# Object Spatial Impedance Achieved by a Multifinger Grasp With Hard-Point Contact

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***Abstract*—This article identifies the space of spatial impedance behaviors that can be achieved by a grasp of a multifinger hand in which each fingertip makes hard-point contact with the held object. The dimension of the space of realizable impedances for a given number of fingers is determined analytically. A set of necessary and sufficient conditions on impedance matrices that can be realized by a given grasp is developed and the physical significance of these conditions is identified. The space of impedances that can be achieved by a hard point grasp is a hyperplane in the space of all possible impedance behaviors. A process to achieve an arbitrary full-rank impedance in the hyperplane with a grasp having the minimum number of fingers is developed. With this process, any spatial impedance behavior in the hyperplane can be attained with a three-finger grasp by properly selecting: first, the locations where fingertips contact the held object, and second, the translational impedance provided by a finger at each of these locations.**

***Index Terms*—Object impedance and admittance, robot finger, robotic grasp.**

## I. Introduction

IN ROBOTIC manipulation, a task-appropriate spatial impedance (a generalization of stiffness, damping, and inertia) facilitates dexterity in performing tasks involving physical interaction between an object held by a manipulator and its environment [1]. A well-designed impedance can provide force regulation and improve relative positioning.

A general impedance behavior associated with a held object can be mathematically characterized as a mapping between the motion (velocity) of the object and the force applied to the object. For small motion, the mapping is linear. For spatial motion, if the force is represented by a 6-vector wrench (force and torque) $\mathbf{w}$ and the motion is represented by a 6-vector twist (translational and angular velocity) $\mathbf{t}$, the impedance can be characterized by a $6 \times 6$ matrix, the impedance matrix $\mathcal{Z}$, which maps motion $\mathbf{t}$ to wrench $\mathbf{w}$

$$\mathbf{w} = \mathcal{Z}\mathbf{t}. \tag{1}$$

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Equivalently, a nonsingular impedance matrix can be expressed by the admittance matrix $\mathcal{Y}$, the inverse of $\mathcal{Z}$

$$\mathbf{t} = \mathcal{Y}\mathbf{w}. \tag{2}$$

A desired impedance can be imparted to a held object by using the following:

1) one (e.g., [2]) or more manipulators (e.g., [3]);
2) a specially designed gripper or end effector (e.g., [4]);
3) a robotic hand.

Due to their high degree of adjustability, robotic hands have a significant advantage. Their ability to both vary the locations where fingertips contact the object and vary the finger impedances allows a hand to change the held object impedance without influencing its pose. The analysis of impedance behaviors that can be achieved by a multifinger grasp is important in understanding and achieving dexterous manipulation.

This work is motivated by the need for the design of robotic grasps so that the held object attains a specified desirable impedance. The grasps considered in this article are associated with multiple fingers in hard-point contact with the object. The space of impedances that can be achieved by a grasp depends on the number of fingers in a hand and the locations of fingertip contact with the object. Thus, in the design of a robotic hand for a task, the number of fingers needed and the range in fingertip positioning are important considerations. Understanding the capabilities and limitations of these types of grasps in achieving a general spatial impedance provides a foundation for the design of robotic hands.

### A. Related Work

Well established important functions of a robot hand [5] include the following:

1) human operability (effective transfer of intended behavior, e.g., [6], [7], [8]);
2) manipulation dexterity (in-hand object repositioning, e.g., [9], [10], [11], [12]);
3) grasp robustness (stable object holding despite disturbances, e.g., [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23]).

The ability to impart a task-appropriate impedance to an object is also an important grasp function.

The analysis (e.g., [24]) of general elastic behavior provided by a serial mechanism and the realization (e.g, [25]) of desired elastic behavior with a serial mechanism have been studied in depth. When a serial mechanism, such as a finger is undeformed, the impedance matrix is symmetric [24], but when the mechanism is displaced from equilibrium, the matrix is nonsymmetric [26], [27], [28]. A significant amount of prior research has investigated the impedance of a given grasp [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41]. In some investigations (e.g., [39], [41]), optimization was used to identify finger configurations and joint stiffnesses that yield an approximation to a desired grasp stiffness matrix.

The minimum number of fingers needed to attain a desired impedance was initially calculated using a dimension counting argument for cases in which the impedance of one joint is either coupled or uncoupled [31], [38] to other joints in the hand. In these calculations, the minimum number of fingers required to achieve a stiffness is deemed to be that for which the number of free variables associated with a given number of elastic fingers exceeds the number of constraints. Since this type of dimensional analysis does not consider the independence of the constraint equations, the results obtained were inaccurate.

In [36], an evaluation of the realizable impedance subspace for different numbers of fingers in a grasp showed that: 1) the dimension of the subspace for a given number of fingers is less than that determined by a counting argument, and 2) the realizable space of impedances for grasps with hard-point contact is constrained regardless of the numbers of fingers in a grasp. These results were interesting observations obtained using rank analyzes of constructed matrices. A rigorous analytical proof of the dimension of the impedance subspace achievable by a grasp was not provided, and the physical significance of the dimension deficiency was not identified. As stated in [36], the unachievable subspace of impedances are "algebraically complicated and are usually quite challenging to interpret, particularly for the spatial cases."

In recent work on *planar* compliance realization, it was shown that *any* specified object compliance can be achieved with a grasp having multiple two-joint fingers [42] or multiple three-joint fingers [43] by properly choosing the configuration and joint stiffnesses of each finger. These planar compliance realization approaches, however, cannot be extended to *spatial* cases due to their higher dimension and increased complexity. For example, the restriction observed in [36] on spatial impedances due to hard-point contact does not exist for planar cases.

### B. Overview

This article provides a detailed evaluation of the space of spatial impedance behaviors that can be achieved by a grasp of a multifinger hand in hard-point contact with an object. This article:

1) provides a rigorous evaluation of the dimension of the space of realizable impedance behaviors that can be achieved by: i) any grasp (a 20-dimensional hyperplane), and ii) a given grasp (hyperplanes of reduced dimension that are determined by the number of fingers and the fingertip contact locations) and identifies the source of the dimension deficiencies observed in [36];
2) identifies the necessary and sufficient conditions for any specified impedance matrix to be realized by a given grasp and describes the physical significance of these conditions;
3) develops a process to achieve any specified impedance behavior (in the 20-dimensional subspace of grasp-realizable behaviors) with a grasp having the minimum number of fingers.

The rest of this article is organized as follows. In Section II, the technical background needed for the analysis of a grasp impedance is presented. In Section III, the space of realizable *symmetric* impedance matrices for a given grasp is analyzed. The dimension of this space for grasps with different numbers of fingers and the source of the dimension deficiency in each case are identified. In Section IV, a physical description of the achievable symmetric impedance subspace associated with grasps having different numbers of fingers is presented. Necessary and sufficient conditions for a Cartesian impedance matrix to be achieved by a grasp are identified. In Section V, the results obtained for symmetric impedance matrices are extended to general *nonsymmetric* cases. In Section VI, implications of the theoretical results and their applications in passive and planar realizations are discussed. A process to realize an arbitrary spatial impedance with a grasp having three fingers is also developed. An example demonstrating this process is provided in Section VII. Finally, Section VIII concludes this article.

## II. Technical Background

In this section, the technical background needed for the analysis of object impedances attained by a grasp with fingers in hard-point contact is provided.

### A. Finger Contact Model

Since a grasp involves interaction of the held object with fingertips, an appropriate physical model of finger-object contact is essential in grasp analysis. Among the different contact models described in [23], hard-point contact is widely used when friction is high and the fingertip/object contact area is small. This article focuses on hard-point contacts because: 1) it is the simplest contact model with practical characteristics (friction is included), and 2) an understanding of the abilities and restrictions of hard-point contact provides a foundation for the analysis of more complicated soft-contact models.

Consider a hand with multiple fingers $n \geq 2$ grasping an object, as shown in Fig. 1. The $i$th finger contacts the object at point $P_i$. For hard-point contact (point contact with friction) [23], the finger is capable of transmitting only the three components of the contact force (no torque) to the body and, therefore, capable of transmitting only translational impedance. Let $\mathbf{Z}_i$ be the $3 \times 3$ symmetric translational (point) impedance matrix of the finger at $P_i$, then

$$\mathbf{f}_i = \mathbf{Z}_i \mathbf{v}_i \tag{3}$$

where $\mathbf{v}_i \in \mathbb{R}^3$ is the translational velocity of the body at $P_i$ and $\mathbf{f}_i \in \mathbb{R}^3$ is the force at the fingertip.

Again note that (3) indicates the translational impedance provided by a single finger to the held object at fingertip contact

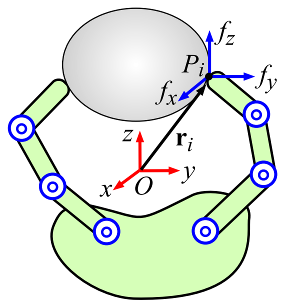


Fig. 1. Object grasped by a robot hand with multiple fingers. Each finger is in hard-point contact with the object to which only force (no torque) is transmitted.

point $P_i$. Each of the $n$ fingers contributes a portion of the general motion (spatial) impedance of the held object that is described in a global coordinate frame. The $6 \times 6$ impedance matrix $\mathcal{Z}_i$ described in this frame is obtained by a screw transformation $\mathbf{T}_i$ of the translational impedance $\mathbf{Z}_i$

$$\mathcal{Z}_i = \mathbf{T}_i \begin{bmatrix} \mathbf{Z}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{T}_i^T \tag{4}$$

where $\mathbf{T}_i$ has the form

$$\mathbf{T}_i = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P}_i & \mathbf{I} \end{bmatrix} \tag{5}$$

and where $\mathbf{I}$ is the $3 \times 3$ identity matrix and $\mathbf{P}_i$ is the $3 \times 3$ skew-symmetric matrix associated with translation from the contact position $P_i$ to the global frame. If $\mathbf{r}_i = (x_i, y_i, z_i)$ describes the position from the global frame to the contact point $P_i$, then the cross product matrix $\mathbf{P}_i$ is

$$\mathbf{P}_i = \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}. \tag{6}$$

Calculating $\mathcal{Z}_i$ in (4) yields

$$\mathcal{Z}_i = \begin{bmatrix} \mathbf{Z}_i & \mathbf{Z}_i\mathbf{P}_i^T \\ \mathbf{P}_i\mathbf{Z}_i & \mathbf{P}_i\mathbf{Z}_i\mathbf{P}_i^T \end{bmatrix}. \tag{7}$$

Since $\mathbf{Z}_i$ is symmetric, $\mathcal{Z}_i$ is symmetric. Since $\mathbf{P}_i$ is skew-symmetric

$$\text{trace}(\mathbf{P}_i\mathbf{Z}_i) = \text{trace}(\mathbf{Z}_i\mathbf{P}_i^T) = 0. \tag{8}$$

In the global coordinate frame, the overall impedance matrix contributed by all fingers is the sum of all $n$ $\mathcal{Z}_i$ in (7)

$$\mathcal{Z} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{H} \end{bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} \mathbf{Z}_i & \mathbf{Z}_i\mathbf{P}_i^T \\ \mathbf{P}_i\mathbf{Z}_i & \mathbf{P}_i\mathbf{Z}_i\mathbf{P}_i^T \end{bmatrix}. \tag{9}$$

Since each matrix $\mathcal{Z}_i$ in (9) satisfies the trace condition of (8), the trace of the off-diagonal blocks in $\mathcal{Z}$ must be zero, i.e.,

$$\text{trace}(\mathbf{B}) = \text{trace}(\mathbf{B}^T) = \sum_{i=1}^{n} \text{trace}(\mathbf{P}_i\mathbf{Z}_i) = 0. \tag{10}$$

Thus, only those symmetric impedance matrices satisfying the trace condition (10) can be achieved by a multifinger hand in hard-point contact. The space of $6 \times 6$ symmetric matrices is 21 dimensional. The trace condition defines a 20-dimensional hyperplane in the 21-dimensional space. A multifinger hand can only achieve behaviors on this 20-dimensional hyperplane regardless of the number of fingers in contact with the object. This restriction is due to the physics of finger contact with an object: hard-point contact only transmits pure force (no torque) to the held object. In fact, this restriction holds for any grasp where each contact provides only a pure force or a pure couple to the object (i.e., no wrench of nonzero finite pitch) [44].

The trace restriction (10) is only a *necessary* condition for an impedance matrix to be realized by a hand. For a given impedance $\mathcal{Z}$ satisfying (10), the realization of the behavior with a grasp requires that: 1) the location of each fingertip $P_i$ (indicated by matrix $\mathbf{P}_i$) in contact with the object be identified, and 2) the $3 \times 3$ translational impedance $\mathbf{Z}_i$ of each finger at the contact point be selected such that (9) is satisfied for the desired impedance $\mathcal{Z}$.

### B. Matrix Vectorization

Similar to the approach described in [36], subsequent analysis is performed on vectorized versions of matrices.

For an arbitrary $m \times m$ symmetric matrix $\mathcal{G}$, since the off-diagonal elements $g_{ij} = g_{ji}$, there are only $m(m+1)/2$ independent variables that are all contained in the upper triangle of $\mathcal{G}$. If we denote $\text{vec}(\mathcal{G})$ as the vector representation of a symmetric matrix $\mathcal{G}$ in the order

$$\text{vec}(\mathcal{G}) = [g_{11}, g_{12}, \ldots, g_{1m}, g_{22}, g_{23}, \ldots, g_{2m}, \ldots, g_{mm}]^T \tag{11}$$

then, $\text{vec}(\mathcal{G})$ is an $m(m+1)/2$ dimensional vector.

Conversely, for an arbitrary $m(m+1)/2$ dimensional vector $\mathbf{g}$, there is a unique $m \times m$ symmetric matrix $\mathcal{G}$ having the corresponding entries. It can be seen that $\text{vec}(\mathcal{G})$ is a bijection between $m \times m$ symmetric matrices and $m(m+1)/2$ dimensional vectors. The inverse of the mapping in (11) for an arbitrary $m(m+1)/2$ dimensional vector $\mathbf{g}$ is defined as $\text{invec}(\mathbf{g})$, which maps vector $\mathbf{g}$ to an $m \times m$ symmetric matrix $\mathcal{G}$.

The $6 \times 6$ impedance matrix $\mathcal{Z}_i$ in (7) linearly depends on the $3 \times 3$ translational impedance matrix $\mathbf{Z}_i$ of the finger at contact location $P_i$. As such, using its vector representation, (7) can be equivalently expressed as

$$\text{vec}(\mathcal{Z}_i) = \mathbf{Q}_i \, \text{vec}(\mathbf{Z}_i) \tag{12}$$

where $\text{vec}(\mathcal{Z}_i) \in \mathbb{R}^{21}$, $\text{vec}(\mathbf{Z}_i) \in \mathbb{R}^{6}$, and the matrix $\mathbf{Q}_i \in \mathbb{R}^{21 \times 6}$ only depends on the finger contact location $P_i(x_i, y_i, z_i)$.

Using (9), the Cartesian impedance $\mathcal{Z}$ provided by the grasp is expressed in vector form as

$$\tilde{\mathbf{z}} = \mathbf{Q}\mathbf{z} \tag{13}$$

where

$$\tilde{\mathbf{z}} = \text{vec}(\mathcal{Z}) \in \mathbb{R}^{21}$$

$$\mathbf{z} = \left[\text{vec}(\mathbf{Z}_1)^T, \text{vec}(\mathbf{Z}_2)^T, \ldots, \text{vec}(\mathbf{Z}_n)^T\right]^T \in \mathbb{R}^{6n}$$

$$\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n] \in \mathbb{R}^{21 \times 6n}.$$

Thus, (13) has a solution if and only if $\tilde{\mathbf{z}}$ is in the column space of $\mathbf{Q}$, $\mathcal{R}(\mathbf{Q})$. To evaluate the space of grasp realizable impedance matrices, the dimension of the column space $\mathcal{R}(\mathbf{Q})$ is further investigated in the following section.

## III. Dimension of the Realizable Impedance Space

In this section, the dimension of $\mathcal{R}(\mathbf{Q})$ is evaluated to determine the dimension of the space of symmetric impedance matrices that can be achieved by a given grasp (which depends on the number of fingertip contacts). Since the trace condition must be satisfied for any number of contacts, only those impedance matrices on the 20-dimensional hyperplane can be achieved, hence

$$\dim \mathcal{R}(\mathbf{Q}) \ \leq \ 20. \tag{14}$$

The impedance provided by a single finger is determined by (7). Since the mapping $\mathbf{Z}_1 \rightarrow \boldsymbol{\mathcal{Z}}_1$ defined by (7) is injective, the $21 \times 6$ matrix $\mathbf{Q}$ for a single finger is full rank. Thus

$$\dim \mathcal{R}(\mathbf{Q}_1) = 6. \tag{15}$$

Therefore, for a given fingertip contact location $P_1$, the space of impedance matrices provided by a single finger (assuming bilateral constraint at contact) is 6 dimensional.

In the following, the dimension of the column space $\mathcal{R}(\mathbf{Q})$ is determined for additional fingers in contact.

### A. Dimension of Realizable Space for Two Fingers

Consider two fingers with fingertip contact at location $P_1$ and $P_2$. If $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are the two $21 \times 6$ matrices associated with the contact locations, then for the grasp

$$\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2] \in \mathbb{R}^{21\times12}$$

$$\mathcal{R}(\mathbf{Q}) = \mathcal{R}([\mathbf{Q}_1, \mathbf{Q}_2]) = \mathcal{R}(\mathbf{Q}_1) + \mathcal{R}(\mathbf{Q}_2)$$

and

$$\begin{aligned}\dim \mathcal{R}(\mathbf{Q}) = \ & \dim \mathcal{R}(\mathbf{Q}_1) + \dim \mathcal{R}(\mathbf{Q}_2) \\ & - \dim\left(\mathcal{R}(\mathbf{Q}_1) \cap \mathcal{R}(\mathbf{Q}_2)\right).\end{aligned} \tag{16}$$

In order to determine the dimension of $\mathcal{R}(\mathbf{Q})$, the intersection of the spaces $\mathcal{R}(\mathbf{Q}_1)$ and $\mathcal{R}(\mathbf{Q}_2)$ needs to be determined.

Suppose that a 21-vector $\tilde{\mathbf{z}}_{12}$ is in $[\mathcal{R}(\mathbf{Q}_1) \cap \mathcal{R}(\mathbf{Q}_2)]$, then, there exists two 6-vectors $\mathbf{z}_1$ and $\mathbf{z}_2$ such that

$$\tilde{\mathbf{z}}_{12} = \mathbf{Q}_1\mathbf{z}_1 = \mathbf{Q}_2\mathbf{z}_2. \tag{17}$$

Denote

$$\begin{aligned}\text{invec}(\tilde{\mathbf{z}}_{12}) &= \boldsymbol{\mathcal{Z}}_{12} \in \mathbb{R}^{6\times6}\\ \text{invec}(\mathbf{z}_1) &= \mathbf{Z}_1 \in \mathbb{R}^{3\times3}\\ \text{invec}(\mathbf{z}_2) &= \mathbf{Z}_2 \in \mathbb{R}^{3\times3}.\end{aligned}$$

Using (7) and (17)

$$\boldsymbol{\mathcal{Z}}_{12} = \begin{bmatrix} \mathbf{Z}_1 & \mathbf{Z}_1\mathbf{P}_1^T \\ \mathbf{P}_1\mathbf{Z}_1 & \mathbf{P}_1\mathbf{Z}_1\mathbf{P}_1^T \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_2 & \mathbf{Z}_2\mathbf{P}_2^T \\ \mathbf{P}_2\mathbf{Z}_2 & \mathbf{P}_2\mathbf{Z}_2\mathbf{P}_2^T \end{bmatrix}. \tag{18}$$

As a consequence, for the intersection of $\mathcal{R}(\mathbf{Q}_1)$ and $\mathcal{R}(\mathbf{Q}_2)$ to exist, it is necessary that

$$\mathbf{Z}_1 = \mathbf{Z}_2, \quad \mathbf{P}_1\mathbf{Z}_1 = \mathbf{P}_2\mathbf{Z}_2$$

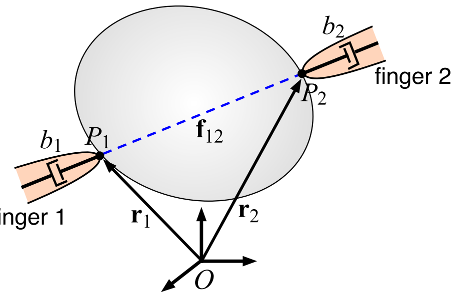


Fig. 2. Intersection of $\mathcal{R}(\mathbf{Q}_1)$ and $\mathcal{R}(\mathbf{Q}_2)$ associated with two-finger contact. If the impedance is algebraic (damping dominated), the intersection corresponds to a 1-dimensional space of linear damping along the line passing through contact locations $P_1$ and $P_2$. The object impedance along this line depends on that provided by finger 1 and that provided by finger 2.

which leads to

$$(\mathbf{P}_1 - \mathbf{P}_2)\mathbf{Z}_1 = \mathbf{0}. \tag{19}$$

Recall that $\mathbf{P}_i$ is the cross-product matrix associated with position $\mathbf{r}_i$ in (6). If $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are the three column vectors of $\mathbf{Z}_1$, and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors indicating the two-finger contact locations $P_1$ and $P_2$, then (19) indicates

$$(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{a}_i = \mathbf{0}, \quad i = 1, 2, 3. \tag{20}$$

Thus, $\mathbf{Z}_1 = \mathbf{Z}_2$ is a rank-1 matrix with each column vector directed along the $(\mathbf{r}_1 - \mathbf{r}_2)$ axis (passing through the two contact points $P_1$ and $P_2$). Mathematically the rank-1 symmetric matrix satisfying (20) can be expressed as

$$\mathbf{Z}_1 = \mathbf{Z}_2 = b\mathbf{f}_{12}\mathbf{f}_{12}^T \tag{21}$$

where $\mathbf{f}_{12}$ is a 3-vector along the direction of $(\mathbf{r}_1 - \mathbf{r}_2)$ and $b$ is a scalar. Note that $\mathbf{P}_1\mathbf{f}_{12} = \mathbf{P}_2\mathbf{f}_{12}$. If $\mathbf{w}_{12}$ is defined as

$$\mathbf{w}_{12} = \begin{bmatrix} \mathbf{f}_{12} \\ \mathbf{P}_1\mathbf{f}_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{12} \\ \mathbf{P}_2\mathbf{f}_{12} \end{bmatrix} \tag{22}$$

then (18) indicates that

$$\boldsymbol{\mathcal{Z}}_{12} = b\mathbf{w}_{12}\mathbf{w}_{12}^T. \tag{23}$$

Since $b$ is an arbitrary scalar

$$\mathcal{R}(\mathbf{Q}_1) \cap \mathcal{R}(\mathbf{Q}_2) = \text{span}\left(\text{vec}(\mathbf{w}_{12}\mathbf{w}_{12}^T)\right) \tag{24}$$

and

$$\dim\left(\mathcal{R}(\mathbf{Q}_1) \cap \mathcal{R}(\mathbf{Q}_2)\right) = 1. \tag{25}$$

Physically, the intersection of the two column spaces $\mathcal{R}(\mathbf{Q}_1)$ and $\mathcal{R}(\mathbf{Q}_2)$ corresponds to those impedance behaviors along the 1-dimensional line of action between the two-finger contact locations $P_1$ and $P_2$. An algebraic (damping dominated) impedance along this line of action can be visualized as two dampers acting in opposition along this direction. The damping behavior of the held object in this direction is determined by a combination of the contribution from each finger, as illustrated in Fig. 2.

Using (16), the dimension of the space $\mathcal{R}(\mathbf{Q})$ is determined to be

$$\dim \mathcal{R}(\mathbf{Q}) = 12 - 1 = 11. \tag{26}$$

Therefore, a two-finger hand can only achieve an 11-dimensional space of impedance behaviors on the 20-dimensional hyperplane.

### *B. Dimension of Realizable Space for Three or More Fingers*

Consider $n \geq 3$ fingers with fingertip contact locations $P_i$ and with matrices $\mathbf{Q}_i \in \mathbb{R}^{21\times 6}$ $(i = 1, 2, \ldots, n)$ defined in (12) for finger $i$. Then

$$\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n] \in \mathbb{R}^{21\times 6n}. \tag{27}$$

Using the results obtained in Section III-A for two fingers, for any two fingers $i$ and $j$

$$\mathcal{R}(\mathbf{Q}_i) \cap \mathcal{R}(\mathbf{Q}_j) = \text{span}\left(\text{vec}(\mathbf{w}_{ij}\mathbf{w}_{ij}^T)\right) \tag{28}$$

where

$$\mathbf{w}_{ij} = \begin{bmatrix} \mathbf{f}_{ij} \\ \mathbf{P}_i\mathbf{f}_{ij} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{ij} \\ \mathbf{P}_j\mathbf{f}_{ij} \end{bmatrix} \tag{29}$$

and $\mathbf{f}_{ij}$ is a 3-vector passing through the contact locations $P_i$ and $P_j$, and

$$\dim\left(\mathcal{R}(\mathbf{Q}_i) \cap \mathcal{R}(\mathbf{Q}_j)\right) = 1. \tag{30}$$

Note that (28)–(30) are the generalizations of (24), (22), and (25) obtained for the two-finger case.

In the generic case (for which any three contact locations are not collinear), the subspaces $\mathcal{R}(\mathbf{Q}_i) \cap \mathcal{R}(\mathbf{Q}_j)$ only intersect at $\mathbf{0}$. Thus, for an $n$-finger ($n \geq 3$) grasp

$$\dim \mathcal{R}(\mathbf{Q}) = \sum_{i=1}^{n} \dim \mathcal{R}(\mathbf{Q}_i) - \sum_{i<j}^{n} \dim\left(\mathcal{R}(\mathbf{Q}_i) \cap \mathcal{R}(\mathbf{Q}_j)\right) \tag{31}$$

Since $\dim \mathcal{R}(\mathbf{Q}_i) = 6$, the first term of (31)

$$\sum_{i=1}^{n} \dim \mathcal{R}(\mathbf{Q}_i) = 6n.$$

Due to (30), the second term of (31)

$$\sum_{i<j}^{n} \dim\left(\mathcal{R}(\mathbf{Q}_i) \cap \mathcal{R}(\mathbf{Q}_j)\right) = C(n,2) = \frac{n!}{2!(n-2)!} = \frac{1}{2}(n-1)n.$$

Thus

$$\dim \mathcal{R}(\mathbf{Q}) = 6n - \frac{1}{2}(n-1)n = \frac{1}{2}(13n - n^2). \tag{32}$$

Note that, since the dimension of $\mathcal{R}(\mathbf{Q})$ cannot exceed 20, the formula in (32) only applies when the number of fingers $n \leq 5$. The dimension of the space $\mathbb{S}$ of realizable impedances by a set of $n$ fingers is given by

$$\begin{aligned} n &= 2, & \dim \mathbb{S} &= 11 \\ n &= 3, & \dim \mathbb{S} &= 15 \\ n &= 4, & \dim \mathbb{S} &= 18 \\ n &\geq 5, & \dim \mathbb{S} &= 20. \end{aligned}$$

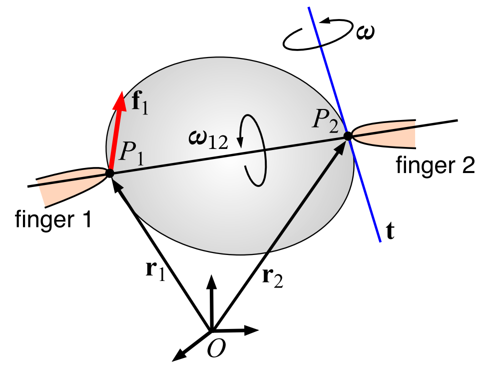


Fig. 3. Impedance achieved by two fingers. A rotation about an arbitrary axis $\boldsymbol{\omega}$ at one-finger contact point results in a pure force $\mathbf{f}_1$ at the other finger contact point. A full-rank impedance cannot be achieved because the grasp is unable to provide rotational impedance along the axis $\boldsymbol{\omega}_{12}$ passing through the two contact locations.

When $n \geq 5$, the realizable space is the entire hyperplane identified in (10).

These results are confirmed using a rank analysis of the $\mathbf{Q}$ matrix for each case. Equivalent results were reported in [36], but the results were based on the rank evaluation without analytical proof and the source of the rank deficiency for each case was not identified.

## IV. Physical Description of the Grasp Achievable Impedance Subspace

The space of achievable impedances by a grasp depends on the number of fingers and the fingertip contact locations on the object. The results of Section III identify the nature and the dimension of the subspace of impedance behaviors that can be achieved by a *given* grasp. This section identifies requirements that must be satisfied by the grasp so that a desired impedance is in the realizable subspace. In other words, the necessary and sufficient conditions for the realization of an impedance with a grasp are identified. The number of fingers in contact with the object determines the conditions.

Consider a given $6 \times 6$ symmetric impedance matrix described in a global frame $Oxyz$ having the partitioned form of (9) and a given grasp for which each fingertip contact location $P_i$ is described in the same global frame by

$$\mathbf{r}_i = [x_i, y_i, z_i]^T. \tag{33}$$

The cross product matrix associated with $\mathbf{r}_i$ is the skew-symmetric matrix $\mathbf{P}_i$ defined in (6).

### *A. Grasp With Two Fingers*

Suppose that the $6 \times 6$ symmetric impedance $\boldsymbol{\mathcal{Z}}$ is achieved by two fingers at contact points $P_1$ and $P_2$. If a pure rotation $\boldsymbol{\omega}$ about point $P_2$ in an arbitrary direction is imposed (as shown in Fig. 3), the twist associated with this pure rotation is described at $O$ as

$$\mathbf{t}_2 = \begin{bmatrix} \mathbf{P}_2\boldsymbol{\omega} \\ \boldsymbol{\omega} \end{bmatrix} = \left[\mathbf{P}_2^T, \mathbf{I}\right]^T \boldsymbol{\omega}. \tag{34}$$

A pure rotation about $P_2$ does not result in any translation at $P_2$ and only yields a pure force $\mathbf{f}_1$ (a no torque wrench) at $P_1$ due to hard-point contact.

Let $\hat{\mathbf{T}}_1$ be the screw transformation from $O$ to $P_1$ [the inverse of $\mathbf{T}_1$ defined in (5)]

$$\hat{\mathbf{T}}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P}_1^T & \mathbf{I} \end{bmatrix}. \tag{35}$$

The wrench expressed in the frame at $P_1$ is

$$\begin{aligned} \mathbf{w}_1 = \hat{\mathbf{T}}_1 \mathcal{Z} \mathbf{t}_2 &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P}_1^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{P}_2 \boldsymbol{\omega} \\ \boldsymbol{\omega} \end{bmatrix} \\ &= \begin{bmatrix} (\mathbf{A}\mathbf{P}_2 + \mathbf{B})\,\boldsymbol{\omega} \\ (\mathbf{P}_1^T \mathbf{A}\mathbf{P}_2 + \mathbf{B}^T\mathbf{P}_2 + \mathbf{P}_1^T\mathbf{B} + \mathbf{H})\,\boldsymbol{\omega} \end{bmatrix}. \end{aligned} \tag{36}$$

As stated previously, the torque at $P_1$ is zero. Thus

$$\left(\mathbf{P}_1^T \mathbf{A}\mathbf{P}_2 + \mathbf{B}^T\mathbf{P}_2 + \mathbf{P}_1^T\mathbf{B} + \mathbf{H}\right)\boldsymbol{\omega} = \mathbf{0}. \tag{37}$$

Since (37) holds for an arbitrary direction of rotation $\boldsymbol{\omega}$, the coefficient matrix that multiplies $\boldsymbol{\omega}$ must be zero, i.e.,

$$\mathbf{P}_1^T \mathbf{A}\mathbf{P}_2 + \mathbf{B}^T\mathbf{P}_2 + \mathbf{P}_1^T\mathbf{B} + \mathbf{H} = [\mathbf{0}]_{3\times 3}. \tag{38}$$

Note that (38) is a matrix equation that contains nine independent scalar equations. Since the trace condition (10) must be satisfied by the Cartesian impedance matrix $\mathcal{Z}$, (38) and (10) together impose 10 constraints on $\mathcal{Z}$. Since the trace condition (10) is not contained in the nine equations of (38), these ten constraints are necessary conditions for an impedance matrix to be achieved by two fingers at specified contact locations. In the following, it is shown that these ten constraints are also sufficient conditions for an impedance matrix to be realized by the two-finger grasp.

Denote $\mathbb{S}_2$ as the collection of all $6 \times 6$ symmetric matrices satisfying the ten constraints identified by both (38) and (10) for two fingers with contact locations $P_1$ and $P_2$. Since these equations impose linear constraints on the entries of the matrices, $\mathbb{S}_2$ is a linear space with

$$\dim \mathbb{S}_2 = 21 - 10 = 11.$$

Consider the matrix $\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2]$ associated with $P_1$ and $P_2$ as defined in (12). If a 21-vector $\text{vec}(\mathcal{Z})$ is in the $\mathcal{Z}$ subspace that can be achieved by two fingers at $P_1$ and $P_2$, then, $\mathcal{Z}$ must be in $\mathbb{S}_2$. Since $\mathbb{S}_2$ and $\mathcal{R}(\mathbf{Q})$ have the same dimension 11

$$\mathcal{Z} \in \mathbb{S}_2 \quad \Longleftrightarrow \quad \text{vec}(\mathcal{Z}) \in \mathcal{R}(\mathbf{Q}).$$

Therefore, any impedance matrix satisfying the ten constraints can be realized by two fingers at the contact locations $P_1$ and $P_2$, which proves the sufficiency of the conditions.

The matrix equation (38) that is expressed in terms of $3 \times 3$ blocks can be equivalently expressed in terms of the $6 \times 6$ impedance matrix as

$$\left[\mathbf{P}_1^T, \mathbf{I}\right] \mathcal{Z} \left[\mathbf{P}_2^T, \mathbf{I}\right]^T = [\mathbf{0}]_{3\times 3}. \tag{39}$$

It can be seen that the constraints expressed in (38) or (39) are symmetric for the 2 points $P_1$ and $P_2$, i.e., if a pure rotation about $P_1$ is considered in the derivation, the same result will be obtained. In summary, we have the following.

*Proposition 1:* A general $6 \times 6$ symmetric impedance $\mathcal{Z}$ partitioned as (9) can be achieved by two fingers at contact locations $P_1$ and $P_2$ if and only if the following two sets of conditions are satisfied:

i) $\text{trace}(\mathbf{B}) = 0$;

ii) $[\mathbf{P}_1^T, \mathbf{I}]\mathcal{Z}[\mathbf{P}_2^T, \mathbf{I}]^T = [\mathbf{0}]_{3\times 3}$.

Also note that, due to hard-point contact, a pure rotation $\boldsymbol{\omega}_{12}$ about the axis passing through both point $P_1$ and $P_2$ results in a zero wrench at each contact location. Thus

$$\mathcal{Z} \begin{bmatrix} \mathbf{P}_1 \boldsymbol{\omega}_{12} \\ \boldsymbol{\omega}_{12} \end{bmatrix} = \mathcal{Z} \begin{bmatrix} \mathbf{P}_2 \boldsymbol{\omega}_{12} \\ \boldsymbol{\omega}_{12} \end{bmatrix} = \mathbf{0}. \tag{40}$$

Since $\boldsymbol{\omega}_{12}$ passes through $P_1$ and $P_2$, it is along the line determined by $(\mathbf{r}_1 - \mathbf{r}_2)$. If $\mathbf{t}_{12}$ is defined as

$$\mathbf{t}_{12} = \begin{bmatrix} \mathbf{P}_2(\mathbf{r}_1 - \mathbf{r}_2) \\ (\mathbf{r}_1 - \mathbf{r}_2) \end{bmatrix} \tag{41}$$

then the screw description of $\boldsymbol{\omega}_{12}$ can be expressed as

$$\begin{bmatrix} \mathbf{P}_1 \boldsymbol{\omega}_{12} \\ \boldsymbol{\omega}_{12} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_2 \boldsymbol{\omega}_{12} \\ \boldsymbol{\omega}_{12} \end{bmatrix} = \gamma \mathbf{t}_{12}$$

where $\gamma \neq 0$ is a scalar. Then, (40) is equivalent to

$$\mathcal{Z}\mathbf{t}_{12} = \mathbf{0}. \tag{42}$$

This constraint [implied by (39) and (10)] has evident physical significance: a two-finger grasp cannot provide any rotational impedance about the axis passing through the two-finger contact points $P_1$ and $P_2$, as shown in Fig. 3. Thus, no full-rank impedance can be achieved by two fingers.

This observation is consistent with that observed in evaluating grasp stability [23].

### *B. Grasp With Three Fingers*

Now consider a three-finger grasp with noncollinear contact locations $P_1$, $P_2$, and $P_3$, as shown in Fig. 4. Denote

$$\mathbf{t}_{ij} = \begin{bmatrix} \mathbf{P}_i(\mathbf{r}_i - \mathbf{r}_j) \\ (\mathbf{r}_i - \mathbf{r}_j) \end{bmatrix} \tag{43}$$

then, $\text{span}(\mathbf{t}_{ij})$ represents all twists associated with pure rotation about the axis passing through points $P_i$ and $P_j$.

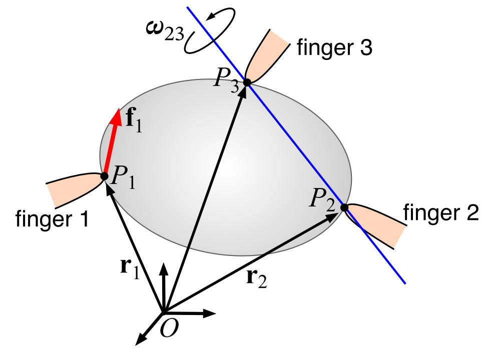

Fig. 4. Impedance achieved by three fingers. A pure rotation about the axis passing points $P_2$ and $P_3$ results in a pure force $\mathbf{f}_1$ at $P_1$.

For example, consider a pure rotation $\boldsymbol{\omega}_{23}$ about the axis passing through $P_2$ and $P_3$, as shown in Fig. 4. This motion, represented by twist $\mathbf{t}_{23}$, only causes a pure force $\mathbf{f}_1$ at $P_1$. Thus, a set of constraints $\mathcal{C}_1$ is obtained

$$\mathcal{C}_1: \quad [\mathbf{P}_1^T, \mathbf{I}]\, \boldsymbol{\mathcal{Z}} \mathbf{t}_{23} = \mathbf{0} \tag{44}$$

where $\mathbf{t}_{23}$ is the twist defined in (43) corresponding to a pure rotation about the axis passing through $P_2$ and $P_3$. Note that the constraint set $\mathcal{C}_1$ provides three independent constraints on $\boldsymbol{\mathcal{Z}}$.

Using the same reasoning for pure rotation twists $\mathbf{t}_{31}$ and $\mathbf{t}_{12}$, respectively, two additional sets of constraints, $\mathcal{C}_2$ and $\mathcal{C}_3$ are obtained

$$\mathcal{C}_2: \quad [\mathbf{P}_2^T, \mathbf{I}]\, \boldsymbol{\mathcal{Z}} \mathbf{t}_{31} = \mathbf{0} \tag{45}$$

$$\mathcal{C}_3: \quad [\mathbf{P}_3^T, \mathbf{I}]\, \boldsymbol{\mathcal{Z}} \mathbf{t}_{12} = \mathbf{0}. \tag{46}$$

Although (44)–(46) provide nine constraints, they are not all independent. In fact, condition (44) implies that wrench $\mathbf{w}_1 = \boldsymbol{\mathcal{Z}}\mathbf{t}_{23}$ is a pure force at $P_1$. Therefore, any pure rotation about an axis passing through $P_1$ must be reciprocal to $\mathbf{w}_1$. For twist $\mathbf{t}_{31}$ representing a pure rotation about the axis passing through $P_1$ and $P_3$

$$\mathbf{t}_{31}^T \boldsymbol{\mathcal{Z}} \mathbf{t}_{23} = 0. \tag{47}$$

Similarly, condition (45) implies

$$\mathbf{t}_{23}^T \boldsymbol{\mathcal{Z}} \mathbf{t}_{31} = 0. \tag{48}$$

Since $\boldsymbol{\mathcal{Z}}$ is symmetric, (47) and (48) are identical conditions and provide the same linear constraint corresponding to a 20-dimensional hyperplane. Thus, the constraint set $\mathcal{C}_1$ and constraint set $\mathcal{C}_2$ contain a common constraint, the hyperplane $\boldsymbol{\mathcal{H}}_{12}$.

In general, any 2 sets of constraints $\mathcal{C}_i$ and $\mathcal{C}_j$ have a common constraint $\boldsymbol{\mathcal{H}}_{ij}$. As such, among the 6 constraints given by $\mathcal{C}_i$ and $\mathcal{C}_j$, only five are independent and among all nine constraints given by (44)–(46) only six are independent. Therefore, if all conditions (44)–(46) are satisfied, the space of realizable impedance matrices is $21 - 6 = 15$ dimensional, which is the dimension of the impedance space (the column space of matrix $\mathbf{Q}$ described in Section III-B) associated with three fingers. Thus, constraints (44)–(46) are also sufficient conditions for an impedance matrix to be achieved by three fingers at the given contact locations.

As stated earlier, the trace condition (10) is a necessary condition for an impedance to be realized with any number of fingers. As proved in Appendix A, the three sets of constraints (44)–(46) imply the trace condition, i.e., if conditions (44)–(46) are satisfied, the trace condition is also satisfied. Thus, when the three constraint sets are considered, a separate trace condition (10) is no longer necessary. Alone, any two sets of constraints in (44)–(46), however, do not imply the trace condition. Since any two sets of constraints $\mathcal{C}_i$ and $\mathcal{C}_j$ provide five independent constraints, if the trace condition (10) is considered in addition, the number of independent constraints is 6, which is sufficient to achieve the impedance with the three-finger contact locations.

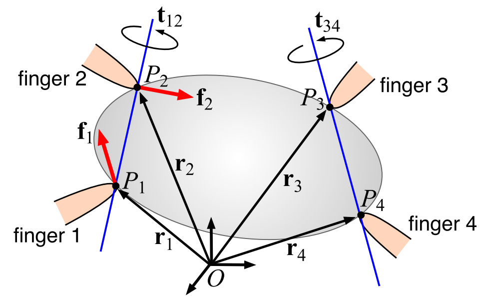


Fig. 5. Impedance achieved by four fingers. A pure rotation twist $\mathbf{t}_{34}$ about the axis passing points $P_3$ and $P_4$ results in a wrench reciprocal to the twist $\mathbf{t}_{12}$ associated with a pure rotation about the axis passing through points $P_1$ and $P_2$.

In summary, we have the following.

*Proposition 2:* A general $6 \times 6$ symmetric impedance $\boldsymbol{\mathcal{Z}}$ can be achieved by three fingers at noncollinear contact locations $P_1$, $P_2$, and $P_3$ if and only if either one of the following conditions is satisfied:

i) all three sets of constraints of (44)–(46);
ii) any two sets of constraints from (44)–(46) and the trace condition (10).

Note that, if Condition (i) of Proposition 2 is considered, since there are only six independent constraints from the nine equations, one can choose to satisfy just six equations. These six equations, however, must be selected from all three sets. If Condition (ii) of the proposition is considered, one can choose any five constraints from any two sets of constraints from (44) to (46) together with the trace condition (10).

### C. Grasp With Four Fingers

Consider a four-finger grasp at contact locations $P_1$, $P_2$, $P_3$, and $P_4$, as shown in Fig. 5. If the object is subjected to a pure rotation twist $\mathbf{t}_{34}$ about the axis passing through points $P_3$ and $P_4$, the motion can only yield a pure force $\mathbf{f}_1$ at $P_1$ and a pure $\mathbf{f}_2$ at $P_2$. Thus, the resultant wrench $\mathbf{w}_{34} = \boldsymbol{\mathcal{Z}}\mathbf{t}_{34}$ must be reciprocal to the twist $\mathbf{t}_{12}$, which corresponds to a pure rotation about the axis passing through points $P_1$ and $P_2$. Thus

$$\mathbf{t}_{12}^T \boldsymbol{\mathcal{Z}} \mathbf{t}_{34} = 0. \tag{49}$$

Similarly, we have

$$\mathbf{t}_{31}^T \boldsymbol{\mathcal{Z}} \mathbf{t}_{24} = 0 \tag{50}$$

$$\mathbf{t}_{14}^T \boldsymbol{\mathcal{Z}} \mathbf{t}_{23} = 0. \tag{51}$$

Each of the three equations of (49)–(51) defines a 20-dimensional hyperplane. In the generic case (for which no three contact locations are collinear), any two hyperplanes are not coincident. Thus, the three constraints (49)–(51) are independent. The space of impedance matrices achievable by the four fingers is the intersection of the three hyperplanes, which is 18 dimensional.

Similar to the proof for the three-finger case (in Appendix A), it can be proved that the trace condition (10) is implied by the three constraints (49)–(51), i.e., if all three constraints are satisfied, the trace condition is also satisfied. Thus, the three constraints are also sufficient conditions for an impedance to be achieved by the four-finger grasp. Since the trace condition is much simpler, one of the three constraints from (49) to (51) can be replaced by the trace condition.

In summary, we have the following.

*Proposition 3:* A general $6 \times 6$ symmetric impedance $\mathcal{Z}$ can be achieved by four fingers at contact locations $P_1$, $P_2$, $P_3$, and $P_4$ (for which any 3 are not collinear) if and only if either one of the following constraint sets is satisfied:

i) the three conditions of (49)–(51);
ii) any two conditions from (49) to (51) and the trace condition (10).

### D. Grasp With Five or More Fingers

As described in Section III-B, for a grasp with five or more fingers, the realizable space of impedances $\mathbb{S}$ is a 20-dimensional linear space. Since the trace condition (10) defines a 20-dimensional linear space (a hyperplane) in the 21-dimensional space of all symmetric impedance matrices, this hyperplane must be identical to $\mathbb{S}$, i.e., any impedance satisfying the trace condition (10) can be achieved by a grasp having five or more fingers. Thus, we have the following.

*Proposition 4:* A general $6 \times 6$ symmetric impedance $\mathcal{Z}$ can be achieved by a grasp having 5 or more fingers (for which no 3 contact locations are collinear) if and only if the trace condition (10) is satisfied.

Note that for a grasp having five or more fingers, the location of each fingertip contact with the object is arbitrary for the generic case in which any three fingertip locations are not collinear.

## V. General Impedance

Suppose that each finger is capable of providing an arbitrary impedance at contact point $P_i$, then the contact Cartesian $3 \times 3$ impedance matrix is not symmetric. In the following, the process used for the symmetric case above is modified for application to the general case.

### A. Trace Condition

Suppose that at contact point $P_i$ the finger impedance is $\mathbf{Z}_i$, then, due to hard-point contact, the corresponding impedance matrix in a global coordinate frame has the same form as (7)

$$\mathcal{Z}_i = \begin{bmatrix} \mathbf{Z}_i & \mathbf{Z}_i\mathbf{P}_i^T \\ \mathbf{P}_i\mathbf{Z}_i & \mathbf{P}_i\mathbf{Z}_i\mathbf{P}_i^T \end{bmatrix}. \tag{52}$$

Note that now, since $\mathbf{Z}_i$ is not symmetric, the $6 \times 6$ matrix $\mathcal{Z}_i$ is not symmetric and the trace condition (10) no longer captures the restriction on the achievable impedances. In the following, an equivalent condition is derived which shows that the sum of the traces of the 2 off-diagonal blocks is zero.

First consider the impedance matrix $\mathcal{Z}_i$ in (52) associated with a single finger at $P_i$. Since a matrix transpose does not change the trace of a matrix

$$\text{trace}(\mathbf{Z}_i\mathbf{P}_i^T) = \frac{1}{2}\left[\text{trace}(\mathbf{Z}_i\mathbf{P}_i^T) + \text{trace}(\mathbf{P}_i\mathbf{Z}_i^T)\right] \tag{53}$$

$$\text{trace}(\mathbf{P}_i\mathbf{Z}_i) = \frac{1}{2}\left[\text{trace}(\mathbf{P}_i\mathbf{Z}_i) + \text{trace}(\mathbf{Z}_i^T\mathbf{P}_i^T)\right]. \tag{54}$$

Summing (53) and (54) yields

$$\begin{aligned} &\text{trace}(\mathbf{Z}_i\mathbf{P}_i^T) + \text{trace}(\mathbf{P}_i\mathbf{Z}_i) \\ &= \frac{1}{2}\left[\text{trace}((\mathbf{Z}_i + \mathbf{Z}_i^T)\mathbf{P}_i^T) + \text{trace}(\mathbf{P}_i(\mathbf{Z}_i + \mathbf{Z}_i^T))\right]. \end{aligned}$$

Since $(\mathbf{Z}_i + \mathbf{Z}_i^T)$ is symmetric and $\mathbf{P}_i$ is skew-symmetric

$$\text{trace}((\mathbf{Z}_i + \mathbf{Z}_i^T)\mathbf{P}_i^T) = \text{trace}(\mathbf{P}_i(\mathbf{Z}_i + \mathbf{Z}_i^T)) = 0.$$

Therefore

$$\text{trace}(\mathbf{Z}_i\mathbf{P}_i^T) + \text{trace}(\mathbf{P}_i\mathbf{Z}_i) = 0.$$

The Cartesian impedance $\mathcal{Z}$ for $n$ fingers is

$$\mathcal{Z} = \sum_{i=1}^{n} \begin{bmatrix} \mathbf{Z}_i & \mathbf{Z}_i\mathbf{P}_i^T \\ \mathbf{P}_i\mathbf{Z}_i & \mathbf{P}_i\mathbf{Z}_i\mathbf{P}_i^T \end{bmatrix}.$$

Thus, the sum of the traces of the 2 off-diagonal blocks is

$$\sum_{i=1}^{n}\left[\text{trace}(\mathbf{Z}_i\mathbf{P}_i^T) + \text{trace}(\mathbf{P}_i\mathbf{Z}_i)\right] = 0.$$

Thus, for a general $6 \times 6$ impedance matrix realized by a set of fingers in hard-point contact having the form

$$\mathcal{Z} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{D} & \mathbf{H} \end{bmatrix} \tag{55}$$

the trace condition

$$\text{trace}(\mathbf{B}) + \text{trace}(\mathbf{D}) = 0 \tag{56}$$

must be satisfied for $\mathcal{Z}$ to be achieved for any number of fingers. This trace restriction was identified in [36] without proof. Similar to the symmetric case, the source of this restriction is due to the physics of hard-point contact: each fingertip only transmits pure force to the held object through the point of contact. The space of general $6 \times 6$ symmetric matrices is 36 dimensional. The linear trace condition (56) defines a 35-dimensional hyperplane in the 36-dimensional space.

### B. Dimension of the Realizable Space

Using matrix vectorization similar to that described in Section II, a general $6 \times 6$ Cartesian impedance matrix $\mathcal{Z}$ can be represented by a 36-vector $\tilde{\mathbf{z}}$ and a general translational $3 \times 3$ matrix $\mathbf{Z}_i$ at location $P_i$ can be represented by a 9-vector. Similar to (13), the vectorized relationship between the object Cartesian impedance and the finger impedance at each contact location can be expressed as

$$\tilde{\mathbf{z}} = \mathbf{Q}\mathbf{z} \tag{57}$$

where the parameters of (13) now have higher dimension

$$\begin{aligned} \tilde{\mathbf{z}} &= \text{vec}(\mathcal{Z}) \in \mathbb{R}^{36} \\ \mathbf{z} &= \left[\text{vec}(\mathbf{Z}_1)^T, \text{vec}(\mathbf{Z}_2)^T, \ldots, \text{vec}(\mathbf{Z}_n)^T\right]^T \in \mathbb{R}^{9n} \\ \mathbf{Q} &= [\mathbf{Q}_1, \mathbf{Q}_2, \ldots, \mathbf{Q}_n] \in \mathbb{R}^{36\times 9n}. \end{aligned}$$

A given impedance $\mathcal{Z}$ can be achieved by an $n$-finger grasp at contact locations $P_i$ $(i = 1, 2, \ldots, n)$ if and only if $\tilde{\mathbf{z}}$ is in the column space of $\mathbf{Q}$, $\mathcal{R}(\mathbf{Q})$. Due to the trace condition (56), $\mathcal{Z}$ must lie on a 35-dimensional hyperplane.

Similar to the symmetric case, for any two fingers $i$ and $j$

$$\mathcal{R}(\mathbf{Q}_i) \cap \mathcal{R}(\mathbf{Q}_j) = \text{span}\left(\text{vec}(\mathbf{w}_{ij}\mathbf{w}_{ij}^T)\right) \tag{58}$$

where $\mathbf{w}_{ij}$ is the 6-vector wrench passing through $P_i$ and $P_j$ defined in (29), and

$$\dim \mathcal{R}(\mathbf{Q}_i) \cap \mathcal{R}(\mathbf{Q}_j) = 1. \tag{59}$$

Note that, unlike the symmetric case (28), the vectorized representation of matrix $\mathbf{w}_{ij}\mathbf{w}_{ij}^T$ in (58) is now 36 dimensional. Similar to the symmetric case (32), the dimension of $\mathcal{R}(\mathbf{Q})$ can be calculated using

$$\dim \mathcal{R}(\mathbf{Q}) = 9n - \frac{1}{2}(n-1)n = \frac{1}{2}(19n - n^2). \tag{60}$$

Since the dimension of $\mathcal{R}(\mathbf{Q})$ cannot exceed 35, the formula in (60) only applies when the number of fingers $n \leq 5$. The dimensions of the space $\mathbb{S}$ achievable by an $n$ finger grasp are summarized

$$\begin{aligned} n &= 2, & \dim \mathbb{S} &= 17 \\ n &= 3, & \dim \mathbb{S} &= 24 \\ n &= 4, & \dim \mathbb{S} &= 30 \\ n &\geq 5, & \dim \mathbb{S} &= 35. \end{aligned}$$

### *C. General Impedance Realization Conditions*

The realization conditions on the impedance matrices can be derived using the same arguments presented in Section IV for the symmetric case. For each number-of-fingers case, the derivation is based on the hard-point contact model for which only pure force can be transmitted at the fingertips.

*1) Grasp With Two Fingers:* For a two-finger grasp at contact locations $P_1$ and $P_2$, a pure rotation about $P_2$ causes a pure force at $P_1$. Thus, (48) must be satisfied

$$\left[\mathbf{P}_1^T, \mathbf{I}\right] \mathcal{Z} \left[\mathbf{P}_2^T, \mathbf{I}\right]^T = [\mathbf{0}]_{3\times 3}. \tag{61}$$

Using the same argument for a pure rotation about $P_1$, we have

$$\left[\mathbf{P}_2^T, \mathbf{I}\right] \mathcal{Z} \left[\mathbf{P}_1^T, \mathbf{I}\right]^T = [\mathbf{0}]_{3\times 3}. \tag{62}$$

Note that since $\mathcal{Z}$ is not symmetric, conditions (61) and (62) are not equivalent. Although the two sets of conditions provide a total of 18 constraints, these constraints are not independent. In fact, it is evident that the two sets of conditions (61) and (62) contain a common constraint

$$\mathbf{t}_{12}^T \mathcal{Z} \mathbf{t}_{12} = 0 \tag{63}$$

which is a hyperplane in the 36-dimensional impedance space. Thus, only 17 of the 18 constraints in (61) and (62) are independent.

As stated in Section IV-A, (42) must be satisfied for two hard-point contact locations. For nonsymmetric $\mathcal{Z}$, condition (42) (which contains six constraints) is not implied by (61) and (62) and the trace condition (56). Thus, at least one constraint from (42) is necessary. Using the same reasoning used for the symmetric case, it can be proved that a general impedance $\mathcal{Z}$ can be achieved by two fingers if and only if conditions (61), (62), (42), and (56) are satisfied. Among all those constraints, only 19 are independent. Thus, in application, one can select any 17 constraints from (61) and (62), and one constraint from (42) together with the trace condition (56) to be the necessary and sufficient conditions on $\mathcal{Z}$ be to realized by two fingers at given contact locations $P_1$ and $P_2$.

Similar to the symmetric case addressed in Section VI-A, no full-rank impedance can be achieved by two fingers.

*2) Grasp With Three Fingers:* For a three-finger grasp at noncollinear contact locations $P_1$, $P_2$, and $P_3$, a pure rotation about the axis passing through 2 points $P_i$ and $P_j$ must cause a pure force at the third point $P_r$. Thus, (44)–(46) must be also satisfied for nonsymmetric impedance matrices.

Now consider an arbitrary rotation $\boldsymbol{\omega}$ about point $P_i$. Similar to (34), the twist associated with this rotation can be expressed as

$$\mathbf{t}_i = \left[\mathbf{P}_i^T, \mathbf{I}\right]^T \boldsymbol{\omega}.$$

Due to hard-point contact, the wrench generated by this twist, $\mathbf{w}_i = \mathcal{Z}\mathbf{t}_i$, is associated with a pure force at the other 2 points $P_j$ and $P_r$ and, therefore, must be reciprocal to the twist $\mathbf{t}_{jr}$ associated with a pure rotation about the axis passing through the other 2 points $P_j$ and $P_r$

$$\mathbf{t}_{jr}^T \mathcal{Z} \left[\mathbf{P}_i^T, \mathbf{I}\right]^T \boldsymbol{\omega} = \mathbf{0}.$$

Since $\boldsymbol{\omega}$ is arbitrary, the coefficient (row) vector $\mathbf{t}_{jr}^T \mathcal{Z}[\mathbf{P}_i^T, \mathbf{I}]^T$ multiplying $\boldsymbol{\omega}$ must be zero, which is equivalent to

$$\left[\mathbf{P}_i^T, \mathbf{I}\right] \mathcal{Z}^T \mathbf{t}_{jr} = \mathbf{0}.$$

In summary, the combination of this with conditions (44)–(46) require that, for a nonsymmetric $\mathcal{Z}$, the following conditions are satisfied:

$$\left[\mathbf{P}_1^T, \mathbf{I}\right] \mathcal{Z} \mathbf{t}_{23} = \mathbf{0}, \quad \left[\mathbf{P}_1^T, \mathbf{I}\right] \mathcal{Z}^T \mathbf{t}_{23} = \mathbf{0} \tag{64}$$

$$\left[\mathbf{P}_2^T, \mathbf{I}\right] \mathcal{Z} \mathbf{t}_{13} = \mathbf{0}, \quad \left[\mathbf{P}_2^T, \mathbf{I}\right] \mathcal{Z}^T \mathbf{t}_{13} = \mathbf{0} \tag{65}$$

$$\left[\mathbf{P}_3^T, \mathbf{I}\right] \mathcal{Z} \mathbf{t}_{12} = \mathbf{0}, \quad \left[\mathbf{P}_3^T, \mathbf{I}\right] \mathcal{Z}^T \mathbf{t}_{12} = \mathbf{0}. \tag{66}$$

As proved in Appendix B, among the 18 equations (64)–(66), only 12 are independent. This set corresponds to the necessary and sufficient conditions for realization of a general impedance with three fingers.

*3) Grasp With Four Fingers:* For a four-finger grasp at contact locations $P_1$, $P_2$, $P_3$, and $P_4$ (for which no 3 are collinear), a pure rotation $\mathbf{t}_{ij}$ causes a pure force $\mathbf{f}_r$ at $P_r$ and a pure force $\mathbf{f}_s$ at $P_s$. Thus, the $\mathbf{w}_{ij}$ caused by $\mathbf{t}_{ij}$ must be reciprocal to twist $\mathbf{t}_{rs}$ responding to a pure rotation about the axis passing through $P_r$ and $P_s$. Thus

$$\mathbf{t}_{12}^T \mathcal{Z} \mathbf{t}_{34} = 0, \quad \mathbf{t}_{34}^T \mathcal{Z} \mathbf{t}_{12} = 0 \tag{67}$$

$$\mathbf{t}_{13}^T \mathcal{Z} \mathbf{t}_{24} = 0, \quad \mathbf{t}_{24}^T \mathcal{Z} \mathbf{t}_{13} = 0 \tag{68}$$

$$\mathbf{t}_{14}^T \boldsymbol{\mathcal{Z}} \mathbf{t}_{23} = 0, \quad \mathbf{t}_{23}^T \boldsymbol{\mathcal{Z}} \mathbf{t}_{14} = 0. \tag{69}$$

These six equations are independent and are the necessary and sufficient conditions for a general impedance to be achieved by four fingers.

*4) Grasp With Five or More Fingers:* Similar to the symmetric case, the 35-dimensional hyperplane defined by (56) is the entire space of impedances that can be realized by a grasp with five or more fingers (for which no 3 contact locations are collinear). Thus, the trace condition (56) is the necessary and sufficient condition for a general impedance to be achieved by five or more fingers.

## VI. Discussion

In this section, implications of the results presented in Sections III–V and their application are discussed. The restrictions identified in Propositions 1–4 are *necessary and sufficient* conditions for the realization of a given symmetric impedance matrix. In other words, a desired impedance can only be realized if the contact locations satisfy the conditions and, if they do, this desired behavior can always be obtained if there are no restrictions on the nature of impedances that can be imposed at the contact location. This assumes that there are no additional restrictions imposed by the following:

1) how the fingertip impedance is to be realized;
2) the size of the hand;
3) the geometry of the held object.

In this section, these task and geometry considerations are addressed. The application of the developed theory to planar impedances is also addressed.

### A. Fingertip Impedance and Object Cartesian Impedance

The results obtained in Sections III–V are based on the assumption that each finger is able to provide an arbitrary $3 \times 3$ symmetric or asymmetric translational impedance matrix at the fingertip contact location. The constraints identified on the Cartesian impedance matrix are valid for all grasps in hard-point contact regardless of whether the impedance at each fingertip is achieved passively or actively, or whether coupling exists between joints.

If the finger impedance is restricted in some way by the type of realization, the constraints presented in Propositions 1–4 are only *necessary* conditions. For these cases, additional conditions are needed to achieve a desired object Cartesian impedance. For example, if each finger only provides passive impedance at a contact location, then additional conditions must be considered to ensure that the finger impedance at each contact location is positive semidefinite.

### B. Finger Contact Locations and the Space of Impedances

Similarly, the necessary and sufficient conditions of Propositions 1–4 are reduced to necessary conditions if there are restrictions on the fingertip contact locations. Checking the conditions for a *given* set of contact locations is not difficult. One would just need to calculate the transformation matrix $\mathbf{P}_i$ for each contact location and substitute them into the corresponding set of equations. To achieve all conditions, one would need to solve a set of (underconstrained) equations to determine the contact location and the fingertip impedance of each finger as described in the following.

The dimension of the realizable impedance space for a given number of fingers and the constraints on the Cartesian impedances are based on a predetermined grasp, i.e., the previously identified fingertip contact locations. If the contact locations change, the realizable impedance space changes (but the dimension of the space does not). Thus, any given impedance matrix may be achieved by properly selecting the contact locations of a grasp with three or more fingers.

### C. Impedance Realization With a Three-Finger Grasp

As previously stated, a full-rank impedance matrix requires at least three fingers. A minimum-finger realization procedure for an arbitrary full-rank impedance matrix is developed as follows.

Consider a three-finger hand having fingertip contact locations

$$\mathbf{r}_i = [x_i, y_i, z_i]^T, \quad i = 1, 2, 3$$

described in the same global frame used to describe a desired symmetric Cartesian impedance matrix $\boldsymbol{\mathcal{Z}}$.

For specified contact locations $\mathbf{r}_i$ $(i = 1, 2, 3)$, the set of realizable impedance matrices is a 15-dimensional subspace of the entire 20-dimensional impedance space achievable by any contact grasp. Because each of the three contact locations $\mathbf{r}_i$ changes in 3-dimensional space, there are nine free variables. By Proposition 2, to achieve the given impedance $\boldsymbol{\mathcal{Z}}$, these nine variables $(x_i, y_i, z_i)$ $(i = 1, 2, 3)$ must be selected so that five equations described in Proposition 2 are satisfied. Since the number of variables is more than the number of equations, there are infinitely many grasps that satisfy the equations (if a solution exists). In the following, a process to select a three-finger grasp that achieves an arbitrary symmetric $6 \times 6$ impedance matrix $\boldsymbol{\mathcal{Z}}$ is presented. The process has two main steps: 1) determine the finger contact locations, and 2) select the finger impedance at each location. Each step is described in more detail as follows.

*1) Determine the Three-Finger Contact Locations:* To simplify the problem, one contact location can be specified arbitrarily. The simplified problem requires that the other two locations (six variables) be selected to satisfy five equations. For possible realization with a grasp, $\boldsymbol{\mathcal{Z}}$ must satisfy the trace condition (10), so if the other two contact locations satisfy Condition (ii) of Proposition 2, the selected impedance can be realized.

For the case in which the location of $P_3$ is selected arbitrarily, $\mathbf{r}_1$ and $\mathbf{r}_2$ must satisfy (44) and (45), respectively. Suppose that, in the coordinate frame at $P_3$, the impedance matrix has the form

$$\hat{\boldsymbol{\mathcal{Z}}} = \begin{bmatrix} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hat{\mathbf{B}}^T & \hat{\mathbf{H}} \end{bmatrix} = \hat{\mathbf{T}}_3 \boldsymbol{\mathcal{Z}} \hat{\mathbf{T}}_3^T \tag{70}$$

where $\hat{\mathbf{T}}_3$ is the $6 \times 6$ coordinate transformation matrix from the global frame to point $P_3$ [the inverse of $\mathbf{T}$ defined in (5)]. Since $\mathbf{P}_3 = \mathbf{0}$ in this coordinate frame, substituting the partitioned $\hat{\boldsymbol{\mathcal{Z}}}$ of (70) into (44) and (45) yields

$$\hat{\mathbf{P}}_1^T \hat{\mathbf{B}} \hat{\mathbf{r}}_2 + \hat{\mathbf{H}} \hat{\mathbf{r}}_2 = \mathbf{0} \tag{71}$$

$$\hat{\mathbf{P}}_2^T \hat{\mathbf{B}} \hat{\mathbf{r}}_1 + \hat{\mathbf{H}} \hat{\mathbf{r}}_1 = \mathbf{0} \tag{72}$$

where $\hat{\mathbf{P}}_i$ is the skew symmetric matrix defined in (6) now associated with the relative positions of contact points $\hat{\mathbf{r}}_i$ and where

$$\hat{\mathbf{r}}_i = \mathbf{r}_i - \mathbf{r}_3 = [\hat{x}_i, \hat{y}_i, \hat{z}_i]^T, \quad i = 1, 2. \tag{73}$$

Note that, since the trace condition (10) is invariant under coordinate transformation, the trace condition for $\hat{\mathbf{B}}$ is satisfied. Also, as shown in Section IV-B, among the six equations (71) and (72), only five are independent. Therefore, there are infinitely many solutions to the set of equations. When the locations $\hat{\mathbf{r}}_i$ relative to $P_3$ are determined, the two contact locations $P_i$ in the global frame can be obtained using (73).

For a given task, the surface of the object can be described by a function $f(x, y, z) = 0$. Since the fingertip contact locations must be on the object surface, each $\mathbf{r}_i$ must satisfy

$$f(x_i, y_i, z_i) = 0. \tag{74}$$

This condition should be considered as a constraint in addition to (71) and (72). If on the object surface $f$, (71) and (72) have no solutions, then, the desired impedance cannot be achieved by any three-finger grasp for the object. A grasp with more fingers (at least 4) should be considered.

Also note that among the sets of solutions to (71) and (72), not every one is suitable for a grasp even if constraint (74) is satisfied. For a stable grasp, a set of contact locations should be judiciously selected from the solution space such that the held object is fully constrained by the fingertips.

*2) Find the Fingertip Impedance at Each Contact Location:* For the obtained three contact locations $\mathbf{r}_i$, matrix $\mathbf{Q}$ in (13) is calculated using (13). By Proposition 2, $\boldsymbol{\mathcal{Z}}$ can be achieved using a three-finger grasp at these contact locations. Thus, $\tilde{\mathbf{z}} = \mathrm{vec}(\boldsymbol{\mathcal{Z}}) \in \mathcal{R}(\mathbf{Q})$ [which guarantees a solution to (13)]. Since $\mathbf{Q} \in \mathbb{R}^{21\times18}$ and $\mathrm{rank}(\mathbf{Q}) = 15$, the number of columns of $\mathbf{Q}$ is greater than its rank. Thus, the set of linear equations (13) has infinitely many solutions. The entire solution space $\mathbb{Z}_q$ is the sum of a particular solution and the null space of $\mathbf{Q}$ (the space of homogeneous solutions).

Choose one solution $\mathbf{z}$ from $\mathbb{Z}_q$

$$\mathbf{z} = [\mathbf{z}_1^T, \mathbf{z}_2^T, \mathbf{z}_3^T]^T, \quad \mathbf{z}_i \in \mathbb{R}^6. \tag{75}$$

The set of finger impedances at contact location $P_i$ associated with this solution $\mathbf{z}$ is

$$\mathbf{Z}_i = \mathrm{invec}(\mathbf{z}_i) \in \mathbb{R}^{3\times3}, \quad i = 1, 2, 3. \tag{76}$$

As such, the given impedance matrix is achieved by the three-finger grasp.

### *D. Planar Impedance*

The results presented in Sections III–V can be used for the planar case for both symmetric and nonsymmetric impedances. In the following, only the symmetric case is considered.

A planar symmetric Cartesian impedance $\boldsymbol{\mathcal{Z}} \in \mathbb{R}^{3\times3}$ can be represented as a 6-vector. The translational impedance $\mathbf{Z}_i$ at each contact location $P_i$ is a $2 \times 2$ symmetric matrix that can be represented as a 3-vector. Based on the results obtained for spatial symmetric impedances, the dimension of the space $\mathbb{S}_p$ associated with planar impedance achievable by $n$ fingers are

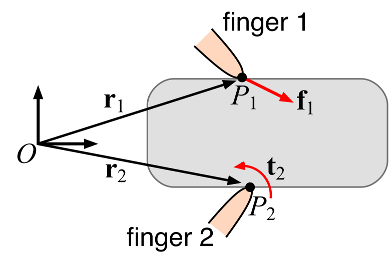


Fig. 6. Planar object grasped by two fingers at contact locations $P_1$ and $P_2$. A pure rotation $\mathbf{t}_2$ about $P_2$ causes a pure force $\mathbf{f}_1$ at $P_1$.

$$n = 2, \quad \dim \mathbb{S}_p = 5$$
$$n \geq 3, \quad \dim \mathbb{S}_p = 6.$$

The necessary and sufficient conditions for a planar impedance to be realized with a grasp are addressed in the following.

*1) Two-Finger Grasp:* Consider a two-finger grasp with contact locations $P_1$ and $P_2$, as shown in Fig. 6. A pure rotation $\mathbf{t}_2$ about $P_2$ causes a pure force $\mathbf{f}_1$ at $P_1$, which yields a result equivalent to (39) derived for the spatial case

$$\left[\mathbf{p}_1^T, 1\right] \boldsymbol{\mathcal{Z}} \left[\mathbf{p}_2^T, 1\right]^T = 0 \tag{77}$$

where $\mathbf{p}_i \in \mathbb{R}^2$ is the translational transformation vector associated with position $\mathbf{r}_i$ of $P_i$ in the global frame

$$\mathbf{p}_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{r}_i = \begin{bmatrix} y_i \\ -x_i \end{bmatrix}. \tag{78}$$

Condition (77) is a necessary and sufficient condition for $\boldsymbol{\mathcal{Z}}$ to be achieved by two fingers at given locations. The realizable impedance space for a given grasp is a 5-dimensional hyperplane in the 6-dimensional space.

Although for given contact locations $P_1$ and $P_2$, only a subspace (hyperplane) of impedances can be achieved, for an arbitrary symmetric $\boldsymbol{\mathcal{Z}}$, one can always find contact locations $P_1$ and $P_2$ such that condition (77) is satisfied. Thus, any $\boldsymbol{\mathcal{Z}}$ can be realized by properly selecting the two-finger contact locations. In fact, it is readily shown that one contact location can be selected arbitrarily.

*2) Three-Finger Grasp:* Since the grasp imposes no constraints on the space of realizable impedance matrices, a three-finger grasp with noncollinear contact locations can achieve all impedances. For any given $\boldsymbol{\mathcal{Z}} \in \mathbb{R}^{3\times3}$, the planar vectorized requirements similar to (13) indicate that there are more variables than required for solution. A solution can be obtained by solving

$$\mathbf{Q}\mathbf{z} = \tilde{\mathbf{z}} \tag{79}$$

where

$$\tilde{\mathbf{z}} = \text{vec}(\boldsymbol{\mathcal{Z}}) \in \mathbb{R}^6$$
$$\mathbf{z} = \left[\mathbf{z}_1^T, \mathbf{z}_2^T, \mathbf{z}_3^T\right]^T \in \mathbb{R}^9, \quad \mathbf{z}_i = \text{vec}(\mathbf{Z}_i) \in \mathbb{R}^3$$
$$\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3] \in \mathbb{R}^{6\times 9}.$$

In the generic case for which the three contact locations $P_1$, $P_2$, and $P_3$ are not collinear, the $6 \times 9$ matrix $\mathbf{Q}$ in (79) is full rank. Thus, (79) always has (infinite many) solutions. Since the equations are linear, a general solution is the sum of a particular solution and a homogeneous solution (in the null space of $\mathbf{Q}$). For this case, a particular solution (with minimum magnitude) can be obtained by

$$\mathbf{z}_p = \text{pinv}(\mathbf{Q})\tilde{\mathbf{z}} \tag{80}$$

where $\text{pinv}(\mathbf{Q})$ is the Moore–Penrose (pseudo) inverse of $\mathbf{Q}$.

For a solution $\mathbf{z} = [\mathbf{z}_1^T, \mathbf{z}_2^T, \mathbf{z}_3^T]^T$, each $2 \times 2$ symmetric translational impedance $\mathbf{Z}_i$ at contact location $P_i$ is obtained by

$$\mathbf{Z}_i = \text{invec}(\mathbf{z}_i) \in \mathbb{R}^{2\times 2}, \quad i = 1, 2, 3. \tag{81}$$

## VII. Example

In this section, two cases are provided to illustrate the developed three-finger grasp impedance selection process. In both cases, the desired symmetric spatial impedance matrix is described in the global frame as

$$\boldsymbol{\mathcal{Z}} = \begin{bmatrix} 20 & -2 & 3 & -5 & -18 & 21 \\ -2 & 10 & 2 & -1 & 2 & -3 \\ 3 & 2 & 25 & -22 & 24 & 3 \\ -5 & -1 & -22 & 26 & -16 & -8 \\ -18 & 2 & 24 & -16 & 56 & -23 \\ 21 & -3 & 3 & -8 & -23 & 30 \end{bmatrix}.$$

Note that the trace condition (10) is satisfied. It is easy to confirm that an arbitrarily selected grasp with finger contact locations at

$$P_1: \quad \mathbf{r}_1 = [2, 3, -2]^T$$
$$P_2: \quad \mathbf{r}_2 = [-1, 2, 3]^T$$
$$P_3: \quad \mathbf{r}_3 = [1, 1, 1]^T$$

does not satisfy conditions (44)–(46). Thus, the given impedance cannot be achieved by this grasp regardless of the impedances of the fingers in contact with the object. In order to attain the impedance, the contact locations must be adjusted to satisfy the conditions of Proposition 2. Two cases are considered. In the first, the process described in Section VI-B is used directly to: 1) select a set of contact locations satisfying the constraints without regard to object geometry, and 2) obtain the fingertip impedance at each contact location. In the second case, the process is modified to consider the shape of the object when selecting the contact locations.

### A. Case 1: Determination of Contact Locations and Fingertip Impedances

For this case, only the contact locations and fingertip impedances are considered. As stated in Section VI-B, one contact location can be arbitrarily selected. Here, $P_3$ is chosen to be at its original location $\mathbf{r}_3 = [1, 1, 1]^T$.

*1) Determine the Contact Locations of Other Two Fingers:* The impedance matrix expressed in the coordinate frame at $P_3$ is

$$\hat{\boldsymbol{\mathcal{Z}}} = \hat{\mathbf{T}}_3 \boldsymbol{\mathcal{Z}} \hat{\mathbf{T}}_3^T = \begin{bmatrix} 20 & -2 & 3 & -10 & -35 & 43 \\ -2 & 10 & 2 & 7 & 6 & -15 \\ 3 & 2 & 25 & -45 & 46 & 4 \\ -10 & 7 & -45 & 99 & -73 & -31 \\ -35 & 6 & 46 & -73 & 179 & -82 \\ 43 & -15 & 4 & -31 & -82 & 112 \end{bmatrix}$$

where $\hat{\mathbf{T}}_3$ is the translational transformation matrix from the global frame to the contact position $P_3$. The $3 \times 3$ block matrices $\hat{\mathbf{B}}$ and $\hat{\mathbf{H}}$ in (71) and (72) are

$$\hat{\mathbf{B}} = \begin{bmatrix} -10 & -35 & 43 \\ 7 & 6 & -15 \\ -45 & 46 & 4 \end{bmatrix}, \quad \hat{\mathbf{H}} = \begin{bmatrix} 99 & -73 & -31 \\ -73 & 179 & -82 \\ -31 & -82 & 112 \end{bmatrix}.$$

Since only five equations in (71) and (72) are independent, one of the six remaining contact location variables can be assigned a value arbitrarily. The variable selected is $\hat{z}_1 = 1$. Since the set of equations is nonlinear (quadratic), MATLAB's nonlinear equation solver "solve" is used to obtain a solution. The solution obtained is

$$\hat{\mathbf{r}}_1 = [0.3230, -2.8329, 1]^T$$
$$\hat{\mathbf{r}}_2 = [-4.6638, -1.4080, 0.3183]^T.$$

The new contact locations $P_1$ and $P_2$ expressed in the original global frame can be calculated using (73). The acceptable set of contact locations described in the global frame is

$$P_1: \quad \mathbf{r}_1 = [1.3230, -1.8329, 2]^T$$
$$P_2: \quad \mathbf{r}_2 = [-3.6638, -0.4080, 1.3183]^T$$
$$P_3: \quad \mathbf{r}_3 = [1, 1, 1]^T.$$

Since the necessary and sufficient conditions of Proposition 2 are satisfied at these contact locations, it is guaranteed that the new grasp can achieve the desired impedance by properly selecting the finger impedances at these locations.

Note that, in determining the contact locations, since the set of equations for three fingers is underconstrained, some variables can be selected arbitrarily.

*2) Determine the Finger Impedance at Each Contact Location:* The given impedance matrix $\boldsymbol{\mathcal{Z}}$ is expressed as a 21-dimensional vector $\tilde{\mathbf{z}}$ (the vectorized $\boldsymbol{\mathcal{Z}}$). Using the new contact locations $P_i$, the corresponding $21 \times 6$ matrix $\mathbf{Q}_i$ is calculated and the $21 \times 18$ matrix $\mathbf{Q} = [\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3]$ in (13) is obtained. Now, in the system of linear equations (13)

$$\mathbf{Q}\mathbf{z} = \tilde{\mathbf{z}}$$

matrix $\mathbf{Q}$ and vector $\tilde{\mathbf{z}}$ are determined. A solution can be obtained by decomposing $\mathbf{z}$ as

$$\mathbf{z} = \mathbf{z}_p + \mathbf{z}_h$$

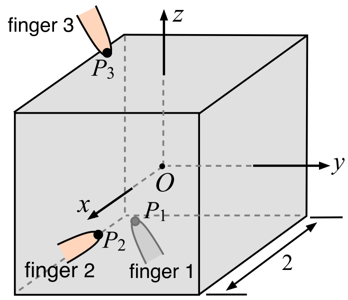


Fig. 7. Determination of the contact locations and fingertip impedances of a three-finger grasp for a specified object. The object is a cube having side length of 2.

where $\mathbf{z}_p$ is a particular solution and $\mathbf{z}_h$ is a homogeneous solution obtained from the null space of $\mathbf{Q}$.

To solve the set of equations, a particular solution $\mathbf{z}_p$ is obtained using MATLAB linear equation solver "linsolve" and the null space of $\mathbf{Q}$ (represented by an $18 \times 3$ matrix $\mathbf{N}_q$) is obtained using MATLAB "null" function. Here, a solution $\mathbf{z}$ is determined by the numerically obtained $\mathbf{z}_p$ and an arbitrarily chosen vector $\mathbf{z}_h$ from the column space of $\mathbf{N}_q$. The $\mathbf{z}$ obtained is an 18-dimensional vector having the form

$$\mathbf{z} = \left[\mathbf{z}_1^T, \mathbf{z}_2^T, \mathbf{z}_3^T\right]^T.$$

Each 6-vector $\mathbf{z}_i$ is converted into a $3 \times 3$ symmetric matrix using (76). The three-finger impedance at the acceptable contact locations are

$$\mathbf{Z}_1 = \begin{bmatrix} 1.0336 & -1.9037 & 8.7220 \\ -1.9037 & 3.1641 & -4.5303 \\ 8.7220 & -4.5303 & 12.6134 \end{bmatrix}$$

$$\mathbf{Z}_2 = \begin{bmatrix} 6.2427 & -6.8393 & -7.5482 \\ -6.8393 & 0.2142 & 1.8477 \\ -7.5482 & 1.8477 & 9.3818 \end{bmatrix}$$

$$\mathbf{Z}_3 = \begin{bmatrix} 12.7237 & 6.7430 & 1.8263 \\ 6.7430 & 6.6216 & 4.6826 \\ 1.8263 & 4.6826 & 3.0047 \end{bmatrix}.$$

With this step, a three-finger grasp with contact locations and fingertip impedances is identified. The result is verified using (9), which confirms that the desired Cartesian impedance $\mathcal{Z}$ is achieved.

### *B. Case 2: Impedance Realization for a Given Object*

Consider a cubic object with sides of length 2. The object is described in the global frame located at its center, as shown in Fig. 7. The desired object impedance (identified prior to Case 1 above) is also described in this frame.

*1) Determine the Contact Locations of Fingers:* Unlike the case in Section VII-A, the contact locations of the three fingers cannot be selected arbitrarily since each contact location must be on the surface of the object. To determine the three fingertip locations capable of realizing the desired object impedance, (44)–(46) must be satisfied. As previously stated, only five equations of the nine equations in (44)–(46) are independent, but these five constraints must be selected from all three sets of equations. In this example, the five independent equations selected are: all three equations from (44), the first equation from (45), and the first equation from (46).

Among the nine location variables $P_i(x_i, y_i, z_i)$, 4 can be selected based on the geometric constraints of the object surface. Mathematically, each $P_i$ must be on one of the six surfaces within the bounds of the body

$$x = \pm 1; \quad \text{or} \quad y = \pm 1; \quad \text{or} \quad z = \pm 1$$

and

$$-1 \le x_i \le 1; \quad -1 \le y_i \le 1; \quad -1 \le z_i \le 1.$$

Note that contact on the object surface only imposes one additional equality constraint for each contact point $P_i$. Therefore, one fingertip coordinate can be selected arbitrarily on the surface of the cube. Here, $P_3$ is chosen to be on the surface $z = 1$ with an arbitrary selected value of $y_3 = 0.6$. Thus

$$\mathbf{r}_3 = [x_3, -0.6, 1]^T.$$

For the other two fingertips, $P_1$ and $P_2$ are selected to be on the surfaces $z = -1$ and $x = 1$, respectively, which yields

$$\mathbf{r}_1 = [x_1, y_1, -1]^T$$

$$\mathbf{r}_2 = [1, y_2, z_2]^T.$$

The five remaining unknown contact location variables $(x_1, y_1, y_2, z_2, x_3)$ must satisfy the five independent equations selected from (44) to (46). Their values can be determined numerically using the nonlinear equation solver in MATLAB. Since the equations are nonlinear algebraic polynomials, the program yields multiple sets of solutions. Here, one set of solutions within the object surface region is selected. The selected results are

$$\mathbf{r}_1 = [-0.8285, -0.9009, -1]^T$$

$$\mathbf{r}_2 = [1, -0.7753, 0.3706]^T$$

$$\mathbf{r}_3 = [-0.8601, -0.6, 1]^T.$$

It can be seen that the three contact points obtained are on the surface of the cubic object.

*2) Determine Finger Impedance at Each Contact Location:* For the determined three contact locations $P_i$s, the fingertip impedance for each finger can be calculated using the same process described in Section VII-A2.

The finger impedances at the three contact locations are calculated to be

$$\mathbf{Z}_1 = \begin{bmatrix} 19.7642 & -3.4088 & -4.1257 \\ -3.4088 & 4.0537 & 0.2841 \\ -4.1257 & 0.2841 & 20.6374 \end{bmatrix}$$

$$\mathbf{Z}_2 = \begin{bmatrix} 0.1217 & 0.7629 & 0.5944 \\ 0.7629 & 4.0669 & 2.1766 \\ 0.5944 & 2.1766 & 4.1545 \end{bmatrix}$$

$$\mathbf{Z}_3 = \begin{bmatrix} 0.1141 & 0.6458 & 6.5312 \\ 0.6458 & 1.8794 & -0.4606 \\ 6.5312 & -0.4606 & 0.2081 \end{bmatrix}.$$

The object impedance associated with the three obtained contact locations $P_i$ and fingertip impedances $\mathbf{Z}_i$ is calculated using (9), which confirms that the desired object impedance $\mathcal{Z}$ is achieved.

## VIII. Summary

In this article, the space of object spatial impedances that can be achieved by a multifinger grasp is evaluated. As shown in [36] and [44], the entire space of impedance matrices attainable by a grasp in hard-point contact is a hyperplane [(10) for symmetric matrices or (56) for general matrices] in the entire impedance space. For a hand with less than five fingers, additional conditions further restrict the realizable space. Necessary and sufficient conditions for a given grasp to realize a specified spatial impedance are presented and physical interpretations of the realization conditions are provided. These conditions are derived with the assumption that each finger is capable of providing an arbitrary impedance at the fingertip. Thus, the developed conditions must be satisfied regardless of 1) whether fingers are fully actuated or underactuated, or 2) whether finger impedances are obtained actively or passively.

Results indicate that an arbitrarily specified impedance matrix satisfying the trace condition [(10) or (56)] can be achieved by a hand having three fingers (the minimum number) by properly choosing: 1) the fingertip contact locations with the object and 2) the finger impedance at each contact location. There are infinitely many sets of acceptable fingertip locations and acceptable finger impedances that will realize a desired object Cartesian impedance for a given number of fingers. In practice, one can choose a set of fingertip locations and finger impedances from the available space based on other considerations. For example, if each finger joint impedance is constrained in a range, then each fingertip impedance must be constrained in a subspace. Another practical consideration is the size and shape of the grasped object relative to the size of the hand. The fingertip locations are constrained to be reachable and on the object surface.

The theory presented in this article explicitly identifies the capabilities and limitations associated with using a multifinger grasp to achieve a desired linear impedance behavior. The results also apply to the less general linear impedance/admittance behaviors (i.e., stiffness/compliance and damping/accommodation matrices) achieved by a grasp performed by a robotic hand. In application, the methods identified can be used as a general guide in the design of a robotic grasp to select:

1) a hand with an appropriate number of fingers;
2) the fingertip contact locations;
3) the fingertip impedance of each finger.

## Appendix A

For a symmetric $\mathcal{Z}$ in the partitioned form of (9) and position vector $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ used in $\mathbf{t}_{ij}$ in the form of (43), (44)–(46) can be expressed as

$$\boldsymbol{\tau}_1 = \left(\mathbf{P}_1^T\mathbf{A}\mathbf{P}_2 + \mathbf{B}^T\mathbf{P}_2 + \mathbf{P}_1^T\mathbf{B} + \mathbf{H}\right)\mathbf{r}_{23} = \mathbf{0} \tag{82}$$

$$\boldsymbol{\tau}_2 = \left(\mathbf{P}_2^T\mathbf{A}\mathbf{P}_3 + \mathbf{B}^T\mathbf{P}_3 + \mathbf{P}_2^T\mathbf{B} + \mathbf{H}\right)\mathbf{r}_{31} = \mathbf{0} \tag{83}$$

$$\boldsymbol{\tau}_3 = \left(\mathbf{P}_3^T\mathbf{A}\mathbf{P}_1 + \mathbf{B}^T\mathbf{P}_1 + \mathbf{P}_3^T\mathbf{B} + \mathbf{H}\right)\mathbf{r}_{12} = \mathbf{0}. \tag{84}$$

In the following, the sum of the three equations $\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3$ is proved to linearly depend on $\text{trace}(\mathbf{B})$.

Using (82)–(84), $\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3$ can be rearranged to be expressed as

$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3 = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \boldsymbol{\sigma}_3 \tag{85}$$

where

$$\begin{aligned} \boldsymbol{\sigma}_1 &= \mathbf{P}_1^T\mathbf{A}\mathbf{P}_2\mathbf{r}_{23} + \mathbf{P}_2^T\mathbf{A}\mathbf{P}_3\mathbf{r}_{31} + \mathbf{P}_3^T\mathbf{A}\mathbf{P}_1\mathbf{r}_{12} \\ \boldsymbol{\sigma}_2 &= (\mathbf{B}^T\mathbf{P}_2 + \mathbf{P}_1^T\mathbf{B})\mathbf{r}_{23} + (\mathbf{B}^T\mathbf{P}_3 + \mathbf{P}_2^T\mathbf{B})\mathbf{r}_{31} \\ &\quad + (\mathbf{B}^T\mathbf{P}_1 + \mathbf{P}_3^T\mathbf{B})\mathbf{r}_{12} \\ \boldsymbol{\sigma}_3 &= \mathbf{H}\mathbf{r}_{23} + \mathbf{H}\mathbf{r}_{31} + \mathbf{H}\mathbf{r}_{12}. \end{aligned} \tag{86}$$

Each of these components is addressed separately.

Since $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, it is easy to see that $\mathbf{r}_{23} + \mathbf{r}_{31} + \mathbf{r}_{12} = \mathbf{0}$. Thus

$$\boldsymbol{\sigma}_3 = \mathbf{H}(\mathbf{r}_{23} + \mathbf{r}_{31} + \mathbf{r}_{12}) = \mathbf{0}.$$

To prove $\boldsymbol{\sigma}_1 = \mathbf{0}$, consider the position of each contact point $\mathbf{r}_i$ $(i = 1, 2, 3)$. First consider $\mathbf{r}_1$. Since $\mathbf{P}_i\mathbf{r}_i = 0$

$$\begin{aligned} \mathbf{r}_1^T\boldsymbol{\sigma}_1 &= \mathbf{r}_1^T\mathbf{P}_3^T\mathbf{A}\mathbf{P}_1(\mathbf{r}_1 - \mathbf{r}_2) - \mathbf{r}_1^T\mathbf{P}_2^T\mathbf{A}\mathbf{P}_3(\mathbf{r}_1 - \mathbf{r}_3) \\ &= -(\mathbf{P}_3\mathbf{r}_1)^T\mathbf{A}(\mathbf{P}_1\mathbf{r}_2) + (\mathbf{P}_2\mathbf{r}_1)^T\mathbf{A}(\mathbf{P}_3\mathbf{r}_1). \end{aligned}$$

Since $\mathbf{A}$ is symmetric, $\mathbf{r}_1^T\boldsymbol{\sigma}_1 = 0$. Using the same reasoning, $\mathbf{r}_2^T\boldsymbol{\sigma}_1 = \mathbf{r}_3^T\boldsymbol{\sigma}_1 = 0$. Since the three contact locations are not collinear, their position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are linearly independent. Thus, $\boldsymbol{\sigma}_1 = \mathbf{0}$.

Only $\boldsymbol{\sigma}_2$ remains, which is shown to be a linear function of $\text{trace}(\mathbf{B})$ as follows.

Substituting $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and $\mathbf{P}_i^T = -\mathbf{P}_i$ into $\boldsymbol{\sigma}_2$ in (86) and rearranging yields

$$\begin{aligned} \boldsymbol{\sigma}_2 = {}& [(\mathbf{P}_1^T\mathbf{B} - \mathbf{B}^T\mathbf{P}_1)\mathbf{r}_2 + \mathbf{P}_2\mathbf{B}\mathbf{r}_1] + [(\mathbf{P}_2^T\mathbf{B} - \mathbf{B}^T\mathbf{P}_2)\mathbf{r}_3 \\ & + \mathbf{P}_3\mathbf{B}\mathbf{r}_2] + [(\mathbf{P}_3^T\mathbf{B} - \mathbf{B}^T\mathbf{P}_3)\mathbf{r}_1 + \mathbf{P}_1\mathbf{B}\mathbf{r}_3]. \end{aligned} \tag{87}$$

Consider the identity

$$\left(\mathbf{P}_j^T\mathbf{B} - \mathbf{B}^T\mathbf{P}_j\right)\mathbf{r}_i = \mathbf{P}_i\left[\text{trace}(\mathbf{B})\mathbf{I} - \mathbf{B}\right]\mathbf{r}_j$$

which can be rearranged as

$$\left(\mathbf{P}_j^T\mathbf{B} - \mathbf{B}^T\mathbf{P}_j\right)\mathbf{r}_i + \mathbf{P}_i\mathbf{B}\mathbf{r}_j = \text{trace}(\mathbf{B})\mathbf{P}_i\mathbf{r}_j. \tag{88}$$

Substituting (88) into (87) yields

$$\boldsymbol{\sigma}_2 = \text{trace}(\mathbf{B})\left(\mathbf{P}_2\mathbf{r}_1 + \mathbf{P}_3\mathbf{r}_2 + \mathbf{P}_1\mathbf{r}_3\right).$$

Thus, the sum of the three equations (44)–(46) is

$$\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3 = \text{trace}(\mathbf{B})\left(\mathbf{P}_2\mathbf{r}_1 + \mathbf{P}_3\mathbf{r}_2 + \mathbf{P}_1\mathbf{r}_3\right).$$

Since $\mathbf{P}_i\mathbf{r}_j = \mathbf{r}_i \times \mathbf{r}_j$ and the three points $P_1$, $P_2$, and $P_3$ are not collinear

$$\mathbf{P}_2\mathbf{r}_1 + \mathbf{P}_3\mathbf{r}_2 + \mathbf{P}_1\mathbf{r}_3 = (\mathbf{r}_1 - \mathbf{r}_2) \times (\mathbf{r}_3 - \mathbf{r}_2) \neq 0.$$

Therefore, if (44)–(46) are satisfied, $\boldsymbol{\tau}_i = \mathbf{0}$ $(i = 1, 2, 3)$, then the trace condition $\text{trace}(\mathbf{B}) = 0$ must be satisfied.

## Appendix B

For a nonsymmetric $\mathcal{Z}$, let $(i, j, k)$ be an arbitrary permutation of $\{1, 2, 3\}$. We show that the following two equations:

$$\left[\mathbf{P}_i^T, \mathbf{I}\right] \mathcal{Z}\mathbf{t}_{jk} = 0 \tag{89}$$

$$\left[\mathbf{P}_j^T, \mathbf{I}\right] \mathcal{Z}^T\mathbf{t}_{ki} = 0 \tag{90}$$

contain a common constraint

$$\mathbf{t}_{ik}^T \mathcal{Z}\mathbf{t}_{jk} = 0. \tag{91}$$

Since $\mathbf{t}_{ik} = [\mathbf{P}_i^T, \mathbf{I}]^T \mathbf{r}_{ik}$ for arbitrary $i \neq k$, (91) can be obtained by multiplying (89) by $\mathbf{r}_{ik}^T$ from the left.

On the other hand, multiplying (90) by $\mathbf{r}_{jk}^T$ from the left yields

$$\mathbf{t}_{jk}^T \mathcal{Z}^T\mathbf{t}_{ik} = 0. \tag{92}$$

It can be seen that (91) and (92) are equivalent because one is the transpose of the other. Thus, (92) is implied by both (89) and (90). Since there are six equations in the form of (92) for different permutations, the number of independent equations in (64)–(66) is: $18 - 6 = 12$.

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