), V(s; t, \tilde{x}, \bar{v}))|}{|X(s; t, \bar{x}, \bar{v}) - X(s; t, \tilde{x}, \bar{v})|^\gamma} |(\bar{v} - \tilde{v})(t - s)|^\gamma \right) \\
&\quad \times (\mathbf{1}_{s > \max\{t^1(\bar{x}, \bar{v}), t^1(\tilde{x}, \bar{v})\}} + \mathbf{1}_{s \leq \min\{t^1(\bar{x}, \bar{v}), t^1(\tilde{x}, \bar{v})\}}) \\
&\quad + \mathbf{1}_{t^1(\bar{x}, \bar{v}) < s \leq t^1(\tilde{x}, \bar{v})} \left(\frac{|f(s, X(s; t, \bar{x}, \bar{v}), \bar{v}) - f(s, x_b(\bar{x}, \bar{v}), \bar{v})|}{|X(s; t, \bar{x}, \bar{v}) - x_b(\bar{x}, \bar{v})|^\gamma} |(\bar{v} - \tilde{v})(t - s)|^\gamma \right. \\
&\quad \left. + \frac{|f(s, x_b(\bar{x}, \bar{v}), R_{x_b(\bar{x}, \bar{v})} \bar{v}) - f(s, x_b(\tilde{x}, \bar{v}), R_{x_b(\tilde{x}, \bar{v})} \bar{v})|}{|R_{x_b(\bar{x}, \bar{v})} \bar{v} - R_{x_b(\tilde{x}, \bar{v})} \bar{v}|^\gamma} |\bar{v} - \tilde{v}|^\gamma \right. \\
&\quad \left. + \frac{|f(s, x_b(\bar{x}, \bar{v}), R_{x_b(\bar{x}, \bar{v})} \bar{v}) - f(s, X(s; t, \tilde{x}, \bar{v}), R_{x_b(\tilde{x}, \bar{v})} \bar{v})|}{|x_b(\bar{x}, \bar{v}) - X(s; t, \tilde{x}, \bar{v})|^\gamma} |(\bar{v} - \tilde{v})(t - s)|^\gamma \right) \\
&\quad + \left(\frac{|f(s, X(s; t, \bar{x}, \bar{v}), R_{x_b(\bar{x}, \bar{v})} \bar{v}) - f(s, x_b(\bar{x}, \bar{v}), R_{x_b(\bar{x}, \bar{v})} \bar{v})|}{|X(s; t, \bar{x}, \bar{v}) - x_b(\bar{x}, \bar{v})|^\gamma} |(\bar{v} - \tilde{v})(t - s)|^\gamma \right. \\
&\quad \left. + \frac{|f(s, x_b(\bar{x}, \bar{v}), \bar{v}) - f(s, x_b(\tilde{x}, \bar{v}), \bar{v})|}{|\bar{v} - \tilde{v}|^\gamma} |\bar{v} - \tilde{v}|^\gamma \right. \\
&\quad \left. + \frac{|f(s, x_b(\bar{x}, \bar{v}), \bar{v}) - f(s, X(s; t, \tilde{x}, \bar{v}), \bar{v})|}{|x_b(\bar{x}, \bar{v}) - X(s; t, \tilde{x}, \bar{v})|^\gamma} |(\bar{v} - \tilde{v})(t - s)|^\gamma \right) \mathbf{1}_{t^1(\bar{x}, \bar{v}) < s \leq t^1(\tilde{x}, \bar{v})}, \tag{6.13}
\end{aligned}$$

where we used specular boundary condition (1.4), $|\bar{v} - \tilde{v}| \lesssim |v - \bar{v}|$ by (2.9). So we get (6.9) from (6.13).

Step 2 (Singular parts) Now, we treat main contributions : (6.6) and (6.8). We will see specular singularity in these estimates. Using specular condition (1.4), we split (6.2) into

$$\begin{aligned}
 (6.2) \leq & \left(\frac{|f(s, X(s; t, x, v), V(s; t, x, v)) - f(s, X(s; t, \tilde{x}, v), V(s; t, x, v))|}{|X(s; t, x, v) - X(s; t, \tilde{x}, v)|^\gamma} \right. \\
 & \times |X(s; t, x, v) - X(s; t, \tilde{x}, v)|^\gamma \mathbf{1}_{s \leq \min\{t^1(x, v), t^1(\tilde{x}, v)\}} \\
 & + \left(\frac{|f(s, X(s; t, \tilde{x}, v), V(s; t, x, v)) - f(s, X(s; t, \tilde{x}, v), V(s; t, \tilde{x}, v))|}{|V(s; t, x, v) - V(s; t, \tilde{x}, v)|^\gamma} \right. \\
 & \times |V(s; t, x, v) - V(s; t, \tilde{x}, v)|^\gamma \mathbf{1}_{s \leq \min\{t^1(x, v), t^1(\tilde{x}, v)\}} \\
 & + \frac{|f(s, X(s; t, x, v), V(s; t, x, v)) - f(s, X(s; t, \tilde{x}, v), V(s; t, \tilde{x}, v))|}{|X(s; t, x, v) - X(s; t, \tilde{x}, v)|^\gamma} \\
 & \left. |x - \tilde{x}|^\gamma \mathbf{1}_{s > \max\{t^1(x, v), t^1(\tilde{x}, v)\}} + A_1 + A_2 + A_3 + A_4 + B_1 + B_2 + B_3 + B_4, \quad (6.14)
 \end{aligned}$$

where

$$\begin{aligned}
 A_1 &:= \left(\frac{|f(s, X(s; t, x, v), R_{x_b(x, v)}v) - f(s, x_b(x, v), R_{x_b(x, v)}v)|}{|X(s; t, x, v) - x_b(x, v)|^\gamma} |v(t^1(x, v) - s)|^\gamma \right. \\
 &+ \frac{|f(s, x_b(x, v), v) - f(s, X(s; t, \tilde{x}, v), v)|}{|x_b(x, v) - X(s; t, \tilde{x}, v)|^\gamma} (|x - \tilde{x}| + |v|(t^1(x, v) - s))^\gamma \Big) \\
 &\times \mathbf{1}_{t^1(\tilde{x}, v) < s \leq t^1(x, v)} \mathbf{1}_{t^1(\tilde{x}, v) > -\infty}, \\
 A_2 &:= \left(\frac{|f(s, X(s; t, x, v), V(s; t, x, v)) - f(s, X(s; t, \mathbf{x}(\tau_-), v), V(s; t, x, v))|}{|X(s; t, x, v) - X(s; t, \mathbf{x}(\tau_-), v)|^\gamma} \right. \\
 &\times |X(s; t, x, v) - X(s; t, \mathbf{x}(\tau_-), v)|^\gamma \\
 &+ \frac{|f(s, X(s; t, \mathbf{x}(\tau_-), v), V(s; t, x, v)) - f(s, X(s; t, \mathbf{x}(\tau_-), v), V(s; t, \mathbf{x}(\tau_-), v))|}{|V(s; t, x, v) - V(s; t, \mathbf{x}(\tau_-), v)|^\gamma} \\
 &\times |V(s; t, x, v) - V(s; t, \mathbf{x}(\tau_-), v)|^\gamma \mathbf{1}_{s \leq \min\{t^1(\mathbf{x}(\tau_-), v), t^1(x, v)\}} \mathbf{1}_{t^1(\tilde{x}, v) = -\infty}, \\
 A_3 &:= \left(\frac{|f(s, X(s; t, x, v), R_{x_b(x, v)}v) - f(s, x_b(x, v), R_{x_b(x, v)}v)|}{|X(s; t, x, v) - x_b(x, v)|^\gamma} |v(t^1(x, v) - s)|^\gamma \right. \\
 &+ \frac{|f(s, x_b(x, v), v) - f(s, X(s; t, \mathbf{x}(\tau_-), v), v)|}{|x_b(x, v) - X(s; t, \mathbf{x}(\tau_-), v)|^\gamma} \\
 &\times (|x - \mathbf{x}(\tau_-)| + |v|(t^1(x, v) - s))^\gamma \Big) \mathbf{1}_{t^1(\mathbf{x}(\tau_-), v) < s \leq t^1(x, v)} \mathbf{1}_{t^1(\tilde{x}, v) = -\infty}, \\
 A_4 &:= \frac{|f(s, X(s; t, \mathbf{x}(\tau_-), v), v) - f(s, X(s; t, \tilde{x}, v), v)|}{|X(s; t, \mathbf{x}(\tau_-), v) - X(s; t, \tilde{x}, v)|^\gamma} \\
 &\times |\mathbf{x}(\tau_-) - \tilde{x}|^\gamma \times \mathbf{1}_{t^1(\tilde{x}, v) < s \leq t^1(x, v)} \mathbf{1}_{t^1(\tilde{x}, v) = -\infty},
 \end{aligned}$$

and

$$B_{1,2,3,4} := \text{interchanging } x \text{ and } \tilde{x} \text{ in } A_{1,2,3,4}, \text{ respectively.}$$

Applying (5.1) and (5.2) in Lemma 5.1, and (5.9) in Lemma 5.3, to (6.14), we obtain (6.6).

For (6.4),

$$\begin{aligned}
 (6.4) \lesssim & \frac{|f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v)) - f(s, X(s; t, \bar{x}, \bar{v}), V(s; t, \bar{x}, v))|}{|X(s; t, \bar{x}, v) - X(s; t, \bar{x}, \bar{v})|^\gamma} \\
 & \times |X(s; t, \bar{x}, v) - X(s; t, \bar{x}, \bar{v})|^\gamma \mathbf{1}_{s \leq \min\{t^1(\bar{x}, v), t^1(\bar{x}, \bar{v})\}} \\
 & + \frac{|f(s, X(s; t, \bar{x}, \bar{v}), V(s; t, \bar{x}, v)) - f(s, X(s; t, \bar{x}, \bar{v}), V(s; t, \bar{x}, \bar{v}))|}{|V(s; t, \bar{x}, v) - V(s; t, \bar{x}, \bar{v})|^\gamma} \\
 & \times |V(s; t, \bar{x}, v) - V(s; t, \bar{x}, \bar{v})|^\gamma \mathbf{1}_{s \leq \min\{t^1(\bar{x}, v), t^1(\bar{x}, \bar{v})\}} \\
 & + \frac{|f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v)) - f(s, X(s; t, \bar{x}, \bar{v}), V(s; t, \bar{x}, v))|}{|X(s; t, \bar{x}, v) - X(s; t, \bar{x}, \bar{v})|^\gamma} \\
 & \times |(v - \bar{v})(t - s)|^\gamma \mathbf{1}_{s > \max\{t^1(\bar{x}, v), t^1(\bar{x}, \bar{v})\}} \\
 & + \frac{|f(s, X(s; t, \bar{x}, \bar{v}), V(s; t, \bar{x}, v)) - f(s, X(s; t, \bar{x}, \bar{v}), V(s; t, \bar{x}, \bar{v}))|}{|V(s; t, \bar{x}, v) - V(s; t, \bar{x}, \bar{v})|^\gamma} \\
 & \times |v - \bar{v}|^\gamma \mathbf{1}_{s > \max\{t^1(\bar{x}, v), t^1(\bar{x}, \bar{v})\}} + C_1 + C_2 + C_3 + C_4 + D_1 + D_2 + D_3 + D_4, \quad (6.15)
 \end{aligned}$$

where

$$\begin{aligned}
 C_1 := & \left(\frac{|f(s, X(s; t, \bar{x}, v), R_{x_b(\bar{x}, v)} v) - f(s, x_b(\bar{x}, v), R_{x_b(\bar{x}, v)} v)|}{|X(s; t, \bar{x}, v) - x_b(\bar{x}, v)|^\gamma} (|v|(t^1(\bar{x}, v) - s))^\gamma \right. \\
 & + \frac{|f(s, x_b(\bar{x}, v), R_{x_b(\bar{x}, v)} v) - f(s, x_b(\bar{x}, v), R_{x_b(\bar{x}, v)} \bar{v})|}{|R_{x_b(\bar{x}, v)} v - R_{x_b(\bar{x}, v)} \bar{v}|^\gamma} |v - \bar{v}|^\gamma \\
 & + \frac{|f(s, x_b(\bar{x}, v), \bar{v}) - f(s, X(s; t, \bar{x}, \bar{v}), \bar{v})|}{|x_b(\bar{x}, v) - X(s; t, \bar{x}, \bar{v})|^\gamma} (|v - \bar{v}| t_b(\bar{x}, v) + |v|(t^1(\bar{x}, v) - s))^\gamma \Big) \\
 & \times \mathbf{1}_{t^1(\bar{x}, \bar{v}) < s \leq t^1(\bar{x}, v)} \mathbf{1}_{t^1(\bar{x}, \bar{v}) > -\infty},
 \end{aligned}$$

$$\begin{aligned}
 C_2 := & \left(\frac{|f(s, X(s; t, \bar{x}, v), V(s; t, \bar{x}, v)) - f(s, X(s; t, \bar{x}, \mathbf{v}(\tau_-)), V(s; t, \bar{x}, v))|}{|X(s; t, \bar{x}, v) - X(s; t, \bar{x}, \mathbf{v}(\tau_-))|^\gamma} \right. \\
 & \times |X(s; t, \bar{x}, v) - X(s; t, \bar{x}, \mathbf{v}(\tau_-))|^\gamma \\
 & + \frac{|f(s, X(s; t, \bar{x}, \mathbf{v}(\tau_-)), V(s; t, \bar{x}, v)) - f(s, X(s; t, \bar{x}, \mathbf{v}(\tau_-)), V(s; t, \bar{x}, \mathbf{v}(\tau_-)))|}{|V(s; t, \bar{x}, v) - V(s; t, \bar{x}, \mathbf{v}(\tau_-))|^\gamma} \\
 & \times |V(s; t, \bar{x}, v) - V(s; t, \bar{x}, \mathbf{v}(\tau_-))|^\gamma \mathbf{1}_{s \leq \min\{t^1(\bar{x}, \mathbf{v}(\tau_-)), t^1(\bar{x}, v)\}} \mathbf{1}_{t^1(\bar{x}, \bar{v}) = -\infty},
 \end{aligned}$$

$$\begin{aligned}
C_3 := & \left(\frac{|f(s, X(s; t, \bar{x}, v), R_{x_b(\bar{x}, v)}v) - f(s, x_b(\bar{x}, v), R_{x_b(\bar{x}, v)}v)|}{|X(s; t, \bar{x}, v) - x_b(\bar{x}, v)|^\gamma} (|v|(t^1(\bar{x}, v) - s))^\gamma \right. \\
& + \frac{|f(s, x_b(\bar{x}, v), R_{x_b(\bar{x}, v)}v) - f(s, x_b(\bar{x}, v), R_{x_b(\bar{x}, v)}\mathbf{v}(\tau_-))|}{|R_{x_b(\bar{x}, v)}v - R_{x_b(\bar{x}, v)}\mathbf{v}(\tau_-)|^\gamma} |v - \mathbf{v}(\tau_-)|^\gamma \\
& + \frac{|f(s, x_b(\bar{x}, v), \mathbf{v}(\tau_-)) - f(s, X(s; t, \bar{x}, \mathbf{v}(\tau_-)), \mathbf{v}(\tau_-))|}{|x_b(\bar{x}, v) - X(s; t, \bar{x}, \mathbf{v}(\tau_-))|^\gamma} \\
& \times (|v - \mathbf{v}(\tau_-)|t_b(\bar{x}, v) + |v|(t^1(\bar{x}, v) - s))^\gamma \mathbf{1}_{t^1(\bar{x}, \mathbf{v}(\tau_-)) < s \leq t^1(\bar{x}, v)} \mathbf{1}_{t^1(\bar{x}, \tilde{v}) = -\infty}, \\
C_4 := & \left(\frac{|f(s, X(s; t, \bar{x}, \mathbf{v}(\tau_-)), \mathbf{v}(\tau_-)) - f(s, X(s; t, \bar{x}, \tilde{v}), \mathbf{v}(\tau_-))|}{|X(s; t, \bar{x}, \mathbf{v}(\tau_-)) - X(s; t, \bar{x}, \tilde{v})|^\gamma} |(\mathbf{v}(\tau_-) - \tilde{v})(t - s)|^\gamma \right. \\
& + \frac{|f(s, X(s; t, \bar{x}, \tilde{v}), \mathbf{v}(\tau_-)) - f(s, X(s; t, \bar{x}, \tilde{v}), \tilde{v})|}{|\mathbf{v}(\tau_-) - \tilde{v}|^\gamma} |\mathbf{v}(\tau_-) - \tilde{v}|^\gamma \Big) \\
& \times \mathbf{1}_{t^1(\bar{x}, \tilde{v}) < s \leq t^1(\bar{x}, v)} \mathbf{1}_{t^1(\bar{x}, \tilde{v}) = -\infty},
\end{aligned}$$

and

$D_{1,2,3,4} :=$ interchanging v and $\tilde{v}$ in $C_{1,2,3,4}$, respectively.

Applying (5.3) and (5.4) in Lemma 5.1, and (5.10) in Lemma 5.3, to (6.15), we obtain (6.8). $\square$

Lemma 6.3. In (6.2)–(6.5), let us replace f into $\nu(f)$ or $\Gamma_{\text{gain}}(f, f)$. Corresponding Lemma 6.1 (with $\nu(f)$ or $\Gamma_{\text{gain}}(f, f)$, instead of f) satisfies the same estimates as (6.6)–(6.9), except that we replace

$$\frac{1}{w(v + \zeta)} \|wf(s)\|_\infty$$

on the RHS of each (6.6)–(6.9), into

$$\frac{\langle v + \zeta \rangle}{w(v + \zeta)} \|wf(s)\|_\infty^2, \quad \text{for } \Gamma_{\text{gain}}(f, f) \text{ case}$$

and

$$\frac{\langle v + \zeta \rangle}{w(v + \zeta)} \|wf(s)\|_\infty, \quad \text{for } \nu(f) \text{ case},$$

respectively.

Proof. It is obvious because both $\nu(f)$ and $\Gamma_{\text{gain}}(f, f)$ also satisfy (1.4) by Lemma 3.5. We omit the proof. $\square$

6.2 | Integrability for $\beta < \frac{1}{2}$

Lemma 6.4. Consider Ω as in Definition 1.2. We $x \in \Omega$ and choose a unique $\hat{z}$ whose backward trajectory hit $\partial\Omega$ vertically, that is, $\hat{z} \cdot \hat{n}(x_{\mathbf{b}}(x, \hat{z})) = -1$. For any plane S which includes x and $x + \hat{z}$, curvature at any point $y \in \partial\Omega \cap S$ is uniformly nonzero of which lower/upper bounds depend only on Ω .

Proof. Imagine a plane which is perpendicular to $\hat{z}$. Then using an angle on the plane (as cylindrical coordinate or spherical coordinate), we can parametrize all possible planes S as S_φ with $[0, 2\pi]$ where $S_0 = S_{2\pi}$. Each cross section $\partial\Omega \cap S_\varphi$ is uniformly convex, so there are finite maximum and minimum of curvature on the curve depending on Ω . Now using compactness of $\varphi \in [0, 2\pi]$, we finish the proof. $\square$

Lemma 6.5 (Integrability).

(i) When $\beta < \frac{1}{2}$

$$\int_{\{\zeta: x_{\mathbf{b}}(x, v+\zeta) \in \partial\Omega\}} \frac{e^{-c|\zeta|^2}}{|\zeta|} \frac{\langle v+\zeta \rangle^r}{|(v+\zeta) \cdot \nabla \xi(x_{\mathbf{b}}(x, v+\zeta))|^{2\beta}} d\zeta \leq C_\beta \langle v \rangle^{r+1-2\beta}. \quad (6.16)$$

(ii) When $0 < \beta \leq \frac{1}{4}$,

$$\int_{\{\zeta: x_{\mathbf{b}}(x, v+\zeta) \in \partial\Omega\}} \frac{e^{-c|\zeta|^2}}{|\zeta|} \frac{\langle v+\zeta \rangle^r}{|(v+\zeta) \cdot \nabla \xi(x_{\mathbf{b}}(x, v+\zeta))|^{2\beta}} \frac{1}{|v+\zeta|^{2\beta}} d\zeta \leq C_\beta \langle v \rangle^{r+1-4\beta}. \quad (6.17)$$

(iii) When $\frac{1}{4} < \beta < \frac{1}{2}$,

$$\int_{\{\zeta: x_{\mathbf{b}}(x, v+\zeta) \in \partial\Omega\}} \frac{e^{-c|\zeta|^2}}{|\zeta|} \frac{\langle v+\zeta \rangle^r}{|(v+\zeta) \cdot \nabla \xi(x_{\mathbf{b}}(x, v+\zeta))|^{2\beta}} \frac{1}{|v+\zeta|^{2\beta}} d\zeta \leq C_\beta \langle v \rangle^r. \quad (6.18)$$

Proof. Let us prove (6.16) first. We consider a fixed point x and $\zeta \in \mathbb{R}^3$ such that $x_{\mathbf{b}}(x, \zeta) \in \partial\Omega$ is well defined. There exist a unique $\hat{z}$ whose backward trajectory hit $\partial\Omega$ vertically, that is, $\hat{z} \cdot \hat{n}(x_{\mathbf{b}}(x, \hat{z})) = -1$. We consider spherical coordinate of $\zeta \in \mathbb{R}^3$ whose $\hat{z}$ is z-axis. Then for each $\varphi \in [0, 2\pi]$, there exists $\theta_g = \theta_g(\varphi) \in [0, \frac{\pi}{2}]$ such that $\zeta_{g,\varphi} \cdot \nabla \xi(x_{\mathbf{b}}(x, \zeta_{g,\varphi})) = 0$, where

$$\zeta_{g,\varphi} := |\zeta|(\sin \theta_g \cos \varphi, \sin \theta_g \sin \varphi, \cos \theta_g),$$

whose spherical component is $(|\zeta|, \theta_g(\varphi), \varphi)$ so that its trajectory grazes on $\partial\Omega$.

Now, let us use S_φ to denote the φ -plane in above coordinate. Since the cross section $\partial\Omega \cap S_\varphi$ with a fixed φ is a two dimensional uniformly convex curve on the cross section,

$$|\zeta \cdot \widehat{n}_\parallel(x_{\mathbf{b}}(x, \zeta_{g,\varphi}))| \leq |\zeta \cdot \widehat{n}_\parallel(x_{\mathbf{b}}(x, \zeta))| \quad (6.19)$$

is obvious, where $\zeta \in \partial\Omega \cap S_\varphi$ and $n_\parallel$ is projection of $\frac{\nabla \xi(x_{\mathbf{b}}(x, \zeta))}{|\nabla \xi(x_{\mathbf{b}}(x, \zeta))|}$ onto $\partial\Omega \cap S_\varphi$. Combining (6.19) and Lemma 4.2, we can derive

$$|\zeta \cdot \nabla \xi(x_{\mathbf{b}}(x, \zeta_{g,\varphi}))| \lesssim_\Omega |\zeta \cdot \nabla \xi(x_{\mathbf{b}}(x, \zeta))|,$$

using similar argument as (4.27), where ζ has spherical coordinates $(|\zeta|, \theta, \varphi)$, $0 \leq \theta \leq \theta_g(\varphi)$ for fixed φ . Then, for given $|v| > 0$,

$$\begin{aligned} & \int_{\{\zeta: x_b(x, \zeta) \in \partial\Omega\}} \frac{e^{-c|v-\zeta|^2}}{|v-\zeta|} \frac{\langle \zeta \rangle^r}{|\zeta \cdot \nabla \xi(x_b(x, \zeta))|^{2\beta}} d\zeta \\ & \lesssim \int_0^{2\pi} \int_0^\infty \int_0^{\theta_g(\varphi)} \frac{e^{-c|v-\zeta|^2}}{|v-\zeta|} \frac{\langle \zeta \rangle^r}{|\zeta \cdot \nabla \xi(x_b(x, \zeta_g(\varphi)))|^{2\beta}} |\zeta|^2 \sin \theta d|\zeta| d\theta d\varphi \\ & \lesssim \sup_{\substack{\hat{v}=(\theta_v, \varphi_v) \\ 0 \leq \theta_v \leq \theta_g(\varphi)}} \int_0^{2\pi} \int_0^\infty \int_0^{\theta_g(\varphi)} \frac{e^{-c|v-\zeta|^2}}{|v-\zeta|} \frac{\langle \zeta \rangle^r}{\sin^{2\beta}(\theta_g(\varphi) - \theta)} \frac{1}{|\zeta|^{2\beta}} |\zeta|^2 \sin \theta d|\zeta| d\theta d\varphi, \end{aligned} \quad (6.20)$$

where we also used $|n_\parallel(x_b(x, \zeta_g(\varphi)))| \gtrsim_\Omega 1$ in the last step, which is true by Lemma 4.2 and Lemma 6.4. (If $v = 0$, (6.16) is obvious.)

Notice that we get optimal $\theta_g(\varphi) = \frac{\pi}{2}$ for all φ only when $x \in \partial\Omega$. Therefore, (6.20) is optimal when $x \in \partial\Omega$ and

$$\text{LHS of (6.16)} \lesssim \sup_{\substack{\hat{v} \in \mathbb{S}^2 \\ \hat{v} \cdot \nabla \xi(x) \leq 0}} \int_{\{\zeta \cdot \nabla \xi(x) \leq 0\}} \frac{e^{-c|v-\zeta|^2}}{|v-\zeta|} \frac{\langle \zeta \rangle^r}{|\zeta \cdot \nabla \xi(x)|^{2\beta}} d\zeta, \quad x \in \partial\Omega, \quad (6.21)$$

which is integration on half space $\{\zeta \in \mathbb{R}^3 : \zeta \cdot \nabla \xi(x) \leq 0\}$ when $x \in \partial\Omega$. To make estimate easier, let us change axis of spherical coordinate. We assign a direction vector in tangential plane of $x \in \partial\Omega$ to $\hat{z}$ axis (so that $\hat{z} \cdot \nabla \xi(x) = 0$) and also assign $-\frac{\nabla \xi(x)}{|\nabla \xi(x)|}$ to $\hat{y}$. Then,

$$\{\zeta : \zeta \cdot \nabla \xi(x) \leq 0\} = \{\zeta = (|\zeta|, \theta, \varphi) : 0 \leq |\zeta| < \infty, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq \pi\}, \quad (6.22)$$

in spherical coordinate and

$$\left| \zeta \cdot \frac{\nabla \xi(x)}{|\nabla \xi(x)|} \right| = |\zeta| \sin \theta \sin \varphi.$$

We write $v = |v|(\sin \theta_v \cos \varphi_v, \sin \theta_v \sin \varphi_v, \cos \theta_v)$, then

$$|v - \zeta|_{\varphi_v} := \sqrt{|v|^2 + |\zeta|^2 - 2|v||\zeta| \cos(\theta - \theta_v)} \leq |v - \zeta|,$$

where $|v - \zeta|_{\varphi_v}$ is 2D distance in a fixed φ_v plane, when both v and ζ have coordinate (θ_v, φ_v) and (θ, φ_v) , respectively. Now we treat v and ζ as like 2D vectors in fixed φ_v plane. Therefore, applying (6.22) to (6.21),

$$\begin{aligned} (6.16) & \lesssim \int_0^\pi \frac{1}{\sin^{2\beta} \varphi} d\varphi \left[\int_0^\infty \int_0^\pi \frac{e^{-c|v-\zeta|_{\varphi_v}^2}}{|v-\zeta|_{\varphi_v}} \langle \zeta \rangle^r \sin^{1-2\beta} \theta \frac{1}{|\zeta|^{2\beta-1}} dA \right] \\ & \lesssim C_\beta \int_0^\infty \int_0^\pi \frac{e^{-c|v-\zeta|_{\varphi_v}^2}}{|v-\zeta|_{\varphi_v}} \langle \zeta \rangle^r \sin^{1-2\beta} \theta \frac{1}{|\zeta|^{2\beta-1}} dA, \quad 2\beta - 1 < 0, \lesssim \langle v \rangle^{r+1-2\beta}, \end{aligned} \quad (6.23)$$

where $dA = |\zeta| d|\zeta| d\theta$ is 2D measure in φ_v plane. This proves (6.16).

Proof for (6.17) is nearly same as (6.16) since $4\beta - 1 \leq 0$. We modify (6.23) to get

$$C_\beta \int_0^\infty \int_0^\pi \frac{e^{-c|v-\zeta|_{\varphi_v}^2}}{|v-\zeta|_{\varphi_v}} \langle \zeta \rangle^r \sin^{1-2\beta} \theta \frac{1}{|\zeta|^{4\beta-1}} dA \lesssim \langle v \rangle^{r+1-4\beta}. \quad (6.24)$$

To prove (6.18), from the LHS of (6.24) with $4\beta - 1 > 0$, we use Hölder inequality with $p < 2$ and $q > 2$ (to be justified below) so that

$$\begin{aligned} (6.16) &\lesssim C_\beta \int_0^\infty \int_0^\pi \frac{e^{-c|v-\zeta|_{\varphi_v}^2}}{|v-\zeta|_{\varphi_v}} \langle \zeta \rangle^r \sin^{1-2\beta} \theta \frac{1}{|\zeta|^{4\beta-1}} dA \\ &\lesssim C_\beta \langle v \rangle^r \left[\iint e^{-\frac{cq}{2}|v-\zeta|_{\varphi_v}^2} \frac{1}{|\zeta|^{(4\beta-1)q}} dA \right]^{\frac{1}{q}} \lesssim C_\beta \langle v \rangle^r, \end{aligned}$$

where $q = q(\beta) > 2$ and $p = p(\beta) < 2$ can be chosen depending on $\beta < \frac{1}{2}$ so that

$$(4\beta - 1)q < 2. \quad \square$$

Corollary 6.6. Let $\beta < \frac{1}{2}$ and $|v - \bar{v}| \leq 1$. We also assume (2.6) and (2.13) with $S_{(x, \bar{x}, v + \zeta)}$, and (2.8) and (2.14) with $S_{(\bar{x}, v, \bar{v}, \zeta)}$. Then, we get the followings,

$$\int_\zeta k_c(v, v + \zeta) \langle v + \zeta \rangle^r \mathcal{T}_{sp}^{2\beta}(x, \bar{x}, v + \zeta) d\zeta \lesssim C_\beta \langle v \rangle^{r+1-2\beta}, \quad (6.25)$$

$$\int_\zeta \mathbf{k}_c(v, \bar{v}, \zeta) \langle v + \zeta \rangle^r \mathcal{T}_{vel}^{2\beta}(\bar{x}, v, \bar{v}, \zeta; t, s) d\zeta \lesssim C_\beta \langle v \rangle^{r+\max\{1-4\beta, 0\}} (1 + \langle v \rangle(t - s)), \quad (6.26)$$

where $\mathcal{T}_{sp}$ and $\mathcal{T}_{vel}$ are defined in (6.10) and (6.11), respectively.

Proof. For (6.25), we apply (5.23) to (6.10) and then use (6.16). For (6.26), we apply (5.23) to (6.11) and then use (6.17) and (6.18). $\square$

6.3 | Uniform estimates for $\mathfrak{S}_{sp,vel}^{2\beta}$

We start from the following

$$\begin{aligned} |f(t, x, v + \zeta) - f(t, \bar{x}, \bar{v} + \zeta)| &\leq e^{-\int_0^t \nu(f)(\tau, X(\tau), V(\tau)) d\tau} |f(0, X(0), V(0)) - f(0, \bar{X}(0), \bar{V}(0))| \\ &+ \int_0^t e^{-\int_s^t \nu(f)(\tau, X(\tau), V(\tau)) d\tau} |\Gamma_{\text{gain}}(f, f)(s, X(s), V(s)) - \Gamma_{\text{gain}}(f, f)(s, \bar{X}(s), \bar{V}(s))| \\ &+ |e^{-\int_0^t \nu(f)(\tau, X(\tau), V(\tau)) d\tau} - e^{-\int_0^t \nu(f)(\tau, \bar{X}(\tau), \bar{V}(\tau)) d\tau}| |f(0, \bar{X}(0), \bar{V}(0))| \\ &+ \int_0^t |e^{-\int_s^t \nu(f)(\tau, X(\tau), V(\tau)) d\tau} - e^{-\int_s^t \nu(f)(\tau, \bar{X}(\tau), \bar{V}(\tau)) d\tau}| |\Gamma_{\text{gain}}(f, f)(s, \bar{X}(s), \bar{V}(s))| ds, \end{aligned}$$

which is trivial by (2.5). Since, $|e^{-a} - e^{-b}| \leq |a - b|$ for $a \geq b \geq 0$, we obtain the following basic estimate,

$$|f(t, x, v + \zeta) - f(t, \bar{x}, \bar{v} + \zeta)| \quad (6.27)$$

$$\leq |f(0, X(0), V(0)) - f(0, \bar{X}(0), \bar{V}(0))| \quad (6.28)$$

$$+ \int_0^t |\Gamma_{\text{gain}}(f, f)(s, X(s), V(s)) - \Gamma_{\text{gain}}(f, f)(s, \bar{X}(s), \bar{V}(s))| \quad (6.29)$$

$$+ \|w_0 f_0\|_\infty \frac{1}{w_0(\bar{v} + \zeta)} \int_0^t |\nu(f)(s, X(s), V(s)) - \nu(f)(s, \bar{X}(s), \bar{V}(s))| ds \quad (6.30)$$

$$+ t \sup_{0 \leq s \leq t} \|w f(s)\|_\infty^2 \frac{1}{\sqrt{w(\bar{v} + \zeta)}} \int_0^t |\nu(f)(s, X(s), V(s)) - \nu(f)(s, \bar{X}(s), \bar{V}(s))| ds, \quad (6.31)$$

where

$$X(s) := X(s; t, x, v + \zeta), \quad V(s) := V(s; t, x, v + \zeta),$$

$$\bar{X}(s) := \bar{X}(s; t, \bar{x}, \bar{v} + \zeta), \quad \bar{V}(s) := \bar{V}(s; t, \bar{x}, \bar{v} + \zeta).$$

Proposition 6.7 (Seminorm estimate). *Suppose the domain is given as in Definition 1.2 and (1.3). For $0 < |(x, v) - (\bar{x}, \bar{v})| \leq 1$ and $\zeta \in \mathbb{R}^3$, there exists $\varpi \gg_{f_0, \beta} 1$ such that*

$$\left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] \lesssim_\beta \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \|w_0 f_0\|_\infty. \quad (6.32)$$

for sufficiently small $T > 0$ such that $\varpi T \ll 1$.

Proof. **Step 1** First, we estimate $\mathfrak{H}_{vel}^{2\beta}$. From definition 2.8, we estimate (we use $\bar{x}$ instead of x to use Lemma 6.1 directly).

$$e^{-\varpi \langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) \frac{|f(t, \bar{x}, v + \zeta) - f(t, \bar{x}, \bar{v} + \zeta)|}{|v - \bar{v}|^{2\beta}} d\zeta, \quad \text{for } |v - \bar{v}| \leq 1. \quad (6.33)$$

We note that $\bar{v} + \zeta$ is well-defined only when (2.8) holds. For given v and $\bar{v}$, however, the set of $\zeta \in \mathbb{R}^3$, where (2.8) does not hold is of measure zero. So we assume (2.8) without loss of generality throughout **Step 1** and can use (6.8) and (6.9) in Lemma 6.1.

For $f(t, \bar{x}, v + \zeta) - f(t, \bar{x}, \bar{v} + \zeta)$ in integrand of (6.33), we use expansion (6.27) by replainc

$$(\bar{X}, \bar{V}) := (\bar{X}(s), \bar{V}(s)) = (X(s; t, \bar{x}, v + \zeta), V(s; t, \bar{x}, \bar{v} + \zeta)), \quad |v - \bar{v}| \leq 1,$$

Now, we apply $e^{-\varpi \langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}(v, \bar{v}, \zeta) \frac{\cdot}{|v - \bar{v}|^{2\beta}} d\zeta$ to each (6.28) – (6.31).

Substep 1-1 In this substep, we consider (6.28). To consider (6.28), we put

$$s = 0, \quad x = \bar{x}. \quad (6.34)$$

in (6.2)–(6.5). Then, it is sufficient to consider only (6.4) and (6.5) only. Since we are dealing with difference of f , let us use notation $(6.4)_f$ and $(6.5)_f$ to stress the function f . Using (6.8), we get

$$\begin{aligned} & e^{-\varpi\langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) \frac{|(6.4)_f|}{|v - \bar{v}|^{2\beta}} d\zeta \\ & \lesssim e^{-\varpi\langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) [t + |v + \zeta|t^2 + (|v + \zeta| + |v + \zeta|^2 t) \mathcal{T}_{vel}(\bar{x}, v, \bar{v}, \zeta; t, 0)]^{2\beta} \\ & \quad \times \left[\frac{1}{\langle v + \zeta \rangle} \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \frac{1}{w_0(v + \zeta)} \|w_0 f_0\|_\infty \right] d\zeta \\ & + e^{-\varpi\langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) [1 + |v + \zeta|t + |v + \zeta|^2 \mathcal{T}_{vel}(\bar{x}, v, \bar{v}, \zeta; t, 0)]^{2\beta} \\ & \quad \times \left[\frac{1}{\langle v + \zeta \rangle^2} \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \frac{1}{w_0(v + \zeta)} \|w_0 f_0\|_\infty \right] d\zeta \\ & \lesssim_\beta \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \|w_0 f_0\|_\infty, \quad (6.35) \end{aligned}$$

where we have used Corollary 6.6.

We similarly apply (6.34) and use (6.9) to get estimate for $(6.5)_f$,

$$\begin{aligned} & e^{-\varpi\langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) \frac{|(6.5)_f|}{|v - \bar{v}|^{2\beta}} d\zeta \lesssim e^{-\varpi\langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) t^{2\beta} \\ & \quad \times \left[\frac{1}{\langle v + \zeta \rangle} \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \left(\langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} \right) + \frac{1}{w_0(v + \zeta)} \|w_0 f_0\|_\infty \right] d\zeta + e^{-\varpi\langle v \rangle^2 t} \\ & \quad \times \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) \left[\frac{1}{\langle v + \zeta \rangle^2} \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \left(\langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} \right) + \frac{1}{w_0(v + \zeta)} \|w_0 f_0\|_\infty \right] d\zeta \\ & \lesssim_\beta \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \|w_0 f_0\|_\infty. \quad (6.36) \end{aligned}$$

Note that the bound of (6.35) also control (6.36).

Hence,

$$e^{-\varpi\langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) \frac{|(6.28)|}{|v - \bar{v}|^{2\beta}} d\zeta \lesssim \text{RHS of (6.35)}. \quad (6.37)$$

Substep 1-2 In this substep, we consider (6.29). We put

$$x = \bar{x}, \quad (6.38)$$

and use (6.8) replacing f into $\Gamma_{\text{gain}}(f, f)$. Using Lemma 6.3, Lemma 3.2, and (3.11) of Lemma 3.4, we obtain

$$\begin{aligned} & e^{-\varpi\langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) \frac{|(6.29)|}{|v - \bar{v}|^{2\beta}} d\zeta \\ & \lesssim \int_0^t e^{-\varpi\langle v \rangle^2(t-s)} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) e^{-\varpi\langle v \rangle^2 s} \\ & \quad \times [t + |v + \zeta|t^2 + (|v + \zeta| + |v + \zeta|^2 t) \mathcal{T}_{\text{vel}}(\bar{x}, v, \bar{v}, \zeta; t, s)]^{2\beta} \\ & \quad \times \left[e^{\varpi\langle v + \zeta \rangle^2 s} \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} e^{-\varpi\langle v \rangle^2 s} \frac{|\Gamma_{\text{gain}}(s, x, v) - \Gamma_{\text{gain}}(s, \bar{x}, v)|}{|x - \bar{x}|^{2\beta}} \right. \\ & \quad \left. + \frac{\langle v + \zeta \rangle}{w(v + \zeta)} \|wf(s)\|_\infty^2 \right] d\zeta ds \\ & + \int_0^t e^{-\varpi\langle v \rangle^2(t-s)} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) \\ & \quad \times e^{-\varpi\langle v \rangle^2 s} [1 + |v + \zeta|t + |v + \zeta|^2 t \mathcal{T}_{\text{vel}}(\bar{x}, v, \bar{v}, \zeta; t, s)]^{2\beta} \\ & \quad \times \left[e^{\varpi\langle v + \zeta \rangle^2 s} \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} e^{-\varpi\langle v \rangle^2 s} \frac{|\Gamma_{\text{gain}}(s, x, v) - \Gamma_{\text{gain}}(s, x, \bar{v})|}{|v - \bar{v}|^{2\beta}} \right. \\ & \quad \left. + \frac{\langle v + \zeta \rangle}{w(v + \zeta)} \|wf(s)\|_\infty^2 \right] d\zeta ds \\ & \lesssim \int_0^t e^{-\varpi\langle v \rangle^2(t-s)} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_{\frac{c}{2}}(v, \bar{v}, \zeta) \\ & \quad \times [t + |v + \zeta|t^2 + (|v + \zeta| + |v + \zeta|^2 t) \mathcal{T}_{\text{vel}}(\bar{x}, v, \bar{v}, \zeta; t, s)]^{2\beta} \|wf(s)\|_\infty \\ & \quad \times \left[\sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} e^{-\varpi\langle v \rangle^2 s} \int_{\mathbb{R}_u^3} k_c(v, v + u) \frac{|f(s, x, v + u) - f(s, \bar{x}, v + u)|}{|x - \bar{x}|^{2\beta}} du \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{\langle v + \zeta \rangle}{w(v + \zeta)} \|wf(s)\|_\infty \Big] d\zeta ds \\
& + \int_0^t e^{-\varpi \langle v \rangle^2(t-s)} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) [1 + |v + \zeta|t + |v + \zeta|^2 \mathcal{T}_{vel}(\bar{x}, v, \bar{v}, \zeta; t, s)]^{2\beta} \\
& \times \|wf(s)\|_\infty \left[\sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} e^{-\varpi \langle v \rangle^2 s} \int_{\mathbb{R}_u^3} \mathbf{k}_c(v, \bar{v}, u) \frac{|f(s, x, v + u) - f(s, x, \bar{v} + u)|}{|v - \bar{v}|^{2\beta}} du \right. \\
& \left. + \left(\frac{1}{\langle v + \zeta \rangle} + \frac{\langle v + \zeta \rangle}{w(v + \zeta)} \right) \|wf(s)\|_\infty \right] d\zeta ds.
\end{aligned}$$

Applying Corollary 6.6 and Lemma 3.2,

$$\begin{aligned}
& e^{-\varpi \langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) \frac{|(6.29)|}{|v - \bar{v}|^{2\beta}} d\zeta \\
& \lesssim_\beta \int_0^t e^{-\frac{\varpi}{2} \langle v \rangle^2(t-s)} ds \|w_0 f_0\|_\infty \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] + \|w_0 f_0\|_\infty^2 \\
& \lesssim_\beta \frac{1}{\varpi} \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] \mathcal{P}_2(\|w_0 f_0\|_\infty), \tag{6.39}
\end{aligned}$$

where $\mathcal{P}_2(\cdot) = 1 + |\cdot| + |\cdot|^2$. Note that we should consider (6.4)_Γ and (6.5)_Γ separately. However, the bound of (6.4)_Γ also control (6.5)_Γ similar as *Substep 1-1*.

Substep 1-3 In this substep, we consider (6.30) and (6.31). We use (6.38) and then, from Lemma 3.3, we have same bound as (6.39), but order of $\|wf(s)\|_\infty$ is reduced by 1. ($v(f)$ is linear.)

$$\begin{aligned}
& e^{-\varpi \langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_c(v, \bar{v}, \zeta) \frac{|(6.30)| + |(6.31)|}{|v - \bar{v}|^{2\beta}} d\zeta \\
& \lesssim \frac{1}{\varpi} \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] \mathcal{P}_3(\|w_0 f_0\|_\infty). \tag{6.40}
\end{aligned}$$

We omit the detail.

Putting (6.37), (6.39), and (6.40) altogether, we conclude

$$\begin{aligned}
& \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \lesssim_\beta \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} \\
& + \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \|w_0 f_0\|_\infty \\
& + \frac{1}{\varpi} \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] \mathcal{P}_2(\|w_0 f_0\|_\infty). \tag{6.41}
\end{aligned}$$

Step 2 We estimate $\mathfrak{S}_{sp}^{2\beta}$. From Definition 2.8, we control

$$e^{-\varpi(v)^2 t} \int_{\mathbb{R}_\zeta^3} k_c(v, v + \zeta) \frac{|f(t, x, v + \zeta) - f(t, \bar{x}, v + \zeta)|}{|x - \bar{x}|^{2\beta}} d\zeta, \quad |x - \bar{x}| \leq 1. \quad (6.42)$$

We note that $\bar{x}$ is well-defined only when (2.6) holds. For given $x, \bar{x}$, and v , the set of $\zeta \in \mathbb{R}^3$, where (2.6) does not hold is of measure zero. So we can assume (2.6) without loss of generality throughout **Step 2** and can use (6.6) and (6.7) in Lemma 6.1.

For $f(t, x, v + \zeta) - f(t, \bar{x}, v + \zeta)$ in integrand of (6.42), we use (6.27) by replacing

$$(\bar{X}, \bar{V}) := (\bar{X}(s), \bar{V}(s)) = (X(s; t, x, v + \zeta), V(s; t, \bar{x}, v + \zeta)), \quad |x - \bar{x}| \leq 1.$$

Now, we apply $e^{-\varpi(v)^2 t} \int_{\mathbb{R}_\zeta^3} k_c(v, v + \zeta) \frac{\cdot}{|x - \bar{x}|^{2\beta}} d\zeta$ to each (6.28) – (6.31).

Substep 2-1 For (6.28), we replace

$$s = 0 \quad (6.43)$$

in (6.2) and (6.3). We use (6.6) then similar as (6.37),

$$\begin{aligned} e^{-\varpi(v)^2 t} \int_{\mathbb{R}_\zeta^3} k_c(v, v + \zeta) \frac{|(6.2)_f|}{|x - \bar{x}|^{2\beta}} d\zeta &\lesssim e^{-\varpi(v)^2 t} \int_{\mathbb{R}_\zeta^3} k_c(v, v + \zeta) [(|v + \zeta| + |v + \zeta|^2 t) \mathcal{T}_{sp}(x, \bar{x}, v + \zeta)]^{2\beta} \\ &\times \left[\frac{1}{\langle v + \zeta \rangle} \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \frac{1}{w_0(v + \zeta)} \|w_0 f_0\|_\infty \right] d\zeta \\ &+ e^{-\varpi(v)^2 t} \int_{\mathbb{R}_\zeta^3} k_c(v, v + \zeta) [(|v + \zeta| + |v + \zeta|^2 t) \mathcal{T}_{sp}(x, \bar{x}, v + \zeta)]^{2\beta} \\ &\times \left[\frac{1}{\langle v + \zeta \rangle^2} \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \frac{1}{w_0(v + \zeta)} \|w_0 f_0\|_\infty \right] d\zeta \\ &\lesssim_\beta \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \|w_0 f_0\|_\infty, \quad (6.44) \end{aligned}$$

where we have used Corollary 6.6.

Substep 2-2 In this substep, we consider (6.29). We put (6.43) and use Lemma 6.3, 3.2, and 3.2 as we did in *Substep 1-2*. Then similar as (6.39), we obtain

$$\begin{aligned} e^{-\varpi(v)^2 t} \int_{\mathbb{R}_\zeta^3} k_c(v, v + \zeta) \frac{|(6.29)|}{|x - \bar{x}|^{2\beta}} d\zeta \\ \lesssim \int_0^t e^{-\varpi(v)^2(t-s)} \int_{\mathbb{R}_\zeta^3} k_{\frac{c}{2}}(v, v + \zeta) [(|v + \zeta| + |v + \zeta|^2 t) \mathcal{T}_{sp}(x, \bar{x}, v + \zeta)]^{2\beta} \|wf(s)\|_\infty \end{aligned}$$

$$\begin{aligned}
& \times \left[\sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} e^{-\varpi \langle v \rangle^2 s} \int_{\mathbb{R}_{\bar{u}}^3} k_c(v, v + u) \frac{|f(s, x, v + u) - f(s, \bar{x}, v + u)|}{|x - \bar{x}|^{2\beta}} du \right. \\
& \quad \left. + \frac{\langle v + \zeta \rangle}{w(v + \zeta)} \|wf(s)\|_\infty \right] d\zeta ds \\
& + \int_0^t e^{-\varpi \langle v \rangle^2 (t-s)} \int_{\mathbb{R}_\zeta^3} k_{\frac{c}{2}}(v, v + \zeta) [(|v + \zeta| + |v + \zeta|^2) \mathcal{T}_{sp}(v, \bar{x}, v + \zeta)]^{2\beta} \|wf(s)\|_\infty \\
& \times \left[\sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} e^{-\varpi \langle v \rangle^2 s} \int_{\mathbb{R}_\zeta^3} \mathbf{k}_{\frac{c}{2}}(v, \bar{v}, u) \frac{|f(s, x, v + u) - f(s, x, \bar{v} + u)|}{|v - \bar{v}|^{2\beta}} \right. \\
& \quad \left. + \left(\frac{1}{\langle v + \zeta \rangle} + \frac{\langle v + \zeta \rangle}{w(v + \zeta)} \right) \|wf(s)\|_\infty \right] d\zeta ds \\
& \lesssim_\beta \frac{1}{\varpi} \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] \mathcal{P}_2(\|w_0 f_0\|_\infty). \tag{6.45}
\end{aligned}$$

Substep 2-3 In this substep, we consider (6.30) and (6.31). Similar as (6.40) in *Substep 1-3*, the bound is same as (6.45) except that the order of $\|wf(s)\|_\infty$ is reduced 1, that is,

$$e^{-\varpi \langle v \rangle^2 t} \int_{\mathbb{R}_\zeta^3} k_c(v, \bar{v}, \zeta) \frac{|(6.30)| + |(6.31)|}{|x - \bar{x}|^{2\beta}} d\zeta \lesssim_\beta \frac{1}{\varpi} \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] \mathcal{P}_3(\|w_0 f_0\|_\infty). \tag{6.46}$$

Putting (6.44), (6.45), and (6.46) altogether, we conclude

$$\begin{aligned}
\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) & \lesssim_\beta \sup_{\substack{v \in \mathbb{R}^3 \\ |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} \\
& + \sup_{\substack{x \in \bar{\Omega} \\ |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \|w_0 f_0\|_\infty \\
& + \frac{1}{\varpi} \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] \mathcal{P}_3(\|w_0 f_0\|_\infty), \tag{6.47}
\end{aligned}$$

for $T \leq T^*$, where T^* is local existence time in Lemma 2.1.

Finally, from (6.41) and (6.47), we finish the proof by choosing sufficiently large $\varpi \gg_{f_0, \beta} 1$. $\square$

7 | REGULARITY ESTIMATE

7.1 | $C_{x,v}^{0,\frac{1}{2}}$ estimates of trajectory

The next geometric lemma asserts that the specular characteristics are basically Hölder regular with an exponent of $1/2$.

Lemma 7.1. *Suppose the domain is given as in Definition 1.2 and (1.3). Let $(x, v), (\bar{x}, \bar{v}) \in \Omega \times \mathbb{R}^3$, and $|(x, v) - (\bar{x}, \bar{v})| \leq 1$, and $\zeta \in \mathbb{R}^3$.*

(1) *We have*

$$\begin{aligned} & \max\{|v + \zeta|, |\bar{v} + \zeta|\} |t_b(x, v + \zeta) - t_b(\bar{x}, \bar{v} + \zeta)| \\ & \lesssim \sqrt{\|\nabla \xi\|_\infty} \left\{ |\bar{x} - x|^{\frac{1}{2}} + \max\{\sqrt{t_b(x, v + \zeta)}, \sqrt{t_b(\bar{x}, \bar{v} + \zeta)}\} |\bar{v} - v|^{\frac{1}{2}} \right\}. \end{aligned} \quad (7.1)$$

(2) *For $s \leq t$*

$$|X(s; t, x, v + \zeta) - X(s; t, \bar{x}, \bar{v} + \zeta)| \lesssim \{1 + \langle v + \zeta \rangle |t - s|\} \left\{ |x - \bar{x}|^{\frac{1}{2}} + |t - s|^{\frac{1}{2}} |v - \bar{v}|^{\frac{1}{2}} \right\}. \quad (7.2)$$

(3) *For $s \leq t$, if*

$$s \leq \min\{t^1(x, v), t^1(\bar{x}, \bar{v})\} \quad \text{or} \quad s > \max\{t^1(x, v), t^1(\bar{x}, \bar{v})\},$$

then

$$|V(s; t, x, v + \zeta) - V(s; t, \bar{x}, \bar{v} + \zeta)| \lesssim |v - \bar{v}| + \langle v + \zeta \rangle \left\{ |x - \bar{x}|^{\frac{1}{2}} + |t - s|^{\frac{1}{2}} |v - \bar{v}|^{\frac{1}{2}} \right\}. \quad (7.3)$$

Proof. In the course of the proof we set $\zeta = 0$ without loss of generality.

Step 1. Proof of (7.1). For simplicity's sake we tentatively denote $t_b = t_b(x, v)$ and $\bar{t}_b = t_b(\bar{x}, \bar{v})$. Without loss of generality we may assume $t_b \geq \bar{t}_b$. For $t_b \leq \bar{t}_b$ we can follow the same argument with obvious modification. We note that $x - \bar{t}_b v \in \Omega$ and therefore $\xi(x - t_b v) - \xi(x - \bar{t}_b v) \geq 0$. Using $\xi(\bar{x} - \bar{t}_b \bar{v}) = 0 = \xi(x - t_b v)$, we derive that

$$0 \leq \xi(x - t_b v) - \xi(x - \bar{t}_b v) = \xi(\bar{x} - \bar{t}_b \bar{v}) - \xi(x - \bar{t}_b v) \leq \|\nabla \xi\|_\infty \{|\bar{x} - x| + |\bar{t}_b| |\bar{v} - v|\}. \quad (7.4)$$

On the other hand, by an expansion,

$$\begin{aligned} \xi(x - t_b v) - \xi(x - \bar{t}_b v) &= \int_{\bar{t}_b}^{t_b} -v \cdot \nabla \xi(x - sv) ds \\ &= \int_{\bar{t}_b}^{t_b} \left\{ -v \cdot \nabla \xi(x - t_b v) + \int_{t_b}^s v \cdot \nabla^2 \xi(x - \tau v) v d\tau \right\} ds \\ &= (-v \cdot \nabla \xi(x_b))(t_b - \bar{t}_b) + \int_{\bar{t}_b}^{t_b} \int_s^{t_b} -v \cdot \nabla^2 \xi(x - \tau v) v d\tau ds. \end{aligned} \quad (7.5)$$

Together with $-v \cdot \nabla \xi(x - t_b v) \geq 0$ and the convexity (1.3), we conclude that

$$\xi(x - t_b v) - \xi(x - \bar{t}_b v) \geq \frac{\theta_\Omega}{2} |v|^2 |t_b - \bar{t}_b|^2. \quad (7.6)$$

From (7.5) and (7.6), we derive that, when $t_b \geq \bar{t}_b$,

$$|v| |t_b - \bar{t}_b| \lesssim \sqrt{\|\nabla \xi\|_\infty} \left\{ \sqrt{|\bar{x} - x|} + \max\{\sqrt{t_b}, \sqrt{\bar{t}_b}\} \sqrt{|\bar{v} - v|} \right\}. \quad (7.7)$$

Now we repeat the procedure (7.4)-(7.7) with some change: For $t_b \geq \bar{t}_b$, as (7.4), we have $0 \leq \xi(\bar{x} - t_b \bar{v}) - \xi(\bar{x} - \bar{t}_b \bar{v}) = \xi(\bar{x} - t_b \bar{v}) - \xi(x - t_b v) \leq \|\nabla \xi\|_\infty \{|\bar{x} - x| + |t_b| |\bar{v} - v|\}$. Then as (7.6) we derive that $\xi(\bar{x} - t_b \bar{v}) - \xi(\bar{x} - \bar{t}_b \bar{v}) \geq \frac{\theta_\Omega}{2} |\bar{v}|^2 |t_b - \bar{t}_b|^2$. Hence we conclude that $|\bar{v}| |t_b - \bar{t}_b|$ has the same bound of (7.7):

$$\frac{\theta_\Omega}{2} |\bar{v}|^2 |t_b - \bar{t}_b|^2 \leq |\xi(\bar{x} - t_b \bar{v}) - \xi(\bar{x} - \bar{t}_b \bar{v})| \leq \|\nabla \xi\|_\infty \{|\bar{x} - x| + \max\{t_b, \bar{t}_b\} |\bar{v} - v|\}. \quad (7.8)$$

Thereby we conclude (7.1) from (7.7) and (7.8).

Step 2: Proof of (7.2). We consider the case of $t_b(x, v) < \infty$ and $t_b(\bar{x}, \bar{v}) < \infty$. Then from (2.4)

$$\begin{aligned} & |X(s; t, x, v) - X(s; t, \bar{x}, \bar{v})| \\ & \lesssim |x - \bar{x}| + |t - s| |v - \bar{v}| + |n(x_b) \cdot v| |t_b - \bar{t}_b| + |t - \bar{t}_b - s| |\bar{v}| |n(x_b) - n(\bar{x}_b)| \\ & \lesssim \{1 + |t - s| \max\{|v|, |\bar{v}|\}\} |x - \bar{x}| + |t - s| |v - \bar{v}| + \max\{|v|, |\bar{v}|\} |t_b - \bar{t}_b|. \end{aligned} \quad (7.9)$$

Now we are using (7.9) and (7.1) to conclude (7.2).

If $t_b(x, v) = \infty$ and $t_b(\bar{x}, \bar{v}) = \infty$ then simply we have

$$|X(s; t, x, v) - X(s; t, \bar{x}, \bar{v})| \leq |x - \bar{x}| + |t - s| |v - \bar{v}|, \quad (7.10)$$

which is bounded as (7.2).

For the rest of case, we bound $|X(s; t, \bar{x}, v) - X(s; t, x, v)|$ and

$|X(s; t, x, \bar{v}) - X(s; t, x, v)|$ separately. We start with $|X(s; t, \bar{x}, v) - X(s; t, x, v)|$. First we consider the case of $t_b(\bar{x}, v) < \infty$ and $t_b(x, v) = \infty$. Recall $\mathbf{x}(\tau)$ in (2.15). From (2.16) there exists $\tau_+ = \tau_+(x, \bar{x}, v)$ such that $-\nabla \xi(x_b(\mathbf{x}(\tau_+), v)) \cdot v = 0$, $t_b(\mathbf{x}(\tau), v) < \infty$ for $\tau \in [0, \tau_+]$, and $t_b(\mathbf{x}(\tau), v) = \infty$ for $\tau \in (\tau_+, 1]$. Equipped with $|X(s; t, \bar{x}, v) - X(s; t, x, v)| \leq |X(s; t, \bar{x}, v) - X(s; t, \mathbf{x}(\tau_+), v)| + |X(s; t, \mathbf{x}(\tau_+), v) - X(s; t, x, v)|$, we have $|X(s; t, \bar{x}, v) - X(s; t, \mathbf{x}(\tau_+), v)| \lesssim (7.9)$ and $|X(s; t, \mathbf{x}(\tau_+), v) - X(s; t, x, v)| \lesssim (7.10)$, which give us the bound of $|X(s; t, \bar{x}, v) - X(s; t, x, v)|$ as in (7.2). The case of $t_b(x, v) < \infty$ and $t_b(\bar{x}, v) = \infty$ can be treated similarly.

Now we bound $|X(s; t, x, \bar{v}) - X(s; t, x, v)|$. Consider first the case of $t_b(x, \bar{v}) < \infty$ and $t_b(x, v) = \infty$. Recall $\mathbf{v}(\tau)$ in (2.15). From (2.16) there exists $\tau_+ = \tau_+(x, v, \bar{v}, 0)$ such that $-\nabla \xi(x_b(x, \mathbf{v}(\tau_+))) \cdot \mathbf{v}(\tau_+) = 0$, $t_b(x, \mathbf{v}(\tau)) < \infty$ for $\tau \in [0, \tau_+]$, and $t_b(x, \mathbf{v}(\tau)) = \infty$ for $\tau \in (\tau_+, 1]$. Then following the previous argument and using (7.9) and (7.10), we easily get the result. Other case $t_b(x, v) < \infty$ and $t_b(x, \bar{v}) = \infty$ also can be treated similarly.

Step 3: Proof of (7.3). We only need to consider the case of

$$s < \min\{t_1(t, x, v), t_1(t, \bar{x}, \bar{v})\}.$$

Otherwise, in the case of $s > \max\{t_1(t, x, v), t_1(t, \bar{x}, \bar{v})\}$, we have $|V(s; t, x, v) - V(s; t, \bar{x}, \bar{v})| = |v - \bar{v}|$. From (2.4) and (7.1)

$$\begin{aligned} |V(s; t, x, v) - V(s; t, \bar{x}, \bar{v})| &\lesssim |v - \bar{v}| + \max\{|v|, |\bar{v}|\}|x_{\mathbf{b}} - \bar{x}_{\mathbf{b}}| \\ &\lesssim \{1 + |t - s| \max\{|v|, |\bar{v}|\}\} |v - \bar{v}| + \max\{|v|, |\bar{v}|\}|x - \bar{x}| + \max\{|v|, |\bar{v}|\}^2 |t_{\mathbf{b}} - \bar{t}_{\mathbf{b}}|. \end{aligned} \quad (7.11)$$

Now using (7.11) and (7.1) we conclude (7.3). $\square$

7.2 | Hölder regularity: Proof of the main theorem

We provide the proof of Theorem 2.9.

Proof of Theorem 2.9. First, note that (2.22) is obvious if $|(x, v) - (\bar{x}, \bar{v})| \geq 1$. If $|(x, v) - (\bar{x}, \bar{v})| \leq 1$, we consider the following steps.

Step 1 Let us assume (2.6) for $v + \zeta$ and (2.8) for $\bar{x}$, respectively.

In this step, we consider (6.28). To consider (6.28), we put $s = 0$ of (6.2)–(6.5) and apply Lemma 7.1 to obtain

$$\begin{aligned} \frac{(6.28)}{|(x, v) - (\bar{x}, \bar{v})|^\beta} &\lesssim [1 + \langle v + \zeta \rangle t]^{2\beta} \\ &\times \left[\frac{1}{\langle v + \zeta \rangle} \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \frac{1}{w_0(v + \zeta)} \|w_0 f_0\|_\infty \right] \\ &+ [1 + \langle v + \zeta \rangle]^{2\beta} \times \left[\frac{1}{\langle v + \zeta \rangle^2} \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \frac{1}{w_0(v + \zeta)} \|w_0 f_0\|_\infty \right] \\ &\lesssim \sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} + \|w_0 f_0\|_\infty. \end{aligned} \quad (7.12)$$

Similarly,

$$\begin{aligned} \frac{(6.29)}{|(x, v) - (\bar{x}, \bar{v})|^\beta} &\lesssim \int_0^t [1 + \langle v + \zeta \rangle(t - s)]^{2\beta} e^{\varpi \langle v + \zeta \rangle^2 s} \|w f(s)\|_\infty \\ &\times \left[\sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} e^{-\varpi \langle v \rangle^2 s} \int_{\mathbb{R}_u^3} k_c(v, v + u) \frac{|f(s, x, v + u) - f(s, \bar{x}, v + u)|}{|x - \bar{x}|^{2\beta}} du + \frac{\langle v + \zeta \rangle}{w(v + \zeta)} \|w f(s)\|_\infty \right] d\zeta ds \\ &+ \int_0^t [1 + \langle v + \zeta \rangle]^{2\beta} e^{\varpi \langle v + \zeta \rangle^2 s} \|w f(s)\|_\infty \end{aligned}$$

$$\begin{aligned}
& \times \left[\sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} e^{-\varpi \langle v \rangle^2 s} \int_{\mathbb{R}^3_\zeta} \mathbf{k}_c(v, \bar{v}, u) \frac{|f(s, x, v + u) - f(s, x, \bar{v} + u)|}{|v - \bar{v}|^{2\beta}} \right. \\
& \quad \left. + \left(\frac{1}{\langle v + \zeta \rangle} + \frac{\langle v + \zeta \rangle}{w(v + \zeta)} \right) \|w f(s)\|_\infty \right] d\zeta ds \\
& \lesssim \langle v + \zeta \rangle^{2\beta} e^{\varpi \langle v + \zeta \rangle^2 t} \left\{ \|w_0 f_0\|_\infty \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] + \|w_0 f_0\|_\infty^2 \right\}, \quad (7.13)
\end{aligned}$$

and

$$\frac{(6.30) + (6.31)}{|(x, v) - (\bar{x}, \bar{v})|^\beta} \lesssim \langle v + \zeta \rangle^{2\beta} e^{\varpi \langle v + \zeta \rangle^2 t} \left\{ \|w_0 f_0\|_\infty^2 \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] + \|w_0 f_0\|_\infty \right\}. \quad (7.14)$$

From (7.12), (7.13), (7.14), and Lemma 6.32, for $|(x, v) - (\bar{x}, \bar{v})| \leq 1$,

$$\begin{aligned}
& \langle v + \zeta \rangle^{-2\beta} e^{-\varpi \langle v + \zeta \rangle^2 t} \frac{|f(t, x, v + \zeta) - f(t, \bar{x}, \bar{v} + \zeta)|}{|(x, v) - (\bar{x}, \bar{v})|^\beta} \\
& \lesssim \|w_0 f_0\|_\infty^2 \left[\sup_{0 \leq s \leq T} \mathfrak{H}_{sp}^{2\beta}(s) + \sup_{0 \leq s \leq T} \mathfrak{H}_{vel}^{2\beta}(s) \right] + \mathcal{P}_3(\|w_0 f_0\|_\infty) \\
& \lesssim \|w_0 f_0\|_\infty^2 \left[\sup_{\substack{v \in \mathbb{R}^3 \\ 0 < |x - \bar{x}| \leq 1}} \langle v \rangle \frac{|f_0(x, v) - f_0(\bar{x}, v)|}{|x - \bar{x}|^{2\beta}} + \sup_{\substack{x \in \bar{\Omega} \\ 0 < |v - \bar{v}| \leq 1}} \langle v \rangle^2 \frac{|f_0(x, v) - f_0(x, \bar{v})|}{|v - \bar{v}|^{2\beta}} \right] \\
& \quad + \mathcal{P}_3(\|w_0 f_0\|_\infty).
\end{aligned}$$

Step 2 (Trivial case) Assume (2.6) or (2.8) do not hold with $v + \zeta$ and $\bar{x}$, respectively. In this case, we cannot split (6.1) into (6.2)–(6.5). Instead, we split

$$\begin{aligned}
& f(s, X(s; t, x, v + \zeta), V(s; t, x, v + \zeta)) - f(s, X(s; t, \bar{x}, \bar{v} + \zeta), V(s; t, \bar{x}, \bar{v} + \zeta)) \\
& \leq [f(s, X(s; t, x, v + \zeta), V(s; t, x, v + \zeta)) - f(s, X(s; t, \bar{x}, v + \zeta), V(s; t, \bar{x}, v + \zeta))] \quad (7.15)
\end{aligned}$$

$$+ [f(s, X(s; t, \bar{x}, v + \zeta), V(s; t, \bar{x}, v + \zeta)) - f(s, X(s; t, \bar{x}, \bar{v} + \zeta), V(s; t, \bar{x}, \bar{v} + \zeta))]. \quad (7.16)$$

When (2.6) does not hold, we can estimate (7.15) similar as (6.7) since $S_{(\bar{x}, \bar{x}, v + \zeta)}$ is not well-defined by (2.7). Similarly, when (2.8) does not hold, we can estimate (7.16) similar as (6.9) since $S_{(\bar{x}, \bar{v}, \bar{v}, \zeta)}$ is not well-defined by (2.9). Therefore, we obtain the same bound (2.22) even if (2.6) for $v + \zeta$ or (2.8) for $\bar{x}$ do not hold. $\square$

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REFERENCES

1. Y. Cao, C. Kim, and D. Lee, *Global strong solutions of the Vlasov-Poisson-Boltzmann system in bounded domains*, Arch. Rational Mech. Anal. **233** (2019), 1027–1130.
2. H. Chen, C. Kim, and Q. Li, *Local well-posedness of vlasov-poisson-boltzmann equation with generalized diffuse boundary condition*, J. Stat. Phys. **179** (2020), 535–631.
3. H. Chen and C. Kim, *Regularity of stationary Boltzmann equation in convex domains*, Arch. Rational Mech. Anal. **244** (2022), 1099–1222.
4. C. Cercignani, R. Illner, and M. Pulvirenti, *The mathematical theory of dilute gases*, Applied Mathematical Sciences, vol. 106, Springer-Verlag, New York, 1994.
5. L. Devillettes and C. Villani, *On the trend to global equilibrium for spatially inhomogeneous kinetic systems: The Boltzmann equation*, Invent. Math. **159** (2005), 245–316.
6. R. DiPerna and P. L. Lions, *On the Cauchy problem for Boltzmann equation: global existence and weak stability*, Ann. Math. **130** (1989), 321–366.
7. M. Do Carmo, *Differential Geometry of Curves and Surfaces*, Dover Publications, Mineola, New York, 2016.
8. R.-J. Duan, F. M. Huang, Y. Wang, and T. Yang, *Global well-posedness of the Boltzmann equation with large amplitude initial data*, Arch. Ration. Mech. Anal. **225** (2017), 375–424.
9. R.-J. Duan, F. M. Huang, Y. Wang, and Z. Zhang, *Effects of soft interaction and non-isothermal boundary upon long-time dynamics of rarefied gas*, Arch. Ration. Mech. Anal. **234** (2019), 925–1006.
10. R.-J. Duan, G. Ko, and D. Lee, *The Boltzmann equation with large-amplitude initial data and specular reflection boundary condition*, arXiv:2011.01503, 2020.
11. R.-J. Duan and Y. Wang, *The Boltzmann equation with large-amplitude initial data in bounded domains*, Adv. Math. **343** (2019), 36–109.
12. R. T. Glassey, *The Cauchy problem in kinetic theory*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1996.
13. Y. Guo, C. Kim, D. Tonon, and A. Trescases, *Regularity of the Boltzmann equation in convex domains*, Invent. Math. **207** (2017), 115–290.
14. Y. Guo, C. Kim, D. Tonon, and A. Trescases, *BV-Regularity of the Boltzmann equation in non-convex domains*, Arch. Rational Mech. Anal. **220** (2016), 1045–1093.
15. Y. Guo, *Classic solutions to the Boltzmann equation for molecules with angular cutoff*, Arch. Rational Mech. Anal. **169** (2003), 305–353.
16. Y. Guo, *The Vlasov-Maxwell-Boltzmann system near Maxwellians*, Invent. Math. **153** (2003), 593–630.
17. Y. Guo, *Decay and continuity of Boltzmann equation in bounded domains*, Arch. Rational Mech. Anal. **197** (2010), 713–809.
18. C. Imbert and L. Silvestre, *Global regularity estimates for the Boltzmann equation without cut-off*, J. Amer. Math. Soc. **35** (2022), 625–703.
19. C. Kim, *Formation and propagation of discontinuity for Boltzmann equation in non-convex domains*, Comm. Math. Phys. **308** (2011), 641–701.
20. C. Kim, *Boltzmann equation with a large external field*, Comm. PDE. **39** (2014), 1393–1423.
21. C. Kim and S. Yun, *The Boltzmann equation near a rotational local Maxwellian*, SIAM J. Math. Anal. **44** (2012), 2560–2598.
22. C. Kim and D. Lee, *Decay of the Boltzmann equation with the specular boundary condition in non-convex cylindrical domains*, Arch. Rational Mech. Anal. **230** (2018), 49–123.
23. C. Kim and D. Lee, *The Boltzmann equation with specular boundary condition in convex domains*, Comm. Pure Appl. Math. **71** (2018), 411–504.
24. G. Ko, C. Kim, and D. Lee, *Dynamical Billiard and a long-time behavior of the Boltzmann equation in general 3D toroidal domains*, Preprint.

25. G. Ko and D. Lee, *On C^2 solution of the free transport equation in a disk*, Kinet. Relat. Models <https://doi.org/10.3934/krm.2022031>.
26. G. Ko, D. Lee, and K. Park, *The large amplitude solution of the Boltzmann equation with soft potential*, J. Differ. Equ. **307** (2022), 297–347.
27. Y. Shizuta and K. Asano, *Global solutions of the Boltzmann equation in a bounded convex domain*, Proc. Japan Acad. **53A** (1977), 3–5.