Systematic Transmission With Fountain Parity Checks for Erasure Channels With Stop Feedback

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***Abstract*—This paper presents new achievability bounds on the maximal achievable rate of variable-length stop-feedback (VLSF) codes operating over a binary erasure channel (BEC) at a fixed message size $M = 2^k$. We provide bounds for two cases: The first case considers VLSF codes with possibly infinite decoding times and zero error probability. The second case limits the maximum (finite) number of decoding times and specifies a maximum tolerable probability of error. Both new achievability bounds are proved by constructing a new VLSF code that employs systematic transmission of the first k message bits followed by random linear fountain parity bits decoded with a rank decoder. For VLSF codes with infinite decoding times, our new bound outperforms the state-of-the-art result for BEC by Devassy *et al.* in 2016. We show that the backoff from capacity reduces to zero as the erasure probability decreases, thus giving a negative answer to the open question Devassy *et al.* posed on whether the 23.4% backoff to capacity at $k = 3$ is fundamental to all BECs. For VLSF codes with finite decoding times, numerical evaluations show that the systematic transmission followed by random linear fountain coding performs better than random linear coding in terms of achievable rates.**

I. Introduction

For point-to-point communications with stop feedback, the blocklength varies because the decoder decides whether to cease transmission at each decoding time by sending as a 1-bit acknowledgement (ACK) or negative acknowledgement (NACK) symbol via a noiseless feedback channel. At the transmitter, the codeword symbols are completely determined by the message. The feedback is only used to instruct the transmitter when to stop, hence the name "stop feedback."

A practical communication paradigm that uses stop feedback is hybrid ARQ [1], [2], where an initial packet is transmitted and ACK/NACK feedback determines whether subsequent packets of incremental redundancy are needed. Hybrid ARQ often allows only a small, finite number of decoding times which occur only after each packet is received. When our paper examines stop feedback with finite number of decoding times, hybrid ARQ is the intended application.

Polyanskiy *et al.* formalized stop-feedback codes as *variable-length stop-feedback (VLSF)* codes in [3] and studied the case where the number of possible decoding times is infinite with stop feedback provided after every symbol transmission. Stop feedback is relatively simple as compared to active feedback approaches such as posterior matching [4], but Polyanskiy *et al.* showed that the maximal achievable rate of VLSF codes with infinite decoding times is significantly larger than fixed-length codes in the nonasymptotic regime with a fixed tolerable error probability. Thus stop feedback harnesses much of the potential non-asymptotic rate benefit of feedback.

This paper focuses on achievable rates for the binary erasure channel (BEC) with stop feedback. Devassy *et al.* obtained the state-of-the-art nonasymptotic achievability [5, Theorem 9] and converse bounds [5, Theorem 5] for VLSF codes operating over a BEC. Constructing VLSF codes for the BEC is equivalent to constructing rateless erasure codes. Motivated by this observation, Devassy *et al.* derived the achievability bound by analyzing a family of random linear fountain codes [6, Chapter 50] of message size 2^k using a rank decoder. The rank decoder keeps track of the rank of the generator matrix that produced the unerased received symbols from the message bits. As soon as the rank equals k, the decoder stops transmission by sending an ACK symbol and reproduces the k-bit message with zero error using the inverse of the generator matrix. The achievability bound [5, Theorem 9] computes the percentage of backoff from capacity as a function of message length k, which attains its maximum of 23.4% at $k = 3$. They posed the question whether this worst-case 23.4% backoff percentage to capacity at $k = 3$ is fundamental.

While Devassy *et al.* examined non-asymptotic behavior when decoding after every symbol, Heidarzadeh *et al.* [7], [8] used sequential differential optimization [9] to explore non-asymptotic behavior for finitely many carefully placed decoding times, as would be used in a practical hybrid ARQ.

Both Devassy *et al.* and Heidarzadeh *et al.* employ random linear coding for all transmitted symbols for a BEC. In contrast, we propose a new coding scheme, namely systematic transmission followed by random linear fountain coding (ST-RLFC). After an initial systematic transmission of the k message bits, the transmitter employs a random linear fountain code to generate parity bits. We use the same rank decoder as Devassy *et al.*

Our contributions in this paper are as follows.

1) For decoding after every symbol, ST-RLFC provides a new achievability bound for VLSF codes with 2^k messages and zero error probability. This bound outperforms Devassy *et al.*'s result [5, Theorem 9]. Regarding Devassy's open question about whether the 23.4% backoff at $k = 3$ message bits is fundamental, our new bound achieves a smaller backoff than 23.4% that decreases with erasure probability.

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2) ST-RLFC also provides new VLSF achievability bounds for VLSF codes constrained to have finite decoding times and a nonzero error probability, showing a similar improvement of ST-RLFC over random linear coding.

The remainder of this paper is organized as follows. Section II introduces the notation and the VLSF code, and presents previously known bounds for VLSF codes operating over a BEC. Section III provides the new VLSF achievability bounds and comparisons with previous results. Section IV concludes the paper. Due to space constraints, only proof sketches for our main results are provided. Detailed proofs can be found in the longer, online version of this paper [10].

II. Preliminaries

A. Notation

Let $\mathbb{N} = \{0, 1, \dots\}$, $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$, $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ be the set of natural numbers, positive integers, and extended natural numbers, respectively. For $i \in \mathbb{N}$, $[i] \triangleq \{1, 2, \dots, i\}$. We use x_i^j to denote a sequence $(x_i, x_{i+1}, \dots, x_j)$, $1 \le i \le j$. We denote by $\boldsymbol{e}_i \in \mathbb{R}^{k \times 1}$ the k-dimensional natural base vector with 1 at index i and 0 everywhere else, $1 \le i \le k$. We denote the distribution of a random variable X by P_X.

B. VLSF Codes

We consider a BEC with input alphabet $\mathcal{X} = \{0, 1\}$, output alphabet $\mathcal{Y} = \{0, ?, 1\}$, and erasure probability $p \in [0, 1)$. A VLSF code for BEC *with possibly finite decoding times* is defined as follows.

Definition 1: An (l, n_1^m, M, ϵ) VLSF code, where $l > 0$, $m \in \mathbb{N}_\infty$, $n_1^m \in \mathbb{N}^m$ satisfying $n_1 < n_2 < \dots < n_m$, $M \in \mathbb{N}_+$, and $\epsilon \in (0, 1)$, is defined by:

1) A finite alphabet $\mathcal{U}$ and a probability distribution P_U on $\mathcal{U}$ defining the common randomness random variable U that is revealed to both the transmitter and the receiver before the start of the transmission.
2) A sequence of encoders $f_n : \mathcal{U} \times [M] \to \mathcal{X}$, $n = 1, 2, \dots, n_m$, defining the channel inputs
$$X_n = f_n(U, W), \tag{1}$$
where $W \in [M]$ is the equiprobable message.
3) A non-negative integer-valued random stopping time $\tau \in \{n_1, n_2, \dots, n_m\}$ of the filtration generated by $\{U, Y^{n_i}\}_{i=1}^m$ that satisfies the average decoding time constraint
$$\mathbb{E}[\tau] \le l. \tag{2}$$
4) m decoding functions $g_{n_i} : \mathcal{U} \times \mathcal{Y}^{n_i} \to [M]$, providing the best estimate of W at time n_i, $i \in [m]$. The final decision $\hat{W}$ is computed at time instant τ, i.e., $\hat{W} = g_\tau(U, Y^\tau)$ and must satisfy the average error probability constraint
$$P_e \triangleq \mathbb{P}[\hat{W} \neq W] \le \epsilon. \tag{3}$$

Comparing to Polyanskiy *et al.*'s VLSF code definition [3], the primary distinctions are two-fold. First, the VLSF code is allowed to have finite decoding times rather than infinite decoding times. As a result, the stopping time is constrained to be one of these decoding times. Second, both the expected blocklength and error probability constraints correspond to the given sequence of decoding times rather than $\mathbb{N}$.

In this paper, we focus on lower bounding the achievable rates of $(l, \mathbb{N}, 2^k, 0)$ VLSF code with $m = \infty$ and $(l, n_1^m, 2^k, \epsilon)$ VLSF code with $m < \infty$. The rate of an (l, n_1^m, M, ϵ) VLSF code is defined by
$$R \triangleq \frac{\log M}{\mathbb{E}[\tau]}. \tag{4}$$

C. Previous Results for VLSF Codes over BECs

For the BEC, the decoder has the ability to identify the correct message whenever only a single codeword is compatible with the unerased channel outputs up to that point. By exploiting this fact and utilizing the RLFC, Devassy *et al.* [5] obtained state-of-the-art achievability bound for zero-error VLSF codes with message size M that is a power of 2.

Theorem 1 (Theorem 9, [5]): For each integer $k \ge 1$, there exists an $(l, \mathbb{N}, 2^k, 0)$ VLSF code for a BEC(p) with
$$l \le \frac{1}{C}\left(k + \sum_{i=1}^{k-1} \frac{2^i - 1}{2^k - 2^i}\right), \tag{5}$$
where $C = 1 - p$ denotes the capacity of the BEC.

Note that the second term in parentheses of (5) is upper bounded by the Erdös-Borwein constant $c \triangleq \sum_{j=1}^{\infty} \frac{1}{2^j - 1} = 1.60669515...$ (OEIS[1]: A065442), because
$$\sum_{i=1}^{k-1} \frac{2^i - 1}{2^k - 2^i} = \sum_{i=1}^{k-1} \frac{1 - 2^{-i}}{2^{k-i} - 1} < \sum_{j=1}^{k-1} \frac{1}{2^j - 1} < c. \tag{6}$$

In [8, Theorem 2], Heidarzadeh *et al.* showed that by constructing random linear codes for which column vectors of the parity check matrix for erased symbols are linearly independent, the average blocklength of the corresponding $(l, \mathbb{N}, 2^k, 0)$ VLSF code is given by $\frac{k+c}{C}$. This indicates that Heidarzadeh's random linear coding scheme performs as well as Devassy's RLFC scheme for sufficiently large message length k.

The state-of-the-art converse bound for $(l, \mathbb{N}, 2^k, 0)$ VLSF codes over a BEC is also obtained by Devassy *et al.* using the method of binary sequential hypothesis testing.

Theorem 2 (Theorem 5 applied to VLSF code, [5]): The minimum average blocklength $l_f^*(M, 0)$ of an $(l, \mathbb{N}, M, 0)$ VLSF code over a BEC(p) is given by
$$l_f^*(M, 0) \ge \frac{\lfloor \log_2 M \rfloor + 2\left(1 - 2^{\lfloor \log_2 M \rfloor - \log_2 M}\right)}{C}. \tag{7}$$

Note that when M is a power of 2, Theorem 2 implies that the converse bound on maximal achievable rate is simply the capacity of the BEC.

[1]The Online Encyclopedia of Integer Sequences (OEIS): https://oeis.org.

III. Achievable Rates of VLSF codes over BECs

In this section, we present a new coding scheme for a BEC called the ST-RLFC, a new achievability bound for zero-error VLSF codes of infinite decoding times, and the comparison with Devassy *et al.*'s result. Finally, we present a new achievability bound for VLSF code with finite decoding times and a comparison of achievability bounds for various numbers of decoding times.

A. ST-RLFC Scheme

Consider transmitting a k-bit message $\boldsymbol{b} = (b_1, b_2, \ldots, b_k) \in \{0,1\}^k$. Let us define the set of nonzero base vectors in $\{0,1\}^k$ by

$$\mathcal{G}_k \triangleq \{\boldsymbol{v} \in \{0,1\}^k : \boldsymbol{v} \neq \boldsymbol{0}\}. \tag{8}$$

Using ST-RLFC scheme, the channel input at time instant $n \in \mathbb{N}_+$ for message $\boldsymbol{b}$ is given by

$$X_n = \begin{cases} b_n, & \text{if } 1 \le n \le k \\ \bigoplus_{i=1}^k g_{n,i} b_i & \text{if } n > k, \end{cases} \tag{9}$$

where $\oplus$ denotes bit-wise exclusive-or (XOR) operator, and $\boldsymbol{g}_n = (g_{n,1}, g_{n,2}, \ldots, g_{n,k})^\top \in \mathcal{G}_k$ is generated at time instant n according to a uniformly distributed random variable $\tilde{U} \in \mathcal{G}_k$. Note that the encoder and decoder share the same common random variable $\tilde{U}$ at time instant $n > k$ so that the decoder can produce the same $\boldsymbol{g}_n$ at time n. For $1 \le n \le k$, both the encoder and decoder simply use the natural base vector $\boldsymbol{e}_n \in \mathbb{R}^{k\times 1}$. For all $\boldsymbol{b} \in \{0,1\}^k$, the procedure (9) specifies the common codebook before the start of transmission, i.e., the common randomness random variable U in Definition 1.

Let Y_n be the received symbol after transmitting X_n over a BEC(p), $p \in [0,1)$. We consider the *rank decoder* [5] that keeps track of the rank of generator matrix G associated with received symbols $Y^n = (Y_1, Y_2, \ldots, Y_n)$. Let $G(n)$ denote the nth column of G. If $Y_n =?$, $G(n) = \boldsymbol{0}$; otherwise, $G(n) = \boldsymbol{g}_n$. Define the stopping time

$$\tau \triangleq \inf\{n \in \mathbb{N} : G(1:n) \text{ has rank } k\}, \tag{10}$$

where $G(i:j)$ denotes the matrix formed by column vectors from time instants i to j, $1 \le i \le j$. Thus, the rank decoder stops transmission at time instant τ and reproduces the k-bit message $\boldsymbol{b}$ using Y^τ and the inverse of $G(1:\tau)$ (namely, by solving k message bits from k linearly independent equations). Clearly, the error probability associated with the ST-RLFC scheme is zero.

B. Achievability of Zero-Error VLSF Codes using ST-RLFC

The ST-RLFC scheme implies the following achievability bound for $(l, \mathbb{N}, 2^k, 0)$ VLSF codes operating over a BEC.

Theorem 3: For a given integer $k \ge 1$, there exists an $(l, \mathbb{N}, 2^k, 0)$ VLSF code for BEC(p), $p \in [0,1)$, with

$$l \le k + \frac{1}{C} \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1-p), \tag{11}$$

where $C = 1 - p$ and

$$F(i; k, 1-p) \triangleq \sum_{j=0}^{i} \binom{k}{j} (1-p)^j p^{k-j} \tag{12}$$

denotes the CDF evaluated at i, $0 \le i \le k$, of a binomial distribution with k trials and success probability $1-p$.

Proof: Let S_n denote the rank of generator matrix G_n. According to the ST-RLFC scheme, at time instant k, the probability mass function (PMF) of S_k is given by the binomial distribution $B(k, 1-p)$.

For $n \ge k$, the behavior of S_n, $n \ge k$, is characterized by the following discrete-time homogeneous Markov chain with $k+1$ states: For $n \ge k$,

$$\mathbb{P}[S_{n+1} = r | S_n = r] = p + \frac{(1-p)(2^r - 1)}{2^k - 1}, \tag{13}$$

$$\mathbb{P}[S_{n+1} = r+1 | S_n = r] = \frac{(1-p)(2^k - 2^r)}{2^k - 1}, \tag{14}$$

where $0 \le r \le k-1$, and $\mathbb{P}[S_{n+1} = k | S_n = k] = 1$. Note that this Markov chain has a single absorbing state $S_n = k$. The time to absorption for this Markov chain follows a discrete phase-type distribution [11, Chapter 2] with initial distribution $[\boldsymbol{\alpha}^\top, \alpha_k]$, where $\boldsymbol{\alpha}^\top \triangleq [\mathbb{P}[S_k = 0], \mathbb{P}[S_k = 1], \ldots, \mathbb{P}[S_k = k-1]]$, $\alpha_k = 1 - \boldsymbol{\alpha}^\top \boldsymbol{1}$, and transfer matrix

$$P = \begin{bmatrix} T & (I-T)\boldsymbol{1} \\ \boldsymbol{0}^\top & 1 \end{bmatrix}, \tag{15}$$

where the entries of $T \in \mathbb{R}^{k\times k}$ are given by

$$T_{i,i} = \mathbb{P}[S_{n+1} = i-1 | S_n = i-1],\ 1 \le i \le k \tag{16a}$$

$$T_{i,i+1} = \mathbb{P}[S_{n+1} = i | S_n = i-1],\ 1 \le i \le k-1 \tag{16b}$$

$$T_{i,j} = 0 \text{ for } j \neq i \text{ and } j \neq i+1,\ 1 \le i,j \le k. \tag{16c}$$

Let random variable X denote the time to absorption state k. It can be shown that the generating function of X is given by $H_X(z) = \alpha_k + z\boldsymbol{\alpha}^\top (I - zT)^{-1}(I-T)\boldsymbol{1}$. Hence, the expected time to absorption state k is given by

$$\mathbb{E}[X] = \frac{\mathrm{d}H_X(z)}{\mathrm{d}z}\Big|_{z=1} = \boldsymbol{\alpha}^\top (I-T)^{-1}\boldsymbol{1}. \tag{17}$$

Therefore, the expected stopping time $\mathbb{E}[\tau]$ is given by

$$\mathbb{E}[\tau] = k + \mathbb{E}[X] \tag{18}$$

$$= k + \frac{1}{C} \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1-p). \tag{19}$$

See Section IV-A in [10] for complete proof. ■

We remark that the new achievability bound (11) is tighter than Devassy's bound (5) and two bounds are equal if $p = 1$ or $k = 1$. An important corollary is that the gap between the achievability and converse is $O(k^{-1})$.

Corollary 1: For a given $k \in \mathbb{N}_+$ and BEC(p), $p \in [0,1]$, it holds that

$$kC + \sum_{i=0}^{k-1} \frac{2^k - 1}{2^k - 2^i} F(i; k, 1-p) \le k + \sum_{i=1}^{k-1} \frac{2^i - 1}{2^k - 2^i}, \tag{20}$$

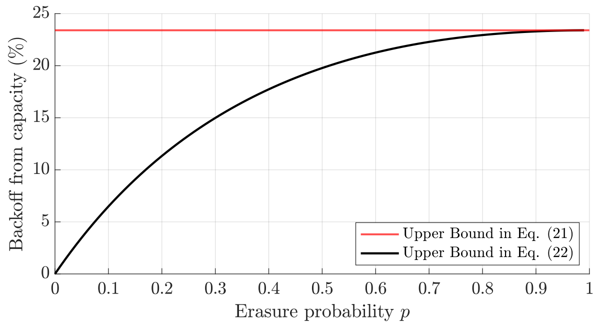

Fig. 1. Percentage of backoff from the capacity of BEC for $k = 3$. The red curve corresponds to a backoff percentage 23.4%.

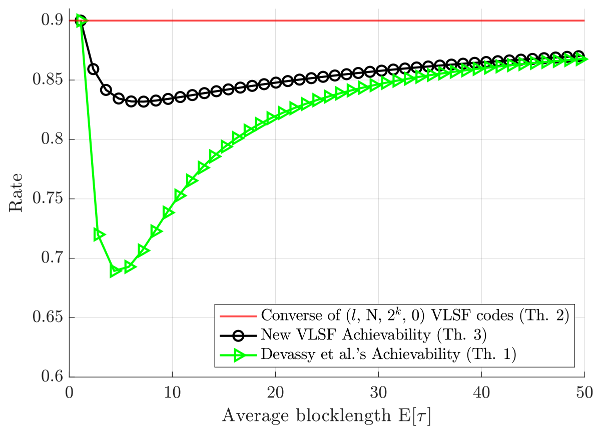

Fig. 2. Rate vs. average blocklength $\mathbb{E}[\tau]$ for $(l, \mathbb{N}, 2^k, 0)$ VLSF codes over BEC(0.1). The markers correspond to integer message length k.

where $C = 1-p$ and $F(i; k, 1-p)$ is given by (12). Equality holds if $p = 1$ or $k = 1$.

Proof: Using $k + \sum_{i=1}^{k-1} \frac{2^i-1}{2^k-2^i} = \sum_{i=0}^{k-1} \frac{2^k-1}{2^k-2^i}$, it can be shown that the RHS of (20) is lower bounded by the LHS of (20). Detailed proof can be found in [10]. ■

For BEC(0) and $k \geq 2$, our new bound (11) reduces to k, whereas Devassy *et al*'s bound (5) is strictly larger than k and approaches $k + c$ for sufficiently large k, where c denotes the Erdös-Borwein constant. Moreover, (5) also implies an upper bound independent of p on the backoff percentage to capacity,

$$1 - \frac{R}{C} \leq 1 - \frac{k}{k + \sum_{i=1}^{k-1} \frac{2^i-1}{2^k-2^i}}. \tag{21}$$

Devassy *et al.* [5] reported that this upper bound attains its maximum 23.4% at $k = 3$ and that the maximum is independent of erasure probability, thus raising the question whether this backoff percentage is fundamental. In contrast, our result in (11) implies a refined upper bound on backoff percentage that is dependent on p,

$$1 - \frac{R}{C} \leq 1 - \frac{k}{kC + \sum_{i=0}^{k-1} \frac{2^k-1}{2^k-2^i} F(i; k, 1-p)}. \tag{22}$$

Fig. 1 shows the comparison of these two upper bounds at $k = 3$. We see that for $k = 3$, the upper bound in (22) is a

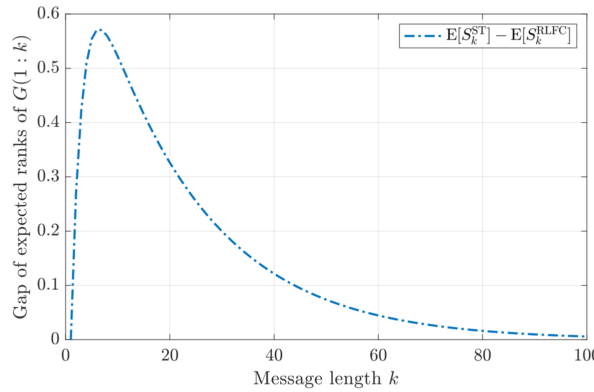

Fig. 3. $\mathbb{E}[S_k^{\mathrm{ST}}] - \mathbb{E}[S_k^{\mathrm{RLFC}}]$ vs. k for BEC(0.1) for k from 1 to 100.

strictly increasing function of p. As $p \to 0$, this upper bound decreases to 0, which closes the backoff from capacity at $k = 3$. As $p \to 1$, the upper bound in (22) increases to the backoff percentage in (21), as shown in Corollary 1. Combining (6), Corollary 1, (21), and (22), we conclude that $C - R = O(k^{-1})$ for both Devassy *et al*'s and our result.

Fig. 2 shows the comparison between the new achievability bound (Thm. 3) and Devassy *et al.*'s bounds (Thms. 1 and 2) for BEC(0.1). As can be seen, for a small message length, the new achievability bound is closer to capacity than Devassy *et al.*'s bound. This is because when k is small, systematic transmission of the uncoded message symbol is more likely to increase rank than transmitting a fountain code symbol. However, as k gets larger, the advantage of systematic transmission over RLFC gradually diminishes. To see this more clearly, let random variables S_k^{ST} and S_k^{RLFC} denote the rank of generator matrix $G(1:k)$ for ST and RLFC, respectively. We use $\mathbb{E}[S_k^{\mathrm{ST}}] - \mathbb{E}[S_k^{\mathrm{RLFC}}]$ as the metric to measure the difference of rank increase rate over time range 1 to k. Note that $\mathbb{E}[S_k^{\mathrm{ST}}] = k(1-p)$ since the rank distribution at time instant k is binomial. $\mathbb{E}[S_k^{\mathrm{RLFC}}]$ can be numerically computed using the one-step transfer matrix P in (15) and initial distribution $[1, \mathbf{0}^\top] \in \mathbb{R}^{1\times(k+1)}$. Fig. 3 shows $\mathbb{E}[S_k^{\mathrm{ST}}] - \mathbb{E}[S_k^{\mathrm{RLFC}}]$ as a function of message length k for BEC(0.1). We see that this gap constantly remains nonnegative, implying that the rank increase rate for ST is always faster than that for RLFC. For sufficiently large k, this gap becomes small, indicating the diminishing advantage of ST over RLFC.

C. Achievability of VLSF Codes with Finite Decoding Times

The ST-RLFC scheme also facilitates an $(l, n_1^m, 2^k, \epsilon)$ VLSF code for a BEC. This code is constructed by using the same ST-RLFC scheme in (9) but a rank decoder that only considers a finite set of decoding times. Specifically, fix $n_1^m \in \mathbb{N}_+$ satisfying $n_1 < n_2 < \cdots < n_m$. For a given $k \in \mathbb{N}_+$ and $\epsilon \in (0,1)$, the rank decoder still shares the same common randomness with the encoder in selecting the base vector $\boldsymbol{g}_n$, except that it adopts the following stopping time:

$$\tau^* \triangleq \inf\{n \in \{n_i\}_{i=1}^m : G(1:n) \text{ has rank } k \text{ or } n = n_m\}. \tag{23}$$

If $\tau \leq n_m$ and $G(1:\tau)$ is full rank, the rank decoder reproduces the transmitted message using Y^τ and the inverse of $G(1:\tau)$. If $\tau = n_m$ and $G(1:n_m)$ is rank deficient, then the rank decoder outputs an arbitrary message.

The ST-RLFC scheme and the modified rank decoder imply an achievability bound for an $(l, n_1^m, 2^k, \epsilon)$ VLSF code.

Theorem 4: Fix $n_1^m \in \mathbb{N}_+^m$ satisfying $n_1 < n_2 < \cdots < n_m$. For any positive integer $k \in \mathbb{N}_+$ and $\epsilon \in (0,1)$, there exists an $(l, n_1^m, 2^k, \epsilon)$ VLSF code for the BEC(p) with

$$l \leq n_m - \sum_{i=1}^{m-1}(n_{i+1}-n_i)\mathbb{P}[S_{n_i}=k], \tag{24}$$

$$\epsilon \leq 1 - \mathbb{P}[S_{n_m}=k], \tag{25}$$

where the random variable S_n denotes the rank of the generator matrix $G(1:n)$ observed by the rank decoder. Specifically, $\mathbb{P}[S_n = k]$ is given by

$$\mathbb{P}[S_n = k] = \begin{cases} 0, & \text{if } n < k \\ 1 - \boldsymbol{\alpha}^\top T^{n-k}\mathbf{1}, & \text{if } n \geq k, \end{cases} \tag{26}$$

where $\boldsymbol{\alpha}^\top \triangleq [\mathbb{P}[S_k=0], \mathbb{P}[S_k=1], \ldots, \mathbb{P}[S_k=k-1]]$ and $T \in \mathbb{R}^{k\times k}$ with entries given by (16).

Proof: The proof essentially builds upon the proof of Theorem 3 with the distiction that the rank decoder adopts a new stopping time given by (23).

Let S_n denote the rank of the generator matrix $G(1:n)$ observed by the rank decoder. The expected stopping time $\mathbb{E}[\tau^*]$ is given by

$$\mathbb{E}[\tau^*] = \sum_{n=0}^{\infty}\mathbb{P}[\tau^* > n] = n_1 + \sum_{i=1}^{m-1}(n_{i+1}-n_i)\mathbb{P}[\tau^* > n_i]$$
$$= n_m - \sum_{i=1}^{m-1}(n_{i+1}-n_i)\mathbb{P}[S_{n_i}=k], \tag{27}$$

At finite blocklength, decoding error only occurs when the rank of the generator matrix $G(1:n_m)$ is still less than k,

$$\epsilon \leq \mathbb{P}[S_{n_m} < k] = 1 - \mathbb{P}[S_{n_m}=k], \tag{28}$$

where $\mathbb{P}[S_n = k]$ can be derived using the discrete phase-type distribution. See Section IV-B in [10] for complete proof. ■

Theorem 4 facilitates an integer program that can be used to compute the achievability bound on rate for all zero-error VLSF codes of message size $M = 2^k$ and m decoding times. Define

$$N(n_1^m) \triangleq n_m - \sum_{i=1}^{m-1}(n_{i+1}-n_i)\mathbb{P}[S_{n_i}=k]. \tag{29}$$

For a given number of decoding times $m \in \mathbb{N}_+$, message length $k \in \mathbb{N}_+$, and a target error probability $\delta \in (0,1)$,

$$\begin{aligned} \min_{n_1^m} \quad & N(n_1^m) \\ \text{s.t.} \quad & 1 - \mathbb{P}[S_{n_m}=k] \leq \delta \end{aligned} \tag{30}$$

Assume $N(a_1^m)$ is the minimum value after solving (30), where a_1^m is the minimizer. Then the achievability bound on rate for a VLSF code of message size 2^k, target error probability δ, and m decoding times is given by $\frac{k}{N(a_1^m)}$. For reasonably small values of m, one can use brute-force method to obtain a_1^m and $N(a_1^m)$.

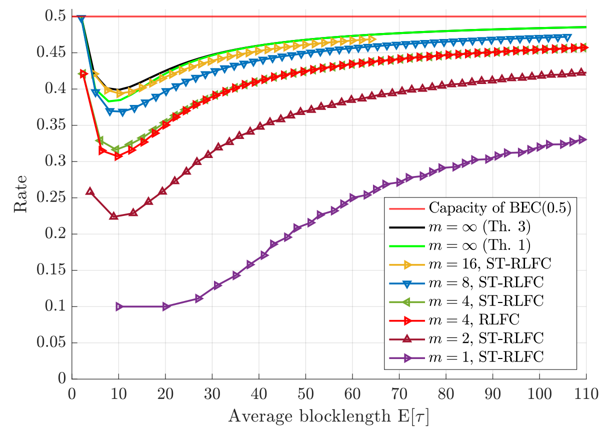

Fig. 4. Rate vs. average blocklength $\mathbb{E}[\tau]$ for $(l, n_1^m, 2^k, \delta)$ VLSF codes operated over the BEC(0.5), where $\delta = 10^{-3}$. The marker corresponds to integer message length k. In this figure, $k \geq 1$ for $m = 1, 2, 4, 8$ and $k \geq 2$ for $m = 16$.

For BEC(0.5), Fig. 4 shows achievability bounds for $(l, n_1^m, 2^k, \delta)$ VLSF codes generated by ST-RLFC, where $m \in \{1, 2, 4, 8, 16\}$ and target error probability $\delta = 10^{-3}$. The achievability bounds associated with $m = \infty$ case for Theorems 1 and 3 are also shown. In addition, we added the achievability bound for $(l, n_1^4, 2^k, \delta)$ VLSF codes generated by RLFC. This bound is obtained using the same analysis for RLFC. We see that when m is small, increasing m can dramatically improve the achievable rate. However, when $m = 16$, the achievable rate closely approaches that for $m = \infty$. We remark that similar effect on achievable rate as a function of the number of decoding times has also been observed in several previous works, e.g., [**?**], [8], [12], [13]. For $m = 4$, Fig. 4 shows a small benefit of ST-RLFC over RLFC. The ST-RLFC advantage over RLFC for small k would be more pronounced for a BEC with a smaller p such as 0.1.

IV. Conclusion

Using the ST-RLFC scheme and the rank decoder, we have shown an improved achievability bound for zero-error VLSF codes of message size $M = 2^k$. The improvement leverages the fact that when k is small, transmitting systematic message symbols at the beginning is more likely to increase the rank of the generator matrix than transmitting fountain code symbols. It remains to be seen how to further close the gap between achievability and converse bounds. In addition, the extension of Theorem 3 to arbitrary message size M remains open.

The ST-RLFC scheme combined with a modified rank decoder facilitates a VLSF code of finite decoding times and bounded error probability. Fig. 4 shows that ST-RLFC still performs better than RLFC for a finite number of decoding times and a bounded error probability. However, a rigorous proof establishing this observation still remains open.

REFERENCES

[1] M. Rice, "Comparative analysis of two realizations for hybrid-ARQ error control," in *1994 IEEE GLOBECOM. Commun.: Communications Theory Mini-Conference Record,*, 1994, pp. 115–119.

[2] C. Lott, O. Milenkovic, and E. Soljanin, "Hybrid ARQ: Theory, state of the art and future directions," in *2007 IEEE Inf. Theory Workshop on Information Theory for Wireless Networks*, 2007, pp. 1–5.

[3] Y. Polyanskiy, H. V. Poor, and S. Verdu, "Feedback in the non-asymptotic regime," *IEEE Trans. Inf. Theory*, vol. 57, no. 8, pp. 4903–4925, Aug. 2011.

[4] O. Shayevitz and M. Feder, "Optimal feedback communication via posterior matching," *IEEE Trans. Inf. Theory*, vol. 57, no. 3, pp. 1186–1222, Mar. 2011.

[5] R. Devassy, G. Durisi, B. Lindqvist, W. Yang, and M. Dalai, "Nonasymptotic coding-rate bounds for binary erasure channels with feedback," in *2016 IEEE Inf. Theory Workshop (ITW)*, 2016, pp. 86–90.

[6] D. J. MacKay, *Information Theory, Inference, and Learning Algorithms*. Cambridge, United Kingdom: Cambridge University Press, 2005.

[7] A. Heidarzadeh, J.-F. Chamberland, P. Parag, and R. D. Wesel, "A systematic approach to incremental redundancy over erasure channels," in *2018 IEEE Int. Sym. Inf. Theory (ISIT)*, Jun. 2018, pp. 1176–1180.

[8] A. Heidarzadeh, J.-F. Chamberland, R. D. Wesel, and P. Parag, "A systematic approach to incremental redundancy with application to erasure channels," *IEEE Trans. Commun.*, vol. 67, no. 4, pp. 2620–2631, Apr. 2019.

[9] K. Vakilinia, S. V. S. Ranganathan, D. Divsalar, and R. D. Wesel, "Optimizing transmission lengths for limited feedback with nonbinary LDPC examples," *IEEE Trans. Commun.*, vol. 64, no. 6, pp. 2245–2257, Jun. 2016.

[10] H. Yang and R. D. Wesel, "Systematic transmission with fountain parity checks for erasure channels with stop feedback," *arXiv preprint*, Jul. 2023. [Online]. Available: https://arxiv.org/abs/2307.14507

[11] M. F. Neuts, *Matrix-Geometric Solutions in Stochastic Models: an Algorithmic Approach*. Baltimore, Maryland, US: The John Hopkins University Press, 1981.

[12] R. C. Yavas, V. Kostina, and M. Effros, "Variable-length feedback codes with several decoding times for the Gaussian channel," in *2021 IEEE Int. Sym. Inf. Theory (ISIT)*, Jul. 2021, pp. 1883–1888.

[13] H. Yang, R. C. Yavas, V. Kostina, and R. D. Wesel, "Variable-length stop-feedback codes with finite optimal decoding times for BI-AWGN channels," in *2022 IEEE Int. Sym. Inf. Theory (ISIT)*, Jun. 2022, pp. 2327–2332.