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Matrix Displacement Convexity Along Density Flows

Yair Shenfeld

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Abstract

A new notion of displacement convexity on a matrix level is developed for density flows arising from mean-field games, compressible Euler equations, entropic interpolation, and semi-classical limits of non-linear Schrödinger equations. Matrix displacement convexity is stronger than the classical notions of displacement convexity, and its verification (formal and rigorous) relies on matrix differential inequalities along the density flows. The matrical nature of these differential inequalities upgrades dimensional functional inequalities to their intrinsic dimensional counterparts, thus improving on many classical results. Applications include turnpike properties, evolution variational inequalities, and entropy growth bounds, which capture the behavior of the density flows along different directions in space.

1. Introduction

The optimal decisions of agents in large populations, the lazy gas experiment of Schrödinger, and the flow of slender jets can all be modeled by systems of coupled partial differential equations of the form

$$\begin{cases} \partial_t \rho_t + \nabla\cdot(\rho_t \nabla\theta_t) = 0, \\ \partial_t\theta_t + \frac{1}{2}|\nabla\theta_t|^2 + \frac{\sigma^2}{2}\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}} + U_t - W * \rho_t - f(\rho_t) = 0, \end{cases} \quad \forall\, t \in [0,\tau]. \tag{1.1}$$

The first equation in (1.1) is the continuity equation of $\rho_t \geq 0$, interpreted as a density over a domain $\Omega \subset \mathbb{R}^n$, driven by a gradient vector field $\nabla\theta_t$. The second equation in (1.1) describes the evolution of the vector field itself via an equation for θ_t, which in turn can depend on the density ρ_t. The boundary conditions of (1.1) will vary based on the model and will usually be a specification of (ρ_0, ρ_τ), or $(\rho_0, \nabla\theta_0)$, or (ρ_0, θ_τ), and so on. The scope of the flows (1.1) will be recalled in Section

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1.2; they include **planning problems** (optimal transport, entropic interpolation, regularization of planning problems), **mean-field games**, **barotropic fluids**, and **semi-classical limits of non-linear Schrödinger equations**. The majority of this work will focus on flows (ρ_t, θ_t) satisfying (1.1) under the assumptions[1] that σ is real, U_t is convex, W is concave, and f is non-decreasing.

1.1. Matrix Displacement Convexity and Intrinsic Dimensional Functional Inequalities

The discovery by McCann [34] of the notion of displacement convexity has had a significant impact on probability, analysis, and geometry. Move specifically, it was shown in [34] that certain functionals are convex along the optimal transport flow. A central example of such a functional is the differential entropy

$$E(t) := \int_\Omega \log \rho_t \, \mathrm{d}\rho_t,$$

which was shown by McCann [34] to be convex (i.e., $t \mapsto E(t)$ is convex) when (ρ_t) is the optimal transport flow. It was later realized that displacement convexity also holds along other density flows. For example, Léonard [32] showed that $E(t)$ is convex when (ρ_t) is the entropic interpolation flow, and Gomes and Seneci [22] showed that $E(t)$ is convex when (ρ_t) is a first-order mean-field game flow. In a certain sense, these results generalize McCann's result (as well as the classical convexity of entropy along heat flows) as will become clear in Section 1.2.

One important application of displacement convexity is its usage in the definition of Ricci curvature for metric measure spaces as developed by Lott-Villani [33] and Sturm [41,42]. Roughly, a metric-measure space is defined to have a non-negative Ricci curvature if the entropy is convex along optimal transport flows over this space. This notion of Ricci curvature coincides with the classical notion when the space is a Riemannian manifold.

There is a stronger curvature condition (going beyond non-negative Ricci curvature) which incorporates the effect of the dimension. Restricting to the flat case, this is the $\mathrm{CD}(0, n)$ curvature-dimension condition of Bakry-Émery [1]. Analogous to the relation between non-negative Ricci curvature and displacement convexity of the entropy, Erbar-Kuwada-Sturm [15] showed that the $\mathrm{CD}(0, n)$ curvature-dimension condition is equivalent (under sufficient regularity) to the concavity of the map

$$t \mapsto e^{-\frac{E(t)}{n}}$$

along the optimal transport flow (which implies the convexity of entropy along the flow). Due to the role of dimension in this notion of convexity it will be dubbed here *dimensional displacement convexity*. The natural question of whether $e^{-\frac{E(t)}{n}}$ is concave along the entropic interpolation flow was settled by Ripani [40], which thus recovered both the result of Erbar-Kuwada-Sturm on flat space, as well as the result of Costa [13] who showed that $e^{-\frac{E(t)}{n}}$ is concave along the heat flow.

[1] Some of these assumptions can in fact be relaxed, cf. Section 4.2.

1.1.1. Matrix Displacement Convexity The main purpose of this work is to develop and prove a new notion of *matrix* displacement convexity which is stronger than dimensional displacement convexity (and thus stronger than classical displacement convexity) along density flows of the form (1.1). To keep the discussion concrete, at this point matrix displacement convexity will be defined just for the entropy (but the extension is clear). Recall that the *entropy production* $S(t)$ associated to a density flow (ρ_t) is defined as

$$S(t) := \partial_t E(t).$$

In the setting of this work (and many others), there is a natural *entropy production matrix* $\mathcal{S}(t)$ which can be defined so that

$$\partial_t E(t) = S(t) = \mathrm{Tr}[\mathcal{S}(t)].$$

Indeed, a simple calculation (cf. Lemma 3.1) shows that when (ρ_t, θ_t) satisfies the continuity equation, the *entropy production matrix* is given by

$$\mathcal{S}(t) = \int_\Omega \nabla\rho_t \otimes_S \nabla\theta_t \, \mathrm{d}x,$$

where $\otimes_S$ is the symmetric tensor product. The *entropy matrix* is defined as

$$\mathcal{E}(t) := \int_0^t \mathcal{S}(s)\, \mathrm{d}s,$$

so that

$$E(t) = E(0) + \mathrm{Tr}[\mathcal{E}(t)].$$

Remark 1.1. When the flow (ρ_t, θ_t) is the optimal transport flow (cf. Example 1.4), one can check that

$$\mathcal{E}(t) = -\int_{\mathbb{R}^n} [\log \nabla\Phi_t(x)]\, \mathrm{d}\rho_0(x),$$

where Φ_t is the optimal transport map (which satisfies $\nabla\Phi_t \succeq 0$) between ρ_0 to ρ_t.

Definition 1.2. The entropy matrix $\mathcal{E}(t)$ is *matrix displacement convex* along a flow (ρ_t, θ_t) if, for any $w \in S^{n-1}$, the function

$$t \mapsto e^{-\langle w, \mathcal{E}(t) w\rangle}$$

is concave.

Note that if $\mathcal{E}(t)$ is matrix displacement convex then $E(t)$ is dimensional displacement convex, and hence displacement convex (cf. Section 1.1.2.)

There are two main inter-related motivations behind Definition 1.2. The first motivation comes from the notion of *intrinsic dimensional functional inequalities*. Consider a flow (ρ_t) which is (approximately) trivial along certain directions in space, that is, its evolution (approximately) takes place on a subspace of low dimension $\ll n$. In such settings, the explicit dependence on the ambient dimension

n in the notion of dimensional displacement convexity of the entropy, formulated as the concavity of $e^{-\frac{E(t)}{n}}$, renders this notion oblivious to the intrinsic dimension of the flow (ρ_t). Consequently, functional inequalities which are derived from dimensional displacement convexity are *dimensional* functional inequalities, in the sense that the ambient dimension n appears explicitly in the inequalities. This dimensional feature is undesirable in high-dimensional settings. On the other hand, in many practical settings, there is a lower-dimensional manifold inside the high-dimensional ambient space to which the objects of interest (approximately) belong (e.g., the *manifold hypothesis*). In order to capture this phenomenon one needs *intrinsic dimensional* functional inequalities where the ambient dimension is absent and which scale like the dimension of the object at hand. Indeed, it will be shown in this work that matrix displacement convexity allows to derive such intrinsic dimensional functional inequalities, which improve on their classical dimensional counterparts by capturing more refined structures of the flow—see Section 1.1.3. This is because controlling the matrix entropy, rather than just its trace, facilitates the analysis of the flow (ρ_t) along different directions in space.

The second motivation behind Definition 1.2 comes back to the discussion of curvature notions. For flat spaces, the $\mathrm{CD}(0, n)$ curvature-dimension condition does not capture the full curvature structure of the space. Indeed, the $\mathrm{CD}(0, n)$ condition implies a zero lower bound on the Ricci tensor, but in flat space one knows that the full Riemann tensor vanishes. More generally, there are important classes of manifolds where information beyond lower bounds on the Ricci tensor is given. One such prominent class in differential geometry is the class of Einstein manifolds with lower bounds on the *sectional* curvature (which includes the sphere and hyperbolic space). What is the correct notion of displacement convexity that captures this type of curvature information? This question was taken up in [28,30], but Definition 1.2 seems to provide an alternative route as will be further explained in Section 1.1.2.

To conclude this section the first result of this paper is stated informally.

Theorem. (Theorem 4.9) *Suppose* (ρ_t, θ_t) *is a nice flow satisfying* (1.1) *and assume that* σ *is real,* U_t *is convex,* W *is concave, and* f *is non-decreasing. Then,* $\mathcal{E}(t)$ *is matrix displacement convex.*

1.1.2. Matrix Differential Inequalities One classical way to deduce convexity is via differential inequalities. The most basic example is expressing the displacement convexity of $E(t)$ along some flow via the differential inequality

$$\partial_t S(t) = \partial_{tt}^2 E(t) \geq 0. \tag{1.2}$$

The *dimensional* displacement convexity of $E(t)$ is equivalent to the differential inequality for the entropy production,

$$\partial_t S(t) \geq \frac{1}{n} S^2(t). \tag{1.3}$$

In particular, comparing (1.2) and (1.3) shows that dimensional displacement convexity is stronger than displacement convexity. It will be shown in this work that the *matrix* displacement displacement convexity of $\mathcal{E}(t)$ is equivalent to the *matrix* differential inequality for the entropy production *matrix*,

$$\partial_t \mathcal{S}(t) \succeq \mathcal{S}^2(t). \tag{1.4}$$

The inequality (1.4) is stronger than (1.3) by the Cauchy-Schwarz inequality. More importantly, the ambient dimension n is absent from (1.4), and having an inequality for the full matrix (rather than just the trace as in (1.3)) allows to control each direction of space separately.

The proof of the matrix displacement convexity of $\mathcal{E}(t)$ will follow by establishing (1.4). In fact, more powerful differential matrix inequalities will be established, which in turn imply new intrinsic dimensional functional inequalities. To state these differential inequalities define the (positive semidefinite) *Fisher information matrix* associated to a flow (ρ_t) as

$$\mathcal{I}(t) := \int_\Omega (\nabla \log \rho_t)^{\otimes 2} \, \mathrm{d}\rho_t,$$

and let

$$\mathcal{T}_\pm(t) := \mathcal{S}(t) \pm \frac{\sigma}{2}\mathcal{I}(t).$$

Theorem. (Theorem 4.8, Theorem 4.9) *Suppose (ρ_t, θ_t) is a nice flow satisfying (1.1) and assume that σ is real, U_t is convex, W is concave, and f is non-decreasing. Then,*

$$\begin{aligned}\partial_t \mathcal{T}_\pm(t) &\succeq \mathcal{T}_\pm^2(t) + \int_\Omega \nabla^2 U_t \, \mathrm{d}\rho_t + \int_\Omega (-\nabla^2 W) * \rho_t \, \mathrm{d}\rho_t + \int_\Omega f'(\rho_t)(\nabla \rho_t)^{\otimes 2} \\ &\succeq \mathcal{T}_\pm^2(t).\end{aligned} \tag{1.5}$$

Consequently,

$$\begin{aligned}\partial_t \mathcal{S}(t) \succeq \mathcal{S}^2(t) + \frac{\sigma^2}{4}\mathcal{I}^2(t) + \int_\Omega \nabla^2 U_t \, \mathrm{d}\rho_t + \int_\Omega (-\nabla^2 W) * \rho_t \, \mathrm{d}\rho_t \\ + \int_\Omega f'(\rho_t)(\nabla \rho_t)^{\otimes 2} \succeq \mathcal{S}^2(t).\end{aligned} \tag{1.6}$$

The proofs of the matrix differential inequalities will rely on integration by parts, which provides a more analytic approach to the study of (1.1) rather than relying on probabilistic representations. The proofs of the matrix differential inequalities also shed light on the question posed at the end of Section 1.1.1. Crucial to the proofs of integration by parts is the exchange of derivatives, which is permitted in the flat case treated in this work. However in the manifold setting, such an exchange of derivatives causes curvature terms to appear. Since the requisite differential inequalities are for *matrices*, it is *sectional* curvature terms which appear, rather than just Ricci terms. Hence, the verification of matrix differential inequalities, and hence matrix displacement convexity and intrinsic dimensional functional inequalities, is intimately tied to curvature information that goes beyond the classical curvature-dimension conditions. A concrete manifestation of this phenomenon can be found in [16] where Eskenazis and the author proved matrix differential inequalities (using different techniques) for heat flows over spaces of constant curvature. These matrix differential inequalities led to Hamilton-type inequalities (which operate on the matrix level and require assumptions on the full Riemann tensor) which imporove on Li-Yau inequalities (which operate on the trace level and only require information on the Ricci tensor). The reader is referred to [16] for further discussion.

1.1.3. Intrinsic Dimensional Functional Inequalities Once the matrix differential inequalities (1.5) and (1.6) are in place one can use known techniques to deduce functional inequalities. As explained above, the *matrical* nature of the differential inequalities leads to the replacement of the (ambient) dimensional functional inequalities by more refined inequalities which capture the intrinsic dimension of the flow (ρ_t), and thus improve on many classical results. These functional inequalities do not apply in the generality of the flows discussed above but apply in many cases of interest. Since the focus of this work is on matrix displacement convexity and the associated matrix differential inequalities, only some intrinsic dimensional functional inequalities will proven to show the power of the method. The reader is referred to the appropriate references for background on the significance of these functional inequalities.

Section 5 contains the intrinsic dimensional functional inequalities which are summarized as follows:

- *Theorem* 5.2. Intrinsic dimensional lower and upper bounds on the growth of the entropy $E(t)$ along flows satisfying (1.1).
- *Theorem* 5.4. Intrinsic dimensional turnpike properties via dissipation of Fisher information along *viscous* flows satisfying (1.1).
- *Theorem* 5.6. Intrinsic dimensional lower and upper bounds on certain costs associated to the flow (1.1) when U_t is independent of t, $W = 0$, and $\sigma \neq 0$. These cost inequalities can also be seen as a generalization of the intrinsic dimensional local logarithmic Sobolev inequalities (Remark 5.7).
- *Theorem* 5.11. Intrinsic dimensional long time asymptotics for cost and energy along entropic interpolation flows.
- *Theorem* 5.12. Intrinsic dimensional evolution variational inequalities along entropic interpolation flows.
- *Theorem* 5.13. Intrinsic dimensional contraction of entropic cost along entropic interpolation flows.

Remark 1.3. This works focuses on the development of matrix displacement convexity, and consequently intrinsic dimensional inequalities, for certain functionals (e.g., entropy) along flows of the form (1.1) in flat spaces. The natural next step is to find other functionals which are matrix displacement convex (analogous to [34]), and to investigate the extension of the results of this paper to curved spaces.

1.2. Examples of Density Flows

To conclude the introduction this section demonstrates the scope of density flows of the form (1.1) via a number of important examples. The first step is to note that the equations in (1.1) have a variational characterization as the Euler-Lagrange equations of the functional

$$(\rho, v) : \partial_t \rho_t + \nabla\cdot(\rho_t v_t) = 0 \quad \mapsto \quad \int_0^\tau \int_\Omega \left[L(t, x, v_t) + \frac{\sigma^2}{8} |\nabla \log \rho_t|^2 + F(\rho_t) + \frac{1}{2} W * \rho_t \right] \mathrm{d}\rho_t \, \mathrm{d}t. \tag{1.7}$$

Here, the Lagrangian L is given by

$$L(t, x, w) := \frac{|w|^2}{2} - U_t(x), \quad t \in [0, \tau], \; x \in \Omega, \; w \in \mathbb{R}^n, \tag{1.8}$$

where $U_t : \Omega \to \mathbb{R}$ is a potential term. The term $W * \rho_t$ stands for the convolution of the density ρ_t with a symmetric interaction potential $W : \Omega \to \mathbb{R}$, and $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is such that $f(r) = F(r) + rF'(r)$ where $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$.

Example 1.4. (Planning problems) Consider the boundary conditions in (1.1) specifying ρ_0 and ρ_τ. The *planning problem* seeks to find the optimal density flow going from ρ_0 to ρ_τ, subject to the minimization of the cost given by (1.7). The optimal flow (ρ_t, θ_t) is given by the equations (1.1).

Optimal transport [43, §5.4]. Taking $U_t = f = W = 0$ and $\sigma = 0$ leads to (ρ_t, θ_t) being the optimal flow minimizing (1.7),

$$\int_0^\tau \int_\Omega \frac{|v_t|^2}{2} \, \mathrm{d}\rho_t \, \mathrm{d}t, \tag{1.9}$$

among all flows satisfying the continuity equation $\partial_t \rho_t + \nabla\cdot(\rho_t v_t) = 0$ with boundary conditions ρ_0 and ρ_τ. The function θ_t evolves according to

$$\partial_t \theta_t + \frac{1}{2} |\nabla \theta_t|^2 = 0. \tag{1.10}$$

In this setting the flow (ρ_t) is a geodesic in the Wasserstein space between ρ_0 and ρ_τ, which is the fluid mechanics formulation by Benamou-Brenier [2] of the optimal transport problem between ρ_0 and ρ_τ.

Heat flow. Taking $U_t = f = W = 0$ and $\sigma \to \infty$ leads to the flow (ρ_t) corresponding to the heat equation

$$\partial_t \rho_t - \frac{1}{2} \Delta \rho_t = 0 \tag{1.11}$$

(with boundary term ρ_τ adjusted appropriately).

Entropic interpolation [7, §4.5]. Taking $U_t = f = W = 0$ leads to (ρ_t, θ_t) being the optimal flow minimizing (1.7),

$$\int_0^\tau \int_\Omega \frac{|v_t|^2}{2} \, \mathrm{d}\rho_t \, \mathrm{d}t + \frac{\sigma^2}{8} \int_0^\tau \int_\Omega |\nabla \log \rho_t|^2 \, \mathrm{d}\rho_t \, \mathrm{d}t, \tag{1.12}$$

among all flows satisfying the continuity equation $\partial_t \rho_t + \nabla\cdot(\rho_t v_t) = 0$ with boundary conditions ρ_0 and ρ_τ. The function θ_t evolves according to

$$\partial_t \theta_t + \frac{1}{2}|\nabla\theta_t|^2 + \frac{\sigma^2}{2}\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}} = 0. \tag{1.13}$$

The flow (ρ_t) is the entropic interpolation between ρ_0 and ρ_τ, which is the same flow of the **Schrödinger bridge problem** and is the dynamic formulation of **entropic optimal transport**. The above fluid dynamics formulation (or stochastic control formulation) of entropic interpolation is due to Chen-Georgiou-Pavon [6] and Gentil-Léonard-Ripani [19]. The entropic interpolation flow encapsulates both the optimal transport flow and the heat flow as the limits $\sigma \to 0$ and $\sigma \to \infty$, respectively.

Regularization of planning problems [23]. Taking $U_t = W = 0$ and $\sigma = 0$ leads to (ρ_t, θ_t) being the optimal flow minimizing (1.7),

$$\int_0^\tau \int_\Omega \frac{|v_t|^2}{2}\,\mathrm{d}\rho_t\,\mathrm{d}t + \int_0^\tau \int_\Omega f(\rho_t)\,\mathrm{d}\rho_t\,\mathrm{d}t, \tag{1.14}$$

among all flows satisfying the continuity equation $\partial_t \rho_t + \nabla\cdot(\rho_t v_t) = 0$ with boundary conditions ρ_0 and ρ_τ. The function f is seen as a *regularization* term of optimal transport and it is often assumed to be non-decreasing (an assumption under which the results of this work apply). The choice

$$f(r) = \epsilon \log r$$

leads to the **entropic regularization of optimal transport** as investigated by Porretta [39].

Example 1.5. (Mean-field games) The theory of mean-field games was developed by Huang-Malhame-Caines [25] and Lasry-Lions [31] to describe Nash equilibrium type concepts for games with large populations of agents. To describe this set up define the Hamiltonian H to be the Legendre transform of the Lagrangian L (in the w variable), so that

$$H(t, x, p) = \frac{|p|^2}{2} + U_t(x), \tag{1.15}$$

and let

$$\mathrm{V}'(x, \rho) := f(\rho(x)) + (W * \rho)(x) \tag{1.16}$$

for a density ρ over Ω. Let (ρ_t, θ_t) be a flow satisfying (1.1) and let $u_t := \theta - \frac{\sigma}{2}\log\rho_t$. Then, the system (1.1) is the combination of a Fokker-Planck equation and a Hamilton-Jacobi equation,

$$\begin{cases} \partial_t \rho_t(x) - \frac{\sigma}{2}\Delta\rho_t(x) + \nabla\cdot\big(\rho_t(x)\partial_p H(t, x, \nabla u_t)\big) = 0, \\ \partial_t u_t(x) + \frac{\sigma}{2}\Delta u_t(x) + H(t, x, \nabla u_t) = \mathrm{V}'(x, \rho_t), \end{cases} \tag{1.17}$$

which describes the following stochastic optimal control problem. Consider an infinite population of agents where each agent evolves its state x_t according to the stochastic differential equation

$$dx_t = v_t(x_t)dt + \sqrt{\sigma}dB_t, \quad x_0 := x \in \Omega \tag{1.18}$$

where v_t is the control chosen by the agent and (B_t) is a standard Brownian motion in $\mathbb{R}^n$. The agent's goal is to minimize

$$\mathbb{E}\left[\int_0^\tau L(t, x_t, v_t)\,\mathrm{d}t + \int_0^\tau \mathrm{V}'(x_t, \rho_t)\,\mathrm{d}t + u_\tau(x_\tau)\right], \tag{1.19}$$

where $\rho_t(x)$ is the density describing the fraction of agents at state x at time t, and u_τ stands for the cost at the final state x_τ. The first term in (1.19) stands for the energy spent by the control v_t, and the second term in (1.19) accounts for the effect of the rest of the population of agents. For example, the common assumption in mean-field games (and in this work) that f is non-decreasing models the agent's aversion to overcrowding. At equilibrium, each agent chooses its control optimally and the resulting density is ρ_t which satisfies the first equation in (1.17). Letting $u_t(x)$ stand for the expected cost that will be incurred by an agent playing optimally, starting at time t at state x, one can show that u_t solves the second equation in (1.17). From a different perspective, $(\rho_t, \nabla u_t)$ can be derived as the optimal solution to the problem of minimizing

$$\int_0^\tau \int_\Omega L(t, x, v_t)\,\mathrm{d}\rho_t(x)\,\mathrm{d}t + \int_0^\tau \int_\Omega \mathrm{V}(x, \rho_t)\,\mathrm{d}\rho_t\,\mathrm{d}t + \int_\Omega u_\tau\,\mathrm{d}\rho_\tau, \tag{1.20}$$

among all flows (ρ_t, v_t) satisfying the continuity equation $\partial_t\rho_t + \nabla\cdot(\rho_t v_t) = 0$ with boundary conditions ρ_0 and u_τ, where

$$\mathrm{V}(x, \rho) := F(\rho(x)) + \frac{1}{2}(W * \rho)(x), \tag{1.21}$$

so that V' is the functional derivative of V (recall $f(r) = F(r) + rF'(r)$). The functional (1.20) is exactly (1.7) and indeed (1.17) is exactly (1.1). Equations (1.17) constitute a *second-order* mean-field game system when $\sigma > 0$ and a *first-order* mean-field game system when $\sigma = 0$. Note that in contrast to the planning problem of Example 1.4, where the boundary conditions were (ρ_0, ρ_τ), in the mean-field game setting the boundary conditions are (ρ_0, u_τ).

Example 1.6. (Barotropic fluids) Let $e : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the internal energy of a fluid and let $p : \mathbb{R} \to \mathbb{R}$ be the pressure function given by $p(r) := e'(r)r^2$. Taking $U_t = W = 0$, $\sigma = 0$, and setting $e = -F$, turns (1.1) (after spatial differentiation of the second equation) into the system of equations

$$\begin{cases} \partial_t\rho_t + \nabla\cdot(\rho_t\nabla\theta_t) = 0, \\ \partial_t\nabla\theta_t + \nabla_{\nabla\theta_t}\nabla\theta_t + \frac{p(\rho_t)}{\rho_t}\nabla\rho_t = 0. \end{cases} \tag{1.22}$$

The system (1.22) describes the *compressible* Euler equations[2] where the pressure depends only on the fluid density ρ_t, which renders the fluid *barotropic* [29, §4.3]. Normally, the pressure should be a non-decreasing function of the density, which translates to $f' \leq 0$ as $-f'(r) = \frac{p'(r)}{r}$. Most of the results of this work only apply to the setting $f' \geq 0$ (which is the relevant setting for the mean-field games of Example 1.5, and in principle can also be used in the planning problem of Example 1.4). However, there are in fact systems of fluid equations where $f' \geq 0$, namely the zero-viscosity limit of the **slender jet equation** where $U_t(x) = gx$ (with $g > 0$ standing for gravity) and $f(r) = -\gamma r^{-\frac{1}{2}}$ (with $\gamma > 0$ standing for the surface tension coefficient) [11,12]. Note that in contrast to Example 1.4 and Example 1.5, the boundary conditions here are usually $(\rho_0, \nabla\theta_0)$.

Example 1.7. (Semi-classical limits of non-linear Schrödinger equations) Consider the equation

$$i\hbar\partial_t\Psi_t + \frac{\hbar^2}{2m}\Delta\Psi_t = U_t\Psi_t - (W * |\Psi_t|^2)\Psi_t - f(|\Psi_t|^2)\Psi_t \tag{1.23}$$

where Ψ is a complex-valued wave function, i is the imaginary unit, m is the mass, and $\hbar$ is the reduced Planck constant. When $f = W = 0$, Equation (1.23) is the standard linear Schrödinger equation with potential U_t (often independent of t). The interaction potential $W(x)$ is often a power law, an inverse power law, or a logarithm in the norm $|x|$, and the non-linearity $f(r)$ is often a polynomial or a logarithm in r. The connection between (1.23) and (1.1) is via the **Madelung transform** [44]: Using the representation

$$\Psi_t(x) := \rho_t^{1/2}(x)e^{i\frac{m}{\hbar}\theta_t(x)},$$

and assuming $|\Psi_t(x)|^2 > 0$ for every $x \in \Omega$ and $t \in [0, \tau]$, the flow (ρ_t, θ_t) satisfy

$$\begin{cases} \partial_t\rho_t + \nabla\cdot(\rho_t\nabla\theta_t) = 0, \\ \partial_t\theta_t + \frac{1}{2}|\nabla\theta_t|^2 - \frac{\hbar^2}{2}\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}} + U_t - W * \rho_t - f(\rho_t) = 0, \end{cases} \tag{1.24}$$

where the units are chosen so that $m = 1$. Equations (1.24) are exactly the same as equations (1.1) with the choice $\sigma = i\hbar$. The term $\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}$ is known as the (non-local) *quantum pressure* or *Bohm potential*. Most of the results in this paper only apply to the case where σ is real so they cannot apply as is to (1.24). However, when taking $\hbar \to 0$, i.e., taking the **semi-classical limit** of (1.23), equations (1.24) formally reduce to equations (1.1) with $\sigma = 0$. In particular, the results of this work apply (at least formally) whenever U_t is convex, W is concave, and $f' \geq 0$. These assumptions cover a number of semi-classical limits of interest:

Semi-classical limit of the linear Schrödinger equation with convex potential. Take $f = W = 0$ and U_t to be convex.

[2] The *incompressible* Euler equations have the additional constraint $\Delta\theta_t = 0$.

Semi-classical limit of *focusing* non-linear Schrödinger equations. Take $U_t = W = 0$ and $f' \geq 0$. Prominent examples are

$$f(r) = \epsilon r, \quad \epsilon > 0;$$

the **semi-classical limit of the focusing *cubic* non-linear Schrödinger equation**, and

$$f(r) = \epsilon \log r, \quad \epsilon > 0;$$

the **semi-classical limit of the focusing *logarithmic* non-linear Schrödinger equation** (cf. entropic regularization of optimal transport in Example 1.4).

For further information on semi-classical limits of non-linear focusing Schrödinger equations see [4,8,24,27,37] and [3,5,14,18].

The results of this work apply to all of the above examples and, in addition, to generalization afforded by considering general flows of the form (1.1).

Remark 1.8. (Quantum drift-diffusion) The *quantum drift-diffusion model* [21] (which for $n = 1$ corresponds to the *Derrida-Lebowitz-Speer-Spohn equation*) is defined by

$$\begin{cases} \partial_t \rho_t + \nabla\cdot(\rho_t \nabla\theta_t) = 0, \\ \theta_t - 2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}} = 0. \end{cases} \tag{1.25}$$

It was shown by Gianazza-Savaré-Toscani [21] that the flow (ρ_t, θ_t) satisfying (1.25) is the gradient flow in Wasserstein space of the Fisher information functional[3], and an important part of the solution theory of (1.25) is the monotonicity of the entropy $E(t)$ and Fisher information $\mathrm{Tr}[\mathcal{I}(t)]$. In Remark 4.10 it is observed that, in addition to the known monotonicity of $\mathrm{Tr}[\mathcal{I}(t)]$, there is also monotonicity (in the positive semidefinite sense) for the Fisher information *matrix* $\mathcal{I}(t)$. This observation is in line with the theme of this work but the flow (1.25) does not fall under the framework of (1.1), and the monotonicity of the Fisher information matrix can anyway be easily deduced from [21], so this observation will not be elaborated beyond Remark 4.10.

Organization of Paper

Section 2 establishes the assumptions, notation, and definitions used in this work. Section 3 contains the derivation of the formulas for the time derivatives of various quantities along the density flows. Section 4 contains the main results of this work where the differential matrix inequalities and matrix displacement convexity are derived. Finally, Section 5 contains the intrinsic dimensional functional inequalities.

[3] In the seminal work of Jordan-Kinderlehrer-Otto [26], building on Otto's gradient flow framework [38], it was shown that the heat equation is the gradient flow in Wasserstein space of the entropy functional.

2. Preliminaries

This section collects the assumptions, notation, and definitions used in this work.

2.1. Assumptions

The existence and regularity theory of density flows of the form (1.1) is highly dependent on the precise form of the partial differential equations and the boundary conditions. Such questions are orthogonal to the topic of this work so will not be addressed here. Instead, sufficient regularity will be assumed to justify the computations. In certain settings, the results of this work are completely rigorous, provided sufficient regularity on the boundary conditions is assumed, while in other settings the computations are formal. In order to avoid distracting from the main point of this work this distinction will not be emphasized.

Definition 2.1. *(Nice flows)*

(1) The domain Ω is assumed to be a convex subset of $\mathbb{R}^n$ with smooth boundary (if the domain is bounded).
(2) The functions $\rho_t(x)$ and $\theta_t(x)$ are classical solutions of (1.1), differentiable in t, twice-differentiable in x, and are finite.
(3) Integration by parts without boundary terms is justified. This entails either fast-enough decay of the flow and its derivative at infinity (when $\Omega = \mathbb{R}^n$), or appropriate boundary conditions when Ω is bounded. A good example to keep in mind is when Ω is a flat torus in $\mathbb{R}^n$.
(4) The density flow ρ_t has a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$, is assumed to be strictly positive, and integrates to 1, $\int_\Omega \mathrm{d}\rho_t(x) = 1$ for all $t \in [0, \tau]$.
(5) The exchange of derivatives and integration is permitted.

2.2. Notation

An absolutely continuous probability measure ν will often be associated with its density with respect to the Lebesgue measure so that $\mathrm{d}\nu = \nu\,\mathrm{d}x$. To alleviate the notation the domain of spatial integrals will be omitted, $\int := \int_\Omega$, and the Lebesgue measure will be omitted as well, $\int := \int \mathrm{d}x$. Often, the x argument of various functions will be omitted, e.g., $\int \nu(x) = \int \nu$, while the time dependence will be kept, e.g., $\int v_t(x)\,\mathrm{d}x = \int v_t$. The metric on $\mathbb{R}^n$ is taken to be the standard Euclidean metric, denoted by $\langle \cdot, \cdot \rangle$, with the associated norm $|\cdot|$. The coordinates of a vector $w \in \mathbb{R}^n$ are denoted by upper scripts, $w = (w^1, \dots, w^n)$. The unit sphere in $\mathbb{R}^n$ is denoted by S^{n-1}. The symmetric tensor product $\otimes_S$ is given by $w \otimes_S w' := \frac{1}{2}[w \otimes w' + w' \otimes w]$ for $w, w' \in \mathbb{R}^n$ where $w \otimes w'$ is the standard tensor product.

Matrix quantities will be denoted by calligraphic fonts, e.g., $\mathcal{M}$, and their traces (scalar) will be denoted by regular fonts, e.g., $M = \mathrm{Tr}[\mathcal{M}]$. The (i, j)th entry of $\mathcal{M}$ is denoted $\mathcal{M}_{ij}$. The transpose of a matrix $\mathcal{M}$ is given by $\mathcal{M}^\mathsf{T}$. The identity matrix on $\mathbb{R}^n$ is denoted by Id. The symbols $\succeq$ and $\preceq$ will stand for the semi-definite order and will be applied only to symmetric matrices.

Time derivatives will be denoted as ∂_t and spatial derivatives will be denoted as $\partial_i := \partial_{x_i}$ and $\partial^2_{ij} := \partial^2_{x_i x_j}$, etc. The spatial gradient and Hessian are denoted ∇, ∇^2, respectively, and $\nabla\cdot$ stands for the divergence of vector fields. Given a vector field v over $\mathbb{R}^n$ denote by ∇v the matrix defined by $(\nabla v)_{ij} = \partial_i v^j$ with $v = (v^1, \dots, v^n)$, and write $\partial_k v := (\partial_k v^1, \dots, \partial_k v^n)$ for $k \in \{1, \dots, n\}$. The first and second derivative of a function η over an interval are denoted by η', η''.

The summation $\sum_{k=1}^n$ will often be written as $\sum_k$.

2.3. Definitions

In the following definitions it is implicitly assumed that the expressions are well-defined. Throughout ν is a density (non-negative function with finite integral) over Ω.

The *differential entropy* of ν is defined as

$$E(\nu) := \int_\Omega \log \nu \, d\nu. \tag{2.1}$$

The *Fisher information matrix* of ν is the symmetric matrix defined as

$$\mathcal{I}(\nu) := -\int_\Omega \nabla^2 \log \nu \; d\nu = \int_\Omega (\nabla \log \nu)^{\otimes 2} \, d\nu, \tag{2.2}$$

with the equality holding by integration by parts. The *Fisher information* of ν is

$$I(\nu) := \mathrm{Tr}[\mathcal{I}(\nu)] = \int_\Omega |\nabla \log \nu|^2 \, d\nu. \tag{2.3}$$

Let $(\rho_t)_{t\in[0,\tau]}$ be a density flow which evolves according to a continuity equation:

$$\partial_t \rho_t + \nabla\cdot(\rho_t v_t) = 0 \quad \forall\, t \in [0, \tau]. \tag{2.4}$$

For $t \in [0, \tau]$ denote the entropy, Fisher information matrix, and Fisher information, respectively, of ρ_t as

$$E(t) := E(\rho_t), \quad \mathcal{I}(t) := \mathcal{I}(\rho_t), \quad I(t) := \mathrm{Tr}[\mathcal{I}(t)]. \tag{2.5}$$

The *entropy production matrix* is defined as

$$\mathcal{S}(t) := \int_\Omega (\nabla \rho_t \otimes_S v_t) \tag{2.6}$$

and the *entropy production* is its trace

$$S(t) := \mathrm{Tr}[\mathcal{S}(t)] = \int_\Omega \langle \nabla \rho_t, v_t \rangle. \tag{2.7}$$

The *matrix entropy* is defined by

$$\mathcal{E}(t) := \int_0^t \mathcal{S}(s) \, ds \tag{2.8}$$

so that

$$E(t) = E(0) + \mathrm{Tr}[\mathcal{E}(t)].$$

Another matrix which will play an important role comes from the driving vector field,

$$\mathcal{V}(t) := \int_{\Omega} (v_t)^{\otimes 2}\, \mathrm{d}\rho_t \tag{2.9}$$

with its trace

$$V(t) := \mathrm{Tr}[\mathcal{V}(t)]. \tag{2.10}$$

Note that $\mathcal{V}(t)$ is symmetric. Finally, the following combinations of matrices will be crucial: Given $\sigma \geq 0$ let

$$\mathcal{T}_{+}(t) := \mathcal{S}(t) + \frac{\sigma}{2}\mathcal{I}(t), \tag{2.11}$$

$$\mathcal{T}_{-}(t) := \mathcal{S}(t) - \frac{\sigma}{2}\mathcal{I}(t). \tag{2.12}$$

The interpretation of $\mathcal{T}_{\pm}(t)$ will become clearer in the subsequent sections.

3. Density Flows

This section derives the evolution equations of key quantities (entropy, entropy production matrix, etc.) along a density flow $(\rho_t)_{t\in[0,\tau]}$ satisfying the continuity equation

$$\partial_t \rho_t + \nabla\cdot(\rho_t v_t) = 0. \tag{3.1}$$

Lemma 3.1 and Lemma 3.2 describe the first derivative along the flow of the entropy $E(t)$ and Fisher information matrix $\mathcal{I}(t)$, respectively, for general flows satisfying (3.1). While these results hold for general flows satisfying a continuity equation, the focus of this paper is on density flows of the form (1.1). Using the identity

$$4\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}} = |\nabla \log \rho_t|^2 + 2\Delta \log \rho_t, \tag{3.2}$$

the flow (1.1) can be written as

$$\partial_t \rho_t + \nabla\cdot(\rho_t \nabla\theta_t) = 0, \tag{3.3}$$

with

$$\partial_t\theta_t + \frac{1}{2}|\nabla\theta_t|^2 + \frac{\sigma^2}{8}\left[|\nabla \log \rho_t|^2 + 2\Delta \log \rho_t\right] + U_t - W * \rho_t - f(\rho_t) = 0,$$
$$\sigma \in \mathbb{R}_{\geq 0} \cup i\mathbb{R}_{\geq 0}. \tag{3.4}$$

For the class of flows (ρ_t, θ_t) satisfying (3.3)-(3.4), Lemma 3.3 provides a formula for the first derivative along the flow of the entropy production matrix $\mathcal{S}(t)$ and, consequently, deduces in Corollary 3.4 a formula for the second derivative of the entropy along the flow. In addition, Lemma 3.5 describes the evolution of $\mathcal{V}(t)$ along the flow, and the evolution of its trace $V(t)$ is given in Corollary 3.6.

The first result describes the time evolution of the entropy of ρ_t.

Lemma 3.1. (1st derivative of entropy $E(t)$) *Suppose* $(\rho_t, v_t)_{t\in[0,\tau]}$ *is a nice flow satisfying (3.1). Then,*

$$\partial_t E(t) = S(t).$$

Proof. From the continuity equation (3.1) and integration by parts,

$$\begin{aligned}\partial_t E(t) &= \int (\partial_t \log \rho_t)\rho_t + \int (\log \rho_t)\partial_t \rho_t = -\int \nabla\cdot(\rho_t v_t) - \int (\log \rho_t)\nabla\cdot(\rho_t v_t)\\ &= 0 + \int \langle \nabla \log \rho_t, v_t\rangle \rho_t = \int \langle \nabla \rho_t, v_t\rangle\\ &= S(t).\end{aligned}$$

□

Next, the time evolution of the Fisher information matrix of ρ_t is derived.

Lemma 3.2. (1st derivative of Fisher information matrix $\mathcal{I}(t)$) *Suppose* $(\rho_t, v_t)_{t\in[0,\tau]}$ *is a nice flow satisfying (3.1). Then,*

$$\partial_t \mathcal{I}(t) = \int_\Omega [\nabla v_t \nabla^2 \log \rho_t]\,\mathrm{d}\rho_t + \int_\Omega [\nabla v_t \nabla^2 \log \rho_t]^\mathsf{T}\,\mathrm{d}\rho_t.$$

Proof. Recall that

$$\mathcal{I}_{ij}(t) = \int (\partial_i \log \rho_t \partial_j \log \rho_t)\rho_t = \int \partial_i \log \rho_t \partial_j \rho_t$$

so that, by exchanging derivatives,

$$\partial_t \mathcal{I}_{ij}(t) = \int \partial_i \left(\frac{\partial_t \rho_t}{\rho_t}\right) \partial_j \rho_t + \int \partial_i \log \rho_t \partial_j \partial_t \rho_t.$$

For a vector field $w = (w^1, \ldots, w^n)$ and $k = 1, \ldots, n$ let $\partial_k w := (\partial_k w^1, \ldots, \partial_k w^n)$. Using the continuity equation (3.1) and exchanging derivatives gives

$$\begin{aligned}\partial_t \mathcal{I}_{ij}(t) &= -\int \partial_i \left(\frac{\nabla\cdot(\rho_t v_t)}{\rho_t}\right) \partial_j \rho_t - \int \partial_i \log \rho_t\, \partial_j \nabla\cdot(\rho_t v_t)\\ &= -\int \frac{\partial_i \nabla\cdot(\rho_t v_t)}{\rho_t}\partial_j \rho_t + \int \frac{\nabla\cdot(\rho_t v_t)}{\rho_t^2}\partial_i \rho_t \partial_j \rho_t - \int \partial_i \log \rho_t\, \partial_j \nabla\cdot(\rho_t v_t)\\ &= -\int [\nabla\cdot\partial_i(\rho_t v_t)]\partial_j \log \rho_t + \int \nabla\cdot(\rho_t v_t)\partial_i \log \rho_t \partial_j \log \rho_t\\ &\quad -\int \partial_i \log \rho_t\, \nabla\cdot\partial_j(\rho_t v_t)\end{aligned}$$

$$= -\sum_k \int \partial^2_{ki}(\rho_t v^k_t)\partial_j \log \rho_t + \sum_k \int \partial_k(\rho_t v^k_t)\partial_i \log \rho_t \partial_j \log \rho_t$$
$$- \sum_k \int \partial_i \log \rho_t \, \partial^2_{kj}(\rho_t v^k_t).$$

Hence, by integration by parts and exchanging derivatives,

$$\begin{aligned}
\partial_t \mathcal{I}_{ij}(t) &= \sum_k \int \partial_i(\rho_t v^k_t)\partial^2_{kj} \log \rho_t - \sum_k \int \rho_t v^k_t [\partial^2_{ik} \log \rho_t \partial_j \log \rho_t \\
&\quad + \partial_i \log \rho_t \partial^2_{jk} \log \rho_t] + \sum_k \int \partial^2_{ik} \log \rho_t \partial_j(\rho_t v^k_t) \\
&= \sum_k \int [v^k_t \partial_i \rho_t + \rho_t \partial_i v^k_t]\partial^2_{kj} \log \rho_t - \sum_k \int v^k_t [\partial^2_{ik} \log \rho_t \partial_j \rho_t \\
&\quad + \partial_i \rho_t \partial^2_{jk} \log \rho_t] + \sum_k \int \partial^2_{ik} \log \rho_t [v^k_t \partial_j \rho_t + \rho_t \partial_j v^k_t] \\
&= \int [\nabla^2 \log \rho_t v_t]_j \partial_i \rho_t + \int [\nabla v_t \nabla^2 \log \rho_t]_{ij} \, \mathrm{d}\rho_t - \int [\nabla^2 \log \rho_t v_t]_i \partial_j \rho_t \\
&\quad - \int [\nabla^2 \log \rho_t v_t]_j \partial_i \rho_t \\
&\quad + \int [\nabla^2 \log \rho_t v_t]_i \partial_j \rho_t + \int [\nabla v_t \nabla^2 \log \rho_t]_{ji} \, \mathrm{d}\rho_t \\
&= \int [\nabla v_t \nabla^2 \log \rho_t]_{ij} \, \mathrm{d}\rho_t + \int [\nabla v_t \nabla^2 \log \rho_t]_{ji} \, \mathrm{d}\rho_t.
\end{aligned}$$

□

The remainder of the section focuses on flows $(\rho_t, \theta_t)_{t\in[0,\tau]}$ satisfying (3.3)-(3.4). Note that under the evolution (3.3)-(3.4), the entropy production matrix $\mathcal{S}(t)$ can also be expressed, by integration by parts, as

$$\mathcal{S}(t) = \int_\Omega (\nabla \rho_t \otimes_S \nabla \theta_t) \, \mathrm{d}x = -\int_\Omega \theta_t \nabla^2 \rho_t \, \mathrm{d}x = -\int_\Omega \nabla^2 \theta_t \, \mathrm{d}\rho_t. \quad (3.5)$$

Lemma 3.3. (1st derivative of entropy production matrix $\mathcal{S}(t)$) *Suppose* $(\rho_t, \theta_t)_{t\in[0,\tau]}$ *is a nice flow satisfying (3.3)-(3.4). Then,*

$$\partial_t \mathcal{S}(t) = \int_\Omega (\nabla^2 \theta_t)^2 \, \mathrm{d}\rho_t + \frac{\sigma^2}{4}\int_\Omega (\nabla^2 \log \rho_t)^2 \, \mathrm{d}\rho_t + \int_\Omega \nabla^2 U_t \, \mathrm{d}\rho_t$$
$$+ \int_\Omega (-\nabla^2 W) * \rho_t \, \mathrm{d}\rho_t + \int_\Omega f'(\rho_t) \frac{(\nabla \rho_t)^{\otimes 2}}{\rho_t} \, \mathrm{d}\rho_t.$$

Proof. By definition

$$\partial_t \mathcal{S}_{ij}(t) = \frac{1}{2}\partial_t \int \partial_i \rho_t \partial_j \theta_t + \frac{1}{2}\partial_t \int \partial_j \rho_t \partial_i \theta_t$$

$$
\begin{aligned}
&= \frac{1}{2}\int \partial_t(\partial_i\rho_t)\partial_j\theta_t + \frac{1}{2}\int \partial_i\rho_t\partial_t(\partial_j\theta_t) + \frac{1}{2}\int \partial_t(\partial_j\rho_t)\partial_i\theta_t \\
&\quad + \frac{1}{2}\int \partial_j\rho_t\partial_t(\partial_i\theta_t) \\
&= \frac{1}{2}\left[A_{ij} + B_{ij} + A_{ji} + B_{ji}\right]
\end{aligned}
$$

where

$$
A_{ij} := \int \partial_t(\partial_i\rho_t)\partial_j\theta_t, \qquad B_{ij} := \int \partial_i\rho_t\partial_t(\partial_j\theta_t).
$$

To compute A_{ij} note that by (3.3),

$$
\begin{aligned}
\partial_t\partial_i\rho_t &= -\partial_i\nabla\cdot(\rho_t\nabla\theta_t) \\
&= -\sum_k \partial_i[\partial_k\theta_t\partial_k\rho_t + \rho_t\partial^2_{kk}\theta_t] \\
&= -\sum_k [\partial^2_{ik}\theta_t\partial_k\rho_t + \partial_k\theta_t\partial^2_{ik}\rho_t + \partial_i\rho_t\partial^2_{kk}\theta_t + \rho_t\partial^3_{kik}\theta_t].
\end{aligned}
$$

Hence, by integration by parts,

$$
\begin{aligned}
A_{ij} &= -\int \sum_k [\partial^2_{ik}\theta_t\partial_k\rho_t + \partial_k\theta_t\partial^2_{ik}\rho_t + \partial_i\rho_t\partial^2_{kk}\theta_t + \rho_t\partial^3_{kik}\theta_t]\partial_j\theta_t \\
&= -\int [\nabla^2\theta_t\nabla\rho_t]_i\partial_j\theta_t - \int [\nabla^2\rho_t\nabla\theta_t]_i\partial_j\theta_t + \sum_k \int \partial_k[\partial_j\theta_t\partial_i\rho_t]\partial_k\theta_t \\
&\quad + \sum_k \int \partial_k[\rho_t\partial_j\theta_t]\partial^2_{ik}\theta_t \\
&= -\int [\nabla^2\theta_t\nabla\rho_t]_i\partial_j\theta_t - \int [\nabla^2\rho_t\nabla\theta_t]_i\partial_j\theta_t + \sum_k \int \partial^2_{jk}\theta_t\partial_i\rho_t\partial_k\theta_t \\
&\quad + \sum_k \int \partial_j\theta_t\partial^2_{ik}\rho_t\partial_k\theta_t \\
&\quad + \sum_k \int \partial_k\rho_t\partial_j\theta_t\partial^2_{ik}\theta_t + \sum_k \int \rho_t\partial^2_{jk}\theta_t\partial^2_{ik}\theta_t \\
&= -\int [\nabla^2\theta_t\nabla\rho_t]_i\partial_j\theta_t - \int [\nabla^2\rho_t\nabla\theta_t]_i\partial_j\theta_t + \int [\nabla^2\theta_t\nabla\theta_t]_j\partial_i\rho_t \\
&\quad + \int [\nabla^2\rho_t\nabla\theta_t]_i\partial_j\theta_t \\
&\quad + \int [\nabla^2\theta_t\nabla\rho_t]_i\partial_j\theta_t + \int (\nabla^2\theta_t)^2_{ij}\,\mathrm{d}\rho_t \\
&= \int [\nabla^2\theta_t\nabla\theta_t]_j\partial_i\rho_t + \int (\nabla^2\theta_t)^2_{ij}\,\mathrm{d}\rho_t.
\end{aligned}
$$

To compute B_{ij} note that by (3.4),

$$\partial_t \partial_j \theta_t = -\partial_j \left\{ \frac{1}{2}|\nabla \theta_t|^2 + \frac{\sigma^2}{8}\left[|\nabla \log \rho_t|^2 + 2\Delta \log \rho_t \right] + U_t - W * \rho_t - f(\rho_t) \right\}$$
$$= -\sum_k \partial_k \theta_t \partial^2_{jk}\theta_t - \frac{\sigma^2}{4}\sum_k \partial_k \log \rho_t \partial^2_{jk} \log \rho_t - \frac{\sigma^2}{4}\sum_k \partial^3_{kjk} \log \rho_t$$
$$- \partial_j U_t + (\partial_j W) * \rho_t + f'(\rho_t)\partial_j \rho_t.$$

Hence, by integration by parts,

$$B_{ij} = -\sum_k \int \partial_k \theta_t \partial^2_{jk} \theta_t \partial_i \rho_t - \frac{\sigma^2}{4}\sum_k \int \partial_k \log \rho_t \partial^2_{jk} \log \rho_t \partial_i \rho_t$$
$$- \frac{\sigma^2}{4}\sum_k \int \partial^3_{kjk} \log \rho_t \partial_i \rho_t$$
$$- \int \partial_j U_t \partial_i \rho_t + \int [(\partial_j W) * \rho_t] \partial_i \rho_t + \int f'(\rho_t)\partial_j \rho_t \partial_i \rho_t$$
$$= -\int [\nabla^2 \theta_t \nabla \theta_t]_j \partial_i \rho_t + \frac{\sigma^2}{4}\sum_k \int [\partial^2_{ik} \log \rho_t \partial^2_{kj} \log \rho_t$$
$$+ \partial_k \log \rho_t \partial^3_{ijk} \log \rho_t]\rho_t - \frac{\sigma^2}{4}\sum_k \int \partial^3_{ijk} \log \rho_t \partial_k \rho_t$$
$$+ \int \partial^2_{ij} U_t \, \mathrm{d}\rho_t - \int (\partial^2_{ij} W) * \rho_t \, \mathrm{d}\rho_t - \int f'(\rho_t)\partial^2_{ij}\rho_t \, \mathrm{d}\rho_t$$
$$- \int f''(\rho_t)\partial_i \rho_t \partial_j \rho_t \, \mathrm{d}\rho_t$$
$$= -\int [\nabla^2 \theta_t \nabla \theta_t]_j \partial_i \rho_t + \frac{\sigma^2}{4}\int (\nabla^2 \log \rho_t)^2_{ij} \, \mathrm{d}\rho_t + \frac{\sigma^2}{4}\sum_k \int \partial^3_{ijk} \log \rho_t \partial_k \rho_t$$
$$- \frac{\sigma^2}{4}\sum_k \int \partial^3_{ijk} \log \rho_t \partial_k \rho_t$$
$$+ \int \partial^2_{ij} U_t \, \mathrm{d}\rho_t - \int (\partial^2_{ij} W) * \rho_t \, \mathrm{d}\rho_t - \int f'(\rho_t)\partial^2_{ij}\rho_t \, \mathrm{d}\rho_t$$
$$- \int f''(\rho_t)\partial_i \rho_t \partial_j \rho_t \, \mathrm{d}\rho_t$$
$$= -\int [\nabla^2 \theta_t \nabla \theta_t]_j \partial_i \rho_t + \frac{\sigma^2}{4}\int (\nabla^2 \log \rho_t)^2_{ij} \, \mathrm{d}\rho_t + \int \partial^2_{ij} U_t \, \mathrm{d}\rho_t$$
$$- \int (\partial^2_{ij} W) * \rho_t \, \mathrm{d}\rho_t$$
$$- \int f'(\rho_t)\partial^2_{ij}\rho_t \, \mathrm{d}\rho_t - \int f''(\rho_t)\partial_i \rho_t \partial_j \rho_t \, \mathrm{d}\rho_t.$$

It follows that

$$\begin{aligned} A_{ij} + B_{ij} &= \int [\nabla^2\theta_t \nabla\theta_t]_j \partial_i \rho_t + \int (\nabla^2\theta_t)_{ij}^2 \,\mathrm{d}\rho_t - \int [\nabla^2\theta_t \nabla\theta_t]_j \partial_i \rho_t \\ &\quad + \frac{\sigma^2}{4} \int (\nabla^2 \log \rho_t)_{ij}^2 \,\mathrm{d}\rho_t \\ &\quad + \int \partial_{ij}^2 U_t \,\mathrm{d}\rho_t - \int (\partial_{ij}^2 W) * \rho_t \,\mathrm{d}\rho_t - \int f'(\rho_t) \partial_{ij}^2 \rho_t \,\mathrm{d}\rho_t \\ &\quad - \int f''(\rho_t) \partial_i \rho_t \partial_j \rho_t \,\mathrm{d}\rho_t \\ &= \int (\nabla^2\theta_t)_{ij}^2 \,\mathrm{d}\rho_t + \frac{\sigma^2}{4} \int (\nabla^2 \log \rho_t)_{ij}^2 \,\mathrm{d}\rho_t + \int (\nabla^2 U_t)_{ij} \,\mathrm{d}\rho_t \\ &\quad - \int (\nabla^2 W)_{ij} * \rho_t \,\mathrm{d}\rho_t \\ &\quad - \int f'(\rho_t)(\nabla^2 \rho_t)_{ij} \,\mathrm{d}\rho_t - \int f''(\rho_t)(\nabla \rho_t)_{ij}^{\otimes 2} \,\mathrm{d}\rho_t. \end{aligned}$$

Analogous argument applies to $A_{ji} + B_{ji}$. Finally, by integration by parts,

$$-\int f'(\rho_t)(\nabla^2\rho_t)_{ij} \,\mathrm{d}\rho_t = \int f''(\rho_t)(\nabla\rho_t)_{ij}^{\otimes 2} \,\mathrm{d}\rho_t + \int f'(\rho_t)(\nabla\rho_t)_{ij}^{\otimes 2} \,\mathrm{d}x$$

so

$$-\int f'(\rho_t)(\nabla^2\rho_t)_{ij} \,\mathrm{d}\rho_t - \int f''(\rho_t)(\nabla\rho_t)_{ij}^{\otimes 2} \,\mathrm{d}\rho_t = \int f'(\rho_t)(\nabla\rho_t)_{ij}^{\otimes 2} \,\mathrm{d}x,$$

which completes the proof. □

Combining Lemma 3.1 and Lemma 3.3 yields:

Corollary 3.4. (2nd derivative of entropy $E(t)$) *Suppose* $(\rho_t, \theta_t)_{t\in[0,\tau]}$ *is a nice flow satisfying (3.3)-(3.4). Then,*

$$\begin{aligned} \partial_{tt}^2 E(t) &= \int_\Omega \mathrm{Tr}[(\nabla^2\theta_t)^2] \,\mathrm{d}\rho_t + \frac{\sigma^2}{4} \int_\Omega \mathrm{Tr}[(\nabla^2 \log \rho_t)^2] \,\mathrm{d}\rho_t + \int_\Omega \Delta U_t \,\mathrm{d}\rho_t \\ &\quad + \int_\Omega (-\Delta W) * \rho_t \,\mathrm{d}\rho_t + \int_\Omega f'(\rho_t) \frac{|\nabla\rho_t|^2}{\rho_t} \,\mathrm{d}\rho_t. \end{aligned}$$

Lemma 3.5. (1st derivative of $\mathcal{V}(t)$) *Suppose* $(\rho_t, \theta_t)_{t\in[0,\tau]}$ *satisfy (3.3)-(3.4). Then,*

$$\partial_t \mathcal{V}_{ij}(t) = \frac{\sigma^2}{4} \partial_t \mathcal{I}_{ij}(t) + \int_\Omega \{U_t - W * \rho_t - f(\rho_t)\} (\partial_i[\rho_t \partial_j \theta_t] + \partial_j[\rho_t \partial_i \theta_t]).$$

Proof. Recall that

$$\mathcal{V}(t) = \int (\nabla\theta_t)^{\otimes 2} \,\mathrm{d}\rho_t$$

so that

$$\partial_t \mathcal{V}_{ij}(t) = \partial_t \int (\partial_i\theta_t\partial_j\theta_t)\rho_t = \int (\partial_i\partial_t\theta_t)\partial_j\theta_t \,\mathrm{d}\rho_t + \int (\partial_j\partial_t\theta_t)\partial_i\theta_t \,\mathrm{d}\rho_t + \int \partial_i\theta_t\partial_j\theta_t\partial_t\rho_t = A_{ij} + A_{ji} + B_{ij}$$

where

$$A_{ij} := \int (\partial_i\partial_t\theta_t)\partial_j\theta_t \,\mathrm{d}\rho_t, \qquad B_{ij} := \int \partial_i\theta_t\partial_j\theta_t\partial_t\rho_t.$$

To compute A_{ij} note that by (3.4),

$$\begin{aligned} A_{ij} &= -\int \partial_i \left\{ \frac{1}{2}|\nabla\theta_t|^2 + \frac{\sigma^2}{8}\left[|\nabla \log \rho_t|^2 + 2\Delta \log \rho_t \right] + U_t - W * \rho_t - f(\rho_t) \right\} \partial_j\theta_t \,\mathrm{d}\rho_t \\ &= -\sum_k \int \left\{ \partial^2_{ik}\theta_t\partial_k\theta_t + \frac{\sigma^2}{4}\partial^2_{ik}\log\rho_t\partial_k\log\rho_t + \frac{\sigma^2}{4}\partial^3_{kik}\log\rho_t \right\} \partial_j\theta_t \,\mathrm{d}\rho_t \\ &\quad - \int \partial_i \left\{ U_t - W * \rho_t - f(\rho_t) \right\} \partial_j\theta_t \,\mathrm{d}\rho_t. \end{aligned}$$

Hence, by integration by parts,

$$\begin{aligned} A_{ij} &= -\int [\nabla^2\theta_t\nabla\theta_t]_i\partial_j\theta_t \,\mathrm{d}\rho_t - \frac{\sigma^2}{4}\int [\nabla^2\log\rho_t\nabla\log\rho_t]_i\partial_j\theta_t \,\mathrm{d}\rho_t \\ &\quad + \frac{\sigma^2}{4}\sum_k \int (\partial^2_{ik}\log\rho_t)[\rho_t\partial^2_{jk}\theta_t + \partial_j\theta_t\partial_k\rho_t] \\ &\quad + \int \{U_t - W * \rho_t - f(\rho_t)\}\partial_i[\rho_t\partial_j\theta_t] \\ &= -\int [\nabla^2\theta_t\nabla\theta_t]_i\partial_j\theta_t \,\mathrm{d}\rho_t - \frac{\sigma^2}{4}\int [\nabla^2\log\rho_t\nabla\log\rho_t]_i\partial_j\theta_t \,\mathrm{d}\rho_t \\ &\quad + \frac{\sigma^2}{4}\int [\nabla^2\log\rho_t\nabla^2\theta_t]_{ij} \,\mathrm{d}\rho_t \\ &\quad + \frac{\sigma^2}{4}\int [\nabla^2\log\rho_t\nabla\log\rho_t]_i\partial_j\theta_t \,\mathrm{d}\rho_t + \int \{U_t - W * \rho_t - f(\rho_t)\}\partial_i[\rho_t\partial_j\theta_t] \\ &= -\int [\nabla^2\theta_t\nabla\theta_t]_i\partial_j\theta_t \,\mathrm{d}\rho_t + \frac{\sigma^2}{4}\int [\nabla^2\log\rho_t\nabla^2\theta_t]_{ij} \,\mathrm{d}\rho_t \\ &\quad + \int \{U_t - W * \rho_t - f(\rho_t)\}\partial_i[\rho_t\partial_j\theta_t]. \end{aligned}$$

Analogously,

$$A_{ji} = -\int [\nabla^2\theta_t\nabla\theta_t]_j\partial_i\theta_t \,\mathrm{d}\rho_t + \frac{\sigma^2}{4}\int [\nabla^2\log\rho_t\nabla^2\theta_t]_{ji} \,\mathrm{d}\rho_t$$

$$+ \int \{U_t - W * \rho_t - f(\rho_t)\}\, \partial_j[\rho_t \partial_i \theta_t].$$

To compute B_{ij} note that by (3.3) and integration by parts,

$$B_{ij} = -\int \partial_i\theta_t \partial_j\theta_t \nabla\cdot(\rho_t \nabla\theta_t) = \sum_k \int \partial_k(\partial_i\theta_t\partial_j\theta_t)\partial_k\theta_t \,\mathrm{d}\rho_t$$
$$= \int [\nabla^2\theta_t \nabla\theta_t]_i \partial_j\theta_t \,\mathrm{d}\rho_t + \int [\nabla^2\theta_t \nabla\theta_t]_j \partial_i\theta_t \,\mathrm{d}\rho_t.$$

It follows that

$$A_{ij} + A_{ji} + B_{ij}$$
$$= \frac{\sigma^2}{4}\int [\nabla^2\theta_t \nabla^2 \log\rho_t + \nabla^2\log\rho_t \nabla^2\theta_t]_{ij} \,\mathrm{d}\rho_t$$
$$+ \int \{U_t - W * \rho_t - f(\rho_t)\}\, (\partial_i[\rho_t\partial_j\theta_t] + \partial_j[\rho_t\partial_i\theta_t]).$$

The proof is complete by Lemma 3.2. □

Combining Lemma 3.5 and (3.3) yields:

Corollary 3.6. (1st derivative of $V(t)$) *Suppose $(\rho_t, \theta_t)_{t\in[0,\tau]}$ is a nice flow satisfying (3.3)-(3.4). Then,*

$$\partial_t V(t) = \frac{\sigma^2}{4}\partial_t I(t) - 2\int_\Omega \{U_t - W * \rho_t - f(\rho_t)\}\, \partial_t\rho_t.$$

4. Matrix Differential Inequalities and Matrix Displacement Convexity

In this section the main matrix differential inequalities of this work are derived. The main result is Theorem 4.1 which provides matrix differential inequalities for $[0,\tau] \ni t \mapsto \mathcal{T}_\pm(t)$, for any flow satisfying (3.3)-(3.4), *provided that* $\sigma \in \mathbb{R}_{\geq 0}$. From Theorem 4.1 it is possible to deduce a matrix differential inequality for $\mathcal{S}(t)$, which is the content of Theorem 4.2. In Section 4.1, a few technical results are collected which show how to obtain bounds on matrices and deduce matrix displacement convexity from matrix differential inequalities. Finally, Section 4.2 apply Theorem 4.1 and Theorem 4.2, together with the results of Section 4.1, to flows of the form (3.3–3.4) under convexity constraints.

The following theorem is the main result of this section and is based on the formulas of Section 3.

Theorem 4.1. (Matrix differential inequalities for $\mathcal{T}_\pm(t)$) *Suppose $(\rho_t, \theta_t)_{t\in[0,\tau]}$ is a nice flow satisfying (3.3)-(3.4) with $\sigma \in \mathbb{R}_{\geq 0}$. Then,*

$$\partial_t\mathcal{T}_\pm(t) \succeq \mathcal{T}_\mp^2(t) + \int_\Omega \nabla^2 U_t \,\mathrm{d}\rho_t + \int_\Omega (-\nabla^2 W) * \rho_t \,\mathrm{d}\rho_t + \int_\Omega f'(\rho_t)\frac{(\nabla\rho_t)^{\otimes 2}}{\rho_t}\,\mathrm{d}\rho_t.$$

Proof. Fix $\sigma \in \mathbb{R}_{\geq 0}$. By Lemma 3.2 and Lemma 3.3,

$$\begin{aligned}
\partial_t \mathcal{T}_{\pm}(t) &= \int (\nabla^2\theta_t)^2 \,\mathrm{d}\rho_t + \frac{\sigma^2}{4}\int (\nabla^2 \log \rho_t)^2 \,\mathrm{d}\rho_t + \int \nabla^2 U_t \,\mathrm{d}\rho_t \\
&\quad + \int (-\nabla^2 W) * \rho_t \,\mathrm{d}\rho_t + \int f'(\rho_t)\frac{(\nabla\rho_t)^{\otimes 2}}{\rho_t}\,\mathrm{d}\rho_t \\
&\quad \pm \frac{\sigma}{2}\int [\nabla^2\theta_t \nabla^2 \log\rho_t + \nabla^2\log\rho_t \nabla^2\theta_t]\,\mathrm{d}\rho_t \\
&= \int \left[\nabla^2\theta_t \pm \frac{\sigma}{2}\nabla^2\log\rho_t\right]^2 \mathrm{d}\rho_t + \int \nabla^2 U_t \,\mathrm{d}\rho_t + \int (-\nabla^2 W) * \rho_t \,\mathrm{d}\rho_t \\
&\quad + \int f'(\rho_t)\frac{(\nabla\rho_t)^{\otimes 2}}{\rho_t}\,\mathrm{d}\rho_t \\
&\succeq \left[\int \left(\nabla^2\theta_t \pm \frac{\sigma}{2}\nabla^2\log\rho_t\right) \mathrm{d}\rho_t\right]^2 + \int \nabla^2 U_t \,\mathrm{d}\rho_t + \int (-\nabla^2 W) * \rho_t \,\mathrm{d}\rho_t \\
&\quad + \int f'(\rho_t)\frac{(\nabla\rho_t)^{\otimes 2}}{\rho_t}\,\mathrm{d}\rho_t \\
&= \mathcal{T}_{\pm}^2(t) + \int \nabla^2 U_t \,\mathrm{d}\rho_t + \int (-\nabla^2 W) * \rho_t \,\mathrm{d}\rho_t + \int f'(\rho_t)\frac{(\nabla\rho_t)^{\otimes 2}}{\rho_t}\,\mathrm{d}\rho_t,
\end{aligned}$$

where the inequality holds by Jensen's inequality. □

By combining the differential inequalities of Theorem 4.1 the following result is deduced.

Theorem 4.2. (Matrix differential inequalities for entropy production matrix $\mathcal{S}(t)$) *Suppose $(\rho_t, \theta_t)_{t\in[0,\tau]}$ is a nice flow satisfying (3.3)-(3.4) with $\sigma \in \mathbb{R}_{\geq 0}$. Then,*

$$\begin{aligned}
\partial_t \mathcal{S}(t) \succeq \mathcal{S}^2(t) + \frac{\sigma^2}{4}\mathcal{I}^2(t) + \int_\Omega \nabla^2 U_t \,\mathrm{d}\rho_t + \int_\Omega (-\nabla^2 W) * \rho_t \,\mathrm{d}\rho_t \\
+ \int_\Omega f'(\rho_t)\frac{(\nabla\rho_t)^{\otimes 2}}{\rho_t}\,\mathrm{d}\rho_t.
\end{aligned}$$

Proof. Since

$$\mathcal{S}(t) = \frac{\mathcal{T}_+(t)}{2} + \frac{\mathcal{T}_-(t)}{2}$$

it follows from Theorem 4.1 that

$$\begin{aligned}
\partial_t \mathcal{S}(t) \succeq & \frac{\mathcal{T}_+^2(t)}{2} + \frac{\mathcal{T}_-^2(t)}{2} + \int \nabla^2 U_t \,\mathrm{d}\rho_t + \int (-\nabla^2 W) * \rho_t \,\mathrm{d}\rho_t \\
& + \int f'(\rho_t)\frac{(\nabla\rho_t)^{\otimes 2}}{\rho_t}\,\mathrm{d}\rho_t.
\end{aligned}$$

The result follows as

$$\frac{\mathcal{T}_+^2(t)}{2} + \frac{\mathcal{T}_-^2(t)}{2} = \frac{1}{2}\left[\mathcal{S}^2(t) + 2\mathcal{S}(t) \otimes_S \frac{\sigma}{2}\mathcal{I}(t) + \frac{\sigma^2}{4}\mathcal{I}^2(t)\right]$$

$$+\frac{1}{2}\left[\mathcal{S}^2(t) - 2\mathcal{S}(t) \otimes_S \frac{\sigma}{2}\mathcal{I}(t) + \frac{\sigma^2}{4}\mathcal{I}^2(t)\right]$$
$$= \mathcal{S}^2(t) + \frac{\sigma^2}{4}\mathcal{I}^2(t).$$

□

Both Theorem 4.1 and Theorem 4.2 provide differential inequalities of the form

$$\partial_t M(t) \succeq M^2(t) + \text{remainder term.} \tag{4.1}$$

In Section 4.2 it will be shown that in many flows of interest the remainder term is nonnegative (in a semidefinite sense), which means that (4.1) implies differential inequalities of the form

$$\partial_t M(t) \succeq M^2(t). \tag{4.2}$$

The following section shows how to take differential inequalities of the form (4.2) and deduce bounds on $M(t)$ as well as obtain matrix displacement convexity.

4.1. Matrix Differential Inequalities and Displacement Convexity

Suppose for the rest of this section that $[0,\tau] \ni t \mapsto M(t)$ is a differentiable function taking values in the set of $n \times n$ symmetric matrices. The first result shows how the differential inequality (4.2) implies bounds on $M(t)$ in terms of $M(0)$.

Remark 4.3. Note that in the following results, the existence time τ of the flow $(M(t))_{t\in[0,\tau]}$ will depend on the value of $M(0)$.

Lemma 4.4. *If*

$$\partial_t M(t) \succeq M^2(t) \quad \forall\, t \in [0,\tau],$$

then, for any $w \in S^{n-1}$,

$$\langle w, M(t)w\rangle \geq \frac{\langle w, M(0)w\rangle}{1 - t\langle w, M(0)w\rangle} \quad \forall\, t \in [0,\tau]. \tag{4.3}$$

Proof. Fix $w \in \mathbb{R}^n$ and let $\eta(t) := \langle w, M(t)w\rangle$. Then, by the Cauchy-Schwarz inequality,

$$\partial_t \eta(t) = \langle w, \partial_t M(t)w\rangle \geq \langle w, M^2(t)w\rangle \geq \langle w, M(t)w\rangle^2 = \eta^2(t).$$

The solution of the ordinary differential equation

$$\partial_t \xi(t) = \xi^2(t) \quad \forall\, t \in [0,\tau] \quad \text{with} \quad \xi(0) = \eta(0)$$

is $\xi(t) := \frac{\eta(0)}{1-t\eta(0)}$. Standard comparison [35] shows that $\eta(t) \geq \xi(t)$ for all $t \in [0,\tau]$, which establishes (4.3). □

Corollary 4.5. *Suppose that*

$$\partial_t M(t) \succeq M^2(t) \quad \forall\, t \in [0, \tau],$$

and let $\{\lambda_i\}_{i=1}^n$ *be the eigenvalues of* $M(0)$. *Then,*

$$\mathrm{Tr}[M(t)] \geq \sum_{i=1}^n \frac{\lambda_i}{1-\lambda_i t}.$$

Proof. Let $\{w_i\}_{i=1}^n$ be the normalized eigenvectors of $M(0)$ corresponding to $\{\lambda_i\}_{i=1}^n$. By Lemma 4.4,

$$\mathrm{Tr}[M(t)] = \sum_{i=1}^n \langle w_i, M(t) w_i \rangle \geq \sum_{i=1}^n \frac{\langle w_i, M(0) w_i \rangle}{1 - t\langle w_i, M(0) w_i\rangle} = \sum_{i=1}^n \frac{\lambda_i}{1-\lambda_i t}.$$

□

Next it is shown how the differential inequality (4.2) implies matrix displacement convexity.

Lemma 4.6. *If*

$$\partial_t M(t) \succeq M^2(t) \quad \forall\, t \in [0, \tau],$$

then, $\int_0^t M(s)\,\mathrm{d}s$ *is matrix displacement convex, that is, for any* $w \in S^{n-1}$, *the function* $\mathrm{c}_w : [0, \tau] \to \mathbb{R}$ *given by*

$$\mathrm{c}_w(t) = \exp\left[-\int_0^t \langle w, M(s) w\rangle\,\mathrm{d}s\right]$$

is concave. Consequently,

$$-\frac{1}{t} \leq \langle w, M(t) w\rangle \leq \frac{1}{\tau - t}. \tag{4.4}$$

Proof. To show the concavity of c_w it suffices to show that $\partial_{tt}^2 \mathrm{c}_w(t) \leq 0$ for every $t \in [0, \tau]$. The first derivative is

$$\partial_t \mathrm{c}_w(t) = -\mathrm{c}_w(t)\langle w, M(t) w\rangle,$$

and the second derivative is nonnegative as

$$\begin{aligned}
\partial_{tt}^2 \mathrm{c}_w(t) &= \mathrm{c}_w(t)\langle w, M(t) w\rangle^2 - \mathrm{c}_w(t)\langle w, \partial_t M(t) w\rangle \\
&\leq \mathrm{c}_w(t)\langle w, M(t) w\rangle^2 - \mathrm{c}_w(t)\langle w, M^2(t) w\rangle \\
&= \mathrm{c}_w(t)\left\{\langle w, M(t) w\rangle^2 - \langle w, M^2(t) w\rangle\right\} \\
&\leq 0,
\end{aligned}$$

where the first inequality holds by the assumption $\partial_t M(t) \succeq M^2(t)$, and the second inequality holds by the Cauchy-Schwarz inequality.

To establish (4.4) follow the argument of [9, §3.3.1] and note that the concavity of c_w implies

$$\partial_t \mathrm{c}_w(\tau) \le \frac{\mathrm{c}_w(\tau) - \mathrm{c}_w(t)}{\tau - t} \le \partial_t \mathrm{c}_w(t) \le \frac{\mathrm{c}_w(t) - \mathrm{c}_w(0)}{t} \le \partial_t \mathrm{c}_w(0).$$

Since $\partial_t \mathrm{c}_w(t) = -\mathrm{c}_w(t)\langle w, M(t)w\rangle$, and $\mathrm{c}_w(t) \ge 0$,

$$-\mathrm{c}_w(t)\langle w, M(t)w\rangle = \partial_t \mathrm{c}_w(t) \le \frac{\mathrm{c}_w(t) - \mathrm{c}_w(0)}{t}$$

is equivalent to

$$\langle w, M(t)w\rangle \ge -\frac{1}{t} + \frac{\mathrm{c}_w(0)}{\mathrm{c}_w(t)t} \ge -\frac{1}{t}.$$

The bound $\langle w, M(t)w\rangle \le \frac{1}{\tau - t}$ follows analogously by using $\frac{\mathrm{c}_w(\tau)-\mathrm{c}_w(t)}{\tau-t} \le \partial_t \mathrm{c}_w(t)$. □

Corollary 4.7. *Suppose*

$$\partial_t M(t) \succeq M^2(t) \quad \forall\, t \in [0, \tau],$$

and let $\{\lambda_i\}_{i=1}^n$ *be the eigenvalues of* $M(0)$. *Then,*

$$\int_0^\tau \mathrm{Tr}[M(t)]\,\mathrm{d}t \ge -\sum_{i=1}^n \log\left(1 - \tau\lambda_i\right).$$

Proof. Taking $t = 0$ in (4.4) gives $\lambda_i \le \frac{1}{\tau}$. Hence, $\lambda_i \le \frac{1}{t}$ for any $t \in [0, \tau]$ which implies $0 \le 1 - t\lambda_i$ for any $t \in [0, \tau]$. In fact, these inequalities are strict since otherwise the left-hand side in Corollary 4.5 is infinite (but by assumption it is finite, cf. Remark 4.3). The result follows by integrating the bound in Corollary 4.5 from $t = 0$ to $t = \tau$. □

4.2. Matrix Differential Inequalities and Displacement Convexity Along Density Flows

This section shows that there are a number of important density flows where the matrix differential inequalities of Theorems 4.1 and 4.2 are of the form

$$\partial_t M(t) \succeq M^2(t) + \text{nonegative term}. \tag{4.5}$$

Hence, Lemma 4.4, Corollary 4.5, Lemma 4.6, and Corollary 4.7 are applicable. The reader should keep in mind Remark 4.3.

Theorem 4.8. (Differential inequalities and matrix displacement convexity for $\mathcal{T}_\pm(t)$) *Suppose* $(\rho_t, \theta_t)_{t\in[0,\tau]}$ *is a nice flow satisfying* (3.3–3.4) *with* $\sigma \in \mathbb{R}_{\geq 0}$, $f'(r) \geq 0$ *for every* $r \in \mathbb{R}_{\geq 0}$, *and* $\int \{\nabla^2 U_t - \nabla^2 W * \rho_t\} \, \mathrm{d}\rho_t \succeq 0$ *for every* $t \in [0, \tau]$. *Then,*

$$\partial_t \mathcal{T}_\pm(t) \succeq \mathcal{T}_\pm^2(t) + \int_\Omega \nabla^2 U_t \, \mathrm{d}\rho_t + \int_\Omega (-\nabla^2 W) * \rho_t \, \mathrm{d}\rho_t + \int_\Omega f'(\rho_t) \frac{(\nabla \rho_t)^{\otimes 2}}{\rho_t} \, \mathrm{d}\rho_t \succeq \mathcal{T}_\pm^2(t) \succeq 0. \tag{4.6}$$

Consequently, for any $w \in S^{n-1}$,

$$\frac{\langle w, \mathcal{T}_\pm(0) w\rangle}{1 - t\langle w, \mathcal{T}_\pm(0) w\rangle} \leq \langle w, \mathcal{T}_\pm(t) w\rangle \quad \forall\, t \in [0, \tau], \tag{4.7}$$

and

$$\sum_{i=1}^n \frac{\lambda_i}{1 - \lambda_i t} \leq \mathrm{Tr}[\mathcal{T}_\pm(t)] \quad \textit{where } \{\lambda_i\}_{i=1}^n \textit{are the eigenvalues of } \mathcal{T}_\pm(0). \tag{4.8}$$

Furthermore, the matrix $\int_0^t \mathcal{T}_\pm(s) \, \mathrm{d}s$ *is matrix displacement convex, that is, for* $w \in S^{n-1}$, *the function* $\mathrm{c}_w : [0, \tau] \to \mathbb{R}$ *given by*

$$\mathrm{c}_w(t) = \exp\left[-\int_0^t \langle w, \mathcal{T}_\pm(s) w\rangle \, \mathrm{d}s\right] \textit{ is concave}. \tag{4.9}$$

Consequently, for every $t \in [0, \tau]$ *and* $w \in S^{n-1}$,

$$-\frac{1}{t} \leq \langle w, \mathcal{T}_\pm(t) w\rangle \leq \frac{1}{\tau - t}, \tag{4.10}$$

and

$$-\sum_{i=1}^n \log(1 - \tau\lambda_i) \leq \int_0^\tau \mathrm{Tr}[\mathcal{T}_\pm(t)] \, \mathrm{d}t \quad \textit{where } \{\lambda_i\}_{i=1}^n \textit{ are the eigenvalues of } \mathcal{T}_\pm(0). \tag{4.11}$$

The implications of Theorem 4.8 to intrinsic dimensional functional inequalities will be derived in Section 5.

The next result is analogous to Theorem 4.8 but applies to $\mathcal{S}(t)$ rather than $\mathcal{T}_\pm(t)$. Its implications to intrinsic dimensional functional inequalities will also be derived in Section 5.

Theorem 4.9. (Differential inequalities and matrix displacement convexity for $\mathcal{S}(t)$) *Suppose* $(\rho_t, \theta_t)_{t\in[0,\tau]}$ *is a nice flow satisfying (3.3)-(3.4) with* $\sigma \in \mathbb{R}_{\geq 0}$, $f'(r) \geq 0$ *for every* $r \in \mathbb{R}_{\geq 0}$, *and* $\int \{\nabla^2 U_t - \nabla^2 W * \rho_t\} \, \mathrm{d}\rho_t \succeq 0$ *for every* $t \in [0, \tau]$. *Then,*

$$\partial_t \mathcal{S}(t) \succeq \mathcal{S}^2(t) + \frac{\sigma^2}{4} \mathcal{I}^2(t) + \int_\Omega \nabla^2 U_t \, \mathrm{d}\rho_t + \int_\Omega (-\nabla^2 W) * \rho_t \, \mathrm{d}\rho_t$$

$$+\int_\Omega f'(\rho_t)\frac{(\nabla\rho_t)^{\otimes 2}}{\rho_t}\,\mathrm{d}\rho_t \succeq \mathcal{S}^2(t) \succeq 0. \tag{4.12}$$

Consequently, for any $w \in S^{n-1}$,

$$\frac{\langle w, \mathcal{S}(0)w\rangle}{1-t\langle w, \mathcal{S}(0)w\rangle} \le \langle w, \mathcal{S}(t)w\rangle \quad \forall\, t \in [0,\tau], \tag{4.13}$$

and

$$\sum_{i=1}^n \frac{\lambda_i}{1-\lambda_i t} \le S(t) \quad \textit{where } \{\lambda_i\}_{i=1}^n \textit{ are the eigenvalues of } \mathcal{S}(0). \tag{4.14}$$

Furthermore, the matrix $\mathcal{E}(t)$ is matrix displacement convex, that is, for any $w \in S^{n-1}$, the function $\mathrm{c}_w : [0,\tau] \to \mathbb{R}$ given by

$$\mathrm{c}_w(t) = e^{-\langle w, \mathcal{E}(t)w\rangle} \textit{ is concave}. \tag{4.15}$$

Consequently, for every $t \in [0,\tau]$ and $w \in S^{n-1}$,

$$-\frac{1}{t} \le \langle w, \mathcal{S}(t)w\rangle \le \frac{1}{\tau - t}, \tag{4.16}$$

and

$$-\sum_{i=1}^n \log(1-\tau\lambda_i) \le \mathrm{Tr}[\mathcal{E}(\tau)] = E(\tau) - E(0) \quad \textit{where } \{\lambda_i\}_{i=1}^n \textit{ are the eigenvalues of } \mathcal{S}(0). \tag{4.17}$$

Remark 4.10. (Quantum drift-diffusion) As mentioned in Remark 1.8, the quantum drift-diffusion model is given by the $(\rho_t, \theta_t)_{t\in[0,\tau]}$ satisfying

$$\begin{cases} \partial_t\rho_t + \nabla\cdot(\rho_t\nabla\theta_t) = 0, \\ \theta_t - 2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}} = 0. \end{cases}$$

In order to compute the derivatives of $E(t)$ it turns out to be convenient to use the identity

$$\nabla\cdot(\rho_t\nabla\theta_t) = \sum_{i,j}\partial_{ij}^2(\rho_t\partial_{ij}^2\log\rho_t).$$

Then, the continuity equation implies (analogous to the proof of Lemma 3.1) that entropy decreases along the flow (ρ_t) since

$$\partial_t E(t) = -\int \mathrm{Tr}[(\nabla^2\log\rho_t)^2]\,\mathrm{d}\rho_t \le 0.$$

For the computation of $\partial_t\mathcal{I}(t)$ apply Lemma 3.2 to write

$$\partial_t\mathcal{I}(t) = \int \left\{\nabla^2\left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right]\nabla^2\log\rho_t + \nabla^2\log\rho_t\nabla^2\left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right]\right\}\mathrm{d}\rho_t,$$

and use

$$\nabla^2 \log \rho_t = 2\frac{\nabla^2 \rho_t^{1/2}}{\rho_t^{1/2}} - 2(\nabla \log \rho_t^{1/2})^{\otimes 2}$$

to get

$$\begin{aligned}\int \nabla^2 \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right] \nabla^2 \log \rho_t \, \mathrm{d}\rho_t &= \int \nabla^2 \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right] 2\frac{\nabla^2 \rho_t^{1/2}}{\rho_t^{1/2}} \, \mathrm{d}\rho_t \\ &\quad - 2\int \nabla^2 \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right] (\nabla \log \rho_t^{1/2})^{\otimes 2} \, \mathrm{d}\rho_t \\ &=: A + B.\end{aligned}$$

Integration by parts shows that

$$\begin{aligned}A_{ij} = &-\int \partial_i \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right] \partial_j \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right] \rho_t \\ &- 2\int \partial_i \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right] \left\{(\nabla^2 \rho_t^{1/2} \nabla \rho_t^{1/2})_j + \Delta\rho_t^{1/2} \partial_j \rho_t^{1/2}\right\},\end{aligned}$$

and

$$B_{ij} = 2\int \partial_i \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right] \left\{\Delta\rho_t^{1/2} \partial_j \rho_t^{1/2} + (\nabla^2 \rho_t^{1/2} \nabla \rho_t^{1/2})_j\right\}.$$

It follows that

$$A_{ij} + B_{ij} = -\int \partial_i \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right] \partial_j \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right] \rho_t$$

and hence

$$\partial_t \mathcal{I}(t) = -2\int \left(\nabla \left[2\frac{\Delta\rho_t^{1/2}}{\rho_t^{1/2}}\right]\right)^{\otimes 2} \mathrm{d}\rho_t \preceq 0,$$

which establishes the monotonicity of the Fisher information *matrix* along the quantum drift-diffusion flow.

5. Intrinsic Dimensional Functional Inequalities

In this section Theorems 4.8 and 4.9 will be used to derive intrinsic dimensional functional inequalities. When the boundary conditions of (1.1) correspond to the planning problem, i.e., $(\rho_0, \rho_\tau) = (\mu_a, \mu_z)$ for densities μ_a, μ_z over Ω, the time symmetry of the problem can be used:

Remark 5.1. (Time symmetry) The variational problem of (1.7) with the boundary conditions (μ_a, μ_z) is time-symmetric. Consequently, if (ρ_t, θ_t) is the optimal flow with boundary conditions (μ_a, μ_z), then the optimal flow with boundary conditions (μ_z, μ_a) is $(\tilde{\rho}_t, \tilde{\theta}_t)$ where $\tilde{\rho}_t := \rho_{\tau-t}$ and $\tilde{\theta}_t := -\theta_{\tau-t}$. Hence, the matrices $\tilde{\mathcal{T}}_\pm, \tilde{\mathcal{S}}, \tilde{\mathcal{I}}$ associated with $(\tilde{\rho}_t, \tilde{\theta}_t)$ satisfy

$$\tilde{\mathcal{S}}(t) = -\mathcal{S}(\tau - t), \quad \tilde{\mathcal{I}}(t) = \mathcal{I}(\tau - t), \quad \tilde{\mathcal{T}}_\pm(t) = -\mathcal{T}_\mp(\tau - t)$$

which implies

$$\partial_t \tilde{\mathcal{T}}_\pm(t) \succeq \tilde{\mathcal{T}}_\pm^2(t), \qquad \partial_t \tilde{\mathcal{S}}(t) \succeq \tilde{\mathcal{S}}^2(t).$$

The first intrinsic dimensional functional inequality describes the growth of the entropy along the flow.

Theorem 5.2. (Entropy growth) *Suppose $(\rho_t, \theta_t)_{t\in[0,\tau]}$ is a nice flow satisfying (3.3–3.4) with $\sigma \in \mathbb{R}_{\geq 0}$, $f'(r) \geq 0$ for every $r \in \mathbb{R}_{\geq 0}$, and $\int\{\nabla^2 U_t - \nabla^2 W * \rho_t\}\,\mathrm{d}\rho_t \succeq 0$ for every $t \in [0, \tau]$. Then,*

$$-\sum_{i=1}^{n} \log\left(1 - \tau\lambda_i(0)\right) \leq E(\tau) - E(0) \tag{5.1}$$

where $\{\lambda_i(t)\}_{i=1}^n$ are the eigenvalues of $\mathcal{S}(t)$. Furthermore, under the planning problem boundary conditions (μ_a, μ_z),

$$-\sum_{i=1}^{n} \log\left(1 - \tau\lambda_i(0)\right) \leq E(\tau) - E(0) \leq \sum_{i=1}^{n} \log\left(1 + \tau\lambda_i(\tau)\right). \tag{5.2}$$

Proof. Inequality (5.1) and the left-hand side of inequality (5.2) is simply (4.17). To get the right-hand side of inequality (5.2), Remark 5.1 is used as follows. By (4.14),

$$\sum_{i=1}^{n} \frac{\tilde{\lambda}_i(0)}{1 - \tilde{\lambda}_i(0)t} \leq \tilde{S}(t) \tag{5.3}$$

where $\{\lambda_i(0)\}_{i=1}^n$ are the eigenvalues of $\tilde{\mathcal{S}}(0) = -\mathcal{S}(\tau)$. By (4.16), $\tilde{\lambda}_i(0) \leq \frac{1}{\tau} \leq \frac{1}{t}$, which implies $1 + t\lambda_i(\tau) = 1 - t\tilde{\lambda}_i(0) \geq 0$. Hence, the integral over $t \in [0, \tau]$ on the left-hand side of (5.3) is equal to $-\sum_{i=1}^n \log\left(1 + \tau\lambda_i(\tau)\right)$. The proof is complete by noting that $\int_0^\tau \tilde{S}(t)\,\mathrm{d}t = -\int_0^\tau S(t)\,\mathrm{d}t$. □

5.1. Viscous Flows

In this section the flow is assumed to be viscous, that is, $\sigma \neq 0$. The first result pertains to the *turnpike property* of a viscous flow $(\rho_t, \theta_t)_{t\in[0,\tau]}$ satisfying (3.3, 3.4). The reader is referred to [9,17,20] for a discussion of the turnpike property, but in this context it suffices to state the formulation of the turnpike property by

Clerc-Conforti-Gentil [9, Theorem 4.9]. They showed that when the flow (ρ_t, θ_t) is the *entropic interpolation flow*,

$$I(t) \le \frac{n}{2t} + \frac{n}{2(\tau - t)}. \tag{5.4}$$

The next result improves on (5.4) by replacing the scalar inequality for the Fisher information by a matrix inequality for the Fisher information matrix, thus disposing of the the ambient dimension n. In addition, the result applies to settings beyond entropic interpolation.

Theorem 5.3. (Turnpike properties via dissipation of Fisher information) *Suppose* $(\rho_t, \theta_t)_{t\in[0,\tau]}$ *is a nice flow satisfying* (3.3–3.4) *with* $\sigma \in \mathbb{R}_{\ge 0}$, $f'(r) \ge 0$ *for every* $r \in \mathbb{R}_{\ge 0}$, *and* $\int \{\nabla^2 U_t - \nabla^2 W * \rho_t\} \, \mathrm{d}\rho_t \succeq 0$ *for every* $t \in [0, \tau]$. *Then,*

$$\mathcal{I}(t) \preceq \frac{1}{\sigma}\left(\frac{1}{t} + \frac{1}{\tau - t}\right) \mathrm{Id}\,.$$

Proof. The proof is analogous to proof of [9, Theorem 4.9]. By (4.10),

$$\mathcal{T}_+(t) \preceq \frac{1}{\tau - t}\,\mathrm{Id} \quad \text{and} \quad -\mathcal{T}_-(t) \preceq \frac{1}{t}\,\mathrm{Id}$$

so

$$\sigma \mathcal{I}(t) = \mathcal{T}_+(t) - \mathcal{T}_-(t) \preceq \left(\frac{1}{t} + \frac{1}{\tau - t}\right) \mathrm{Id}\,.$$

□

The remainder of the results of this section are restricted to flows of the form

$$\begin{cases} \partial_t \rho_t + \nabla\cdot(\rho_t \nabla\theta_t) = 0, \quad \rho_0 = \mu_a, \;\; \rho_\tau = \mu_z, \\ \partial_t \theta_t + \frac{1}{2}|\nabla\theta_t|^2 + \frac{\sigma^2}{8}\left[|\nabla \log \rho_t|^2 + 2\Delta \log \rho_t\right] + U - f(\rho_t) = 0, \quad \sigma \ne 0, \end{cases} \tag{5.5}$$

so that the potential assumed to be independent of time, i.e.,

$$U_t = U \quad \forall\, t \in [0, \tau],$$

and the interaction term W is assumed to vanish. Under these assumptions an *energy* can be defined which is constant along the flow. Begin by defining

$$\begin{aligned} O(t) &:= \int_\Omega H(x, \nabla\theta_t(x))\, \mathrm{d}\rho_t(x) - \frac{\sigma^2}{8} I(t) - \int_\Omega F(\rho_t)\, \mathrm{d}\rho_t \\ &= \int_0^\tau \int_\Omega \left[\frac{1}{2}|\nabla\theta_t|^2 + U - \frac{\sigma^2}{8}|\nabla \log \rho_t|^2 - F(\rho_t)\right] \mathrm{d}\rho_t\, \mathrm{d}t \end{aligned} \tag{5.6}$$

where the Hamiltonian H is given by

$$H(x, p) := \frac{|p|^2}{2} + U(x)$$

and where F satisfies

$$f(r) = F(r) + r F'(r).$$

Lemma 5.4. (Preservation of energy) *Suppose* $(\rho_t, \theta_t)_{t\in[0,\tau]}$ *is a nice flow satisfying* (5.5). *Then, the energy* $O(t)$ *is constant along* $t \in [0, \tau]$.

Proof. First note that

$$\int_\Omega H(x, \nabla\theta_t(x))\,\mathrm{d}\rho_t(x) = \frac{1}{2}V(t) + \int_\Omega U\,\mathrm{d}\rho_t.$$

By Corollary 3.6,

$$\partial_t \frac{1}{2}V(t) = \frac{\sigma^2}{8}\partial_t I(t) - \int \{U - f(\rho_t)\}\,\partial_t\rho_t$$

while, on the other hand,

$$\partial_t \int F(\rho_t)\,\mathrm{d}\rho_t = \int \rho_t F'(\rho_t)\partial_t\rho_t + \int F(\rho_t)\partial_t\rho_t = \int f(\rho_t)\partial_t\rho_t.$$

It follows that $\partial_t O(t) = 0$. □

In light of Lemma 5.4 define

$$O_\tau := O(t) \quad \text{for any } t \in [0, \tau]. \tag{5.7}$$

Next define the *cost*

$$\begin{aligned} C_\tau &:= \int_0^\tau \int_\Omega \left[L(x, \nabla\theta_t(x)) + \frac{\sigma^2}{8}|\nabla \log \rho_t(x)|^2 + F(\rho_t(x))\right]\,\mathrm{d}\rho_t(x)\,\mathrm{d}t \\ &= \int_0^\tau \int_\Omega \left[\frac{1}{2}|\nabla\theta_t|^2 - U + \frac{\sigma^2}{8}|\nabla \log \rho_t|^2 + F(\rho_t)\right]\,\mathrm{d}\rho_t\,\mathrm{d}t, \end{aligned} \tag{5.8}$$

where the Lagrangian L is given by

$$L(x, w) = \frac{|w|^2}{2} - U(x).$$

The relation between the cost C_τ, the entropy E, the energy O_τ, and the the matrix $\mathcal{T}_\pm$ is captured by the following lemma:

Lemma 5.5. *Suppose* $(\rho_t, \theta_t)_{t\in[0,\tau]}$ *is a nice flow satisfying* (5.5). *Then,*

$$\frac{\sigma}{2}\int_0^\tau \mathrm{Tr}[\mathcal{T}_\pm(t)]\,\mathrm{d}t = \frac{\sigma}{2}[E(\tau) - E(0)] \pm [C_\tau - \tau O_\tau] \mp 2\int_0^\tau \int_\Omega [F(\rho_t) - U]\,\mathrm{d}\rho_t\,\mathrm{d}t.$$

Proof. By definition

$$C_\tau - \tau O_\tau = \frac{\sigma^2}{4}\int_0^\tau I(t)\,\mathrm{d}t + 2\int_0^\tau \int [F(\rho_t) - U]\,\mathrm{d}\rho_t\,\mathrm{d}t,$$

so

$$\int_0^\tau \mathrm{Tr}[\mathcal{T}_\pm(t)]\,\mathrm{d}t = \int_0^\tau S(t)\,\mathrm{d}t \pm \frac{\sigma}{2}\int_0^\tau I(t)\,\mathrm{d}t = E(\tau) - E(0) \pm \frac{\sigma}{2}\int_0^\tau I(t)\,\mathrm{d}t$$

$$= \frac{2}{\sigma}\left\{\frac{\sigma}{2}[E(\tau) - E(0)] \pm \frac{\sigma^2}{4}\int_0^\tau I(t)\,\mathrm{d}t\right\}$$
$$= \frac{2}{\sigma}\left\{\frac{\sigma}{2}[E(\tau) - E(0)] \pm [C_\tau - \tau O_\tau] \mp 2\int_0^\tau \int [F(\rho_t) - U]\,\mathrm{d}\rho_t\,\mathrm{d}t\right\}.$$

□

With Lemma 5.5 in hand the following intrinsic dimensional functional inequality for the combination of cost, entropy, and energy can be proved.

Theorem 5.6. (Cost inequalities) *Suppose* $(\rho_t, \theta_t)_{t\in[0,\tau]}$ *is a nice flow satisfying* (5.5) *with* $\sigma > 0$, $f'(r) \geq 0$ *for every* $r \in \mathbb{R}_{\geq 0}$, *and* $\int \nabla^2 U\,\mathrm{d}\rho_t \succeq 0$ *for every* $t \in [0, \tau]$. *Then,*

$$-\frac{\sigma}{2}\sum_{i=1}^n \log(1 - \tau\lambda_i(0))$$
$$\leq \frac{\sigma}{2}[E(\tau) - E(0)] \pm [C_\tau - \tau O_\tau] \mp 2\int_0^\tau \int_\Omega [F(\rho_t) - U]\,\mathrm{d}\rho_t\,\mathrm{d}t \quad (5.9)$$

where $\{\lambda_i(t)\}_{i=1}^n$ *are the eigenvalues of* $\mathcal{T}_\pm(t)$. *Furthermore, under the planning problem boundary conditions* (μ_a, μ_z),

$$-\frac{\sigma}{2}\sum_{i=1}^n \log(1 - \tau\lambda_i(0))$$
$$\leq \frac{\sigma}{2}[E(\tau) - E(0)] \pm [C_\tau - \tau O_\tau] \mp 2\int_0^\tau \int_\Omega [F(\rho_t) - U]\,\mathrm{d}\rho_t\,\mathrm{d}t$$
$$\leq \frac{\sigma}{2}\sum_{i=1}^n \log(1 + \tau\lambda_i(\tau)), \quad (5.10)$$

where $\{\lambda_i(t)\}_{i=1}^n$ *are the eigenvalues of* $\mathcal{T}_\pm(t)$.

Proof. Inequality (5.9) and the left-hand side of inequality (5.10) follows from (4.11) and Lemma 5.5. For the right-hand side of inequality (5.10), use (4.8) and Remark 5.1 to get

$$\sum_{i=1}^n \frac{\lambda_i(0)}{1 - t\lambda_i(0)} \leq \mathrm{Tr}[\mathcal{T}_\pm(t)] \quad \text{and} \quad \sum_{i=1}^n \frac{\tilde{\lambda}_i(0)}{1 - t\tilde{\lambda}_i(0)} \leq \mathrm{Tr}[\tilde{\mathcal{T}}_\mp(t)] = -\mathrm{Tr}[\mathcal{T}_\pm(\tau - t)],$$

where $\{\lambda_i(0)\}_{i=1}^n$ are the eigenvalues of $\mathcal{T}_\pm(0)$ and $\{\tilde{\lambda}_i(0)\}_{i=1}^n$ are the eigenvalues of $\tilde{\mathcal{T}}_\mp(0) = -\mathcal{T}_\pm(\tau)$. Hence,

$$\sum_{i=1}^n \frac{\lambda_i(0)}{1 - t\lambda_i(0)} \leq \mathrm{Tr}[\mathcal{T}_\pm(t)] \quad \text{and} \quad \mathrm{Tr}[\mathcal{T}_\pm(\tau - t)] \leq \sum_{i=1}^n \frac{\lambda_i(\tau)}{1 + t\lambda_i(\tau)}.$$

Using

$$\int_0^\tau \mathrm{Tr}[\mathcal{T}_\pm(t)]\,\mathrm{d}t = \int_0^\tau \mathrm{Tr}[\mathcal{T}_\pm(\tau - t)]\,\mathrm{d}t,$$

and Lemma 5.5, gives

$$\begin{aligned}
&\frac{\sigma}{2}\sum_{i=1}^n \int_0^\tau \frac{\lambda_i(0)}{1-t\lambda_i(0)}\,\mathrm{d}t \\
&\quad\le \frac{\sigma}{2}[E(\mu_z)-E(\mu_a)] \pm [C_\tau - \tau O_\tau] \mp 2\int_0^\tau \int_\Omega [F(\rho_t)-U]\,\mathrm{d}\rho_t\,\mathrm{d}t \\
&\quad\le \frac{\sigma}{2}\sum_{i=1}^n \int_0^\tau \frac{\lambda_i(\tau)}{1+t\lambda_i(\tau)}\,\mathrm{d}t.
\end{aligned} \tag{5.11}$$

The next step is to note that by (4.10), $-\frac{1}{t} \le -\frac{1}{\tau} \le \lambda_i(\tau)$, which implies that $0 \le 1 + t\lambda_i(\tau)$. Hence, the integral over $t \in [0, \tau]$ of the right-hand side of (5.11) is equal to $\frac{\sigma}{2}\sum_{i=1}^n \log(1+\tau\lambda_i(\tau))$. Similarly, (4.10) gives $\lambda_i(0) \le \frac{1}{\tau} \le \frac{1}{t}$, which implies $1 - t\lambda_i(0) \ge 0$. Hence, the integral over $t \in [0, \tau]$ of the left-hand side of (5.11) is equal to $-\frac{\sigma}{2}\sum_{i=1}^n \log(1-\tau\lambda_i(0))$. □

Remark 5.7. (Intrinsic dimensional local logarithmic Sobolev inequalities) The inequalities of Theorem 5.6 can be viewed as a generalization of the intrinsic dimensional local logarithmic Sobolev inequalities for the Euclidean heat semigroup [16, Equations (29) and (30)]. In particular, consider the entropic interpolation setting with $U_t = W = f = 0$ and $\sigma = 1$. Fix $x \in \mathbb{R}^n$ and take $\mu_a := \delta_x$ and $\mathrm{d}\mu_z(y) := h(y)p_\tau(x, y)$ where $h \ge 0$ and p_τ is the heat kernel associated with the Euclidean heat semigroup P_τ. Then, using the explicit expression for (ρ_t) in [10, Remark 4.1], one can formally derive the inequalities of [16, Equations (29) and (30)] for the function h evaluated at x:

$$\mathrm{P}_\tau(h\log h) - \mathrm{P}_\tau h \log \mathrm{P}_\tau h \le \frac{\tau}{2}\mathrm{P}_\tau(\Delta h) + \frac{\mathrm{P}_\tau h}{2}\log\det\left\{\mathrm{Id} - \tau\frac{\mathrm{P}_\tau(h\nabla^2\log h)}{\mathrm{P}_\tau h}\right\}, \tag{5.12}$$

$$\frac{\tau}{2}\mathrm{P}_\tau(\Delta h) - \frac{1}{2}\mathrm{P}_\tau h\log\det(\mathrm{Id} + \tau\nabla^2\log \mathrm{P}_\tau h) \le \mathrm{P}_\tau(h\log h) - \mathrm{P}_\tau h\log \mathrm{P}_\tau h. \tag{5.13}$$

In [10], this entropic interpolation flow was used to prove the *dimensional* local log-Sobolev inequalities which are weaker than the *intrinsic dimensional* local log-Sobolev inequalities (5.12)-(5.13).

5.2. Entropic Interpolation Flows

In this section the flow will be assumed to be the entropic interpolation flow,

$$\begin{cases}\partial_t\rho_t + \nabla\cdot(\rho_t\nabla\theta_t) = 0, \quad \rho_0 = \mu_a,\ \rho_\tau = \mu_z, \\ \partial_t\theta_t + \frac{1}{2}|\nabla\theta_t|^2 + \frac{\sigma^2}{8}\left[|\nabla\log\rho_t|^2 + 2\Delta\log\rho_t\right] = 0, \quad \sigma \in \mathbb{R}_{\ge 0},\end{cases} \tag{5.14}$$

over the domain $\Omega = \mathbb{R}^n$. In contrast to Section 5.1 where the energy preserved along the flow is a *scalar* quantity, in the entropic interpolation setting a *matrix* quantity is preserved as well. Define the *matrix energy* as

$$\mathcal{O}(t) := \frac{1}{2}\mathcal{V}(t) - \frac{\sigma^2}{8}\mathcal{I}(t). \tag{5.15}$$

Lemma 5.8. (Preservation of matrix energy) *Suppose that* (ρ_t, θ_t) *is a nice flow satisfying* (5.14). *Then, the energy matrix* $\mathcal{O}(t)$ *is constant along* $t \in [0, \tau]$.

Proof. The proof is immediate by Lemma 3.5 which states

$$\partial_t \mathcal{V}(t) = \frac{\sigma^2}{4}\partial_t \mathcal{I}(t).$$

□

In light of Lemma 5.8 set

$$\mathcal{O}_\tau(\mu_a, \mu_z) := \mathcal{O}(t) \quad \forall\, t \in [0, \tau]. \tag{5.16}$$

In the setting of entropic interpolation a *matrix* cost can be defined as

$$\begin{aligned} \mathcal{C}_\tau(\mu_a, \mu_z) &= \int_0^\tau \int_{\mathbb{R}^n} \left[\frac{1}{2}(\nabla\theta_t)^{\otimes 2} + \frac{\sigma^2}{8}(\nabla \log \rho_t)^{\otimes 2}\right] \mathrm{d}\rho_t \, \mathrm{d}t \\ &= \int_0^\tau \left[\frac{1}{2}\mathcal{V}(t) + \frac{\sigma^2}{8}\mathcal{I}(t)\right] \mathrm{d}t \end{aligned} \tag{5.17}$$

with its trace

$$C_\tau(\mu_a, \mu_z) := \mathrm{Tr}[\mathcal{C}_\tau(\mu_a, \mu_z)]. \tag{5.18}$$

The relation between the matrix cost and the matrix energy is captured by the following lemma, but first a remark is in order.

Remark 5.9. (Time scaling) Define

$$\mathcal{A}_\tau(\mu_a, \mu_z) := \inf_{(\rho_t, v_t)_{t\in[0,1]}} \int_0^1 \int_{\mathbb{R}^n} \left[\frac{1}{2}(v_t)^{\otimes 2} + \tau^2 \frac{\sigma^2}{8}(\nabla \log \rho_t)^{\otimes 2}\right] \mathrm{d}\rho_t \, \mathrm{d}t, \tag{5.19}$$

where the minimum is over flows satisfying the continuity equations with boundary conditions:

$$\partial_t \rho_t + \nabla\cdot(\rho_t v_t) = 0 \quad \forall\, t \in [0, \tau], \quad \rho_0 = \mu_a, \ \rho_1 = \mu_z.$$

Then, the Euler-Lagrange equations of (5.19) are

$$\begin{cases} \partial_t \tilde{\rho}_t + \nabla\cdot(\tilde{\rho}_t \nabla\tilde{\theta}_t) = 0, \quad \rho_0 = \mu_a, \ \rho_\tau = \mu_z, \\ \partial_t \tilde{\theta}_t + \frac{1}{2}|\nabla\tilde{\theta}_t|^2 + \tau^2 \frac{\sigma^2}{8}\left[|\nabla \log \tilde{\rho}_t|^2 + 2\Delta \log \tilde{\rho}_t\right] = 0, \end{cases} \tag{5.20}$$

and it is easy to see that $(\tilde{\rho}_t, \tilde{\theta}_t) = (\rho_{\tau t}, \tau\theta_{\tau t})$ where (ρ_t, θ_t) satisfy (5.14).

Lemma 5.10. *Suppose that* (ρ_t, θ_t) *is a nice flow satisfying* (5.14). *Assume that* $\tau \mapsto \partial_\tau \mathcal{C}_\tau(\mu_a, \mu_z)$ *is differentiable. Then,*

$$\partial_\tau \mathcal{C}_\tau(\mu_a, \mu_z) = -\mathcal{O}_\tau(\mu_a, \mu_z).$$

Proof. By the envelope theorem [36] and Remark 5.9,

$$\partial_\tau \mathcal{A}_\tau(\mu_a, \mu_z) = \int_0^1 \int_{\mathbb{R}^n} \tau \frac{\sigma^2}{4} (\nabla \log \tilde{\rho}_t)^{\otimes 2} \, \mathrm{d}\tilde{\rho}_t$$

where $(\tilde{\rho}_t)_{t\in[0,1]}$ is the optimal flow in $\mathcal{A}_\tau(\mu_a, \mu_z)$ given by $\tilde{\rho}_t = \rho_{\tau t}$. Hence, by the change of variables $t \mapsto \frac{t}{\tau}$,

$$\partial_\tau \mathcal{A}_\tau(\mu_a, \mu_z) = \int_0^\tau \int_{\mathbb{R}^n} \frac{\sigma^2}{4} (\nabla \log \rho_t)^{\otimes 2} \, \mathrm{d}\rho_t = \frac{\sigma^2}{4} \int_0^\tau \mathcal{I}(t) \, \mathrm{d}t = \mathcal{C}_\tau(\mu_a, \mu_z) \\ -\tau \mathcal{O}_\tau(\mu_a, \mu_z).$$

On the other hand, changing variables $t \mapsto \tau t$ shows that

$$\tau \mathcal{C}_\tau(\mu_a, \mu_z) = \mathcal{A}_\tau(\mu_a, \mu_z)$$

so it follows that

$$\mathcal{C}_\tau(\mu_a, \mu_z) - \tau \mathcal{O}_\tau(\mu_a, \mu_z) = \partial_\tau \mathcal{A}_\tau(\mu_a, \mu_z) = \partial_\tau[\tau \mathcal{C}_\tau(\mu_a, \mu_z)] = \mathcal{C}_\tau(\mu_a, \mu_z) \\ +\tau \partial_\tau \mathcal{C}_\tau(\mu_a, \mu_z),$$

which implies the result. □

To set up the first main result of this section recall the definition of the matrix entropy (2.8) and define

$$\mathcal{E}_\tau(\mu_a, \mu_z) := \mathcal{E}(\tau) \tag{5.21}$$

so that

$$\mathrm{Tr}[\mathcal{E}_\tau(\mu_a, \mu_z)] = E(\mu_z) - E(\mu_a). \tag{5.22}$$

The following result is the intrinsic dimensional improvement of [9, Theorem 4.6] by Clerc-Conforti-Gentil, and is proved similarly.

Theorem 5.11. (Large time asymptotics for cost and energy) *Suppose that* (ρ_t, θ_t) *is a nice flow satisfying* (5.14). *Then,*

$$-\mathcal{O}_\tau(\mu_a, \mu_z) \preceq \frac{\sigma}{2} \frac{1}{\tau} \, \mathrm{Id}, \tag{5.23}$$

and, consequently,

$$\mathcal{C}_\tau(\mu_a, \mu_z) \preceq \mathcal{C}_1(\mu_a, \mu_z) + \left(\frac{\sigma}{2} \log \tau\right) \mathrm{Id}\,. \tag{5.24}$$

Moreover,

$$\int \left[\nabla \theta_t + \frac{\sigma}{2} \nabla \log \rho_t\right]^{\otimes 2} \mathrm{d}\rho_t \preceq \frac{\sigma \mathcal{E}_\tau(\mu_a, \mu_z) + 2\mathcal{C}_1(\mu_a, \mu_z) + (\sigma \log \tau) \, \mathrm{Id}}{\tau - t}. \tag{5.25}$$

Proof. To prove (5.23) note that

$$0 \preceq \int \left[\nabla\theta_t + \frac{\sigma}{2}\nabla \log \rho_t\right]^{\otimes 2} \mathrm{d}\rho_t = \mathcal{V}(t) + \sigma \mathcal{S}(t) + \frac{\sigma^2}{4}\mathcal{I}(t) \tag{5.26}$$

which implies

$$-\frac{1}{2}\mathcal{V}(t) - \frac{1}{2}\frac{\sigma^2}{4}\mathcal{I}(t) \preceq \frac{1}{2}\sigma \mathcal{S}(t). \tag{5.27}$$

Adding $\frac{\sigma^2}{4}\mathcal{I}(t)$ to both sides of (5.27) gives

$$\frac{\sigma^2}{8}\mathcal{I}(t) - \frac{1}{2}\mathcal{V}(t) \preceq \frac{1}{2}\sigma\mathcal{S}(t) + \frac{\sigma^2}{4}\mathcal{I}(t) = \frac{\sigma}{2}\mathcal{T}_+(t)$$

which is equivalent to

$$-\mathcal{O}_\tau(\mu_a, \mu_z) \preceq \frac{\sigma}{2}\mathcal{T}_+(t).$$

Taking $t = 0$ gives

$$-\mathcal{O}_\tau(\mu_a, \mu_z) \preceq \frac{\sigma}{2}\mathcal{T}_+(0) \preceq \frac{\sigma}{2}\frac{1}{\tau}\,\mathrm{Id}$$

where the last inequality holds by (4.10). This establishes (5.23).

To prove (5.24) integrate (5.23) over t from 0 to τ and use Lemma 5.10 to get

$$\begin{aligned}\mathcal{C}_\tau(\mu_a, \mu_z) - \mathcal{C}_1(\mu_a, \mu_z) &= \int_1^\tau \partial_s \mathcal{C}_\tau(\mu_a, \mu_z)\,\mathrm{d}s = \int_1^\tau -\mathcal{O}_\tau(\mu_a, \mu_z)\,\mathrm{d}s \\ &\preceq \frac{\sigma}{2}\left(\int_1^\tau \frac{1}{s}\,\mathrm{d}s\right)\mathrm{Id} = \left(\frac{\sigma}{2}\log\tau\right)\mathrm{Id}\,.\end{aligned}$$

Finally, to prove (5.25) note that by (5.26), Lemma 3.5, and Theorem 4.1,

$$\partial_t \int \left[\nabla\theta_t + \frac{\sigma}{2}\nabla \log \rho_t\right]^{\otimes 2} \mathrm{d}\rho_t = \sigma\partial_t\left[\mathcal{S}(t) + \frac{\sigma}{2}\mathcal{I}(t)\right] = \sigma\partial_t\mathcal{T}_+(t) \succeq \sigma\mathcal{T}_+^2(t) \succeq 0,$$

which shows that $[0, \tau] \ni s \mapsto \int \left[\nabla\theta_s + \frac{\sigma}{2}\nabla \log \rho_s\right]^{\otimes 2} \mathrm{d}\rho_s$ is non-decreasing. Hence,

$$\begin{aligned}(\tau - t)\int \left[\nabla\theta_t + \frac{\sigma}{2}\nabla \log \rho_t\right]^{\otimes 2} \mathrm{d}\rho_t &\preceq \int_t^\tau \int \left[\nabla\theta_s + \frac{\sigma}{2}\nabla \log \rho_s\right]^{\otimes 2} \mathrm{d}\rho_s\,\mathrm{d}s \\ \preceq \int_0^\tau \int \left[\nabla\theta_s + \frac{\sigma}{2}\nabla \log \rho_s\right]^{\otimes 2} \mathrm{d}\rho_s\,\mathrm{d}s &= \int_0^\tau \left[\mathcal{V}(s) + \sigma\mathcal{S}(s) + \frac{\sigma^2}{4}\mathcal{I}(s)\right]\mathrm{d}s.\end{aligned}$$

Since

$$\int_0^\tau \left[\mathcal{V}(s) + \frac{\sigma^2}{4}\mathcal{I}(s)\right]\mathrm{d}s + \sigma\int_0^\tau \mathcal{S}(s)\,\mathrm{d}s = 2\mathcal{C}_\tau(\mu_a, \mu_z) + \sigma\mathcal{E}_\tau(\mu_a, \mu_z),$$

the bound (5.25) follows by applying (5.24). □

The next result is the intrinsic dimensional improvement of the evolution variational inequality for the entropic cost of Ripani [40, Corollary 11], and is proved similarly.

Theorem 5.12. (Evolution variational inequality) *Fix $\tau = 1$, $\sigma = \sqrt{2}$, and suppose that (ρ_t, θ_t) is a nice flow satisfying (5.14). Let (P_t) be the heat semigroup in $\mathbb{R}^n$ and suppose that $t \mapsto C_1(\mu_a, \mathrm{P}_t\mu_z)$ is differentiable. Then, for any normalized basis $\{w_i\}_{i=1}^n$ of $\mathbb{R}^n$ and fixed $t \in [0, 1]$,*

$$\partial_t C_1(\mu_a, \mathrm{P}_t\mu_z) \leq \frac{1}{2}\sum_{i=1}^n \left[1 - e^{\langle w_i, \mathcal{E}_1(\mu_a, \mathrm{P}_t\mu_z) w_i\rangle}\right].$$

Proof. Let $\{w_i\}_{i=1}^n$ be any normalized basis of $\mathbb{R}^n$ and fix $t \in [0, 1]$. Consider the entropic interpolation flow betwen μ_a and $\mathrm{P}_t\mu_z$ so, by (4.15), $\mathrm{c}_{w_i}(t) := e^{-\langle w_i, \mathcal{E}(t) w_i\rangle}$ is concave, and hence,

$$\partial_t \mathrm{c}_{w_i}(1) \leq \mathrm{c}_{w_i}(1) - \mathrm{c}_{w_i}(0). \tag{5.28}$$

Inequality (5.28) is equivalent to

$$-\mathrm{c}_{w_i}(1)\langle w_i, \mathcal{S}(1) w_i\rangle \leq \mathrm{c}_{w_i}(1) - 1,$$

which upon rearrangement gives

$$-\langle w_i, \mathcal{S}(1) w_i\rangle \leq 1 - (\mathrm{c}_{w_i}(1))^{-1}. \tag{5.29}$$

Using the definition of $\mathrm{c}_{w_i}(1)$, and summing over i in (5.29), gives

$$\begin{aligned} -\partial_t E(t)|_{t=1} &= -\operatorname{Tr}[\mathcal{S}(1)] = -\sum_{i=1}^n \langle w_i, \mathcal{S}(1) w_i\rangle \leq \sum_{i=1}^n \left[1 - e^{\langle w_i, \mathcal{E}(1) w_i\rangle}\right] \\ &= \sum_{i=1}^n \left[1 - e^{\langle w_i, \mathcal{E}_1(\mu_a, \mathrm{P}_t\mu_z) w_i\rangle}\right]. \end{aligned} \tag{5.30}$$

By [40, Theorem 9],

$$-\frac{1}{2}\partial_t E(t)|_{t=1} = \partial_t C_1(\mu_a, \mathrm{P}_t\mu_z)|_{t=0}$$

so (5.30) implies

$$\partial_t C_1(\mu_a, \mathrm{P}_t\mu_z)|_{t=0} \leq \frac{1}{2}\sum_{i=1}^n \left[1 - e^{\langle w_i, \mathcal{E}_1(\mu_a, \mathrm{P}_t\mu_z) w_i\rangle}\right]. \tag{5.31}$$

By the semigroup property, inequality (5.31) can be applied at any t to yield the result. □

Finally, the last result is the intrinsic dimensional improvement of the entropic cost contraction of Ripani [40, Corollary 13], and is proved similarly.

Theorem 5.13. (Contraction of entropy cost) *Fix* $\tau = 1$, $\sigma = \sqrt{2}$, *and suppose that* (ρ_t, θ_t) *is a nice flow satisfying* (5.14). *Let* (P_t) *be the heat semigroup in* $\mathbb{R}^n$ *and suppose that* $t \mapsto C_1(\mu_a, \mathrm{P}_t\mu_z)$ *is differentiable. Then,*

$$C_1(\mathrm{P}_\tau\mu_a, \mathrm{P}_\tau\mu_z) \le C_1(\mu_a, \mu_z) - \sum_{i=1}^n \int_0^\tau \sinh^2\left(\frac{\langle w_i, \mathcal{E}_1(\mathrm{P}_t\mu_a, \mathrm{P}_t\mu_z)w_i\rangle}{2}\right).$$

Proof. Fix $s \in [0, 1]$ and let $\{w_i\}_{i=1}^n$ be any normalized basis of $\mathbb{R}^n$. Applying Theorem 5.12 with $\mu_a \mapsto \mathrm{P}_s\mu_a$ gives

$$\partial_t C_1(\mathrm{P}_s\mu_a, \mathrm{P}_t\mu_z) \le \frac{1}{2}\sum_{i=1}^n \left[1 - e^{\langle w_i, \mathcal{E}_1(\mathrm{P}_s\mu_a, \mathrm{P}_t\mu_z)w_i\rangle}\right]. \tag{5.32}$$

On the other hand, by time symmetry (Remark 5.1),

$$\partial_t C_1(\mathrm{P}_t\mu_z, \mu_a) \le \frac{1}{2}\sum_{i=1}^n \left[1 - e^{\langle w_i, \mathcal{E}_1(\mu_a, \mathrm{P}_t\mu_z)w_i\rangle}\right],$$

and switching the roles of μ_a and μ_z thus gives

$$\partial_t C_1(\mathrm{P}_t\mu_a, \mu_z) \le \frac{1}{2}\sum_{i=1}^n \left[1 - e^{\langle w_i, \mathcal{E}_1(\mu_z, \mathrm{P}_t\mu_a)w_i\rangle}\right].$$

Taking $\mu_z \mapsto \mathrm{P}_s\mu_z$ yields

$$\partial_t C_1(\mathrm{P}_t\mu_a, \mathrm{P}_s\mu_z) \le \frac{1}{2}\sum_{i=1}^n \left[1 - e^{\langle w_i, \mathcal{E}_1(\mathrm{P}_s\mu_z, \mathrm{P}_t\mu_a)w_i\rangle}\right], \tag{5.33}$$

and adding (5.32) and (5.33) shows that

$$\begin{aligned}&\partial_t C_1(\mathrm{P}_s\mu_a, \mathrm{P}_t\mu_z) + \partial_t C_1(\mathrm{P}_t\mu_a, \mathrm{P}_s\mu_z)\\ &\quad\le \frac{1}{2}\sum_{i=1}^n \left[2 - e^{\langle w_i, \mathcal{E}_1(\mathrm{P}_s\mu_a, \mathrm{P}_t\mu_z)w_i\rangle} - e^{\langle w_i, \mathcal{E}_1(\mathrm{P}_s\mu_z, \mathrm{P}_t\mu_a)w_i\rangle}\right].\end{aligned}$$

Taking $s = t$ yields

$$\begin{aligned}\partial_t C_1(\mathrm{P}_t\mu_a, \mathrm{P}_t\mu_z) &\le \frac{1}{2}\sum_{i=1}^n \left[2 - e^{\langle w_i, \mathcal{E}_1(\mathrm{P}_t\mu_a, \mathrm{P}_t\mu_z)w_i\rangle} - e^{\langle w_i, \mathcal{E}_1(\mathrm{P}_t\mu_z, \mathrm{P}_t\mu_a)w_i\rangle}\right]\\ &= \sum_{i=1}^n \left[1 - \left\{\frac{e^{\langle w_i, \mathcal{E}_1(\mathrm{P}_t\mu_a, \mathrm{P}_t\mu_z)w_i\rangle} + e^{-\langle w_i, \mathcal{E}_1(\mathrm{P}_t\mu_a, \mathrm{P}_t\mu_z)w_i\rangle}}{2}\right\}\right],\end{aligned}$$

where time symmetry (Remark 5.1) was used to write $\tilde{S}(t) = -S(1-t)$ and hence $\mathcal{E}_1(\mathrm{P}_t\mu_z, \mathrm{P}_t\mu_a) = -\mathcal{E}_1(\mathrm{P}_t\mu_a, \mathrm{P}_t\mu_z)$. Integrating over t from 0 to 1, and using $\cosh r = \frac{e^r + e^{-r}}{2}$, gives

$$C_1(\mathrm{P}_\tau\mu_a, \mathrm{P}_\tau\mu_z) - C_1(\mu_a, \mu_z) = \sum_{i=1}^n \int_0^\tau [1 - \cosh(\langle w_i, \mathcal{E}_1(\mathrm{P}_t\mu_a, \mathrm{P}_t\mu_z)w_i\rangle)]\,\mathrm{d}t.$$

Finally, since $\sinh^2 r = \frac{\cosh(2r)-1}{2}$,

$$C_1(\mathrm{P}_\tau\mu_a, \mathrm{P}_\tau\mu_z) \leq C_1(\mu_a, \mu_z) - \sum_{i=1}^{n} \int_0^\tau \sinh^2\left(\frac{\langle w_i, \mathcal{E}_1(\mathrm{P}_t\mu_a, \mathrm{P}_t\mu_z) w_i\rangle}{2}\right).$$

□

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Y. Shenfeld
Division of Applied Mathematics,
Brown University,
Providence
RI
USA.
e-mail: Yair_Shenfeld@Brown.edu

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