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The extremals of Stanley's inequalities for partially ordered sets

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A B S T R A C T

Stanley's inequalities for partially ordered sets establish important log-concavity relations for sequences of linear extensions counts. Their extremals however, i.e., the equality cases of these inequalities, were until now poorly understood with even conjectures lacking. In this work, we solve this problem by providing a complete characterization of the extremals of Stanley's inequalities. Our proof is based on building a new "dictionary" between the combinatorics of partially ordered sets and the geometry of convex polytopes, which captures their extremal structures.

Contents

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1. Introduction

1.1. Log-concave sequences

Finite sequences of numbers $\{a_i\}_{i=1}^n$ often serve as a powerful way to encode properties of algebraic, geometric, and combinatorial objects: a_i can stand for the ith coefficient of a Schur polynomial, the dimension of the ith cohomology group of a toric variety, or the number of i-elements independent sets of a matroid, etc. The properties and interrelations of the elements of the sequence $\{a_i\}_{i=1}^n$ provide valuable information about the underlying mathematical objects. Here we focus on *log-concavity* relations:

$$a_i^2 \geq a_{i-1}a_{i+1} \quad \text{for all } i = 2, \dots, n-1,$$

which are tied to notions of *positivity* and *unimodality* [2,3,20,21,15,1]. The question that motivates our work is the following: Suppose a log-concave sequence $\{a_i\}_{i=1}^n$, whose elements stand for some algebraic/geometric/combinatorial properties of a mathematical object, satisfies

$$a_j^2 = a_{j-1}a_{j+1} \quad \text{for some } \textit{fixed} \text{ index } j.$$

What can we deduce about the underlying object? This question of identifying the *extremals* of the sequence $\{a_i\}_{i=1}^n$ is fundamental for a number of reasons. At the very basic

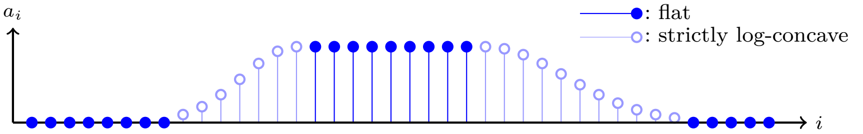


Fig. 1.1. The extremals of this log-concave sequence (cf. (1.2)) are such that $a_j^2 = a_{j-1}a_{j+1} \Rightarrow a_{j-1} = a_j = a_{j+1}$, corresponding to the flat parts of the sequence. The width of each of the flat parts can be characterized as well. This precise description of the shape of the sequence cannot be obtained from the log-concavity property alone.

level, the structure of the extremals is a basic property of the sequence which we ought to understand. More concretely, information about the extremals can provide information about the shape of the sequence which cannot be inferred from the log-concavity property alone: see Fig. 1.1. Additionally, if one wishes to improve on the log-concavity property by having $a_i^2 - a_{i-1}a_{i+1} \geq d_i$ for some non-trivial $d_i \geq 0$, then usually understanding the extremals of $\{a_i\}$, and hence the vanishing of d_i, is a necessary first step. From a different perspective, there are interesting questions related to combinatorial interpretations and computational complexity of the difference $a_i^2 - a_{i-1}a_{i+1}$, where characterizing the vanishing condition $a_i^2 = a_{i-1}a_{i+1}$ is a basic question [13,14,6].

Establishing that a given sequence, which arises in an algebraic/geometric/combinatorial setting, is log-concave is a difficult problem, with many remaining open questions. In recent years, major advances were achieved on the fronts of proving log-concavity relations for various important sequences in combinatorics [11,12,5]. These approaches rely on building "dictionaries" between combinatorial and geometric-algebraic objects, and then using (or taking inspiration from) already-known log-concavity relations in the geometric-algebraic settings. What is missing, however, are the analogous dictionaries between the *extremals* arising in the combinatorial and geometric-algebraic settings. In this work, we take a step towards bridging this gap by focusing on the correspondence between combinatorics and *convex* geometry due to R. Stanley in the context of partially ordered sets. We will build such a dictionary and, as a consequence, completely characterize the extremal structures arising in Stanley's inequalities [18]. The question of the characterization of these extremals was already raised by Stanley, but even conjectures on these extremals were lacking. As we will see, this is for a good reason since, surprisingly, the extremal structures of our combinatorial sequences will display the richness and subtle nature of their geometric counterparts.

1.2. Stanley's inequalities

Let $\bar{\alpha} = \{y_1, \dots, y_{n-k}\} \cup \{x_1, \dots, x_k\}$ be a partially ordered set (poset) of n elements with a fixed chain $x_1 < \cdots < x_k$ of length k. The set of linear extensions of $\bar{\alpha}$ is the set of bijections of $\bar{\alpha}$ into $[n] := \{1, \dots, n\}$ which are order-preserving:

$$\mathcal{N} := \{\text{bijections } \sigma : \bar{\alpha} \to [n] : w \leq z \Rightarrow \sigma(w) \leq \sigma(z) \ \forall\ w, z \in \bar{\alpha}\}.$$

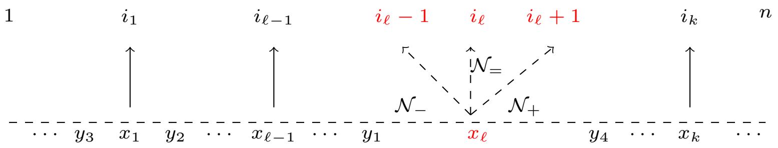


Fig. 1.2. Every linear extension sends x_j to i_j whenever $j \neq \ell$. But x_ℓ is sent to one of the locations $i_\ell - 1,\ i_\ell,\ i_\ell + 1$, depending on whether the linear extension is in $\mathcal{N}_-, \mathcal{N}_=, \mathcal{N}_+$, respectively.

We are interested in linear extensions which send the elements in the chain $x_1 < \cdots < x_k$ into *fixed* locations. Fix $1 \le i_1 < \cdots < i_k \le n$ and fix $\ell \in [k]$ such that $i_{\ell-1} + 1 < i_\ell < i_{\ell+1} - 1$. For $\circ \in \{-, =, +\}$, let

$$\mathcal{N}_\circ := \{\sigma \in \mathcal{N} : \sigma(x_j) = i_j \ \forall j \in [k]\backslash\{\ell\} \text{ and } \sigma(x_\ell) = i_\ell + 1_\circ\},$$

where $1_\circ := 1_{\{\circ \text{ is } +\}} - 1_{\{\circ \text{ is } -\}}$. In words, whenever $j \neq \ell$, x_j is placed at i_j, and when $j = \ell$, x_ℓ is placed at one of the locations in $\{i_\ell - 1, i_\ell, i_\ell + 1\}$, depending on the sign of $\circ \in \{-, =, +\}$; see Fig. 1.2.

In [18, Theorem 3.2], Stanley showed that

$$|\mathcal{N}_=|^2 \ge |\mathcal{N}_-||\mathcal{N}_+|, \tag{1.1}$$

thus resolving a conjecture of Chung, Fishburn and Graham [10]. To see the relation to log-concave sequences consider the case $k = 1$ and set

$$a_i := |\{\sigma \in \mathcal{N} : \sigma(x_1) = i\}|, \quad i \in [n]. \tag{1.2}$$

Then, (1.1) amounts to the statement that the sequence $\{a_i\}$ is log-concave. For the general case $k \ge 1$, (1.1) is a log-concavity statement about multi-index sequences.

The goal of this work is to provide a complete characterization of the equality cases of (1.1) for any k. That is, we will answer the following question: If

$$|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|, \tag{1.3}$$

what can we deduce about the poset $\bar{\alpha}$?

To gain some intuition for the extremals of Stanley's inequalities (1.1) let us start with a trivial observation: If $\{y_1, \dots, y_{n-k}\}$ are all incomparable to x_ℓ, then $|\mathcal{N}_-| = |\mathcal{N}_=| = |\mathcal{N}_+|$, which yields equality in (1.1). In the same vein, consider the following example which is slightly less trivial.

Example 1.1. Suppose the poset $\bar{\alpha}$ satisfies

$$\{z \in \bar{\alpha} : z < x_\ell \text{ and } z \not< x_{\ell-1}\} \cup \{z \in \bar{\alpha} : z > x_\ell \text{ and } z \not> x_{\ell+1}\} = \varnothing. \tag{1.4}$$

Then, given any $\sigma \in \cup_{\circ\in\{-,=,+\}}\mathcal{N}_\circ$, we can permute (some of) the locations of the elements $\{\sigma^{-1}(i_\ell-1), \sigma^{-1}(i_\ell), \sigma^{-1}(i_\ell+1)\}$ without violating any constraints. For example, given $\sigma \in \mathcal{N}_+$, the elements $\sigma^{-1}(i_\ell-1)$, $\sigma^{-1}(i_\ell)$ must be incomparable to $x_\ell = \sigma^{-1}(i_\ell+1)$ since, as $i_{\ell-1}+1 < i_\ell < i_{\ell+1}-1$, the converse would violate (1.4). Hence, we can exchange the locations of $\{\sigma^{-1}(i_\ell-1), \sigma^{-1}(i_\ell+1)\}$ or $\{\sigma^{-1}(i_\ell), \sigma^{-1}(i_\ell+1)\}$. It follows that

$$|\mathcal{N}_=| = |\mathcal{N}_-| = |\mathcal{N}_+|, \tag{1.5}$$

which in particular implies (1.3).

The mechanism (1.4) is wasteful since it is *global* in nature. It controls all the elements between $x_{\ell-1}$ and $x_{\ell+1}$, even though we are concerned only with the elements which are close to x_ℓ in the sense that they are located in $i_\ell-1$, i_ℓ, $i_\ell+1$. Instead, we expect (1.3) to hold as soon as the mechanism (1.4) occurs only on a *local* scale. To make this idea precise we make the following definition regarding elements that are close to x_ℓ.

Definition 1.2. Fix $\ell \in [k]$ such that $i_{\ell-1}+1 < i_\ell < i_{\ell+1}-1$, and given $\circ \in \{-,=,+\}$, fix $\sigma \in \mathcal{N}_\circ$. The *companions* of $x_\ell = \sigma^{-1}(i_\ell+1_\circ)$ are $\sigma^{-1}(i_j)$ for $i_j \in \{i_\ell-1, i_\ell, i_\ell+1\}\backslash\{i_\ell+1_\circ\}$, where $1_\circ := 1_{\{\circ \text{ is } +\}} - 1_{\{\circ \text{ is } -\}}$. The companion lower in ranking is the *lower companion* and the companion higher in ranking is the *upper companion*.

For example, with $\circ$ being $-$, the companions of $x_\ell = \sigma^{-1}(i_\ell-1)$ are $\sigma^{-1}(i_\ell)$ and $\sigma^{-1}(i_\ell+1)$. The lower companion is $\sigma^{-1}(i_\ell)$ and the upper companion is $\sigma^{-1}(i_\ell+1)$.

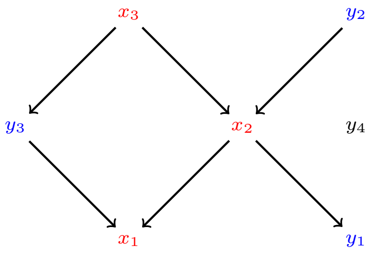
1.3. The extremals of Stanley's inequalities

The characterization of the extremals of Stanley's inequalities will be in terms of the companions of x_ℓ as defined in Definition 1.2. On a finer resolution, there are two distinct classes of posets which in turn have different types of extremals. The two classes of posets will be called *supercritical* and *critical*, a terminology which will become clear later. The precise definitions are deferred to Definition 2.11, but for now, we will simply note that a supercritical poset is always critical, but the converse is false. (There are further classes which reduce to the supercritical and critical classes. They will be handled in Section 6, see also Theorem 1.6.)

Theorem 1.3. ***(Supercritical extremals of Stanley's inequalities)***
Suppose the poset $\bar{\alpha}$ is supercritical. The following are equivalent:

(i) $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$.
(ii) $|\mathcal{N}_-| = |\mathcal{N}_=| = |\mathcal{N}_+|$.
(iii) For every linear extension in $\mathcal{N}_-\cup\mathcal{N}_=\cup\mathcal{N}_+$, both companions of x_ℓ are incomparable to x_ℓ.

$\bar{\alpha}$ =



$$\mathcal{N}_- = \{y_1x_1x_2y_3y_4x_3y_2, \quad y_1x_1x_2y_4y_3x_3y_2, \quad y_1x_1x_2y_2y_3x_3y_4, \quad y_1x_1x_2y_3y_2x_3y_4\},$$

$$\mathcal{N}_= = \{y_1x_1y_3x_2y_4x_3y_2, \quad y_1x_1y_4x_2y_3x_3y_2, \quad y_1x_1y_3x_2y_2x_3y_4, \quad y_4x_1y_1x_2y_3x_3y_2\},$$

$$\mathcal{N}_+ = \{y_1x_1y_4y_3x_2x_3y_2, \quad y_1x_1y_3y_4x_2x_3y_2, \quad y_4x_1y_1y_3x_2x_3y_2, \quad y_4x_1y_3y_1x_2x_3y_2\}.$$

Fig. 1.3. Top: Hasse diagram (arrows point from smaller to larger elements) of poset in Example 1.4. **Bottom**: Collections of linear extensions of poset in Example 1.4.

Theorem 1.3 provides a number of insights into the extremals of (1.3). Part (ii) of the theorem (which held in (1.5)) is non-trivial, and even surprising, since it puts heavy constraints on the ways in which $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$ can occur. A priori, we could have a geometric progression where $|\mathcal{N}_-| = ab^{c-1}$, $|\mathcal{N}_=| = ab^c$, $|\mathcal{N}_+| = ab^{c+1}$, for some $a, b, c > 0$, which would yield the equality

$$|\mathcal{N}_=|^2 = a^2b^{2c} = (ab^{c-1})(ab^{c+1}) = |\mathcal{N}_-||\mathcal{N}_+|.$$

Theorem 1.3(ii) excludes this possibility. On the other hand, despite the information provided by (ii), it sheds no light on the mechanism which yield equality in (1.1). In contrast, Theorem 1.3(iii) provides the mechanism behind the extremals: The companions of x_ℓ, under any linear extension in $\bigcup_{\circ\in\{-,=,+\}}\mathcal{N}_\circ$, must be incomparable to x_ℓ. Hence, the positions of x_ℓ and both of its companions can be swapped, which leads to part (ii). Note that (iii) is a *local* condition which controls only the immediate companions of x_ℓ, unlike (1.4). The power of Theorem 1.3 lies in the statement that this mechanism is the *only* mechanism behind the extremals of Stanley's inequalities for supercritical posets.

The characterization of Theorem 1.3 is very clean and one might hope that it applies to every poset. This hope is quickly shattered:

Example 1.4. Let $\bar{\alpha} = \{y_1, y_2, y_3, y_4, x_1, x_2, x_3\}$ with the relations

$$x_1 < x_2 < x_3, \quad y_1 < x_2, \quad x_2 < y_2, \quad x_1 < y_3 < x_3.$$

Set $\ell = 2$, and $i_1 = 2$, $i_2 = 4$, $i_3 = 6$. One can check that $|\mathcal{N}_-| = |\mathcal{N}_=| = |\mathcal{N}_+| = 4$ so that Theorem 1.3(ii) holds. On the other hand, Theorem 1.3(iii) is false since y_1, y_2 are comparable to x_2 but can appear as companions of x_2 under linear extensions in $\mathcal{N}_- \cup \mathcal{N}_= \cup \mathcal{N}_+$. See Fig. 1.3.

Our next result goes beyond Theorem 1.3 and characterizes the extremals of critical posets.

Theorem 1.5. *(**Critical extremals of Stanley's inequalities**)*
Suppose the poset $\bar{\alpha}$ is critical. The following are equivalent:

(i) $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$.
(ii) $|\mathcal{N}_-| = |\mathcal{N}_=| = |\mathcal{N}_+|$.
(iii) *For every linear extension in $\mathcal{N}_- \cup \mathcal{N}_= \cup \mathcal{N}_+$, at least one companion of x_ℓ is incomparable to x_ℓ. In addition, there exist nonnegative numbers $\mathrm{N}_1, \mathrm{N}_2$ such that:*
- *For any fixed $\circ \in \{-, =, +\}$,*

$$|\{\sigma \in \mathcal{N}_\circ : \textit{only the lower companion of } x_\ell \textit{ is incomparable to } x_\ell\}|$$
$$= \mathrm{N}_1 = |\{\sigma \in \mathcal{N}_\circ : \textit{only the upper companion of } x_\ell \textit{ is incomparable to } x_\ell\}|.$$

- $|\{\sigma \in \mathcal{N}_\circ$ *: both companions of x_ℓ are incomparable to x_ℓ*$\}| = \mathrm{N}_2 \quad \forall \circ \in \{-, =, +\}$.

Let us compare and contrast Theorem 1.3 and Theorem 1.5. The conclusion in part (ii) that the equality (1.3) necessitates $|\mathcal{N}_-| = |\mathcal{N}_=| = |\mathcal{N}_+|$ remains true for supercritical and critical posets. But the mechanisms, i.e., part (iii), for this phenomenon are different. Clearly, Theorem 1.3(iii) is a stronger condition since it trivially implies the condition in Theorem 1.5(iii). For critical posets, the conclusion that only 0 comparable companions are allowed (namely Theorem 1.3(iii)) is relaxed into the statement that 0 or 1 comparable companions are allowed. But in order to get $|\mathcal{N}_-| = |\mathcal{N}_=| = |\mathcal{N}_+|$, there must be a balance between those linear extensions with 1 comparable companion, which is the content of the second part of Theorem 1.5(iii).

Our formulation of Theorem 1.3 and Theorem 1.5 mirrors the analogous distinction in convex geometry between supercritical and critical (cf. Theorem 1.10). However, our proofs provide us with a stronger statement which encompass both Theorem 1.3 and Theorem 1.5.

Theorem 1.6. *(**Extremals of Stanley's inequalities**) Suppose $\bar{\alpha}$ is a poset such that $|\mathcal{N}_=| > 0$. Then, the following hold:*

- *The conclusions of Theorem 1.5 remain true. In addition, given any $\sigma \in \mathcal{N}_\circ$, for any $\circ \in \{-, =, +\}$, where one of the companions is comparable to x_ℓ, we have that the lower and upper companions are incomparable to each other.*
- *If $\bar{\alpha}$ is supercritical then the conclusions of Theorem 1.3 remain true.*

Theorem 1.6 improves upon Theorem 1.3 and Theorem 1.5 by showing that the conclusions of Theorem 1.5 hold even under the assumption $|\mathcal{N}_=| > 0$. In addition, Theorem 1.6

provides further information on the structure of the linear extensions. The only case not covered by Theorem 1.6 is when $|\mathcal{N}_{=}| = 0$, which is in fact trivial and will be characterized later (Theorem 5.3).

Remark 1.7. (**Poset characterization**) There is a way to reformulate Theorem 1.3(iii) so that the characterization of the extremals is given in terms of conditions on the poset itself rather than on the set of its linear extensions:

$$\begin{aligned} \forall\, y < x_\ell :\quad & \exists\, s(y) \in \{0,\dots,k+1\} \text{ s.t. } y < x_{s(y)} \\ & \text{and } |\{z \in \bar{\alpha} : y < z < x_{s(y)}\}| > i_{s(y)} - i_\ell, \\ \forall\, y > x_\ell :\quad & \exists\, r(y) \in \{0,\dots,k+1\} \text{ s.t. } y > x_{r(y)} \\ & \text{and } |\{z \in \bar{\alpha} : x_{r(y)} < z < y\}|| > i_\ell - i_{r(y)}; \end{aligned} \tag{1.6}$$

see Proposition 7.5. Here, x_0 (res. x_{k+1}) is the added element with the property that it is smaller (res. bigger) than any other element in $\bar{\alpha}$. The formulation (1.6) can be useful in practice since, given a standard description of a poset, (1.6) is easier to check. On the other hand, the formulation of Theorem 1.3(iii) is more compatible with our dictionary, which is more natural to formulate in terms of conditions on the linear extensions of the poset. In the first version of this manuscript we wrote that "It is an interesting problem to find an analogue of (1.6) for critical posets." However, since the first version of our work was made public, Chan and Pak [6, Theorem 1.3] proved a remarkable result on the computational complexity of the characterization of Stanley's inequalities, which in particular implies that a poset characterization of Stanley's inequalities of the form (1.6) would contradict fundamental conjectures in computational complexity [6, §3.5].

Remark 1.8. ($k = 1$) The characterization of the extremals of Stanley's inequalities when $k = 1$ was done in [17, §15]. It turns out that, when $k = 1$, the poset must be supercritical and the characterization of [17, §15] in this case is the same as Theorem 1.3 and Remark 1.7. While our proofs take much inspiration from the work [17], the new phenomena of critical posets necessitated the development of many new ideas (see Fig. 1.4). For example, the dictionary constructed in [17, §15] was in terms of the poset itself (as in Remark 1.7), rather than its linear extensions. But when progressing to critical posets, the approach of [17, §15] no longer works (especially in light of [6, Theorem 1.3]), while our dictionary, which is in terms of a linear extensions description, is suitable for these more subtle and rich extremals.

Let us also mention that, when $k = 1$, Chan and Pak, using their *combinatorial atlas* method [4], provided a *linear-algebraic* proof of Stanley's inequalities and characterized their extremals, thus avoiding any use of convex geometry; see also the proof for width two posets by Chan, Pak, and Panova [8]. However, their approach does not currently extend to the case $k > 1$.

Remark 1.9. ($k = 2$) Using our techniques, Chan and Pak [6, Lemma 9.1] showed that, in fact, the conclusion of Theorem 1.3 remains true whenever $k = 2$. Per Remark 1.8, the same holds true for $k = 1$. It follows that Example 1.4, where the conclusion of Theorem 1.3 is no longer valid, is sharp in terms of k.

1.4. Dictionaries between convex geometry and combinatorics

Stanley's proof of (1.1) relies on a remarkable correspondence that he found between mixed volumes of certain convex polytopes and linear extensions counts. Once this correspondence is established, the inequality (1.1) follows from a deep log-concavity result in convex geometry: The Alexandrov-Fenchel inequality. We will start this section by reviewing Stanley's proof of the inequality (1.1), and then move to the discussion of its extremals.

1.4.1. The Alexandrov-Fenchel inequality

We start with some preliminaries from convex geometry; our standard reference is [16]. Given convex bodies (non-empty compact convex sets) $C, C' \subseteq \mathbb{R}^{n-k}$ and scalars $\lambda, \lambda' \geq 0$, we define their sum as

$$\lambda C + \lambda' C' := \{\lambda x + \lambda' y : x \in C, y \in C'\}.$$

The volume of a sum of convex bodies behaves as a polynomial: Given a positive integer p, convex bodies $C_1, \ldots, C_p \subseteq \mathbb{R}^{n-k}$, and scalars $\lambda_1, \ldots, \lambda_p \geq 0$, we have

$$\mathrm{Vol}_{n-k}(\lambda_1 C_1 + \cdots + \lambda_p C_p) = \sum_{1 \leq j_1, \ldots, j_{n-k} \leq p} \mathsf{V}_{n-k}(C_{j_1}, \ldots, C_{j_{n-k}}) \lambda_{j_1} \cdots \lambda_{j_{n-k}},$$

where the coefficients $\mathsf{V}_{n-k}(C_{j_1}, \ldots, C_{j_{n-k}})$ are called *mixed volumes*. These geometric objects generalize the notions of volume, surface area, mean width, etc. The *Alexandrov-Fenchel inequality* [16, §7.3] states that sequences of mixed volumes are log-concave: For any convex bodies $C_1, \ldots, C_{n-k} \subset \mathbb{R}^{n-k}$,

$$\mathsf{V}_{n-k}(C_1, C_2, C_3, \ldots, C_{n-k})^2 \geq \mathsf{V}_{n-k}(C_1, C_1, C_3, \ldots, C_{n-k}) \mathsf{V}_{n-k}(C_2, C_2, C_3, \ldots, C_{n-k}). \tag{1.7}$$

Stanley's proof of (1.1) relies on the identification of the poset $\bar{\alpha}$ with polytopes $K_0, \ldots, K_k$. We defer the explicit construction of these polytopes for later (Section 2), and for now denote by $\mathcal{K}$ a certain collection of these polytopes containing $n - k - 2$ of them. The key point is the identities

$$\begin{aligned} |\mathcal{N}_-| &= (n-k)!\, \mathsf{V}_{n-k}(K_\ell, K_\ell, \mathcal{K}), \\ |\mathcal{N}_=| &= (n-k)!\, \mathsf{V}_{n-k}(K_{\ell-1}, K_\ell, \mathcal{K}), \\ |\mathcal{N}_+| &= (n-k)!\, \mathsf{V}_{n-k}(K_{\ell-1}, K_{\ell-1}, \mathcal{K}). \end{aligned} \tag{1.8}$$

With the representation (1.8) in hand, the inequality (1.1) is equivalent to

$$\mathsf{V}_{n-k}(K_{\ell-1}, K_\ell, \mathcal{K})^2 \geq \mathsf{V}_{n-k}(K_\ell, K_\ell, \mathcal{K})\mathsf{V}_{n-k}(K_{\ell-1}, K_{\ell-1}, \mathcal{K}), \tag{1.9}$$

which follows immediately from (1.7).

Stanely's proof of (1.1) is the only proof currently known. Hence, a natural route towards the characterization of the extremals of Stanley's inequalities would require:

- Characterization of the extremals of the Alexandrov-Fenchel inequality.
- Dictionary between the extremals of the Alexandrov-Fenchel inequality and the extremals of Stanley's inequalities.

For arbitrary convex bodies, the characterization of the extremals of (1.7) is a long-standing open problem [16, §7.6]. But when the bodies are *polytopes*, this problem was recently solved by the second-named author and Van Handel [17]. Thus, the work [17] takes care of the first item and our work here is dedicated to the second item.

To build intuition regarding the correspondence between the extremal structures of posets and polytopes, let us revisit Example 1.1. As will be evident (see (2.3)), the identity (1.4) holds if, and only if, $K_{\ell-1} = K_\ell$. In this case it is clear that equality will be attained in (1.9). But as we saw in Theorem 1.3 and Theorem 1.5, equality can be attained in Stanley's inequalities under much weaker conditions than those captured by Example 1.1. It follows that equality holds in (1.9) under conditions which are much weaker than $K_{\ell-1} = K_\ell$. The characterization of these conditions is the topic of the next section.

1.4.2. The extremals of the Alexandrov-Fenchel inequality for convex polytopes

The terminology of supercritical and critical posets comes in fact from the analogous terminology in the characterization of the extremals of the Alexandrov-Fenchel inequality for convex polytopes as introduced in [17]—-the precise definitions of supercriticality and criticality is deferred to Definition 2.5. In the sequel, $B \subseteq \mathbb{R}^{n-k}$ always stands for the unit ball, and the notions of $(B, \mathcal{K})$-*extreme normal directions* and $\mathcal{K}$-*degenerate pairs*, which will be used in the subsequent theorem, will be given in Definition 2.4 and Definition 2.7, respectively.

Theorem 1.10. ***(Extremals of the Alexandrov-Fenchel inequality for convex polytopes,*** *[17])*

- *Suppose $\mathcal{K}$ is supercritical. Then,*

$$\mathsf{V}_{n-k}(K_{\ell-1}, K_\ell, \mathcal{K})^2 = \mathsf{V}_{n-k}(K_\ell, K_\ell, \mathcal{K})\mathsf{V}_{n-k}(K_{\ell-1}, K_{\ell-1}, \mathcal{K}),$$

if, and only if, up to dilation and translation, the supporting hyperplanes of $K_{\ell-1}$ and K_ℓ agree in all $(B, \mathcal{K})$-extreme normal directions.

Geometry	Dictionary	Combinatorics
Criticality of polytopes (Definition 2.5)	Section 5 (Proposition 5.7)	Criticality of posets (Definition 2.11)
Projection ([16, Theorem 5.3.1])	Section 6 (Remark 6.1)	Splitting (Definition 6.2)
Criticality of splitting pairs (Definition 7.6)	Section 7	Mixing of splitting pairs (Fig. 7.1)
Maximal collection of polytopes ([17, section 9.1])	Section 7 (Proposition 7.8)	Maximal splitting pair (Definition 7.7)
Extreme normal directions	Section 8	First- and second-neighbors
Translation and dilation	Sections 9-10	Chains of poset
Critical subspace (Equation (10.1))	Section 10	Critical subposet (Equation (10.1))

Fig. 1.4. Dictionary between geometry of polytopes and combinatorics of posets.

- *Suppose $\mathcal{K}$ is critical. Then,*

$$\mathsf{V}_{n-k}(K_{\ell-1}, K_\ell, \mathcal{K})^2 = \mathsf{V}_{n-k}(K_\ell, K_\ell, \mathcal{K})\mathsf{V}_{n-k}(K_{\ell-1}, K_{\ell-1}, \mathcal{K}),$$

if, and only if, there exist $0 \le d < \infty$ $\mathcal{K}$-degenerate pairs $(P_1, Q_1), \dots, (P_d, Q_d)$, such that, up to dilation and translation, the supporting hyperplanes of $K_{\ell-1} + \sum_{j=1}^d Q_j$ and $K_\ell + \sum_{j=1}^d P_j$ agree in all $(B, \mathcal{K})$-extreme normal directions.

The complicated structure of the $(B, \mathcal{K})$-extreme normal directions (see Fig. 2.1) is what gives rise to the richness of the extremals. If the supporting hyperplanes of $K_{\ell-1}$ and K_ℓ agree in *every direction* on the sphere S^{n-k-1}, then, up to dilation and translation, $K_{\ell-1}$ and K_ℓ are identical. This is an example where a *global* mechanism (supporting hyperplanes of $K_{\ell-1}, K_\ell$ agree everywhere) gives rise to equality in (1.7). Theorem 1.10 provides a *local* mechanism for equality in (1.7) (supporting hyperplanes of $K_{\ell-1}, K_\ell$ agree only in very few directions), and furthermore, establishes that this local mechanism is the *only* mechanism for the extremal structures of the Alexandrov-Fenchel inequality.

1.4.3. Dictionary for extremals

A priori, it is not at all clear that the complications and richness of the extremals of (1.7) would also arise in our very specific family of polytopes. Indeed, in the case $k = 1$, only the supercritical extremals appear. Remarkably, not only does this complexity arise, but we can provide a clean and intuitive characterization of the extremals arising in Stanley's inequalities for critical posets. At the core of our work is a powerful dictionary which translates between the *extremal* properties of convex polytopes and partially ordered sets. We discover *new* extreme normal directions, and in addition, introduce numerous new key ideas: *closure*, *splitting pairs*, *mixing*, *critical subposet*, to name just a few. It will be best to introduce these ideas at the appropriate places in the paper; Section 4 will contain a brief outline of our proof. We refer to Fig. 1.4 for a quick summary of the main components in our dictionary, and recommend that the reader revisit this table from time to time.

1.5. Organization of paper

We start in Section 2 by reviewing the connection between partially ordered sets and convex geometry. In Section 3 we develop a number of tools (*decompositions, closure*) that are used throughout the paper and also prove the sufficiency parts of Theorem 1.3 and Theorem 1.5. Section 4 provides a brief outline of the proofs of the main results. Section 5 sets the first building block of our dictionary by showing the equivalences between notions of criticality for posets and polytopes. Section 6 introduces the idea of *splitting* and characterizes the extremals of the *subcritical* posets. Section 7 introduces the idea of *mixing* which is at the heart of our proofs and applies it to *splitting pairs*. In Section 8 we add to our dictionary the combinatorial characterization of the extreme normal directions. We complete the proof of Theorem 1.3 in Section 9 and the proofs of Theorem 1.5 and Theorem 1.6 in Section 10. At the end of the paper we include a Notation Appendix for the convenience of the reader.

2. Preliminaries

In this section we review some basics about posets and convex geometry, as well as introduce the notation we use throughout the paper. We review the connection between posets and mixed volumes, and state the characterization of the extremals of the Alexandrov-Fenchel inequality for (convex) polytopes. In addition, we provide the criticality definitions for polytopes and posets.

We use the notation $\leq, <, =, \geq, >, \sim$ to describe the relations in a poset, where $\sim$ stands for the comparability relation,[1] and by $\not\leq, \not<, \neq, \not\geq, \not>, \nsim$ to describe their negations. Given integers $p \leq q$ we write

$$[\![p, q]\!] := \{p, p+1, \dots, q-1, q\}. \tag{2.1}$$

Fix positive integers k, n, with $k \leq n$, and consider the poset $\bar{\alpha}$, of size n,

$$\bar{\alpha} = \{y_1, \dots, y_{n-k}, x_1, \dots, x_k\},$$

where $x_1 < x_2 < \cdots < x_k$ is a chain. Let

$$\alpha = \{y_1, \dots, y_{n-k}\}$$

be the induced poset of size $n-k$ obtained from $\bar{\alpha}$ by removing the chain. To simplify the notation we add two elements x_0, x_{k+1} to $\bar{\alpha}$ with the property that x_0 is smaller than any element in $\bar{\alpha}$ while x_{k+1} is bigger than any element in $\bar{\alpha}$. Note that this allows us to consider the case $k = 0$.

[1] Note that $\sim$ is not a transitive property.

Let $\mathcal{N}$ be the set of all linear extensions of $\bar{\alpha}$, that is,

$$\mathcal{N} = \{\text{bijections } \sigma : \bar{\alpha} \to [n] : w \leq z \Rightarrow \sigma(w) \leq \sigma(z) \ \forall \ w, z \in \bar{\alpha}\},$$

with the convention that $\sigma(x_0) = 0$ and $\sigma(x_{k+1}) = n + 1$ for any $\sigma \in \mathcal{N}$. Fix $\ell \in [k] := \{1, \ldots, k\}$ and fix $i_1 < i_2 < \cdots < i_k \in [n]$, with the property $i_{\ell-1}+1 < i_\ell < i_{\ell+1}-1$, and let $i_0 := 0,\ i_{k+1} := n+1$. We define the following sets of linear extensions, $\mathcal{N}_-, \mathcal{N}_=, \mathcal{N}_+ \subseteq \mathcal{N}$,

$$\begin{aligned}
\mathcal{N}_- &:= \{\sigma \in \mathcal{N} : \sigma(x_\ell) = i_\ell - 1 \quad \text{and} \quad \sigma(x_m) = i_m \ \forall \ m \in [k]\backslash\{\ell\}\},\\
\mathcal{N}_= &:= \{\sigma \in \mathcal{N} : \sigma(x_\ell) = i_\ell \quad \text{and} \quad \sigma(x_m) = i_m \ \forall \ m \in [k]\backslash\{\ell\}\},\\
\mathcal{N}_+ &:= \{\sigma \in \mathcal{N} : \sigma(x_\ell) = i_\ell + 1 \quad \text{and} \quad \sigma(x_m) = i_m \ \forall \ m \in [k]\backslash\{\ell\}\},
\end{aligned}$$

so Stanley's inequalities read

$$|\mathcal{N}_=|^2 \geq |\mathcal{N}_-||\mathcal{N}_+|. \tag{2.2}$$

2.1. Posets and polytopes

Fundamental to our approach towards the extremals of (2.2) is the connection, due to Stanley [18], between posets and convex polytopes. We start with the definition of an *order polytope*: Given $\beta \subseteq \alpha$ we let $\mathbb{R}^\beta := \{t \in \mathbb{R}^{n-k} : t_j = 0 \text{ for } y_j \notin \beta\}$ and define the order polytope $O_\beta \subseteq \mathbb{R}^\beta \subseteq \mathbb{R}^\alpha$ by

$$O_\beta := \{t \in \mathbb{R}^\beta : t_j \in [0,1] \ \forall y_j \in \beta, \text{ and } t_u \leq t_v \text{ if } y_u \leq y_v \ \forall y_u, y_v \in \beta\}.$$

The order polytope encodes important properties of the poset, e.g., the volume of O_α is proportional to the number of linear extensions of α [19, Corollary 4.2]. Let us recall some basic facts about order polytopes, which will require the following poset notions. A *maximal* (res. *minimal*) element $y \in \alpha$ is such that there exists no $z \in \alpha$, different than y, satisfying $y < z$ (res. $z < y$). Given a set $\beta \subseteq \alpha$ we define $\beta^\uparrow$ (res. $\beta^\downarrow$) to be the set of maximal (res. minimal) elements of β. Given a relation $\star \in \{\leq, <, =, \geq, >, \sim, \not\leq, \not<, \neq, \not\geq, \not>, \nsim\}$ and $y \in \beta$ we let

$$\beta_{\star y} := \{z \in \beta : z \star y\},$$

and, similarly, given relations $\star, * \in \{\leq, <, =, \geq, >, \sim, \not\leq, \not<, \neq, \not\geq, \not>, \nsim\}$, and $y, y' \in \beta$, we write

$$\beta_{\star y, * y'} := \{z \in \beta : z \star y \text{ and } z * y'\}.$$

An element $z \in \beta$ *covers* $y \in \beta$ if $z \in \beta^\downarrow_{>y}$. We say that β is an *upper set* (res. *lower set*) if $\alpha_{>y} \subseteq \beta$ (res. $\alpha_{<y} \subseteq \beta$), for every $y \in \beta$.

The next result provides information about the face structure of order polytopes based on the poset notions just introduced.

Lemma 2.1. *([19, §1]) For any $\beta \subseteq \alpha$ we have $\dim O_\beta = |\beta|$. The $(|\beta|-1)$-dimensional faces of O_β are precisely the following subsets of O_β:*

(i) $O_\beta \cap \{t_j = 0\}$ *for* $y_j \in \beta^\downarrow$.
(ii) $O_\beta \cap \{t_j = 1\}$ *for* $y_j \in \beta^\uparrow$.
(iii) $O_\beta \cap \{t_u = t_v\}$ *for* $y_u, y_v \in \beta$ *such that* y_v *covers* y_u *in* β.

Hyperplane sections of order polytopes will play a crucial role for us: Given $i \in [\![0,k]\!]$, define the polytopes in $\mathbb{R}^{n-k}$,

$$K_i := \{t \in O_\alpha : t_j = 0 \text{ if } y_j < x_i,\ t_j = 1 \text{ if } y_j > x_{i+1},\ \text{for all } y_j \in \alpha\}. \tag{2.3}$$

While we defined the polytopes $\{K_i\}$ as hyperplane sections of order polytopes, they are in fact nothing but translations of certain order polytopes. To see this relation we start with the next lemma whose proof is a matter of checking the definitions. In the sequel, given $\beta \subseteq \alpha$ let $1_\beta := \sum_{y_j \in \beta} e_j$, with $\{e_j\}_{j\in\beta}$ denoting the standard basis of $\mathbb{R}^\beta$.

Lemma 2.2. *Let $\beta, \beta' \subseteq \alpha$ be disjoint sets where β is an upper set and β' is a lower set. Then,*

$$O_{\alpha\setminus(\beta\cup\beta')} + 1_\beta = \{t \in O_\alpha : t_j = 0 \textit{ if } y_j \in \beta' \textit{ and } t_j = 1 \textit{ if } y_j \in \beta\},$$

where we view $\mathbb{R}^{\alpha\setminus(\beta\cup\beta')}$ as a subset of $\mathbb{R}^\alpha \cong \mathbb{R}^{n-k}$.

We can now write $\{K_i\}_{i\in[\![0,k]\!]}$ as translates of order polytopes. For $i \in [\![0,k]\!]$ define

$$\beta_i := \alpha\setminus(\alpha_{<x_i} \cup \alpha_{>x_{i+1}}), \tag{2.4}$$

with the convention that $\beta_i = \varnothing$ if $i < 0$ or $i > k$; for $S \subseteq [\![0,k]\!]$ set $\beta_S := \cup_{i\in S}\beta_i$. The interpretation of β_i is as the set of elements which can potentially be ordered between x_i and x_{i+1}. Then, applying Lemma 2.2, with the disjoint upper and lower sets $\beta = \alpha_{>x_{i+1}}$, $\beta' = \alpha_{<x_i}$, shows that

$$K_i = O_{\beta_i} + 1_{\alpha_{>x_{i+1}}} \text{ for } i \in [\![0,k]\!]. \tag{2.5}$$

As an example of β_S, which will be useful later, the following result handles the set $S := [\![0,r]\!] \cup [\![s,k]\!]$.

Lemma 2.3. *For any $r \le s$,*

$$\beta_{[\![0,r]\!]\cup[\![s,k]\!]} = \alpha\setminus\alpha_{>x_{r+1},<x_s} = \beta_r \cup \beta_s \cup \alpha_{<x_{r+1}} \cup \alpha_{>x_s}.$$

Proof. The second identity is clear so we focus on the first identity. Let $j_0 := -1,\ 0 \leq j_1 < \cdots < j_p \leq k,\ j_{p+1} := k+1$. We claim that

$$\bigcap_{q=1}^{p}(\alpha_{<x_{j_q}} \cup \alpha_{>x_{j_q+1}}) = \bigcup_{q=0}^{p} \alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}. \tag{2.6}$$

$\subseteq$: Let $y \in \bigcap_{q=1}^{p}(\alpha_{<x_{j_q}} \cup \alpha_{>x_{j_q+1}})$ so that, for each $q \in [\![1,p]\!]$, either $y < x_{j_q}$ or $y > x_{j_q+1}$. Let q' be the largest q such that $y > x_{j_q+1}$. Then, y is not bigger than $x_{j_{(q'+1)}+1}$, which means that $y < x_{j_{(q'+1)}}$, as $y \in \alpha_{<x_{j_{(q'+1)}}} \cup \alpha_{>x_{j_{(q'+1)}}+1}$ (this is trivially true if $q' = p$). Hence, $y \in \alpha_{>x_{j_{q'}+1},<x_{j_{(q'+1)}}}$.

$\supseteq$: Let $y \in \alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}$ for some $q \in [\![0,p]\!]$. Then, for any $q' \leq q$, $y > x_{j_{q'}+1}$ and, for any $q' > q$, $y < x_{j_{q'}}$. Hence, $y \in \bigcap_{q=0}^{p+1}(\alpha_{<x_{j_q}} \cup \alpha_{>x_{j_q+1}}) = \bigcap_{q=1}^{p}(\alpha_{<x_{j_q}} \cup \alpha_{>x_{j_q+1}})$.

We now to turn to the proof of the lemma. Let $j_0 := -1, j_{p+1} := k+1$, and $\{j_1, \ldots, j_p\} := \{0, \ldots, r, s, \ldots, k\}$. We have

$$\beta_{[\![0,r]\!]\cup[\![s,k]\!]} = \bigcup_{q=1}^{p} \beta_{j_q} = \bigcup_{q=1}^{p}\left(\alpha \Big\backslash \left(\alpha_{<x_{j_q}} \cup \alpha_{>x_{j_q+1}}\right)\right) = \alpha \Big\backslash \bigcap_{q=1}^{p}\left(\alpha_{<x_{j_q}} \cup \alpha_{>x_{j_q+1}}\right)$$

$$\underset{(2.6)}{=} \alpha \Big\backslash \bigcup_{q=0}^{p} \alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}.$$

Whenever $j_q \neq r$, $j_q + 1 = j_{q+1}$, so $\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}} = \varnothing$. It follows that $\bigcup_{q=0}^{p} \alpha_{>x_{j_q+1},<x_{j_{(q+1)}}} = \alpha_{>x_{r+1},<x_s}$, which completes the proof. $\square$

2.2. Posets and mixed volumes

The connection between the polytopes $\{K_i\}_{i\in[\![0,k]\!]}$ and $|\mathcal{N}_-|, |\mathcal{N}_=|, |\mathcal{N}_+|$, which leads to Stanley's proof of (2.2), goes through the notion of mixed volumes; we refer to [16] as the standard reference for the theory of convex bodies. Given convex bodies (nonempty compact convex sets) $C, C' \subseteq \mathbb{R}^{n-k}$, and scalars $\lambda, \lambda' \geq 0$, we define their sum as

$$\lambda C + \lambda' C' := \{\lambda x + \lambda' y : x \in C, y \in C'\}.$$

The volume of a sum of convex bodies behaves as a polynomial: Given convex bodies $C_1, \ldots, C_p \subseteq \mathbb{R}^{n-k}$, and scalars $\lambda_1, \ldots, \lambda_p \geq 0$, we have [16, Theorem 5.1.7],

$$\mathrm{Vol}_{n-k}(\lambda_1 C_1 + \cdots + \lambda_p C_p) = \sum_{j_1,\ldots,j_{n-k}\in[\![1,p]\!]} \mathsf{V}_{n-k}(C_{j_1}, \ldots, C_{j_{n-k}})\lambda_{j_1}\cdots\lambda_{j_{n-k}}.$$

The coefficients $\mathsf{V}_{n-k}(C_{j_1}, \ldots, C_{j_{n-k}})$, which are nonnegative, symmetric, and multilinear in their arguments, are called *mixed volumes*. Stanley's proof of (2.2) relies on the following identification of $|\mathcal{N}_-|, |\mathcal{N}_=|, |\mathcal{N}_+|$ with mixed volumes [18, Theorem 3.2]. For $m \in [\![0,k]\!]$ let

$$\mathcal{K}_m := (\underbrace{K_m, \ldots, K_m}_{i_{m+1}-i_m-1}).$$

Then,

$$|\mathcal{N}_-| = (n-k)!\mathsf{V}_{n-k}(\mathcal{K}_0, \mathcal{K}_1, \ldots, \underbrace{K_{\ell-1}, \ldots, K_{\ell-1}}_{i_\ell-1-i_{\ell-1}-1}, \underbrace{K_\ell, \ldots, K_\ell}_{i_{\ell+1}-(i_\ell-1)-1}, \mathcal{K}_{\ell+1}, \ldots, \mathcal{K}_k),$$

$$|\mathcal{N}_=| = (n-k)!\mathsf{V}_{n-k}(\mathcal{K}_0, \mathcal{K}_1, \ldots, \underbrace{K_{\ell-1}, \ldots, K_{\ell-1}}_{i_\ell-i_{\ell-1}-1}, \underbrace{K_\ell, \ldots, K_\ell}_{i_{\ell+1}-i_\ell-1}, \mathcal{K}_{\ell+1}, \ldots, \mathcal{K}_k),$$

$$|\mathcal{N}_+| = (n-k)!\mathsf{V}_{n-k}(\mathcal{K}_0, \mathcal{K}_1, \ldots, \underbrace{K_{\ell-1}, \ldots, K_{\ell-1}}_{i_\ell+1-i_{\ell-1}-1}, \underbrace{K_\ell, \ldots, K_\ell}_{i_{\ell+1}-(i_\ell+1)-1}, \mathcal{K}_{\ell+1}, \ldots, \mathcal{K}_k).$$

To shorten the notation, let

$$\mathcal{K} := (\mathcal{K}_0, \mathcal{K}_1, \ldots, \underbrace{K_{\ell-1}, \ldots, K_{\ell-1}}_{i_\ell-i_{\ell-1}-2}, \underbrace{K_\ell, \ldots, K_\ell}_{i_{\ell+1}-i_\ell-2}, \mathcal{K}_{\ell+1}, \ldots, \mathcal{K}_k),$$

to get

$$\begin{aligned} |\mathcal{N}_-| &= (n-k)!\mathsf{V}_{n-k}(K_\ell, K_\ell, \mathcal{K}), \\ |\mathcal{N}_=| &= (n-k)!\mathsf{V}_{n-k}(K_{\ell-1}, K_\ell, \mathcal{K}), \\ |\mathcal{N}_+| &= (n-k)!\mathsf{V}_{n-k}(K_{\ell-1}, K_{\ell-1}, \mathcal{K}). \end{aligned} \tag{2.7}$$

With the representation (2.7) in hand, we get that the inequality (2.2) is equivalent to

$$\mathsf{V}_{n-k}(K_{\ell-1}, K_\ell, \mathcal{K})^2 \geq \mathsf{V}_{n-k}(K_{\ell-1}, K_{\ell-1}, \mathcal{K})\mathsf{V}_{n-k}(K_\ell, K_\ell, \mathcal{K}).$$

The latter inequality follows immediately from the Alexandrov-Fenchel inequality [16, Theorem 7.3.1]: For any convex bodies $C_1, \ldots, C_{n-k} \subseteq \mathbb{R}^{n-k}$ we have

$$\mathsf{V}_{n-k}(C_1, C_2, C_3, \ldots, C_{n-k})^2 \geq \mathsf{V}_{n-k}(C_1, C_1, C_3, \ldots, C_{n-k})\mathsf{V}_{n-k}(C_2, C_2, C_3, \ldots, C_{n-k}). \tag{AF}$$

This completes Stanley's proof of (2.2). Since our goal in this paper is to understand the equality cases of (2.2), the above discussion naturally leads to the investigation of the equality cases of the Alexandrov-Fenchel inequality itself.

2.3. The extremals of the Alexandrov-Fenchel inequality for convex polytopes

We start with the *support function* associated to a convex body: Given a convex body $C \subseteq \mathbb{R}^{n-k}$ we define $h_C : S^{n-k-1} \to \mathbb{R}$ by

$$h_C(\mathrm{u}) := \sup_{x \in C} \langle \mathrm{u}, x \rangle, \quad \text{for } \mathrm{u} \in S^{n-k-1}.$$

The support function evaluated at u gives the distance to the origin of the hyperplane orthogonal to u supporting C. The support function respects the summation of convex bodies in the sense that

$$h_{\lambda C+\lambda' C'} = \lambda h_C + \lambda' h_{C'},$$

for any convex bodies $C, C' \subseteq \mathbb{R}^{n-k}$ and scalars $\lambda, \lambda' \geq 0$. The function h_C completely describes C in the sense that two convex bodies are the same if their support functions are identical. That is, $C = C'$ if $h_C(\mathrm{u}) = h_{C'}(\mathrm{u})$ for every $\mathrm{u} \in S^{n-k-1}$. Since mixed volumes are invariant under translations, and scale proportionally with dilations, it is clear that equality holds in (AF) whenever there exist $a \geq 0$ and $\mathrm{v} \in \mathbb{R}^{n-k}$ such that $h_{C_1}(\mathrm{u}) = h_{aC_2+\mathrm{v}}(\mathrm{u})$ for every $\mathrm{u} \in S^{n-k-1}$. However, the difficulty in characterizing the extremals of the Alexandrov-Fenchel inequality stems from the fact that equality can be attained in (AF) even if h_{C_1} and $h_{aC_2+\mathrm{v}}$ agree on a very small subset of S^{n-k-1}. The complete characterization of the extremals of (AF) has been open for decades. But in the case of *polytopes*, which is the setting relevant to Stanley's inequalities, the problem was completely settled in [17]. In order to present the results of [17] we need some definitions. In the sequel, $B \subseteq \mathbb{R}^{n-k}$ always stands for the unit ball. Given a polytope $C \subseteq \mathbb{R}^{n-k}$ and $\mathrm{u} \in S^{n-k-1}$ we write

$$F(C, \mathrm{u}) := \{x \in C : \langle \mathrm{u}, x\rangle = h_C(\mathrm{u})\},$$

for the face of C in the direction u. We recall [16, Theorem 1.7.2] that

$$F(C + C', \mathrm{u}) = F(C, \mathrm{u}) + F(C', \mathrm{u}), \tag{2.8}$$

for any convex bodies C, C' and $\mathrm{u} \in S^{n-k-1}$.

Definition 2.4. Let $\mathcal{C} := (C_3, \dots, C_{n-k})$ be a nonempty collection of polytopes in $\mathbb{R}^{n-k}$. A vector $\mathrm{u} \in S^{n-k-1}$ is a $(B, \mathcal{C})$-*extreme normal direction* if, for any $\mathcal{C}' \subseteq \mathcal{C}$,

$$\dim\left(\sum_{C \in \mathcal{C}'} F(C, \mathrm{u})\right) \geq |\mathcal{C}'|.$$

One example of $(B, \mathcal{C})$-extreme normal directions can be found in Fig. 2.1. The definition of $(B, \mathcal{C})$-extreme normal directions plays a crucial role in the characterization of the extremals of the Alexandrov-Fenchel inequality for convex polytopes. For example, it follows from [17] that if $C_1, \dots, C_{n-k}$ are *full-dimensional* polytopes in $\mathbb{R}^{n-k}$, then, equality holds in (AF) if, and only if, there exist $a \geq 0$ and $\mathrm{v} \in \mathbb{R}^{n-k}$ such that

$$h_{C_1}(\mathrm{u}) = h_{aC_2+\mathrm{v}}(\mathrm{u}) \quad \text{for every } (B, \mathcal{C})\text{-extreme normal directions u.}$$

In the setting of Stanley's inequalities, the full-dimensionality assumption does not hold so we need the full power of the results of [17]. This requires a few definitions.

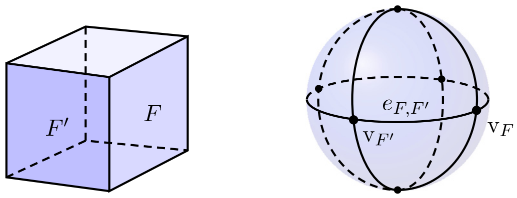


Fig. 2.1. Extreme normal directions associated to the cube. The vectors $\mathrm{v}_F, \mathrm{v}_{F'} \in S^2$ are the unit normals of the facets F, F', and the line $e_{F,F'}$ is the shortest geodesic between the nodes $\mathrm{v}_F, \mathrm{v}_{F'}$. The (Ball, Cube)-extreme normal directions comprises of the nodes and arcs in this embedded graph on the sphere S^2.

Definition 2.5. Let $\mathcal{C}$ be a nonempty collection of polytopes in $\mathbb{R}^{n-k}$.

- The collection $\mathcal{C}$ is *subcritical* if, for any collection $\mathcal{C}' \subseteq \mathcal{C}$, $\dim\left(\sum_{C\in\mathcal{C}'} C\right) \geq |\mathcal{C}'|$. A collection $\mathcal{C}' \subseteq \mathcal{C}$ is *sharp-subcritical* if $\dim\left(\sum_{C\in\mathcal{C}'} C\right) = |\mathcal{C}'|$.
- The collection $\mathcal{C}$ is *critical* if, for any nonempty collection $\mathcal{C}' \subseteq \mathcal{C}$, $\dim\left(\sum_{C\in\mathcal{C}'} C\right) \geq |\mathcal{C}'| + 1$. A collection $\mathcal{C}' \subseteq \mathcal{C}$ is *sharp-critical* if $\dim\left(\sum_{C\in\mathcal{C}'} C\right) = |\mathcal{C}'| + 1$.
- The collection $\mathcal{C}$ is *supercritical* if, for any nonempty collection $\mathcal{C}' \subseteq \mathcal{C}$, $\dim\left(\sum_{C\in\mathcal{C}'} C\right) \geq |\mathcal{C}'| + 2$.

The origin of the above definition is the following lemma, which characterizes the conditions under which mixed volumes are positive [16, Theorem 5.1.8].

Lemma 2.6. *(**Positivity of mixed volumes**) Let $C_1, \ldots, C_{n-k}$ be convex bodies in $\mathbb{R}^{n-k}$. Then, $\mathsf{V}_{n-k}(C_1, \ldots, C_{n-k}) > 0$ if, and only if,*

$$\dim\left(\sum_{C\in\mathcal{C}'} C\right) \geq |\mathcal{C}'| \quad \textit{for every collection } \mathcal{C}' \subseteq \{C_i\}_{i\in[\![1,n-k]\!]}.$$

For example, if the collection of polytopes $\mathcal{C} := (C_3, \ldots, C_{n-k})$ in (AF) is not subcritical, then Lemma 2.6 shows that equality holds in (AF) for trivial reasons: both sides of the inequality are zero. If $\mathcal{C}$ is subcritical with a sharp-subcritical collection, then the equality cases of (AF) can be reduced to the equality cases of the Alexandrov-Fenchel inequality in a lower dimension; we refer to [17] for details. The difficult equality cases of (AF) are the supercritical and, to a much larger degree, the critical collections. The following definition is needed for the characterization of the critical extremals of (AF).

Definition 2.7. Let $\mathcal{C} = (C_3, \ldots, C_{n-k})$ be a collection of polytopes in $\mathbb{R}^{n-k}$ and let (P, Q) be a pair of convex bodies in $\mathbb{R}^{n-k}$. The pair (P, Q) is a $\mathcal{C}$-*degenerate pair* if P is not a translate of Q,

$$\mathsf{V}_{n-k}(P, Q, \mathcal{C}) = 0, \quad \text{and} \quad \mathsf{V}_{n-k}(P, B, \mathcal{C}) = \mathsf{V}_{n-k}(Q, B, \mathcal{C}).$$

Theorem 2.8. *([17, Theorem 2.13, Corollary 2.16]) Let $C_1, \ldots, C_{n-k}$ be polytopes in $\mathbb{R}^{n-k}$ and let $\mathcal{C} := (C_3, \ldots, C_{n-k})$.*

- *Suppose $\mathcal{C}$ is supercritical. Then, equality holds in (AF) if, and only if, there exist $a \geq 0$ and $\mathrm{v} \in \mathbb{R}^{n-k}$ such that*

$$h_{C_1}(\mathrm{u}) = h_{aC_2+\mathrm{v}}(\mathrm{u}) \quad \textit{for all } (B,\mathcal{C})\textit{-extreme normal directions } \mathrm{u}.$$

- *Suppose $\mathcal{C}$ is critical. Then, equality holds in (AF) if, and only if, there exist $a \geq 0$, $\mathrm{v} \in \mathbb{R}^{n-k}$, and a number $0 \leq d < \infty$ of $\mathcal{C}$-degenerate pairs $(P_1, Q_1), \ldots, (P_d, Q_d)$, such that*

$$h_{C_1+\sum_{j=1}^d Q_j}(\mathrm{u}) = h_{aC_2+\mathrm{v}+\sum_{j=1}^d P_j}(\mathrm{u}) \quad \textit{for all } (B,\mathcal{C})\textit{-extreme normal directions } \mathrm{u}.$$

2.3.1. The extremals of Stanley's inequalities

The crux of our work lies in understanding how to apply Theorem 2.8 in our setting in order to get a *combinatorial* characterization of the equality cases of (2.2). For convenience and future reference, let us explicitly write Theorem 2.8 in our setting.

Theorem 2.9.

- *Suppose $\mathcal{K}$ is supercritical. Then, $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$ holds, if, and only if, there exist $a \geq 0$ and $\mathrm{v} \in \mathbb{R}^{n-k}$ such that*

$$h_{K_{\ell-1}}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}}(\mathrm{u}) \quad \textit{for all } (B,\mathcal{K})\textit{-extreme normal directions } \mathrm{u}.$$

- *Suppose $\mathcal{K}$ is critical. Then, $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$ holds, if, and only if, there exist $a \geq 0$, $\mathrm{v} \in \mathbb{R}^{n-k}$, and a number $0 \leq d < \infty$ of $\mathcal{K}$-degenerate pairs $(P_1, Q_1), \ldots, (P_d, Q_d)$, such that*

$$h_{K_{\ell-1}+\sum_{j=1}^d Q_j}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}+\sum_{j=1}^d P_j}(\mathrm{u}) \quad \textit{for all } (B,\mathcal{K})\textit{-extreme normal directions } \mathrm{u}.$$

Our proof proceeds by induction on k. The base case $k = 0$ is trivial as equality in (2.2) cannot occur because $|\mathcal{N}_-| = |\mathcal{N}_+| = 0$ while $|\mathcal{N}_=| = |\mathcal{N}|$. Hence, Theorem 1.3 and Theorem 1.5 hold trivially when $k = 0$. From here on we assume that $k \geq 1$ and that equality holds in (2.2):

$$\begin{aligned} &|\mathcal{N}_=|^2 = |\mathcal{N}_+||\mathcal{N}_-| \\ \iff \quad &\mathsf{V}_{n-k}(K_{\ell-1}, K_\ell, \mathcal{K})^2 = \mathsf{V}_{n-k}(K_{\ell-1}, K_{\ell-1}, \mathcal{K})\mathsf{V}_{n-k}(K_\ell, K_\ell, \mathcal{K}). \end{aligned}$$

Assumption 2.10. Theorem 1.3 and Theorem 1.5 hold true for $k - 1$.

We conclude this section by introducing the notions of criticality for posets. The relations between the criticality notions of Definition 2.5 and the following Definition 2.11 is given in Section 5.

Definition 2.11. Let $\bar{\alpha} = \{y_1, \ldots, y_{n-k}\} \cup \{x_1, \ldots, x_k\}$ be a poset, with a fixed chain $x_1 < \cdots < x_k$, and fix $1 \leq i_1 < \cdots < i_k \leq n$ such that $i_{\ell-1} + 1 < i_\ell < i_{\ell+1} - 1$ for some fixed $\ell \in [k]$. Suppose that $|\mathcal{N}_=| > 0$.

- The poset $\bar{\alpha}$ is *supercritical* if, for any integer $p \geq 1$ and $j_0 := -1 < j_1 < \cdots < j_p < k+1 =: j_{p+1}$, such that $i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ are positive for any $q \in [p]$, we have

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| \leq |\{q \in [p] : j_q \in \{\ell-1,\ell\}\}| - 2 + \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} (i_{j_{(q+1)}} - i_{j_q+1} - 1).$$

- The poset $\bar{\alpha}$ is *critical* if, for any integer $p \geq 1$ and $j_0 := -1 < j_1 < \cdots < j_p < k+1 =: j_{p+1}$, such that $i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ are positive for any $q \in [p]$, we have

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| \leq |\{q \in [p] : j_q \in \{\ell-1,\ell\}\}| - 1 + \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} (i_{j_{(q+1)}} - i_{j_q+1} - 1).$$

To get some intuition for Definition 2.11 note that when $|\mathcal{N}_=| > 0$ we have

$$|\bar{\alpha}_{>x_{r+1},<x_s}| \leq i_s - i_{r+1} - 1 \quad \forall\, r \leq s.$$

Hence, criticality is captured in Definition 2.11 by checking the tightness of the above bound. (Equivalent and more transparent definitions of (super)criticality of posets are given in [7, §10.7].)

Finally, let us remark that the case $k = 1$ is always supercritical, where we use that $|\mathcal{N}_-|, |\mathcal{N}_=|, |\mathcal{N}_+|$ are positive, as $|\mathcal{N}_=| > 0$ and $|\mathcal{N}_=|^2 = |\mathcal{N}_+||\mathcal{N}_-|$.

3. Linear extensions

In this section we introduce a number of ideas and tools that will simplify the proofs of our main results. Section 3.1 presents *decompositions* of $\mathcal{N}_-, \mathcal{N}_=, \mathcal{N}_+$. Section 3.2 uses the above decompositions to prove the sufficiency part of Theorem 1.3 and Theorem 10.1 (Proposition 3.2), and introduces conditions which are equivalent to Theorem 1.3 and Theorem 10.1 (Lemma 3.3). Finally, Section 3.3 introduces the technical tool of *closure* where relations are added to the poset $\bar{\alpha}$ based on linear extensions.

3.1. Decompositions of linear extensions

Fix $\circ \in \{-, =, +\}$ and $\star, * \in \{\nsim, \sim\}$. Recall Definition 1.2 and let

$$\mathcal{N}_\circ(\star, *) := \{\sigma \in \mathcal{N}_\circ : \text{ lower companion } \star\, x_\ell \text{ and upper companion } *\, x_\ell\}.$$

It is clear that we have the disjoint decompositions,

$$\begin{aligned}
|\mathcal{N}_-| &= |\mathcal{N}_-(\nsim, \nsim)| + |\mathcal{N}_-(\nsim, \sim)| + |\mathcal{N}_-(\sim, \nsim)| + |\mathcal{N}_-(\sim, \sim)|, \\
|\mathcal{N}_=| &= |\mathcal{N}_=(\nsim, \nsim)| + |\mathcal{N}_=(\nsim, \sim)| + |\mathcal{N}_=(\sim, \nsim)| + |\mathcal{N}_=(\sim, \sim)|, \qquad (3.1) \\
|\mathcal{N}_+| &= |\mathcal{N}_+(\nsim, \nsim)| + |\mathcal{N}_+(\nsim, \sim)| + |\mathcal{N}_+(\sim, \nsim)| + |\mathcal{N}_+(\sim, \sim)|.
\end{aligned}$$

The next result shows that, regardless of whether equality holds in (2.2), certain relations between terms in (3.1) always hold.

Lemma 3.1. *For any poset $\bar{\alpha}$ the following hold:*

(i) $|\mathcal{N}_-(\nsim, \nsim)| = |\mathcal{N}_=(\nsim, \nsim)| = |\mathcal{N}_+(\nsim, \nsim)|$.
(ii) $|\mathcal{N}_-(\nsim, \sim)| = |\mathcal{N}_=(\nsim, \sim)|$.
(iii) $|\mathcal{N}_=(\sim, \nsim)| = |\mathcal{N}_+(\sim, \nsim)|$.
(iv) $|\mathcal{N}_-(\sim, \nsim)| \leq |\mathcal{N}_-(\nsim, \sim)|$.
(v) $|\mathcal{N}_+(\nsim, \sim)| \leq |\mathcal{N}_+(\sim, \nsim)|$.

Proof. (i) We show $|\mathcal{N}_-(\nsim, \nsim)| = |\mathcal{N}_=(\nsim, \nsim)|$; the argument for $|\mathcal{N}_=(\nsim, \nsim)| = |\mathcal{N}_+(\nsim, \nsim)|$ is analogous. Let $\pi_{i_{\ell-1}, i_\ell} : [n] \to [n]$ be the permutation that swaps the positions of $i_{\ell-1}$ and i_ℓ. We claim that defining $\pi_{i_{\ell-1}, i_\ell}(\sigma) := \pi_{i_{\ell-1}, i_\ell} \circ \sigma$, for $\sigma \in \mathcal{N}_-(\nsim, \nsim)$, yields a bijection $\pi_{i_{\ell-1}, i_\ell} : \mathcal{N}_-(\nsim, \nsim) \to \mathcal{N}_=(\nsim, \nsim)$. That $\pi_{i_{\ell-1}, i_\ell}(\mathcal{N}_-(\nsim, \nsim)) \subseteq \mathcal{N}_=(\nsim, \nsim)$ follows from the fact that x_ℓ is incomparable to the element placed in i_ℓ so their positions can be swapped. Hence, to conclude that $\pi_{i_{\ell-1}, i_\ell}$ is a bijection it suffices to show that $\pi_{i_{\ell-1}, i_\ell}$ is invertible and that its inverse $\pi^{-1}_{i_{\ell-1}, i_\ell}$ satisfies $\pi^{-1}_{i_{\ell-1}, i_\ell}(\mathcal{N}_=(\nsim, \nsim)) \subseteq \mathcal{N}_-(\nsim, \nsim)$. The inverse $\pi^{-1}_{i_{\ell-1}, i_\ell}$ exists since $\pi^{-1}_{i_{\ell-1}, i_\ell} = \pi_{i_{\ell-1}, i_\ell}$. That $\pi_{i_{\ell-1}, i_\ell}(\mathcal{N}_=(\nsim, \nsim)) \subseteq \mathcal{N}_-(\nsim, \nsim)$ is clear.

(ii) Analogous argument to (i).

(iii) Analogous argument to (i).

(iv) Let $\pi_{i_\ell, i_\ell+1} : [n] \to [n]$ be the permutation that swaps the positions of i_ℓ and $i_{\ell+1}$. We claim that defining $\pi_{i_\ell, i_{\ell+1}}(\sigma) := \pi_{i_\ell, i_{\ell+1}} \circ \sigma$, for $\sigma \in \mathcal{N}_-(\sim, \nsim)$, yields an injection $\pi_{i_\ell, i_{\ell+1}} : \mathcal{N}_-(\sim, \nsim) \to \mathcal{N}_-(\nsim, \sim)$. Indeed, fix $\sigma \in \mathcal{N}_-(\sim, \nsim)$, so $\sigma(x_\ell) = i_\ell - 1$, and let $y_u := \sigma^{-1}(i_\ell), y_v := \sigma^{-1}(i_\ell + 1)$ so that, by the definition of $\mathcal{N}_-(\sim, \nsim)$, $x_\ell < y_u$ and $y_v \nsim x_\ell$. We cannot have $y_u < y_v$ since that would imply $x_\ell < y_u < y_v$ contradicting $y_v \nsim x_\ell$. Since $y_u = \sigma^{-1}(i_\ell), y_v = \sigma^{-1}(i_\ell + 1)$, we cannot have $y_v < y_u$ so we must have $y_u \nsim y_v$. It follows that swapping the positions of y_u and y_v in σ yields the linear extension $\pi_{i_\ell, i_{\ell+1}}(\sigma) \in \mathcal{N}_-(\nsim, \sim)$.

(v) Analogous argument to (iv). □

3.2. Sufficiency

The decompositions given in Section 3.1 help us prove the sufficiency of the conditions of Theorem 1.3(iii) and Theorem 1.5(iii).

Proposition 3.2. ***(Sufficient conditions)***

(a) Theorem 1.3(ii) $\Longrightarrow$ Theorem 1.3(i) and Theorem 1.5(ii) $\Longrightarrow$ Theorem 1.5(i).
(b) Theorem 1.5(iii) $\Longrightarrow$ Theorem 1.5(ii).
(c) Theorem 1.3(iii) $\Longrightarrow$ Theorem 1.5(iii) $\Longrightarrow$ Theorem 1.3(ii).

Proof. (a) Immediate.
(b) The conditions in Theorem 1.5(iii) read

$$
\begin{aligned}
&|\mathcal{N}_-(\sim,\sim)| = |\mathcal{N}_=(\sim,\sim)| = |\mathcal{N}_+(\sim,\sim)| = 0,\\
&|\mathcal{N}_-(\nsim,\sim)| = |\mathcal{N}_-(\sim,\nsim)| = \mathrm{N}_1,\\
&|\mathcal{N}_=(\nsim,\sim)| = |\mathcal{N}_=(\sim,\nsim)| = \mathrm{N}_1,\\
&|\mathcal{N}_+(\nsim,\sim)| = |\mathcal{N}_+(\sim,\nsim)| = \mathrm{N}_1,\\
&|\mathcal{N}_-(\nsim,\nsim)| = |\mathcal{N}_=(\nsim,\nsim)| = |\mathcal{N}_+(\nsim,\nsim)| = \mathrm{N}_2.
\end{aligned}
$$

Hence, (3.1) reads

$$
\begin{aligned}
|\mathcal{N}_-| &= \mathrm{N}_2 + \mathrm{N}_1 + \mathrm{N}_1 + 0 = \mathrm{N}_2 + 2\mathrm{N}_1,\\
|\mathcal{N}_=| &= \mathrm{N}_2 + \mathrm{N}_1 + \mathrm{N}_1 + 0 = \mathrm{N}_2 + 2\mathrm{N}_1,\\
|\mathcal{N}_+| &= \mathrm{N}_2 + \mathrm{N}_1 + \mathrm{N}_1 + 0 = \mathrm{N}_2 + 2\mathrm{N}_1,
\end{aligned}
$$

which is the statement in Theorem 1.5(ii).
(c) The first implication is immediate and the second implication follows from (b). □

In order to prove Theorem 1.3 and Theorem 1.5 it remains to show that Theorem 1.3(i) $\Longrightarrow$ Theorem 1.3(iii) and Theorem 1.5(i) $\Longrightarrow$ Theorem 1.5(iii). To this end, the following conditions will suffice.

Lemma 3.3.

(a) The conditions in Theorem 1.3(iii) hold if, and only if,

$$|\mathcal{N}_=(\nsim,\sim)| = |\mathcal{N}_=(\sim,\nsim)| = |\mathcal{N}_=(\sim,\sim)| = 0.$$

(b) Suppose $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$*. The conditions in Theorem 1.5(iii) hold if, and only if,*

$$|\mathcal{N}_-(\sim,\sim)| = |\mathcal{N}_+(\sim,\sim)| = 0.$$

Proof. We start with proof of (a). The "only if" part is clear. To prove the "if" part, assume that

$$|\mathcal{N}_=(\nsim,\sim)| = |\mathcal{N}_=(\sim,\nsim)| = |\mathcal{N}_=(\sim,\sim)| = 0,$$

which by (3.1) implies

$$|\mathcal{N}_=| = |\mathcal{N}_=(\nsim,\nsim)|.$$

On the other hand, Lemma 3.1(i) yields

$$N' := |\mathcal{N}_-(\nsim,\nsim)| = |\mathcal{N}_=(\nsim,\nsim)| = |\mathcal{N}_+(\nsim,\nsim)|,$$

so (3.1) reads

$$\begin{aligned}
|\mathcal{N}_-| &= N' + |\mathcal{N}_-(\nsim,\sim)| + |\mathcal{N}_-(\sim,\nsim)| + |\mathcal{N}_-(\sim,\sim)|,\\
|\mathcal{N}_=| &= N',\\
|\mathcal{N}_+| &= N' + |\mathcal{N}_+(\nsim,\sim)| + |\mathcal{N}_+(\sim,\nsim)| + |\mathcal{N}_+(\sim,\sim)|.
\end{aligned}$$

Stanley's inequality (2.2),

$$|\mathcal{N}_=|^2 \geq |\mathcal{N}_-||\mathcal{N}_+|,$$

implies that all the terms other than N' must vanish, which completes the proof.

We now prove (b). The 'only if" part is clear. To prove the "if" part, assume that

$$|\mathcal{N}_-(\sim,\sim)| = |\mathcal{N}_+(\sim,\sim)| = 0.$$

Using Lemma 3.1(i-iii), set

$$\begin{aligned}
N' &:= |\mathcal{N}_-(\nsim,\nsim)| = |\mathcal{N}_=(\nsim,\nsim)| = |\mathcal{N}_+(\nsim,\nsim)|,\\
N'_a &:= |\mathcal{N}_-(\nsim,\sim)| = |\mathcal{N}_=(\nsim,\sim)|,\\
N'_b &:= |\mathcal{N}_=(\sim,\nsim)| = |\mathcal{N}_+(\sim,\nsim)|,
\end{aligned}$$

so (3.1) reads

$$\begin{aligned}|\mathcal{N}_-| &= N' + N'_a + |\mathcal{N}_-(\sim,\nsim)|,\\ |\mathcal{N}_=| &= N' + N'_a + N'_b + |\mathcal{N}_=(\sim,\sim)|,\\ |\mathcal{N}_+| &= N' + N'_b + |\mathcal{N}_+(\nsim,\sim)|.\end{aligned}$$

By Lemma 3.1(iv-v),

$$|\mathcal{N}_-(\sim,\nsim)| \leq N'_a \quad \text{and} \quad |\mathcal{N}_+(\nsim,\sim)| \leq N'_b$$

so

$$\begin{aligned}|\mathcal{N}_-| &= N' + N'_a + |\mathcal{N}_-(\sim,\nsim)| \leq N' + 2N'_a,\\ |\mathcal{N}_=| &= N' + N'_a + N'_b + |\mathcal{N}_=(\sim,\sim)| \geq N' + N'_a + N'_b,\\ |\mathcal{N}_+| &= N' + N'_b + |\mathcal{N}_+(\nsim,\sim)| \leq N' + 2N'_b.\end{aligned}$$

Hence,

$$\begin{aligned}&(N' + 2N'_a)(N' + 2N'_b) = (N' + N'_a + N'_b)^2 - (N'_a - N'_b)^2 \leq (N' + N'_a + N'_b)^2\\ &\leq |\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+| \leq (N' + 2N'_a)(N' + 2N'_b).\end{aligned}$$

It follows that all of the above inequalities are in fact equalities. In particular,

$$|\mathcal{N}_=(\sim,\sim)| = 0, \tag{3.2}$$

$$N'_a = N'_b, \tag{3.3}$$

$$|\mathcal{N}_-(\sim,\nsim)| = N'_a, \tag{3.4}$$

$$|\mathcal{N}_+(\nsim,\sim)| = N'_b. \tag{3.5}$$

The identity (3.2), together with the assumption $|\mathcal{N}_-(\sim,\sim)| = |\mathcal{N}_+(\sim,\sim)| = 0$, implies that every linear extension in $\mathcal{N}_\circ$, for any $\circ \in \{-,=,+\}$, has either 0 or 1 comparable companions to x_ℓ. It remains to show that there exist nonnegative numbers $\mathrm{N}_1, \mathrm{N}_2$ such that

$$\begin{aligned}&|\mathcal{N}_-(\sim,\nsim)| = |\mathcal{N}_=(\sim,\nsim)| = |\mathcal{N}_+(\sim,\nsim)| = |\mathcal{N}_-(\nsim,\sim)| = |\mathcal{N}_=(\nsim,\sim)| = |\mathcal{N}_+(\nsim,\sim)| = \mathrm{N}_1,\\ &|\mathcal{N}_-(\nsim,\nsim)| = |\mathcal{N}_=(\nsim,\nsim)| = |\mathcal{N}_+(\nsim,\nsim)| = \mathrm{N}_2.\end{aligned}$$

The first part follows since

$$\begin{aligned}&|\mathcal{N}_-(\sim,\nsim)| \underset{(3.4)}{=} N'_a := |\mathcal{N}_-(\nsim,\sim)| \underset{\text{Lemma 3.1(ii)}}{=} |\mathcal{N}_=(\nsim,\sim)| \underset{(3.3)}{=} N'_b := |\mathcal{N}_=(\sim,\nsim)|\\ &\underset{\text{Lemma 3.1(iii)}}{=} |\mathcal{N}_+(\sim,\nsim)| \underset{(3.5)}{=} |\mathcal{N}_+(\nsim,\sim)| =: \mathrm{N}_1,\end{aligned}$$

and the second part follows by Lemma 3.1(i). □

We conclude the section with a corollary of the above lemmas, which will be needed for the proof of Theorem 1.6. (Note that the assumption in the following result that $\bar{\alpha}$ is critical can be relaxed to $|\mathcal{N}_=| > 0$, cf. Section 10.)

Corollary 3.4. *Let $\bar{\alpha}$ be a critical poset such that $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$, and assume that Theorem 1.3 and Theorem 1.5 hold true. Fix $\circ \in \{-, =, +\}$ and $\sigma \in \mathcal{N}_\circ(\sim, \nsim) \cup \mathcal{N}_\circ(\nsim, \sim)$. Then, the upper and lower companions are incomparable to each other.*

Proof. We start by establishing the claim in the case where $\circ$ is equal to $-$. Fix $\sigma \in \mathcal{N}_-(\sim, \nsim)$. If the upper and lower companions are comparable to each other, then, by transitivity, $\sigma \in \mathcal{N}_-(\sim, \sim)$, which is a contradiction. On the other hand, the proof of Lemma 3.3 shows that $|\mathcal{N}_-(\nsim, \sim)| = |\mathcal{N}_-(\sim, \nsim)|$. Hence, the map $\pi_{i_\ell, i_\ell+1} : \mathcal{N}_-(\sim, \nsim) \to \mathcal{N}_-(\nsim, \sim)$ defined in the proof of Lemma 3.1(iv) is a bijection. It follows that the upper and lower companions in any $\sigma \in \mathcal{N}_-(\nsim, \sim)$ cannot be comparable to each other, or else they will also be comparable to each other in $\pi_{i_\ell, i_\ell+1}(\sigma) \in \mathcal{N}_-(\sim, \nsim)$, which is a contradiction.

Analogous argument works when $\circ$ is equal to $+$. In the case when $\circ$ is equal to $=$, we note that Lemma 3.1(ii-iii) gives bijections $\mathcal{N}_-(\nsim, \sim) \to \mathcal{N}_=(\nsim, \sim)$ and $\mathcal{N}_+(\sim, \nsim) \to \mathcal{N}_=(\sim, \nsim)$, so we can argue as above to conclude that the upper and lower companions are incomparable. □

3.3. Closure

Since we are interested in the extremals of (2.2), it is beneficial to add relations to $\bar{\alpha}$ which are compatible with $\mathcal{N}_-, \mathcal{N}_=, \mathcal{N}_+$, while leaving these sets invariant.

Definition 3.5. Denote by $\mathrm{Cl}(\bar{\alpha})$ (the *closure* of $\bar{\alpha}$) the poset with the same elements as $\bar{\alpha}$ and with the partial order on $\mathrm{Cl}(\bar{\alpha})$ given by

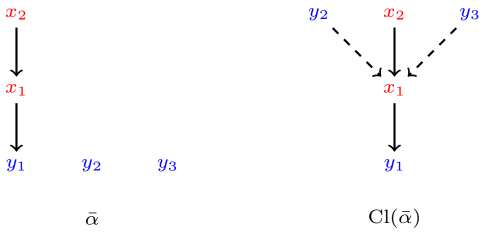
$$w < z \quad \text{if and only if} \quad \sigma(w) < \sigma(z) \;\forall\; \sigma \in \mathcal{N}_- \cup \mathcal{N}_= \cup \mathcal{N}_+.$$

Let

$$\mathcal{N}^{\mathrm{cl}} := \{\text{bijections } \sigma : \mathrm{Cl}(\bar{\alpha}) \to [n] : w \le z \Rightarrow \sigma(w) \le \sigma(z) \;\forall\; w, z \in \mathrm{Cl}(\bar{\alpha})\},$$

with the analogous $\mathcal{N}^{\mathrm{cl}}_\circ(\star, *)$ for $\circ \in \{-, =, +\}$ and $\star, * \in \{\nsim, \sim\}$.

We first need to check that Definition 3.5 is well-defined. Indeed, if $z_1, z_2, z_3 \in \mathrm{Cl}(\bar{\alpha})$ are such that $z_1 < z_2$ and $z_2 < z_3$ in $\mathrm{Cl}(\bar{\alpha})$, then, by definition, $\sigma(z_1) < \sigma(z_2)$ and $\sigma(z_2) < \sigma(z_3)$ for every $\sigma \in \bigcup_{\circ \in \{-,=,+\}} \mathcal{N}_\circ$, so $\sigma(z_1) < \sigma(z_2) < \sigma(z_3)$. It follows that $z_1 < z_3$ in $\mathrm{Cl}(\bar{\alpha})$.



$$\mathcal{N}_- = \{y_1x_1x_2y_2y_3,\ y_1x_1x_2y_3y_2\}$$

$$\mathcal{N}_= = \{y_1x_1y_2x_2y_3,\ y_1x_1y_3x_2y_2\}$$

$$\mathcal{N}_+ := \{y_1x_1y_2y_3x_2,\ y_1x_1y_3y_2x_2\}$$

Fig. 3.1. Hasse diagram (arrows point from smaller to larger elements) of posets in Example 3.7, together with their (identical) sets of linear extensions, showing that new relations can occur under the closure operation.

Let us now show that the relations in $\mathrm{Cl}(\bar{\alpha})$ are compatible with the relations in $\bar{\alpha}$.

Lemma 3.6. *If $z_1 < z_2$ in $\bar{\alpha}$ then $z_1 < z_2$ in $Cl(\bar{\alpha})$. If $z_1 \nsim z_2$ in $Cl(\bar{\alpha})$ then $z_1 \nsim z_2$ in $\bar{\alpha}$.*

Proof. If $z_1 < z_2$ in $\bar{\alpha}$, then $\sigma(z_1) < \sigma(z_2)$ for every $\sigma \in \bigcup_{\circ \in \{-,=,+\}} \mathcal{N}_\circ$, so $z_1 < z_2$ in $\mathrm{Cl}(\bar{\alpha})$. The contrapositive of this statement is that if $z_1 \nsim z_2$ in $\mathrm{Cl}(\bar{\alpha})$ then $z_1 \nsim z_2$ in $\bar{\alpha}$. $\square$

While the closure operation is compatible with the relations in $\bar{\alpha}$, it can introduce new relations as the following example demonstrates.

Example 3.7. Let $\bar{\alpha} = \{x_1, x_2, y_1, y_2, y_3\}$, so $k = 2$ and $n = 5$, and suppose that the only relations are $x_1 < x_2$ and $y_1 < x_1$. Let $i_1 = 2, i_2 = 4$ and $l = 2$, and note that $i_{\ell-1} + 1 = i_1 + 1 = 3 < 4 = i_2 = i_\ell < 5 = (n+1) - 1 = i_{\ell+1} - 1$. Let us show that, in $\mathrm{Cl}(\bar{\alpha})$, $x_1 < y_2$ and $x_1 < y_3$, relations which do not hold in $\bar{\alpha}$. Indeed, take any $\sigma \in \mathcal{N}_- \cup \mathcal{N}_= \cup \mathcal{N}_+$ and note that $\sigma(x_1) = i_1 = 2$ so, since $y_1 < x_1$, we must have $\sigma(y_1) = 1$. Thus, $\sigma(y_2), \sigma(y_3) > 2$, and hence, in $\mathrm{Cl}(\bar{\alpha})$, $x_1 < y_2$ and $x_1 < y_3$. See Fig. 3.1.

The next result shows that our basic objects of interest remain more-or-less invariant under the closure operation. To simplify the notation, let (i) (res. (ii)) stand for the conditions in Theorem 1.3(i) and Theorem 1.5(i) (res. Theorem 1.3(ii) and Theorem 1.5(ii)), and let ($\mathrm{iii}_{\mathrm{supcrit}}$) (res. ($\mathrm{iii}_{\mathrm{crit}}$)) stand for the conditions in Theorem 1.3(iii) (res. Theorem 1.5(iii)). We use an upper script "cl" for the corresponding notation when $\mathrm{Cl}(\bar{\alpha})$, rather than $\bar{\alpha}$, is used.

Proposition 3.8. *The set $Cl(\bar{\alpha})$ is a poset satisfying*

(a) $\mathcal{N}_\circ^{cl} = \mathcal{N}_\circ$ *for every* $\circ \in \{-,=,+\}$.
(b) $(i^{cl}) \Longleftrightarrow (i)$,
(c) $(ii^{cl}) \Longleftrightarrow (ii)$,
(d) $(iii^{cl}_{supcrit}) \Longrightarrow (iii_{supcrit})$ *and* $(iii^{cl}_{crit}) \Longrightarrow (iii_{crit})$.

Proof. $\mathrm{Cl}(\bar{\alpha})$ is indeed a poset since irreflexivity is immediate and transitivity was checked after Definition 3.5.

(a) We show that $\mathcal{N}_=^{\mathrm{cl}} = \mathcal{N}_=$; the proof that $\mathcal{N}_-^{\mathrm{cl}} = \mathcal{N}_-$ and $\mathcal{N}_+^{\mathrm{cl}} = \mathcal{N}_+$ is analogous. We start by observing that since Lemma 3.6 yields "$w < z$ in $\bar{\alpha}$ implies $w < z$ in $\mathrm{Cl}(\bar{\alpha})$", it follows that "$\sigma \in \mathcal{N}_=^{\mathrm{cl}}$ implies $\sigma \in \mathcal{N}_=$". Conversely, let $\sigma \in \mathcal{N}_=$ so it suffices to show that $\sigma \in \mathcal{N}^{\mathrm{cl}}$. The latter holds since if $w < z$ in $\mathrm{Cl}(\bar{\alpha})$, then it must be, by the definition of $\mathrm{Cl}(\bar{\alpha})$, that $\sigma(w) < \sigma(z)$, and hence $\sigma \in \mathcal{N}^{\mathrm{cl}}$.

(b) Follows trivially from (a).

(c) Follows trivially from (a).

(d) We show that

$$\begin{aligned} &|\mathcal{N}_=^{\mathrm{cl}}(\nsim, \sim)| = |\mathcal{N}_=^{\mathrm{cl}}(\sim, \nsim)| = |\mathcal{N}_=^{\mathrm{cl}}(\sim, \sim)| = 0 \\ \Longrightarrow \quad &|\mathcal{N}_=(\nsim, \sim)| = |\mathcal{N}_=(\sim, \nsim)| = |\mathcal{N}_=(\sim, \sim)| = 0, \end{aligned} \tag{3.6}$$

which proves $(\mathrm{iii}^{\mathrm{cl}}_{\mathrm{supcrit}}) \Longrightarrow (\mathrm{iii}_{\mathrm{supcrit}})$ by Lemma 3.3(a). To establish (3.6) we show $|\mathcal{N}_=(\nsim, \sim)| = 0$; the proof of $|\mathcal{N}_=(\sim, \nsim)| = 0$ and $|\mathcal{N}_=(\sim, \sim)| = 0$ is analogous. Suppose $|\mathcal{N}_=(\nsim, \sim)| > 0$ so there exists $\sigma \in \mathcal{N}_=$ such that $\sigma(x_\ell) = i_\ell$ and $x_\ell < \sigma^{-1}(i_\ell + 1)$ in $\bar{\alpha}$. By (a), $\sigma \in \mathcal{N}_=^{\mathrm{cl}}$, and by Lemma 3.6, $x_\ell < \sigma^{-1}(i_\ell + 1)$ in $\mathrm{Cl}(\bar{\alpha})$. It follows that $\sigma \in \mathcal{N}_=^{\mathrm{cl}}(\nsim, \sim) \cup \mathcal{N}_=^{\mathrm{cl}}(\sim, \sim)$, which is a contradiction.

Next we show

$$|\mathcal{N}_-^{\mathrm{cl}}(\sim, \sim)| = |\mathcal{N}_+^{\mathrm{cl}}(\sim, \sim)| = 0 \quad \Longrightarrow \quad |\mathcal{N}_-(\sim, \sim)| = |\mathcal{N}_+(\sim, \sim)| = 0. \tag{3.7}$$

Since $(\mathrm{iii}^{\mathrm{cl}}_{\mathrm{crit}}) \Longrightarrow (\mathrm{i}^{\mathrm{cl}}_{\mathrm{crit}})$ by Proposition 3.2(a–b), and since $(\mathrm{i}^{\mathrm{cl}}) \Longleftrightarrow (\mathrm{i})$ by part (a), the proof will be complete by Lemma 3.3(b).

To establish (3.7) we show that $|\mathcal{N}_+^{\mathrm{cl}}(\sim, \sim)| = 0 \Rightarrow |\mathcal{N}_+(\sim, \sim)| = 0$; the proof of $|\mathcal{N}_-^{\mathrm{cl}}(\sim, \sim)| = 0 \Rightarrow |\mathcal{N}_-(\sim, \sim)| = 0$ is analogous. Indeed, if $|\mathcal{N}_+(\sim, \sim)| > 0$ then there exists $\sigma \in \mathcal{N}_+$ such that $\sigma(x_\ell) = i_\ell + 1$ and $\sigma^{-1}(i_\ell - 1), \sigma^{-1}(i_\ell)$ are both smaller than x_ℓ in $\bar{\alpha}$. By (a), $\sigma \in \mathcal{N}_+^{\mathrm{cl}}$, and by Lemma 3.6, $\sigma^{-1}(i_\ell - 1), \sigma^{-1}(i_\ell)$ are both smaller than x_ℓ in $\mathrm{Cl}(\bar{\alpha})$. It follows that $\sigma \in \mathcal{N}_+^{\mathrm{cl}}(\sim, \sim)$. In other words, $|\mathcal{N}_+(\sim, \sim)| > 0 \Rightarrow |\mathcal{N}_+^{\mathrm{cl}}(\sim, \sim)| > 0$, which is the contrapositive of what we want to show. $\square$

4. Proof outline

In this section we outline the proof of the characterization of the extremals of Stanley's inequalities. The first step is to understand how we use the closure procedure. We have the following equivalences:

$$\begin{array}{ccccccc} (\mathrm{i}^{\mathrm{cl}}) & \underset{\text{Thms. 9.1, 10.1 + Lem. 3.3}}{\Longrightarrow} & (\mathrm{iii}^{\mathrm{cl}}) & \underset{\text{Prop. 3.2(b-c)}}{\Longrightarrow} & (\mathrm{ii}^{\mathrm{cl}}) & \underset{\text{trivial}}{\Longrightarrow} & (\mathrm{i}^{\mathrm{cl}}) \\ \Updownarrow \text{Prop. 3.8(b)} & & \Downarrow \text{Prop. 3.8(d)} & & \Updownarrow \text{Prop. 3.8(c)} & & \Updownarrow \text{Prop. 3.8(b)} \end{array}$$

$$\text{(i)} \qquad\qquad \text{(iii)} \underset{\text{Prop. 3.2(b-c)}}{\Longrightarrow} \text{(ii)} \underset{\text{trivial}}{\Longrightarrow} \text{(i)}$$

The only implication that has not been proven thus far is $(\mathrm{i}^{\mathrm{cl}}) \Longrightarrow (\mathrm{iii}^{\mathrm{cl}})$, which will follow from Theorem 9.1, Theorem 10.1, and Lemma 3.3. Hence, from here on we may assume:

Assumption 4.1.

$$\bar{\alpha} = \mathrm{Cl}(\bar{\alpha}).$$

Note that Remark 1.7, which is proven in Proposition 7.5, does not require Assumption 4.1. The first extremals we need to characterize are those arising in the trivial case $|\mathcal{N}_=| = 0$, which we dispose of in Theorem 5.3. Assuming that $|\mathcal{N}_=| > 0$, the characterization of (1.3) is divided to three types of classes, *subcritical*, *supercritical*, and *critical*. By subcritical we mean that $\mathcal{K}$ is subcritical. The supercritical and critical settings were defined in Definition 2.5 and Definition 2.11.

The characterization of the **subcritical extremals** relies on the *splitting mechanism* (Definition 6.2 and Proposition 6.4). The idea is that if $\mathcal{K}$ is truly subcritical, rather than critical, we can reduce the problem to the extremals of a poset with a shorter chain $\{x_i\}$. Arguing by induction, we then characterize the subcritical extremals (Theorem 6.6).

For the **supercritical extremals**, the starting point is Theorem 2.9 which yields that $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$ holds, if, and only if, there exist $a \geq 0$ and $\mathrm{v} \in \mathbb{R}^{n-k}$ such that

$$h_{K_{\ell-1}}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}}(\mathrm{u}) \quad \text{for all } (B,\mathcal{K})\text{-extreme normal directions u.} \tag{4.1}$$

The identity (4.1) constitutes a system of equations (one equation for each u) and the goal is to interpret these equations as combinatorial constraints on the poset $\bar{\alpha}$. Hence, the first important step is to find enough $(B,\mathcal{K})$–extreme normal directions which can be described combinatorially. This is achieved in Section 8 (Proposition 8.2(a-d)) by using the *mixing* phenomenon (Section 7.2). Once these directions are found in Section 8, Section 9 is dedicated to plugging these directions back into (4.1) and analyzing the outcomes. The second important step is to show that the scalar a and the vector v in (4.1) satisfy $a = 1$ and $\mathrm{v}_j = 0$ for certain j's. The identity (4.1) then further simplifies and provides the bulk of the desired characterization of the extremals (Theorem 9.1). We explain in Section 9 how to control a and v.

The starting point for the **critical extremals** is again Theorem 2.9, but now we need to use its second part which states that $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$ holds, if, and only if, there exist $a \geq 0$, $\mathrm{v} \in \mathbb{R}^{n-k}$, and a number $0 \leq d < \infty$ of $\mathcal{K}$-degenerate pairs $(P_1, Q_1), \ldots, (P_d, Q_d)$, such that

$$h_{K_{\ell-1}+\sum_{j=1}^d Q_j}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}+\sum_{j=1}^d P_j}(\mathrm{u}) \quad \text{for all } (B,\mathcal{K})\text{-extreme normal directions u.}$$

The presence of the degenerate pairs causes great difficulties (which are not just technical since, as we saw, new extremals do indeed arise for critical posets). The first key idea to resolve these problems is to find a sub-poset of $\bar{\alpha}$ on which we have more-or-less a supercritical behavior. From a geometric standpoint, this corresponds to finding a subspace $E^{\perp}$ such that

$$h_{K_{\ell-1}}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}}(\mathrm{u}) \quad \text{for all } (B,\mathcal{K})\text{-extreme normal directions u which are contained in } E^{\perp}. \tag{4.2}$$

The identification of $E^{\perp}$ and its properties relies on the mixing properties of the *maximal splitting pair* (Section 7.10). Even after identifying $E^{\perp}$ we face the problem that (4.2) provides less constraints than (4.1) due to the restriction to the subspace $E^{\perp}$. Hence, we cannot derive enough combinatorial constraints on $\bar{\alpha}$. The solution is to find even more $(B,\mathcal{K})$-extreme normal directions which were not needed for supercritical posets (Proposition 8.2(e-h)). With these new directions in hand, Section 10 proceeds roughly as Section 9 to show that $a = 1$ and $\mathrm{v}_j = 0$ for certain j's. This description is an oversimplification since the situation is in fact much more delicate. It is precisely this delicacy which leads to the new extremals for critical posets.

5. Notions of criticality

In this section we start building our dictionary between convex geometry and combinatorics. The first building block is a correspondence between geometric and combinatorial notions of criticality, which will be used throughout this work. Section 5.1 starts with the easiest correspondence (Lemma 5.1), which connects the linear spans of polytopes in $\mathcal{K}$ with subsets of $\bar{\alpha}$. Consequently, we characterize the trivial extremals which appear when $|\mathcal{N}_=| = 0$ (Theorem 5.3). Section 5.2 is dedicated to the equivalences between geometric and combinatorial notions of criticality (Proposition 5.7), and their consequences on sharp-subcritical and sharp-critical collections (Lemmas 5.10, 5.11).

5.1. The trivial extremals

We start with some notation. Given a convex body C let $\mathrm{aff}(C)$ stand for the affine hull of C, and let $\mathrm{Lin}(C)$ stand for the vector space obtained by the translation of $\mathrm{aff}(C)$ to the origin, i.e., $\mathrm{Lin}(C) := \mathrm{aff}(C) - c_0 = \mathrm{span}(C - c_0)$, for any $c_0 \in C$. Given a collection $\mathcal{C}$ of convex bodies, it is immediate to see that

$$\mathrm{Lin}\left(\sum_{C\in\mathcal{C}} C\right) = \mathrm{span}\left((\mathrm{Lin}(C_1), \dots, \mathrm{Lin}(C_{|\mathcal{C}|})\right). \tag{5.1}$$

The following lemma relates the combinatorics of subsets of α to the linear spans of the polytopes in $\{K_i\}$.

Lemma 5.1. *Let $j_0 := -1 < j_1 < \cdots < j_p < k+1 =: j_{p+1}$ and set*

$$\mathcal{K}' := (\underbrace{K_{j_1}, \dots, K_{j_1}}_{\kappa_1}, \dots, \underbrace{K_{j_p}, \dots, K_{j_p}}_{\kappa_p}),$$

where $\kappa_1, \dots, \kappa_p$ are positive integers. Then,

$$\mathrm{Lin}\left(\sum_{K \in \mathcal{K}'} K\right) = \mathbb{R}^{\beta_{\{j_1,\dots,j_p\}}},$$

and, consequently,

$$\dim\left(\sum_{K \in \mathcal{K}'} K\right) = n - k - \sum_{q=0}^{p} |\alpha_{>x_{j_q+1}, <x_{j_{(q+1)}}}|.$$

Proof. Combining (5.1) and (2.5) shows that $\mathrm{Lin}\left(\sum_{K\in\mathcal{K}'} K\right) = \mathbb{R}^{\beta_{\{j_1,\dots,j_p\}}}$. It follows that

$$\dim\left(\sum_{K \in \mathcal{K}'} K\right) = \left|\bigcup_{q=1}^{p} \beta_{j_q}\right| = \left|\bigcup_{q=1}^{p} \alpha\backslash(\alpha_{<x_{j_q}} \cup \alpha_{>x_{j_q+1}})\right| = \left|\alpha \Big\backslash \bigcap_{q=1}^{p}(\alpha_{<x_{j_q}} \cup \alpha_{>x_{j_q+1}})\right|.$$

The proof is complete by (2.6), and by noting that the sets $\{\alpha_{>x_{j_q+1}, <x_{j_{(q+1)}}}\}_{q\in[\![0,p]\!]}$ are disjoint. □

As a first application of Lemma 5.1, we dispose of the trivial extremals. Before doing so, we present the following definition which will be used throughout the paper.

Definition 5.2. A pair (r, s) is *splitting* if $0 \le r+1 < s \le k+1$ and $(r+1, s) \ne (0, k+1)$. A splitting pair (r, s) is an *ℓ-splitting pair* if $r+1 < \ell < s$.

Theorem 5.3. *(**Trivial extremals**) We have $|\mathcal{N}_=| = 0$ if, and only if, there exists a splitting pair (r, s) such that*

$$|\bar{\alpha}_{>x_{r+1}, <x_s}| > i_s - i_{r+1} - 1.$$

Proof. $\Longleftarrow$: Suppose there exists a splitting pair (r, s) such that

$$|\bar{\alpha}_{>x_{r+1}, <x_s}| > i_s - i_{r+1} - 1.$$

Every $\sigma \in \mathcal{N}_=$ must satisfy $\sigma(z) \in [\![i_{r+1}+1, i_s - 1]\!]$ for every $z \in \bar{\alpha}_{>x_{r+1}, <x_s}$. Since $|[\![i_{r+1}+1, i_s-1]\!]| = (i_s - 1) - (i_{r+1}+1) + 1 = i_s - i_{r+1} - 1 < |\bar{\alpha}_{>x_{r+1}, <x_s}|$, we see that no such σ can exist.

$\Longrightarrow$: If $|\mathcal{N}_=| = 0$ then, by (2.7) and Lemma 2.6, there exist $0 \le j_1 < \cdots < j_p \le k$, and positive integers $\kappa_1, \dots, \kappa_p$, with $\kappa_q \le i_{j_q+1} - i_{j_q} - 1$ for $q \in [p]$, such that, with

$$\mathcal{K}' = (\underbrace{K_{j_1}, \dots, K_{j_1}}_{\kappa_1}, \dots, \underbrace{K_{j_p}, \dots, K_{j_p}}_{\kappa_p}) \subseteq (K_{\ell-1}, K_\ell, \mathcal{K}),$$

we have

$$\dim\left(\sum_{K\in\mathcal{K}'} K\right) < |\mathcal{K}'|.$$

Let $j_0 := -1,\ j_{p+1} := k+1$ and use Lemma 5.1 to get

$$\dim\left(\sum_{K\in\mathcal{K}'} K\right) = n - k - \sum_{q=0}^{p} |\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}|.$$

On the other hand,

$$\begin{aligned}
|\mathcal{K}'| = \sum_{q=1}^{p} \kappa_q \le \sum_{q=1}^{p} [i_{j_q+1} - i_{j_q} - 1] = n - k - i_{j_{p+1}} + i_{j_0+1} + k + 1 + \sum_{q=1}^{p} (i_{j_q+1} - i_{j_q} - 1) \\
= n - k - \left(\sum_{q=1}^{p+1} i_{j_q}\right) + \left(\sum_{q=0}^{p} i_{j_q+1}\right) + j_{p+1} - j_0 - (p+1) \\
= n - k - \left(\sum_{q=0}^{p} i_{j_{(q+1)}}\right) + \left(\sum_{q=0}^{p} i_{j_q+1}\right) + j_{p+1} - j_0 - (p+1) \\
= n - k - \sum_{q=0}^{p} (i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1).
\end{aligned}$$

It follows that

$$\sum_{q=0}^{p} (i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1) < \sum_{q=0}^{p} |\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}|. \tag{5.2}$$

Since

$$|\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}| = 1_{\{j_q+1<j_{(q+1)}\}}\, |\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}|,$$

$$i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1 = 1_{\{j_q+1<j_{(q+1)}\}}\, (i_{j_{(q+1)}} - i_{j_q+1} - j_{(q+1)} + j_q + 1),$$

the inequality (5.2) is equivalent to

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}}(i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1) < \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}}|\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}|.$$

Using

$$1_{\{j_q+1<j_{(q+1)}\}}\,|\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}} \backslash \alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}| = j_{(q+1)} - j_q - 2,$$

we get that (5.2) is equivalent to

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}}(i_{j_{(q+1)}} - i_{j_q+1} - 1) < \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}}|\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}|.$$

Hence, there must exist a pair $(j_q + 1, j_{(q+1)})$, with $j_q + 1 < j_{(q+1)}$, such that

$$|\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| > i_{j_{(q+1)}} - i_{j_q+1} - 1.$$

Since $(j_q + 1, j_{(q+1)}) \neq (0, k + 1)$, because $j_q + 1 = 0 \Rightarrow q = 0$ so $j_{(q+1)} = j_1 \leq j_p < j_{p+1} = k + 1$, we conclude that there exists a splitting pair (r, s) such that

$$|\bar{\alpha}_{>x_{r+1},<x_s}| > i_s - i_{r+1} - 1. \quad \square$$

Remark 5.4. Theorem 5.3 is the same as the result of Chan, Pak, and Panova in [9, Theorem 1.12], where it was proved using purely combinatorial arguments.

In light of Theorem 5.3 we assume from here on that $|\mathcal{N}_=| > 0$. Note that $|\mathcal{N}_=| > 0$ implies, by (2.7) and Lemma 2.6, that $\mathcal{K}$ is subcritical. To summarize:

Assumption 5.5.

$$|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|, \quad |\mathcal{N}_=| > 0, \quad \text{and} \quad \mathcal{K} \text{ is subcritical.}$$

Remark 5.6. For future reference, we note that under Assumption 5.5, $\bar{\alpha}$ cannot be totally ordered. Indeed, if $\bar{\alpha}$ is totally ordered, then at least two elements in $\{|\mathcal{N}_=|, |\mathcal{N}_-|, |\mathcal{N}_+|\}$ are zero. But since $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$, that would imply that $|\mathcal{N}_=| = 0$.

5.2. Equivalences of criticality notions

The next result is at the base of the correspondence between criticality notions in our geometric and combinatorial settings, namely, the equivalence between Definition 2.5 and Definition 2.11.

Proposition 5.7. *Fix a nonnegative integer c. The following are equivalent.*

(1) For any integer $p \geq 1$ and $j_0 := -1 < j_1 < \cdots < j_p < k+1 =: j_{p+1}$ such that $i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ are positive for any $q \in [p]$, it holds that with any

$$\mathcal{K}' := (\underbrace{K_{j_1}, \ldots, K_{j_1}}_{\kappa_1}, \ldots, \underbrace{K_{j_p}, \ldots, K_{j_p}}_{\kappa_p}) \subseteq \mathcal{K},$$

where $\kappa_q \leq i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ are positive integers, we have

$$\dim\left(\sum_{K \in \mathcal{K}'} K\right) \geq |\mathcal{K}'| + c.$$

(2) For any integer $p \geq 1$, $j_0 := -1 < j_1 < \cdots < j_p < k+1 =: j_{p+1}$ such that $i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ are positive for any $q \in [p]$, it holds that

$$\sum_{q=0}^{p} 1_{\{j_q+1 < j_{(q+1)}\}} \, |\bar{\alpha}_{>x_{j_q+1}, <x_{j_{(q+1)}}}| \leq |\{q \in [p] : j_q \in \{\ell-1,\ell\}\}| - c$$
$$+ \sum_{q=0}^{p} 1_{\{j_q+1 < j_{(q+1)}\}} \, (i_{j_{(q+1)}} - i_{j_q+1} - 1).$$

The proof of Proposition 5.7 follows the logic of the proof of Theorem 5.3, but it is more complicated since we now work with collections $\mathcal{K}' \subseteq \mathcal{K}$, rather than $\mathcal{K}' \subseteq (K_{\ell-1}, K_\ell, \mathcal{K})$. This leads to the presence of the term $1_{j_{(q+1)}=\ell} + 1_{j_q+1=\ell}$ in the proof below.

Proof of Proposition 5.7. Fix

$$\mathcal{K}' := (\underbrace{K_{j_1}, \ldots, K_{j_1}}_{\kappa_1}, \ldots, \underbrace{K_{j_p}, \ldots, K_{j_p}}_{\kappa_p}) \subseteq \mathcal{K},$$

where $p \geq 1$, $j_0 := -1 < j_1 < \cdots < j_p < k+1 =: j_{p+1}$, and $0 < \kappa_q \leq i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$. By Lemma 5.1,

$$\dim\left(\sum_{K \in \mathcal{K}'} K\right) = n - k - \sum_{q=0}^{p} |\alpha_{>x_{j_q+1}, <x_{j_{(q+1)}}}|.$$

On the other hand, using $\ell \notin \{0, k+1\}$, and arguing as in the proof of Theorem 5.3,

$$|\mathcal{K}'| = \sum_{q=1}^{p} \kappa_q \leq \sum_{q=1}^{p} \left(i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}\right)$$
$$= n - k - \sum_{q=0}^{p} \left(i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1 + 1_{j_{(q+1)}=\ell} + 1_{j_q+1=\ell}\right).$$

Hence, given c, we have that

$$\dim\left(\sum_{K\in\mathcal{K}'} K\right) < |\mathcal{K}'| + c,$$

if and only if

$$\sum_{q=0}^{p} |\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}| > -c + \sum_{q=0}^{p}\left(i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1 + 1_{j_{(q+1)}=\ell} + 1_{j_q+1=\ell}\right). \tag{5.3}$$

Conversely, if (5.3) holds, then we may take $\mathcal{K}'$ to be such that $\kappa_q = i_{j_q+1} - i_{j_q} - 1 - 1_{j_q\in\{\ell-1,\ell\}}$ for every q, to get $|\mathcal{K}'| = n - k - \sum_{q=0}^{p}\left(i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1 + 1_{j_{(q+1)}=\ell} + 1_{j_q+1=\ell}\right)$. We may then conclude that $\dim\left(\sum_{K\in\mathcal{K}'} K\right) < |\mathcal{K}'| + c$. Hence, we get

$$\dim\left(\sum_{K\in\mathcal{K}'} K\right) \geq |\mathcal{K}'| + c \quad\Longleftrightarrow\quad (5.4),$$

where

$$\sum_{q=0}^{p} |\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}| \leq -c + \sum_{q=0}^{p}\left(i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1 + 1_{j_{(q+1)}=\ell} + 1_{j_q+1=\ell}\right). \tag{5.4}$$

Since

$$\sum_{q=0}^{p}(1_{j_{(q+1)}=\ell} + 1_{j_q+1=\ell}) = |\{q\in[p] : j_q \in \{\ell-1,\ell\}\}|,$$

and

$$|\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}| = 1_{\{j_q+1<j_{(q+1)}\}}\,|\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}|,$$
$$i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1 = 1_{\{j_q+1<j_{(q+1)}\}}(i_{j_{(q+1)}} - i_{j_q+1} - j_{(q+1)} + j_q + 1),$$

the inequality (5.4) is equivalent to

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}}\,|\alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}|$$
$$\leq |\{q\in[p] : j_q\in\{\ell-1,\ell\}\}| - c + \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}}\,\left(i_{j_{(q+1)}} - i_{j_q+1} - j_{q+1} + j_q + 1\right),$$

$$= |\{q \in [p] : j_q \in \{\ell - 1, \ell\}\}| - c$$
$$+ \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} \left(i_{j_{(q+1)}} - i_{j_q+1} - 1 - j_{(q+1)} + j_q + 2\right).$$

Using

$$1_{\{j_q+1<j_{(q+1)}\}} |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}} \backslash \alpha_{>x_{j_q+1},<x_{j_{(q+1)}}}| = j_{(q+1)} - j_q - 2,$$

we find that (5.4) is equivalent to

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| \le |\{q \in [p] : j_q \in \{\ell - 1, \ell\}\}| - c$$
$$+ \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} (i_{j_{(q+1)}} - i_{j_q+1} - 1). \quad \square$$

In contrast to Proposition 5.7, the next lemma, which treats the opposite inequality of Proposition 5.7, holds for a *fixed* $\mathcal{K}'$.

Lemma 5.8. *Fix an integer $p \ge 1$ and $j_0 := -1 < j_1 < \cdots < j_p < k + 1 =: j_{p+1}$ such that $i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ are positive for any $q \in [p]$. Let*

$$\mathcal{K}' := (\underbrace{K_{j_1}, \dots, K_{j_1}}_{\kappa_1}, \dots, \underbrace{K_{j_p}, \dots, K_{j_p}}_{\kappa_p}) \subseteq \mathcal{K},$$

where κ_q are integers such that $0 < \kappa_q \le i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ for all $q \in [p]$, be such that

$$\dim \left(\sum_{K \in \mathcal{K}'} K \right) \le |\mathcal{K}'| + c.$$

Then,

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| \ge |\{q \in [p] : j_q \in \{\ell - 1, \ell\}\}| - c$$
$$+ \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} (i_{j_{(q+1)}} - i_{j_q+1} - 1).$$

Proof. We proceed as in the proof of Proposition 5.7 and use

$$|\mathcal{K}'| = \sum_{q=1}^{p} \kappa_q \le \sum_{q=1}^{p} \left(i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}\right),$$

to reason about a fixed collection $\mathcal{K}'$. $\square$

As a consequence of Lemma 5.8, we get the following combinatorial information about *sharp* collections.

Lemma 5.9. *Fix $c \geq 0$, an integer $p \geq 1$, and $j_0 := -1 < j_1 < \cdots < j_p < k+1 =: j_{p+1}$ such that $i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ are positive for any $q \in [p]$. Suppose there exist*

$$\mathcal{K}' := (\underbrace{K_{j_1}, \ldots, K_{j_1}}_{\kappa_1}, \ldots, \underbrace{K_{j_p}, \ldots, K_{j_p}}_{\kappa_p}) \subseteq \mathcal{K},$$

where κ_q are integers such that $0 < \kappa_q \leq i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ for all $q \in [p]$, such that

$$\dim\left(\sum_{K \in \mathcal{K}'} K\right) = |\mathcal{K}'| + c.$$

Then,

$$|\{q \in [p] : j_q \in \{\ell-1,\ell\}\}| \leq c.$$

Proof. The assumption $\dim\left(\sum_{K\in\mathcal{K}'} K\right) = |\mathcal{K}'| + c$ implies $\dim\left(\sum_{K\in\mathcal{K}'} K\right) \leq |\mathcal{K}'| + c$, so by Lemma 5.8,

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| \geq |\{q \in [p] : j_q \in \{\ell-1,\ell\}\}| - c + \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} (i_{j_{(q+1)}} - i_{j_q+1} - 1). \tag{5.5}$$

On the other hand, since $|\mathcal{N}_=| > 0$, we have

$$1_{\{j_q+1<j_{(q+1)}\}} |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| \leq 1_{\{j_q+1<j_{(q+1)}\}} (i_{j_{(q+1)}} - i_{j_q+1} - 1) \tag{5.6}$$

(because $|[\![i_{j_q+1}+1, i_{j_{(q+1)}} - 1]\!]| \leq i_{j_{(q+1)}} - i_{j_q+1} - 1$), so

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| \leq \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} (i_{j_{(q+1)}} - i_{j_q+1} - 1). \tag{5.7}$$

Combining (5.5) and (5.7) we get

$$|\{q \in [p] : j_q \in \{\ell-1,\ell\}\}| \leq c. \quad \square$$

We are now ready to characterize the sharp-(sub)critical collections. We start with the sharp-subcritical collections.

Lemma 5.10. ***(Sharp-subcritical collections)*** *Fix an integer $p \geq 1$, and $j_0 := -1 < j_1 < \cdots < j_p < k+1 =: j_{p+1}$ such that $i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ are positive for any $q \in [p]$. Suppose that*

$$\mathcal{K}' := (\underbrace{K_{j_1}, \ldots, K_{j_1}}_{\kappa_1}, \ldots, \underbrace{K_{j_p}, \ldots, K_{j_p}}_{\kappa_p}) \subseteq \mathcal{K},$$

where κ_q are integers such that $0 < \kappa_q \leq i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ for all $q \in [p]$, is sharp-subcritical. Then,

$$\forall\, q \in [p]: \quad j_q \notin \{\ell-1, \ell\} \quad \textit{and} \quad 1_{\{j_q+1<j_{(q+1)}\}}\, |\bar{\alpha}_{>x_{j_q+1}, <x_{j_{(q+1)}}}| \\ = 1_{\{j_q+1<j_{(q+1)}\}}\, (i_{j_{(q+1)}} - i_{j_q+1} - 1).$$

Proof. Take $c = 0$ in Lemma 5.9 to get

$$|\{q \in [p] : j_q \in \{\ell-1, \ell\}\}| = 0. \tag{5.8}$$

Since $\dim\left(\sum_{K \in \mathcal{K}'} K\right) \leq |\mathcal{K}'|$, and $\mathcal{K}$ is subcritical, applying Lemma 5.8 and Proposition 5.7, with $c = 0$, yields

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}}\, |\bar{\alpha}_{>x_{j_q+1}, <x_{j_{(q+1)}}}| = \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}}\, (i_{j_{(q+1)}} - i_{j_q+1} - 1).$$

By (5.6), it follows that, for every $0 \leq j_q \leq k+1$,

$$1_{\{j_q+1<j_{(q+1)}\}}\, |\bar{\alpha}_{>x_{j_q+1}, <x_{j_{(q+1)}}}| = 1_{\{j_q+1<j_{(q+1)}\}}\, (i_{j_{(q+1)}} - i_{j_q+1} - 1). \quad \square$$

We now turn to the sharp-critical collections. The assumption made in the following lemma does not follow automatically from the fact that $\mathcal{K}$ is sharp-critical. Rather, we will be able to make this assumption only after Section 6, and the motivation behind this assumption can be found in Theorem 6.6. The proof, however, is similar in spirit to the rest of this section so it is included here.

Lemma 5.11. ***(Sharp-critical collections)*** *Suppose $|\bar{\alpha}_{>x_{r+1}, <x_s}| \leq i_s - i_{r+1} - 2$ for every splitting pair (r, s). Fix an integer $p \geq 1$, and $j_0 := -1 < j_1 < \cdots < j_p < k+1 =: j_{p+1}$ such that $i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ are positive for any $q \in [p]$. Then, every*

$$\mathcal{K}' := (\underbrace{K_{j_1}, \ldots, K_{j_1}}_{\kappa_1}, \ldots, \underbrace{K_{j_p}, \ldots, K_{j_p}}_{\kappa_p}) \subseteq \mathcal{K},$$

where κ_q are integers such that $0 < \kappa_q \leq i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$ for all $q \in [p]$, satisfying

$$\dim\left(\sum_{K\in\mathcal{K}'} K\right) = |\mathcal{K}'| + 1,$$

must be of the form

$$\mathcal{K}' = (\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_{r-1}, \mathcal{K}_r, \mathcal{K}_s, \mathcal{K}_{s+1}, \ldots, \mathcal{K}_k),$$

where (r,s) is an ℓ-splitting pair satisfying

$$|\bar{\alpha}_{>x_{r+1},<x_s}| = i_s - i_{r+1} - 2.$$

Proof. First note that $(j_q+1, j_{(q+1)}) \neq (0, k+1)$ because $j_q + 1 = 0 \Rightarrow q = 0$ so $j_{(q+1)} = j_1 \leq j_p < j_{p+1} = k+1$. The assumption $|\bar{\alpha}_{>x_{r+1},<x_s}| \leq i_s - i_{r+1} - 2$ for every splitting pair (r,s) implies that

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| \leq \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} \left(i_{j_{(q+1)}} - i_{j_q+1} - 2\right)$$
$$= -\left|\{q \in [p] : j_q + 1 < j_{(q+1)}\}\right| + \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} (i_{j_{(q+1)}} - i_{j_q+1} - 1).$$

On the other hand, since $\dim\left(\sum_{K\in\mathcal{K}'} K\right) \leq |\mathcal{K}'| + 1$, applying Lemma 5.8 with $c = 1$ yields

$$\sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} \left|\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}\right| \geq \left|\{q \in [p] : j_q \in \{\ell-1,\ell\}\}\right| - 1 + \sum_{q=0}^{p} 1_{\{j_q+1<j_{(q+1)}\}} \left(i_{j_{(q+1)}} - i_{j_q+1} - 1\right). \tag{5.9}$$

We conclude that

$$\left|\{q \in [p] : j_q \in \{\ell-1,\ell\}\}\right| + \left|\{q \in [p] : j_q + 1 < j_{(q+1)}\}\right| \leq 1. \tag{5.10}$$

Since

$$\left|\{q \in [p] : j_q + 1 < j_{(q+1)}\}\right| = 0 \quad \Longrightarrow \quad \{j_1, \ldots, j_p\} = \{1, \ldots, k\},$$

we get

$$|\{q \in [p] : j_q + 1 < j_{(q+1)}\}| = 0 \quad \Longrightarrow \quad |\{q \in [p] : j_q \in \{\ell - 1, \ell\}\}| = 2.$$

Hence, (5.10) can hold if, and only if,

$$|\{q \in [p] : j_q \in \{\ell - 1, \ell\}\}| = 0 \quad \text{and} \quad |\{q \in [p] : j_q + 1 < j_{(q+1)}\}| = 1.$$

It follows that

$$\mathcal{K}' = (\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_{r-1}, \mathcal{K}_r, \mathcal{K}_s, \mathcal{K}_{s+1}, \ldots, \mathcal{K}_k),$$

where (r, s) is an ℓ-splitting pair. Finally, plugging in $|\{q \in [p] : j_q \in \{\ell-1, \ell\}\}| = 0$ into (5.9), and using that (r, s) is the only pair $(j_q, j_{(q+1)})$ satisfying $j_q + 1 < j_{(q+1)}$, yields

$$|\bar{\alpha}_{>x_{r+1},<x_s}| \geq -1 + [i_s - i_{r+1} - 1] = i_s - i_{r+1} - 2.$$

On the other hand, by assumption, $|\bar{\alpha}_{>x_{r+1},<x_s}| \leq i_s - i_{r+1} - 2$, so we conclude

$$|\bar{\alpha}_{>x_{r+1},<x_s}| = i_s - i_{r+1} - 2. \quad \square$$

Remark 5.12. In the proof of Lemma 5.11 we only used the condition $\dim \left(\sum_{K \in \mathcal{K}'} K\right) \leq |\mathcal{K}'| + 1$, so the reader might wonder why we assume that $\mathcal{K}'$ is sharp-critical. By Assumption 5.5, the only other possibility would be for $\mathcal{K}'$ to be sharp-subcritical, but this is impossible by Lemma 5.10 and the assumption $|\bar{\alpha}_{>x_{r+1},<x_s}| \leq i_s - i_{r+1} - 2$ for every splitting pair (r, s).

6. Splitting and the subcritical extremals

In this section we introduce the *splitting* mechanism for posets, which is connected to a reduction to lower dimensional extremals. Consequently, we characterize the subcritical extremals (Theorem 6.6). To motivate the splitting mechanism recall that, by Lemma 5.10, we know that every sharp-subcritical collection

$$\mathcal{K}' := (K_{j_1}, \ldots, K_{j_1}, \ldots, K_{j_p}, \ldots, K_{j_p}) \subseteq \mathcal{K},$$

must satisfy

$$\forall\, q \in [p]: \quad j_q \notin \{\ell - 1, \ell\} \quad \text{and} \quad 1_{\{j_q+1<j_{(q+1)}\}}\, |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| = 1_{\{j_q+1<j_{(q+1)}\}}\, (i_{j_{(q+1)}} - i_{j_q+1} - 1).$$

Fix an index j_q such that $j_q \notin \{\ell - 1, \ell\}$ and $j_q + 1 < j_{(q+1)}$, so that

$$|\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| = i_{j_{(q+1)}} - i_{j_q+1} - 1.$$

Since $|[\![i_{j_q+1}+1, i_{j_{(q+1)}}-1]\!]| = i_{j_{(q+1)}} - i_{j_q+1} - 1$, we must have

$$\bar{\alpha}_{\geq x_{j_q+1}, \leq x_{j_{(q+1)}}} \overset{\text{bijection}}{\mapsto} [\![i_{j_q+1}, i_{j_{(q+1)}}]\!]$$

under any linear extension. This means that the poset $\bar{\alpha}$ can be *split* by factoring out the poset $\bar{\alpha}_{\geq x_{j_q+1}, \leq x_{j_{(q+1)}}}$, so that we are left with a poset with a shorter chain. We will show that $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$ implies that equality holds in Stanley's inequalities *also for the poset with the shorter chain.* We may then resort to our induction hypothesis that the extremals in the case where the chain size is $< k$ were already characterized.

Remark 6.1. The splitting mechanism described in this section can be viewed as a combinatorial equivalence of the projection formula for mixed volumes [16, Theorem 5.3.1]. This is another building block of our dictionary between geometry and combinatorics.

We now proceed to formalize the above splitting mechanism.

Definition 6.2. The *split* of $\bar{\alpha}$, based on a splitting pair (r, s), is given by defining posets $\bar{\alpha}_1, \bar{\alpha}_2$ as

$$\bar{\alpha}_1 := \bar{\alpha}_{\geq x_{r+1}, \leq x_s} \quad \text{and} \quad \bar{\alpha}_2 := (\bar{\alpha} \backslash \bar{\alpha}_1) \cup \{x\},$$

where the relations for x are defined via $x * z$, for $* \in \{<, >\}$ and $z \in \bar{\alpha} \backslash \bar{\alpha}_1$, if, and only if, there exists $w \in \bar{\alpha}_1$ such that $w * z$.[2]

Let (r, s) be a splitting pair satisfying $\ell \notin \{r+1, s\}$. We will define the analogues of $\mathcal{N}_-, \mathcal{N}_=, \mathcal{N}_+$ associated with the posets $\bar{\alpha}_1, \bar{\alpha}_2$. This requires distinguishing between two cases: (1) $x_\ell \in \{x_{r+2}, \dots, x_{s-1}\}$ and (2) $x_\ell \in \{x_1, \dots, x_r\} \cup \{x_{s+1}, \dots, x_k\}$; note that $x_\ell \notin \{x_{r+1}, x_s\}$ by assumption.[3] For $\iota = 1, 2$ let

$$\mathcal{N}^\iota := \{\text{bijections } \sigma : \bar{\alpha}_\iota \to [|\bar{\alpha}_\iota|] : w \leq z \Rightarrow \sigma(w) \leq \sigma(z) \ \forall \ w, z \in \bar{\alpha}_i\},$$

and, given $\circ \in \{-, =, +\}$, let $1_\circ := 1_{\{\circ \text{ is } +\}} - 1_{\{\circ \text{ is } -\}}$.

Case (1). For $\circ \in \{-, =, +\}$ set

$$\mathcal{N}^1_\circ := \{\sigma \in \mathcal{N}^1 : \sigma(x_j) = i_j - i_{r+1} + 1_{j=\ell} 1_\circ \text{ for } j \in [\![r+1, s]\!]\},$$

$$\mathcal{N}^2_\circ := \{\sigma \in \mathcal{N}^2 : \sigma(x_j) = i_j \text{ for } j \in [\![0, r]\!],\ \sigma(x) = i_{r+1},$$
$$\text{and } \sigma(x_j) = i_j - (i_s - i_{r+1}) \text{ for } j \in [\![s+1, k+1]\!]\};$$

[2] The new element x should be thought of as a compression of $\bar{\alpha}_1$ into one element, namely x. The relations for x are consistent since we cannot have $w_1 < z < w_2$ for $w_1, w_2 \in \bar{\alpha}_1, z \in \bar{\alpha} \backslash \bar{\alpha}_1$ because this would imply that $x_r \leq z \leq x_s$, and hence $z \in \bar{\alpha}_1$, which is a contradiction.

[3] We use the convention $\{x_a, \dots, x_z\} = \varnothing$ when $z < a$; e.g., $\{x_1, \dots, x_{r-2}\} = \varnothing$ when $r = 0$.

note that the definition of $\mathcal{N}_\circ^2$ is independent of $\circ$.

Case (2). For $\circ \in \{-,=,+\}$ set

$$\mathcal{N}_\circ^1 := \{\sigma \in \mathcal{N}^1 : \sigma(x_j) = i_j - i_{r+1} \text{ for } j \in [\![r+1, s]\!]\},$$

and

$$\mathcal{N}_\circ^2 := \{\sigma \in \mathcal{N}^2 : \sigma(x_j) = i_j + 1_{\{j=\ell\}}1_\circ \text{ for } j \in [\![0, r]\!],\ \sigma(x) = i_{r+1}, \\ \text{and } \sigma(x_j) = i_j - (i_s - i_{r+1}) + 1_{j=\ell}1_\circ \text{ for } j \in [\![s+1, k+1]\!]\};$$

note that the definition of $\mathcal{N}_\circ^1$ is independent of $\circ$.

Before exploiting the splitting mechanism we start with a quick observation.

Lemma 6.3. *For every splitting pair* (r, s)*,*

$$|\bar{\alpha}_{>x_{r+1},<x_s}| \le i_s - i_{r+1} - 1 - 1_{r+1=\ell} - 1_{s=\ell}. \tag{6.1}$$

Proof. The converse of Theorem 5.3 yields

$$|\bar{\alpha}_{>x_{r+1},<x_s}| \le i_s - i_{r+1} - 1 \quad \text{for every splitting pair } (r, s).$$

Hence, it suffices to consider the case where either $r+1 = \ell$ or $s = \ell$. Suppose $r+1 = \ell$; the case $s = \ell$ is proven analogously. Then, every $\sigma \in \mathcal{N}_+$ (which must exist since $|\mathcal{N}_=| > 0 \Rightarrow |\mathcal{N}_+| > 0$ as $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$) satisfies $\sigma(x_{r+1}) = i_{r+1} + 1$ and $\sigma(x_s) = i_s$. Hence, given $z \in \bar{\alpha}_{>x_{r+1},<x_s}$, the number of available spots for $\sigma(z)$ is $|[\![i_{r+1}+2, i_s-1]\!]| = (i_s - 1) - (i_{r+1} + 2) + 1 = i_s - i_{r+1} - 2$. □

Proposition 6.4. *Fix a splitting pair* (r, s) *satisfying* $\ell \notin \{r+1, s\}$*, and let* $\bar{\alpha}_1, \bar{\alpha}_2$ *be the split based on* (r, s)*. One of the following must occur:*

(i) $|\mathcal{N}_=^\iota|^2 = |\mathcal{N}_-^\iota||\mathcal{N}_+^\iota|$ *for every* $\iota \in \{1, 2\}$*.*
(ii) $|\bar{\alpha}_{>x_{r+1},<x_s}| \le i_s - i_{r+1} - 2$*.*

Proof. We will prove the proposition under the assumption that case (1) occurs; the proof for case (2) is analogous. Note that under case (1) we trivially have $|\mathcal{N}_=^2|^2 = |\mathcal{N}_-^2||\mathcal{N}_+^2|$ since $\mathcal{N}_\circ^2$ is independent of $\circ$.

It suffices to show that if (ii) is false then (i) is true. This will be proven by showing that if (ii) is false, then, for any $\circ \in \{-,=,+\}$,

$$|\mathcal{N}_\circ| = |\mathcal{N}_\circ^1||\mathcal{N}_\circ^2|, \tag{6.2}$$

where we recall that $\mathcal{N}_\circ^2$ is independent of $\circ$. Plugging (6.2) into $|\mathcal{N}_=|^2 = |\mathcal{N}_-|\mathcal{N}_+|$ gives $|\mathcal{N}_=^1|^2|\mathcal{N}_\circ^2|^2 = |\mathcal{N}_-^1||\mathcal{N}_+^1||\mathcal{N}_\circ^2|^2$. Canceling $|\mathcal{N}_\circ^2|$ on both sides ($|\mathcal{N}_\circ^2| > 0$ since $|\mathcal{N}_=| > 0$) gives (i).

We now turn to prove (6.2) under the assumption that (ii) is false. By (6.1), (ii) being false is equivalent to $|\bar{\alpha}_{>x_{r+1},<x_s}| = i_s - i_{r+1} - 1$, i.e., $|\bar{\alpha}_1| = i_s - i_{r+1} + 1$. We will prove (6.2) by constructing a bijection $b : \mathcal{N}_\circ \to \mathcal{N}^1_\circ \times \mathcal{N}^2_\circ$ for $\circ \in \{-, =, +\}$. Fix $\circ \in \{-, =, +\}$ and define a map b via $b = (b_1, b_2)$, with $b_1 : \mathcal{N}_\circ \to \mathcal{N}^1_\circ,\ b_2 : \mathcal{N}_\circ \to \mathcal{N}^2_\circ$, where we set, for each $\sigma \in \mathcal{N}_\circ$,

$$\text{For } z \in \bar{\alpha}_1: \quad b_1(\sigma)(z) = \sigma(z) - i_{r+1},$$

$$\text{For } z \in \bar{\alpha}_2: \quad b_2(\sigma)(z) = \begin{cases} \sigma(z) \text{ if } \sigma(z) \in [\![0, i_{r+1} - 1]\!], \\ i_{r+1} \text{ if } z = x, \\ \sigma(z) - (i_s - i_{r+1}) \text{ if } \sigma(z) \in [\![i_s + 1, n + 1]\!]. \end{cases}$$

We will first check that, given $\sigma \in \mathcal{N}_\circ$, $b_1(\sigma) \in \mathcal{N}^1_\circ$ and $b_2(\sigma) \in \mathcal{N}^2_\circ$. We will then construct a map $b' : \mathcal{N}^1_\circ \times \mathcal{N}^2_\circ \to \mathcal{N}_\circ$ and show that $b \circ b' = b' \circ b = \mathrm{Id}$, completing the proof. That $b_1(\sigma) \in \mathcal{N}^1_\circ$ and $b_2(\sigma) \in \mathcal{N}^2_\circ$ follows from the definitions of $\mathcal{N}^1_\circ, \mathcal{N}^2_\circ$ and the fact that $\sigma \in \mathcal{N}_\circ$. The map $b' : \mathcal{N}^1_\circ \times \mathcal{N}^2_\circ \to \mathcal{N}_\circ$ is defined by taking $\sigma_\iota \in \mathcal{N}^\iota_\circ$, for $\iota = 1, 2$, and setting, for $z \in \bar{\alpha}$,

$$b'(\sigma_1, \sigma_2)(z) = \begin{cases} \sigma_2(z) & \text{if } z \in \bar{\alpha}_2 \text{ and } \sigma_2(z) \in [\![0, i_{r+1} - 1]\!] \\ \sigma_1(z) + i_{r+1} & \text{if } z \in \bar{\alpha}_1 \\ \sigma_2(z) + (i_s - i_{r+1}) & \text{if } z \in \bar{\alpha}_2 \text{ and } \sigma_2(z) \in [\![i_{r+1} + 1, |\bar{\alpha}_2|]\!] \end{cases}.$$

To see that $b'(\sigma_1, \sigma_2) \in \mathcal{N}_\circ$ we first need to check that given $z < w$ we have $b'(\sigma_1, \sigma_2)(w) < b'(\sigma_1, \sigma_2)(z)$. If $w, z \in \bar{\alpha}_1$ or $w, z \in \bar{\alpha}_2$, this follows from $\sigma_i \in \mathcal{N}^\iota_\circ$, for $\iota = 1, 2$, so it remains to check $w \in \bar{\alpha}_1,\ z \in \bar{\alpha}_2$ and $w \in \bar{\alpha}_2,\ z \in \bar{\alpha}_1$; we check the first case and the second case is analogous. Suppose that $w \in \bar{\alpha}_1$ and $z \in \bar{\alpha}_2$. Then, we must have $\sigma_2(z) \in [\![0, i_{r+1} - 1]\!]$ since, by the definition of x, $w > z \Rightarrow x > z$ and $\sigma_2(x) = i_{r+1}$. Hence, $b'(\sigma_1, \sigma_2)(w) = \sigma_1(w) + i_{r+1} > \sigma_2(z) = b'(\sigma_1, \sigma_2)(z)$. Now that we know that $b'(\sigma_1, \sigma_2)$ respects the relations of $\bar{\alpha}$, in order to show that $b'(\sigma_1, \sigma_2) \in \mathcal{N}_\circ$, it remains to check that $b'(\sigma_1, \sigma_2)(x_j) = i_j + 1_{j=\ell} 1_\circ$ for all $1 \le j \le k$. This follows immediately from the definitions of $\mathcal{N}^\iota_\circ$ for $\iota \in \{1, 2\}$ and $\circ \in \{-, =, +\}$. Finally, that $b \circ b' = b' \circ b = \mathrm{Id}$ follows from the construction of b and b'. □

The next result provides a *geometric* characterization under which the case in Proposition 6.4(i) occurs.

Lemma 6.5. *Let $\mathcal{K}' \subseteq \mathcal{K}$ be a sharp subcritical collection. Then, there exists a splitting pair (r, s) satisfying $\ell \notin \{r + 1, s\}$, with a corresponding split $\bar{\alpha}_1, \bar{\alpha}_2$, such that $K_r, K_s \in \mathcal{K}'$ and $|\mathcal{N}^\iota_=|^2 = |\mathcal{N}^\iota_-||\mathcal{N}^\iota_+|$ for every $\iota \in \{1, 2\}$.*

Proof. By Lemma 5.10,

$$\mathcal{K}' = (\mathcal{K}_{j_1}, \ldots, \mathcal{K}_{j_p}),$$

where $j_0 := -1$, $0 \leq j_1 < \cdots < j_p \leq k$, $j_{p+1} := k+1$, $\kappa_q \leq i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$, and $p \in [n-k-2]$, must satisfy

$$\forall\, q \in [p]: \quad j_q \notin \{\ell-1,\ell\} \quad \text{and} \quad 1_{\{j_q+1<j_{(q+1)}\}}\, |\bar{\alpha}_{>x_{j_q+1},<x_{j_{(q+1)}}}| \\ = 1_{\{j_q+1<j_{(q+1)}\}}\, [i_{j_{(q+1)}} - i_{j_q+1} - 1].$$

Note that, for any $0 \leq q \leq p$, $(j_q+1, j_{(q+1)}) \neq (0, k+1)$. Indeed, for the latter to occur we need to have $p = 1$ and $q = 0$, but then $(j_0+1, j_{(0+1)}) = (0, j_p) \neq (0, k+1)$ as $j_p < k+1$. We now show that there exists $0 \leq q' \leq p$ such that $(j_{q'}+1, j_{(q'+1)})$ is a splitting pair. Indeed, if not, then $j_q + 1 = j_{(q+1)}$ for every $0 \leq q \leq p$ so we get $j_1 = 0, j_2 = 1, \ldots, j_{p+1} = k+1$ which contradicts $j_q \notin \{\ell-1,\ell\}$. Setting $r := j_{q'}$, $s := j_{(q'+1)}$, we get a splitting pair (r,s) such that $K_r, K_s \in \mathcal{K}'$ and $|\bar{\alpha}_{>x_{r+1},<x_s}| = i_s - i_{r+1} - 1$. By Proposition 6.4, we must have $|\mathcal{N}_=^\iota|^2 = |\mathcal{N}_-^\iota||\mathcal{N}_+^\iota|$ for every $\iota \in \{1,2\}$. □

Using Lemma 6.5, the characterization of the subcritical extremals of Stanley's inequalities now follows.

Theorem 6.6. ***(Subcritical extremals)***
Suppose that $\mathcal{K}$ has a sharp-subcritical collection. Then there exists a splitting pair (r,s) such that the associated posets split $\bar{\alpha}_1, \bar{\alpha}_2$ satisfies $|\mathcal{N}_=^\iota|^2 = |\mathcal{N}_-^\iota||\mathcal{N}_+^\iota|$ for every $\iota \in \{1,2\}$.

Our induction hypothesis Assumption 2.10 is that Theorem 1.3 and Theorem 1.5 hold for $k-1$. Hence, without loss of generality we may assume from now on that

$$\text{For all splits } \bar{\alpha}_1, \bar{\alpha}_2: \quad |\mathcal{N}_=^\iota|^2 \neq |\mathcal{N}_-^\iota||\mathcal{N}_+^\iota| \quad \forall\, \iota \in \{1,2\}. \tag{6.3}$$

By Theorem 6.6, the assumption (6.3) implies that $\mathcal{K}$ is critical. Further, by Proposition 6.4,

$$|\bar{\alpha}_{>x_{r+1},<x_s}| \leq i_s - i_{r+1} - 2 \quad \text{for every splitting pair } (r,s) \text{ satisfying } \ell \notin \{r+1, s\},$$

so using in addition Lemma 6.3, we get

$$|\bar{\alpha}_{>x_{r+1},<x_s}| \leq i_s - i_{r+1} - 2 \quad \text{for every splitting pair } (r,s).$$

Putting everything together we assume from now on:

Assumption 6.7. The collection $\mathcal{K}$ is critical and

$$|\bar{\alpha}_{\geq x_{r+1},\leq x_s}| \leq i_s - i_{r+1} \quad \text{for every splitting pair } (r,s).$$

7. Mixing

Under the current assumptions, we know that $\bar{\alpha}$ cannot be totally ordered (Remark 5.6). In this section, we develop the notion of *mixing* which takes advantage of the fact that $\bar{\alpha}$ must have some incomparable elements. The level of mixing will depend on the criticality notions developed in Section 5, which will be further developed in the current section. We begin with Section 7.1 which characterizes the locations where elements of the poset can be placed. We then introduce in Section 7.2 the notions of criticality and maximality for splitting pairs. Finally, Section 7.3 provides information on the mixing properties of splitting pairs.

7.1. Range

A fixed element $y \in \alpha$ can only be placed in a limited number of locations under any linear extension. For example, if α is totally ordered, there would be only one such location. We start by defining a few quantities associated to y which will provide information on the possible placements of y under linear extensions.

Definition 7.1. Given $y \in \alpha$ let $i_{\max}(y)$ be the maximum index such that $y > x_{i_{\max}(y)}$ and let $i_{\min}(y)$ be the minimum index such that $y < x_{i_{\min}(y)}$. Set

$$l_\circ(y) := \max_{r \le i_{\max}(y)} (i_r^\circ + |\bar{\alpha}_{>x_r, \le y}|) \quad \text{and} \quad u_\circ(y) := \min_{s \ge i_{\min}(y)} (i_s^\circ - |\bar{\alpha}_{\ge y, <x_s}|),$$

where

$$i_j^\circ := i_j + 1_{j=\ell} 1_\circ,$$

and let

$$m_{\min}^\circ(y) := \min_{\sigma \in \mathcal{N}_\circ} \sigma(y) \quad \text{and} \quad m_{\max}^\circ(y) := \max_{\sigma \in \mathcal{N}_\circ} \sigma(y).$$

Note that i_j° is the location where x_j is placed under every linear extension in $\mathcal{N}_\circ$. Hence, for any choice of $r \le i_{\max}(y)$ (res. $s \ge i_{\min}(y)$), y must be placed at a location at least as large (res. small) as $i_r^\circ + |\bar{\alpha}_{>x_r, \le y}|$ (res. $i_s^\circ - |\bar{\alpha}_{\ge y, <x_s}|$).

Definition 7.1 immediately implies the following relations between $l_\circ$ (res. $u_\circ$) for $\circ \in \{-, =, +\}$:

Lemma 7.2. *Fix $y \in \alpha$. Then,*

(i) $l_=(y) - 1 \le l_-(y) \le l_=(y) \le l_+(y) \le l_=(y) + 1$.
(ii) *If $i_{\max}(y) < \ell$, then $l_-(y) = l_=(y) = l_+(y)$.*
(iii) $u_=(y) - 1 \le u_-(y) \le u_=(y) \le u_+(y) \le u_=(y) + 1$.

(iv) If $i_{\min}(y) > \ell$, then $u_-(y) = u_=(y) = u_+(y)$.

The next result provides necessary and sufficient conditions for an element of the poset to be placed at a specific location under linear extensions.

Lemma 7.3. *Fix $y \in \alpha$, $\circ \in \{-, =, +\}$, and $i \in [n]$. There exists $\sigma \in \mathcal{N}_\circ$ with $\sigma(y) = i$ if, and only if, $i \in [\![l_\circ(y), u_\circ(y)]\!]$ and $i \neq i_m^\circ$ for any $m \in [k]$.*

Proof. $\Longrightarrow$: Fix $\sigma \in \mathcal{N}_\circ$ such that $\sigma(y) = i$. Since $y \neq x_m$ for all $m \in [k]$ it follows that $i \neq i_m^\circ$. We now show $i \leq u_\circ(y)$; the argument for $i \geq l_\circ(y)$ is analogous. Given any $s \geq i_{\min}(y)$, every element $z \in \bar{\alpha}_{>y,<x_s}$ must satisfy $i = \sigma(y) < \sigma(z) < \sigma(x_s)$. Hence, $\sigma(z)$ can take on only $\sigma(x_s) - i - 1$ possible values, which means that $|\bar{\alpha}_{>y,<x_s}| \leq \sigma(x_s) - i - 1$. In other words, $i \leq \sigma(x_s) - |\bar{\alpha}_{\geq y,<x_s}| = i_s^\circ - |\bar{\alpha}_{\geq y,<x_s}|$. The latter holds for any $s \geq i_{\min}(y)$ which shows $i \leq u_\circ(y)$.

$\Longleftarrow$: The assumption $i \neq i_m^\circ$ for any $m \in [k]$ implies that we can choose $m \in [k]$ such that $i_m^\circ < i < i_{m+1}^\circ$. Consider the poset $\bar{\alpha}' := \bar{\alpha}$ with the relabeling

$$x'_j = x_j \text{ for } j \in [\![1, m]\!], \quad x'_{m+1} = y, \quad x'_j = x_{j-1} \text{ for } j \in [\![m+2, k+1]\!],$$
$$i'_j = i_j^\circ \text{ for } j \in [\![1, m]\!], \quad i'_{m+1} = i, \quad i'_j = i_{j-1}^\circ \text{ for } j \in [\![m+2, k+1]\!].$$

To complete the proof it suffices to show that there exists a linear extension σ' of $\bar{\alpha}'$ satisfying $\sigma'(x'_j) = i'_j$ for all $j \in [\![1, k+1]\!]$. By Theorem 5.3, it suffices to show that

$$|\bar{\alpha}'_{>x'_{r+1},<x'_s}| \leq i'_s - i'_{r+1} - 1 \quad \text{for all } 0 \leq r+1 < s \leq k+1. \tag{7.1}$$

When $r+1 \neq m+1, s \neq m+1$, (7.1) holds by the assumption $|\mathcal{N}_\circ| > 0$ for all $\circ \in \{-, =, +\}$ and Theorem 5.3. The case $r+1 = m+1 = s$ is impossible since $r+1 < s$. It remains to check the cases $r+1 = m+1, s \neq m+1$ and $r+1 \neq m+1, s = m+1$. We verify (7.1) in the case $s = m+1$; the proof for the case $r+1 = m+1$ is analogous. When $s = m+1$, (7.1) is equivalent to

$$|\bar{\alpha}_{>x_{r+1},<y}| = |\bar{\alpha}'_{>x'_{r+1},<x'_s}| \leq i'_s - i'_{r+1} - 1 = i - i_{r+1}^\circ - 1. \tag{7.2}$$

When $r+1 \leq i_{\max}(y)$, (7.2) holds since, by assumption, $i - i_{r+1}^\circ - 1 \geq l_\circ(y) - i_{r+1}^\circ - 1$, so (7.2) holds by the definition of $l_\circ(y)$. When $i_{\max}(y) < r+1 < s = m+1$, $\bar{\alpha}_{>x_{r+1},<y} = \varnothing$ because if there exists $x_{r+1} < z < y$, that would imply $x_{r+1} < y$, which contradicts the maximality of $i_{\max}(y)$. Hence, (7.2) is equivalent to $0 \leq i - i_{r+1}^\circ - 1$, which holds since $i_{r+1}^\circ \leq i_m^\circ < i$, where the last inequality holds by the definition of m. $\square$

Lemma 7.3 immediately implies:

Corollary 7.4. *Fix $y \in \alpha$ and $\circ \in \{-, =, +\}$. Then,*

$$l_\circ(y) \leq m_{\min}^\circ(y) \quad \text{and} \quad m_{\max}^\circ(y) \leq u_\circ(y).$$

A second corollary of Lemma 7.3 is the proof of Remark 1.7. Note that Assumption 4.1 is not needed for the following result.

Proposition 7.5. *The condition in Theorem 1.3(iii) is equivalent to*

$$\forall\, y < x_\ell \;\exists\, s(y) \in [\![0, k+1]\!] \textit{ s.t. } y < x_{s(y)} \textit{ and } |\bar{\alpha}_{>y,<x_{s(y)}}| > i_{s(y)} - i_\ell,$$

and

$$\forall\, y > x_\ell \;\exists\, r(y) \in [\![0, k+1]\!] \textit{ s.t. } y > x_{r(y)} \textit{ and } |\bar{\alpha}_{>x_{r(y)},<y}| > i_\ell - i_{r(y)}.$$

Proof. By Lemma 3.3(a), the conditions in Theorem 1.3(iii) are equivalent to: $\sigma^{-1}(i_\ell - 1) \not\sim x_\ell$ and $\sigma^{-1}(i_\ell + 1) \not\sim x_\ell \;\forall\, \sigma \in \mathcal{N}_=$. We start by showing that

$$\begin{aligned}&\forall y < x_\ell \;\exists\, s(y) \in [\![0, k+1]\!] \text{ s.t. } y < x_{s(y)} \text{ and } |\bar{\alpha}_{>y,<x_{s(y)}}| > i_{s(y)} - i_\ell \\ &\Longleftrightarrow \\ &\sigma^{-1}(i_\ell - 1) \not\sim x_\ell \;\forall\, \sigma \in \mathcal{N}_=;\end{aligned}$$

The equivalence $\forall\, y > x_\ell \;\exists\, r(y) \in [\![0, k+1]\!]$ s.t. $y > x_{r(y)}$ and $|\bar{\alpha}_{>x_{r(y)},<y}| > i_\ell - i_{r(y)} \Longleftrightarrow \sigma^{-1}(i_\ell + 1) \not\sim x_\ell \;\forall\, \sigma \in \mathcal{N}_=$ is analogous.

Indeed, the statement $\sigma^{-1}(i_\ell - 1) \not\sim x_\ell \;\forall\, \sigma \in \mathcal{N}_=$ is equivalent to the statement that for all $y < x_\ell$, there exists no $\sigma \in \mathcal{N}_=$ such that $\sigma(y) = i_\ell - 1$. We will show that the latter is equivalent to $u_=(y) < i_\ell - 1$, which completes the proof. To see this equivalence, note that if $u_=(y) < i_\ell - 1$, then Lemma 7.3 implies that exists no $\sigma \in \mathcal{N}_=$ such that $\sigma(y) = i_\ell - 1$. Conversely, suppose there exists no $\sigma \in \mathcal{N}_=$ such that $\sigma(y) = i_\ell - 1$, so, by Lemma 7.3, $i_\ell - 1 \neq [\![l_=(y), u_=(y)]\!]$. Note that, by Lemma 7.3, $u_=(y) \le i_\ell - 1$ as $y < x_\ell$. Hence, the possibility of $i_\ell - 1 < l_=(y) \le u_=(y)$ cannot occur, which means that $i_\ell - 1 \neq [\![l_=(y), u_=(y)]\!] \Rightarrow u_=(y) < i_\ell - 1$, as claimed. □

7.2. Introduction to mixing

When $\bar{\alpha}$ is totally ordered we have, for any splitting pair (r, s),

$$\bar{\alpha}_{\ge x_{r+1},\le x_s} \overset{\text{bijection}}{\mapsto} [\![i_{r+1}, i_s]\!]$$

under any linear extension $\sigma \in \bigcup_{\circ\in\{-,=,+\}} \mathcal{N}_\circ$. But under the current assumptions, $\bar{\alpha}$ is not totally ordered (Remark 5.6), which means that a certain amount of *mixing* must occurs; see Definition 7.9 for a precise statement. In Section 7.3 we will show that there is at least one mixed element (Lemma 7.10) for any splitting pair (r, s). When the splitting pair is in addition an ℓ-splitting pair we characterize the exact number of mixed element, which depends on the criticality level of the pair:

Definition 7.6. An ℓ-splitting pair (r, s) is *supercritical* if $\mathcal{K}' := (\mathcal{K}_0, \ldots, \mathcal{K}_r, \mathcal{K}_s, \ldots, \mathcal{K}_k)$ satisfies $\dim\left(\sum_{K\in\mathcal{K}'} K\right) \geq |\mathcal{K}'| + 2$, and is *sharp-critical* if $\dim\left(\sum_{K\in\mathcal{K}'} K\right) = |\mathcal{K}'| + 1$.

We show in Section 7.3 how the above notion of criticality is related to the number of mixed elements (Lemma 7.13). The sharp-critical ℓ-splitting pairs give rise to the following unique pair which will play an important role in the characterization of the extremals of the critical posets.

Definition 7.7. Let $(r_\iota, s_\iota)_\iota$ be the sharp-critical ℓ-splitting pairs, where we assume that at least one such pair exists. The *maximal splitting pair* $(r_{\max}, s_{\min})$ is given by $r_{\max} := \max_\iota r_\iota$ and $s_{\min} := \min_\iota s_\iota$. Associated to the maximal splitting pair are

$$\begin{aligned} &\mathcal{K}_{\max} := (\mathcal{K}_0, \ldots, \mathcal{K}_{r_{\max}}, \mathcal{K}_{s_{\min}}, \ldots, \mathcal{K}_k), \\ &\beta_{\max} := \beta_{[\![0,r_{\max}]\!]\cup[\![s_{\min},k]\!]}, \quad \text{and} \quad \alpha\backslash\beta_{\max} = \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}, \end{aligned} \tag{7.3}$$

where the last identity follows from Lemma 2.3.

The notion of the maximal splitting pair in Definition 7.7 is tied to the notion of maximal sharp-critical collections introduced [17, section 9.1], as part of the characterization of the extremals of the Alexandrov-Fenchel inequality for critical polytopes. In particular, a sharp-critical collection $\mathcal{K}' \subseteq \mathcal{K}$ is *maximal* if, for any $\mathcal{K}' \subsetneq \mathcal{K}'' \subseteq \mathcal{K}$, we have $\dim\left(\sum_{K\in\mathcal{K}''} K\right) \geq |\mathcal{K}''| + 2$. In other words, any addition of polytopes to $\mathcal{K}'$ destroys its sharp-critical nature. The next result explains the connection between these two notions of maximality.

Proposition 7.8. *Suppose there exists a sharp-critical collection. Then, $\mathcal{K}_{\max}$ is the only maximal sharp-critical collection.*

Proof. We start by recalling that all sharp-critical maximal collections of $\mathcal{K}$ must be disjoint [17, Lemma 9.2]. By assumption there exists a sharp-critical collection $\mathcal{K}'$ so let $\mathcal{K}_*$ be the (necessarily unique) maximal sharp-critical collection containing $\mathcal{K}'$. On the other hand, Lemma 5.11 shows that any two sharp-critical collections of $\mathcal{K}$ have a non-trivial intersection. It follows that $\mathcal{K}_*$ is the only maximal sharp-critical collection in $\mathcal{K}$. Next we show that

$$\mathcal{K}_* = \{\text{union of all sharp-critical collections}\} = \mathcal{K}_{\max},$$

where the second identity follows from Lemma 5.11, which completes the proof. Indeed, clearly, $\mathcal{K}_* \subseteq \bigcup\{\text{sharp-critical collection}\}$ since $\mathcal{K}_*$ is a sharp-critical collection. If $\bigcup\{\text{sharp-critical collection}\}$ a strictly greater than $\mathcal{K}_*$, i.e., it contains a polytope K not in $\mathcal{K}_*$, then there exists a sharp-critical collection $\mathcal{K}''$ such that $K \in \mathcal{K}''$. Let $\mathcal{K}_{**}$ be the (necessarily unique) maximal sharp-critical collection containing $\mathcal{K}''$. Then $\mathcal{K}_{**} \neq \mathcal{K}_*$ (as

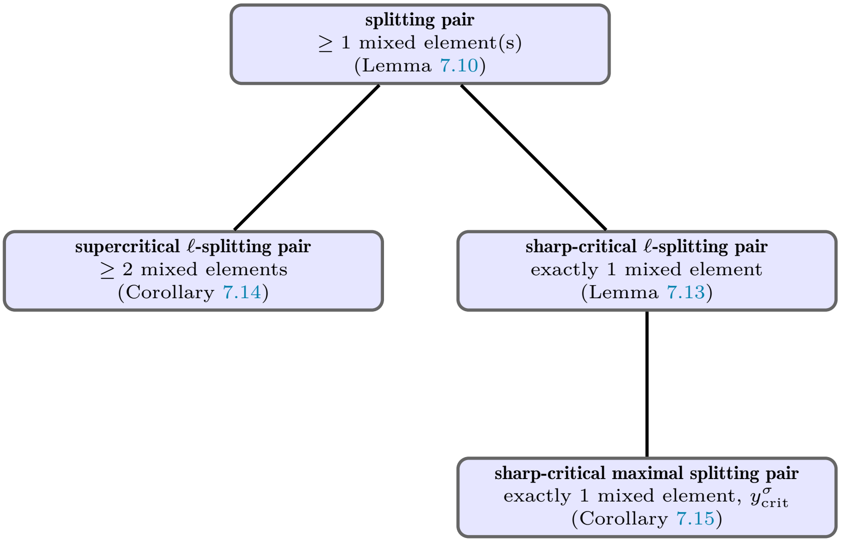


Fig. 7.1. A summary of the mixing results from Section 7.3.

$K \in \mathcal{K}_{**}$ but $K \notin \mathcal{K}_*$), which contradicts the fact $\mathcal{K}_*$ is the only maximal sharp-critical collection. □

We conclude the section by introducing notation that will be used throughout the paper. Let

$$[\![i_j, i_{j+1}]\!]^\circ := [\![i_j^\circ, i_{j+1}^\circ]\!] = [\![i_j + 1_{j=\ell}1_\circ, i_{j+1} + 1_{j+1=\ell}1_\circ]\!]. \tag{7.4}$$

We use this notation when constants are added as well, for example, $[\![i_j+1, i_{j+1}-1]\!]^\circ := [\![i_j^\circ + 1, i_{j+1}^\circ - 1]\!]$.

7.3. Mixing properties of splitting pairs

In this section we analyze the mixing properties of splitting pairs—see Fig. 7.1 for a summary. We start by making the definition of a mixed element precise (recall (2.4)):

Definition 7.9. Fix a splitting pair (r, s) and $\sigma \in \mathcal{N}_\circ$ for $\circ \in \{-, =, +\}$. An element $y^\sigma \in \beta_r \cup \beta_s$ is a *mixed element* if $\sigma(y^\sigma) \in [\![i_{r+1}, i_s]\!] \backslash \{i_{r+1}, \dots, i_s\}$.

Our first result in this section is on the existence of mixed elements.

Lemma 7.10. *Fix a splitting pair (r, s) and $\sigma \in \mathcal{N}_=$. There exists a mixed element $y^\sigma \in \beta_r \cup \beta_s$ such that $\sigma(y^\sigma) \in [\![i_{r+1}, i_s]\!] \backslash \{i_{r+1}, \dots, i_s\}$.*

Proof. Recall that $|\bar{\alpha}_{\geq x_{r+1},\leq x_s}| \leq i_s - i_{r+1}$ by Assumption 6.7, which is equivalent to $|\alpha_{>x_{r+1},<x_s}| \leq i_s - i_{r+1} - (s-(r+1)) - 1$. Fix $\sigma \in \mathcal{N}_=$. If there exists no $y^\sigma \in \beta_r \cup \beta_s$ with $\sigma(y^\sigma) \in [\![i_{r+1}, i_s]\!] \backslash \{i_{r+1}, \ldots, i_s\}$, then, by Lemma 2.3,

$$\begin{aligned}|\alpha_{>x_{r+1},<x_s}| &= |\alpha\backslash(\beta_r \cup \beta_s \cup \alpha_{<x_{r+1}} \cup \alpha_{>x_s})| \geq |[\![i_{r+1}, i_s]\!]\backslash\{i_{r+1}, \ldots, i_s\}| \\ &= i_s - i_{r+1} + 1 - (s-(r+1)+1) = i_s - i_{r+1} - (s-(r+1)),\end{aligned}$$

which is a contradiction. □

Corollary 7.11. *For every $0 \leq j \leq k$, $i_j + 1 < i_{j+1}$.*

Proof. If $k = 1$ then the corollary holds by the assumption $i_\ell < i_{\ell+1} - 1$. Otherwise, note that $(r,s) = (j-1, j+1)$ is a splitting pair. Fix $\sigma \in \mathcal{N}_=$ and note that Lemma 7.10 implies that there exists $y^\sigma \notin \bar{\alpha}_{\geq x_j,\leq x_{j+1}}$ with $\sigma(y^\sigma) \in [\![i_j, i_{j+1}]\!]$. The first condition gives $y^\sigma \notin \{x_j, x_{j+1}\}$, so $\sigma(y^\sigma) \notin \{i_j, i_{j+1}\}$. We conclude that $[\![i_j+1, i_{j+1}-1]\!] = [\![i_j, i_{j+1}]\!] \setminus \{i_j, i_{j+1}\}$ is nonempty. □

Next we move to the mixing properties of ℓ-splitting pairs. This requires the following simple result.

Lemma 7.12.

- *Fix $j \in [\![0,k]\!]$. For every $\sigma \in \mathcal{N}_=$, $[\![i_j+1, i_{j+1}-1]\!] \subseteq \sigma(\beta_j)$ and, for every $S \subseteq [\![0,k]\!]$, $\bigcup_{j\in S}[\![i_j+1, i_{j+1}-1]\!] \subseteq \sigma(\beta_S)$.*
- *Fix $j \in [\![0,k]\!]\backslash\{\ell-1,\ell\}$ and $\circ \in \{-,+\}$. For every $\sigma \in \mathcal{N}_\circ$, $[\![i_j+1, i_{j+1}-1]\!]^\circ \subseteq \sigma(\beta_j)$ and, for every $S \subseteq [\![0,k]\!]\backslash\{\ell-1,\ell\}$, $\bigcup_{j\in S}[\![i_j+1, i_{j+1}-1]\!]^\circ \subseteq \sigma(\beta_S)$.*

Proof.
- Fix $\sigma \in \mathcal{N}_=$. We will show that $\sigma(y) \in [\![i_j+1, i_{j+1}-1]\!] \Rightarrow y \in \beta_j$ which implies $[\![i_j+1, i_{j+1}-1]\!] \subseteq \sigma(\beta_j)$; the statement about S follows by taking unions. If $\sigma(y) \in [\![i_j+1, i_{j+1}-1]\!]$, then clearly $y \in \alpha$ and $\sigma(x_j) = i_j < \sigma(y) < i_{j+1} = \sigma(x_{j+1})$. Hence, neither $y < x_j$ nor $y > x_{j+1}$ can occur. It follows that $y \in \beta_j$.
- The proof is the same as for the first part where we use that $j \notin \{\ell-1,\ell\} \Rightarrow \sigma(x_j) = i_j$ and $\sigma(x_{j+1}) = i_{j+1}$. □

We now show how the mixing properties of ℓ-splitting pairs are related to their criticality properties.

Lemma 7.13. *Fix an ℓ-splitting pair (r,s), let*

$$\mathcal{K}' := (\mathcal{K}_0, \ldots, \mathcal{K}_r, \mathcal{K}_s, \ldots, \mathcal{K}_k),$$

and set

$$c := \dim\left(\sum_{K\in\mathcal{K}'} K\right) - |\mathcal{K}'|.$$

Then, for any fixed $\sigma \in \mathcal{N}_\circ$, for $\circ \in \{-,=,+\}$, there are exactly c distinct mixed elements $y_1^\sigma,\dots,y_c^\sigma \in \beta_r \cup \beta_s$ satisfying $\sigma(y_1^\sigma),\dots,\sigma(y_c^\sigma) \in [\![i_{r+1}, i_s]\!]\backslash\{i_{r+1},\dots,i_s\}$.

Proof. By Lemma 5.1,

$$|\beta_{[\![0,r]\!]\cup[\![s,k]\!]}| = \dim\left(\sum_{K\in\mathcal{K}'} K\right) \quad \text{and} \quad |\mathcal{K}'| = |\cup_{j\in[\![0,r]\!]\cup[\![s,k]\!]} [\![i_j+1, i_{j+1}-1]\!]|.$$

On the other hand, applying Lemma 7.12 to $S := [\![0,r]\!] \cup [\![s,k]\!]$ yields $\cup_{j\in[\![0,r]\!]\cup[\![s,k]\!]}[\![i_j + 1, i_{j+1} - 1]\!] \subseteq \sigma(\beta_{[\![0,r]\!]\cup[\![s,k]\!]})$. Hence, there are exactly c distinct elements $\{y_i^\sigma\}_{i\in[c]}$ satisfying $y_i^\sigma \in \beta_{[\![0,r]\!]\cup[\![s,k]\!]}$ and $\sigma(y_i^\sigma) \notin \cup_{j\in[\![0,r]\!]\cup[\![s,k]\!]}[\![i_j + 1, i_{j+1} - 1]\!]$. Now recall that $\beta_{[\![0,r]\!]\cup[\![s,k]\!]} = \beta_r \cup \beta_s \cup \alpha_{<x_{r+1}} \cup \alpha_{>x_s}$ (Lemma 2.3), and note that $\sigma(y_i^\sigma) \notin \cup_{j\in[\![0,r]\!]\cup[\![s,k]\!]}[\![i_j + 1, i_{j+1} - 1]\!]$ implies that $y_i^\sigma \in \beta_r \cup \beta_s$. □

Corollary 7.14. *Let (r, s) be a supercritical ℓ-splitting pair. Then, for any $\sigma \in \mathcal{N}_\circ$, for $\circ \in \{-,=,+\}$, there are $c \geq 2$ distinct mixed elements $y_1^\sigma,\dots,y_c^\sigma \in \beta_r \cup \beta_s$ satisfying $\sigma(y_1^\sigma),\dots,\sigma(y_c^\sigma) \in [\![i_{r+1}, i_s]\!]\backslash\{i_{r+1},\dots,i_s\}$.*

Note that Corollary 7.14 is an improvement on Lemma 7.10 in the setting of supercritical ℓ-splitting pairs, as it guarantees the existence of two distinct mixed elements rather than one. In addition, because Corollary 7.14 specializes to ℓ-splitting pairs it can handle $\mathcal{N}_\circ$, for any $\circ \in \{-,=,+\}$, while Lemma 7.10 applies only to $\mathcal{N}_=$.

We conclude this section by specializing to the setting where the ℓ-splitting pair is maximal. Since the maximal splitting pair is sharp-critical, Lemma 7.13 immediately gives that we have exactly one mixed element.

Corollary 7.15. *Fix $\circ \in \{-,=,+\}$ and $\sigma \in \mathcal{N}_\circ$. There exists a unique mixed element y_{crit}^σ satisfying $y_{crit}^\sigma \in \beta_{r_{\max}} \cup \beta_{s_{\min}}$ and $\sigma(y_{crit}^\sigma) \in [\![i_{r_{\max}+1}, i_{s_{\min}}]\!]\backslash\{i_{r_{\max}+1},\dots,i_{s_{\min}}\}$.*

8. The extreme normal directions

Once Assumption 6.7 is set in place, we are ready, in principle, to apply Theorem 2.8. However, Theorem 2.8 characterizes the extremals *geometrically* in terms of the $(B,\mathcal{K})$-extreme normal directions so a *combinatorial* interpretation of these vectors is needed. The goal of this section is to characterize, combinatorially, a sufficient number of the $(B,\mathcal{K})$-extreme normal directions so that Theorem 2.8 can be applied.

We recall that $\{e_j\}_{j\in[n-k]}$ is the standard basis of $\mathbb{R}^{n-k}$ and, for $u, v \in [n-k]$ distinct, we let $e_{uv} := \frac{e_u-e_v}{\sqrt{2}}$ and $o_{uv} := \frac{e_u+e_v}{\sqrt{2}}$. We also recall the definition (2.4):

$$\beta_i := \alpha\backslash(\alpha_{<x_i} \cup \alpha_{>x_{i+1}}).$$

The next result characterizes certain faces of the polytopes $\{K_i\}$.

Lemma 8.1. *Fix $i \in [\![0, k]\!]$. We have,*

(i) For $y_j \notin \beta_i$, $\mathrm{Lin}(F(K_i, \pm e_j)) = \mathbb{R}^{\beta_i}$, and for $y_u, y_v \notin \beta_i$, $\mathrm{Lin}(F(K_i, \pm e_{uv})) = \mathbb{R}^{\beta_i}$.
(ii) For $y_j \in \beta_i$, $\mathrm{Lin}(F(K_i, -e_j)) = \mathbb{R}^{\beta_i \setminus \alpha_{\leq y_j}}$ and $\mathrm{Lin}(F(K_i, e_j)) = \mathbb{R}^{\beta_i \setminus \alpha_{\geq y_j}}$.
(iii) For $y_u, y_v \in \beta_i$ such that y_v covers y_u in α, $\mathrm{Lin}(F(K_i, e_{uv})) = \mathbb{R}^{\beta_i \setminus \{y_u, y_v\}} \oplus \mathrm{span}(o_{uv})$.

Proof. We start by recalling (2.5):

$$K_i = O_{\beta_i} + 1_{\alpha_{>x_{i+1}}} \text{ for } i \in [\![0, k]\!]$$

so that

$$\mathrm{Lin}(F(K_i, \mathrm{u})) = \mathrm{Lin}(F(O_{\beta_i}, \mathrm{u})) \quad \forall\, \mathrm{u} \in S^{n-k-1}.$$

(i) Let $\mathrm{u} \in \{\pm e_j\}$ so, since $h_{O_{\beta_i}}(\mathrm{u}) = 0$ as $y_j \notin \beta_i$, we get that $\mathrm{Lin}(F(K_i, \mathrm{u})) = O_{\beta_i} \cap \{t_j = 0\} = O_{\beta_i}$, where the last equality holds as $y_j \notin \beta_i$. Similarly, let $\mathrm{u} \in \{\pm e_{uv}\}$ so, since $h_{O_{\beta_i}}(\mathrm{u}) = 0$ as $y_u, y_v \notin \beta_i$, we get that $\mathrm{Lin}(F(K_i, \mathrm{u})) = O_{\beta_i} \cap \{t_u = t_v\} = O_{\beta_i}$, where the last equality holds as $y_u, y_v \notin \beta_i$. The proof is complete as $\dim O_{\beta_i} = |\beta_i|$ (Lemma 2.1).
(ii) Since $h_{O_{\beta_i}}(-e_j) = 0$, we get $\mathrm{Lin}(F(K_i, -e_j)) = O_{\beta_i} \cap \{t_j = 0\} = O_{\beta_i \setminus \alpha_{\leq y_j}}$ where the last equality holds as $y_j \in \beta_i$. Analogously, since $h_{O_{\beta_i}}(e_j) = 1$ (because $y_j \in \beta_i$), we get $\mathrm{Lin}(F(K_i, e_j)) = O_{\beta_i} \cap \{t_j = 1\} = O_{\beta_i \setminus \alpha_{\geq y_j}}$.
(iii) Since $y_u \leq y_v$ we have $h_{O_{\beta_i}}(e_{uv}) = 0$, so $\mathrm{Lin}(F(K_i, e_{uv})) = O_{\beta_i} \cap \{t_u = t_v\}$. Since y_v covers y_u, it follows from Lemma 2.1(iii) that $\dim(\mathrm{Lin}(F(K_i, e_{uv}))) = |\beta_i| - 1$. On the other hand, since $\mathrm{Lin}(F(K_i, e_{uv})) \perp e_{uv}$, we have $\mathrm{Lin}(F(K_i, e_{uv})) \subseteq \mathbb{R}^{\beta_i} \cap e_{uv}^{\perp} = \mathbb{R}^{\beta_i \setminus \{y_u, y_v\}} \oplus \mathrm{span}(o_{uv})$. The proof is complete since $\dim(\mathbb{R}^{\beta_i \setminus \{y_u, y_v\}} \oplus \mathrm{span}(o_{uv})) = |\beta_i| - 1$. $\square$

The following proposition, which is the main result of this section, characterizes combinatorially some of the $(B, \mathcal{K})$-extreme normal directions. We remark that the $(B, \mathcal{K})$-extreme normal directions given in Proposition 8.2(*e*–*h*) will be used only for the characterization of the extremals of sharp-critical posets.

Proposition 8.2. *The following vectors are $(B, \mathcal{K})$-extreme normal directions:*

(a) For each fixed $0 \leq m \leq \ell$: $-e_j$ for any j such that $y_j \in \alpha_{>x_m}$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_j) = i_m + 1$.
(b) For each fixed $\ell \leq m \leq k+1$: e_j for any j such that $y_j \in \alpha_{<x_m}$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_j) = i_m - 1$.

(c) e_{uv} *for any* u, v *such that* $y_u < y_v$ *and there exists* $\sigma \in \mathcal{N}_=$ *satisfying* $\sigma(y_u) + 1 = \sigma(y_v)$.
(d) e_{uv} *for any* u, v *such that* $y_u < y_v$ *and there exists* $\sigma \in \mathcal{N}_=$ *satisfying* $\sigma(y_u) = i_\ell - 1$ *and* $\sigma(y_v) = i_\ell + 1$.
(e) For each fixed $r_{\max} + 1 \le m \le \ell - 1$*:* $-e_j$ *for any* j *such that* $y_j \in \alpha_{>x_m}$ *and there exists* $\sigma \in \mathcal{N}_=$ *satisfying* $\sigma(y_j) = i_m + 2$.
(f) For each fixed $\ell + 1 \le m \le s_{\min}$*:* e_j *for any* j *such that* $y_j \in \alpha_{<x_m}$ *and there exists* $\sigma \in \mathcal{N}_=$ *satisfying* $\sigma(y_j) = i_m - 2$.
(g) $-e_j$ *for any* j *such that* $y_j \in \alpha_{>x_{\ell-1}}$ *and there exists* $\sigma \in \mathcal{N}_+$ *satisfying* $\sigma(y_j) = i_{\ell-1} + 2$.
(h) e_j *for any* j *such that* $y_j \in \alpha_{<x_{\ell+1}}$ *and there exists* $\sigma \in \mathcal{N}_-$ *satisfying* $\sigma(y_j) = i_{\ell+1} - 2$.

Note that parts (a–b), which suffice for the supercritical posets, provide information about nearest neighbors of x_m, while parts (e–f), which are needed for the critical posets, provide information about second-nearest neighbors of x_m.

Proof. (of Proposition 8.2) By Definition 2.4, we need to show that, whenever u is one of the vectors in the proposition, we have, for any collection $\mathcal{K}' \subseteq \mathcal{K}$,

$$\dim\left(\sum_{K\in\mathcal{K}'} F(K, \mathrm{u})\right) \ge |\mathcal{K}'|.$$

Let $j_0 := -1 < 0 \le j_1 < \cdots < j_p \le k < k+1 =: j_{p+1}$ and $\kappa_1, \ldots, \kappa_p$, with $0 \le \kappa_q \le i_{j_q+1} - i_{j_q} - 1 - 1_{j_q \in \{\ell-1,\ell\}}$, for $j_q \in [\![0, k]\!]$, and set

$$\mathcal{K}' := (\underbrace{K_{j_1}, \ldots, K_{j_1}}_{\kappa_1}, \ldots, \underbrace{K_{j_p}, \ldots, K_{j_p}}_{\kappa_p}),$$
$$J := \{j_1, \ldots, j_p\}.$$

For notational simplicity we set

$$\mathrm{I}_j := [\![i_j + 1, i_{j+1} - 1]\!] \quad \text{for } j \in [\![0, k]\!], \quad \mathrm{I}_S := \cup_{j_q \in S} \mathrm{I}_{j_q} \quad \text{for } S \subset [\![0, k]\!]; \tag{8.1}$$

for example,

$$\mathrm{I}_{[\![r+1,s]\!]} = [\![i_{r+1}, i_s]\!] \backslash \{i_{r+1}, \ldots, i_s\}.$$

Note that

$$|\mathrm{I}_J| - 1_{\ell-1\in J} - 1_{\ell\in J} \ge |\mathcal{K}'|,$$

because $0 \le \kappa_q \le i_{j_q+1} - i_{j_q} - 1 - 1_{j_q\in\{\ell-1,\ell\}}$ and since $\mathrm{I}_{j_q} = i_{j_q+1} - i_{j_q} - 1$.

(a) Fix $0 \leq m \leq \ell$ and consider $\sigma \in \mathcal{N}_=$ such that $\sigma(y_j) = i_m + 1$ where j is such that $y_j \in \alpha_{>x_m}$. Let

$$\gamma_{j_q} := \begin{cases} \beta_{j_q} & \text{if } y_j \notin \beta_{j_q} \\ \beta_{j_q} \backslash \alpha_{\leq y_j} & \text{if } y_j \in \beta_{j_q} \end{cases},$$

and $\gamma_J := \cup_{j_q \in J} \gamma_q$. By Lemma 8.1(i–ii),

$$\mathrm{Lin}(F(K_{j_q}, -e_j)) = \mathbb{R}^{\gamma_{j_q}} \quad \text{for all} \quad j_q \in J,$$

so, by (5.1),

$$\mathrm{Lin}\left(F\left(\sum_{K \in \mathcal{K}'} K, -e_j\right)\right) = \mathbb{R}^{\gamma_J}.$$

It follows that

$$\dim\left(\sum_{K \in \mathcal{K}'} F(K, -e_j)\right) = |\gamma_J|,$$

so it remains to show that $|\gamma_J| \geq |\mathcal{K}'|$. Since $|\mathrm{I}_J| - 1_{\ell-1 \in J} - 1_{\ell \in J} \geq |\mathcal{K}'|$, it will suffice to show that

$$|\gamma_J| \geq |\mathrm{I}_J| - 1_{\ell-1 \in J} - 1_{\ell \in J},$$

which requires the following claim.

Claim 8.3.

(i) For $j_q \neq m$, $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.
(ii) For $j_q = m$, $\mathrm{I}_m \backslash \{i_m + 1\} \subseteq \sigma(\gamma_m)$.

Proof. (i) We need to consider the cases $y_j \notin \beta_{j_q}$ and $y_j \in \beta_{j_q}$. If $y_j \notin \beta_{j_q}$ then the result holds by Lemma 7.12. Suppose $y_j \in \beta_{j_q}$. Then, we must have $m < j_q$; otherwise, $j_q < m$ (by assumption $j_q \neq m$) so $x_{j_q+1} \leq x_m < y_j$, but this implies $y_j \notin \beta_{j_q}$, which is a contradiction. Now let y be any element such that $\sigma(y) \in \mathrm{I}_{j_q}$, which by Lemma 7.12, implies that $y \in \beta_{j_q}$. Since $\sigma(y) \geq i_{j_q} + 1 > i_m + 1 = \sigma(y_j)$, we can conclude that, in fact, $y \in \beta_{j_q} \backslash \alpha_{\leq y_j} = \gamma_{j_q}$. To summarize, $\sigma(y) \in \mathrm{I}_{j_q} \Rightarrow y \in \gamma_{j_q}$, which shows $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.

(ii) We need to consider the cases $y_j \notin \beta_m$ and $y_j \in \beta_m$. Suppose $y_j \notin \beta_m$. By Corollary 7.11, $i_m + 1 < i_{m+1}$ so $\sigma(y_j) = i_m + 1 \in \mathrm{I}_m \subseteq \sigma(\beta_m)$, where we used Lemma 7.12. This contradicts $y_j \notin \beta_m$ so we are left to consider $y_j \in \beta_m$. Let

y be any element such that $\sigma(y) \in \mathrm{I}_m \backslash \{i_m + 1\} = [\![i_m + 2, i_{m+1} - 1]\!]$. Then, $y \in \beta_m \backslash \alpha_{\leq y_j}$ since, by Lemma 7.12, $y \in \beta_m$, but we also have $\sigma(y) \geq i_m + 2 > i_m + 1 = \sigma(y_j)$. To summarize, $\sigma(y) \in \mathrm{I}_m \backslash \{i_m + 1\} \Rightarrow y \in \beta_m \backslash \alpha_{\leq y_j}$, which shows $\mathrm{I}_m \backslash \{i_m + 1\} \subseteq \sigma(\gamma_m)$. □

In order to use Claim 8.3 in the proof of $|\gamma_J| \geq |\mathrm{I}_J| - 1_{\ell-1 \in J} - 1_{\ell \in J}$, we distinguish between two cases: $m \notin J$ and $m \in J$. If $m \notin J$, then taking a union over $j_q \in J$ in Claim 8.3 gives $\mathrm{I}_J \subseteq \sigma(\gamma_J)$, so $|\gamma_J| \geq |\mathrm{I}_J| \geq |\mathrm{I}_J| - 1_{\ell-1 \in J} - 1_{\ell \in J}$, as desired.

Suppose then that $m \in J$. Taking a union over $j_q \in J$ in Claim 8.3 gives $\mathrm{I}_J \backslash \{i_m + 1\} \subseteq \sigma(\gamma_J)$. Hence, if $\ell \in J$, we have $|\gamma_J| \geq |\mathrm{I}_J| - 1 \geq |\mathrm{I}_J| - 1_{\ell-1 \in J} - 1_{\ell \in J}$, which completes the proof. It remains to consider the case $m \in J$ and $\ell \notin J$:

Choose the largest $0 \leq b \leq p$ such that $j_b < \ell$, so $j_b < \ell < j_{b+1}$, and, in particular, (j_b, j_{b+1}) is an ℓ-splitting pair. By Lemma 7.10, there exists $y^\sigma \in \beta_{j_b} \cup \beta_{j_b+1}$ such that $\sigma(y^\sigma) \in \mathrm{I}_{[\![j_b+1, j_{b+1}-1]\!]}$. Since $m = j_q < \ell$ for some $0 \leq q \leq p$, and since b is the largest element in $[\![0, p]\!]$ such that $j_b < \ell$, we have $q \leq b$, and hence $m \leq j_b$. It follows that $\sigma(y_j) = i_m + 1 < i_{j_b+1} + 1 \leq \sigma(y^\sigma)$, and, in particular, $y^\sigma \notin \alpha_{\leq y_j}$. Hence, $y^\sigma \in (\beta_{j_b} \backslash \alpha_{\leq y_j}) \cup (\beta_{j_{b+1}} \backslash \alpha_{\leq y_j}) \subseteq \gamma_{j_b} \cup \gamma_{j_{b+1}} \subseteq \gamma_J$, so $(\mathrm{I}_J \backslash \{i_m + 1\}) \cup \{\sigma(y^\sigma)\} \subseteq \sigma(\gamma_J)$. Finally, $\sigma(y^\sigma) \notin \mathrm{I}_J$ because J and $[\![j_b + 1, j_{b+1} - 1]\!]$ do not intersect, which completes the proof since it implies that $|\gamma_J| \geq |(\mathrm{I}_J \backslash \{i_m + 1\}) \cup \{\sigma(y^\sigma)\}| \geq |\mathrm{I}_J| - 1 + 1 = |\mathrm{I}_J| \geq |\mathrm{I}_J| - 1_{\ell-1 \in J} - 1_{\ell \in J}$.

(b) The proof is analogous to part (a).

(c) Fix u, v such that there exist $y_u < y_v$ with $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_u) + 1 = \sigma(y_v)$. For $j_q \in J$, let

$$\gamma_{j_q} := \begin{cases} \beta_{j_q} & \text{if } y_u, y_v \notin \beta_{j_q} \\ \beta_{j_q} \backslash \{y_u, y_v\} & \text{if } y_u, y_v \in \beta_{j_q}, \\ \beta_{j_q} \backslash \alpha_{\geq y_u} & \text{if } y_u \in \beta_{j_q}, y_v \notin \beta_{j_q}, \\ \beta_{j_q} \backslash \alpha_{\leq y_v} & \text{if } y_u \notin \beta_{j_q}, y_v \in \beta_{j_q}. \end{cases}$$

We start by describing the faces of $\{K_{j_q}\}_{j_q \in J}$ in the directions $\{e_{uv}\}$.

Claim 8.4. For every $j_q \in J$,

$$\mathrm{Lin}(F(K_{j_q}, e_{uv})) = \begin{cases} \mathbb{R}^{\gamma_{j_q}} \oplus \mathrm{span}(o_{uv}) & \text{if } y_u, y_v \in \beta_{j_q}, \\ \mathbb{R}^{\gamma_{j_q}} & \text{otherwise.} \end{cases}$$

Proof. There are four cases to consider:

- $y_u, y_v \in \beta_{j_q}$: The claim follows from Lemma 8.1(iii).
- $y_u, y_v \notin \beta_{j_q}$: The claim follows from Lemma 8.1(i).
- $y_u \in \beta_{j_q}, y_v \notin \beta_{j_q}$: We will show that $\mathrm{Lin}(F(K_{j_q}, e_{uv})) = \mathrm{Lin}(F(K_{j_q}, e_u))$, and the claim will then follow from Lemma 8.1(ii). Indeed, the assumption $y_v \notin \beta_{j_q}$

implies that $y_v \in \alpha_{<x_{j_q}} \cup \alpha_{>x_{j_q+1}}$. But $y_v \notin \alpha_{<x_{j_q}}$ because, otherwise, $y_u < y_v < x_{j_q}$, which contradicts the assumption $y_u \in \beta_{j_q}$. Hence, $y_v > x_{j_q+1}$ so, by the definition (2.3) of K_{j_q}, $t_v = 1$ for any $t \in O_\alpha$. Since $\sup_{t_u \in [0,1]} t_u = 1$ (as $y_u \in \beta_{j_q} \Rightarrow y \not< x_{j_q}$), it follows that $h_{K_{j_q}}(e_{uv}) = \sup_{t_u \in [0,1]} \frac{t_u - t_v}{\sqrt{2}} = 0$, and hence

$$\mathrm{Lin}(F(K_{j_q}, e_{uv})) = K_{j_q} \cap \{t_u = t_v\} = K_{j_q} \cap \{t_u = 1\} = \mathrm{Lin}(F(K_{j_q}, e_u)),$$

as needed.

- $y_u \notin \beta_{j_q}, y_v \in \beta_{j_q}$: The argument is analogous to the previous case: $y_u \in \beta_{j_q}$ and $y_v \notin \beta_{j_q}$. □

Next we prove the analogue of Claim 8.3.

Claim 8.5. Choose $m \in [\![0, k+1]\!]$ such that $i_m < \sigma(y_u) < \sigma(y_v) < i_{m+1}$.

(i) For $j_q \neq m$, $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.
(ii) For $j_q = m$, $\mathrm{I}_m \backslash \{\sigma(y_u), \sigma(y_v)\} \subseteq \sigma(\gamma_m)$.

Proof. We need to consider the four cases (1) $y_u, y_v \in \beta_{j_q}$, (2) $y_u, y_v \notin \beta_{j_q}$, (3) $y_u \in \beta_{j_q}, y_v \notin \beta_{j_q}$, and (4) $y_u \notin \beta_{j_q}, y_v \in \beta_{j_q}$.

(i) Case (1): For any y such that $\sigma(y) \in \mathrm{I}_{j_q}$, we have $y \in \beta_{j_q}$, by Lemma 7.12, and $y \notin \{y_u, y_v\}$, since $\sigma(y_u), \sigma(y_v) \in \mathrm{I}_m$, and $\mathrm{I}_m \cap \mathrm{I}_{j_q} = \varnothing$ as $m \neq j_q$. Hence, $y \in \beta_{j_q} \backslash \{y_u, y_v\} = \gamma_{j_q}$, so we conclude $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.
Case (2): Since $\gamma_{j_q} = \beta_{j_q}$, Lemma 7.12 implies $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.
Case (3): For any y such that $\sigma(y) \in \mathrm{I}_{j_q}$, we have $y \in \beta_{j_q}$, by Lemma 7.12. On the other hand, the proof of Claim 8.4 showed that $y_v > x_{j_q+1}$, so the assumption on m implies that $j_q < m$, which means that $\sigma(y) < i_{j_q+1} \leq i_m < \sigma(y_u)$. In particular, $y \notin \alpha_{\geq y_u}$ so we conclude that $y \in \beta_{j_q} \backslash \alpha_{\geq y_u} = \gamma_{j_q}$. It follows that $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.
Case (4) is analogous to case (3).

(ii) Case (1): For any $y \in \mathrm{I}_m \backslash \{\sigma(y_u), \sigma(y_v)\}$, Lemma 7.12 implies that $y \in \beta_m \backslash \{y_u, y_v\} = \gamma_m$, which implies that $\mathrm{I}_m \backslash \{\sigma(y_u), \sigma(y_v)\} \subseteq \sigma(\gamma_m)$.
Case (2): Since $\gamma_m = \beta_m$, Lemma 7.12 implies $\mathrm{I}_m \backslash \{\sigma(y_u), \sigma(y_v)\} \subseteq \sigma(\gamma_m)$.
Case (3): As shown in part (i) case (3), we must have $j_q < m$ so this case cannot occur.
Case (4) is analogous to case (3). □

Choose $m \in [\![0, k+1]\!]$ such that $i_m < \sigma(y_u) < \sigma(y_v) < i_{m+1}$. To complete the proof we distinguish between two cases: $m \notin J$ and $m \in J$. Suppose $m \notin J$. By (5.1) and Claim 8.4, $\mathbb{R}^{\gamma_J} \subseteq \mathrm{Lin}\left(\sum_{K \in \mathcal{K}'} F(K, e_{uv})\right)$, so $\dim\left(\sum_{K \in \mathcal{K}'} F(K, e_{uv})\right) \geq |\gamma_J|$. On the other hand, by Claim 8.5 and as $m \notin J$, $|\gamma_J| \geq |\mathrm{I}_J|$. We conclude

$$\dim\left(\sum_{K\in\mathcal{K}'}F(K,e_{uv})\right)\geq|\mathrm{I}_J|\geq|\mathrm{I}_J|-1_{\ell-1\in J}-1_{\ell\in J}\geq|\mathcal{K}'|,$$

which completes the proof.

Suppose that $m\in J$. By the definition of m, $\sigma(y_u),\sigma(y_v)\in\mathrm{I}_m$, so Lemma 7.12 implies that $y_u,y_v\in\beta_m$. By Claim 8.4, it follows that $F(K_m,e_{uv})=\mathbb{R}^{\gamma_m}\oplus\mathrm{span}(o_{uv})$. On the other hand, for any $j_q\in J$, by the definition of γ_{j_q}, we have $y_u,y_v\notin\gamma_{j_q}$. Hence, $\mathbb{R}^{\gamma_{j_q}}\cap\mathrm{span}(o_{uv})=\{0\}$ for all $j_q\in J$, and in particular, $\mathbb{R}^{\gamma_J}\cap\mathrm{span}(o_{uv})=\{0\}$. It follows from (5.1) that

$$\mathrm{Lin}\left(\sum_{K\in\mathcal{K}'}F(K,e_{uv})\right)=\mathbb{R}^{\gamma_J}\oplus\mathrm{span}(o_{uv}),$$

and

$$\dim\left(\sum_{K\in\mathcal{K}'}F(K,e_{uv})\right)=|\gamma_J|+1.$$

We now consider separately the cases $\ell\in J$ and $\ell\notin J$. Suppose $\ell\in J$. By Claim 8.5, $|\gamma_J|\geq|\mathrm{I}_J|-2$ so

$$|\gamma_J|+1\geq|\mathrm{I}_J|-1\geq|\mathrm{I}_J|-1-1_{\ell-1\in J}\geq|\mathcal{K}'|,$$

which completes the proof. It remains to consider the case $m\in J$ and $\ell\notin J$:

Choose the largest $b\in[\![0,p]\!]$ such that $j_b<\ell$, so $j_b<\ell<j_{b+1}$, and, in particular, (j_b,j_{b+1}) is an ℓ-splitting pair. By Lemma 7.10, there exists $y^\sigma\in\beta_{j_b}\cup\beta_{j_{b+1}}$ with $\sigma(y^\sigma)\in\mathrm{I}_{[\![j_b+1,j_{b+1}-1]\!]}$. We will show that

$$y^\sigma\in\gamma_{j_b}\cup\gamma_{j_{b+1}}. \tag{8.2}$$

Assume for now that (8.2) holds. Then, $(\mathrm{I}_J\backslash\{\sigma(y_u),\sigma(y_v)\})\cup\{\sigma(y^\sigma)\}\subseteq\sigma(\gamma_J)$. On the other hand, arguing as in part (a) for the case $m\in J,\ell\notin J$, we have $\sigma(y^\sigma)\notin\mathrm{I}_J$. Hence, $|\gamma_J|+1\geq|\mathrm{I}_J|$, so $\dim\left(\sum_{K\in\mathcal{K}'}F(K,e_{uv})\right)\geq|\mathrm{I}_J|\geq|\mathrm{I}_J|-1_{\ell-1\in J}-1_{\ell\in J}\geq|\mathcal{K}'|$, which completes the proof.

It remains to prove (8.2). We will show $y^\sigma\in\beta_{j_b}\Rightarrow y^\sigma\in\gamma_{j_b}$, and the argument for $y^\sigma\in\beta_{j_{b+1}}\Rightarrow y^\sigma\in\gamma_{j_{b+1}}$ is analogous. Since $y^\sigma\in\beta_{j_b}\cup\beta_{j_{b+1}}$, (8.2) will follow. Suppose then that $y^\sigma\in\beta_{j_b}$ so our task is to show that $y^\sigma\in\gamma_{j_b}$. There are two cases to consider: $j_b\geq m$ and $j_{b+1}\leq m$; we will consider the case $j_b\geq m$ and the argument for the case $j_{b+1}\leq m$ is analogous.

Let us start by showing that γ_{j_q} cannot be equal to $\beta_{j_q}\backslash\alpha_{\geq y_u}$. Indeed, the latter occurs only if $y_u\in\beta_{j_q},y_v\notin\beta_{j_q}$, in which case, either $y_v<x_{j_b}$ or $y_v>x_{j_b+1}$. If $y_v<x_{j_b}$, then $y_u<y_v<x_{j_b}$ which contradicts $y_u\in\beta_{j_q}$. If $y_v>x_{j_b+1}$, then $\sigma(x_{j_b+1})<\sigma(y_v)<i_{m+1}=\sigma(x_{m+1})$, which contradicts $m\leq j_b$. We conclude that

$\gamma_{j_q} \in \{\beta_{j_q}, \beta_{j_q}\backslash\{y_u, y_v\}, \beta_{j_q}\backslash\alpha_{\leq y_u}\}$, and since $y^\sigma \in \beta_{j_b}$, it suffices to show that $y^\sigma \notin \{y_u, y_v\}$ and $y^\sigma \notin \alpha_{\leq y_u}$. To see that $y^\sigma \notin \{y_u, y_v\}$, note that $\sigma(y^\sigma) \in \mathrm{I}_{[\![j_b+1, j_{b+1}-1]\!]}$ while $\sigma(y_u), \sigma(y_v) \in \mathrm{I}_m$. Since $m \leq j_b$, $\mathrm{I}_{[\![j_b+1, j_{b+1}-1]\!]} \cap \mathrm{I}_m = \varnothing$ so $y^\sigma \notin \{y_u, y_v\}$. To see that $y^\sigma \notin \alpha_{\leq y_u}$, note that, since $m \leq j_b$, $\sigma(y_u) < i_{m+1} \leq i_{j_b+1} < \sigma(y^\sigma)$, where the last inequality holds as $\sigma(y^\sigma) \in \mathrm{I}_{[\![j_b+1, j_{b+1}-1]\!]}$.

(d) Fix u, v such that there exist $y_u < y_v$ with $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_u) = i_\ell - 1$ and $\sigma(y_v) = i_\ell + 1$. For $j_q \in J$ we let γ_{j_q} be as in part (c). We start by showing that Claim 8.4 holds here as well.

Claim 8.6. For every $j_q \in J$,

$$\mathrm{Lin}(F(K_{j_q}, e_{uv})) = \begin{cases} \mathbb{R}^{\gamma_{j_q}} \oplus \mathrm{span}(o_{uv}) & \text{if } y_u, y_v \in \beta_{j_q}, \\ \mathbb{R}^{\gamma_{j_q}} & \text{otherwise.} \end{cases}$$

Proof. The proof is the same as the proof of Claim 8.4, but we need to check that, when $y_u, y_v \in \beta_{j_q}$, y_v covers y_u in α. The latter must be true since, otherwise, there exists $z \in \alpha$ such that $y_u < z < y_v$, so $i_\ell - 1 = \sigma(y_u) < \sigma(z) < \sigma(y_v) = i_\ell + 1$. This implies $z = x_\ell$, which contradicts $z \in \alpha$. □

Next we prove the analogue of Claim 8.5.

Claim 8.7.

(i) For $j_q \notin \{\ell - 1, \ell\}$, $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.
(ii) For $j_q = \ell - 1$, $\mathrm{I}_{\ell-1}\backslash\{i_\ell - 1\} \subseteq \sigma(\gamma_{\ell-1})$.
(iii) For $j_q = \ell$, $\mathrm{I}_\ell\backslash\{i_\ell + 1\} \subseteq \sigma(\gamma_\ell)$.

Proof. We need to consider the four cases (1) $y_u, y_v \in \beta_{j_q}$, (2) $y_u, y_v \notin \beta_{j_q}$, (3) $y_u \in \beta_{j_q}, y_v \notin \beta_{j_q}$, and (4) $y_u \notin \beta_{j_q}, y_v \in \beta_{j_q}$.

(i) Case (1): For any y such that $\sigma(y) \in \mathrm{I}_{j_q}$, we have $y \notin \{y_u, y_v\}$ since $\sigma(y_u), \sigma(y_v) \notin \mathrm{I}_{j_q}$ (because $j_q \notin \{\ell - 1, \ell\}$). Hence, by Lemma 7.12, $y \in \beta_{j_q}\backslash\{y_u, y_v\} = \gamma_{j_q}$, so we conclude $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.

Case (2): By Lemma 7.12, $\mathrm{I}_{j_q} \subseteq \beta_{j_q} = \gamma_{j_q}$ so $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.

Case (3): We start by showing that $j_q < \ell$. Indeed, suppose for contradiction that $j_q \geq \ell$. Since $y_v \notin \beta_{j_q}$, we have that either $y_v < x_{j_q}$ or $y_v > x_{j_q+1} \geq x_{\ell+1}$. We cannot have $y_v > x_{j_q+1} \geq x_{\ell+1}$, since $\sigma(y_v) = i_\ell + 1 < i_{\ell+1} = \sigma(x_{\ell+1})$. Hence, we must have $y_u < y_v < x_{j_q}$, which contradicts $y_u \in \beta_{j_q}$. We conclude that $j_q < \ell$. The assumption $j_q \notin \{\ell - 1, \ell\}$ implies that in fact $j_q < \ell - 1$. Hence, for any y such that $\sigma(y) \in \mathrm{I}_{j_q}$, we have $y \in \beta_{j_q}\backslash\alpha_{\geq y_u} = \gamma_{j_q}$, because $\sigma(y) < i_{j_q+1} \leq i_{\ell-1} < i_\ell - 1 = \sigma(y_u)$. It follows that $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.

Case (4) is analogous to case (3).

(ii) Case (1): For any y such that $\sigma(y) \in \mathrm{I}_{\ell-1}\backslash\{i_\ell - 1\}$, we have $y \notin \{y_u, y_v\}$ so, by Lemma 7.12, $\mathrm{I}_{\ell-1}\backslash\{i_\ell - 1\} \subseteq \sigma(\gamma_{\ell-1})$.

Case (2): By Lemma 7.12, $\mathrm{I}_{\ell-1} \subseteq \sigma(\beta_{\ell-1}) = \sigma(\gamma_{\ell-1})$ so $\mathrm{I}_{\ell-1} \subseteq \sigma(\gamma_{\ell-1})$.

Case (3): For any y such that $\sigma(y) \in \mathrm{I}_{\ell-1}\backslash\{i_\ell-1\}$, we have $y \in \beta_{\ell-1}\backslash\alpha_{\geq y_u} = \gamma_{\ell-1}$, because, by the definition of $\mathrm{I}_{\ell-1}\backslash\{i_\ell-1\}$, $\sigma(y) < i_\ell-1 = \sigma(y_u)$. It follows that $\mathrm{I}_{\ell-1}\backslash\{i_\ell - 1\} \subseteq \sigma(\gamma_{\ell-1})$.

Case (4) is analogous to case (3).

(iii) The argument is analogous to (ii). □

By (5.1) and Claim 8.6, $\mathbb{R}^{\gamma_J} \subseteq \mathrm{Lin}\left(\sum_{K\in\mathcal{K}'} F(K, e_{uv})\right)$, so $\dim\left(\sum_{K\in\mathcal{K}'} F(K, e_{uv})\right) \geq |\gamma_J|$. By Claim 8.7, using the fact that $\{\mathrm{I}_{j_q}\}_{j_q\in J\backslash\{\ell-1,\ell\}}, \mathrm{I}_{\ell-1}, \mathrm{I}_\ell$ are disjoint, we have

$$|\gamma_J| \geq \sum_{j_q\in J} [|\mathrm{I}_{j_q}| - 1_{j_q=\ell-1} - 1_{j_q=\ell}] = |\mathrm{I}_J| - 1_{\ell-1\in J} - 1_{\ell\in J} \geq |\mathcal{K}'|,$$

which completes the proof.

(e) Fix $r_{\max} + 1 \leq m \leq \ell - 1$ and consider $\sigma \in \mathcal{N}_=$ such that $\sigma(y_j) = i_m + 2$ where j is such that $y_j \in \alpha_{>x_m}$. By Corollary 7.11, $\sigma(y_j) = i_m + 2 \leq i_{m+1} = \sigma(x_{m+1})$, and since $\sigma(y_j) \neq \sigma(x_{m+1})$ (as $y_j \neq x_{m+1}$), we get that $\sigma(y_j) = i_m + 2 < i_m + 3 \leq \sigma(x_{m+1}) = i_{m+1}$. It follows that $i_m + 1 < i_{m+1} - 1$, so $\sigma(y_j) \in \mathrm{I}_m$.

For $j_q \in J$, let γ_{j_q} be as in part (a), and note that an analogous argument yield

$$\dim\left(\sum_{K\in\mathcal{K}'} F(K, -e_j)\right) = |\gamma_J|,$$

and

Claim 8.8.

(i) For $j_q \neq m$, $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.
(ii) For $j_q = m$, $\mathrm{I}_m\backslash\{i_m + 1, i_m + 2\} \subseteq \sigma(\gamma_m)$.

In order to complete the proof we distinguish between two cases: $m \notin J$ and $m \in J$. The proof of the case $m \notin J$ is the same as in part (a). Suppose that $m \in J$ and consider the following cases:

- $\ell - 1, \ell \in J$: The proof is complete since $|\mathcal{K}'| \leq |\mathrm{I}_J| - 1_{\ell-1\in J} - 1_{\ell\in J} = |\mathrm{I}_J| - 2$, and since Claim 8.8 yields $|\gamma_J| \geq |\mathrm{I}_J| - 2$.
- $\ell - 1 \in J$ and $\ell \notin J$: Since $\ell \notin J$, there is an index j_b such that $j_b = \ell - 1$ and $j_{b+1} > \ell$, and note that (j_b, j_{b+1}) is a splitting pair. Note that since $m \leq \ell-1$, and $m \in J$, we must have $m \leq j_b$. By Lemma 7.10, there exists $y^\sigma \in \beta_{j_b} \cup \beta_{j_b+1}$ such that $\sigma(y^\sigma) \in \mathrm{I}_{[\![j_b+1, j_{b+1}-1]\!]}$. Suppose $y^\sigma \in \beta_{j_b}$; the proof for the case $y^\sigma \in \beta_{j_b+1}$ is

analogous. Since $\sigma(y^\sigma) > i_{j_b+1} \geq i_{m+1} > \sigma(y_j)$, we get $y^\sigma \in \beta_{j_b} \backslash \alpha_{\leq y_j} \subseteq \gamma_{j_b} \subseteq \gamma_J$. Hence,

$$(\mathrm{I}_J \backslash \{i_m + 1, i_m + 2\}) \cup \{\sigma(y^\sigma)\} \subseteq \sigma(\gamma_J).$$

Since $\sigma(y^\sigma) \in \mathrm{I}_{[\![j_b+1, j_{b+1}-1]\!]}$, we have $\sigma(y^\sigma) \notin \mathrm{I}_J$ (because $j_b = \ell - 1$ and $\ell \notin J$ so the indices $\{j_b + 1, \ldots, j_{b+1} - 1\} = [\![j_b + 1, j_{b+1} - 1]\!]$ are not in J), so we get that $|\gamma_J| \geq |\mathrm{I}_J| - 1 = |\mathrm{I}_J| - 1_{\ell-1 \in J} - 1_{\ell \in J} \geq |\mathcal{K}'|$.

- $\ell - 1 \notin J$ and $\ell \in J$: The proof is analogous to the case $\ell - 1 \in J$ and $\ell \notin J$.
- $\ell - 1, \ell \notin J$: Since $\ell - 1, \ell \notin J$, we can choose b to be an index such that $j_b < \ell - 1 < \ell < j_{b+1}$, or the largest index such $j_b < \ell - 1 < \ell$, and note that (j_b, j_{b+1}) is an ℓ-splitting pair. Note that since $m \leq \ell - 1$, and $m \in J$, we must have $m \leq j_b$. Consider the collection

$$\mathcal{K}'' := (\mathcal{K}_0, \ldots, \mathcal{K}_{j_b}, \mathcal{K}_{j_b+1}, \ldots, \mathcal{K}_k)$$

and note that, by Assumption 6.7, $\mathcal{K}''$ is critical. We claim that $\mathcal{K}''$ is in fact supercritical. Indeed, if $\mathcal{K}''$ is sharp-critical, then $j_b \leq r_{\max}$. But $j_b \geq m > r_{\max}$, so we get a contradiction. Since $\mathcal{K}''$ is supercritical, and since (j_b, j_{b+1}) is an ℓ-splitting pair, Corollary 7.14 provides two distinct $y^\sigma, z^\sigma \in \beta_{j_b} \cup \beta_{j_{b+1}}$, with $\sigma(y^\sigma), \sigma(z^\sigma) \in \mathrm{I}_{[\![j_b+1, j_{b+1}-1]\!]}$, from which it follows that

$$\mathrm{I}_{[\![0,j_b]\!] \cup [\![j_{b+1},k]\!]} \cup \{\sigma(y^\sigma), \sigma(z^\sigma)\} \subseteq \sigma(\beta_{[\![0,j_b]\!] \cup [\![j_{b+1},k]\!]}).$$

Suppose that $y^\sigma \in \beta_{j_b}$; the case $y^\sigma \in \beta_{j_{b+1}}$ is analogous. Since $m \leq j_b$, $\sigma(y^\sigma) > i_{j_b+1} \geq i_{m+1} > \sigma(y_j)$, so we can conclude that $y^\sigma \in \beta_{j_b} \backslash \alpha_{\leq y_j} \subseteq \gamma_{j_b} \subseteq \gamma_J$. Analogous argument shows that $z^\sigma \in \gamma_J$. By Claim 8.8, it follows that

$$(\mathrm{I}_J \backslash \{i_m + 1, i_m + 2\}) \cup \{\sigma(y^\sigma), \sigma(z^\sigma)\} \subseteq \sigma(\gamma_J).$$

Since $\sigma(y^\sigma), \sigma(z^\sigma) \in \mathrm{I}_{[\![j_b+1, j_{b+1}-1]\!]}$, we have $\sigma(y^\sigma), \sigma(z^\sigma) \notin \mathrm{I}_J$ (because b satisfies $j_b < \ell - 1 < \ell < j_{b+1}$, or the maximal $j_b < \ell - 1$, so the indices $\{j_b + 1, \ldots, j_{b+1} - 1\} = [\![j_b + 1, j_{b+1} - 1]\!]$ are not in J). On the other hand, because $m \in J$ and $i_m + 2 < i_{m+1}$, we have $i_m + 1, i_m + 2 \in \mathrm{I}_J$. It follows that $(\mathrm{I}_J \backslash \{i_m + 1, i_m + 2\}) \cup \{\sigma(y^\sigma), \sigma(z^\sigma)\}| = |\mathrm{I}_J|$, and hence, $|\gamma_J| \geq |\mathrm{I}_J| \geq |\mathcal{K}'|$.

(f) The proof is analogous to part (e).

(g) Consider $\sigma \in \mathcal{N}_+$ such that $\sigma(y_j) = i_{\ell-1} + 2$ where j is such that $y_j \in \alpha_{>x_{\ell-1}}$. By Corollary 7.11, $\sigma(x_{\ell-1}) < i_{\ell-1} + 2 = \sigma(y_j) < i_\ell + 1 = \sigma(x_\ell)$, so we conclude that $y_j \in \beta_{\ell-1}$. For $j_q \in J$ let γ_{j_q} be as in part (a), and note that an analogous argument yields $\dim\left(\sum_{K \in \mathcal{K}'} F(K, -e_j)\right) = |\gamma_J|$. We start with the analogue of Claim 8.7.

Claim 8.9.

(i) For $j_q \notin \{\ell-1, \ell\}$, $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.
(ii) For $j_q = \ell - 1$, $(\mathrm{I}_{\ell-1} \cup \{i_\ell\})\backslash\{i_{\ell-1}+1, i_{\ell-1}+2\} \subseteq \sigma(\gamma_{\ell-1})$.
(iii) For $j_q = \ell$, $\mathrm{I}_\ell\backslash\{i_\ell+1\} \subseteq \sigma(\gamma_\ell)$.

Proof. There two cases to consider: (1) $y_j \notin \beta_{j_q}$ and (2) $y_j \in \beta_{j_q}$.

(i) Case (1): By Lemma 7.12, $\mathrm{I}_{j_q} \subseteq \sigma(\beta_{j_q}) = \sigma(\gamma_{j_q})$.
Case (2): First we note that $j_q \geq \ell - 1$ since, otherwise, $y_j > x_{\ell-1} \geq x_{j_q+1}$ which contradicts $y_j \in \beta_{j_q}$. Since $j_q \notin \{\ell-1, \ell\}$, it follows that in fact $\ell < j_q$. Hence, for any y such that $\sigma(y) \in \mathrm{I}_{j_q}$, we have $\sigma(y) > \sigma(x_{j_q}) \geq \sigma(x_{\ell+1}) = i_{\ell+1} > i_{\ell-1} + 2 = \sigma(y_j)$, so that $y \notin \alpha_{\leq y_j}$. It follows that $y \in \gamma_{j_q}$, so we conclude $\mathrm{I}_{j_q} \subseteq \sigma(\gamma_{j_q})$.
(ii) Case (1) cannot occur since we have shown that $y_j \in \beta_{\ell-1}$.
Case (2): Every y such that $\sigma(y) \in (\mathrm{I}_{\ell-1} \cup \{i_\ell\})\backslash\{i_{\ell-1}+1, i_{\ell-1}+2\}$ satisfies $\sigma(x_{\ell-1}) < \sigma(y) < \sigma(x_\ell)$, so $y \in \beta_{\ell-1}$. Further, $\sigma(y) > i_{\ell-1} + 2 = \sigma(y_j)$, so $y \notin \alpha_{\leq y_j}$. It follows that $y \in \gamma_{\ell-1}$, so we conclude $(\mathrm{I}_{\ell-1} \cup \{i_\ell\})\backslash\{i_{\ell-1}+1, i_{\ell-1}+2\} \subseteq \sigma(\gamma_{\ell-1})$.
(iii) Case (1): Every y such that $\sigma(y) \in \mathrm{I}_\ell\backslash\{i_\ell+1\}$ satisfies $\sigma(x_\ell) = i_\ell + 1 < \sigma(y) < i_{\ell+1} = \sigma(x_{\ell+1})$, so $y \in \beta_\ell = \gamma_\ell$. We conclude that $\mathrm{I}_\ell\backslash\{i_\ell+1\} \subseteq \sigma(\gamma_\ell)$.
Case (2): Every y such that $\sigma(y) \in \mathrm{I}_\ell\backslash\{i_\ell+1\}$ satisfies $\sigma(x_\ell) = i_\ell + 1 < \sigma(y) < i_{\ell+1} = \sigma(x_{\ell+1}$, so $y \in \beta_\ell$. Further, $\sigma(y) > i_\ell + 1 > i_{\ell-1} + 2 = \sigma(y_j)$, so $y \notin \alpha_{\leq y_j}$. It follows that $y \in \gamma_\ell$, so we conclude $\mathrm{I}_\ell\backslash\{i_\ell+1\} \subseteq \sigma(\gamma_\ell)$. □

By (5.1) $\mathrm{Lin}\left(\sum_{K\in\mathcal{K}'} F(K, -e_j)\right) = \mathbb{R}^{\gamma_J}$, so $\dim\left(\sum_{K\in\mathcal{K}'} F(K, -e_j)\right) = |\gamma_J|$. By Claim 8.9, using the fact that $\{\mathrm{I}_{j_q}\}_{j_q\in J\backslash\{\ell-1,\ell\}}, \mathrm{I}_{\ell-1}, \mathrm{I}_\ell$ are disjoint, it suffices to show that $|(\mathrm{I}_{\ell-1} \cup \{i_\ell\})\backslash\{i_{\ell-1}+1, i_{\ell-1}+2\}| = |\mathrm{I}_{\ell-1}| - 1$, and that $|\mathrm{I}_\ell\backslash\{i_\ell+1\}| = |\mathrm{I}_\ell| - 1$, since then

$$|\gamma_J| \geq \sum_{j_q\in J} [|\mathrm{I}_{j_q}| - 1_{j_q=\ell-1} - 1_{j_q=\ell}] = |\mathrm{I}_J| - 1_{\ell-1\in J} - 1_{\ell\in J} \geq |\mathcal{K}'|,$$

which completes the proof. To see that $|(\mathrm{I}_{\ell-1} \cup \{i_\ell\})\backslash\{i_{\ell-1}+1, i_{\ell-1}+2\}| = |\mathrm{I}_{\ell-1}| - 1$, we note that $|\mathrm{I}_{\ell-1} \cup \{i_\ell\}| = |\mathrm{I}_{\ell-1}| + 1$, and that $i_{\ell-1}+1, i_{\ell-1}+2 \in \mathrm{I}_{\ell-1} \cup \{i_\ell\}$, because $i_{\ell-1}+1 < i_{\ell-1}+2 \leq i_\ell$, by Corollary 7.11. Hence, $|(\mathrm{I}_{\ell-1} \cup \{i_\ell\})\backslash\{i_{\ell-1}+1, i_{\ell-1}+2\}| = (|\mathrm{I}_{\ell-1}| + 1) - 2 = |\mathrm{I}_{\ell-1}| - 1$. Finally, it is clear that $|\mathrm{I}_\ell\backslash\{i_\ell+1\}| = |\mathrm{I}_\ell| - 1$, since $i_\ell + 1 \in \mathrm{I}_\ell$.

(h) The proof is analogous to part (g). □

9. Supercritical posets

In this section we complete the characterization of the extremals of Stanley's inequalities for supercritical posets. The following result, together with Proposition 3.2, Lemma 3.3, Proposition 3.8, and Proposition 5.7, complete the proof of Theorem 1.3.

Theorem 9.1. *Suppose that $\mathcal{K}$ is supercritical and that $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$. Then,*

$$|\mathcal{N}_=(\nsim, \sim)| = |\mathcal{N}_=(\sim, \nsim)| = |\mathcal{N}_=(\sim, \sim)| = 0.$$

In order to prove Theorem 9.1, we will invoke Theorem 2.9 and use the extreme normal directions found in Proposition 8.2(a–d). Theorem 2.9 tells us that there exist $a \geq 0$ and $\mathrm{v} \in \mathbb{R}^{n-k}$ such that

$$h_{K_{\ell-1}}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}}(\mathrm{u}) \quad \text{for all } (B, \mathcal{K})\text{-extreme normal directions u.} \tag{9.1}$$

The following results derive constraints from (9.1) on the allowed a and v. We start with v.

Proposition 9.2.

(a) *For each fixed $0 \leq m \leq \ell - 1$: $\mathrm{v}_j = 0$ for any j such that $y_j \in \alpha_{>x_m}$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_j) = i_m + 1$.*

(b) *For each fixed $\ell + 1 \leq m \leq k + 1$: $\mathrm{v}_j = 1 - a$ for any j such that $y_j \in \alpha_{<x_m}$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_j) = i_m - 1$.*

(c) *$\mathrm{v}_u = \mathrm{v}_v$ for any u, v such that $y_u < y_v$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_u)+1 = \sigma(y_v)$.*

(d) *$\mathrm{v}_u = \mathrm{v}_v$ for any u, v such that $y_u < y_v$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_u) = i_\ell - 1$ and $\sigma(y_v) = i_\ell + 1$.*

Proof. (a) By Proposition 8.2(a), $-e_j$ is a $(B, \mathcal{K})$-extreme normal direction, so by (9.1), $h_{K_{\ell-1}}(-e_j) = ah_{K_\ell}(-e_j) - \mathrm{v}_j$. Since $\sigma(y_j) = i_m + 1$, and $m \leq \ell - 1$, we have $\sigma(y_j) = i_m + 1 \leq i_{\ell-1} + 1 < i_\ell, i_{\ell+1}$, so $y_j \notin \alpha_{>x_\ell} \cup \alpha_{>x_{\ell+1}}$. Hence, it follows from (2.5) that $h_{K_{\ell-1}}(-e_j) = h_{K_\ell}(-e_j) = 0$. We conclude that $\mathrm{v}_j = 0$.

(b) By Proposition 8.2(b), e_j is a $(B, \mathcal{K})$-extreme normal direction, so by (9.1), $h_{K_{\ell-1}}(-e_j) = ah_{K_\ell}(-e_j) + \mathrm{v}_j$. Since $\sigma(y_j) = i_m - 1$, and $m \geq \ell + 1$, we have $\sigma(y_j) = i_m - 1 \geq i_{\ell+1} - 1 > i_\ell, i_{\ell-1}$, so $y_j \notin \alpha_{<x_{\ell-1}} \cup \alpha_{<x_\ell}$. Hence, it follows from (2.5) that $h_{K_{\ell-1}}(e_j) = 1$ and $ah_{K_\ell}(e_j) + \mathrm{v}_j = a + \mathrm{v}_j$. We conclude that $\mathrm{v}_j = 1 - a$.

(c) By Proposition 8.2(c), e_{uv} is a $(B, \mathcal{K})$-extreme normal direction, so by (9.1), $h_{K_{\ell-1}}(e_{uv}) = ah_{K_\ell}(e_{uv}) + \frac{1}{\sqrt{2}}(\mathrm{v}_u - \mathrm{v}_v)$. We will show that $h_{K_{\ell-1}}(e_{uv}) = h_{K_\ell}(e_{uv}) = 0$, from which we can conclude $\mathrm{v}_u = \mathrm{v}_v$. We will show that $h_{K_{\ell-1}}(e_{uv}) = 0$; the proof of $h_{K_\ell}(e_{uv}) = 0$ is analogous. We distinguish between the following cases:

Case (1): $y_u, y_v \in \beta_{\ell-1}$. By (2.5), $h_{K_{\ell-1}}(e_{uv}) = 0$ since $t_u \le t_v$ for $t \in O_{\beta_{\ell-1}}$, and equality is attained with $t = 0$.

Case (2): $y_u \in \beta_{\ell-1}, y_v \notin \beta_{\ell-1}$, or $y_u \notin \beta_{\ell-1}, y_v \in \beta_{\ell-1}$. See the proof of Claim 8.4.

Case (3): $y_u, y_v \notin \beta_{\ell-1}$. Since there exists $\sigma \in \mathcal{N}_=$ with $\sigma(y_u) + 1 = \sigma(y_v)$, the assumption $y_u, y_v \notin \beta_{\ell-1}$ implies that either $y_u, y_v < x_{\ell-1}$, or $y_u, y_v > x_\ell$. Hence, either $t_u = t_v = 1$, or $t_u = t_v = 0$ for any $t \in K_{\ell-1}$, so, in particular, $h_{K_{\ell-1}}(e_{uv}) = 0$.

(d) The proof is analogous to part (c), where we note that $y_u \notin \beta_{\ell-1}$ cannot occur. □

While Proposition 9.2(a–b) took care of elements neighboring x_m's, the next result takes care of elements that are at the bottom (res. the top) of the poset.

Lemma 9.3. *For any $y_j \in \alpha$: If $m^=_{\min}(y_j) < i_\ell$ then $\mathrm{v}_j = 0$, and if $m^=_{\max}(y_j) > i_\ell$ then $\mathrm{v}_j = 1 - a$.*

Proof. We prove that $m^=_{\max}(y_j) > i_\ell \Rightarrow \mathrm{v}_j = 1 - a$; the proof of $m^=_{\min}(y_j) < i_\ell \Rightarrow \mathrm{v}_j = 0$ is analogous.

Set $y_{j_0} := y_j$ and construct the sequence $y_{j_0} < y_{j_1} < \cdots < y_{j_p}$, for some $p < \infty$, iteratively, according to the algorithm below. The sequence will be constructed so that $y_{j_i} \in \alpha$ for every $i \in [\![0, p]\!]$, $\mathrm{v}_{j_i} = \mathrm{v}_{j_{i+1}}$ for all $i \in [\![0, p-1]\!]$, and $\mathrm{v}_{j_p} = 1 - a$. Clearly, it will then follow that $\mathrm{v}_j = \mathrm{v}_{j_0} = 1 - a$, completing the proof.

Assume that the sequence $y_{j_0} < y_{j_1} < \cdots < y_{j_i}$ has been constructed. Set $M := m^=_{\max}(y_{j_i})$, and note that $i_\ell < m^=_{\max}(y_{j_0}) \le M$. Consider the following two cases:

- $M \ne i_m - 1$ for every $\ell < m$: Choose $\sigma \in \mathcal{N}_=$ such that $\sigma(y_{j_i}) = M$ (such a σ must exist by the definition of M) and set $y_{j_{i+1}} := \sigma^{-1}(M + 1)$. We first show that $M + 1 \ne i_m$ for any $m \in [\![0, k]\!]$. Indeed, by assumption $M + 1 \ne i_m$ for every $\ell < m$, and if $m \le \ell$, then $i_m \le i_\ell < M + 1$. It follows that $y_{j_{i+1}} \in \alpha$. Next we show that $y_{j_i} < y_{j_{i+1}}$. Indeed, otherwise, by the definition of M, y_{j_i} and $y_{j_{i+1}}$ must be incomparable, so we can swap the positions of y_{j_i} and $y_{j_{i+1}}$ in σ to get $\sigma' \in \mathcal{N}_=$ such that $\sigma'(y_{j_i}) = M + 1$, which contradicts the maximality of M. We conclude that $y_{j_i} < y_{j_{i+1}}$. Finally, by Proposition 9.2(c), $\mathrm{v}_{j_i} = \mathrm{v}_{j_{i+1}}$.
- $M = i_m - 1$ for some $\ell < m$: In this case, the sequence will be terminated with $p := i$. Note that Corollary 7.11 implies that $y_{j_i} \in \alpha$, since $M = i_m - 1$. We will show that $\sigma(y_{j_i}) < \sigma(x_m)$ for all $\sigma \in \cup_{\circ\in\{-,=,+\}} \mathcal{N}_\circ$. Then, by Assumption 4.1, it follows that $y_{j_i} < x_m$ so, by Proposition 9.2(b), $\mathrm{v}_{j_i} = 1 - a$. To show that $\sigma(y_{j_i}) < \sigma(x_m)$ for all $\sigma \in \cup_{\circ\in\{-,=,+\}} \mathcal{N}_\circ$, suppose for contradiction otherwise, which means that there exists $\sigma \in \mathcal{N}_\circ$, for some $\circ \in \{-,=,+\}$, such that $\sigma(y_{j_i}) > \sigma(x_m) = i_m$. Set $q := \sigma(y_{j_i})$. We will show that Lemma 7.3 can be applied with y_{j_i}, $=$, and q, to yield $\sigma' \in \mathcal{N}_=$ such that $\sigma'(y_{j_i}) = q$, contradicting the maximality of M (since $q > i_m > i_m - 1 = M$).

  To apply Lemma 7.3 to y_{j_i}, $=$, and q, we need to check that all of the conditions of the lemma are satisfied. Applying the lemma to y_{j_i}, $\circ$, and q, we get $q \le u_\circ(y_{j_i})$, and by Lemma 7.2 (as $i_{\min}(y_{j_i}) > m > \ell$), we get $q \le u_\circ(y_{j_i}) = u_=(y_{j_i})$. On the other hand, by Corollary 7.4, $l_=(y_{j_i}) \le m^=_{\max}(y_{j_i}) = M = i_m - 1 < q$. We conclude that the condition $q \in [\![l_=(y_{j_i}), u_=(y_{j_i})]\!]$ holds. Finally, we show that $q \ne i_r + 1_= = i_r$ for any $r \in [\![1,k]\!]$. Indeed, if $q = i_r$ for some $r \in [\![1,k]\!]$, then $i_r = q > i_m$, which implies $\ell < m < r$, and hence $\sigma(x_r) = i_r$ as $r \ne \ell$. It follows that $\sigma(y_{j_i}) = q = \sigma(x_r)$, contradicting $y \in \alpha$. $\square$

Next we move to a.

Lemma 9.4. $a = 1$.

Proof. Fix $\sigma \in \mathcal{N}_=$ and set $y_u^\sigma := \sigma^{-1}(i_\ell - 1)$, $y_v^\sigma := \sigma^{-1}(i_\ell + 1)$. There are a few cases to check:

- $y_u^\sigma \nsim x_\ell$: If $m^=_{\max}(y_u^\sigma) > i_\ell$, then, since $m^=_{\min}(y_u^\sigma) \le \sigma(y_u^\sigma) < i_\ell$, Lemma 9.3 implies that $\mathrm{v}_u = 0$ and $\mathrm{v}_u = 1 - a$ so $a = 1$. Suppose then that $m^=_{\max}(y_u^\sigma) < i_\ell$. We claim that $u_=(y_u^\sigma) \le i_\ell$. Indeed, otherwise, $u_=(y_u^\sigma) \ge i_\ell + 1 \ge \sigma(y_u^\sigma) \ge l_=(y_u^\sigma)$. Hence, since $i_\ell + 1 \ne \sigma(x_m)$ for any $m \in [\![1,k]\!]$, Lemma 7.3 implies that there exists $\sigma' \in \mathcal{N}_=$ satisfying $\sigma'(y_u^\sigma) = i_\ell + 1$, which contradicts the maximality of $m^=_{\max}(y_u^\sigma) < i_\ell$. Now, since $u_=(y_u^\sigma) \le i_\ell$, there must exist b, with $y_u^\sigma < x_b$, such that $i_b^\circ - |\bar\alpha_{\ge y_u^\sigma, <x_b}| \le i_\ell$. It follows that $|\bar\alpha_{\ge y_u^\sigma, <x_b}| \ge i_b - i_\ell$, where we used $i_b^= = i_b$. Fix $z \in \bar\alpha_{>y_u^\sigma, <x_b}$, and note that $z \ne x_\ell$, since otherwise $x_\ell > y_u^\sigma$, which contradicts the assumption $y_u^\sigma \nsim x_\ell$. In particular, since $\sigma(x_\ell) = i_\ell$, we have $\sigma(z) \ne i_\ell$. Since $i_\ell - 1 = \sigma(y_u^\sigma) < \sigma(z) < \sigma(x_b) = i_b$, we conclude that $\sigma(z) \in [\![i_\ell + 1, i_b - 1]\!]$. The size of $[\![i_\ell + 1, i_b - 1]\!]$ is $i_b - i_\ell - 1$, so combining $|\bar\alpha_{>y_u^\sigma, <x_b}| \ge i_b - i_\ell - 1$, with $\sigma(z) \in [\![i_\ell+1, i_b-1]\!]$ for every $z \in \bar\alpha_{>y_u^\sigma, <x_b}$, shows that $\sigma(\bar\alpha_{>y_u^\sigma, <x_b}) = [\![i_\ell+1, i_b-1]\!]$. In particular, since $\sigma(y_v^\sigma) = i_\ell + 1$, we get that $y_v^\sigma \in \bar\alpha_{>y_u^\sigma, <x_b}$, so $y_v^\sigma < y_u^\sigma$. It follows from Proposition 9.2(c) that $\mathrm{v}_u = \mathrm{v}_v$. Since $m^=_{\min}(y_u^\sigma) < i_\ell$, and $m^=_{\max}(y_v^\sigma) > i_\ell$, Lemma 9.3 yields $0 = \mathrm{v}_u = \mathrm{v}_v = 1 - a$. We conclude that $a = 1$.
- $y_v^\sigma \nsim x_\ell$: Analogous to the case $y_u^\sigma \nsim x_\ell$.
- $y_u^\sigma < x_\ell$ and $x_\ell < y_v^\sigma$: By Proposition 9.2(d), $\mathrm{v}_u = \mathrm{v}_v$. Since $y_u^\sigma < x_\ell$, we have $m^=_{\min}(y_u^\sigma) < i_\ell$ so, by Lemma 9.3, $\mathrm{v}_u = 0$. Similarly, since $x_\ell < y_v^\sigma$, we have $m^=_{\max}(y_v^\sigma) > i_\ell$ so, by Lemma 9.3, $\mathrm{v}_v = 1 - a$. We conclude that $0 = \mathrm{v}_u = \mathrm{v}_v = 1 - a$, so $a = 1$. $\square$

We are now ready to prove Theorem 9.1.

Proof. (of Theorem 9.1) We will show that

$$\forall\, \sigma \in \mathcal{N}_= : \quad \sigma^{-1}(i_\ell - 1) \nsim x_\ell \quad \text{and} \quad \sigma^{-1}(i_\ell + 1) \nsim x_\ell, \tag{9.2}$$

which is equivalent to $|\mathcal{N}_=(\nsim, \sim)| = |\mathcal{N}_=(\sim, \nsim)| = |\mathcal{N}_=(\sim, \sim)| = 0$.

Let $y_j \in \alpha$ be any element such that there exists $\sigma \in \mathcal{N}_=$ with $\sigma(y_j) = i_\ell + 1$; the proof for elements $y_j \in \alpha$ with $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_j) = i_\ell - 1$ is analogous. Since $m^=_{\max}(y_j) > i_\ell$, Lemma 9.3 yields $\mathrm{v}_j = 1 - a = 0$, where the last equality follows from Lemma 9.4. Assume for contradiction that x_ℓ is comparable to y_j, which, by the assumption $\sigma(y_j) = i_\ell + 1$, means that $x_\ell < y_j$. By Proposition 8.2(a), $-e_j$ is a $(B, \mathcal{K})$-extreme normal direction so, by (9.1), $h_{K_{\ell-1}}(-e_j) = h_{K_\ell}(-e_j)$. Since $x_\ell < y_j$, we have $h_{K_{\ell-1}}(-e_j) = -1$. On the other hand, $i_\ell < \sigma(y_j) = i_\ell + 1 < i_{\ell+1}$, so $y_j \in \beta_\ell$. By (2.5), $h_{K_\ell}(-e_j) = 0 \neq -1 = h_{K_{\ell-1}}(-e_j)$, so we have arrived at the desired contradiction. □

10. Critical posets

In this section we complete the characterization of the extremals of Stanley's inequalities for critical posets (as well as Theorem 1.6). We will assume that $\mathcal{K}$ is sharp-critical since, otherwise, we reduce back to the supercritical setting. We note that the assumption that $\mathcal{K}$ is sharp-critical implies, by Proposition 7.8, that the maximal sharp-critical collection $\mathcal{K}_{\max}$, with its associated splitting pair $(r_{\max}, s_{\min})$, exists. The following result (Theorem 10.1), together with Proposition 3.2, Lemma 3.3, Proposition 3.8, and Proposition 5.7, complete the proof of Theorem 1.5.

The proof of Theorem 1.6 follows by Corollary 3.4, and by applying Theorem 6.6 repeatedly until arriving at a critical subposet. Once a critical subposet is reached, Theorem 1.5 can be applied to the critical subposet, together with the bijection construction in the proof of Proposition 6.4, to conclude that the results of Theorem 1.5 hold for the original poset as well.

Theorem 10.1. *Suppose that $\mathcal{K}$ is sharp-critical and that $|\mathcal{N}_=|^2 = |\mathcal{N}_-||\mathcal{N}_+|$. Then,*

$$|\mathcal{N}_-(\sim,\sim)| = |\mathcal{N}_+(\sim,\sim)| = 0.$$

10.1. The critical subspace

We now enter the critical territory so the equation

$$h_{K_{\ell-1}}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}}(\mathrm{u}) \quad \text{for all } (B,\mathcal{K})\text{-extreme normal directions u},$$

which held for supercritical posets, is no longer valid. Instead, we only have

$$h_{K_{\ell-1}+\sum_{j=1}^d Q_j}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}+\sum_{j=1}^d P_j}(\mathrm{u}) \quad \text{for all } (B,\mathcal{K})\text{-extreme normal directions u},$$

where $(P_1, Q_1), \dots, (P_d, Q_d)$ are $\mathcal{K}$-degenerate pairs. Our approach to this problem is to find a subspace $E^\perp$, on which we do in fact have $h_{K_{\ell-1}}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}}(\mathrm{u})$ for all $(B, \mathcal{K})$-extreme normal directions $\mathrm{u} \in E^\perp$. Since we now require that the $(B, \mathcal{K})$-extreme normal directions are contained in $E^\perp$, we will need more of them in order to derive enough constraints to characterize the extremals of critical posets. These extreme normal directions are the ones given in Proposition 8.2(e–h). We define the subspace $E^\perp$ by

$$E^\perp := \mathbb{R}^{\alpha\setminus\beta_{\max}}, \tag{10.1}$$

where we recall (7.3). We call the subspace E the *critical subspace* and note that, by Lemma 5.1, $\mathrm{Lin}(\mathcal{K}_{\max}) = \mathbb{R}^{\beta_{\max}} = E$. The following result explains the connection between $\mathcal{K}$-degenerate pairs and E.

Lemma 10.2. *Let (P, Q) be a $\mathcal{K}$-degenerate pair. Then,* $\mathrm{Lin}(P), \mathrm{Lin}(Q) \subseteq E$.

Proof. The result follows by [17, Lemma 9.6] and Proposition 7.8. □

When we restrict to the subspace $E^\perp$, we are in the supercritical case in the following sense:

Lemma 10.3. *There exist $a \geq 0$ and $\mathrm{v} \in S^{n-k-1}$ such that*

$$h_{K_{\ell-1}}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}}(\mathrm{u}) \quad \text{for all } (B,\mathcal{K})\text{-extreme normal directions } \mathrm{u} \text{ which are contained in } E^\perp.$$

Proof. Let $\mathrm{u} \in E^\perp$ be a $(B, \mathcal{K})$-extreme normal direction. By Theorem 2.9

$$h_{K_{\ell-1}+\sum_{j=1}^d Q'_j+\sum_{j=1}^d q_j}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}+\sum_{j=1}^d P'_j+\sum_{j=1}^d p_j}(\mathrm{u}),$$

where $(P_j, Q_j)_{j\in[\![1,d]\!]}$ are $\mathcal{K}$-degenerate pairs and $P'_j = P_j - p_j$, $Q'_j = Q_j - q_j$ where $p_j \in P_j$, $q_j \in Q_j$ are fixed. Hence, with $\mathrm{v}' := v + \sum_{j=1}^d p_j - \sum_{j=1}^d q_j$, $\tilde{P} := \sum_{j=1}^d P'_j$, and $\tilde{Q} := \sum_{j=1}^d Q'_j$, we have

$$h_{K_{\ell-1}+\tilde{Q}}(\mathrm{u}) = h_{aK_\ell+\mathrm{v}'+\tilde{P}}(\mathrm{u}).$$

Since $\tilde{P}, \tilde{Q} \subseteq E$ and $\mathrm{u} \in E^\perp$, we have $h_{\tilde{Q}}(\mathrm{u}) = h_{\tilde{P}}(\mathrm{u}) = 0$. Relabeling $\mathrm{v}' \to \mathrm{v}$ completes the proof. □

10.2. The critical extremals

In order to prove Theorem 10.1 we need to prove the analogues of Proposition 9.2, Lemma 9.3, and Lemma 9.4, as well as some additional results. Roughly speaking, on

$$\alpha\setminus\beta_{\max} = \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}},$$

we have a supercritical behavior. Indeed, the proof of the following result is analogous to the proof of Proposition 9.2 once we use the full power of Proposition 8.2, Lemma 10.3, and restrict to $y_j, y_u, y_v \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$, rather than allowing for all $y_j, y_u, y_v \in \alpha$.

Proposition 10.4. *For any $y_j, y_u, y_v \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$:*

(a) For each fixed $0 \le m \le \ell - 1$: $\mathrm{v}_j = 0$ for any j such that $y_j \in \alpha_{>x_m}$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_j) = i_m + 1$.
(b) For each fixed $\ell + 1 \le m \le k + 1$: $\mathrm{v}_j = 1 - a$ for any j such that $y_j \in \alpha_{<x_m}$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_j) = i_m - 1$.
(c) $\mathrm{v}_u = \mathrm{v}_v$ for any u, v such that $y_u < y_v$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_u)+1 = \sigma(y_v)$.
(d) $\mathrm{v}_u = \mathrm{v}_v$ for any u, v such that $y_u < y_v$ and there exists $\sigma \in \mathcal{N}_=$ satisfying $\sigma(y_u) = i_\ell - 1$ and $\sigma(y_v) = i_\ell + 1$.
(e) For each fixed $r_{\max} \le m \le \ell - 1$: $\mathrm{v}_j = 0$ for any j such that $y_j \in \alpha_{>x_m}$ and there exists $\sigma \in \mathcal{N}_=$ satisfying either $\sigma(y_j) = i_m + 1$ or $\sigma(y_j) = i_m + 2$.
(f) For each fixed $\ell + 1 \le m \le s_{\min}$: $\mathrm{v}_j = 1 - a$ for any j such that $y_j \in \alpha_{<x_m}$ and there exists $\sigma \in \mathcal{N}_=$ satisfying either $\sigma(y_j) = i_m - 1$ or $\sigma(y_j) = i_m - 2$.
(g) $\mathrm{v}_j = 0$ for any j such that $y_j \in \alpha_{>x_{\ell-1}}$ and there exists $\sigma \in \mathcal{N}_+$ satisfying $\sigma(y_j) = i_{\ell-1} + 2$.
(h) $\mathrm{v}_j = 1 - a$ for any j such that $y_j \in \alpha_{<x_{\ell+1}}$ and there exists $\sigma \in \mathcal{N}_-$ satisfying $\sigma(y_j) = i_{\ell+1} - 2$.

Towards the proofs of the analogues of Lemma 9.3 and Lemma 9.4 we recall Corollary 7.15, together with some of its immediate consequences.

Corollary 10.5. *Fix $\circ \in \{-, =, +\}$ and $\sigma \in \mathcal{N}_\circ$. There exists a unique mixed element y^σ_{crit} satisfying $y^\sigma_{crit} \in \beta_{r_{\max}} \cup \beta_{s_{\min}}$ and $\sigma(y^\sigma_{crit}) \in [\![i_{r_{\max}+1}, i_{s_{\min}}]\!] \backslash \{i_{r_{\max}+1}, \dots, i_{s_{\min}}\}$. In particular, any other element $y \ne y^\sigma_{crit}$ satisfying $\sigma(y) \in [\![i_{r_{\max}+1}, i_{s_{\min}}]\!]$ must satisfy $y \in \bar{\alpha}_{\ge x_{r_{\max}+1}, \le x_{s_{\min}}}$. Furthermore, y^σ_{crit} satisfies either $y^\sigma_{crit} \not\ge x_{r_{\max}+1}$ or $y^\sigma_{crit} \not\le x_{s_{\min}}$. If $y^\sigma_{crit} \not\ge x_{r_{\max}+1}$, then $y^\sigma_{crit} \not\ge y$ for any $y \in \bar{\alpha}_{\ge x_{r_{\max}+1}, \le x_{s_{\min}}}$. Analogously, if $y^\sigma_{crit} \not\le x_{s_{\min}}$, then $y^\sigma_{crit} \not\le y$ for any $y \in \bar{\alpha}_{\ge x_{r_{\max}+1}, \le x_{s_{\min}}}$.*

The following result is the analogue of Lemma 9.3 where again we restrict to $y_j \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$ rather than allowing for all $y_j \in \alpha$.

Lemma 10.6. *For any $y_j \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$: If $m^=_{\min}(y_j) < i_\ell$ then $\mathrm{v}_j = 0$, and if $m^=_{\max}(y_j) > i_\ell$ then $\mathrm{v}_j = 1 - a$.*

Proof. We prove that $m^=_{\max}(y_j) > i_\ell \Rightarrow \mathrm{v}_j = 1 - a$; the proof of $m^=_{\min}(y_j) < i_\ell \Rightarrow \mathrm{v}_j = 0$ is analogous.

Set $y_{j_0} := y_j \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$ and construct the sequence $y_{j_0} < y_{j_1} < \cdots < y_{j_p}$, for some $p < \infty$, iteratively, according to the algorithm below. The sequence will be constructed so that $y_{j_i} \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$ for every $i \in [\![0, p]\!]$, $\mathrm{v}_{j_i} = \mathrm{v}_{j_{i+1}}$ for all $i \in [\![0, p-1]\!]$, and $\mathrm{v}_{j_p} = 1 - a$. Clearly, it will then follow that $\mathrm{v}_j = \mathrm{v}_{j_0} = 1 - a$, completing the proof.

Assume that the sequence $y_{j_0} < y_{j_1} < \cdots < y_{j_i}$ has been constructed. Set $M := m^{=}_{\max}(y_{j_i})$ and note that $i_\ell < m^{=}_{\max}(y_{j_0}) \le M < i_{s_{\min}}$. Let b be the index satisfying $i_b < M < i_{b+1}$ so that $\ell \le b \le s_{\min} - 1$. Consider the following two cases:

- $M \notin \{i_m - 2, i_m - 1\}$ for every $\ell < m$. Choose $\sigma \in \mathcal{N}_=$ such that $\sigma(y_{j_i}) = M$ (such a σ must exist by the definition of M) and set $y_r = \sigma^{-1}(M+1), y_s = \sigma^{-1}(M+2)$. Note that $i_{b+1} \notin \{M+1, M+2\}$ since $M \notin \{i_m-2, i_m-1\}$ for every $\ell < m$, so in particular, we can take $b+1 = m$ (using $b+1 > \ell$). Hence, we have $i_b < M, M+1, M+2 < i_{b+1}$, so $M, M+1, M+2 \in [\![i_b + 1, i_{b+1} - 1]\!]$, and hence $y_r, y_s \in \alpha$. Note that $y_{j_i} < y_r$ since otherwise their positions in σ can be swapped to contradict the maximality of M. Further, $M, M+1, M+2 \in [\![i_b + 1, i_{b+1} - 1]\!] \Rightarrow \sigma(y_r), \sigma(y_s) \in [\![i_b + 1, i_{b+1} - 1]\!] \subseteq \mathrm{I}_{[\![r_{\max}+1, s_{\min}-1]\!]}$, where the last containment holds since $b \le s_{\min} - 1$ (as shown above), and since $r_{\max} + 1 \le b$ (because $r_{\max} + 1 < \ell \le b$ as $(r_{\max}, s_{\min})$ is an ℓ-splitting pair). Corollary 10.5 now yields $y_r, y_s \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}} \cup \{y^\sigma_{\text{crit}}\}$. We now choose $y_{j_{i+1}}$ as follows:
  1. If $y_r \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}}$ set $y_{j_{i+1}} := y_r$. Then we see that $y_{j_i} < y_{j_{i+1}}$ and that $y_{j_{i+1}} \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}}$ so Proposition 10.4(c) yields $\mathrm{v}_{j_{i+1}} = \mathrm{v}_{j_i}$.
  2. If $y_r = y^\sigma_{\text{crit}}$, then $y_s \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}}$. If $y^\sigma_{\text{crit}} \not> x_{r_{\max}+1}$, then $y^\sigma_{\text{crit}} \not> y_{j_i}$, a contradiction. Otherwise, $y^\sigma_{\text{crit}} \not< x_{s_{\min}}$, so $y^\sigma_{\text{crit}} \not< y_s$. Hence, we can swap the positions of $y_r = y^\sigma_{\text{crit}}$ and y_s, which reduces to (1).

- $M \in \{i_m - 2, i_m - 1\}$ for some $\ell < m$. In this case the sequence will be terminated with $p := i$. Arguing as in the analogous case in Lemma 9.3, we get that $y_{j_i} < x_m$. Note that $m = b + 1 \le s_{\min}$ (the last inequality was shown above), so since $\ell + 1 \le m \le s_{\min}$, Proposition 10.4(f) yields $\mathrm{v}_{j_i} = 1 - a$. □

The following result can be viewed as a continuation of Lemma 10.6. To ease the notation we will use

$$\mathrm{I}_j := [\![i_j + 1, i_{j+1} - 1]\!] \quad \text{and} \quad \mathrm{I}_S := \cup_{j \in S} \mathrm{I}_j \quad \text{for} \quad S \subseteq [\![0, k]\!]. \tag{10.2}$$

Lemma 10.7. *For any $y_j \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}}$: If $\min_{\circ \in \{-,=,+\}} m^\circ_{\min}(y_j) < i_\ell + 1_\circ$ then $\mathrm{v}_j = 0$, and if $\max_{\circ \in \{-,=,+\}} m^\circ_{\max}(y_j) > i_\ell + 1_\circ$ then $\mathrm{v}_j = 1 - a$.*

Proof. We will prove $\max_{\circ \in \{-,=,+\}} m^\circ_{\max}(y_j) > i_\ell + 1_\circ \Rightarrow \mathrm{v}_j = 1 - a$; the proof of $\min_{\circ \in \{-,=,+\}} m^\circ_{\min}(y_j) < i_\ell + 1_\circ \Rightarrow \mathrm{v}_j = 0$ is analogous. Fix $\circ \in \{-,=,+\}$ and $\sigma \in \mathcal{N}_\circ$ such that $\sigma(y_j) > \sigma(x_\ell) = i_\ell + 1_\circ$. There are three cases to consider:

(1) $\circ$ is $=$. We have $m^{=}_{\max}(y_j) \ge \sigma(y_j)$ and by assumption $\sigma(y_j) > \sigma(x_\ell) = i_\ell$. Hence, $m^{=}_{\max}(y_j) > i_\ell$ and the proof is complete by Lemma 10.6.

(2) $\circ$ is $+$. Let $q := \sigma(y_j) > i_\ell + 1$. We are going to apply Lemma 7.3 with $y_j, =$, and q so we will check its conditions. Since $i_{\min}(y_j) > \ell$, Lemma 7.2 and Corollary 7.4 yield $u_=(y_j) = u_+(y_j) \geq m^+_{\max}(y_j) \geq q$ and $l_=(y_j) \leq l_+(y_j) \leq m^+_{\min}(y_j) \leq \sigma(y_j) = q$, so we conclude that $q \in [\![l_=(y_j), u_=(y_j)]\!]$. Next we show that $q \neq i_m$ for any $m \in [k]$. Indeed, if $m \leq \ell$ then $i_m \leq i_\ell < q$, and if $m > \ell$, then $i_m = q$ implies $\sigma(y_j) = \sigma(x_m)$, which is impossible since $y_j \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}} \subseteq \alpha$. It follows from Lemma 7.3 that there exists $\sigma' \in \mathcal{N}_=$ such that $\sigma'(y_j) = q$. It follows that $m^=_{\max}(y_j) \geq \sigma'(y_j) = q > i_\ell$, and the proof is complete by Lemma 10.6.

(3) $\circ$ is $-$. If $m^=_{\max}(y_j) > i_\ell$ we are done by Lemma 10.6. Suppose then that $m^=_{\max}(y_j) < i_\ell$ (note that $m^=_{\max}(y_j) = i_\ell$ is impossible).

Claim 10.8. $m^-_{\max}(y_j) = i_\ell = \sigma(y_j)$.

Proof. Suppose for contradiction that $m^-_{\max}(y_j) \geq i_\ell + 1$, so there must exist $\sigma_1 \in \mathcal{N}_-$ with $\sigma_1(y_j) \geq i_\ell + 1$. Since $i_{\min}(y_j) > \ell$, Lemma 7.2 and Corollary 7.4 yield $u_=(y_j) = u_-(y_j) \geq m^-_{\max}(y_j) \geq \sigma_1(y_j) = i_\ell + 1$. On the other hand, by Corollary 7.4 and the assumption $m^=_{\max}(y_j) < i_\ell$, we have $l_=(y_j) \leq m^=_{\min}(y_j) \leq m^=_{\max}(y_j) < i_\ell$, so we conclude that $i_\ell + 1 \in [\![l_=(y_j), u_=(y_j)]\!]$. By Corollary 7.11, $i_m \neq i_\ell + 1$ for any $m \in [k]$ so Lemma 7.3 implies that there exists $\sigma_2 \in \mathcal{N}_=$ satisfying $\sigma_2(y_j) = i_\ell + 1$, which contradicts $m^=_{\max}(y_j) < i_\ell$. We conclude that $m^-_{\max}(y_j) \leq i_\ell$. Since, by assumption, $\sigma(y_j) > \sigma(x_\ell) = i_\ell - 1$ we get $m^-_{\max}(y_j) = \sigma(y_j) = i_\ell$. $\square$

Let y_v be such that $\sigma(y_v) = i_\ell + 1$ and note that $y_v \in \alpha$ by Corollary 7.11. We must have $y_j < y_v$ since if $y_j \nsim y_v$ (by Claim 10.8 it is impossible to have $y_v < y_j$), then we can swap the positions of y_j and y_v in σ to get $\sigma_3 \in \mathcal{N}_-$ satisfying $\sigma_3(y_j) = i_\ell + 1$, which contradicts Claim 10.8. Next we show that there exists $\sigma' \in \mathcal{N}_=$ satisfying $\sigma'(y_j) = i_\ell - 1$ and $\sigma'(y_v) = i_\ell + 1$. Indeed, since we assume $m^=_{\max}(y_j) < i_\ell$, we have that, for any $\sigma_4 \in \mathcal{N}_=$, $\sigma_4(y_j) < \sigma_4(x_\ell)$. Hence, since by the assumption $\sigma(y_j) > \sigma(x_\ell)$, we must have $y_j \nsim x_\ell$. Swapping the positions of y_j and x_ℓ in σ yields σ', where we used Claim 10.8.

We will now analyze the element y_v. Since $\sigma'(y_v) = i_\ell + 1$ we see that $\sigma'(y_v) \in \mathrm{I}_{[\![r_{\max}+1, s_{\min}-1]\!]}$ because $(r_{\max}, s_{\min})$ is an ℓ-splitting pair. Hence, Corollary 10.5 yields that either $y_v = y^\sigma_{\mathrm{crit}}$ or $y_v \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}}$. Consider both cases:

(a) $y_v \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}}$. Since $y_j < y_v$, and since there exists $\sigma' \in \mathcal{N}_=$ satisfying $\sigma'(y_j) = i_\ell - 1$ and $\sigma'(y_v) = i_\ell + 1$, Proposition 10.4(d) yields $\mathrm{v}_j = \mathrm{v}_v$. On the other hand, $\mathrm{v}_v = 1 - a$ by Lemma 10.6 since $m^=_{\max}(y_v) > i_\ell$ as $\sigma'(y_v) = i_\ell + 1$. We conclude that $\mathrm{v}_j = 1 - a$, which proves the lemma.

(b) $y_v = y^\sigma_{\mathrm{crit}}$. Since $y_v > y_j$ and $y_j \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}}$ (as we cannot have $y_j = y^\sigma_{\mathrm{crit}}$), we have $y^\sigma_{\mathrm{crit}} > x_{r_{\max}+1}$. Hence, we must have $y^\sigma_{\mathrm{crit}} \not< x_{s_{\min}}$. Let z be such that $\sigma(z) = i_\ell + 2$ and note that $\sigma'(z) = i_\ell + 2$ as well (since σ' was obtained from σ by swapping the positions of y_j and x_ℓ in σ). If $\sigma'(z) \in \mathrm{I}_{[\![r_{\max}+1, s_{\min}-1]\!]}$, then, by Corollary 10.5, since $z \neq y^\sigma_{\mathrm{crit}}$, we must have $z \in \alpha_{>x_{r_{\max}+1}, <x_{s_{\min}}}$. Recall that

$y^{\sigma}_{\text{crit}} \not\leq x_{s_{\min}}$ so $y^{\sigma}_{\text{crit}} \not\leq z$. Hence, we can swap $y_v = y^{\sigma}_{\text{crit}}$ and z to reduce to the case (a).

Suppose then that $\sigma'(z) \notin \mathrm{I}_{[\![r_{\max}+1,s_{\min}-1]\!]}$:

Claim 10.9. If $\sigma'(z) \notin \mathrm{I}_{[\![r_{\max}+1,s_{\min}-1]\!]}$ then $z = x_{\ell+1}$ and $i_{\ell+1} = i_\ell + 2$.

Proof. Since $\sigma'(z) = i_\ell + 2 \notin \mathrm{I}_{[\![r_{\max}+1,s_{\min}-1]\!]}$ we get that $i_\ell + 2 \notin \mathrm{I}_\ell$ (because $r_{\max}+1 < \ell < s_{\min}$ as $(r_{\max}, s_{\min})$ is an ℓ-splitting pair). Hence, $i_\ell + 2 \geq i_{\ell+1}$ (since $i_\ell+2 \leq i_\ell$ is impossible). On the other hand, Corollary 7.11 yields $i_\ell+1 < i_{\ell+1} \leq i_\ell+2$ so we conclude $i_\ell + 2 = i_{\ell+1}$. Since $\sigma' \in \mathcal{N}_=$ we also conclude that $z = x_{\ell+1}$. □

Since $(r_{\max}, s_{\min})$ is an ℓ-splitting pair, we have that either $s_{\min} = \ell + 1$ or $s_{\min} > \ell + 1$. If $s_{\min} = \ell + 1$ then, since by assumption $y_j \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$, we have $y_j < x_{s_{\min}} = x_{\ell+1}$. Since, by Claim 10.8 and Claim 10.9, $\sigma(y_j) = i_\ell = i_{\ell+1} - 2$, Proposition 10.4(h) shows that $\mathrm{v}_j = 1 - a$.

Suppose then that $s_{\min} > \ell + 1$. Consider the set

$$\gamma := \{y \in \alpha : \sigma'(y) \in \mathrm{I}_{[\![\ell+1,s_{\min}-1]\!]}, y \not> x_{l+1}\}.$$

We claim that γ is nonempty. Indeed, since $(\ell, s_{\min})$ is a splitting pair, Lemma 7.10 yields $y^{\sigma'} \in \beta_\ell \cup \beta_{s_{\min}}$ such that $\sigma'(y^{\sigma'}) \in \mathrm{I}_{[\![\ell+1,s_{\min}-1]\!]}$. We must have that either $y^{\sigma'} \not\leq x_{s_{\min}}$ or $y^{\sigma'} \not\geq x_{l+1}$. We cannot have $y^{\sigma'} \not\leq x_{s_{\min}}$ since $r_{\max}+1 < \ell+1$ and $y^{\sigma'} \neq y^{\sigma}_{\text{crit}}$ (as $\sigma'(y^{\sigma}_{\text{crit}}) = i_\ell + 1 \notin \mathrm{I}_{[\![\ell+1,s_{\min}-1]\!]} \ni \sigma'(y^{\sigma'})$) imply $y^{\sigma'} \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$. Hence, $y^{\sigma'} \in \gamma$. Now pick $y \in \gamma^{\downarrow}$, which exists as γ is nonempty. Note that $y \in \gamma^{\downarrow}$ implies that $y \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$ because $\sigma'(y) \in \mathrm{I}_{[\![\ell+1,s_{\min}-1]\!]} \subseteq \mathrm{I}_{[\![r_{\max}+1,s_{\min}-1]\!]}$ yields, by Corollary 10.5, $y \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}} \cup \{y^{\sigma}_{\text{crit}}\}$, and $y \neq y^{\sigma}_{\text{crit}}$ since $\sigma'(y^{\sigma}_{\text{crit}}) = i_\ell + 1 \notin \mathrm{I}_{[\![\ell+1,s_{\min}-1]\!]} \ni \sigma'(y)$.

We will show next that the positions of y and y_v can be swapped in both σ and σ' to yield valid linear extensions in $\mathcal{N}_-, \mathcal{N}_=$, respectively. This completes the proof since we reduce back to 3(a).

Let us now verify that the swaps yield valid linear extensions. We will show the validity of the swap of σ; the argument for σ' is analogous since by construction σ and σ' are the same up to the swap of y_j and x_ℓ. Suppose this swap violated some relation so that there exists w such that $\sigma(y_v) = i_\ell+1 < \sigma(w) < \sigma(y)$, satisfying either $y_v < w$ or $w < y$. We cannot have $y^{\sigma}_{\text{crit}} = y_v < w$ because $\sigma(w) \in [\![i_{r_{\max}+1}, i_{s_{\min}}]\!]$ implies, by Corollary 10.5, that $w \leq x_{s_{\min}}$ (as $w \neq y^{\sigma}_{\text{crit}}$). But then $y^{\sigma}_{\text{crit}} = y_v < w \leq x_{s_{\min}}$, which contradicts $y^{\sigma}_{\text{crit}} \not\leq x_{s_{\min}}$, as was shown at the beginning of (3). We also cannot have $w < y$ since, otherwise, $w \not\geq x_{l+1}$ by the definition of γ. But if $w \notin \alpha$, then $w = x_r$ for some $r \geq l+1$ (as $i_\ell+1 < \sigma(w)$) which implies $w \geq x_{l+1}$, a contradiction. On the other hand, if $w \in \alpha$, then combined with $\sigma(w) \in [\![i_{l+1}, i_{s_{\min}}]\!]$ we have that $\sigma(w) \in \mathrm{I}_{[\![l+1,s_{\min}-1]\!]}$. Hence, $w \in \gamma$, which contradicts $y \in \gamma^{\downarrow}$. □

Next we move to proving the analogue of Lemma 9.4. We will again use the notation (10.2).

Lemma 10.10. $a = 1$.

Proof. We will show that there exists $y_j \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$ such that $y_j \nsim x_\ell$. This will complete the proof since, by Assumption 4.1, $y_j \nsim x_\ell$ implies that there exist $\sigma, \sigma' \in \cup_{\circ\in\{-,=,+\}}\mathcal{N}_\circ$ satisfying $\sigma(y_j) > \sigma(x_\ell)$ and $\sigma'(y_j) < \sigma'(x_\ell)$. Applying Lemma 10.7 yields $0 = \mathrm{v}_j = 1 - a$ so $a = 1$.

We now show that there exists $y_j \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$ such that $y_j \nsim x_\ell$. Suppose for contradiction that such y_j does not exist. Then, for any $y \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$, we must have either $y < x_\ell$ or $y > x_\ell$. In particular, we have the disjoint union

$$\alpha_{>x_{r_{\max}+1},<x_{s_{\min}}} = [\alpha_{>x_{r_{\max}+1},<x_{s_{\min}}} \cap \alpha_{<x_\ell}] \cup [\alpha_{>x_{r_{\max}+1},<x_{s_{\min}}} \cap \alpha_{>x_\ell}]. \quad (10.3)$$

Let us show that

$$\begin{aligned} &|\alpha_{>x_{r_{\max}+1},<x_{s_{\min}}} \cap \alpha_{>x_\ell}| \le |\mathrm{I}_{[\![\ell,s_{\min}-1]\!]}| - 1 \quad \text{and} \\ &|\alpha_{>x_{r_{\max}+1},<x_{s_{\min}}} \cap \alpha_{<x_\ell}| \le |\mathrm{I}_{[\![r_{\max}+1,\ell-1]\!]}| - 1; \end{aligned} \quad (10.4)$$

we prove the first inequality and the proof of the second inequality is analogous. Given any $\sigma \in \mathcal{N}_+$ and $y \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}} \cap \alpha_{>x_\ell}$ we have $i_\ell + 1 < \sigma(y) < i_{s_{\min}}$ so $\sigma(y) \in \mathrm{I}_{[\![\ell,s_{\min}-1]\!]} \setminus \{i_l + 1\}$. It follows that $|\alpha_{>x_{r_{\max}+1},<x_{s_{\min}}} \cap \alpha_{>x_\ell}| \le |\mathrm{I}_{[\![\ell,s_{\min}-1]\!]}| - 1$ as desired. By (10.3) and (10.4) we now get

$$|\alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}| \le |\mathrm{I}_{[\![r_{\max}+1,\ell-1]\!]}| + |\mathrm{I}_{[\![\ell,s_{\min}-1]\!]}| - 2 = |\mathrm{I}_{[\![r_{\max}+1,s_{\min}-1]\!]}| - 2. \quad (10.5)$$

However, by Lemma 7.13, $|\alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}| = |\mathrm{I}_{[\![r_{\max}+1,s_{\min}-1]\!]}| - |\{\text{mixed elements}\}|$. Hence, the number of mixed elements is at least 2 which means that the maximal splitting pair is supercritical, which contradicts Proposition 7.8. □

We are now ready to prove Theorem 10.1.

Proof of Theorem 10.1. We start by proving the analogue of (9.2).

Lemma 10.11. *Let* $y \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$.

(a) If there exists $\sigma \in \mathcal{N}_=$ *such that either* $\sigma(y) = i_\ell - 1$ *or* $\sigma(y) = i_\ell + 1$*, then* $y \nsim x_\ell$.
(b) If there exists $\sigma \in \mathcal{N}_- \cup \mathcal{N}_+$ *such that* $\sigma(y) = i_\ell$*, then* $y \nsim x_\ell$.

Proof. (a) We proceed as in the proof of Theorem 9.1 where we use Lemma 10.3 rather than (9.1).

(b) Let $y \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$ be such that there exists $\sigma \in \mathcal{N}_-$ with $\sigma(y) = i_\ell$; the proof for the case $\sigma \in \mathcal{N}_+$ is analogous. Since we cannot have $y < x_\ell$ it suffices to show that $y \not> x_\ell$. Suppose for contradiction that $y > x_\ell$. By Lemma 7.3, $l_-(y) \le i_\ell$ so by Lemma 7.2 $l_=(y) \le i_\ell + 1$. On the other hand, for any $\sigma' \in \mathcal{N}_=$, Corollary 7.4 yields $i_\ell = \sigma'(x_\ell) < \sigma'(y) \le m^=_{\max}(y) \le u_=(y)$ so $u_=(y) \ge i_\ell + 1$. Since $i_\ell + 1 \ne i_m$ for any $m \in [k]$ (by Corollary 7.11), Lemma 7.3 yields $\sigma'' \in \mathcal{N}_=$ such that $\sigma''(y) = i_\ell + 1$. By part (a), $y \not\sim x_\ell$, which contradicts $y > x_\ell$. $\square$

We now prove $|\mathcal{N}_+(\sim,\sim)| = 0$; the proof of $|\mathcal{N}_-(\sim,\sim)| = 0$ is analogous. Suppose for contradiction that $|\mathcal{N}_+(\sim,\sim)| > 0$ so there exists $\sigma \in \mathcal{N}_+$ such that $y_u := \sigma^{-1}(i_\ell - 1)$ and $y_v := \sigma^{-1}(i_\ell)$ satisfy $y_u, y_v < x_\ell$. Since $i_\ell - 1, i_\ell \in \mathrm{I}_{[\![r_{\max}+1, s_{\min}-1]\!]}$ (because $(r_{\max}, s_{\min})$ is an ℓ-splitting pair so $i_{r_{\max}+1} \le i_{\ell-1} < i_\ell - 1$ by Corollary 7.11), Corollary 10.5 yields $y_u, y_v \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}} \cup \{y^\sigma_{\text{crit}}\}$. Consider the following two cases:

If $y_v \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$, then, by Lemma 10.11(b), $y_v \not\sim x_\ell$ which contradicts $y_v < x_\ell$.

If $y_v = y^\sigma_{\text{crit}}$, we have $y_u \in \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$. Then, because $y^\sigma_{\text{crit}} < x_\ell < x_{s_{\min}}$, we must have $y^\sigma_{\text{crit}} \not\gtrsim x_{r_{\max}+1}$, which implies $y_v = y^\sigma_{\text{crit}} \not\gtrsim y_u$. Hence, we can swap the positions of y_u and y_v in σ to reduce to the previous case. $\square$

Notation index

- $[p] := \{1, \ldots, p\}$ for positive integers p.
- $[\![p, q]\!] := \{p, p+1, \ldots, q-1, q\}$ for integers $p \le q$; (2.1).
- $\bar{\alpha} = \{y_1, \ldots, y_{n-k}, x_0, x_1, \ldots, x_k, x_{k+1}\}$ and $\alpha = \{y_1, \ldots, y_{n-k}\}$ where x_0 (res. x_{k+1}) is smaller (res. bigger) than every element in $\bar{\alpha}$.
- $i_0 = 0$ and $i_{k+1} = n + 1$. $j_0 = -1$ and $j_{p+1} = k + 1$.
- $1_\circ = 1_{\{\circ \text{ is } +\}} - 1_{\{\circ \text{ is } -\}}$ for $\circ \in \{-, =, +\}$.
- $\beta_i = \alpha \backslash (\alpha_{<x_i} \cup \alpha_{>x_{i+1}})$ and $\beta_S = \cup_{i \in S} \beta_i$; (2.4).
- $i_{\max}(y)$ (res. $i_{\min}(y)$) is the maximum (res. minimum) number such that $y > x_{i_{\max}(y)}$ (res. $y < x_{i_{\min}(y)}$); Definition 7.1.
- $l_\circ(y) := \max_{r \le i_{\max}(y)}(i_r + 1_\circ + |\bar{\alpha}_{>x_r,<y}| + 1)$ and $u_\circ(y) := \min_{s \ge i_{\min}(y)}(i_s + 1_\circ - |\bar{\alpha}_{>y,<x_s}| - 1)$; Definition 7.1.
- $m^\circ_{\min}(y) = \min_{\sigma \in \mathcal{N}_\circ} \sigma(y)$ and $m^\circ_{\max}(y) = \max_{\sigma \in \mathcal{N}_\circ} \sigma(y)$ for $\circ \in \{-, =, +\}$ and $y \in \alpha$; Definition 7.1.
- $i^\circ_j := i_j + 1_{j=\ell} 1_\circ$; Definition 7.1.
- $r_{\max} = \max_\iota r_\iota$ and $s_{\min} = \min_\iota s_\iota$ where (r_ι, s_ι) are the sharp-critical ℓ-splitting pairs; Definition 7.7.
- y^σ_{crit}; Corollary 7.15.
- $\mathcal{K}_{\max}$, $\beta_{\max} := \beta_{[\![0,r_{\max}]\!] \cup [\![s_{\min},k]\!]}$, $\alpha \backslash \beta_{\max} = \alpha_{>x_{r_{\max}+1},<x_{s_{\min}}}$, and $E^\perp := \mathbb{R}^{\alpha \backslash \beta_{\max}}$; (7.3), (10.1).
- $[\![i_j, i_{j+1}]\!]^\circ := [\![i^\circ_j, i^\circ_{j+1}]\!] = [\![i_j + 1_{j=\ell} 1_\circ, i_{j+1} + 1_{j+1=\ell} 1_\circ]\!]$ and $[\![i_j + 1, i_{j+1} - 1]\!]^\circ := [\![i^\circ_j + 1, i^\circ_{j+1} - 1]\!]$; (7.4).
- $\mathrm{I}_{j_q} := [\![i_j + 1, i_{j+1} - 1]\!]$ for $j_q \in [\![0, k]\!]$, $\mathrm{I}_J := \cup_{j_q \in J} \mathrm{I}_{j_q}$; (8.1), (10.2).

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