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ABSTRACT

In recent years, there has been a growing interest in low-Reynolds-number, unsteady flight applications, leading to renewed scrutiny of the Kutta condition. As an alternative, various methods have been proposed, including the combination of potential flow with the triple-deck boundary layer theory to introduce a viscous correction for Theodorsen's unsteady lift. In this research article, we present a dynamical system approach for the total circulatory unsteady lift generation of a pitching airfoil. The system's input is the pitching angle, and the output is the total circulatory lift. By employing the triple-deck boundary layer theory instead of the Kutta condition, a new nonlinearity in the system emerges, necessitating further investigation to understand its impact on the unsteady lift model. To achieve this, we utilize the describing function method to represent the frequency response of the total circulatory lift. Our analysis focuses on a pitching flat plate about the mid-chord point. The results demonstrate that there is an additional phase lag due to viscous effects, which increase as the reduced frequency increases, the Reynolds number decreases, and/or the pitching amplitude increases.

NOMENCLATURE

a	Hinge location along the airfoil chord
A_α	Angle of attack amplitude
b	Half chord
B_e	Scaled version of the trailing edge singularity strength
B_s	The strength of the trailing edge singularity
B_v	Unsteady triple-deck viscous correction
$C(k)$	Theodorsen function (lift frequency response function)
C_L	Lift coefficient
C_{Lc}	Total circulatory lift coefficient
C_{LP}	Potential flow lift coefficient
C_{LQS}	Quasi-steady lift coefficient
C_{Ls}	Viscous steady flow lift coefficient
$H_n^{(m)}$	Hankel function of the mth kind of order n
$I_{()}$	Imaginary part of ()
i	$\sqrt{-1}$
k	Reduced frequency
N_{B_v}	The describing function of the triple-deck nonlinearity
N_{C_L}	The describing function of the total circulatory lift
P	Pressure
R	Reynolds number
$R_{()}$	Imaginary part of ()
t	Time variable
U	Free stream velocity
v_p	The airfoil's normal velocity
$v_{1/2}$	The normal velocity at the mid-chord
$v_{3/4}$	The normal velocity at the three-quarter-chord point
α	Angle of attack
α_e	Scaled angle of attack
α_s	Steady angle of attack or actual angle of attack
ε	$R^{-1/8}$
λ	Blasius skin friction coefficient
ρ	Density
ω	Oscillation frequency

I. INTRODUCTION

The classical theory of unsteady aerodynamics has witnessed significant advancements in recent years due to the emergence of some modern applications such as bio-inspired flights (flapping flight), highly flexible aircraft, etc. These developments are primarily based on the assumptions introduced by Prandtl[1] and Birnbaum,[2] regarding the flow around a thin airfoil. These assumptions, valid for high Reynolds numbers and small angles of attack, involve the shedding of vorticity sheets from the trailing edge, and the flow outside these sheets is considered irrotational. Additionally, they asserted that vorticity generation is a result of flow unsteadiness. These assumptions formed the basis for many analytical theories of linear aerodynamics, where an ideal flow is only governed by the Laplace equation, which is a linear equation enabling the use of linear properties like superposition. Consequently, these assumptions were employed in both steady theories (e.g., the thin airfoil theory[3,4] and the lifting surface theory[5,6]) and unsteady ones (e.g., Wagner's lift step response,[7] Theodorsen's lift frequency response,[8] and the contributions of Von Karman and Sears[9]). Moreover, these assumptions continue to be central to recent developments in the field such as those by Yongliang *et al.*,[10] Pullin and Wang,[11] Michelin and Smith,[12] Ramesh *et al.*,[13] Ramesh *et al.*,[14] Taha *et al.*,[15] Yan *et al.*,[16] Zakaria *et al.*,[17] Xia and Mohseni,[18] and Hussein *et al.*[19]

In addition to the aforementioned assumptions, researchers typically adopted more assumptions to make the analysis of unsteady aerodynamics tractable: they assumed small disturbance variations to the mean flow (i.e., flat wake assumption), and they replaced the thin airfoil and the wake with singularities of vorticity which satisfy Laplace's equation everywhere except at the surface of singularities (i.e., they are weak solutions). In order to find a unique solution of the vorticity strength (hence, the generated lift), three conditions are needed. However, only two are available: the no penetration boundary condition (the airfoil surface is considered a flow streamline) and the conservation of total circulation. The third condition is considered mainly through observation of the flow around a thin airfoil with a sharp trailing edge by Martin Kutta (1902), and it is the famous Kutta condition. It asserts that there is no flow around the trailing edge (i.e., the flow stagnation point is at the point in the cylinder domain corresponding to the trailing edge). It is basically a singularity removal condition; Crighton[20] defined it as a condition of least singularity. The Kutta condition is an appropriate representation for steady flow as the stagnation point lies at the trailing edge in the cylinder domain. In contrast, the stagnation point in an unsteady impulsive flow starts at the upper surface of the airfoil ahead of the trailing edge and moves downstream along the upper surface until it reaches the trailing edge at the steady state condition.[21]

Numerous papers in the literature have criticized the application of the Kutta condition to unsteady flows, prompting the need for corrections: Woolston and Castile,[22] Rott and George,[23] Abramson and Chu,[24–26] Henry,[27] Chu,[28] Shen and Crimi,[29] Orszag and Crow,[30] Basu and Hancock,[31] Daniels,[32] Satyanarayana and Davis,[33] Bass *et al.*,[34] Ansari *et al.*,[35] Pitt Ford and Babinsky,[36] Hemati *et al.*,[37] Xia and Mohseni,[18] Taha and Rezaei,[38] Zhu *et al.*,[39] Gonzalez and Taha,[40] and Taha and Gonzalez.[41,42] To correct for the Kutta condition in unsteady applications, Taha and Rezaei[38,43] developed a viscous extension of the classical theory of unsteady aerodynamics. They utilized the unsteady boundary layer triple-deck theory, originally developed by Brown and Daniels[44] and Brown and Cheng,[45] to provide a viscous extension of Theodorsen's lift frequency response without relying on the Kutta condition.

In this paper, we revisit the viscous extension of Theodorsen's lift frequency response model.[38] Instead of linearizing the theory to obtain a linear frequency response as previously done by Taha and Rezaei,[38] we construct a describing function of the nonlinear dynamics of the viscous unsteady lift. Notably, a linear frequency response is independent of the input signal's amplitude (e.g., angle of attack). With the constructed describing function, we investigate the effects of pitching amplitude on the frequency response of the unsteady lift at different Reynolds numbers.

II. UNSTEADY LIFT MODEL

The extended Theodorsen's unsteady lift model comprises three components:[38] potential-flow noncirculatory lift, potential-flow circulatory lift, and a viscous correction implemented through the triple-deck boundary layer theory.

A. Potential-flow model

Consider an arbitrarily-deforming thin airfoil subjected to a uniform stream U, as depicted in Fig. 1. The pressure distribution on the airfoil's upper surface can be represented by a series solution satisfying the Kutta condition, as explained by Robinson and Laurmann[46]

$$P(\theta,t) - P_\infty = \rho\left[\frac{1}{2}a_0(t)\tan\frac{\theta}{2} + \sum_{n=1}^{\infty} a_n(t)\sin\theta\right], \tag{1}$$

where $x = b\cos\theta$ and a_0 is the leading-edge singularity term. The series coefficients are determined using the no penetration boundary condition as follows.

The plate's normal velocity is expressed in terms of the plate motion kinematics in a Fourier series with coefficients b_n,

$$v_p(\theta,t) = \frac{1}{2}b_0(t) + \sum_{n=1}^{\infty} b_n(t)\cos n\theta. \tag{2}$$

Hence, the coefficients a_n of the pressure series can then be written in terms of b_n (which are known from the plate motion) as follows:

$$a_n(t) = \frac{b}{2n}\dot{b}_{n-1}(t) + Ub_n(t) - \frac{b}{2n}\dot{b}_{n+1}(t), \quad \forall\, n \geq 1. \tag{3}$$

For the a_0 term, an integral equation needs be solved, which is challenging for arbitrary time-varying motion. However, it has been solved for common inputs, e.g., harmonic motion. In this study of the nonlinear behavior of Theodorsen's viscous extension, we focus on simple harmonic motion, with the airfoil's normal velocity expressed in terms of a Fourier series,

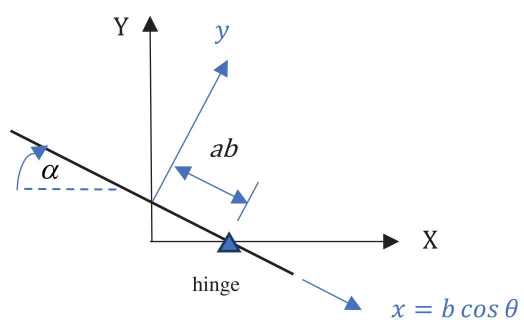

FIG. 1. An airfoil subject to a sinusoidal time varying pitching.

$$v_p(\theta,t) = \overline{V}_p(\theta)e^{i\omega t}, \quad \overline{V}_p(\theta) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n \cos n\theta. \tag{4}$$

Consequently, a_0 can be expressed as

$$a_0(t) = U\left[(B_0 + B_1)\,C(k) - B_1\right]e^{i\omega t}, \tag{5}$$

where $C(k)$ is Theodorsen's frequency response function, which depends on the reduced frequency $k = \frac{\omega b}{U}$ and can be evaluated using the Hankel function $H_n^{(m)}$ of the mth kind of order n,

$$C(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + i\,H_0^{(2)}(k)}. \tag{6}$$

Finally, the potential-flow lift coefficient can be written as

$$C_{LP} = -\frac{\pi}{U^2}\,(a_0 + a_1), \tag{7}$$

which, according to Theodorsen,[8] can be divided into circulatory and noncirculatory components.

B. Viscous model (the triple-deck boundary layer theory)

Over the years, various extensions to the original boundary layer theory developed by Prandtl[47] have been proposed. One such theory is the triple-deck boundary layer theory, devised to resolve the flow near the trailing edge. It divides the flow around a flat plate into three regimes, as explained by Messiter[48] and shown in Fig. 2: (i) the upper deck is composed of irrotational flow; (ii) the main deck contains rotational but inviscid (no viscous forces) flow; and (iii) the lower deck where Prandtl's boundary layer equation applies. The triple-deck theory matches the Blasius boundary layer[49] upstream of the trailing edge with Goldstein's shear layer[50] downstream of the edge, resolving the discontinuity in the viscous boundary condition from zero slip on the plate to zero stress on the wake centerline. This approach provides detailed flow information in the vicinity of the trailing edge down to the Kolmogorov length scale.

1. Steady triple-deck

For steady flow over a flat plate at an angle of attack, Euler's equation has infinitely many solutions, each with a different value of circulation over the plate.[40] The velocity distribution corresponding to this family of solutions can be expressed as

$$\frac{u}{U} = 1 + \alpha_s\sqrt{\frac{1-\tilde{x}}{1+\tilde{x}}} - \frac{B_s}{\sqrt{1-\tilde{x}^2}}, \tag{8}$$

where B_s is the correction to Kutta's circulation, which leads to a singularity at the trailing edge. Note that any correction to the Kutta condition must allow for a singularity at the trailing edge within the framework of ideal flow. That is, B_s represents the strength of the trailing edge singularity, corresponding to each solution in the family. According to the Kutta condition, the solution with $B_s = 0$ is selected by nature. However, the Kutta condition is replaced in this study by the triple-deck theory.

According to the triple-deck theory, the strength of the trailing edge singularity can be calculated as[51]

$$B_s = 2\varepsilon^3\lambda^{-5/4}\,B_e(\alpha_e)\,\alpha_s, \tag{9}$$

where

$\lambda = 0.332$ is the Blasius skin friction coefficient.
$\varepsilon = R^{-1/8}$, where R is the Reynolds number.
α_s is the steady angle of attack.
$\alpha_e = \varepsilon^{-1/2}\lambda^{-9/8}\alpha_s$ is the scaled angle of attack.

B_e is a scaled version of the trailing edge singularity strength, which is determined by the numerical solution of Prandtl's partial differential equations in the lower deck. Chow and Melnik[52] expressed B_e as a nonlinear function of the scaled angle of attack α_e, as shown in Fig. 3.

Hence, the viscous steady lift coefficient can be written in terms of the trailing edge singularity as

$$C_{L_s} = 2\pi(\sin\alpha_s - B_s). \tag{10}$$

Chow and Melnik[52] concluded that the trailing edge stall angle of attack α_e is equal to 0.47; at this value, the flow will separate from the

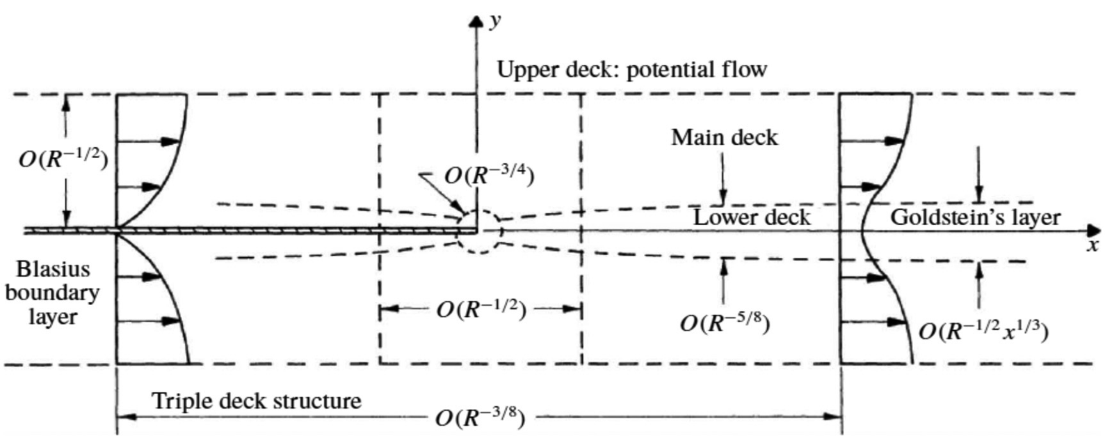

FIG. 2. The triple-deck structure and various flow regimes.[48]

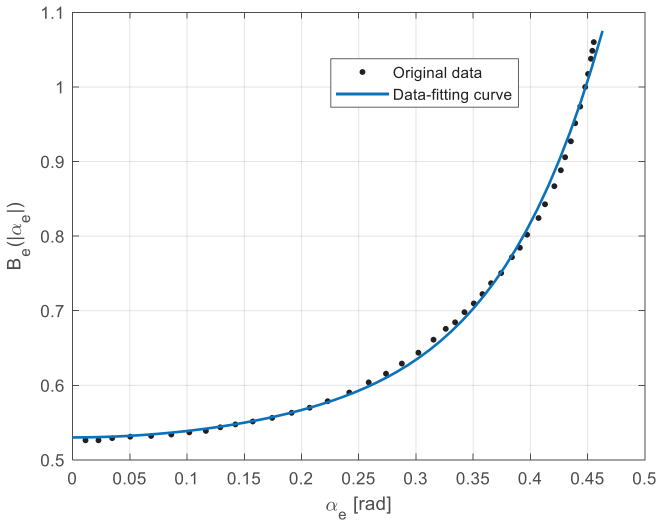

FIG. 3. The trailing edge scaled singularity B_e as function of the scaled angle of attack α_e.

upper surface of the flat plate, limiting the actual angle of attack to the range $\alpha_s = 3.1\,°–4.2\,°$ for Reynolds number in the range of $10^4–10^6$.

2. Unsteady triple-deck

Brown and Daniels[44] were the first to develop an unsteady version of the triple-deck theory for high-frequency ω and low-amplitude oscillations of a flat plate, corresponding to a range of reduced frequencies k from 5 to 15 for Reynolds numbers R in the range of $10^4–10^6$ (where k is of order $R^{1/4}$). However, these reduced frequencies are too large for engineering applications. Therefore, Brown and Cheng[45] obtained the unsteady triple-deck solution for a more practical range of reduced frequencies $0 < k \ll R^{1/4}$. Taha and Rezaei[38] utilized their solution to develop a viscous extension of the classical theory of unsteady aerodynamics, as explained in Sec. II A. They focused on deriving a viscous extension of Theodorsen's linear frequency response, resulting in a Reynolds-number-dependent linear frequency response. However, a linear frequency response is independent of the input amplitude (e.g., angle of attack). In this study, we adopt the concept of describing function for weakly nonlinear dynamical systems[53] to investigate the nonlinear effects of pitching amplitude on the frequency response at different angles of attack. Figure 4 provides a schematic representation of the contribution of this paper in comparison to the efforts of Taha and Rezaei[38] and Theodorsen.[8]

3. Viscous correction

In the context of a pitching flat plate mapped to a circular cylinder through standard conformal mapping, Taha and Rezaei[38] introduced additional circulation Γ_v at the center of the cylinder as a correction to Kutta's circulation. As a result, this circulation creates a singularity of strength B_v in the pressure distribution at the trailing edge. That is, the potential-flow series (1) is modified with the viscous contribution B_v as

$$P(\theta,t) - P_\infty = \rho\left[\frac{1}{2}a_0(t)\tan\frac{\theta}{2} + \sum_{n=1}^{\infty} a_n(t)\sin n\theta + \frac{1}{2}B_v(t)\left(\cot\frac{\theta}{2} + a_{0_v}\tan\frac{\theta}{2}\right)\right]. \tag{11}$$

To determine the viscous correction B_v, Taha and Rezaei[38] employed the unsteady triple-deck theory, which yielded the expression:

$$B_v(t) = -2\varepsilon^3\lambda^{-5/4}\left(\frac{1}{2}a_0(t) + 2\sum_{n=1}^{\infty} n a_n(t)\right)B_e(\alpha_e), \tag{12}$$

and $\alpha_e(t)$ is given by

$${}_e(t) = \varepsilon^{-1/2}\lambda^{-9/8}\underbrace{\frac{1}{U^2}\left|\frac{1}{2}a_0(t) + 2\sum_{n=1}^{\infty} n a_n(t)\right|}_{\alpha_s(t)}, \tag{13}$$

where α_s is the equivalent instantaneous steady angle of attack. Then, the total lift coefficient can be expressed as sum of two contributions,

$$C_L = \underbrace{\frac{-\pi}{U^2}(a_0 + a_1)}_{\text{Potential}} + \underbrace{\frac{-\pi}{U^2}B_v(1 + a_{0v})}_{\text{Viscous correction}}, \tag{14}$$

where the coefficient a_{0v} is given by $[a_{0v} = 2\,C(k) - 1]$.[45]

This formulation allows us to investigate the influence of viscous effects on the total lift coefficient. The viscous correction B_v provides valuable insights into the nonlinear dynamics of the system and the impact of the additional circulation at the trailing edge. The triple-deck theory, in conjunction with the unsteady extension proposed by Taha and Rezaei,[38] has proven to be effective in capturing the essential aspects of the viscous unsteady circulatory lift coefficient.

III. NONLINEAR ANALYSIS AND DESCRIBING FUNCTION FORMULATION

In linear systems theory, the frequency response method is a powerful tool used to study the steady state response of linear systems to sinusoidal inputs. When a linear system $G(i\omega)$ is excited by a harmonic input $u(t) = Ae^{i\omega t}$, the steady state output $y(t)$ is expressed as

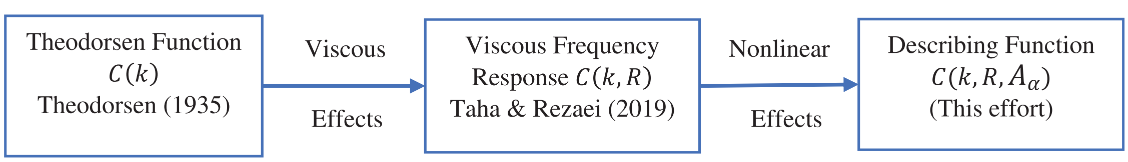

FIG. 4. A schematic diagram showing the contribution of this paper in comparison to the viscous extension of Taha and Rezaei[38] and the linear frequency response of Theodorsen.[8]

$y(t) = A|G(i\omega)|e^{i\omega t + \angle G(i\omega)}$. However, the frequency response method cannot be directly applied to nonlinear systems.[54] Fortunately, the concept of frequency response can be extended to weakly nonlinear systems using the describing function (DF) method.

The describing function method is an approximate approach that predicts the nonlinear behavior of a dynamical system in response to a sinusoidal input while retaining the appealing properties of the frequency response method. For weakly nonlinear systems forced by a sinusoidal input $u(t) = Ae^{i\omega t}$, the DF method considers only the fundamental harmonic of the output.[54–56] It should be noted that the describing function approach is a particular class of the more general framework of harmonic balance.[55,56]

In this section, we present the construction of the describing function for the viscous lift dynamics of a flat plate pitching around its mid-chord point, subject to a sinusoidal time-varying angle of attack (pitching angle), in the following form:

$$\alpha(t) = A_\alpha e^{i\omega t}, \tag{15}$$

where A_α is an angle of attack amplitude, ω is the frequency, and the positive angle of attack is clockwise. In this representation, the actual angle of attack is given by the real part of Eq. (15).

Recall that for a pitching motion according to Eq. (15), the flat plate's normal velocity can be written as

$$v_p(\theta, t) = -\dot{\alpha}(b\cos\theta - ab) - U\alpha, \quad -b \le x \le b, \tag{16}$$

where $\dot{\alpha}$ is the pitching rate and ab is the distance from mid-chord to the hinge location, as shown in Fig. 1. Then, the kinematics of the flat plate can be expressed using Eqs. (2) and (16) in terms of α, $\dot{\alpha}$ as

$$b_0(t) = 2v_{1/2}(t), \tag{17a}$$

$$b_1(t) = -b\dot{\alpha}(t) \quad \text{and} \quad b_n(t) = 0 \,\forall\, n > 1, \tag{17b}$$

where $v_{1/2}(t)$ is the normal velocity at the mid-chord. Hence, for the harmonic motion (15), we can find the series coefficients of the pressure distribution on the upper surface of the flat plate using Eqs. (3)–(5),

$$a_0 = U\Big(2v_{3/4}(t)\, C(k) + b\dot{\alpha}\Big), \tag{18a}$$

$$a_1 = b\big(\dot{v}_{1/2}(t) - U\dot{\alpha}(t)\big), \tag{18b}$$

$$a_2 = -\frac{b^2\ddot{\alpha}(t)}{4} \quad \text{and} \quad a_n = 0 \,\forall\, n > 2, \tag{18c}$$

where $v_{3/4}(t) = V_{3/4}e^{i\omega t}$ is the normal velocity at the three-quarter-chord point. Finally, the total lift coefficient can be written as

$$C_L = \underbrace{\frac{-\pi}{U^2} b\dot{v}_{1/2}(t)}_{\text{potential noncirculatory}} + \overbrace{\underbrace{2\pi\alpha_{3/4}(t)C(k)}_{\text{potential circulatory}} - \underbrace{2\pi\tilde{B}_v(t)C(k)}_{\text{viscous correction}}}^{\text{ciculatory lift coefficient}}, \tag{19}$$

where $\tilde{B}_v(t) = B_v/U^2$, $\alpha_{3/4}(t) = -\frac{v_{3/4}(t)}{U} = \overline{\alpha_{3/4}}\, e^{i\omega t}$ is the angle of attack at the three-quarter-chord point as recommended by the Pistolesi theorem (see p. 80 of Ref. 57 by Schlichting). Note that for the harmonic motion $\alpha(t) = A_\alpha e^{i\omega t}$, complex values may appear in the lift coefficient expression of Eq. (19), due to complex numbers multiplication $[e^{i\omega t}\, C(k)]$. But only the real part of these values is considered.

Figure 5 shows a block diagram of the total circulatory lift dynamics; the dynamics of the unsteady lift is considered as a dynamical system with the angle of attack being the input and the total circulatory lift coefficient being the output.[58] Furthermore, Fig. 5 clearly indicates that the term B_v and the total circulatory lift coefficient exhibit nonlinear dynamics. Consequently, when the input α oscillates at frequency ω, it generates not only the fundamental harmonic at ω but also higher harmonics. However, the describing function approximation technique focuses solely on the fundamental harmonic. Interestingly, the describing function can be constructed for each nonlinear block separately or for the entire nonlinear system. In this paper, we construct two describing functions: (i) for the triple-deck viscous nonlinearity (i.e., between α_s and $\tilde{B}_v$) and (ii) for the entire system between the angle of attack and the total circulatory lift. Similar to Theodorsen's function $C(k)$, we normalize the unsteady lift by the quasi-steady lift in our definition of the describing function.

To make the analysis more tractable, we performed curve fitting to find an algebraic expression for B_e as a function of α_e. Moreover, we retained only even powers of α_e in the curve fitting expression, to account for the absolute-value function of α_e in Eq. (13),

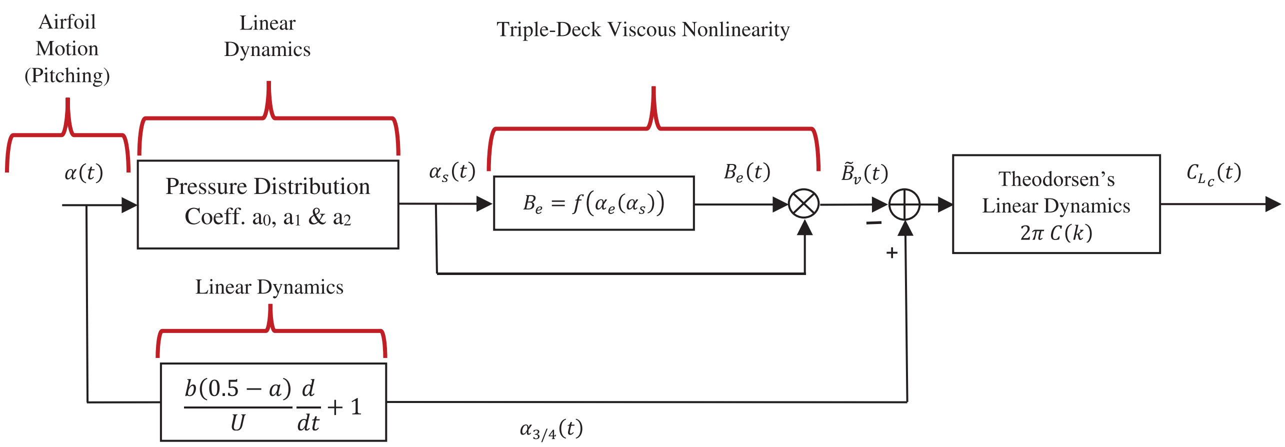

FIG. 5. The total circulatory lift dynamics block diagram.

$$B_e(|\alpha_e|) = 36.63\,\alpha_e^6 + 0.8598\,\alpha_e^2 + 0.5301. \tag{20}$$

Figure 3 shows that the performed fitting is accurate.

The input and the first harmonic of the output of the dynamical system were written in the form $(R + iI)e^{i\omega t}$. The describing functions are then defined as the ratio between the phasor representation of the frequency ω of the output to that of the input, i.e., the ratio between the complex amplitudes of the coefficients of the fundamental harmonic $e^{i\omega t}$ of the output to the input.

Figure 6 illustrates a flow chart outlining the construction of the describing function. Based on this notation, we define the describing function of the triple-deck viscous nonlinearity as

$$N_{B_v} = \frac{(R_{B_v} + iI_{B_v})}{(R_a + iI_a)} \tag{21}$$

and the describing function of the total circulatory lift coefficient as

$$N_{C_L} = \frac{(R_{C_L} + iI_{C_L})}{(R_{QS} + iI_{QS})} \tag{22}$$

where all the complex amplitudes depend on the pitching amplitude A_α, the reduced frequency k, and the Reynolds number R.

IV. DERIVATION AND VALIDATION OF THE DESCRIBING FUNCTIONS

A. Describing function of the triple-deck viscous nonlinearity

Proceeding through the flow chart in Fig. 6, we can represent χ as follows:

$$\chi = A_\alpha\left[\underbrace{\left[-U^2R_k + \omega b\left(\frac{1}{2} - a\right)I_k + b^2\omega^2\left(\frac{1}{4} - 2ab\right)\right]}_{R_\chi} + i\underbrace{\left[b\omega U\left(-\left(\frac{1}{2} - a\right)R_k - \frac{7}{2}\right) - U^2 I_k\right]}_{I_\chi}\right]e^{i\omega t}, \tag{23}$$

where R_k and I_k represent the real and imaginary parts of Theodorsen's frequency response function, respectively. Next, we express α_e as

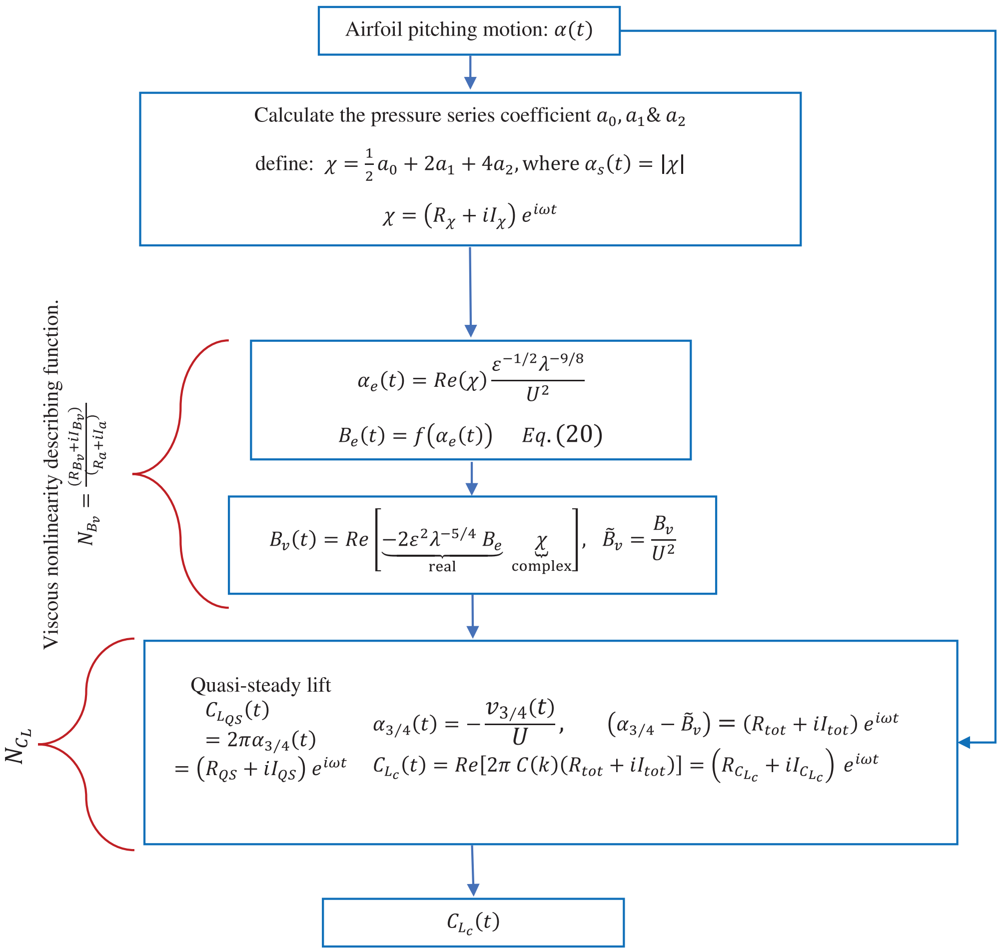

FIG. 6. The describing function flow chart.

$$\alpha_e = [R_\chi \cos\omega t - I_\chi \sin\omega t]\frac{\varepsilon^{-\frac{1}{2}}\lambda^{-\frac{9}{8}}}{U^2}. \tag{24}$$

After substituting α_e in Eq. (20) and using it to find B_v from Eq. (12), we perform mathematical manipulations to compute the amplitudes of the fundamental harmonic, given by

$$B_v(\mathrm{t}) = Re\Big[\Big(-2\varepsilon^3\lambda^{-\frac{5}{4}}\Big)((R_c + iI_c)\cos\omega t + (R_s + iI_s)\sin\omega t)\Big], \tag{25}$$

where R_c, I_c, R_s, and I_s are polynomials in R_χ and I_χ. Finally, to calculate the describing function of the triple-deck nonlinearity, we represent B_v in phasor form as

$$B_v(\mathrm{t}) = Re\Big[\Big(-2\varepsilon^3\lambda^{-\frac{5}{4}}\Big)(R_c - iR_s)e^{i\omega t}\Big] \tag{26}$$

and the DF is then given by

$$N_{B_v} = \frac{\Big(-2\varepsilon^3\lambda^{-\frac{5}{4}}\Big)(R_c - iR_s)}{(R_\chi + iI_\chi)}. \tag{27}$$

In addition, the describing function of the triple-deck nonlinearity exhibits a constant phase of $-180°$, which can be explained using Eq. (13) where the output of the describing function N_{B_v} can be represented as the real value B_e multiplied by the complex input χ,

$$B_v(\mathrm{t}) = \underbrace{-2\varepsilon^3\lambda^{-5/4}}_{\text{constant}}\ \underbrace{B_e(\alpha_e)}_{\text{real}}\ \underbrace{\chi}_{\text{complex}}.$$

To validate this approximation of B_v by the first harmonic only, we conduct simulations of the exact dynamics, as given in the block diagram in Fig. 5, and compare the resulting response to the fundamental harmonic expression given by Eq. (27). Figure 7 illustrates this comparison for the viscous correction B_v at various reduced frequencies. All simulations are performed at a pitching amplitude of $A_\alpha = 1°$ and a Reynolds number of $R = 10^4$. For a wide range of reduced frequencies ($k < 0.5$), the two responses are indistinguishable. Slight differences start to emerge at $k = 1$. This validation confirms that, for small angles of attack, the viscous nonlinearity is weak enough to admit a describing function approach even at low-Reynolds numbers (down to 10^4) and high frequencies (up to $k = 1$).

B. Describing function of the total circulatory lift coefficient

Considering the flow chart in Fig. 6, we express the fundamental harmonic of the total circulatory lift as

$$\begin{aligned} C_{L_c}(\mathrm{t}) = 2\pi\, Re[((R_{L_c}R_k - I_{L_c}I_k) + i(R_kI_{L_c} + R_kI_{L_c}))\cos\omega t \\ +((R_{Ls}R_k - I_{L_s}I_k) + i(R_kI_{L_s} + I_kR_{Ls}))\sin\omega t], \end{aligned} \tag{28}$$

where

$$R_{L_c} = A_\alpha - \frac{R_c}{U^2}, \tag{29a}$$

$$I_{L_c} = \frac{b\omega A_\alpha}{U}\left(\frac{1}{2} - a\right) - I_c, \tag{29b}$$

$$R_{L_s} = -\frac{b\omega A_\alpha}{U}\left(\frac{1}{2} - a\right) - R_s, \tag{29c}$$

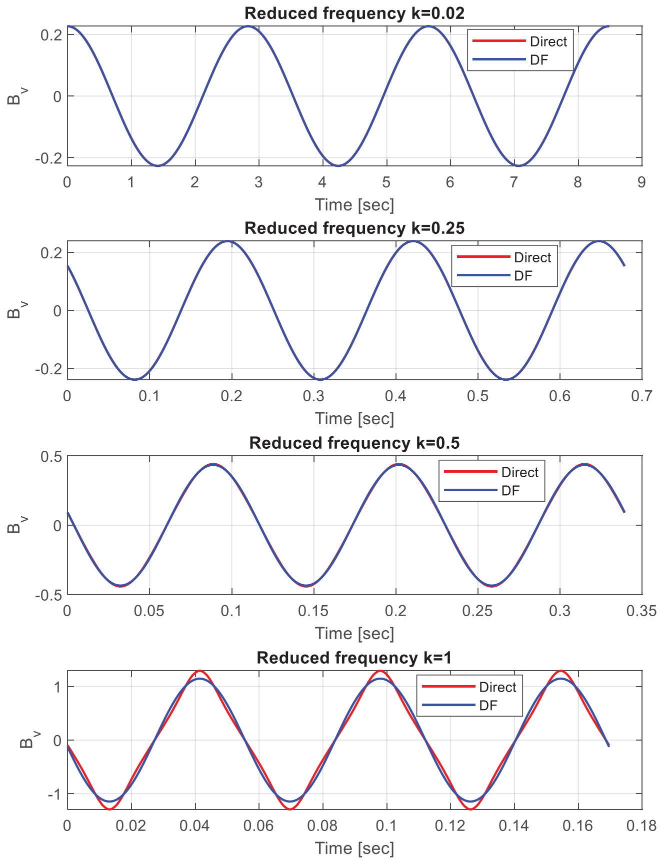

FIG. 7. Triple-deck viscous nonlinearity validation for a pitching flat plate about mid-chord point for different reduced frequencies k at $A_\alpha = 1°$, $R = 10^4$.

$$I_{L_s} = A_\alpha - I_s. \tag{29d}$$

Then, we represent the fundamental harmonic of the total circulatory lift in phasor form as

$$C_{L_c}(\mathrm{t}) = Re\big[2\pi((R_{L_c}R_k - I_{L_c}I_k) - i(R_{Ls}R_k - I_{L_s}I_k))e^{i\omega t}\big]. \tag{30}$$

Note that the time signal is given by the real part.

The quasi-steady lift is calculated as

$$_{L_{QS}}(\mathrm{t}) = Re\left[2\pi A_\alpha\left(1 + i\frac{b\omega}{U}\left(\frac{1}{2} - a\right)\right)e^{i\omega t}\right]. \tag{31}$$

Finally, the describing function of the total circulatory lift is given by

$$N_{C_L} = \frac{(R_{L_c}R_k - I_{L_c}I_k) - i(R_{Ls}R_k - I_{L_s}I_k)}{A_\alpha\left(1 + i\frac{b\omega}{U}\left(\frac{1}{2} - a\right)\right)}. \tag{32}$$

Figure 8 provides validation of the describing function fundamental frequency approach for the total circulatory lift at different reduced frequencies. All simulations are performed at a pitching amplitude of $A_\alpha = 1°$ and a Reynolds number of $R = 10^4$. The describing function (DF) approximate response of the total circulatory lift is even closer to the exact response than that of B_v, as shown in Fig. 8. The slight

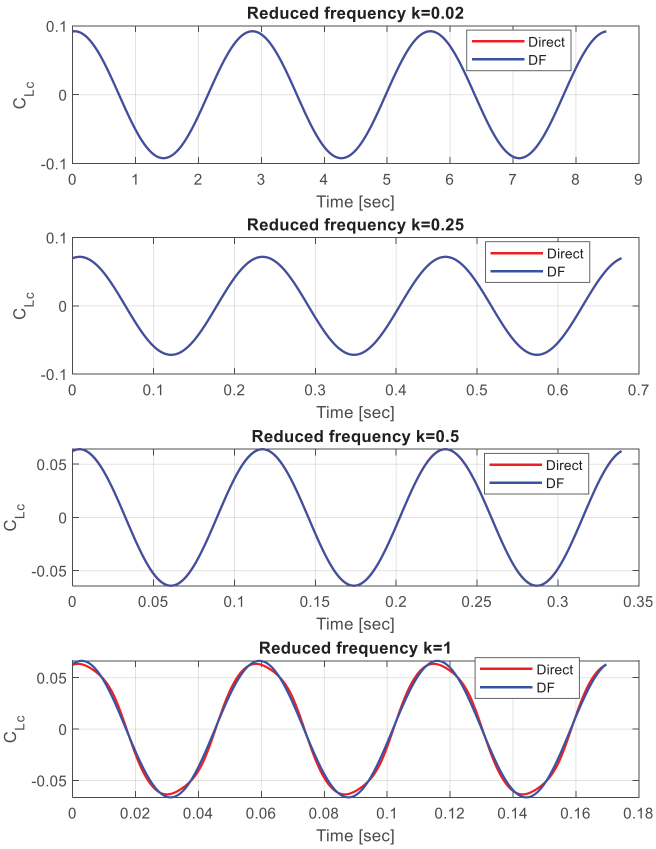

FIG. 8. Total circulatory lift comparison for a pitching flat plate about mid-chord point for different reduced frequencies k at $A_\alpha = 1^\circ$, $R = 10^4$.

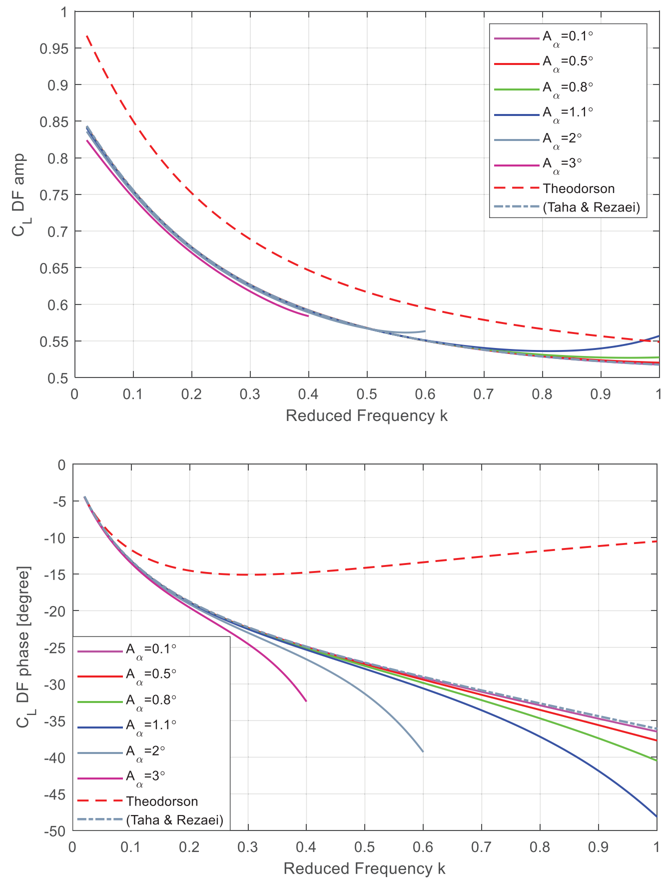

FIG. 9. The describing function of the total circulatory lift coefficient of a pitching flat plate about the mid-chord point at $R = 10^4$ and different pitching amplitudes.

differences observed at $k = 1$ for B_v are further reduced in the case of C_{L_c}. This outcome is intuitively expected due to the low-pass filter effect of Theodorsen's function between B_v and C_{L_c} (see Fig. 5), which attenuates the higher harmonics of B_v.

V. RESULTS AND DISCUSSION

The main advantage of the describing function of a weakly nonlinear system compared to the frequency response of the linearized version is its ability to capture the dependence on the input amplitude. Thus, for our dynamical system that represents the viscous unsteady circulatory lift coefficient, the describing function approach captures dependence on the pitching amplitude A_α, in addition to the conventional dependence on reduced frequency k and Reynolds number R.

Therefore, the obtained describing functions of the total circulatory lift for a pitching flat plate about mid-chord point at different pitching amplitudes and different Reynolds numbers are presented in Figs. 9 and 10. The former displays the obtained DFs for various pitching amplitudes at $R = 10^4$, while the later shows the obtained DFs for different Reynolds numbers at $A_\alpha = 0.8^\circ$. Figure 9 also shows the linear frequency response of Taha and Rezaei[38] for comparison, which closely matches our describing function at low pitching amplitude ($A_\alpha = 0.1^\circ$). The selected range for the pitching amplitude is between 0.1° and -3° over the range $k = 0$–1. However, the considered range of frequency is smaller for larger pitching amplitudes to avoid flow separation at the trailing edge. The selected range for the Reynolds number is between 10^3 and 10^6 to illustrate the effects of the viscous correction on total circulatory lift as the flow approaches inviscid conditions. It is important to note that the triple-deck theory and the unsteady extension of Taha and Rezaei,[38] which form the basis of the current work, are valid only up to trailing edge stall. Even within this range of small angles of attack, there are considerable nonlinear viscous effects, as shown below.

Observing Fig. 9, it is evident that the viscous effects introduce a considerable phase lag to the total circulatory lift coefficient, which increases as the reduced frequency increases at all pitching amplitudes. Moreover, this additional lag further amplifies with increasing the pitching amplitude. Furthermore, the viscous effects result in a reduction in the amplitude of the total circulatory lift coefficient. It is interesting (perhaps counter-intuitive) to observe a significant deviation (reduction) from the amplitude of Theodorsen's frequency response even at very small pitching amplitudes (down to 0.1°). This deviation is due to the fact that the ideal flow outside the boundary layer is no longer flowing over a smooth body, but rather over the body plus the boundary layer. In other words, viscosity leads to a reduction in the lift curve slope (even at small angles of attack).[40,41] Figure 10 demonstrates

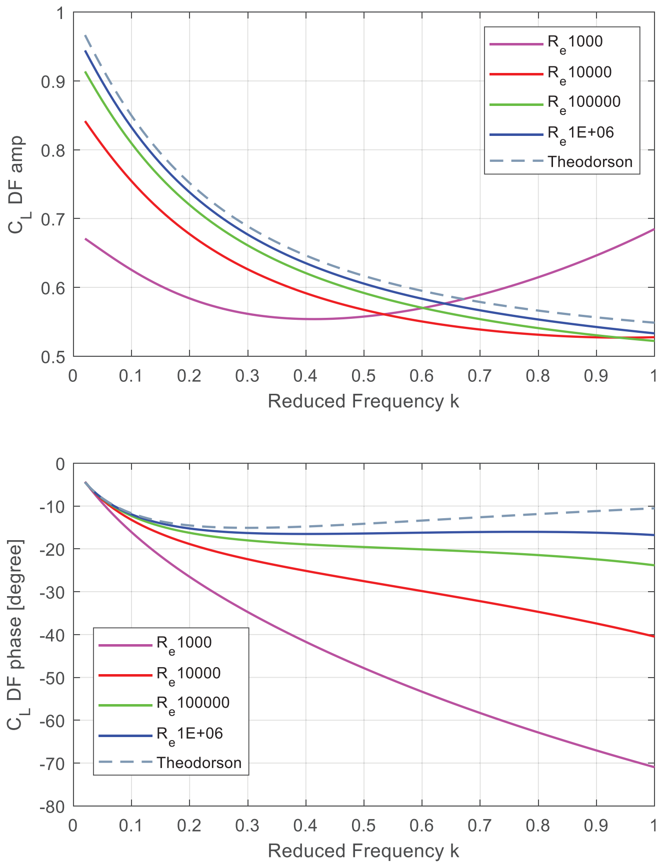

FIG. 10. The describing function of the total circulatory lift coefficient of a pitching flat plate about the mid-chord point at $A_\alpha = 0.8°$ and different Reynolds numbers.

that these viscous effects (larger lag and smaller amplitude) are exaggerated as the Reynolds number decreases, which is intuitively expected; for a very large Reynolds number, Theodorsen's frequency response is recovered. The smaller the Reynolds number, the larger the boundary layer thickness, and consequently, the larger the deviation from potential-flow results. It should be noted that, as the Reynolds number gets smaller, the inertial effects (i.e., convective acceleration and curvature) are no longer dominant, and viscous effects (e.g., viscous dissipation) play a more prominent role in the picture.[42] Finally, it is worth noting that the resulting describing functions over the considered Reynolds number range at $A_\alpha = 0.1°$ are similar to those presented in Fig. 10 at $A_\alpha = 0.8°$. So, we opt to present only one figure.

These findings align with previous results in the literature. For instance, Chu and Abramson[25] proposed to add 10° phase lag to Theodorsen's function at the reduced frequency $k = 0.5$. Similarly, Bass *et al.*[34] suggested adding a phase lag of 30° over the range of $0.5 < k < 10$ and Reynolds numbers R ranging from 6500 to 26500. Thus, it is widely recognized that Theodorsen's inviscid model underestimates the lag of the unsteady lift dynamics. These insights hold particular significance for flutter analysis, where the phase difference between the airfoil oscillations and unsteady aerodynamic loads plays a pivotal role in defining the flutter boundary.[59,60] The provided

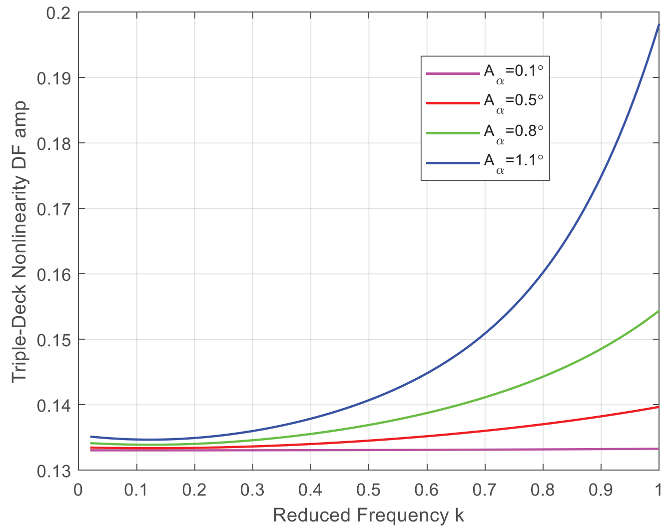

FIG. 11. The describing function of triple-deck nonlinearity of a pitching flat plate about the mid-chord point at $R = 10^4$ and different pitching amplitudes.

describing function formulation allows nonlinear analysis of aeroelastic systems pre-, during, and post-flutter.

We also present the describing functions (DFs) of the triple-deck nonlinearity for a pitching flat plate about the mid-chord point at different pitching amplitudes and Reynolds numbers in Figs. 11 and 12, respectively. The former illustrates the obtained DFs for various pitching amplitudes at $R = 10^4$, while the later shows the obtained DFs for different Reynolds numbers at $A_\alpha = 0.8°$. Upon analyzing Fig. 11, it becomes apparent that the amplitude of the triple-deck nonlinearity experiences an exponential increase as both the reduced frequency and the pitching amplitude increase. In contrast, Fig. 12 reveals that the amplitude of the triple-deck nonlinearity exhibits a direct correlation

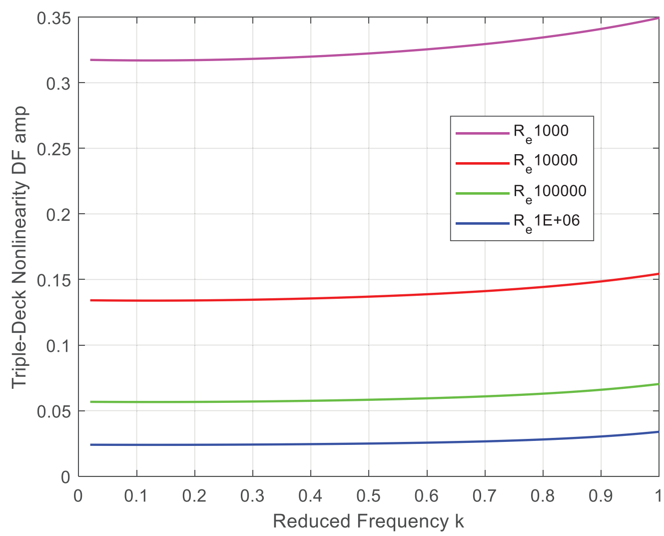

FIG. 12. The describing function of triple-deck nonlinearity of a pitching flat plate about the mid-chord point at $A_\alpha = 0.8°$ and different Reynolds numbers.

solely with the Reynolds number; it is almost independent of k. The triple-deck nonlinear element between α_s and B_v has a phase lag of -180° (i.e., out of phase). However, since B_v contributes in negative sign to C_{L_c} as shown in Eq. (19) and Fig. 5, the triple-deck nonlinearity *per se* does not add any phase lag to the circulatory lift coefficient.

VI. CONCLUSION

In this study, we investigated the nonlinear dynamics of viscous unsteady lift on a harmonically pitching flat plate. We employed the concept of describing function to capture the nonlinear behavior of the system, particularly the triple-deck viscous nonlinearity and the total circulatory lift coefficient. The results from simulations show good agreement between the describing function-based approach and the exact dynamics, confirming the efficacy of the describing function approximation.

For the triple-deck viscous nonlinearity, we have derived the describing function in terms of the reduced frequency k, the Reynolds number R, and the pitching amplitude A_α. By employing this approach, we successfully demonstrated that the amplitude of the triple-deck nonlinearity is significantly influenced by the concurrent increase in both the reduced frequency and the pitching amplitude.

Similarly, we constructed the describing function for the entire aerodynamic system, which relates the total circulatory lift as an output to the pitching angle (i.e., angle of attack) as an input. The obtained describing functions reveal the considerable phase lag caused by viscous effects in the total circulatory lift coefficient, which increases with higher reduced frequencies and pitching amplitudes. Additionally, viscous effects lead to a reduction in the amplitude of the circulatory lift coefficient—a finding consistent with previous results in the literature.

These results hold great significance for flutter analysis and studies of aeroelastic systems, where the phase difference between airfoil oscillations and unsteady aerodynamic loads plays a critical role in determining the flutter boundary. The proposed describing function formulations provide valuable tools for nonlinear analysis of aeroelastic systems pre-, during, and post-flutter. For example, it is worth investigating, using the presented results, how the robustness measures (e.g., gain margin and phase margin)[61,62] of a given flight controller may change due to viscous unsteady aerodynamic effects, which are typically neglected in the control design process.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to ~~disclose.~~

Author Contributions

Yasmin Selim: Formal analysis (lead); Methodology (lead); Software (lead); Validation (lead); Writing – original draft (lead); Writing – review & editing (equal). **Haithem Taha:** Formal analysis (supporting); Funding acquisition (lead); Methodology (supporting); Supervision (lead); Writing – review & editing (equal). **Gamal El-Bayoumi:** Formal analysis (supporting); Methodology (supporting); Supervision (supporting).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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