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Letter to the Editor

A sharp sufficient condition for mobile sampling in terms of surface density

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ABSTRACT

We provide a surface density threshold to guarantee mobile sampling in terms of the surface density of the set. This threshold is sharp if the Fourier transform is supported in either a ball or a cube, and further examples in the two-dimensional case where the result is sharp are given.

1. Introduction

This letter builds upon [9] and answers a question left open in that paper. For a set $\Gamma \subset \mathbb{R}^d$ of locally finite $\mathcal{H}^{d-1}$-measure, the mobile sampling problem concerns whether there exists a constant $C > 0$ such that

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq C \int_{\Gamma} |f|^2 d\mathcal{H}^{d-1}$$

for every function f in $L^2(\mathbb{R}^d)$ whose Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{2\pi i \langle x,\xi\rangle} dm_d(x)$ is supported in an origin symmetric convex set K.

This problem has been quite heavily studied in the last ten years, see e.g. [1,2,7,8,12] and references therein, following foundational work by Unnikrishnan and Vetterli, who formulated the problem precisely and coined the term mobile sampling. These papers contain a number of *precise* results characterizing mobile sampling sets within a variety of special families of curves and surfaces. Unnikrishnan and Vetterli [13,14] also introduced the surface density as analog of lower Beurling density for discrete sets that featured in classical results of Beurling and Kahane. The lower surface density $\mathbf{D}^-(\Gamma)$ of a set $\Gamma \subset \mathbb{R}^d$ is defined by

$$\mathbf{D}^-(\Gamma) = \liminf_{r\mapsto\infty} \inf_{x\in\mathbb{R}^d} \frac{\mathcal{H}^{d-1}(\Gamma \cap B(x,r))}{m_d(B(x,r))}.$$

The goal of [9] was to provide a general sufficient condition for mobile sampling in terms of the lower surface density of Γ alone that is valid for a large class of surfaces, in the spirit of one-dimensional results of Beurling and Kahane [4,11]. Unlike the one-dimensional case, one cannot expect a necessary condition given in terms of density for every curve Γ– see Proposition 4.1 in [7]. A general sufficient condition for sampling by discrete sets is given by Beurling's covering theorem [5].

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In [9], it was shown that there is a constant A_d such that for if $\Gamma \subset \mathbb{R}^d$ is a 'regular' surface satisfying $\mathbf{D}^-(\Gamma) > A_d \mathbf{W}(K)$, then it must be mobile sampling set. Here $\mathbf{W}(K)$ is the mean width of the symmetric convex set K. The value of A_d in [9] was $A_d = \frac{\omega_d}{\omega_{d-1}} \frac{3d^2}{2d+4}$, where ω_d is the volume of the d-dimensional unit ball, and it was left as an open problem if this constant could be improved. In this letter we resolve this issue and provide the sharp value of the constant $A_d = \frac{d}{2}\frac{\omega_d}{\omega_{d-1}}$. All relevant definitions will be given in the next section.

Theorem 1.1. *Suppose that* Γ *is* φ*-regular and*

$$\mathbf{D}^-(\Gamma) > \varphi(0) \cdot A_d \cdot \mathbf{W}(K), \text{ where } A_d = \frac{d}{2}\frac{\omega_d}{\omega_{d-1}}.$$

For every $1 \le p \le \infty$*, there exists a constant* $C > 0$ *such that*

$$\Bigg(\int_{\mathbb{R}^d} |f|^p dm_d\Bigg)^{1/p} \le C\Bigg(\int_{\Gamma} |f|^p d\mathcal{H}^{d-1}\Bigg)^{1/p} \tag{1.1}$$

for every $f \in L^p(\mathbb{R}^d)$ *whose (distributional) Fourier transform is supported in* K*.*

In Section 5 of [9] it was already shown that $A_d = \frac{d}{2}\frac{\omega_d}{\omega_{d-1}}$ is the sharp constant when $K = [-1,1]^d$. It is plausible that this constant could be sharp for a large class of origin symmetric convex sets. We are able to show that for $d = 2$ (in which case $A_2 = \pi/2$), this constant is sharp for any convex set K which is $\pi/2$-symmetric, which means that

$$(x_1, x_2) \in K \iff (-x_2, x_1) \in K.$$

This class of symmetric convex sets contains all ℓ^p-balls for $1 \le p \le \infty$.

Theorem 1.2. *Suppose* $d = 2$*, and* K *is a* $\pi/2$*-symmetric convex set. For every* $\delta > 0$*, there is a function* f *with* $f(0) = \|f\|_\infty = 1$*,* $\mathrm{supp}(\widehat{f}) \subset K$*, and a* φ*-regular set* Γ *with* $\varphi(0) = 1$*,* $\Gamma \subset \{f = 0\}$*, and*

$$\mathbf{D}^-(\Gamma) > \left(\frac{\pi}{2} - \delta\right)\mathbf{W}(K).$$

Additionally, we can show that in any dimension, the constant $A_d = \frac{d}{2}\frac{\omega_d}{\omega_{d-1}}$ is sharp for the Euclidean ball $B(0,1)$.

Proposition 1.3. *For every* $\delta > 0$*, there is a function* f *with* $f(0) = \|f\|_\infty = 1$*,* $\mathrm{supp}(\widehat{f}) \subset B(0,1)$*, and a* φ*-regular set* Γ *with* $\varphi(0) = 1$*,* $\Gamma \subset \{f = 0\}$*, and*

$$\mathbf{D}^-(\Gamma) > 2\left(\frac{d}{2}\frac{\omega_d}{\omega_{d-1}} - \delta\right).$$

In order to prove Theorem 1.1 we prove an improved bound on the density of the zero set of a Paley-Weiner class function (Proposition 3.1 below). Compared with [9], the main new tool is a modification of an averaging trick which has appeared before in studying the zero sets of analytic functions [3,10]. We consider it of independent interest that the technique provides the sharp bound when executed properly in the case when K is a Euclidean ball in all dimensions, and a wide class of convex sets when $d = 2$.

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2. Notation

For a positive integer k, let ω_k denote the volume of the k-dimensional unit ball in $\mathbb{R}^k$. Recall that $\omega_k = \frac{\pi^{k/2}}{\Gamma(k/2+1)}$. Let $E \subset \mathbb{R}^d$, we define

$$\mathcal{H}^k(E) = \lim_{\delta\to 0^+} \inf\Big\{\omega_k \sum_j r_j^k : E \subset \cup_j B(x_j, r_j) \text{ and } r_j \le \delta\Big\}.$$

Restricting $\mathcal{H}^k$ to a k-dimensional plane, $\mathcal{H}^k = m_k$, where m_k is the k-dimensional Lebesgue measure. Furthermore, $\mathcal{H}^{d-1}(\mathbb{S}^{d-1}) = d\omega_d$.

Let $K \subset \mathbb{R}^d$, be an origin symmetric compact convex set for $d \ge 2$. We set $\mathcal{PW}_p(K)$ to be the collection of functions in $L^p(\mathbb{R}^d)$ whose distributional Fourier transform is supported in K.

We define regular sets (and measures) in the same way as given in [9]. Let us suppose $\varphi : [0,1) \mapsto [0,\infty)$ is function continuous at 0. We say a measure μ is φ-regular if $\mu(B(x,r)) \le \varphi(r)\omega_{d-1}r^{d-1}$ for every $x \in \mathbb{R}^d$ and $r \in (0,1)$. We say a closed set $E \subset \mathbb{R}^d$ is called φ-regular if the measure $\mathcal{H}^{d-1}|_E$ is φ-regular.

For an origin-symmetric convex set K, we denote $\mathbf{W}(K)$ the mean width, which is given by $\mathbf{W}(K)=\frac{2}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})}\int_{\mathbb{S}^{d-1}} h_K(\theta)d\mathcal{H}^{d-1}(\theta)$, where $h_K(\theta) = \max_{x\in K}\langle x,\theta\rangle$ is the support function. If $B(0,R)$ denotes the origin-centered ball with radius R then $\mathbf{W}(B(0,R)) = 2R$, and $\mathbf{W}([-R,R]^d) = \frac{2R\omega_{d-1}}{\omega_d}$ for (see, e.g., Section 5 of [9]).

3. The proof of Theorem 1.1

Proposition 3.1. *If $f \in \mathcal{PW}_\infty(K)$ satisfies $\|f\|_\infty \le 1$ and $|f(0)| > 0$, then*

$$\limsup_{R\to\infty} \frac{\mathcal{H}^{d-1}(B(0,R)\cap\{f=0\})}{\omega_d R^d} \le A_d \cdot \mathbf{W}(K), \tag{3.1}$$

where $A_d = \frac{d}{2}\frac{\omega_d}{\omega_{d-1}}$.

The path from Proposition 3.1 to Theorem 1.1 follows the same lines as in [9], and so we briefly outline the process here. First, we note that (3.1) implies the averaged estimate

$$\limsup_{R\mapsto\infty} \frac{1}{\omega_d R^d}\int_0^R \mathcal{H}^{d-1}(B(0,r)\cap\{f=0\})\frac{dr}{r} \le \frac{A_d}{d}\mathbf{W}(K), \tag{3.2}$$

for the same value of A_d (this corresponds to an improved version of Proposition 3.1 of [9]).[1] Now, following Section 3 of [9] line-for-line with this new value of constant A_d yields the following improved version of Proposition 3.2 of [9]:

Proposition 3.2. *Fix a $\delta > 0$, $R_0 > 0$. There exists $\varepsilon > 0$ such that for every φ-regular set Γ and $f \in \mathcal{PW}_\infty(K)$ satisfies $\|f\|_\infty \le 1$ and $|f(0)| > 1/2$, there exists $R \ge R_0$ such that*

$$\frac{1}{\omega_d R^d}\int_0^R \mathcal{H}^{d-1}(\Gamma\cap B(x,r)\cap\{|f|\le\varepsilon\})\frac{dr}{r} \le \varphi(0)\Big(\frac{A_d}{d}\mathbf{W}(K)+\delta\Big).$$

With this result in hand, one completes the proof of Theorem 1.1 in precisely the same manner as in Section 3.4 of [9]. We now return to give the proof of Proposition 3.1.

4. Proof of Proposition 3.1

Proof of Proposition 3.1. For $\theta \in \mathbb{S}^{d-1}$, $R > 0$ and $\varepsilon \in (0,1/4)$, we consider the quantity

$$V_\theta = \int_{B(0,R)}\int_0^{\varepsilon R} \mathrm{card}(\{|s|\le t : f(x+s\theta)=0\})\frac{dt}{t}dm_d(x).$$

For ease of notation, start by fixing $\theta = (0,0,....,0,1) \in \mathbb{S}^{d-1}$ and consider $x = (x', x_d) \in \mathbb{R}^d$ where $x' \in \mathbb{R}^{d-1}$. Denote by $f_{x'}(t) = f(x',t)$, which has its one-dimensional distributional Fourier transform supported in the interval $[-2\pi h_K(\theta), 2\pi h_K(\theta)]$. Therefore, $f_{x'}$ extends to an entire function in $\mathbb{C}$ and $|f_{x'}(t+si)| \le e^{2\pi h_K(\theta)|s|}$ for $t,s\in\mathbb{R}$. Therefore, Jensen's formula yields that for any $x \in \mathbb{R}^d$,

$$\begin{aligned}\int_0^{\varepsilon R} &\mathrm{card}(\{|s|\le t : f_{x'}(x_n+s)=0\})\frac{dt}{t}\\ &\le \frac{1}{2\pi}\int_0^{2\pi} 2\pi h_K(\theta)\varepsilon R|\sin(\varphi)|d\varphi + \log\Big(\frac{1}{|f_{x'}(x_d)|}\Big)\\ &= 4\varepsilon R h_K(\theta) + \log\left(\frac{1}{|f(x)|}\right).\end{aligned}$$

[1] To derive (3.2), fix $R_0 > 1$ and split the integral $\frac{1}{\omega_d R^d}\int_0^R = \frac{1}{\omega_d R^d}\int_0^{R_0}\cdots + \frac{1}{\omega_d R^d}\int_{R_0}^R \cdots$. Proposition 3.1 of [9] (for instance) ensures that the first integral is of the order $O(\frac{R_0^d}{R^d})$, while Proposition 3.1 above implies that for every $\varepsilon > 0$, the second integral is $\le \frac{A_d}{d}\mathbf{W}(K)+\varepsilon$ if R_0 is large enough.

Substituting this bound into the definition of V_θ yields

$$V_\theta \le 4\epsilon R^{d+1}\omega_d h_K(\theta) + \int_{B(0,R)} \log\left(\frac{1}{|f(x)|}\right) dm_d(x). \tag{4.1}$$

On the other hand, V_θ equals

$$\int_{B^{d-1}(0,R)} \int_{-\sqrt{R^2-|x'|^2}}^{\sqrt{R^2-|x'|^2}} \int_0^{\epsilon R} \mathrm{card}(\{|s| \le t : f_{x'}(x_d+s)=0\})\frac{dt}{t} dm_1(x_d) dm_{d-1}(x').$$

Fix $x' \in B^{(d-1)}(0,R)$, and define a locally finite Borel measure μ on $\mathbb{R}$ via

$$\mu = \sum_{s\in\mathbb{R}\,:\, f_{x'}(s)=0} \delta_s.$$

Then

$$\begin{aligned}\int_{-\sqrt{R^2-|x'|^2}}^{\sqrt{R^2-|x'|^2}} \int_0^{\epsilon R}\int_{-t}^{t} d\mu(x_d+s)\frac{dt}{t} dm_1(x_d) \\ = \int_{-\sqrt{R^2-|x'|^2}-\epsilon R}^{\sqrt{R^2-|x'|^2}+\epsilon R} \int_0^{\epsilon R} \int_{-\sqrt{R^2-|x'|^2}}^{\sqrt{R^2-|x'|^2}} \mathbf{1}_{\{r-t\le x_d\le r+t\}} dm_1(x_d)\frac{dt}{t} d\mu(r).\end{aligned}$$

Now, note that if $r \in [-\sqrt{R^2-|x'|^2}+\epsilon R, \sqrt{R^2-|x'|^2}-\epsilon R]$, and $t \in (0,\epsilon R)$, then

$$[r-t, r+t] \subset [-\sqrt{R^2-|x'|^2}, \sqrt{R^2-|x'|^2}]$$

and so

$$\begin{aligned}\int_{-\sqrt{R^2-|x'|^2}}^{\sqrt{R^2-|x'|^2}} \int_0^{\epsilon R}\int_{-t}^{t} d\mu(x_d+s)\frac{dt}{t} dm_1(x_d) \\ \ge 2\epsilon R \cdot \mu([-\sqrt{R^2-|x'|^2}+\epsilon R, \sqrt{R^2-|x'|^2}-\epsilon R]).\end{aligned} \tag{4.2}$$

Observing that

$$\sqrt{R^2-|x'|^2} - \epsilon R \ge \sqrt{(1-2\epsilon)R^2-|x'|^2} \text{ provided that } |x'| \le \sqrt{1-2\epsilon}\cdot R,$$

we infer that, with $\theta = (0,0,\dots,1)$, V_θ is at least

$$2\epsilon R \cdot \int_{B^{(d-1)}(0,\sqrt{1-2\epsilon}\cdot R)} \mathrm{card}(\{|r| \le \sqrt{(1-2\epsilon)R^2-|x'|^2} : f(x'+r)=0\}) dm_{d-1}(x').$$

Therefore, with a suitable rotation, we find that for every $\theta \in \mathbb{S}^{d-1}$,

$$V_\theta \ge 2\epsilon R \int_{\theta^\perp\cap B(0,\sqrt{1-2\epsilon}\cdot R)} \mathrm{card}(B(0,\sqrt{1-2\epsilon}\cdot R)\cap\{f=0\}\cap \ell_{y,\theta}) dm_{d-1}(y),$$

where $\ell_{y,\theta}$ is the line through y with direction θ. Combining this with (4.1) therefore yields

$$\begin{aligned}\int_{\theta^\perp\cap B(0,\sqrt{1-2\epsilon}\cdot R)} \mathrm{card}(B(0,\sqrt{1-2\epsilon}\cdot R)\cap\{f=0\}\cap \ell_{y,\theta}) dm_{d-1}(y) \\ \le 2h_K(\theta)\omega_d R^d + \frac{1}{2\epsilon R}\int_{B(0,R)} \log\left(\frac{1}{|f(x)|}\right) dm_d(x).\end{aligned} \tag{4.3}$$

The Crofton formula (e.g. [6, 3.2.26]) states that for any set $E \subset \mathbb{R}^d$ that is $(d-1)$-rectifiable,

$$\mathcal{H}^{d-1}(E) = \frac{1}{2\omega_{d-1}} \int_{\mathbb{S}^{(d-1)}} \int_{\theta^\perp} \mathrm{card}(E\cap \ell_{y,\theta}) dm_{d-1}(y) d\mathcal{H}^{d-1}(\theta).$$

Whence, integrating (4.3) over $\mathbb{S}^{d-1}$ with respect to the $\mathcal{H}^{d-1}$ measure yields that

$$\frac{\mathcal{H}^{(d-1)}(B(0,\sqrt{1-2\varepsilon}R)\cap\{f=0\})}{\omega_d R^d}$$
$$\leq \Big\{ \frac{d\omega_d}{2\omega_{d-1}}\mathbf{W}(K) + \frac{d}{2\omega_{d-1}}\frac{1}{2\varepsilon R^{d+1}}\int\limits_{B(0,R)}\log\Big(\frac{1}{|f(x)|}\Big)dm_d(x)\Big\}.$$

Regarding the second term on the right-hand side of this inequality, it follows from work of Ronkin [10] on functions with completely regular growth (see Lemma 4.4 of [9] for a concise proof) that

$$\lim_{R\to\infty}\frac{1}{R^{d+1}}\int\limits_{B(0,R)}\log\Big(\frac{1}{|f(x)|}\Big)dm_d(x)=0.$$

Therefore,

$$\limsup_{R\to\infty}\frac{\mathcal{H}^{(d-1)}(B(0,\sqrt{1-2\varepsilon}R)\cap\{f=0\})}{\omega_d R^d}\leq\frac{d\omega_d}{2\omega_{d-1}}\mathbf{W}(K).$$

Letting $\varepsilon\to 0$ completes the proof of the proposition. □

5. The sharpness of the bound

5.1. The general construction

Assume that K is a origin symmetric strictly convex body. For $x\in\partial K$ we put $\nu(x)$ to be the outward pointing unit normal vector to K.

Let us recall the polar body

$$K^\circ=\{y\in\mathbb{R}^d : \langle x,y\rangle\leq 1 \text{ for every } x\in K\},$$

which satisfies that

$$\|x\|_{K^\circ} := \inf\{\lambda\geq 0 : x\in\lambda K\}=h_K(x).$$

For a bounded continuous function $g:\partial K\to[0,\infty)$, put

$$\langle g\rangle=\frac{1}{\mathcal{H}^{d-1}(\partial K)}\int\limits_{\partial K}g(x)d\mathcal{H}^{d-1}(x),\text{ and }\mu=\frac{1}{\mathcal{H}^{d-1}(\partial K)}\sup_{y\in K^\circ}\int\limits_{\partial K}|\langle\nu(x),y\rangle|g(x)d\mathcal{H}^{d-1}(x).$$

Our first goal is to prove the following

Proposition 5.1. *For every $\varepsilon>0$, there exists*

1. *a continuous function $\varphi:[0,\infty)\to[0,\infty)$ with $\varphi(0)=1$, and*
2. *a bounded function f with $|f(0)|=\|f\|_\infty=1$, $\mathrm{supp}(\widehat{f})\subset K$,*
3. *a φ-regular set $\Gamma\subset\mathbb{R}^d$ with $\Gamma\subset\{f\equiv 0\}$,*

such that

$$\mathbf{D}^-(\Gamma)\geq\frac{2\langle g\rangle}{\mu}-\varepsilon.$$

Proof. For $N\in\mathbb{N}$, select $x_1,\dots,x_N$ uniformly and independently on ∂K, and consider the associated vectors $\nu_n=\nu(x_n)$ for $n=1,\dots,N$. For $\alpha>0$, consider the function

$$f(x)=\prod_{n=1}^{N}\cos\Big(2\pi\frac{\alpha g(x_n)}{N}\langle x,\nu_n\rangle\Big).$$

Observe that $f(0)=\|f\|_\infty=1$, and the Fourier transform of f is the N-fold convolution of the factors $\frac{1}{2}\big(\delta_{\frac{\alpha g(x_n)}{N}\theta_n}+\delta_{-\frac{\alpha g(x_n)}{N}\nu_n}\big)$, and therefore

$$\mathrm{supp}(\widehat{f})\subset\Big\{\frac{\alpha}{N}\sum_{n=1}^{N}g(x_n)\varepsilon_n\nu_n : \varepsilon_n\in\{-1,1\}\Big\}.$$

Our goal is to find (the largest) α to ensure that $\mathrm{supp}(\widehat{f}) \subset K$. Recall that, for any $x \in \mathbb{R}^d$ $\|x\|_K = \sup_{y\in\partial K^\circ}\langle x, y\rangle$, and $x \in K$ if and only if $\|x\|_K \le 1$. Therefore, we want to select $\alpha > 0$ so that, for any $y \in \partial K^\circ$,

$$\alpha \sup_{\varepsilon_n\in\{-1,1\}} \frac{1}{N}\sum_{n=1}^{N} g(x_n)\varepsilon_n\langle v_n, y\rangle \le 1,$$

or, in other words,

$$\alpha \cdot \frac{1}{N}\sum_{n=1}^{N} |\langle v_n, y\rangle| g(x_n) \le 1.$$

For $y \in \partial K^\circ$, the random variable $X_{n,y} = |\langle v_n, y\rangle| g(x_n)$ has mean

$$\mu_y = \frac{1}{\mathcal{H}^{d-1}(\partial K)}\int_{\partial K} |\langle \mu(x), y\rangle| g(x) d\mathcal{H}^{d-1}(x),$$

and its variance is may be crudely bounded independently of n in terms of the geometry of the convex body K and the $L^\infty(\partial K)$ norm of g. Since, $X_{n,y}$ are independent, Chebyshev's inequality yields that

$$\mathrm{Prob}\Big(\Big|\frac{1}{N}\sum_{n=1}^{N} |\langle v_n, y\rangle| g(x_n) - \mu_y\Big| > \delta\Big) \le \frac{C(K,g)}{N\delta^2}. \tag{5.1}$$

Now, for $v_1, \dots, v_N$ fixed on $\mathbb{S}^{d-1}$, the function $y \mapsto \frac{1}{N}\sum_{n=1}^{N} |\langle v_n, y\rangle| g(x_n)$ is Lipschitz continuous with Lipschitz constant bounded by $K := \|g\|_\infty$. Similarly, the function μ_y is Lipschitz continuous with constant $\le K$. Elementary volume considerations ensure that the set ∂K° can be covered by $C(\delta/K)^{-(d-1)}$ balls $B(y_j, \delta/K)$ with $y_j \in \partial K^\circ$. Therefore, if $\max_{y\in\partial K^\circ}\Big|\frac{1}{N}\sum_{n=1}^{N} |\langle v_n, y\rangle| g(x_n) - \mu_y\Big| > 3\delta$, then there must exist j with $\Big|\frac{1}{N}\sum_{n=1}^{N} |\langle v_n, y_j\rangle| g(x_n) - \mu_{y_j}\Big| > \delta$. Consequently, (5.1) ensures that

$$\mathrm{Prob}\Big(\max_{y\in\partial K^\circ}\Big|\frac{1}{N}\sum_{n=1}^{N} |\langle v_n, y\rangle| g(x_n) - \mu_y\Big| > 3\delta\Big)$$
$$\le \sum_j \mathrm{Prob}\Big(\Big|\frac{1}{N}\sum_{n=1}^{N} |\langle v_n, y_j\rangle| g(x_n) - \mu_{y_j}\Big| > \delta\Big) \to 0 \text{ as } N \to \infty.$$

Since $\mu = \sup_{y\in\partial K^\circ} \mu_y$, we conclude that for any $\delta > 0$, if N is chosen sufficiently large then there exist $x_1, \dots, x_N \in \partial K$ such that for any $\varepsilon_n \in \{-1, 1\}$,

$$\Big\|\frac{1}{N}\sum_{n=1}^{N} g(x_n)\varepsilon_n v_n\Big\|_K \le \mu + 3\delta, \tag{5.2}$$

and so if $\alpha = \frac{1}{\mu+3\delta}$, then $\mathrm{supp}(\widehat{f}) \subset K$. Repeating the Chebyshev inequality argument if necessary, we may additionally ensure that

$$\frac{1}{N}\sum_{n=1}^{N} g(x_n) \ge \frac{1}{\mathcal{H}^{d-1}(\partial K)}\int_{\partial K} g(x) d\mathcal{H}^{d-1}(x) - \delta = \langle g\rangle - \delta.$$

On the other hand, each factor $f_n(x) = \cos\big(2\pi\frac{\alpha}{N} g(x_n)\langle x, v_n\rangle\big)$ satisfies

$$\lim_{R\to\infty} \frac{\mathcal{H}^{d-1}(\{f_n = 0\} \cap B(0, R))}{\omega_d R^d} = \frac{2\alpha g(x_n)}{N}.$$

(See Lemma 5.1 of [9].) Since there are N factors f_n, and the nodal sets of each f_n intersect in a set of dimension $d-2$, we have that,

$$\lim_{R\to\infty} \frac{\mathcal{H}^{d-1}(\{f = 0\} \cap B(0, R))}{\omega_d R^d} = 2\alpha\frac{1}{N}\sum_{n=1}^{N} g(x_n) \ge \frac{2(\langle g\rangle - \delta)}{(\mu + 3\delta)}.$$

We cannot immediately conclude Lemma 5.1 as the set $\{f = 0\}$ is not φ-regular for a function φ with $\lim_{t\to 0^+}\varphi(t) = 1$. However by removing small regions where any of the planes in the sets $\{f_n = 0\}$ intersect, we obtain a set Γ that is φ-regular for some φ with $\lim_{t\to 0^+}\varphi(t) = 1$, and such that Γ has surface density at least $2\frac{\langle g\rangle}{\mu} - C'\delta$, where C' is an absolute constant, and $\{f \equiv 0\} \supset \Gamma$. This concludes the proof of Proposition 5.1. □

We first use Proposition 5.1 to prove Proposition 1.3.

Proof of Proposition 1.3. In order to verify Proposition 1.3 from Proposition 5.1, we take $g \equiv 1$ and recall the following well-known computation (see, for instance the end of Section 5 of [9]): if $|v| = 1$, then

$$\int_{\mathbb{S}^{d-1}} |\langle \theta, v\rangle| d\mathcal{H}^{d-1}(\theta) = 2\omega_{d-1}.$$

Since $\mathbf{W}(B(0,1)) = 2$, we conclude that $\mathbf{D}^{-}(\Gamma) \geq \frac{d\omega_d}{2\omega_{d-1}}\mathbf{W}(B(0,1)) - \varepsilon$, as required. □

5.2. Sharpness for any $\pi/2$-symmetric convex body if $d = 2$

We now prove Theorem 1.2 as a consequence of Proposition 5.1. Recall that a convex body is called $\pi/2$-symmetric if it is symmetric under a rotation by $\pi/2$, i.e.

$$(x_1, x_2) \in K \iff (-x_2, x_1) \in K$$

This condition implies origin symmetry, and that $h_K(\theta_1, \theta_2) = h_K(-\theta_2, \theta_1)$ for $(\theta_1, \theta_2) \in \mathbb{S}^1$.

By an approximation argument, in proving Theorem 1.2, we may assume that K is strictly convex. We again will set $g \equiv 1$ in the statement of Proposition 5.1, and calculate, for $\theta \in \partial K^{\circ}$

$$\frac{1}{\mathcal{H}^1(\partial K)}\int_{\partial K} |\langle v_x, \theta\rangle| d\mathcal{H}^1(x) = \frac{2}{\mathcal{H}^1(\partial K)}\int_{(\partial K)_+} \langle v_x, \theta\rangle d\mathcal{H}^1(x)$$

where $(\partial K)_+ = \{x \in \partial K : \langle v_x, \theta\rangle \geq 0\}$.

Denote by $z_{\pm} \in \partial K$ the two points that satisfy $v_{z_{\pm}} \perp \theta$, and put $\theta^{\perp} = (-\theta_2, \theta_1)$. Since K is origin symmetric, $z_- = -z_+$, so the line segment $[z_-, z_+] \subset K$ contains 0, and has length $2|\nabla h_K(\frac{\theta^{\perp}}{|\theta|})|$. Put v to be a unit vector normal to the direction of the line segment $[z_-, z_+]$. Then the divergence theorem implies that

$$\int_{(\partial K)_+} \langle v_x, \theta\rangle d\mathcal{H}^1(x) = \int_{[z_-, z_+]} |\langle v, \theta\rangle| d\mathcal{H}^1 = 2|\nabla h_K(\tfrac{\theta^{\perp}}{|\theta|})||\langle v, \theta\rangle|.$$

It is an elementary geometry exercise to see that

$$|\nabla h_K(\tfrac{\theta^{\perp}}{|\theta|})||\langle v, \theta\rangle| = h_K(\tfrac{\theta^{\perp}}{|\theta|})|\theta|.$$

(Indeed, this boils down to the following fact: For $A > B > 0$ denote by T the right angle triangle with vertices $(0,0)$, $(0,B)$ and $(\sqrt{A^2 - B^2}, 0)$, then the outward unit vector to the hypotenuse of T has its first component equal B/A.)

Since $h_K(\frac{\theta^{\perp}}{|\theta|})|\theta| = h_K(\theta^{\perp})$ the quantity μ appearing in Proposition 5.1 equals

$$\mu = \sup_{\theta \in \partial K^{\circ}} \frac{4h_K(\theta^{\perp})}{\mathcal{H}^1(\partial K)}.$$

When combined with the classical fact that $\mathcal{H}^1(\partial K) = \pi \mathbf{W}(K)$, we see that, for any $\varepsilon > 0$, Proposition 5.1 ensures that there is a bounded function with Fourier transform supported in K, that vanishes on a regular set of density at least

$$\frac{\pi}{2}\frac{\mathbf{W}(K)}{\sup_{\theta \in \partial K^{\circ}} h_K(\theta^{\perp})} - \varepsilon.$$

To this point we have not used the $\frac{\pi}{2}$-symmetric assumption, but finally we observe that if K is $\frac{\pi}{2}$-symmetric then $\sup_{\theta \in \partial K^{\circ}} h_K(\theta^{\perp}) = 1$ (insofar as it implies that $h_K(\theta) = h_K(\theta^{\perp})$), and this completes the proof.

Data availability

No data was used for the research described in the article.

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