ZDM – Mathematics Education (2024) 56:199–210
https://doi.org/10.1007/s11858-023-01545-9

ORIGINAL PAPER

Trends, insights, and developments in research on the teaching and learning of algebra

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Accepted: 29 December 2023 / Published online: 9 February 2024

Abstract
This paper addresses the recent body of research in algebra and algebraic thinking from 2018 to 2022. We reviewed 74 journal articles and identified four clusters of content areas: (a) literal symbols and symbolizing, (b) equivalence and the equal sign, (c) equations and systems, and (d) functions and graphing. We present the research on each of these content clusters, and we discuss insights on effective teaching practices and the social processes supporting algebraic reasoning. The research base shows that incorporating algebraic thinking into the elementary grades, emphasizing analytic and structural thinking processes, and emphasizing covariational reasoning supports students' meaningful learning of core algebraic ideas. We close with a discussion of the major theoretical contributions emerging from the past five years, offering suggestions for future research.

1 Trends, insights, and developments in research on the teaching and learning of algebra

Within the last two decades there has been a shift from the dichotomy of arithmetic and algebra, with scholars advocating for a longitudinal approach to teaching and learning algebra beginning with formal schooling. This shift also marked a departure from researching students' shortcomings in algebraic thinking to instead adopting a competency perspective that explores what students *can* do and understand (Stephens et al., 2017). Accordingly, we approach algebra as a continuous progression up the grade bands with increased sophistication. Considering research on the teaching, learning, and theory of algebra in grades 3—12 from 2018—2022, we identify (a) trends in what the body of research has investigated over the past five years; (b) what major findings and insights were gained, and consequently; (c) what changes emerged in the teaching and learning of algebra. We close by providing suggestions for future research.

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2 Theoretical frameworks for algebraic reasoning and thinking

In 2008 Kaput proposed a framework for algebraic reasoning that included two core aspects: (a) systematically symbolizing generalizations of regularities and constraints, and (b) syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems. These core aspects are expressed in three strands: (1) algebra as the study of structures and systems abstracted from computations and relations; (2) algebra as the study of functions, relations, and joint variation, and (3) algebra as the application of a cluster of modeling languages. Strand (1) addresses generalized arithmetic, including generalizing about number properties, relationships, and computation strategies. Strand (2) acknowledges the central role functions play in algebra, and strand (3) entails three types of modeling: number or quantity-specific modeling, modeling generalizations, typically with variables, and making comparisons with other models and situations.

In recent years, there has been increased attention to algebraic reasoning, thinking, and early algebra. This work began in earnest with a PME Research Forum and an ICMI thematic working group dedicated to early algebra (Kieran, 2022), and was followed by researchers advancing definitions of early algebraic thinking (e.g., Britt & Irwin, 2011; Carraher & Schliemann, 2007; Kieran, 2004). In 2022, Kieran published a framework for early algebraic thinking. Although it addresses early algebra, whereas this review includes secondary studies on formal algebra, it nevertheless offers a way to structure the studies in the field. Kieran's framework articulates three types of algebraic thinking—analytic, structural, and functional—with generalization being the common thread across the types. The first two types, analytic and structural thinking, are rooted in generalized arithmetic. Analytic thinking entails thinking about indeterminate quantities as if they were known, and its main focus is equations, equalities, and equivalence. Kieran's second type, structural thinking, entails making use of relationships as instantiations of properties, and includes seeing relations, properties, and structure within numbers, operations, and expressions. These two types of thinking intersect with, but also differentiate, aspects of Kaput's first strand of algebra as the study of structures and systems abstracted from computations and relations. Kieran's third type is functional thinking, like Kaput's second strand (the study of functions, relations, and joint variation). The functional thinking type addresses recursive, covariational, and correspondence thinking. These distinctions reflect more recent trends in the field which differentiate covariational from correspondence approaches. In the next section, we describe how we leveraged the ideas in these frameworks to identify relevant articles for review.

3 The process of identifying articles

Aiming to explore the state of the research conducted on the teaching and learning of algebra over the past five years, we began with a set of inclusion and exclusion criteria to identify articles for review. We included articles (a) published in peer-reviewed, high-quality journals; (b) published between 2018 and 2022; (c) situated at the K-12 level; and (d) whose focus was the teaching and learning of algebra. Our exclusion criteria were (a) literature reviews or policy documents; (b) practitioner papers, book chapters, or proceedings; (c) studies conducted at the college level; (d) articles in which algebra was the context, but that did not address the teaching or learning of algebra; (d) studies about teachers' beliefs or knowledge, rather than teaching practices; and (e) textbook analyses without accompanying student data.

We then compiled a comprehensive list of journals in which mathematics education research articles are published, as taken from five sources: (1) The Vanderbilt University Library list of key journals in mathematics education,[1] (2) the Mathematics Education Research Group (MERG) journal list,[2] (3) the Education Bureau of Hong Kong,[3] (4) the University of Central Florida Library list of key journals for mathematics education,[4] and (5) the Center for Research and Training in Mathematics Education journal list.[5] From those sources, we compiled a list of 40 possible journals for inclusion, and then searched for each journal in the Web of Science to identify its Impact Factor (IF) and its Journal Citation Indicator (JCI). Impact Factors are measures of the frequency with which an average article in a journal has been cited in a particular year, and the JCI measures the average citation impact for papers published in the prior three-year period. It is normed against 1.0, so values greater than 1 indicate a higher-than-average impact (Crea et al., 2023). Thirteen of the journals were not indexed by the Web of Science. For the remaining 27 journals, we included those that had either an IF, a JCI > 1.0, or both, which yielded 18 journals. We also included three of the 13 journals not indexed by the Web of Science, due to their prominence in mathematics education: *For the Learning of Mathematics*, the *Journal of Urban Mathematics Education*, and the *Australian Journal of Education*. The resulting 21 journals (see Table 1 in Electronic Supplementary Material) constituted the database for our article search.

Restricting our search results to our journal list, we then searched on the Web of Science, ERIC, Google Scholar, and the individual journal indices, running four searches with the following terms: (a) algebra + learning (202 results); (b) algebra + learning AND equals OR function OR graph OR equation OR variable (1038 results); (c) algebra + teaching (91 results); and (d) algebra + teaching AND equals OR function OR graph OR equation OR variable (239 results). We then eliminated duplicates, and excluded any article that did not meet our inclusion criteria or met our exclusion criteria. This yielded 73 articles. To further cull, we then excluded articles situated at the K-2 level (corresponding to ages 5—7), as well as articles that were published by the same author(s) and reported close to identical results. Applying these additional criteria yielded a total of 60 articles for our review. To ensure that articles about algebraic thinking processes were also included in the sample, we then ran additional searches with the following terms: (a) algebraic thinking (11 new results); (b) algebraic reasoning (9 new results); and (c) algebraic thinking processes (1 new result). After eliminating the studies that did not meet our inclusion criteria, our final sample included 74 articles, which appeared in a total of 13 journals (Table 1).

[1] https://researchguides.library.vanderbilt.edu/mathed/journals.

[2] https://sites.google.com/site/ditmerg/journals.

[3] https://www.edb.gov.hk/attachment/en/curriculum-development/kla/ma/res/journal_e.pdf.

[4] https://guides.ucf.edu/education-mathematics/journals.

[5] https://ued.uniandes.edu.co/portfolio/ranking-of-mathematics-education-journals/.

4 What has the body of literature investigated over the past 5 years?

The reviewed articles report findings from 22 countries, providing a rich international perspective, with five studies offering cross-national comparisons. Thirty-two studies were early algebra studies, situated in grades 3–5 (corresponding to ages 8–11), and 44 studies were algebra studies, situated in grades 6–12 (corresponding to ages 11 – 18). We found four clusters of content areas: (a) literal symbols and symbolizing, (b) equivalence and the equal sign, (c) equations and systems, and (d) functions and graphing. Table 2 in the Electronic Supplementary Materials depicts the studies according to grade level and algebra content; note that some studies addressed more than one content cluster, and a few studies did not fall within any specific content cluster; thus, the totals do not add to 74.

We considered the studies through three lenses. The first distinguishes studies addressing the conditions supporting algebraic reasoning, such as curricular treatments, task features, or technology use, versus studies addressing instructional features, such as social interaction or teacher practices. We also distinguished studies investigating reasoning or performance as measured at one point in time, through written assessments, interviews, or observations, from studies characterizing change over time. These latter studies employed interventions, design experiments, or learning trajectories. The third lens distinguishes studies characterizing reasoning from those measuring performance. Few articles explicitly defined reasoning or learning, but we found many studies characterizing students' concepts, meanings, and mental operations. These articles relied on a variety of theoretical frameworks, including semiotics, constructivism, objectification, and quantitative reasoning, among others. Attempts to understand students' reasoning requires methods that offer access to student thinking, such as clinical or task-based interviews, often combined with classroom observation or design-experiment data. Given the in-depth nature of the qualitative analysis required to create models of student reasoning, these studies often had small participant sizes. Moreover, finding evidence of learning can be difficult to achieve methodologically, and may require both access to students' thinking over extended periods of time and multiple triangulated forms of evidence.

A separate group of studies collapsed learning and performance, treating performance on written measures as a proxy for learning. Certainly, learning and performance are related, and improvements in performance may reflect changes in students' concepts or operations. However, changes in performance may not always mean that students' operations have undergone metamorphic accommodations (Steffe, 1991); such changes could also reflect a local behavior shift that might not always entail meaningful concept change, much less persist or be generalizable to other contexts (De Bock et al., 2011). A couple of the studies concerned with performance did draw on theoretical frameworks such as structural mapping theories or Kaput's conceptual framework for algebra, but the majority did not articulate a theory of learning. These studies treated learning as observable, demonstrated through increased correctness on written measures. Evidence to support claims of improved performance rely on experimental or quasi-experimental designs and statistically significant gains from pretest to posttest (Fig. 1).

Figure 1 shows that the body of empirical research is heavily weighted towards studies investigating the conditions supporting understanding and learning, with fewer investigating the features of instruction. The number of studies on each arrow only addresses those that intersect across the categories, and thus do not include all 74 papers. The majority addressed students' change over time, even though in some of these studies the duration of the intervention was as short as one or two sessions. About a third of those studies assessed students' performance prior to and after an intervention to evaluate its effectiveness. These findings are consistent with Inglis and Foster's (2018) description of the "experimental migration", in which a sizable amount of research is still being conducted within the experimental psychology research program. Few studies examined the role of social interaction or effective teaching practices, and they were mostly conducted at the secondary level in the content of functions and graphing.

Next, we present the major findings and insights gained from the research reviewed. We organize these findings first by the four content areas, and within each area we consider the studies in light of a) Kieran's (2022) types of algebraic thinking, b) whether they were situated in early algebra or algebra, c) whether they addressed extant reasoning versus change over time, d) their focus on learning versus performance, e) their contributions to theory, and f) practical findings for instruction. We then consider the field's recent findings on effective teaching practices and social processes for supporting reasoning, before turning to a summary of the field's theoretical advancements.

5 Major findings and insights gained

5.1 Findings about literal symbols and symbolizing

Literal symbols play several roles in algebra, including understanding a literal symbol as a generalized number, as a fixed unknown, and as a varying quantity; these roles are

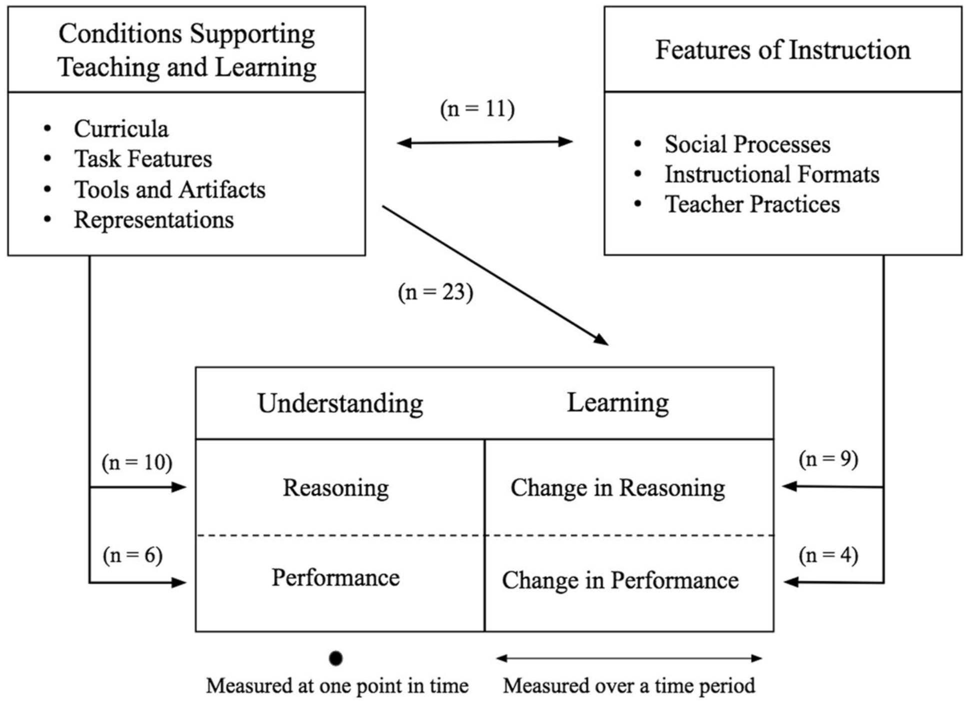

Fig. 1 Relationships between conditions supporting understanding and learning, features of instruction, and reasoning and performance

subsumed under the term indeterminates (as per Radford, 2018). Many of these studies either address Kieran's (2022) analytic thinking type, as they considered students' abilities to treat indeterminate quantities as if they were known, or Kieran's (2022) structural thinking type, through investigations of students' abilities to consider and express properties of number and operation. For instance, the two studies we found investigating students' extant reasoning about literal symbols in early algebra took the structural thinking approach. Xolocotzin et al. (2022) examined how students expressed rules and operations, finding that their "intuitions about symbolic notations were diverse and did not necessarily focus on using letters" (p. 1375). Lenz (2022) investigated children's relational thinking in tasks with the literal symbol as unknown, finding that even young children can indicate relationships between indeterminate quantities when they have physical materials as supports.

More studies addressed learning or improved performance in early algebra. These findings are mixed but point to the promise of interventions to support students' abilities to use and interpret literal symbols to represent indeterminate quantities, attending to Kieran's analytic thinking type (Ayala-Altamirano & Molina, 2020; Blanton et al., 2019a; Pang & Sunwoo, 2022; Papadopoulus & Patsiala, 2019). For instance, Ayala-Altamirano and Molina (2020) found that third graders can represent indeterminates symbolically, mirroring Blanton et al.'s (2019a) findings that children can use literal symbols to represent fixed unknowns, varying quantities, and generalized patterns. Blanton and colleagues implemented a longitudinal intervention, but researchers have found that even brief interventions can help students learn to represent relationships symbolically (Papadopoulus & Patsiala, 2019; Xie & Cai, 2022). In contrast to these findings, however, others found no progression in students' use of literal symbols; students either rejected literal symbols or held meanings for them as labels or objects (Ayala-Altamirano & Molina, 2020; Ayala-Altamirano et al., 2022).

Most of the secondary studies addressed students' reasoning or performance and can be situated in Kieran's structural thinking type. They identified the presence of the natural number bias, or considered how the presence of indexical expressions affected students' use of literal symbols (Christou et al., 2022; Soneira et al., 2018). In one learning study, Hackenberg et al. (2021) took both an analytic and structural thinking approach to have middle-school students relate two quantitative unknowns, finding that this supported their abilities to make sense of indeterminacy. First introducing literal symbols as measured and indeterminate supported the construction of variable before then reasoning structurally about varying quantities. However, even without intervention, students can improve in their abilities to correctly use variables (Sharpe, 2019).

These studies offer both theoretical and practical outcomes. Building on Tall et al.'s (2001) process and concept views of symbols and Hoch and Dreyfus's (2004) notion of structure sense, Lenz (2022) differentiated a number-oriented approach from a structure-oriented approach, in which the former focuses on specific numbers while the latter considers the entire task, its subsets, and their interrelationships. This mirrors Kieran's (2022) structural thinking type and shows the power of this type of thinking for developing relationships between variables. Hackenberg et al. (2021) also offered a contribution to theory by articulating the role that students' multiplicative concepts play in their understanding and use of literal symbols. Leveraging Steffe's (1992) construct of units coordination, which involves inserting the units of one composite unit across the units of another composite unit, the authors identified the importance of reasoning with three levels of units to construct meaning for quantitative unknowns.

Taken as a whole, the studies show that children can think in sophisticated ways about literal symbols and variable notation. It is important to emphasize the connections between numbers and literal symbols early (Christou et al., 2022), and then provide sustained experiences over time (Papadopoulos & Patsiala, 2019; Xie & Cai, 2022). Recommendations for supporting students' understanding of literal symbols include using natural language terms such as "many" to help develop general expressions (Ayala-Altamirano & Molina, 2022), using problem statements with implicit indexicality (Soneira et al., 2018), making implicit variables explicit (Kilhamm et al., 2022), and using contexts with varying quantities, not just unknowns (Christou et al., 2022). Tasks that involve true/false sentences (Ayala-Altamirano & Molina, 2020) and that ask students to draw and relate unknowns appear to be useful (Hackenberg et al., 2021), and students should have opportunities to consider non-natural unknowns (Christou et al., 2022).

6 Findings about equivalence and the equal sign

A central component of early algebra addresses children's understanding of the equals sign and its role in equations. Research in this area is grounded in Kieran's (2022) analytic thinking type, in which equations and equalities are the focus. Students' conceptions about the equals sign have been well documented, with research distinguishing between an operational understanding (in which the equals sign is seen as indicating a computation to be made) from a relational understanding (in which the equals sign indicates the numeric equivalence of two expressions) (e.g., Carpenter et al., 2003). More recently, research suggests that elementary students can simultaneously hold both meanings (Lee & Pang, 2021; Madej, 2022).

Studies investigating students' acquisition of the relational meaning of the equals sign is optimistic, suggesting that brief interventions can be effective (e.g., Bajwa & Perry, 2021; Radford, 2022). For instance, Donovan and colleagues (2022) found that an online intervention focused on the sameness (relational) and substitution (the replacement of one representation with another) meanings enhanced students' abilities to produce equals sign definitions that relied on those meanings. However, this finding should be tempered with Madej's (2022) caution that students may struggle to think relationally even when able to provide a relational definition. Kieran and Martinez-Hernandez (2022) investigated students' evolution from calculational to structural transformations of equality. When asked to rewrite equalities without calculating the total of each side, they began to decompose numerical expressions and describe truth values in terms of sameness. The authors realized that computational underpinnings may be central to structure-based transformational work, a finding that straddles both the analytic and structural thinking types (Kieran, 2022). Similarly, Tondorf and Prediger (2022) found that fifth-grade students could engage in structural transformations after a short intervention. In doing so, they introduced a new theoretical construct, that of restructuring equivalence, as a way of transferring relational understanding to transformations.

The bulk of research on equality is situated in early algebra. One secondary study identified high school students' lack of relational meaning for the equals sign (Soneira, 2022), and another found that seventh-grade students could produce a relational definition, but that this did not necessarily imply a relational understanding (Sumpter & Sowenhielm, 2022). These findings suggest a continuum between the operational and relational understandings, rather than a binary distinction between the two, and echo's Lee and Pang's (2021) findings that students can hold both conceptions simultaneously.

These studies suggest that sameness-relational and substitutive-relational meanings should be continually elaborated during and after elementary school. The substitution meaning is important for understanding the equals sign, and although Sumpter and Lowenhielm (2022) found that it could be treated alongside sameness, Donovan and colleagues (2022) suggested that a sameness conception may be a necessary precursor. Merely highlighting the relational meaning of the equals sign is not sufficient; students should also reason in contexts that necessitate relational meanings (Lee & Pang, 2021). Teachers can leverage story problems, concrete or visual contexts, appropriate representations, and tasks requiring students to rewrite equalities to encourage relational meanings (Kieran & Martinez-Herandez, 2022; Radford, 2022; Schifter & Russell, 2022). However, contextual supports should not be too grounded; they should leave room for discussion, interpretation, and productive struggle (Bajwa & Perry, 2021).

7 Findings about equations, equation solving, and systems of equations

Understanding and solving equations and systems is analytic thinking, but studies addressing these issues could conceivably entail all three of Kieran's (2022) types of algebraic thinking. For instance, students can use and generalize properties of number and operations to make sense of equations and their solutions, which entails structural thinking, and when students consider equations as representations of, for instance, the roots of a function, they may be leveraging functional thinking. The bulk of the research we reviewed in this area occurred at the secondary level, except for two studies that we have already mentioned. Xie and Cai (2022) investigated the effect of early arithmetic strategies on fifth grade students' equation solving strategies and found that students performed better when unknowns were represented with brackets instead of letters. After formal instruction on equation solving, some students were able to apply an inversing and formal (performing the same operation on both sides) combined strategy, but only a minority obtained the correct answer. They suggested that arithmetic inversing strategies could interfere with students' learning of the formal strategy. Radford (2022) also found that children can solve simple equations, and that naming operations with terms such as "removing" can support concept formation.

The secondary studies addressed students' strategies and performance in solving equations and systems, taking both analytic and structural thinking approaches (e.g., Chval et al., 2021). Jiang and colleagues (2022) studied strategies across Sweden, Finland, and Spain and found that a curricular emphasis on standard strategies influenced students' approaches to equations. If students saw the standard algorithm to be more important than strategy flexibility, they were more likely to apply it even if it was less appropriate. Those studying strategies for solving systems of equations found connections between students' arithmetic and algebraic reasoning; students with more sophisticated arithmetic strategies could transcend guess-and-check methods to develop algebraic strategies (Demattè & Furinghetti, 2022; Zwanch, 2022).

Studies investigating students' learning have considered the effects of digital games and tools, such as Dragonbox, computer algebra systems, and guided interactive diagrams. These studies are promising, suggesting that supporting students in making translations across representational systems can support equation solving (Fonger, 2019; Naftaliev & Yerushalmy, 2022). However, it is important to provide instruction that scaffolds connections between algebraic objects and the objects in digital tools, games, or representations (Kapon et al., 2019).

This body of work offers new theoretical constructs, such as a framework for meaningfulness in representational fluency (Fonger, 2019), the introduction of the notion of representational sense (Lepak et al., 2018), and, similar to Hackenberg et al.'s (2021) extension of the units coordination construct into literal symbols, an examination of the need for coordinating three levels of units to solve word problems that can be modeled by systems of equations (Zwanch, 2022). These studies also suggest implications for instruction. At the elementary level, emphasizing flexibility can support the use of appropriate solution methods (Jiang et al., 2022). Learning to solve equations as acts of sensemaking is critical, which can be encouraged by building on students' existing understanding of arithmetic operations (Chval et al., 2021; Xie & Cai, 2022), having students discuss and compare solutions (Jiang et al., 2022), fostering attention to connections across representations (Fonger, 2019; Naftaliev & Yerushalmy, 2022), encouraging reflection on relationships between unknowns (Zwanch, 2022), and leveraging meaningful contexts (Jiang et al., 2022; Kapon et al., 2019; Zwanch, 2022).

8 Findings about functions and graphing

Studies about functions and their graphs fall within Kieran's (2022) functional thinking type and Kaput's (2008) strand of algebra as the study of functions, relations, and joint variation. However, as in the other sections, many studies also addressed aspects of analytic and structural thinking. The work on functions occurred in both early algebra and algebra, which is a shift from older work situated mostly at the secondary level. Stephens et al. (2017) argued that not only is there a growing body of evidence that children can successfully reason about functional relationships, but that introducing functional thinking in the elementary grades can support more successful reasoning in secondary school. Since then, researchers have considered elementary students' abilities to generalize functional relationships and to develop and interpret graphs of functions. These studies show promise for supporting children's function reasoning, but also point to some challenges. Studies have demonstrated students' success in generalizing function rules (Pang & Sunwoo, 2022), as well as difficulties (Pinto et al., 2022; Xolocotzin et al., 2022). Correspondence approaches (developing a rule relating x and y) with function tables were more common than covariational approaches (coordinating changes in y with associated changes in x) (Pinto et al., 2022; Xolocotzin et al., 2022). Numerical representation can be a useful bridge to developing an awareness of mathematical structure to generalize function relationships, a perspective consistent with Kieran's structural thinking type (Pinto et al., 2022; Pittalis, 2022; Stephens et al., 2021), as can patterning tasks (Montenegro et al., 2018; Stephens et al., 2021; Walkoe & Levin, 2020).

At the secondary level, a related group of studies comparing outcomes across countries have investigated how curricular and pedagogical treatments affect students' abilities to understand and generalize function relationships, either by examining curricular frameworks (Hemmi et al., 2021) or student work (Ayalon & Wilkie, 2019, 2020; Watson et al., 2018). Three studies in particular examined student responses from Israel, Australia, and England and documented Israeli students' strong performance in figural pattern generalization, conceptions of function, and ability to relate correspondence and covariation (Ayalon & Wilkie, 2019, 2020; Watson et al., 2018). Introducing the function concept formally in middle school, which occurs in the Israeli curriculum, appears to support students' pattern generalization and functional thinking, as well as help students avoid some of the difficulties highlighted in earlier literature, such as attending only to *y*-values in tables, remaining stuck in recursive approaches, and engaging in inappropriate proportional reasoning (e.g., Stacey, 1989; Orton et al., 1999; Van Dooren et al., 2005). These findings show that an explicit emphasis on functional approaches, beginning in middle school, can foster an understanding of function as a relation between variables.

Other studies addressing generalization show some challenges at the middle-school level, with students confusing, for instance, linear and quadratic variables (Ramírez et al., 2022). However, interventions aimed at helping students identify structural similarity can foster generalization of function relationships (Hunter et al., 2022; Wilkie, 2020, 2022). Figural patterns are particularly useful for supporting students' development of symbolic generalizations, as are encouraging multiple ways to represent relationships (Hunter et al., 2022; Wilkie, 2022). As a whole, these studies show us the value of approaching functional reasoning in a manner that emphasizes structural thinking (Kieran, 2022).

Secondary studies also addressed student's construction and interpretation of graphs, learning about function families, and covariational reasoning. They found that students' symbol sense was connected to correctness in graphing (Kop et al., 2020, 2021), and that students can hold both iconic and scientific meanings for graphs (Lingefjärd & Farahani, 2018; Patterson & McGraw, 2018). Worked examples and digital technologies are effective in supporting students' graphing skills (Barbiere et al., 2019; Günster & Weigand, 2020).

Several of the studies examined functions from a covariation perspective (Ellis et al., 2020; Fonger et al., 2020; Johnson & McClintock, 2018). For instance, Ellis and colleagues (2020) introduced the theoretical construct of scaling-continuous reasoning, a form of covariation relying on an image of zooming, which supported productive thinking about rates of change. There have also been other theoretical contributions to come from this body of literature, including Pittalis's (2022) theoretical framework for arithmetic-algebraic structure sense, Günster and Weigand's (2020) Function-Operation-Matrix for task construction to develop students' functional thinking, and Díaz-Berrios and Martínez-Planell's (2022) genetic decomposition of exponential and logarithmic functions. This body of work also introduced learning trajectories documenting students' learning of quantitative and covariational relationships (e.g., Kafetzopoulos & Psycharis, 2022), as well as quadratic functions (Fonger et al., 2020), exponential and logarithmic functions (Díaz-Berrios & Martínez-Planell, 2022), and finite-to-finite functions (Eames et al., 2021).

These studies point to the value of introducing functional thinking early, informally in elementary school with formal definitions in middle school (Ayalon & Wilkie, 2019, 2020; Pinto et al., 2022). Natural language, embodied activities, and visual patterns can serve as vehicles for generalizing function relations (Ayalon & Wilkie, 2019; Duijzer et al., 2019; Pinto et al., 2022; Wilkie, 2020). Input–output models can also support the construction of function rules from data (Watson et al., 2018), and encouraging students to create rather than just interpret graphs can support their graphical reasoning (Duijzer et al., 2019). Recent studies emphasize covariation as a promising route for developing an understanding of function (Ellis et al., 2020; Fonger et al., 2020; Peck, 2020). Dynamic contexts are useful for helping students think about rates as coordinated changes, and digital tools can help students correlate changes in linked quantities (Ellis et al., 2020; Fonger et al., 2020; Kafetzopoulos & Psycharis, 2022; Patterson & McGraw, 2018).

9 Findings on effective teaching practices and social processes supporting reasoning

Although a substantial body of research has accrued over the past five years examining the effects of curriculum, tasks, and digital tools on students' learning and performance, there is less research on teaching practices and social processes. One exception is a pair of studies by Litke (2020a, 2020b). Drawing on large data sets of videos of ninth grade US algebra lessons, Litke (2020a) identified five types of learning opportunities teachers enacted: giving meaning to procedures, moving towards procedural flexibility, making connections across representations, building connections across topics, and making connections explicit between more and less familiar ideas. Litke (2020b) also found that, consistent with prior research, most of the lessons were teacher-led and emphasized procedures, with few opportunities for student exploration and few connections across topics or representations. However, many teachers did support procedural flexibility, and taken together, these findings suggest that identifying and building on existing instructional features to gradually improve their quality and prevalence could be one potentially fruitful avenue for improving algebra instruction.

Others have addressed links between reasoning and instruction. Fonger et al. (2020) examined how instructional supports engendered students' learning of quadratic growth, identifying productive moves such as pressing for justification, but also emphasizing the importance of attending to the interaction between task design, teacher moves, and group norms. Hohensee and colleagues (2022) investigated how teachers' instructional approaches to quadratic functions influenced students' backwards transfer to linear functions. They linked changes in students' reasoning to the teachers' practices, such as reversing the steps of quadratic functions or reasoning about landmark features. Kop et al. (2021) showed that teachers can help students gain insight into algebraic formulas through graphing by focusing on recognition and qualitative reasoning, modeling expert thinking and providing examples, and offering overviews and reflection questions. Collaborating with teachers following a lesson sequence focused on (a) noticing regularities, (b) articulating conjectures, (c) representing examples, (d) constructing representation-based arguments, and (e) comparing and contrasting operations, Schifter and Russell (2022) found that students could deepen their understanding of mathematical structure through opportunities to create and analyze representations.

A second under-researched topic is the role of social interaction in supporting learning. Wilkie (2022) studied a pair of students generalizing quadratic figural patterns and identified several productive interactions, including explaining one's reasoning, building on a partner's idea, disagreeing, amending responses, and verifying another's generalization. Taking the classroom as the unit of analysis, others have attended to interaction to detail how learning is mediated through task design, discourse, multimodal communication, artifact use, and representations (Hunter, 2022; Montenegro et al., 2018; Lee & Pang, 2021; Peck, 2020; Radford, 2022). These studies offer insight into the social processes supporting students' reasoning, insights that could not be garnered from studies focused solely on cognitive data.

Although the above studies highlight the complexity of investigating relationships between instruction, classroom interaction, and student learning, future research needs to attend to these features if we are to improve algebra instruction in classrooms at scale. In particular, the extent to which improving the quality and prevalence of specific instructional moves affects student learning remains an open question.

10 Theoretical advances in research on algebraic thinking

In addition to the above-described empirical contributions, the recent body of research also offered significant theoretical advancements. In addition to Kieran's (2022) conceptualization of early algebraic thinking across the dimensions of analytic thinking, structural thinking, and functional thinking, other studies have also extended research to the early grades and include a variety of perspectives, approaches, and tools. Focusing on children's pre-instructional experiences, Walkoe and Levin (2020) offered a complementary conceptualization of the development of algebraic thinking. They theorized that children construct algebra-relevant cognitive resources, called seeds of algebraic thinking, through their repeated experiences with the world. These seeds are then refined and reorganized across contexts and levels of formal schooling (Levin & Walkoe, 2022).

Computational thinking has emerged as an important skill, and Brating and Kilhamn (2021) explored the intersection between computational and algebraic thinking in terms of their respective representation systems. Examining three examples of programming activities, they discussed the similarities and differences in how variables, equality, functions, and algorithms are represented in computer programs and algebraic notation. Despite the syntactic similarities between programming languages and algebraic notation, they revealed differences in the meaning of several concepts, giving rise to possible difficulties these differences might present to students.

Quantitative reasoning has also gained prominence in recent years. Thompson (2011) defined quantitative reasoning as constructing and operating with quantities and their relationships. Quantities are schemes composed of one's conception of an attribute of an object, such as a person's height, an appropriate unit, such as centimeters, and a process for assigning a numerical value to the attribute. Quantitative operations entail conceiving a new quantity in relation to other already-conceived quantities, such as comparing how much taller one person is than another (an additive comparison). Researchers have since considered students' conceptions of literal symbols as unknown and varying quantities (Blanton et al., 2019a; Hackenberg et al., 2021; Lenz, 2022), students' construction of graphs as representations of changes in quantities (Patterson & McGraw, 2018), and have shown the promise of supporting an emerging understanding of function relationships through quantitative reasoning approaches (Ellis et al., 2020; Fonger et al., 2020; Kafetzopoulos & Psycharis, 2022).

Much of the research reported in our four content areas have addressed algebraic thinking processes, such as reasoning with mathematical structure and relationships, generalizing, representing, and justifying. In particular, an interest in using and representing structure has led to a variety of theoretical contributions, and speaks to the utility of Kieran's (2022) framework for both early algebra and algebra. For instance, Pittalis (2022) developed and validated a model of young students' arithmetic-algebraic structure sense, and Coles and Ahn (2022) proposed an extension to Radford's (2014) third aspect of algebraic thinking (analyticity), *the* *notion of structuring a mathematical space,* as a form of proto-analytical thinking that includes elements of deductive reasoning.

Extending the structural dimension beyond natural numbers, Pearn and colleagues (2022) linked the fractional competence of young students and their progressive shift to algebraic thinking through two frameworks, the classification framework for reverse fraction tasks and the emerging algebraic reasoning framework. Relatedly, Vlassis and Demonty (2022) extended the structural dimension to negative integers and found that an ability to view the subtraction operation as a 'transformation' involving the single use of the minus sign accounts for students' success in operations with negatives. Tondorf and Prediger (2022), on the other hand, bridged relational and transformational characterizations of restructuring equivalence, connecting the equivalence of numerical-symbolic and figural representations. Theoretical developments about representing and generalizing include the meaningfulness in representational fluency framework (Fonger, 2019) and a learning progression for algebraic generalization developed and validated by Stephens et al. (2021).

11 Discussion: emerging changes and future directions

The literature over the past five years shows that a decades-long effort to incorporate early algebraic thinking into the elementary grades, emphasize analytic and structural thinking processes, and emphasize quantitative and covariational reasoning is beginning to pay off. The body of research we reviewed reflects a rich international perspective and includes several cross-national comparison studies that offer valuable insights about the effects of curricular and instructional shifts on student understanding (Ayalon & Wilkie, 2019, 2020; Jiang et al., 2022; Watson et al., 2018). Together, these studies show that commonly documented difficulties decrease when curriculum and instruction emphasize covariation beginning in elementary school, introduce formal function concepts in middle school, and emphasize linear relationships with non-zero constants. These findings are encouraging, showing the effectiveness of a curricular shift to a greater emphasis on functions in the early grades. Relatedly, we also found a stronger emphasis on function research compared to the other three content area clusters, with 45% of the articles (33 out of 74) addressing functions and graphing. This distinction was particularly noteworthy in the algebra articles, compared to a more balanced distribution across the four content clusters in early algebra, which is unsurprising given the greater emphasis on function in secondary mathematics.

Additionally, longitudinal studies show that an emphasis on attention to the core algebraic thinking processes supports significant performance gains for elementary students (e.g., Blanton et al., 2019a, 2019b). A greater focus on nuances in understandings of equivalence, on function relationships, and on generalization are also yielding promising outcomes for students' reasoning. Research into students' acquisition of the relational meaning of the equal sign, for instance, shows that even brief interventions can support powerful understandings of equivalence (Donovan et al., 2022; Radford, 2022). The field has also seen a greater emphasis on covariation and on quantitative reasoning (Thompson, 2011), the latter representing an approach to algebraic thinking not addressed in either Kaput's (2008) or Kieran's (2022) frameworks. Covariation was a particular focus in the secondary studies in our sample, and together they suggest benefits for students' reasoning about linear, quadratic, and exponential functions, rates of change, slope and other features of graphs, and the development of correspondence rules. The literature also shows increased attention to learning trajectories and progressions, which historically have been situated at the elementary level (Ellis, 2022). In the past five years we have seen learning trajectories documenting students' learning about slope (Peck, 2020), quantitative and covariational relationships (Kafetzopoulos & Psycharis, 2022), quadratic growth (Fonger et al., 2020), finite-to-finite functions (Eames et al., 2021), and generalization (Stephens et al., 2021). Future research could extend this work into other algebra topics.

The findings from this body of literature suggest a number of potential directions for future research. Given the increased emphasis on early algebra, future longitudinal studies should examine the effects of these interventions on students' reasoning and performance in middle school and beyond. We also need studies that, while not longitudinal, nevertheless examine student learning over longer periods of time. Several articles in our sample called for longer-term interventions (e.g., Ayala-Altamirano & Molina, 2020; Donovan et al., 2022), as interventions of just a few sessions are often not sufficient for engendering lasting change. Such studies could also triangulate more data sources, including clinical interviews and classroom data, for richer models of student thinking. A drawback of many of the studies in our sample was that the data were limited to written responses, sometimes only from one task, limiting the generalizability of outcomes. In particular, there is a need for more classroom studies to scale up findings on the effects of task features, digital tools, and instructional techniques. Finally, the field needs a greater emphasis on teaching studies, particularly the instruction of specific algebra topics and studies that address the social aspects of learning, so that we can better understand how to support teachers in implementing high-quality algebra instruction.

Supplementary Information The online version contains supplementary material available at https://doi.org/10.1007/s11858-023-01545-9.

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*The papers included in the review have been marked with one asterisk, along with their number in brackets.

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