SUPPORT VARIETIES WITHOUT THE TENSOR PRODUCT PROPERTY

PETTER ANDREAS BERGH, JULIA YAEL PLAVNIK, SARAH WITHERSPOON

Abstract. We show that over a perfect eld, every non-semisimple nite tensor category with nitely generated cohomology embeds into a larger such category where the tensor product property does not hold for support varieties.

1. Introduction

In the two recent papers [2, 3], we studied support varieties in the setting of nite tensor categories. When the cohomology of such a category is nitely generated—as conjectured by Etingof and Ostrik to be always true—then the varieties contain much homological information on the objects, and the theory resembles that for support varieties over group algebras and more general cocommutative Hopf algebras.

In [3], we focused on the tensor product property for support varieties. That is, given a nite tensor category C with nitely generated cohomology, we studied conditions under which the equality

$$V_C(X \otimes Y) = V_C(X) \cap V_C(Y)$$

holds for all objects $X, Y \in C$. It is well known that there are non-braided nite tensor categories where this property does not hold, as observed, for example, in [1, 12]. However, we showed in [3] that when the category is braided, the tensor product property holds for all objects if and only if it holds between indecomposable periodic objects. In general, the tensor product property is potentially a useful tool if one for example wants to use support varieties to classify the thick tensor ideals in the stable category, although there are examples of such classications in situations where the property fails; see, for example, [1, 8, 9].

In this paper, we show that when the ground eld is perfect, then every non-semisimple nite tensor category C with nitely generated cohomology embeds into one such cate-gory D where the tensor product property does not hold. This is true even if the tensor product property does hold in C. The category D that we construct is a crossed product category that is not braided; along the way we collect facts about such crossed product categories that may be of independent interest. It remains an open question whether the tensor product property always holds in the braided case.

Acknowledgments. P.A. Bergh would like to thank the organizers of the Representation Theory program hosted by the Centre for Advanced Study at The Norwegian ~~Academy of~~ Science and Letters, where he spent parts of fall 2022. J.Y. Plavnik was

2020 Mathematics Subject Classication. 16E40, 16T05, 18M05, 18M15.
Key words and phrases. Finite tensor categories; support varieties; tensor product property.

partially supported by NSF grant DMS-2146392 and by Simons Foundation Award 889000 as part of the Simons Collaboration on Global Categorical Symmetries. J.Y.P. would like to thank the hospitality and excellent working conditions at the Department of Mathematics at Universität Hamburg, where she has carried out part of this research as an Experienced Fellow of the Alexander von Humboldt Foundation. S.J. Witherspoon was partially supported by NSF grant 2001163.

All the authors would like to thank the anonymous referee for suggestions that made the paper better.

2. Preliminaries

We x a eld k that is not necessarily algebraically closed, together with a nite tensor k-category (C ;
; 1) in the sense of [6]. This means that C is a locally nite k-linear abelian category, with a nite set of isomorphism classes of simple objects. Moreover, every object admits a projective cover, and hence also a minimal projective resolution. Furthermore, there is a bifunctor
from C C to C , associative up to functorial isomorphisms, and called the tensor product. There is also a unit object 1 with respect to the tensor product, and (C ;
; 1) is a monoidal category. In particular, the tensor product satises the so-called pentagon axiom; see [6, Section 2.1]. The unit object is simple, and the monoidal structure is compatible with the abelian structure in that the tensor product is bilinear on morphisms. Finally, every object admits a left and a right dual in the sense of [6, Section 2.10], so that C is rigid as a monoidal category.

The rigidity of C has important consequences; we mention three of them. First of all, by [6, Proposition 4.2.1], the tensor product is biexact, that is, exact in each argument. Secondly, by [6, Proposition 4.2.12], the projective objects form a two-sided ideal in C , so that the tensor product between a projective object and any other object is again projective. Finally, by [6, Proposition 6.1.3], the projective and the injective objects of C are the same, so that the category is actually quasi-Frobenius.

Given objects M; N 2 C , we denote the graded k-vector space $^1_{n=0}$ $\mathrm{Ext}^n_C(M;N)$ by $\mathrm{Ext}_C(M;N)$. With the usual Yoneda product as multiplication, the space $\mathrm{Ext}_C(M;M)$ becomes a graded k-algebra, and of particular interest is the algebra $\mathrm{Ext}_C(1;1)$. This is the cohomology ring of C , and denoted by H(C). By [13, Theorem 1.7], this is a graded-commutative k-algebra. Since the tensor product is exact in the rst argument, the functor

M induces a homomorphism

$$H(C) \xrightarrow{'_M} \mathrm{Ext}_C(M;M)$$

of graded k-algebras, turning $\mathrm{Ext}_C(M;M)$ into a left and a right H(C)-module. Now since $\mathrm{Ext}_C(M;N)$ is a left $\mathrm{Ext}_C(N;N)$-module and a right $\mathrm{Ext}_C(M;M)$-module (again using the Yoneda product), we see that it is both a left and a right module over H(C), via $'_N$ and $'_M$, respectively. However, by [3, Lemma 2.2] the two module actions coincide for homogeneous elements, up to a sign. In particular, it makes no dierence whether we view $\mathrm{Ext}_C(M;M)$ as a left or as a right module over H(C).

Since the cohomology ring is graded-commutative, the graded k-algebra dened by q

$$H(C) = \begin{cases} H(C) & \text{if the characteristic of k is two,} \\ H^2(C) & \text{if not} \end{cases}$$

is commutative in the ordinary sense. We denote by $\mathfrak{m}_0$ the ideal $H^+(C)$ of this ring, that is, the ideal of $H^\bullet(C)$ generated by the homogeneous elements of positive degree. This is a maximal ideal, since $H^0(C) = \mathrm{Hom}_C(1, 1)$ is a field; it is a division ring since the unit object is simple, and commutative by the above discussion.

Definition. The support variety of an object $M \in C$ is

$$V_C(M) = \{\mathfrak{m}_0\} \cup \{\mathfrak{m} \in \operatorname{MaxSpec} H^\bullet(C) \mid \operatorname{Ker}\varphi_M \subseteq \mathfrak{m}\}$$

Note that the presence of $\mathfrak{m}_0$ in the definition of support varieties is superfluous when-ever M is nonzero, for then this maximal ideal automatically contains the homogeneous ideal $\operatorname{Ker}\varphi_M$. Without any finiteness condition on the cohomology of C, these support varieties may not contain any important homological information, and so we make the following definition.

Definition. The finite tensor category C satisfies the finiteness condition **Fg** if the cohomology ring $H(C)$ is finitely generated, and $\operatorname{Ext}_C^*(M, M)$ is a finitely generated $H(C)$-module for every object $M \in C$.

By [2, Remark 3.5], one can replace $H(C)$ by $H^\bullet(C)$ in this definition; the two versions are equivalent. It was conjectured by Etingof and Ostrik in [7] that every finite tensor category satisfies **Fg**, and this conjecture is still open. As shown in [2], when this finiteness condition holds, then the theory of support varieties becomes quite powerful, as in the classical case for modules over group algebras of finite groups.

In this paper, we are concerned with the question of whether support varieties respect tensor products, in the following sense.

Definition. The finite tensor category C satisfies the tensor product property for sup-port varieties if $V_C(M \otimes N) = V_C(M) \cap V_C(N)$ for all objects $M, N \in C$.

This definition makes perfect sense without assuming that C satisfies **Fg**. By [2, Proposition 3.3(v)], the inclusion $V_C(M \otimes N) \supseteq V_C(M) \cap V_C(N)$ always holds when C is braided, that is, when for all objects $M, N \in C$ there are functorial isomorphisms $b_{M,N} \colon M \otimes N \to N \otimes M$ that satisfy the hexagonal identities in [6, Definition 8.1.1]. In [1] and [12], examples are given of finite tensor categories where the tensor product property does not hold, in fact not even the above inclusion. These examples are then necessarily non-braided. It is an open question whether the tensor product property always holds in the braided case, or under the stronger requirement that C is symmetric, that is, when the braiding isomorphisms satisfy $b_{N,M} \circ b_{M,N} = 1_{M \otimes N}$ for all $M, N \in C$. Other than categories of modules of some types of Hopf algebras, the only case that has been completely settled is when the ground field is algebraically closed and of characteristic zero; over such a field, every symmetric finite tensor category satisfies the tensor product property, by [3, Theorem 4.9]. The proof provided relies on Deligne's classification of such categories as certain skew group algebras, from [5].

By [3, Theorem 3.6], when C is braided and satisfies **Fg**, the tensor product property holds if and only if the following holds for all $M, N \in C$: if $V_C(M) \cap V_C(N)$ is not trivial, that is, if $V_C(M) \cap V_C(N) \neq \{\mathfrak{m}_0\}$, then $M \otimes N$ is not projective. Consequently, if the tensor product property does not hold, then there must exist two nonprojective objects M, N whose tensor product $M \otimes N$ is projective, but for which $V_C(M) \cap V_C(N)$ is not trivial. They must be nonprojective since the variety of a projective object is necessarily trivial; see the paragraph following [2, Definition 3.1]. In the following result, we show that at least such a pair of objects with $M = N$ cannot exist.

Proposition 2.1. *Let k be a field and $(\mathcal{C}, \otimes, 1)$ a braided finite tensor k-category. Then an object $M \in \mathcal{C}$ is projective if and only if the n-fold tensor product $M^{\otimes n}$ is projective for some $n \geq 1$.*

Proof. If M is projective, then so is every tensor product $M^{\otimes n}$, since the projective objects form an ideal in $\mathcal{C}$. Conversely, suppose that $M^{\otimes n}$ is projective for some $n \geq 2$. Since $\mathcal{C}$ is rigid, the object M admits a left dual M^*, which implies that there exist morphisms

$$M \to M \otimes M^* \otimes M \to M$$

whose composition equals the identity on M; see [6, Definition 2.10.1]. Tensoring with $M^{\otimes(n-2)}$, and using the fact that $\mathcal{C}$ is braided, we obtain morphisms

$$M^{\otimes(n-1)} \to M^{\otimes n} \otimes M^* \to M^{\otimes(n-1)}$$

whose composition equals the identity on $M^{\otimes(n-1)}$. This implies that $M^{\otimes(n-1)}$ is a direct summand of $M^{\otimes n} \otimes M^*$, which is a projective object since $M^{\otimes n}$ is. Consequently, the object $M^{\otimes(n-1)}$ is also projective. Repeating the process, we eventually end up with M, which must then be projective.

Let us now in the last part of this section recall a construction that will play an important role in the main result. Suppose that $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ are two finite tensor k-categories. Their Deligne tensor product, denoted $\mathcal{C} \boxtimes \mathcal{D}$, is a k-linear abelian category that is universal with respect to right exact bifunctors on $\mathcal{C} \times \mathcal{D}$. In other words, there is a bifunctor $T \colon \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ of k-linear abelian categories, right exact in both variables, with the property that for every bifunctor $F \colon \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ of k-linear abelian categories, the following hold: if F is also right exact in both variables, then there exists a unique right exact functor $F' \colon \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$ of k-linear abelian categories, with the property that the diagram

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{D} \arrow[rr, "T"] \arrow[dr, "F"'] & & \mathcal{C} \boxtimes \mathcal{D} \arrow[dl, "F'"] \\
& \mathcal{A} &
\end{tikzcd}$$

commutes. The Deligne tensor product was introduced in [4]; it exists, is unique up to equivalence, and is again a finite tensor category. Moreover, the bifunctor T is actually exact in both variables; for details, we refer to [6, Sections 1.11 and 4.6].

Given objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, it is standard to denote the image in $\mathcal{C} \boxtimes \mathcal{D}$ of the object $(C, D) \in \mathcal{C} \times \mathcal{D}$ by $C \boxtimes D$. When we restrict the tensor product in $\mathcal{C} \boxtimes \mathcal{D}$ to such objects, we are basically using the original tensor products. Thus if $C, C' \in \mathcal{C}$ and $D, D' \in \mathcal{D}$, then

$$(C \boxtimes D) \otimes (C' \boxtimes D') = (C \otimes_{\mathcal{C}} C') \boxtimes (D \otimes_{\mathcal{D}} D')$$

where $\otimes$ denotes the tensor product in $\mathcal{C} \boxtimes \mathcal{D}$. The unit object in $\mathcal{C} \boxtimes \mathcal{D}$ is $1_{\mathcal{C}} \boxtimes 1_{\mathcal{D}}$. Moreover, there is an isomorphism

$$\mathrm{Hom}_{\mathcal{C} \boxtimes \mathcal{D}}(C \boxtimes D; C' \boxtimes D') \simeq \mathrm{Hom}_{\mathcal{C}}(C; C') \otimes_k \mathrm{Hom}_{\mathcal{D}}(D; D')$$

of vector spaces, and using this, one can show that

$$\mathrm{Ext}^*_{\mathcal{C}\boxtimes\mathcal{D}}(\mathbf{1}_{\mathcal{C}}\boxtimes\mathbf{1}_{\mathcal{D}};\mathbf{1}_{\mathcal{C}}\boxtimes\mathbf{1}_{\mathcal{D}}) \simeq \mathrm{Ext}^*_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}};\mathbf{1}_{\mathcal{C}}) \otimes_k \mathrm{Ext}^*_{\mathcal{D}}(\mathbf{1}_{\mathcal{D}};\mathbf{1}_{\mathcal{D}})$$

as graded k-algebras for $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}$ and $\mathbf{1}_{\mathcal{D}} \in \mathcal{D}$. In particular, there is an isomorphism

$$H^*(\mathcal{C}\boxtimes\mathcal{D}) \simeq H^*(\mathcal{C}) \otimes_k H^*(\mathcal{D})$$

of cohomology rings. Therefore, if the categories $\mathcal{C}$ and $\mathcal{D}$ both satisfy **Fg**, then we see immediately that $H^*(\mathcal{C}\boxtimes\mathcal{D})$ is finitely generated, so that at least half of **Fg** also holds for $\mathcal{C}\boxtimes\mathcal{D}$. However, if the ground field k is perfect, then by [10, Lemma 5.3] the Deligne tensor product satisfies **Fg** if and only if it holds for both $\mathcal{C}$ and $\mathcal{D}$. Moreover, in this situation, the Krull dimension of $H^*(\mathcal{C}\boxtimes\mathcal{D})$ is the sum of the Krull dimensions of $H^*(\mathcal{C})$ and $H^*(\mathcal{D})$.

3. The main result

In this main section, we show that every finite tensor category that satisfies **Fg** embeds into a finite tensor category that also satisfies **Fg**, but for which the tensor product property does not hold. The construction of the bigger category uses the Deligne tensor product, as well as the notion of crossed product categories that we recall next. As before, we fix a field k and a finite tensor k-category $(\mathcal{C};\otimes;\mathbf{1})$.

Suppose that a finite group G acts on $\mathcal{C}$ by tensor autoequivalences. This means that there exists a monoidal functor $\mathrm{Mon}(G) \to \mathrm{Aut}^{\otimes}(\mathcal{C})$, where $\mathrm{Aut}^{\otimes}(\mathcal{C})$ is the monoidal category of tensor autoequivalences on $\mathcal{C}$, and $\mathrm{Mon}(G)$ is the monoidal category whose objects are the elements of G, the only morphisms are the identity maps, and the monoidal product is the multiplication in G. For an element $\sigma \in G$, we denote by σ the corresponding tensor autoequivalence on $\mathcal{C}$, so that the action of σ on an object $M \in \mathcal{C}$ is $\sigma(M)$. Note that if $\tau \in G$ is another element, then by definition there is a coherent isomorphism $\sigma(\tau(-)) \simeq \sigma\tau$ in $\mathrm{Aut}^{\otimes}(\mathcal{C})$.

Following [14] and [11], when G acts on $\mathcal{C}$ as above, we define the crossed product category $\mathcal{C} \rtimes G$ as follows. As a k-linear abelian category, it is G-graded, and equal to $\mathcal{C}$ in each degree. Thus the objects in $\mathcal{C} \rtimes G$ are of the form $\oplus_{\sigma\in G}(M_\sigma;\sigma)$, with M_σ an object in $\mathcal{C}$ for each $\sigma \in G$, and a morphism from $\oplus_{\sigma\in G}(M_\sigma;\sigma)$ to $\oplus_{\sigma\in G}(N_\sigma;\sigma)$ is a sum $\oplus_{\sigma\in G}(f_\sigma;\sigma)$, where $f_\sigma \colon M_\sigma \to N_\sigma$ is a morphism in $\mathcal{C}$. We define the tensor product on homogeneous objects and morphisms by

$$(M;\sigma) \otimes (N;\tau) = (M \otimes \sigma(N);\sigma\tau);$$
$$(f;\sigma) \otimes (g;\tau) = (f \otimes \sigma(g);\sigma\tau):$$

In this way, the crossed product category becomes a G-graded finite tensor category, with unit object $(\mathbf{1};e)$, where e is the identity element of G. The construction is in some sense a categorification of skew group algebras. Note that $\mathcal{C}$ embeds as a finite tensor category into $\mathcal{C} \rtimes G$, via the assignment $M \mapsto (M;e)$, for $M \in \mathcal{C}$.

As an abelian category, the crossed product category is a Deligne product. Namely, let Vec_G be the category of G-graded finite dimensional vector spaces over k, and consider the functor $T \colon \mathcal{C} \boxtimes \mathrm{Vec}_G \to \mathcal{C} \rtimes G$ defined as follows. The image of an object $(M;V)$ is $\oplus_{\sigma\in G}(M^{\dim V_\sigma};\sigma)$, where M^n denotes the direct sum of n copies of M. Given a morphism $M \to N$ in $\mathcal{C}$, the image of the corresponding morphism $(M;V) \to (N;V)$ is the obvious morphism from $\oplus_{\sigma\in G}(M^{\dim V_\sigma};\sigma)$ to $\oplus_{\sigma\in G}(N^{\dim V_\sigma};\sigma)$. Finally, suppose that $: V \to W$ is a morphism in Vec_G, that is, a tuple $(\ )_{\in G}$ with each $: V \to W$ a linear transformation. Fixing bases for V and W, we may view as a matrix (c_{ij}) with each $c_{ij} \in k$, from which we obtain a corresponding morphism $M^{\dim V} \to M^{\dim W}$ in C given by the matrix $(c_{ij} 1_M)$. One now checks that T is a well dened bifunctor of k-linear abelian categories, and right exact in each variable. Moreover, given any k-linear abelian category A together with a k-linear bifunctor $F : C \ \mathrm{Vec}_G \to A$ which is right exact in each variable, we can construct a right exact functor $F^0 : C \rtimes G \to A$ as follows. Given an object $_{\in G}(M;)$ in $C \rtimes G$, let V be the G-graded vector space which is just k in each degree, and dene

$$F^0(_{\in G}(M;)) = F\ (_{\in G} M; V):$$

A morphism

$$_{\in G}(f;): \ _{\in G}(M;) \to \ _{\in G}(N;)$$

in $C \rtimes G$ induces a morphism between $(_{\in G} M; V)$ and $(_{\in G} N; V)$ in $C \ \mathrm{Vec}_G$, and we dene $F\ (_{\in G}(f;))$ to be the image under F of the latter. One now checks that F is a well dened functor of k-linear abelian categories, and that the diagram

$$\begin{tikzcd} C \ \mathrm{Vec}_G \arrow[r, "T"] \arrow[dr, "F^0"'] & C \rtimes G \arrow[d, "F"] \\ & A \end{tikzcd}$$

commutes. Furthermore, one checks that F^0 is unique with this property. This shows that $C \rtimes G$ is the Deligne product $C \ \mathrm{Vec}_G$ as an abelian category but not as a nite tensor category when we view Vec_G as a fusion category. After all, the monoidal structure in $C \ \mathrm{Vec}_G$ does not use the categorical G-action on C.

Since $C \rtimes G = C \ \mathrm{Vec}_G$ as a k-linear abelian category, the cohomology ring $H(C \rtimes G)$ is isomorphic to the tensor product $H(C) \ _k H(\mathrm{Vec}_G)$; this does not use the monoidal structures in the categories involved. Now as Vec_G is a fusion category, its cohomology ring is trivial, and so $H(C \rtimes G) \cong H(C)$. Consequently, when Fg holds for either C or $C \rtimes G$, then at least the cohomology ring of the other category is nitely generated. However, the following lemma shows that Fg holds for one of the categories if and only if it holds for the other. Moreover, the support varieties for the objects of $C \rtimes G$ are just unions of support varieties over C.

Lemma 3.1. Let k be a eld, $(C; ; 1)$ a nite tensor k-category with a categorical action from a nite group G, and $C \rtimes G$ the corresponding crossed product category. Then the following hold.

(1) There is an isomorphism $H(C \rtimes G) \cong H(C)$ of cohomology rings.

(2) C satises Fg if and only if $C \rtimes G$ does.

(3) If $_{\in G}(M;)$ is an object in $C \rtimes G$, then

$$V_{C \rtimes G}(_{\in G}(M;)) = \bigcup_{\in G} V_C(M)$$

when we use the isomorphism from (1) to replace $H(C \rtimes G)$ by $H(C)$.

Proof. We saw an argument for (1) above, but we now give an elementary argument for both (1) and (2). Namely, since the morphisms in $C \rtimes G$ respect the G-grading, the cohomology of C o G takes place in each individual degree. The projective objects are of the form 2G(P;), with each P projective in C , and a (minimal) projective reso-lution of an object 2G(M;) is of the form 2G(P;), with each P a (minimal) projective resolution of M. Therefore, given another object 2G(N;), there is a natural isomorphism

(y) Ext$_{C o G}$ (2G(M;); 2G(N;)) ' M 2G Ext$_C$ (M; N) ;

which is an isomorphism of rings when 2G(M;) = 2G(N;). Note that since the unit object in C o G is (1; e), it follows immediately that H (C o G) ' H (C), proving (1).

Suppose that C satises Fg. Then by (1) the cohomology ring H(C o G) is nitely generated. If X = 2G(M;) is an object of C o G, then using the above isomor-phism (y), we see that the cohomology ring H (C o G) acts on Ext$_{C o G}$ (X; X) in a way that respects the G-grading. That is, the action is induced by the action of H(C) on each Ext$_C$ (M; M). Since the latter is a nitely generated H(C)-module for each 2 G, we see that Ext$_{C o G}$ (X; X) is nitely generated as a module over H (C o G), and so C o G satises Fg. Conversely, if the crossed product category satises Fg, then H(C) is nitely generated by (1) again. Moreover, if M is an object of C , then Ext$_{C o G}$ ((M; e); (M; e)) is a nitely generated H(C o G)-module. Using the isomor-phism (y), we then see that Ext$_C$ (M; M) is nitely generated as a module over H(C), so that C satises Fg. This proves (2).

For (3), we use again that the cohomology of C o G respects the G-grading. Given an object (M;) 2 C o G concentrated in degree , consider the composition

H(C) ! H(C o G) $_{(M;)}$! Ext$_{C oG}$((M;); (M;)) ! Ext$_C$ (M; M)

of graded ring homomorphisms, where the outer ones are the isomorphisms from (y). The composition equals '$_M$, that is, the homomorphism M. Thus when we compute support varieties by using H (C), we see that V$_{C oG}$((M;)) = V$_C$ (M). For an arbitrary object 2G(M;) of C o G ,we then see that

V$_{C oG}$ (2G(M;)) = [2G V$_{C oG}$ ((M;)) = [2G V$_C$ (M);

since support varieties respect direct sums by [2, Proposition 3.3(i)].

The group G acts on the crossed product category C o G by tensor autoequivalences in a natural way. Namely, for an element 2 G, the action on objects and morphisms in C o G is given by

(2G(M;)) = 2G (M); 1 ; (2G(f;)) = 2G (f); 1 ;

where we have used the notation to denote the tensor autoequivalences on both C and C o G. The following result shows that when the tensor product property holds for C , then a twisted version holds for the crossed product category.

Proposition 3.2. Let k be a eld, and (C ; ; 1) a non-semisimple nite tensor k-category that satises the tensor product property for support varieties. Furthermore, let

G be nite group acting on C by tensor autoequivalences. Then for any objects (M;) and (N;) of C o G, concentrated in degrees and , the following holds:

$$V_{C\,oG}((M;)$$
$$(N;)) = V_{C\,oG}((M;)) \setminus V_{C\,oG}((N;))$$

Proof. By the denition of the tensor product in C o G and Lemma 3.1(3), we have

$V_{C\,oG}((M;)$ (N ;)) = = = = =

$V_{C\,oG}(M$
$(N);)\ V_C\ (M$
$(N))$
$V_C(M) \setminus V_C((N))$
$V_{C\,oG}((M;)) \setminus V_{C\,oG}\ \ (N);\ {}^1V_{C\,oG}$
$((M;)) \setminus V_{C\,oG}((N;))$

In general, it is not always the case that $V_{C\,oG}((N;))$ is equal to $V_{C\,oG}(N;)$, or equivalently (by Lemma 3.1(3)), that $V_C(N)$ is equal to $V_C((N))$. Therefore, the above proposition may be used to construct examples where the tensor product property does not hold. However, it turns out that it is in fact not necessary to assume that the tensor product property holds for C to construct such examples. Inspired by the twisted version of the tensor product property given in the proposition, we formalize such a class of examples in a larger context next. Specically, we will combine the Deligne tensor product with a crossed product of a specic kind. As we shall see, when the nite tensor category C that we start with is not semisimple (that is, not a fusion category), then the nite tensor category that we construct turns out not to satisfy the tensor product property.

Let C_2 = fe; g be the multiplicative group with two elements, where e is the identity. Consider the twisting map : C C ! C C given by interchanging the factors, that is, mapping an object (M; N) to (N; M), and similarly for morphisms. This is a bilinear functor, and exact in each variable. Composing with the biexact structure bifunctor T : C C ! C C , we use the universal property of the Deligne tensor product to obtain a unique right exact functor : C C ! C C making the diagram

$$\begin{array}{ccc} C\ C & \xrightarrow{\quad T \quad} & C\ C\ T \\ & \searrow \quad \nearrow & \\ & C\ C & \end{array}$$

commute. The functors T and are monoidal, hence so is , making it a functor of nite tensor categories. Moreover, from the diagram above we obtain

$$T = \ T\ = T\ \ = T$$

and from the universal property of T we may conclude that is the identity. Thus is an autoequivalence of order two, and there is a monoidal functor

$$\mathrm{Mon}(C_2)\ \ !\ \ \mathrm{Aut}$$
$$(C\ \ C)$$

mapping to . We shall say that C_2 acts on C C by interchanging factors, since for objects C; C 2 C there is an equality

$$(C \quad C^0) = \quad T \quad (C; C^0) \quad = T \quad (C^0; C) \quad = C^0 \quad C$$

We may now form the crossed product category $(C \ C) \text{ o } C_2$. When the ground eld k is perfect and C satises Fg, then as mentioned in Section 2, the Deligne tensor product $C \ C$ also satises Fg, by [10, Lemma 5.3]. Then in turn so does $(C \ C) \text{ o } C_2$, by Lemma 3.1(2). The following theorem, our main result, shows that if C is not a fusion category, that is, not semisimple, then $(C \ C) \text{ o } C_2$ does not satisfy the tensor product property for support varieties.

Theorem 3.3. Let k be a perfect eld and $(C; \ ; 1)$ a non-semisimple nite tensor k-category that satises Fg. Furthermore, let C_2 be the multiplicative group of order two, acting on $C \ C$ by interchanging factors. Then the nite tensor k-category $(C \ C) \text{ o } C_2$ satises Fg, but not the tensor product property for support varieties.

Proof. For simplicity, we denote the crossed product category $(C \ C) \text{ o } C_2$ by D. In the course of the proof, we shall be using the tensor products in all the three categories C, $C \ C$, and D. To distinguish them, we therefore denote them by $,$ $_1$, and $_2$, respectively.

We saw in the paragraph preceding the theorem that D satises Fg. Now, since C is not semisimple, we may choose a nonprojective object $M \ 2 \ C$, for example the unit object; if 1 were projective, then so would be every object $N \ 2 \ C$, since $N \ ' \ N \ 1$ and the projectives form an ideal. Choose a projective object $P \ 2 \ C$ for which there exists an epimorphism $P \ ! \ M$; there exists such an object since C has enough projectives. Note that P is nonzero since M is not projective. Let us denote the object $(P \ M; \ )$ of D by just X, where is the element of C_2 of order two. We shall show that

$$V_D(X \ _2 X) = V_D(X)$$

and consequently that the tensor product property for support varieties in D does not hold, since trivially $V_D(X) \setminus V_D(X) = V_D(X)$.

By [2, Corollary 4.2], since C satises Fg and M is not projective, the support variety $V_C(M)$ is not trivial. Then by [2, Proposition 6.2], the k-vector space $\mathrm{Ext}^n_C(M; M)$ is nonzero for innitely many $n \ 1$. Consider now the object $P \ M$ of $C \ C$. At the end of Section 2, we saw that there is an isomorphism

$$\mathrm{Ext}_{C \ C}(P \ M; P \ M) \ ' \ \mathrm{Ext}_C(P; P) \ _k \mathrm{Ext}_C(M; M)$$

of k-vector spaces, and so since P is nonzero we see that $\mathrm{Ext}^n_{C \ C}(P \ M; P \ M)$ must be nonzero for innitely many $n \ 1$. The Deligne product $C \ C$ satises Fg (again from the paragraph preceding the theorem), hence by using [2, Proposition 6.2 and Corollary 4.2] again we see that $P \ M$ is not projective in $C \ C$. This implies that $X = (P \ M; \ )$ is not projective in D, as explained in the proof of Lemma 3.1. Consequently, the support variety $V_D(X)$ is not trivial, again by [2, Corollary 4.2].

Now consider the object $X \ _2 X$. By denition of the tensor product in D, we obtain

$$
\begin{aligned}
X \ _2 X &= (P \ M) \ _1 (P \ M); \ ^2 = ((P \ M) \ _1 (M \ P); e) \\
&= ((P \ M) \ (M
\end{aligned}
$$

P); e)

where e is the identity element of C_2. Let us denote the objects P
M and M
P in C by Q_1 and Q_2, respectively; these are both projective, since the projective objects form

an ideal. As in the previous paragraph, there is an isomorphism

$$\begin{aligned}\mathrm{Ext}_{\mathcal{C}\boxtimes\mathcal{C}}(Q_1 \boxtimes Q_2; Q_1 \boxtimes Q_2) &\simeq \mathrm{Ext}_{\mathcal{C}}(Q_1; Q_1)\\ &\otimes_k \mathrm{Ext}_{\mathcal{C}}(Q_2; Q_2)\end{aligned}$$

of k-vector spaces, and so since Q_1 and Q_2 are projective in $\mathcal{C}$, we conclude this time that $\mathrm{Ext}^n_{\mathcal{C}\boxtimes\mathcal{C}}(Q_1 \boxtimes Q_2; Q_1 \boxtimes Q_2) = 0$ for all $n \geq 1$. Therefore, by [2, Proposition 6.2 and Corollary 4.2], the object $Q_1 \boxtimes Q_2$ is projective in $\mathcal{C} \boxtimes \mathcal{C}$. Again, as explained in the proof of Lemma 3.1, we now see that $X \otimes_2 X = (Q_1 \boxtimes Q_2; e)$ is projective in $\mathcal{D}$, hence the support variety $V_{\mathcal{D}}(X \otimes_2 X)$ is trivial. This shows that $V_{\mathcal{D}}(X \otimes_2 X) = V_{\mathcal{D}}(X)$.

In general, each factor in a Deligne tensor product embeds into it, with a structure preserving functor. Thus if $\mathcal{C}$ and $\mathcal{D}$ are finite tensor categories, then $\mathcal{C}$ embeds (as a finite tensor category) into $\mathcal{C} \boxtimes \mathcal{D}$ via $\mathcal{C} \to \mathcal{C} \boxtimes 1_{\mathcal{D}}$, and similarly for morphisms. Using this, we see that $\mathcal{C}$ embeds as a finite tensor category into $(\mathcal{C} \boxtimes \mathcal{C}) \rtimes C_2$ via $\mathcal{C} \to (\mathcal{C} \boxtimes 1; e)$. Consequently, Theorem 3.3 shows that over a perfect field, any finite tensor category that satisfies **Fg** embeds into one that also satisfies **Fg**, but not the tensor product property for support varieties—even when the tensor product property does hold for the original category.

Corollary 3.4. *Let k be a perfect field and $(\mathcal{C}; \otimes_{\mathcal{C}}; 1_{\mathcal{C}})$ a non-semisimple finite tensor k-category that satisfies* **Fg**. *Then $(\mathcal{C}; \otimes_{\mathcal{C}}; 1_{\mathcal{C}})$ embeds as a finite tensor category into a finite tensor k-category $(\mathcal{D}; \otimes_{\mathcal{D}}; 1_{\mathcal{D}})$ that also satisfies* **Fg**, *but not the tensor product property for support varieties.*

We end the paper with the following remark, and an open question.

Remark 3.5. (1) In the proof of Theorem 3.3, we constructed an object X in the crossed product category $\mathcal{D} = (\mathcal{C} \boxtimes \mathcal{C}) \rtimes C_2$, with the property that X is not projective, whereas the tensor product $X \otimes_2 X$ is (here $\otimes_2$ denotes the tensor product in $(\mathcal{C} \boxtimes \mathcal{C}) \rtimes C_2$, as in the proof). When the ground field k is algebraically closed, then this does not actually need the finiteness condition **Fg**; it only requires the original category $\mathcal{C}$ to be non-semisimple.

To see this, suppose first that $\mathcal{C}_1$ and $\mathcal{C}_2$ are finite tensor categories over such a field k, and take two nonzero objects $U \in \mathcal{C}_1; V \in \mathcal{C}_2$. Since k is algebraically closed, the simple objects of the Deligne product $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ are the objects $S_1 \boxtimes S_2$, where S_i is a simple object of $\mathcal{C}_i$. There is an isomorphism

$$\begin{aligned}\mathrm{Ext}_{\mathcal{C}_1\boxtimes\mathcal{C}_2}(U \boxtimes V; S_1 \boxtimes S_2) &\simeq \mathrm{Ext}_{\mathcal{C}_1}(U; S_1)\\ &\otimes_k \mathrm{Ext}_{\mathcal{C}_2}(V; S_2)\end{aligned}$$

of k-vector spaces, and so it follows that $U \boxtimes V$ is projective in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ if and only if both U and V are projective.

Returning to the proof of Theorem 3.3, start with a non-projective object $M \in \mathcal{C}$, and an epimorphism $P \to M$, with P projective in $\mathcal{C}$. In the proof, we used support varieties to show that the object $X = (P \boxtimes M;)$ is not projective in $\mathcal{D}$, but that $X \otimes_2 X$ is. However, when k is algebraically closed, then from the above we see that $P \boxtimes M$ is not projective in $\mathcal{C} \boxtimes \mathcal{C}$, and then $X = (P \boxtimes M;)$ is not projective in $\mathcal{D}$. On the other hand, in the last part of the proof we saw that the tensor product $X \otimes_2 X$ is of the form $(Q_1 \boxtimes Q_2; e)$, where Q_1 and Q_2 are projective in $\mathcal{C}$. Then using the above once more, we see that $Q_1 \boxtimes Q_2$ is projective in $\mathcal{C} \boxtimes \mathcal{C}$, and consequently $X \otimes_2 X = (Q_1 \boxtimes Q_2; e)$ is projective in $\mathcal{D}$.

(2) The crossed product category $(C\ \ C) \text{ o } C_2$ from Theorem 3.3 is not braided. This can be seen directly from the proof, by involving Proposition 2.1: the object X from the proof is not projective in $(C\ \ C) \text{ o } C_2$, but the tensor product X $_2 X$ is. One can also convince oneself in a more direct way. Namely, let M be an object in C, and denote by $_1$ the tensor product in $C\ \ C$, again as in the proof of Theorem 3.3. Then

$$
\begin{aligned}
(M\ \ 1;\ )\ {}_2\,(1\ \ 1;\ ) &= (M\ \ 1)\ {}_1\,(1\ \ 1);{}^2 \\
&= ((M\ \ 1)\ {}_1\,(1\ \ 1); e) = (M\ \ 1; e)
\end{aligned}
$$

whereas

$$
\begin{aligned}
(1\ \ 1;\ )\ {}_2\,(M\ \ 1;\ ) &= (1\ \ 1)\ {}_1\,(M\ \ 1);{}^2 = ((1\ \ 1)\ {}_1\,(1\ \ M); e) \\
&= (1\ \ M; e)
\end{aligned}
$$

The objects $(M\ 1; e)$ and $(1\ M; e)$ are isomorphic in $(C\ \ C) \text{ o } C_2$ if and only if the objects $M\ \ 1$ and $1\ \ M$ are isomorphic in $C\ \ C$. This is not the case in general.

In light of the remark, we ask the following question.

Question. Does every braided nite tensor category that satises Fg also satisfy the tensor product property for support varieties?

References

[1] D.J. Benson, S. Witherspoon, Examples of support varieties for Hopf algebras with noncommutative tensor products, Arch. Math. (Basel) 102 (2014), no. 6, 513{520.

[2] P.A. Bergh, J. Plavnik, S. Witherspoon, Support varieties for nite tensor categories: complexity, realization, and connectedness, J. Pure Appl. Algebra 225 (2021), no. 9, Paper No. 106705, 21 pp.

[3] P.A. Bergh, J. Plavnik, S. Witherspoon, Support varieties for nite tensor categories: the tensor product property, arXiv:2306.16082.

[4] P. Deligne, Categories tannakiennes, (French) [Tannakian categories], in The Grothendieck Festschrift, Vol. II, 111{195, Progr. Math., 87, Birkhauser Boston, Boston, MA, 1990.

[5] P. Deligne, Categories tensorielles, (French) [Tensor categories], Mosc. Math. J. 2 (2002), no. 2, 227{248.

[6] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, Tensor categories, Mathematical Surveys and Monographs 205, American Mathematical Society, Providence, RI, 2015.

[7] P. Etingof, V. Ostrik, Finite tensor categories, Mosc. Math. J. 4 (2004), no. 3, 627{654, 782{783.

[8] D.K. Nakano, K.B. Vashaw, M.T. Yakimov, On the spectrum and support theory of a nite tensor category, Math. Ann. (2023), https://doi.org/10.1007/s00208-023-02759-8.

[9] C. Negron, J. Pevtsova, Hypersurface support and prime ideal spectra for stable categories, Ann. K-Theory 8 (2023), no. 1, 25{79.

[10] C. Negron, J. Plavnik, Cohomology of nite tensor categories: duality and Drinfeld centers, Trans. Amer. Math. Soc. 375 (2022), 2069{2112.

[11] D. Nikshych, Non-group-theoretical semisimple Hopf algebras from group actions on fusion categories, Selecta Math. (N.S.) 14 (2008), no. 1, 145{161.

[12] J. Plavnik, S. Witherspoon, Tensor products and support varieties for some noncocommutative Hopf algebras, Algebras and Rep. Th. 21 (2018), no. 2, 259{276.

[13] M. Suarez-Alvarez, The Hilton-Eckmann argument for the anti-commutativity of cup products, Proc. Amer. Math. Soc. 132 (2004), no. 8, 2241{2246.

[14] D. Tambara, Invariants and semi-direct products for nite group actions on tensor categories, J. Math. Soc. Japan 53 (2001), no. 2, 429{456.

Petter Andreas Bergh, Institutt for matematiske fag, NTNU, N-7491 Trondheim, Norway

Email address: petter.bergh@ntnu.no

Julia Yael Plavnik, Department of Mathematics, Indiana University, Bloomington, Indiana 47405, USA, & Fachbereich Mathematik, Universität Hamburg, Hamburg 20146, Germany

Email address: jplavnik@iu.edu

Sarah Witherspoon, Department of Mathematics, Texas A&M University, College Station, Texas 77843, USA

Email address: sjw@math.tamu.edu