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A Novel Approach for Data-Free, Physics-Informed Neural Networks in Fluid Mechanics Using the Principle of Minimum Pressure Gradient

Ahmed Atallah *
Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA, 92093, USA

Abdelrahman A Elmaradny † and Haithem E Taha ‡
Department of Mechanical and Aerospace Engineering, University of California Irvine, Irvine, CA, 92697, USA

Fluid dynamics traditionally relies on the solution of the Navier-Stokes equations – a set of non-linear partial differential equations that represent Newton's second law for fluid motion. However, these equations can be challenging to solve numerically for complex flow scenarios. This paper introduces a novel approach, applying Physics-Informed Neural Networks (PINNs) to fluid mechanics problems by implementing the principle of minimum pressure gradient (PMPG), which asserts that an incompressible flow evolves from one instant to another such that the total magnitude of the pressure gradient in the domain is minimized. Leveraging Gauss' principle of least constraint, this method bypasses the need for a direct solution of the Navier-Stokes equations, and instead, translates the fluid mechanics problem into a minimization problem. We demonstrate the effectiveness of this data-free method by solving the problem of flow around a cylinder, resulting in a model that aligns well with the expected physics, and showing a negligible deviation from Euler's equation in a scenario of an inviscid, and incompressible flow.

I. Introduction

The study of fluid mechanics is of great significance, as it plays a vital role in numerous applications ranging from atmospheric and ocean dynamics to the design of transport and energy systems. Despite the advancements in this field, solving fluid dynamics problems remains a challenging task due to the complex nature of the governing equations, namely, the Navier-Stokes equations. The nonlinearity and coupling of these equations present substantial challenges, particularly for cases involving high-Reynolds number and turbulent flows. Traditionally, fluid mechanics problems have been solved either analytically or through computational fluid dynamics (CFD) simulations. While analytic solutions provide exact results, they are often limited to simplified cases. CFD simulations, on the other hand, offer the capability to model complex real-world scenarios, but they can be computationally expensive and often require a significant degree of expertise especially for turbulent flows. Such complexities in CFD underline the necessity for new approaches that can offer more precise, efficient, and comprehensive solutions for fluid flow analysis.

The recent revolution in the use of neural networks and machine learning tools has spawned the term Physics-Informed Neural Networks (PINNs) [1, 2]. PINNs represent a fusion of deep learning and physics-based modeling, offering a novel pathway to tackle complex fluid dynamics problems. In this approach, the laws of physics, encapsulated in differential equations like the Navier-Stokes, get directly integrated into the learning process of neural networks. This integration enables the networks to satisfy underlying physical principles, ensuring that the learned solutions are not only data-driven but also conform to the laws governing fluid flow. This approach finds a spatio-temporal solution that minimizes the residuals in the governing equations, boundary conditions, initial conditions, and measurements. That is, the physics problem is turned into a minimization problem: a least-squares problem. While the residual of the equation is an intuitive cost, which is the standard cost in PINNs, minimizing residuals (i.e., least-squares) results in a system of equations that have more solutions than the original equation [3].

On the other hand, the recently developed principle of minimum pressure gradient (PMPG) [4, 5] naturally provides the cost function that Nature seeks to minimize for incompressible flows; the principle asserts that an incompressible flow evolves from one time instant to another such that the total magnitude of the pressure gradient over the domain is

*PhD Candidate, Department of Mechanical and Aerospace Engineering, University of California San Diego, and AIAA Student Member
†PhD Student, Department of Mechanical and Aerospace Engineering, University of California Irvine
‡Assosiate Professor, Department of Mechanical and Aerospace Engineering, University of California Irvine, and AIAA Associate Fellow

minimized. It was proved that Navier-Stokes' equation is the first-order necessary condition for minimizing the pressure gradient. That is, a candidate flow that minimizes the pressure gradient at every instant, is guaranteed to naturally satisfy Navier-Stokes' equation. Hence, this principle turns a fluid mechanics problem into a minimization problem where the cost function is the total magnitude of the pressure gradient over the domain. If the flow field is parameterized by some parameters $P_1, ..., P_n$, the values (or the dynamics) of these parameters can be easily obtained by minimizing the cost.

The above discussion points to an unequivocal connection between PINNs and the PMPG for the mechanics of incompressible fluids; Simply, the PMPG provides the suitable cost for the PINN in this case. In return, the PINN formulation provides a convenient framework for applying the PMPG to large-scale problems where a parameterization of the flow field from physical intuition may not be clear; the PINN-formulation provides a natural and general parameterization of flow field, where the flow parameters $P_1, ..., P_n$ are the neural network parameters (e.g., weights and biases). In this work, we combine PINNs and PMPG and propose a PMPG-based PINN formulation. We show that this new approach is successful in solving inviscid 2D flow over a cylinder, without relying on training data. Moreover, the developed platform demonstrates promise in enhancing noisy data within incompressible flow fields. In this case, the network is trained to minimize deviations from the original field while also adhering to the physics of the problem, effectively eliminating noise-induced components. By including conservation of mass and momentum constraints, alongside boundary conditions during training, the noisy flow field data are corrected/filtered. A practical application involves filtering data obtained from Particle Image Velocimetry (PIV) measurements, underscoring the potential of PINNs to integrate physical knowledge into neural network-based modeling.

This paper is organized as follows. Section II presents the theoretical background, including the discussion of the governing equations and the equivalence to the Principle of Minimum Pressure Gradient in addition to the PINNs framework. In section III, we present the PMPG-based PINN formulation. Section IV details the numerical example of the inviscid incompressible flow over a 2D cylinder, showing the results and discussion obtained using our approach. Finally, Section V provides concluding remarks and future directions.

II. Theoretical Background

A. Governing Equations of Incompressible Flows

As a branch of dynamics, fluid dynamics follows the principles of conservation of mass and momentum. The continuity equation represents the conservation of mass. The mathematical formulation for the conservation of mass in the case of incompressible flows can be expressed as follows:

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

Here, $\mathbf{u}$ represents the velocity field vector.

The Navier-Stokes equations describe the conservation of momentum. They are a set of three coupled partial differential equations that articulate how the velocity field of a fluid evolves with respect to time. These equations account for the effects of pressure, viscosity, and external forces on the fluid flow. They are written as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} \tag{2}$$

where p is the pressure, ρ is the fluid density, and ν is the kinematic viscosity.

Applying the above equations subject to the appropriate boundary conditions such as the no-penetration and no-slip on the surface of the walls in addition to the initial conditions specific to the problem statement gives us the solution to the flow. Usually, solving these equations requires sophisticated mathematical and computational techniques due to their nonlinear and complex nature especially when turbulent effects are prevalent.

B. Principle of Minimum Pressure Gradient (PMPG)

In this inquiry, we leverage the recently developed Principle of Minimum Pressure Gradient (PMPG). Diverging from the traditional approach of employing Newtonian mechanics to derive the Navier-Stokes equations, we underscore the application of variational principles to describe the dynamics of the fluid flow. This departure from established methodologies is notable, given that previous attempts utilizing variational techniques were confined to ideal fluids within the purview of Hamilton's principle of least action [3, 6–8].

The recent advancement in fluid analysis relying on Gauss' principle of least constraint [9] in the recent works of Taha et al. [4, 5, 10] may provide a substantial progression that surpasses the limitations inherent in earlier applications of variational methods in fluid dynamics. In contrast to Hamilton's principle of least action, Gauss' principle of least constraint can account for non-conservative dissipative forces that don't come from a potential such as viscous forces. This remarkable difference provides a foundation for converting fluid mechanics problems into minimization problems, where the objective is to minimize Nature's cost function: the pressure gradient. The referred principle was able to solve the century-old problem of how lift over the airfoils without the need for a closure condition (e.g., a Kutta-like condition) even for smooth shapes without sharp trailing edges. [11–13].

The forces imposed on a body in analytical mechanics that lead to motion can be divided into (a) impressed or driving forces donated as $\mathbf{F}$ and (b) constraint forces donated as $\mathbf{R}$. The motion equation can be written as:

$$m\mathbf{a} = \mathbf{F} + \mathbf{R} \tag{3}$$

where $\mathbf{a}$ is the inertial acceleration. The constraint forces do not drive the motion, rather their sole role is to impose a constraint on the motion to maintain a certain geometric or kinematic trajectory. These forces are passive or workless and classic examples of those are the tension in a string for a pendulum or the normal forces for a body moving on a surface. Gauss' principle states that Nature minimizes the magnitude of the constrained forces $\mathbf{R}$ hence naming the principle least constraint. In mathematical formulation by Jacobi;

$$Z = \frac{1}{2}\sum m\Big(\mathbf{a} - \frac{\mathbf{F}}{m}\Big)^2 \tag{4}$$

Z must be minimum at every instant. The edges of relying on Gauss' principle of least constraint over using other variational principles are that it accounts for any arbitrary forces $\mathbf{F}$ even non-conservative ones. In addition, it can be applied instantaneously without the need to integrate over time. Moreover, it is a true minimum principle, in contrast to the mere stationary principle of least action (see [5] for more details).

Applying this principle for fluid mechanics where our inertial acceleration $\mathbf{a}$ becomes $\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}$, whereas the arbitrary driving force is the viscous term. In their work, Taha et al. [4, 5] proved that for an incompressible flow in the absence of an external pressure gradient, the pressure gradient term is workless hence, the only constraint force $\mathbf{R}$ whose sole role is to impose the continuity constraint i.e, the divergence-free condition (1) coming from the conservation of mass in addition to the no penetration boundary conditions dictated by the geometry:

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \text{on } \delta\Omega \tag{5}$$

Reformulating (4) for incompressible flows, and a continuum of particles is equivalent to minimizing the pressure gradient (the constraint force) $\mathcal{A}$ over the fluid domain (or the Appellian) at every instant of time:

$$\mathcal{A} = \frac{1}{2}\int_{\Omega} \rho\Big(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\nabla^2\mathbf{u}\Big)^2 d\mathbf{x} \tag{6}$$

This principle derived from fundamental principles of analytical mechanics provides an efficient way to transform a fluid mechanics problem into a minimization problem. Importantly, the PMPG can be applied independent of the Navier-Stokes equations and even extends to fluids subject to arbitrary forcing. It becomes hence a matter of parameterizing the flow with suitable parameters and use the standard tools of optimization.

C. Physics-Informed Neural Networks (PINNs)

Physics-Informed Neural Networks (PINNs) have emerged as an innovative blend of deep learning and physics-based modeling, playing a pivotal role in computational science, especially in fluid dynamics. The fundamental concept of PINNs involves integrating physical laws, often represented as differential equations, directly into the neural network architecture. This integration ensures that the network's predictions are not just data-driven but also conform to the physical principles governing the system. The latter is achieved by introducing a loss term, called the physics-informed loss, into the training process. This loss term is a function of the residuals of the governing equations when evaluated at the model's outputs.

Mathematically, the PINN model is formulated as follows:

$$\mathbf{y}(\mathbf{x}, t; \boldsymbol{\theta}) = F_{NN}(\mathbf{x}, t; \boldsymbol{\theta}) \tag{7}$$

Here, $\mathbf{y}$ is the output of the model, $\mathbf{x}$ is the input, t is time, and $\boldsymbol{\theta}$ represents the parameters of the neural network F_{NN}. The parameters $\boldsymbol{\theta}$ are determined by the training of the neural network to minimize the total loss function $\mathcal{L}$, where:

$$\mathcal{L} = \mathcal{L}_{data} + \lambda \mathcal{L}_{physics}, \tag{8}$$

Where $\mathcal{L}$ is composed of a data loss term $\mathcal{L}_{data}$ and a physics-informed loss term $\mathcal{L}_{physics}$, balanced by a factor λ. This cost function ensures that the PINN learns from the data while adhering to the physical constraints imposed by the governing equations.

The integration of physical principles in the form of differential equations enables PINNs to make predictions that adhere to the laws of physics. This feature makes them particularly suitable for fluid dynamics problems, where they can provide accurate predictions even in complex scenarios with sparse or noisy data. Additionally, the neural network's ability to learn complex patterns allows PINNs to adapt to a wide range of fluid dynamics challenges, from simulating turbulent flows to predicting behaviors around intricate geometries.

III. PMPG-based Physics-Informed Neural Networks

With the recent advancements in Neural Networks and their use in solving physics problems, emerged potential for solving fluid mechanics problems without relying on any data. The emergence of "Data-free" Physics Constrained Neural Networks represents a recent and intriguing direction of research in the field of Neural Networks for solving physics problems. Unlike traditional approaches that rely on partial differential equations and data-driven optimization, this variant of PINNs explores the minimization of a physical quantity as a replacement for data training inputs. For example, minimizing the potential energy in the works of Nguyen et al. demonstrated the applicability of this approach in structural mechanical problems [14], while Goswami utilized variational energy minimization for solving brittle fracture problems [15]. However, the question remained as to whether a similar framework could be established for fluid mechanics problems.

The integration of the Principle of Minimum Pressure Gradient (PMPG) with Physics-Informed Neural Networks (PINNs) represents step forward in this direction. This hybrid approach, PMPG-based PINNs, leverages the strengths of both methodologies to address complex fluid dynamics problems more effectively, exploiting a Nature-minimizing cost function.

In PMPG-based PINNs, the principle of minimum pressure gradient is integrated into the neural network's learning process as part of the physics-informed loss function. This integration not only guides the neural network to adhere to physical laws but also to specifically focus on minimizing the pressure gradient in the fluid flow. In this approach, we reformulate the Navier-Stokes equations in terms of the velocity field and its derivatives, effectively avoiding the explicit calculation of the pressure field. The network's architecture and training process remain similar to standard PINNs, but the cost function is augmented to include a term representing the PMPG:

$$\mathcal{L}_{PMPG} = \sum \left| \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \mathbf{u} \right|^2 \tag{9}$$

Note that for brevity and clarity, the notation $(\mathbf{x}, t; \boldsymbol{\theta})$, which typically indicates spatial coordinates, time, and neural network parameters, has been omitted from the velocity term. In this formulation, $\mathbf{u}(\mathbf{x}, t; \boldsymbol{\theta})$ represents the predicted velocity field output by the neural network parametrized by weights and biases of the neural network. The PMPG loss thus calculates the squared norm of the pressure gradient, aiming to minimize it over the flow domain in accordance with the PMPG.

The incorporation of PMPG into PINNs offers several advantages. Firstly, it provides a more targeted approach to learning fluid dynamics by focusing on a key characteristic of fluid behavior. This focus can lead to more accurate models, particularly in scenarios involving complex flow patterns. Secondly, it can potentially reduce the computational complexity by narrowing the scope of the optimization problem. As opposed to standard PINNs, where separate neural networks are often employed to model both the velocity field $\mathbf{u}$ and the pressure field p, PMPG-based PINNs introduce a more streamlined approach. In this method, the focus shifts predominantly to the velocity field $\mathbf{u}$ and its derivatives. By focusing on the velocity field, PMPG-based PINNs provide an efficient and less computationally demanding route to model complex fluid dynamics, making them particularly advantageous in scenarios where traditional computational methods might struggle.

By formulating a sum of loss functions that incorporate boundary conditions, and other physical constraints, the network can be guided towards satisfying the PMPG (Figure1). The infinite pools of solutions are narrowed down to a

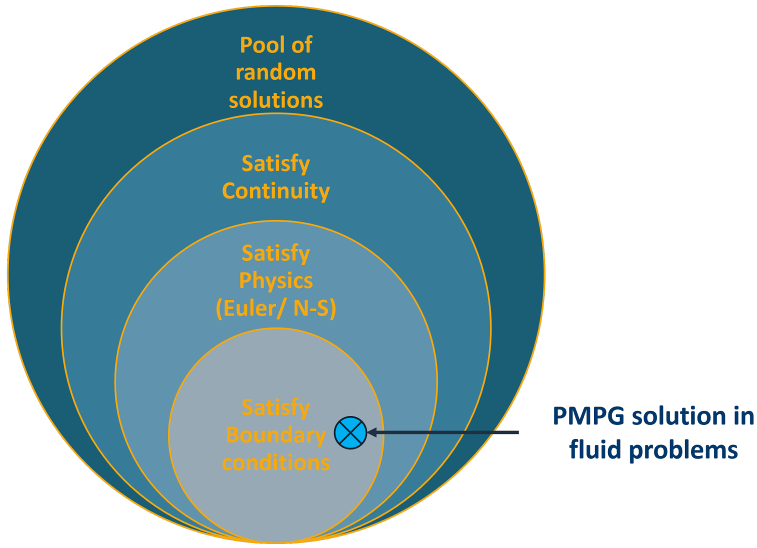

Fig. 1 A Venn diagram for how the PCNN will search for the optimal model narrowing down the possible solutions based on boundary and physics constraints and then minimizing the Pressure Gradient to find the correct solution

smaller pool by different layers of constraints. We constrain our solver to abide by the governing conservation laws, including conservation of mass and geometric aspects of the flow such as boundary conditions, and then we minimize the pressure gradient to converge to the solution without relying on any external data. This enables the formulation of an unconstrained optimization problem where the minimum pressure gradient is sought. Leveraging the power of neural networks and autodifferentiation to calculate the derivates, this approach holds promise for accurately solving fluid mechanics problems in a data-independent manner. This approach also eliminates the need to calculate the pressure field, thus avoiding complications associated with pressure-velocity coupling in the traditional Navier-Stokes equation solver.

IV. Numerical Example

As a numerical example to demonstrate the effectiveness of the PMPG-based PINN approach, we focus on the classic problem of steady, inviscid flow around a cylinder in a two-dimensional domain. Despite its extensive study in the past, this problem continues to be of relevance in various applications, ranging from aircraft wing flow to blood flow in arteries.

A. Total Loss Function

The flow domain around the cylinder is represented using a two-dimensional grid of 5000 randomly spaced points in the exterior of a cylinder with a unity radius in a squared domain of edge 5 times the cylinder radius, as shown in figure 2. The neural networks are trained to predict the stream function as an output with the inputs as the $x - y$ positions of the collocation points. The use of the stream function for building the neural network automatically satisfies the continuity equation 1. By differentiating the stream function, the velocity field components are obtained as follows:

$$u_x = \frac{\partial \psi}{\partial y}, \qquad u_y = -\frac{\partial \psi}{\partial x}. \tag{10}$$

The boundary conditions are defined such that the radial velocity component is zero on the cylinder surface, and the flow far from the cylinder is horizontal with a uniform velocity of unity. Both conditions are imposed as a hard constraint using the penalty method detailed in IV.B.

In the case of steady flow, the PMPG loss function of inviscid flow (9) is reduced to minimize the convective acceleration over the domain:

$$\mathcal{L}_{PMPG}(\mathbf{x};\boldsymbol{\theta}) = \sum |(\mathbf{u}\cdot\nabla)\mathbf{u}(\mathbf{x};\boldsymbol{\theta})|^2 \tag{11}$$

In other words, the PMPG is enforced during training by minimizing the convective accelerations from the predicted velocity field.

In order to impose the conservation of momentum, we take the curl of the momentum (Euler inviscid equation). In the case of steady flow, the curl of the convective acceleration is set to zero utilizing the vector calculus identity that the curl of the pressure gradient (a potential) equals zero, i.e.,

$$\nabla \times (\mathbf{u} \cdot \nabla)\mathbf{u} = 0 \tag{12}$$

This condition is also imposed as a hard constraint using the penalty method.

Taking into account all of the above, the total loss function is defined by

$$\mathcal{L}(\mathbf{x};\boldsymbol{\theta}) = \lambda_1 \mathcal{L}_{PMPG}(\mathbf{x};\boldsymbol{\theta}) + \lambda_2 \mathcal{L}_{\text{No penetration}}(\mathbf{x};\boldsymbol{\theta}) + \lambda_3 \mathcal{L}_{\text{Far Field boundary}}(\mathbf{x};\boldsymbol{\theta}) + \lambda_4 \mathcal{L}_{\text{Curl of acceleration}}(\mathbf{x};\boldsymbol{\theta}) \tag{13}$$

Here, $\mathcal{L}_{PMPG}$, $\mathcal{L}_{\text{No penetration}}$, $\mathcal{L}_{\text{Far Field boundary}}$, $\mathcal{L}_{\text{Curl of acceleration}}$ (i.e., momentum) are, respectively, the losses due to PMPG, no penetration boundary conditions, far-field boundary conditions, and the curl of acceleration; λ_1, λ_2, λ_3, and λ_4 are the corresponding hyperparameters. The boundary conditions and conservation of momentum are imposed using the penalty method and the conservation of mass is automatically satisfied due to the use of stream function in training.

B. Penalty Method

One of the most common challenges with a problem involving different cost functions is the trade-off between each term. How the solver chooses which term to favor in case of conflicts highly depends on the weight of the multiplier assigned to it λ. In some cases, some terms require being fully satisfied such as boundary conditions and governing conservation laws.

The penalty method is a technique commonly used to impose hard constraints in training neural networks by incorporating them into the loss function. Constraints are mathematical conditions that restrict the feasible solution space of a problem. In the context of Neural Networks, constraints can be applied to enforce desired properties or limitations on the model's outputs or parameters.

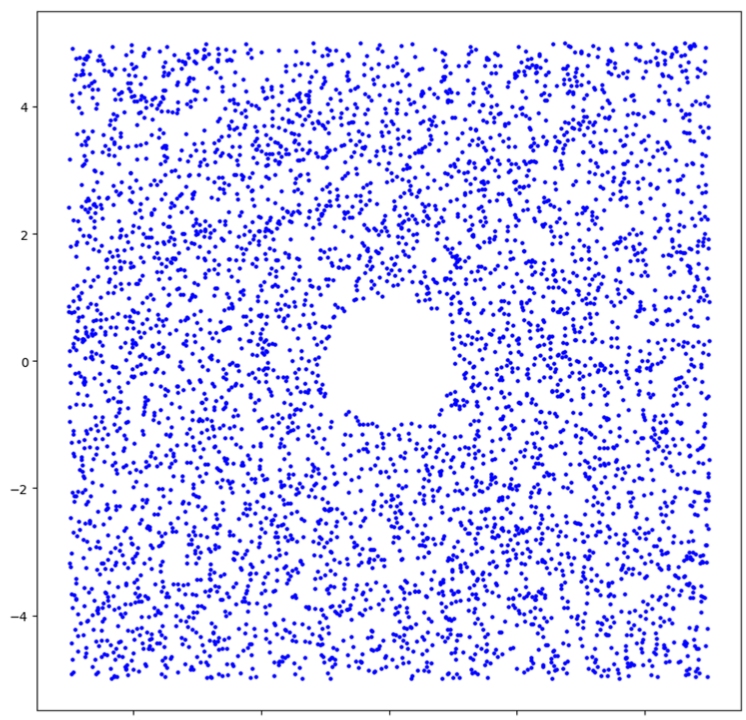

Fig. 2 Random collocation points

Mathematically, the penalty method introduces a penalty term in the loss function that quantifies the violation of the constraints. The penalty term is typically formulated as a function of the constraint violation, and its magnitude increases as the violation becomes larger. By including this penalty term, the Neural Networks is incentivized to minimize the constraint violation during the training process [16].

In order to impose constraints on neural networks, the penalty method is commonly employed, wherein a penalty term is introduced to the loss function. Let us consider a neural network model with an input vector $\mathbf{x}$ and parameters denoted as $\boldsymbol{\theta}$ and an objective function given by $\mathcal{L}(\mathbf{x};\boldsymbol{\theta})$. Additionally, a constraint function $C(\mathbf{x};\boldsymbol{\theta})$ is defined to specify the desired constraints on the parameters.

To incorporate the constraint violation, the penalty method modifies the loss function by adding a penalty term. The modified loss function can be expressed as:

$$\mathcal{L}_{\text{penalty}}(\mathbf{x},\boldsymbol{\theta}) = \mathcal{L}(\mathbf{x};\boldsymbol{\theta}) + \lambda\mathcal{P}(C(\mathbf{x};\boldsymbol{\theta})) \tag{14}$$

Here, λ represents the penalty coefficient controlling the importance of the constraint, and $\mathcal{P}(\cdot)$ denotes a penalty function that quantifies the violation of the constraint. The choice of the penalty function depends on the specific nature of the constraint being imposed.

For instance, when imposing a constraint on the parameter values to satisfy a feasible range, a quadratic penalty function can be employed. In this case, the modified loss function is given by:

$$\mathcal{L}_{\text{penalty}}(\mathbf{x},\boldsymbol{\theta}) = \mathcal{L}(\mathbf{x};\boldsymbol{\theta}) + \lambda\left(C(\mathbf{x};\boldsymbol{\theta}) - \text{feasible\_range}\right)^2 \tag{15}$$

where feasible_range denotes the allowable range of parameter values.

During the training process, the neural network is trained to minimize the modified loss function $\mathcal{L}_{\text{penalty}}(\mathbf{x};\boldsymbol{\theta})$. By incorporating the penalty term, the network is encouraged to find parameter values that optimize the objective while simultaneously satisfying the imposed constraint. The penalty coefficient λ determines the trade-off between the primary objective and the constraint violation. In the current model, λ was set to 10^6 and the feasible_range was set to 0.01.

The RMS error magnitude of boundary conditions was 0.7% and maximum error of 1.1% illustrating how efficient and accurate the application of the penalty method was. The accuracy can be further enhanced by increasing λ and decreasing feasible_range, but the error magnitudes for this problem were more than acceptable for our work.

Through iterative training, the neural network learns to strike a balance between minimizing the objective and adhering to the constraint, resulting in a model that satisfies the desired constraints while achieving commendable performance on the primary task.

This highlights how the penalty method is an effective approach for imposing constraints in neural networks by modifying the loss function to include a penalty term that quantifies the constraint violation. It was used in imposing the no penetration boundary condition and the far field horizontal flow. The same technique was implemented to satisfy the conservation of momentum by penalizing the curl of acceleration term in this problem.

C. Neural Network Architecture

Our implementation of PMPG-based PINNs uses Python's PyTorch library. The neural network consists of an input layer composed of two neurons (for spatial inputs (x, y)), two hidden 50-neuron layers activated by the Tanh function, and an output layer with one neuron for the predicted stream function value. Training occurs through the *Adam* optimization algorithm aiming to minimize the total loss function defined in (13).

D. Results and Discussion

The performance of the proposed PMPG-based PINNs in solving the inviscid, steady flow around a cylinder problem was quite satisfactory. This subsection provides a detailed discussion of the results and the implications of these findings.

The model accurately captured the flow characteristics around the cylinder. By avoiding explicit calculation of pressure, we achieved considerable computational efficiency without sacrificing accuracy. Our model's stream function output was compared to the theoretical exact stream function ψ, defined in polar coordinates by:

$$\psi(r,\theta) = U_\infty\left(r - \frac{a^2}{r}\right)\sin\theta \tag{16}$$

Here, r donates the distance from the origin, θ the angle from the x-axis, U_∞ the free-stream velocity, a the radius of the cylinder. The trained model errors compared to the theoretical equation are listed in Table 1.

Table 1 Error magnitudes

Metric	Value
Mean Error magnitude	0.017
RMS Error magnitude	0.023
Maximum Absolute Error	0.133

The Root Mean Square (RMS) error, was found to be 2.3% which is deemed quite satisfactory, given the simple structure of the Neural Network and its minimal computational requirements.

Figure 3 shows the velocity field of our model which when inspected visually is consistent with our physical comprehension of the problem.

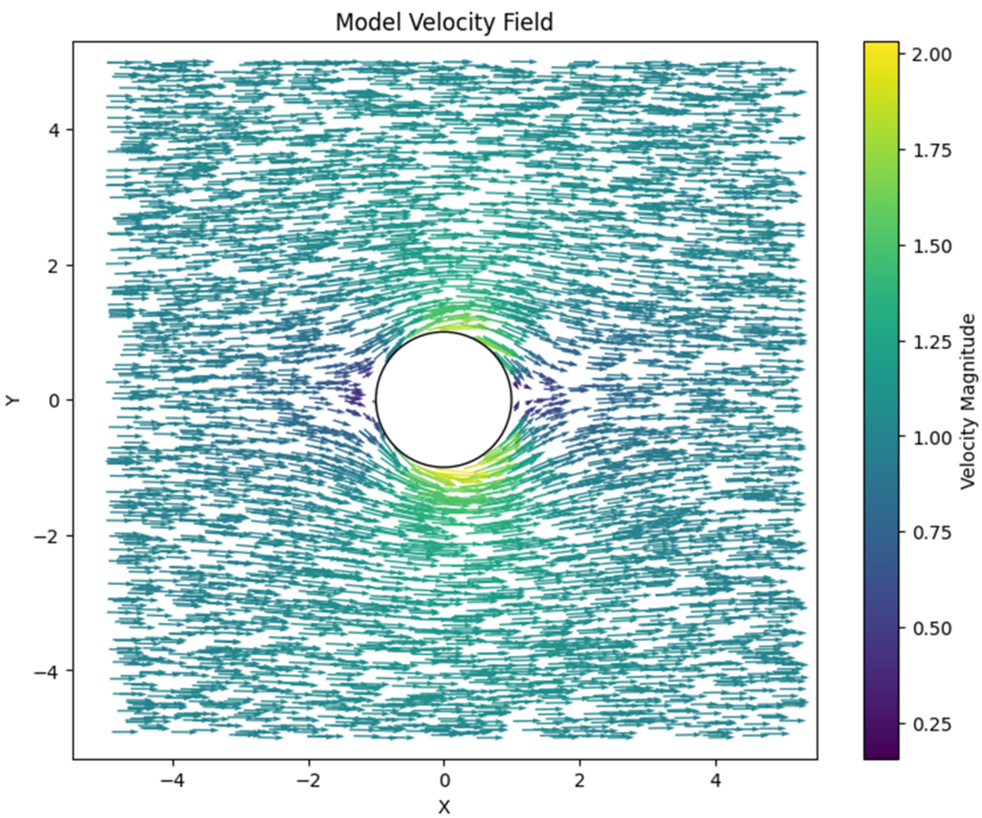

Fig. 3 Model velocity field

Several physics checks were implemented to validate our model. For an ideal inviscid, steady flow, the curl of convective acceleration should theoretically be zero especially after being enforced using the penalty method. In addition to that, the continuity equation was also satisfied with a negligible divergence of the generated velocity field as a result of choosing the stream function to be our neural network model. The vorticity of the flow was also calculated and is expected to be almost zero as well due to the absence of viscous forces effects. Table 2 shows the divergence of the velocity field, the curl of convective acceleration, and the velocity field as a physical check for our model. The resulting small errors provide some confidence in the fidelity of the model.

Consistently delivering strong performance for the numerical example of inviscid flow over a cylinder, our model demonstrates robustness and accuracy. The utilization of PMPG-based PINNs has the potential to advance fluid flow modeling, enhancing efficiency and flexibility in the process. Furthermore, these results pave the way for further exploration of PINNs in addressing more complex fluid dynamics problems.

Table 2 **Summary of physical checks**

Metric	Value
Mean Divergence of velocity	-1.93×10^{-10}
RMS Magnitude of Divergence of velocity	4.8×10^{-8}
Mean Curl of convective acceleration	0.0005
RMS Magnitude of Curl of convective acceleration	0.01741
Mean Vorticity	−0.00864
RMS Magnitude of Vorticity	0.04385

V. Conclusion

We introduced a new approach to modeling fluid flows using Physics-Informed Neural Networks (PINNs) guided by the Principle of Minimum Pressure Gradient (PMPG). The advantages of this methodology are threefold. Firstly, it eliminates the requirement for extensive data during training; our approach stands out as a data-free model by imposing constraints on the pool of solutions to single out the naturally selected candidate using variational principles of analytical mechanics. Secondly, the computational costs are significantly reduced due to the absence of the need for training a separate pressure model, presenting a more resource-friendly solution. Lastly, the minimization of a true physical quantity, guided by the PMPG, not only aligns our model with the fundamental principles of fluid dynamics but also capitalizes on Nature's intrinsic cost function. Through successful application to an inviscid, steady flow around a cylinder , our findings validate the efficacy of this approach. While our study focused on a relatively simple fluid mechanics problem, the results suggest that the proposed method holds promise for generalization to different boundary conditions and fluid flow scenarios. However, the full extent of its applicability to more complex fluid dynamics problems, such as turbulent flows or multi-phase flows, remains a subject for future exploration.

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