EFFECTIVE FRONTS OF POLYGON SHAPES IN TWO DIMENSIONS*

WENJIA JING†, HUNG V. TRAN‡, AND YIFENG YU§

Abstract. We study the effective fronts of first order front propagations in two dimensions ($n = 2$) in the periodic setting. Using PDE-based approaches, we show that for every $\alpha \in (0, 1)$, the class of centrally symmetric polygons with rational vertices (i.e., vectors in $\bigcup_{\lambda\in\mathbb{R}} \lambda\mathbb{Z}^2$) and nonempty interior is admissible as effective fronts for front speeds in $C^{1,\alpha}(\mathbb{T}^2, (0, \infty))$. This result can also be formulated in the language of stable norms corresponding to periodic metrics in $\mathbb{T}^2$. Similar results were known long time ago when $n \geq 3$ for front speeds in $C^\infty(\mathbb{T}^n, (0, \infty))$. The two dimensional case is much more subtle due to topological restrictions. In fact, for given $C^{1,1}(\mathbb{T}^2, (0, \infty))$ front speeds, the effective front is C^1 and hence cannot be a polygon. Our regularity requirements on front speeds are hence optimal. To the best of our knowledge, this is the first time that polygonal effective fronts have been constructed in two dimensions.

Key words. Homogenization; front propagation; effective Hamiltonian; effective fronts; centrally symmetric polygons; stable norm; limit shape

MSC codes. 35B40, 37J51, 49L25

1. Introduction.

In this paper, we are concerned with fine properties of the effective fronts of first order front propagations in oscillatory periodic environment in two dimensions. The front propagation is modeled by the following Hamilton-Jacobi equation with oscillatory periodic coefficient:

$$\begin{cases} u^\varepsilon_t + a\left(\frac{x}{\varepsilon}\right)|Du^\varepsilon| = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u^\varepsilon(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here, $\varepsilon \in (0, 1)$, $g \in \mathrm{BUC}(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$ is the initial data, where $\mathrm{BUC}(\mathbb{R}^n)$ is the space of bounded, uniformly continuous functions on $\mathbb{R}^n$. The coefficient $a : \mathbb{R}^n \to (0, \infty)$ determines the normal velocity in the underlying front propagation model. Throughout the paper we deal with a that is continuous, $\mathbb{Z}^n$-periodic and non-constant positive function. Denote by $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ the n-dimensional flat torus. We then write $a \in C(\mathbb{T}^n, (0, \infty))$.

We now give a minimalistic review of the literature on the qualitative homogenization of (1.1), which fits in the classical and standard framework (see [18, 10, 21]). As $\varepsilon \to 0$, the solution u^ε of (1.1) converges, locally uniformly on $\mathbb{R}^n \times [0, \infty)$, to the solution of the effective (homogenized) problem:

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.2}$$

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Here, $\overline{H}$ is the effective Hamiltonian determined by the Hamiltonian $H(x,p)=a(x)|p|$ of (1.1) in a nonlinear way as follows. For each $p\in\mathbb{R}^n$, $\overline{H}(p)$ is the unique real number such that the following equation admits a continuous viscosity solution

$$a(y)|p+Dv_p(y)|=\overline{H}(p) \quad \text{in } \mathbb{T}^n. \tag{1.3}$$

This is the well-known cell (ergodic) problem. Although $\overline{H}(p)$ is unique, v_p is not unique in general even up to additive constants. It is known that $\overline{H}(p)$ has the following inf-max representation formula (see e.g., [21])

$$\overline{H}(p)=\inf_{\phi\in C^\infty(\mathbb{T}^n)}\max_{y\in\mathbb{T}^n} a(y)|p+D\phi(y)| = \inf_{\phi\in C^1(\mathbb{T}^n)}\max_{y\in\mathbb{T}^n} a(y)|p+D\phi(y)|. \tag{1.4}$$

Clearly, $\overline{H}$ is convex, even, and positively homogeneous of degree 1. We sometime write $\overline{H}=\overline{H}_a$ to emphasize the dependence on the function a. Due to those properties of $\overline{H}_a$, its 1-sublevel set

$$S_a := \{p\in\mathbb{R}^n \,:\, \overline{H}_a(p)\le 1\}$$

belongs to $\mathcal{W}$, which denotes the collection of all convex compact sets in $\mathbb{R}^n$ that are centrally symmetric with nonempty interior.

Due to the 1-homogeneity of $\overline{H}_a$, it is determined by S_a. By *effective front* we mean the dual set D_a of S_a, defined by

$$D_a := \{q\in\mathbb{R}^n \,:\, q\cdot p\le 1,\ \forall\, p\in S_a\}. \tag{1.5}$$

D_a is also convex and it is known to be the large time average limit of the so-called reachable set that arises in the control representation of (1.1). See more discussions in Remark 1.3.

As in general homogenization theory, the function a in (1.1) models the periodic environment that rules the front propagation in the microscopic scale. In the limit as $\varepsilon\to 0$, the homogenized problem (1.2) captures the effects of the oscillatory periodic environment on front propagations in the macroscopic level. From both mathematical and practical point of views, it is very important and interesting to characterize certain finer details of the effective Hamiltonian $\overline{H}$, or equivalently, those of the set S_a in the current setting. For example, in combustion literature, the well-known G-equation is often used as another front propagation model, and the effective burning velocity associated to it is sometimes taken to be isotropic for convenience (see [17]). This strongly motivates the following question.

Question 1.1. For what kind of $W\in\mathcal{W}$ does there exist $a\in C(\mathbb{T}^n,(0,\infty))$ such that $S_a=W$?

Here, we set a in the regularity class $C(\mathbb{T}^n,(0,\infty))$ since this is most common in the homogenization theory of Hamilton-Jacobi equations. In environments of composite materials, piecewise constant functions (or more generally, $L^\infty(\mathbb{T}^n,(0,\infty))$ functions) are probably more suitable.

The above question is often called the realization problem, which remains largely open. Below we summarize what is known so far. Most of them were formulated in equivalent forms in terms of β functions in the Aubry-Mather theory or in terms of the stable norms of periodic metrics in geometry; see Remark 1.3.

1. When $n\ge 3$, all centrally symmetric polytopes with rational vertices and nonempty interior are realizable for some front speed $a\in C^\infty(\mathbb{T}^n,(0,\infty))$.

This was first studied in the seminal work [13], and completed by [3, 2, 16, 15] via different approaches. Hence, the set of realizable convex sets are dense in $\mathcal{W}$. Very little is known about finer properties of S_a except along some irrational directions (see [7] for instance).

2. When $n = 2$, the result in [8] implies that ∂S_a is C^1 if $a \in C^{1,1}(\mathbb{T}^2)$ due to some topological restrictions in two dimensional spaces and due to the fact that the initial value problem of the ODE system $\dot{\xi} = V(\xi)$ has a unique solution for $V \in \mathrm{Lip}(\mathbb{R}^2)$. If we assume further that $a \in C^\infty(\mathbb{T}^2, (0,\infty))$, then there are other restrictions on S_a: for example, [4] yields that S_a cannot be a strictly convex set other than a disk.

A very natural question arising from (2) above is: for $n = 2$, are all centrally symmetric polygons with rational slopes and nonempty interior realizable if we lower the regularity of a to $C^{1,\alpha}(\mathbb{T}^2, (0,\infty))$ for $\alpha \in (0,1)$? See [6, 5] for more questions on possible shapes and differentiability properties of S_a. Hereafter, a polygon is said to be *centrally symmetric with rational slopes* if it can be expressed as

$$P = \{p \in \mathbb{R}^n \,:\, \max_{1\le i\le m} |q_i \cdot p| \le 1\} \tag{1.6}$$

for m given rational vectors $\{q_i\}_{i=1}^m \subset \mathbb{R}^n$.

The following is our main result, which gives an affirmative answer to the above question.

Theorem 1.2. *Assume that* $n = 2$*. Then, for any* $\alpha \in (0,1)$ *and for any centrally symmetric polygon* P *with rational slopes and nonempty interior, there exists* $a \in C^{1,\alpha}(\mathbb{T}^2, (0,\infty))$ *such that*

$$S_a = P.$$

Remark 1.3. We can formulate the result of Theorem 1.2 in the language of stable norms corresponding to periodic metrics as follows. We view $a \in C(\mathbb{T}^2, (0,\infty))$ as a $\mathbb{Z}^2$-periodic function in $\mathbb{R}^2$, and it defines a Riemannian metric

$$g = \frac{1}{a(x)}(dx_1^2 + dx_2^2)$$

on $\mathbb{R}^2$ that is clearly periodic. Let $d_a(\cdot,\cdot)$ denote the distance function induced by this metric. The *stable norm* associated to g, or a, is well defined by

$$\|v\|_a = \lim_{\lambda\to\infty} \frac{d_a(0,\lambda v)}{\lambda}, \qquad v \in \mathbb{R}^2.$$

See [6] for more background. In particular, it was proved there that, for all $v \in \mathbb{R}^2$,

$$|\lambda\|v\|_a - d_a(0,\lambda v)| \le C$$

for a universal constant $C > 0$. In view of the connection between the stable norm $\|\cdot\|_a$ and the effective Hamiltonian $\overline{H}_a$, given a polygon P with rational slopes $\{q_i\}_{i=1}^m$, $S_a = P$ means that the unit ball $\overline{B}_1^a$ of the stable norm satisfies

$$\overline{B}_1^a = \mathrm{conv}\left(\{\pm q_i \,:\, 1 \le i \le m\}\right).$$

Here, $\mathrm{conv}(E)$ is the convex hull of a set E, i.e., the smallest convex set that contains E. It is standard to check that $\overline{B}_1^a$ is the dual set of S_a in $\mathbb{R}^n$; see (1.5). In [15], we wrote $\overline{B}_1^a$ as D_a. Thus, Theorem 1.2 implies that all centrally symmetric polygons with rational vertices and nonempty interior are realizable as unit balls of the stable norms associated to some $a \in C^{1,\alpha}(\mathbb{T}^2, (0,\infty))$.

Remark 1.4. Thanks to [23, Theorem 1.1], for $a \in C(\mathbb{T}^2, (0,\infty))$, ∂S_a cannot have an edge of irrational slope. Thus, our main result, combined with [23], shows that a polygon is an effective front for some $a \in C(\mathbb{T}^2, (0,\infty))$ if and only if it is centrally symmetric with rational vertices and nonempty interior.

In view of the duality between S_a and $\overline{B}_1^a$, Question 1.1 can also be reformulated in terms of unit balls of stable norms.

Question 1.5. For what kind of $W \in \mathcal{W}$ does there exist $a \in C(\mathbb{T}^n, (0,\infty))$ such that $\overline{B}_1^a = W$?

We mention that it was proved in [20] that there exists $a \in C^\infty(\mathbb{T}^2, (0,\infty))$ such that

$$\{\pm q_i \,:\, 1 \le i \le m\} \subset \partial \overline{B}_1^a, \tag{1.7}$$

i.e., the stable norm is partially prescribed. Our PDE based approach also provides a very simple proof of this fact. See Remark 2.3 at the end of Section 2.

It is worth mentioning that Questions 1.1 and 1.5 also appear in the first passage percolation literature, where the unit ball of the stable norm is called the limit shape. In the general stationary ergodic setting, that is $a : \mathbb{R}^n \times \Omega \to (0,\infty)$ being a stationary ergodic random field, it was proved in [9] that the limit shape exists and is a deterministic convex compact set in $\mathbb{R}^n$. Then, it was shown in [12] that any symmetric compact convex set C with nonempty interior is a limit shape corresponding to some stationary ergodic a. However, when a is restricted to the independent and identically distributed (i.i.d.) setting, Question 1.5 is completely open, and it is of great interests to study properties of the limit shape. We refer the readers to [1] for detailed discussions and a list of extremely interesting open problems. For example, it is expected that the n-dimensional cube is not a possible limit shape in the i.i.d. setting.

As an immediate consequence of Theorem 1.2, we obtain the following result, which also follows from the less delicate inclusion (1.7).

COROLLARY 1.6. *The two collections*

$$\left\{S_a \,:\, a \in C^\infty(\mathbb{T}^2, (0,\infty))\right\} \quad \textit{and} \quad \left\{\overline{B}_1^a \,:\, a \in C^\infty(\mathbb{T}^2, (0,\infty))\right\}$$

are both proper dense sets in $\mathcal{W}$.

Our proof of Theorem 1.2 is done by explicit construction and relies on the characterization (1.4). Similar to the proof in [15] for higher dimensional cases ($n \ge 3$), a rough idea to construct a is sort of clear: form a network of curves pointing to the rational directions $\{q_i\}_{i=1}^m$, and assign values of a appropriately in this network. The curves in this network serve as highways so that proper assignment of values of a here guarantees that q_i's are in the effective front, proving the lower bound. Let a be very small away from the network of highways so that, to check the upper bound, we can choose appropriate test functions in (1.4) and still concentrate on behaviors close to the network. In three dimensions this strategy is easy to carry out since we can easily choose disjoint straight lines as highways pointing to the directions $\{q_i\}_{i=1}^m$, thanks to the availability of space. In two dimensions, however, those highways always intersect and it is very delicate to design a near the intersection points to make everything still compatible.

We would like to mention that this paper belongs to an ongoing project of systematic studies of inverse problems in periodic homogenization of Hamilton-Jacobi equations (see [19, 14, 22]).

The rest of the paper is organized as follows. The proof of Theorem 1.2 is given in Section 2. Some auxiliary results are given in Appendix A.

2. Proof of Theorem 1.2. Let P be a centrally symmetric polygon with rational slopes $\{q_i\}_{i=1}^m$ of the form (1.6). Since $n = 2$, we can assume that the rational vectors $\{q_i\}_{i=1}^m \subset \mathbb{R}^2$ are arranged clockwise; see Figure 1. For each $i = 1, .., m$, there is a unique real number $\lambda_i > 0$, and a unique irreducible integer vector $(k_i, \ell_i) \in \mathbb{Z}^2$ so that

$$q_i = \lambda_i\,(k_i, \ell_i).$$

Note that by the definition (1.6), $\{q_i\}_{i=1}^m$ form normal vectors of half of the faces of P. By symmetry, we order the other half by

$$q_{m+i} = -q_i, \qquad 1 \le i \le m.$$

Let p_i be the vertex of P between q_i and q_{i+1} for $1 \le i \le 2m-1$. Then the vertices $\{p_i\}_{i=1}^m$ of P (in fact, half of them) are determined by

$$p_i \cdot q_i = p_i \cdot q_{i+1} = 1 \quad \text{and} \quad \max_{\substack{j\neq i,i+1\\ 1\le j\le m}} |q_j \cdot p_i| < 1, \qquad 1 \le i \le m. \tag{2.1}$$

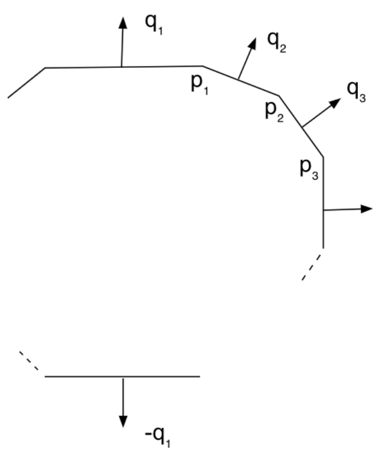

Fig. 1: Polygon P with vertices $p_1, p_2, \ldots, p_{2m}$

Lemma 2.1. *Suppose that* $\xi \in C^1([0,T], \mathbb{R}^2)$ *satisfies that*

$$\xi(T) - \xi(0) = (k, \ell) \in \mathbb{Z}^2.$$

Then for $\frac{1}{\lambda} := \int_0^T \frac{1}{a(\xi(t))} |\dot{\xi}(t)|\, dt$, *we have*

$$\overline{H}(p) \ge \lambda\, p \cdot (k, \ell), \qquad \forall p \in \mathbb{R}^2.$$

Proof. Without loss of generality we assume $\lambda > 0$. Owing to the inf-max formula (1.4), it suffices to show that for any fixed $p \in \mathbb{R}^2$ and for any $\phi \in C^\infty(\mathbb{T}^2)$,

$$M := \max_{x\in\mathbb{R}^2} a(x)|p + D\phi(x)| \geq \lambda\, p \cdot (k,\ell).$$

Let $u(x) = p \cdot x + \phi(x)$. Then, we compute and check

$$p \cdot (k,\ell) = u(\xi(T)) - u(\xi(0)) = \int_0^T (p + D\phi(\xi(t)) \cdot \dot{\xi}(t)\, dt \leq \frac{M}{\lambda}.$$

The desired inequality follows immediately. ∎

Proof of Theorem 1.2. Given a polygon $P \subset \mathbb{R}^2$ with rational slopes $\{q_i\}_{i=1}^M$, let $\{p_i\}_{i=1}^M$ (and the vectors opposite to them) be the vertices of P. Given $\alpha \in (0,1)$, our goal is to construct a speed function $a \in C^{1,\alpha}(\mathbb{T}^2,(0,\infty))$ so that the associated effective Hamiltonian $\overline{H} = \overline{H}_a$ satisfies the following properties:

(i) for all $p \in \mathbb{R}^2$, and for all $i = 1, 2, \dots, m$,

$$\overline{H}(p) \geq \max_{1\leq i\leq m} |q_i \cdot p|. \tag{2.2}$$

(ii) for all $i = 1, 2, \dots, m$,

$$\overline{H}(p_i) \leq 1. \tag{2.3}$$

In view of (1.6), the first inequality shows $\{\overline{H}(p) \leq 1\} \subset P$, i.e., $S_a \subset P$. On the other hand, since P is the convex hull of $\{\pm p_i\}_{i=1}^M$ and $\overline{H}(p)$ is convex and even, the second inequality above implies $\overline{H}(p) \leq 1$ for all $p \in P$, i.e., $P \subset S_a$. This would prove Theorem 1.2.

Before diving into details, let us first present the basic idea of constructing the weight function a: for $1 \leq i \leq m$ and each prescribed direction q_i, choose a suitable periodic curve ξ_i with rotation vector parallel to q_i that form a network. By assigning suitable values of $a(x)$ along ξ_i, we can use Lemma 2.1 to obtain (2.2) in a rather straightforward way. The other inequality (2.3) is more subtle. To achieve that, for each i, the key is to build a periodic smooth subsolution v_i of

$$a(x)|p_i + Dv_i| \leq 1 \quad \text{on } \mathbb{R}^2.$$

Moreover, we need both ξ_i and ξ_{i+1} to be gradient flows of $u_i = p_i \cdot x + v_i$ associated to the common vertex p_i (see the paragraph below for motivations). Due to the two dimensional topological restrictions, ξ_i and ξ_{i+1} have to be tangent at their intersections. Accordingly, we need to let u_i equal to a function with multiple gradient flows starting from the intersection points. This is why we cannot simply use straight lines for ξ_i as in the three dimensional case, and the lower regularity of $a(x)$ is necessary. In order to glue the pieces together, we choose $u_i = u_j$ near intersection points $(\xi_i+\mathbb{Z}^2)\cap(\xi_j+\mathbb{Z}^2)$ and then properly extend to a neighborhood of the network using methods in the Appendix, which provides the value of $a(x)$ near the network by denoting $a(x) = \frac{1}{|Du_i|}$ near $\xi_i + \mathbb{Z}^2$ for each i. Finally, for each i, extend u_i to $\mathbb{R}^2$ and adjust $a(x)$ to make it close to 0 away from the network, which will ensure that u_i is a subsolution in the whole plane.

We would like to remark that the above construction method is more or less necessary. In fact, suppose that $a(x)$ is a continuous function with given form of $\overline{H}$.

By the classical Aubry-Mather theory and suitable approximations, we can show that for each $i = 1, 2, .., m$, there exists a Lipschitz continuous periodic viscosity solution v_i to

$$a(x)|p_i + Dv_i| = 1 \qquad \text{on } \mathbb{R}^2$$

and v_i has two periodic gradient flows ξ_i and ξ_{i+1} with rotation vectors parallel to q_i and q_{i+1} respectively.

Step 1. Creation of a suitable network. First we choose m lines $\{L_i\}_{i=1}^m$ in $\mathbb{R}^2$ such that L_i is parallel to q_i and, when projected to $\mathbb{T}^2$, no three lines intersect at the same point. Then, by (2.1), for every two distinct points x and y on L_i, we have that

$$|p_i \cdot (x-y)| > \max_{\substack{j \neq i-1, i \\ 1 \leq j \leq m}} |p_j \cdot (x-y)|.$$

Consider all integer translations of L_i, which form a network $\bigcup_{i=1}^m \left(L_i + \mathbb{Z}^2\right)$. Let

$$I = \text{the collection of all intersection points in } \bigcup_{i=1}^m \left(L_i + \mathbb{Z}^2\right).$$

Note that the intersection set I is $\mathbb{Z}^2$-periodic. Denote

$$d = \min\{|x-y| : \ x \neq y, \ x, y \in I\}.$$

Due to the rationality of q_i's, $I/\mathbb{Z}^2$ is finite and $d > 0$.

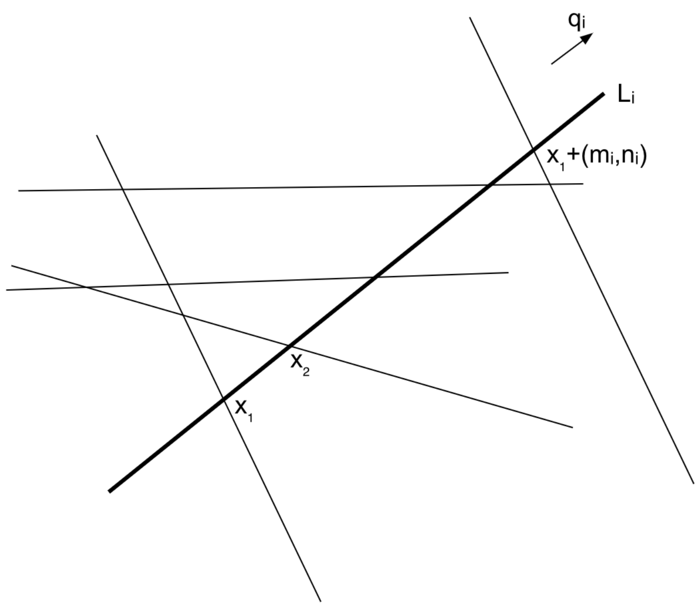

Fig. 2: Intersection points on L_i

Next, in a small neighborhood of each fixed intersection point in I, we perturb the two corresponding intersecting lines a bit to create gradient flows of an appropriate function. Since this is purely local, by linear transformations and translations, it suffices to show how to perform this procedure in a neighborhood of the origin $(0,0)$ provided that L_1, L_2 are the x_1-axis and x_2-axis, respectively.

Let $\alpha \in (0,1)$ as fixed in Theorem 1.2, pick $k \in \mathbb{N}$ so that

$$\alpha \leq 1 - \frac{1}{2k}.$$

Consider the potential function

$$u(x_1,x_2) = C_k \left(\frac{x_1^{4k}}{C_k} + x_2^2 \right)^{1-\frac{1}{4k}} + 2x_1,$$

where $C_k > 1$ is a positive constant to be determined. Clearly, $u \in C^{1,1-\frac{1}{2k}}(\mathbb{R}^2)$ and is C^2 away from the origin. We say a curve $\gamma : J \subset \mathbb{R} \to \mathbb{R}^2$ is a gradient flow of the potential function u if it solves

$$\dot\gamma(t) = Du(\gamma(t)), \qquad t \in J,$$

where J is the maximal open interval of the solution and, without loss of generality, we may assume $0 \in J$. Moreover, we say a gradient flow passes through the origin if $\gamma(0) = 0$.

To modify L_1 and L_2 locally near the intersection point (here set to be the origin $(0,0)$), we show that for the potential function u defined above, there are infinitely many gradient flows of it passing through the origin. We can then keep L_1, for instance, while replacing L_2 by (the image of) a different gradient flow of u near the origin.

LEMMA 2.2. *Fix $C_k > 2k(4k+1)$. Then, u has infinitely many distinct gradient flows passing through the origin.*

Proof. Consider the curve $\gamma_1(t) = (f(t),0)$, where f is the unique solution to

$$\begin{cases} \dot f(t) = 2 + C_k^{\frac{1}{4k}}(4k-1)f(t)^{4k-2}, \\ f(0) = 0. \end{cases}$$

Apparently, $\gamma_1 : J \to \mathbb{R}^2$ is a gradient flow of u passing through the origin, where J is the maximal open interval containing 0 of the solution.

To prove the lemma, it suffices to show that if $\xi(t) = (x_1(t), x_2(t)) : J \to \mathbb{R}^2$ is a gradient flow of u and

$$\xi(0) \in D := \{(a,b) \,:\, 0 < a < 1,\ 0 < b < a^{2k}\},$$

then

$$\xi(J \cap (-\infty,0)) \cap (0,\infty)^2 \subset D.$$

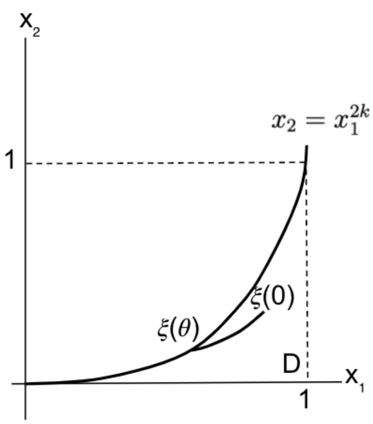

Fig. 3: Graph of ξ in D

Note that $x_1(t)$ and $x_2(t)$ are both increasing within D, the above statement would imply that ξ must intersect the flow γ_1 (the x_1 axis) at the origin. By translation in time, we obtain a different gradient flow of u passing through the origin.

Now suppose the statement fails, then there exists $\theta < 0$ such that

$$0 < x_2(\theta) = x_1^{2k}(\theta) \quad \text{and} \quad 0 < x_2(t) < x_1^{2k}(t) < 1 \quad \text{for } t \in (\theta, 0).$$

At θ,

$$\frac{C_k x_1^{2k-1}(\theta)}{1+4k} < \frac{u_{x_2}(x_1(\theta), x_2(\theta))}{u_{x_1}(x_1(\theta), x_2(\theta))} = \frac{x_2'(\theta)}{x_1'(\theta)} \leq 2k x_1^{2k-1}(\theta).$$

This contradicts the choice of C_k. The proof is complete. ∎

By this construction, we are able to form m periodic curves $\{\tilde{L}_i\}_{i=1}^m$ and their integer translations such that, for some small $r \in (0, \frac{d}{10})$,

1. $\tilde{L}_i = L_i$ away from the set $I_r = \{x \in \mathbb{R}^2 : d(x, I) \leq r\}$;
2. the set of intersection points remains the same, i.e., for $i \neq j$ and any integer vector $v \in \mathbb{Z}^2$,
$$\tilde{L}_i \cap (\tilde{L}_j + v) = L_i \cap (L_j + v);$$
Equivalently, $\tilde{L}_i \cap \tilde{L}_j = L_i \cap L_j$ when projected to $\mathbb{T}^2$.
3. given $i \neq j$ and an integer vector $v \in \mathbb{Z}^2$, if $\tilde{L}_i$ and $\tilde{L}_j + v$ intersect at $x = x_{i,j,v}$, then there exists a $C^{1,\alpha}$ function $u = u_{i,j,v}$ in $B_{\frac{r}{2}}(x)$ such that
   - $|Du(x)| \geq 1$ in $B_{\frac{r}{2}}(x)$;
   - within $B_{\frac{r}{2}}(x)$, $\tilde{L}_i$ and $\tilde{L}_j + v$ are two gradient flows of u that only intersect at x;
   - (periodicity) if two intersection points $x_{i,j,v} = x_{i',j',v'} + w$ for some $w \in \mathbb{Z}^2$, then
   $$u_{i,j,v}(x + w) = u_{i',j',v'}(x) \quad \text{for } x \in B_r(x_{i',j',v'}).$$
   This says that u is well defined on $I_{\frac{r}{2}}$ when being projected to the flat torus $\mathbb{T}^2$.

The perturbed network is henceforth denoted by

$$\Gamma = \bigcup_{1\le i\le m} (\tilde{L}_i + \mathbb{Z}^2).$$

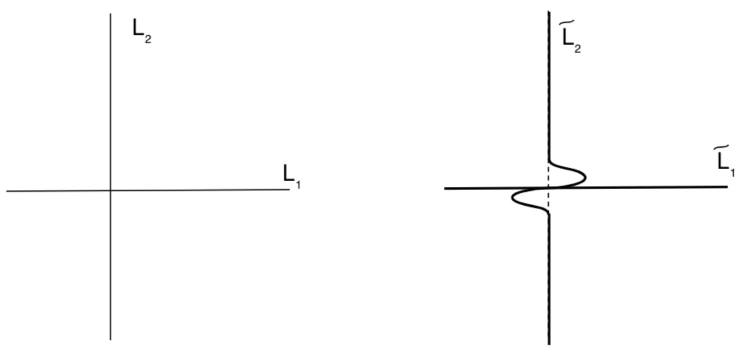

Fig. 4: Local perturbation at the intersection of L_1 and L_2

Step 2. Initial choice of a_0**.** We can choose $r_0 \in (0, \frac{r}{2})$ and $a_0 \in C^{1,\alpha}(\mathbb{T}^2, (0,\infty))$ such that a_0 is C^∞ away from the set I and satisfies the following conditions.

1. for each given intersection point $x = x_{i,j,v} \in I$ and the associated function $u = u_{i,j,v}$ from the above

$$a_0(y) = \frac{1}{|Du(y)|} \qquad \text{for } x \in B_{r_0}(x);$$

2. for every two intersection points x, y on $\tilde{L}_i$ for $1 \le i \le m$ (i.e., $x, y \in \tilde{L}_i \cap I$), the weighted length $l_i(x,y)$ between x and y along $\tilde{L}_i$ satisfies

$$l_i(x,y) := \int_0^1 \frac{1}{a_0(\xi(t))} |\dot{\xi}(t)|\, dt = |p_i \cdot (x-y)|. \tag{2.4}$$

Here, $\xi : [0,1] \to \tilde{L}_i$ is an arbitrary parametrization of $\tilde{L}_i$ between x and y. In particular, the weighted length of each period (i.e., from x to $x + (k_i, \ell_i)$) of $\tilde{L}_i$ is $\frac{1}{\lambda_i}$. The existence of a_0 is clear provided $r > 0$ is small enough. By Lemma 2.1,

$$\overline{H}_{a_0}(p) \ge \max_{1\le i\le m} |q_i \cdot p|, \qquad \forall p \in \mathbb{R}^2. \tag{2.5}$$

For $i = 1, 2, ..., m$, let $\xi_i : \mathbb{R} \to \tilde{L}_i$ be the smooth reparametrization of $\tilde{L}_i$ such that

$$|\dot{\xi}_i(t)| = \frac{1}{a_0(\xi_i(t))} \qquad \text{for } t \in \mathbb{R}.$$

For each $\delta > 0$ and $i = 1, 2, \ldots, m$, define $\tilde{L}_{i,\delta} = \{x : d(x, \tilde{L}_i) < \delta\}$ and let

$$\Gamma_\delta = \{x \in \mathbb{R}^2 : d(x,\Gamma) < \delta\} = \bigcup_{i=1}^m (\tilde{L}_{i,\delta} + \mathbb{Z}^2).$$

Owing to Lemma A.1 and the periodicity of Γ, there exists a universal $\delta_0 \in (0, r_0)$ such that for each $i = 1, 2, ..., m$, there exists $w_i \in C^{1,\alpha}(\tilde{L}_{i,\delta_0})$ such that w_i is C^∞ away from intersection points and

1. $\dot{\xi}_i(t) = Dw_i(\xi_i(t))$ for all $t \in \mathbb{R}$, i.e., ξ_i is the gradient flow of w_i;
2. $Dw_i(x) = Du_{i,j,v}(x)$ for $x \in B_{\delta_0}(x_{i,j,v})$, for each intersection point $x_{i,j,v}$ along $\tilde{L}_i$;
3. $\inf_{x \in \tilde{L}_{i,\delta_0}} |Dw_i(x)| > 0$.

Then, for $x \in \Gamma_{\delta_0}$, we define

$$a_0(x) = \frac{1}{|Dw_i(x - v)|} \quad \text{ if } x - v \in \tilde{L}_{i,\delta_0} \text{ for } v \in \mathbb{Z}^2.$$

Extend a_0 so that it belongs to $C^{1,\alpha}(\mathbb{T}^2, (0, \infty))$ and is smooth away from I.

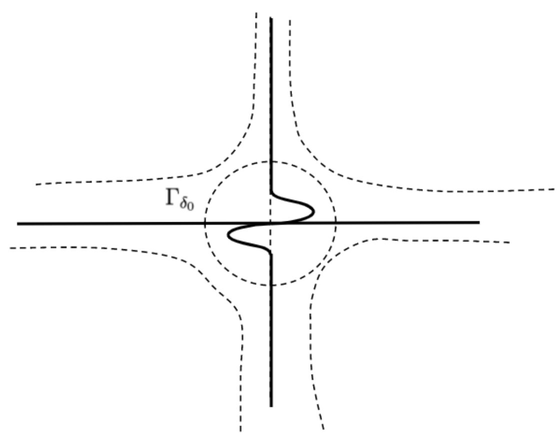

Fig. 5: Part of Γ_{δ_0}

Step 3. Adjustments of a_0. Next we need to construct $\tilde{a} \in C^{1,\alpha}(\mathbb{T}^2, (0, \infty))$ that is smooth away from I,

$$\tilde{a} = a_0 \quad \text{ on } \Gamma,$$

and, for $1 \le i \le m$, $\overline{H}_{\tilde{a}}(p_i) \le 1$. Since $\tilde{a}$ agree with a_0 of the previous section along $\tilde{L}_i$'s, the property (2.4) and, by Lemma 2.1, the inequality (2.5) are preserved. Hence, both (2.2) and (2.3) hold for $\tilde{a}$. This finishes the proof of Theorem 1.2.

Note that, owing to (2.1), for given $i \in \{1, 2, .., m\}$, the following points hold

- for $j = i, i + 1$ and two intersection points $x, y \in L_j$,

$$|p_i \cdot x - p_i \cdot y| = l_j(x, y);$$

- for $j \ne i, i + 1$ and every two distinct intersection points $x, y \in L_j$,

$$|p_i \cdot x - p_i \cdot y| = |p_i \cdot (x - y)| \le \max_{l \ne j-1,j} |p_l \cdot (x - y)| < |p_j \cdot (x - y)| = l_j(x, y).$$

In light of Remark A.2 and the periodicity of Γ, there exists $\mu_0 \in (0, \delta_0)$ such that for each $i = 1, 2, ..., m$, there exists a function $\tilde{u}_i \in C^{1,\alpha}(\Gamma_{\mu_0})$ such that

$$\begin{cases} \tilde{u}_i \in C^{1,\alpha}(\Gamma_{\delta_0}),\ \tilde{u}_i \in C^\infty(\Gamma_{\mu_0}\backslash I), \\ \inf_{\Gamma_{\delta_0}} |D\tilde{u}_i| > 0, \\ \tilde{u}_i - p_i \cdot x \quad \text{is } \mathbb{Z}^2\text{-periodic in } \Gamma_{\mu_0}, \\ |D\tilde{u}_i| \le |Dw_i| \quad \text{in } \Gamma_{\mu_0}, \end{cases}$$

and for any intersection point $x = x_{j,k,v} \in I$,

$$D\tilde{u}_i = Dw_j = Du_{j,k,v} \quad \text{in } B_{\mu_0}(x_{i,j,v}).$$

We extend $\tilde{u}_i - p_i \cdot x$ to $v_i \in C^{1,\alpha}(\mathbb{T}^2)$ such that v_i is C^2 away from I, and for $u_i = p_i \cdot x + v_i$,

$$u_i = \tilde{u}_i \quad \text{on } \Gamma_{\frac{\mu_0}{2}}.$$

Now let

$$K_1 = \max_{1\le i\le m} \max_{x\in\mathbb{R}^2} |Du_i(x)| \quad \text{and} \quad K_2 = \max_{x\in\mathbb{R}^2} a_0(x).$$

Choose $\phi(x) \in C^\infty(\mathbb{T}^2, (0,1])$ such that

$$\phi(x) = \begin{cases} 1 & \text{for } x \in \Gamma_{\frac{\mu_0}{4}}, \\ \frac{1}{K_1(1+K_2)} & \text{for } x \in \mathbb{R}^2\backslash\Gamma_{\frac{\mu_0}{2}}. \end{cases}$$

Finally, let

$$\tilde{a}(x) = \phi(x) a_0(x) \quad \text{for } x \in \mathbb{R}^2.$$

Then, for $i = 1, 2, ..., m$,

$$\begin{cases} \tilde{a}(x)|p + Dv_i(x)| \le \tilde{a}(x)|Dw_i(x)| = \phi(x) \le 1 & \text{for } x \in \Gamma_{\frac{\mu_0}{2}}, \\ \tilde{a}(x)|p + Dv_i(x)| = \frac{a_0(x)|Du_i(x)|}{K_1(1+K_2)} \le 1 & \text{for } x \in \mathbb{R}^2\backslash\Gamma_{\frac{\mu_0}{2}}, \end{cases}$$

which says

$$\max_{x\in\mathbb{R}^2} \tilde{a}(x)|p + Dv_i(x)| = \max_{x\in\mathbb{R}^2} \tilde{a}(x)|Du_i(x)| \le 1.$$

By the inf-max formula (1.4), for $1 \le i \le m$,

$$\overline{H}_{\tilde{a}}(p_i) \le 1.$$

This verifies that $\tilde{a}$ constructed above has the desired properties, and the proof of Theorem 1.2 is completed. ∎

Remark 2.3. Our method also provides a simple proof of the following result in [20]: there exists $a \in C^\infty(\mathbb{T}^2, (0,\infty))$ such that

$$\{\pm q_i \,:\, 1 \le i \le m\} \subset \partial \overline{B}_1^a. \tag{2.6}$$

In fact, to prove this claim, no gradient matching is needed at the intersections. Steps 1 and 2 in the proof of Theorem 1.2 are not needed. Below we give some adaptions to get (2.6). We use the straight line network $\cup_{i=1}^m \left(L_i + \mathbb{Z}^2\right)$ directly.

1. Pick $a \in C^\infty(\mathbb{T}^2,(0,\infty))$ such that $a = 1$ in a small neighborhood of I, and (2.4) holds with a, L_i in place of $a_0, \tilde{L}_i$, respectively.
2. In Step 3 of the proof of Theorem 1.2, choose u_i as

$$u_i(x) = \frac{q_i}{|q_i|} \cdot (x - x_{i,j,v}) + p_i \cdot x_{i,j,v}$$

near each intersection point $x_{i,j,v}$. Then, using the method of characteristics (see [11, Chapter 3] for instance), we extend $u_i(x)$ to a smooth function on Γ_δ for some $\delta > 0$ such that

$$a(x)|Du_i(x)| = 1 \quad \text{in } \Gamma_\delta.$$

3. Finally, following the same arguments in Step 3 of the proof of Theorem 1.2, we can conclude.

Appendix A. Some auxiliary lemmas.

LEMMA A.1. *Suppose that $\gamma : [0,1] \to \mathbb{R}^2$ is a smooth curve satisfying that*
1. $\min_{t\in[0,1]} |\dot\gamma(t)| > 0$ *and* $\gamma(t_1) \neq \gamma(t_2)$ *for* $t_1 \neq t_2$*;*
2. *there exist* $r > 0$ *and* $u_0, u_1 \in C^\infty(\mathbb{R}^2)$ *such that*

$$\begin{cases} \dot\gamma(t) = Du_0(\gamma(t)) & \text{for } t \in [0,r], \\ \dot\gamma(t) = Du_1(\gamma(t)) & \text{for } t \in [1-r,1]. \end{cases}$$

Then, there exist $\delta > 0$, an open neighborhood U of γ, and $u \in C^\infty(U)$ such that

$$\begin{cases} \inf_U |Du| > 0, \\ Du = Du_0 & \text{in } B_\delta(\gamma(0)), \\ Du = Du_1 & \text{in } B_\delta(\gamma(1)) \end{cases}$$

and

$$\dot\gamma(t) = Du(\gamma(t)) \quad \text{for } t \in [0,1].$$

The proof of the above lemma is standard, and we leave it as an exercise for the interested readers.

Remark A.2. Consider the same set-up of Lemma A.1. Let $a(\gamma(t)) = \frac{1}{|Du(\gamma(t))|}$ for $t \in [0,1]$, and

$$M = \int_0^1 \frac{1}{a(\gamma(t))} |\dot\gamma(t)|\, dt = u(\gamma(1)) - u(\gamma(0)).$$

For each $r \in (-M,M)$, let $\tau > 0$ be sufficiently small, and choose $h \in C^\infty(\mathbb{R})$ so that

$$\begin{cases} h(t) = t & \text{for } t \in [0, \frac{\tau}{2}], \\ h(t) = r + t - M & \text{for } t \in [M - \frac{\tau}{2}, M], \\ |h'(t)| \le 1 & \text{for all } t \in [0,M]. \end{cases}$$

Then, $u_r = h(u - u(\gamma(0))) + u(\gamma(0))$ satisfies that

$$u_r(\gamma(0)) = u(\gamma(0)), \quad u_r(\gamma(1)) = u(\gamma(0)) + r.$$

Moreover, we also have $|Du_r| \le |Du|$ in U and

$$Du_r(x) = Du(x) \quad \text{for } x \in B_\mu(\gamma(0)) \cup B_\mu(\gamma(1))$$

for some $\mu > 0$.

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