

Dynamic Tolling in Arc-based Traffic Assignment Models

Chih-Yuan Chiu¹, Chinmay Maheshwari¹, Pan-Yang Su¹, and Shankar Sastry¹

Abstract—Tolling in traffic networks offers a popular measure to minimize overall congestion. Existing toll designs primarily focus on congestion in *route-based* traffic assignment models (TAMs), in which travelers make a single route selection from source to destination. However, these models do not reflect real-world traveler decisions because they preclude deviations from a chosen route, and because the enumeration of all routes is computationally expensive. To address these limitations, our work focuses on *arc-based* TAMs, in which travelers sequentially select individual arcs (or edges) on the network to reach their destination. We first demonstrate that *marginal pricing*, a tolling scheme commonly used in route-based TAMs, also achieves socially optimal congestion levels in our arc-based formulation. Then, we use *perturbed best response* dynamics to model the evolution of travelers’ arc selection preferences over time, and a marginal pricing scheme to capture the social planner’s adaptive toll updates in response. We prove that our adaptive learning and marginal pricing dynamics converge to a neighborhood of the socially optimal loads and tolls. We then present empirical results that verify our theoretical claims.

I. INTRODUCTION

Mitigating congestion on transportation networks is a key concern in urban planning, since the selfish behavior of individual drivers often significantly increases driving time and pollution levels. Congestion pricing (tolling) is an increasingly popular tool for regulating traffic flows ([1, 2]). The design of tolls that can effectually induce socially optimal traffic loads requires a realistic traffic assignment model (TAM) that captures travelers’ routing preferences.

The classical literature on congestion pricing [3–5] often considers *route-based TAMs*, in which travelers make a single route selection at the origin node of the network, and do not deviate from their selected route until they reach the destination node. However, route-based modeling often requires enumerating all routes in a network, which may be computationally impractical, and do not capture correlations between the total costs of routes that share arcs. To address these issues, this work uses an *arc-based TAM* [6–11] to capture travelers’ routing decisions. In this framework, travelers navigate through a traffic network by sequentially selecting among outgoing edges at each intermediate node. Designing tolls for arc-based TAMs is relatively under-studied, with the only exception of [11] where the authors show that, similar to route based TAMs, marginal tolling also achieves social optimality in arc-based TAMs.

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¹Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720 (emails: {chihyuan.chiu, chinmay.maheshwari, pan-yang.su, sastry} at berkeley dot edu).

The basic philosophy of toll design is to steer the equilibrium behavior of agents towards social optimality by adding external incentives to their utility functions. However, a key assumption in this setting is that agents always adopt the equilibrium behavior, regardless of the incentives applied. This is not realistic, as real-world agents typically update their strategies from their initial strategies based on repeated interactions, only eventually converging to an equilibrium outcome [12]. While there exist learning rules for route-based TAMs which provably converge to the equilibrium strategies [13, 14], the development of analogous learning mechanisms for arc-based TAMs is relatively recent, e.g., in [10], which introduces a perturbed best response based dynamics. Consequently, it is necessary to study tolling in the presence of such dynamic adaptation rules by travelers.

Many prior works design tolls in dynamic environments by using reinforcement learning to iteratively update the toll on each arc. Chen et al. formulated the toll design problem as a Markov Decision Process (MDP) with high-dimensional state and action spaces, and apply a novel policy gradient algorithm to dynamically design tolls [15]. Mirzaei et al. used policy gradient methods to design incremental tolls on each link based on the difference between the observed and free-flow travel times [16]. Qiu et al. cast dynamic tolling into the framework of cooperative multi-agent reinforcement learning, and then applies graph convolutional networks to tractably solve the problem [17]. Likewise, Wang et al. use a cooperative actor-critic algorithm to tractably update a dynamic tolling scheme [18]. However, these methods operate on high-dimensional spaces, and are thus often computationally expensive. Moreover, they typically lack theoretical guarantees of convergence. The work most closely related to ours is [13] which studies dynamic tolling on parallel-link networks.

In this work, we study tolling in the arc-based TAM detailed in [10]. We show that there exists a unique toll that induces socially optimal congestion levels. Furthermore, we propose an adaptive tolling dynamics that steers the travelers’ routing preferences towards socially optimal congestion levels on the network. Specifically, we implement *marginal cost tolling*, via a discrete-time dynamic tolling scheme that adjusts tolls on arcs, with the following key features:

- 1) Tolls are adjusted at each time step towards the direction of the current marginal cost of travel latency.
- 2) Tolls are updated at a much slower rate compared to the rate at which travelers update arc selections at each non-destination node (timescale separation).
- 3) The toll update of each arc only depends on “local information” (in particular, the flow on each arc), and

does not require the traffic authority to access the demands of travelers elsewhere on the network.

This form of adaptive tolling was first introduced in [13] to study dynamic tolling scheme for parallel-link networks. This work extends the scope of that tolling scheme to bidirectional traffic networks, in the context of arc-based TAMs.

We show that the tolling dynamics converges to a neighborhood of a fixed toll vector, the corresponding equilibrium flows of which we prove to be socially optimal. We also show that the travelers' arc selections converge to a neighborhood of this socially optimal equilibrium flow. Our proof is based on the constant step-size two-timescale stochastic approximation theory [19], which allows us to decouple the toll and arc selection dynamics, and establish their convergence via two separate Lyapunov-based proofs. Although marginal tolling provably leads to socially efficient traffic allocation in a route-based TAM framework [5], to the best of our knowledge, this work presents the first marginal tolling scheme that induces socially optimal traffic flows in an arc-based setting.

The rest of the paper is outlined as follows: In Section II we present the transportation network model we consider in this work and summarize the required preliminaries from [10] on arc-based TAM. Furthermore, we also introduce the equilibrium concept we consider in this work, along with the notion of social optimality. In Section III, we present properties of the optimal tolls which induce social optimality in this setup. In Section IV, we introduce the tolling dynamics and present the convergence results. In Section V, we present a numerical study which corroborate the theoretical findings of this paper. Finally we conclude this paper in VI and present some directions of future research.

Notation: For each positive integer $n \in \mathbb{N}$, we denote $[n] := \{1, \dots, n\}$. For each $i \in [n]$ in an Euclidean space \mathbb{R}^n , we denote by e_i the i -th standard unit vector. Finally, let $\mathbf{1}\{\cdot\}$ denote the indicator function, which returns 1 if the input is true and 0 otherwise.

II. SETUP

Consider a traffic network described by a directed graph $G_O = (I_O, A_O)$, where I_O and A_O denote nodes and arcs, respectively. An example is shown in Figure 1 (top left); note that G_O can contain bidirectional arcs. Let the *origin nodes* and *destination nodes* be two disjoint subsets of I_O . To simplify our exposition, we assume that I_O contains only one origin $o \in I$ and one destination $d \in I$, although the results presented below straightforwardly extend to the multiple origin-destination-pair scenario. Travelers navigate through the network, from origin o to destination d , by sequentially selecting arcs at every intermediate node. This process produces congestion on each arc, which in turn determines travel times. The *cost* on each arc is then obtained by summing the travel time and toll. Specifically, each arc $a \in A_O$ is associated with a toll $p_a \in \mathbb{R}^{|A_O|}$, and a positive, strictly increasing *latency function* $s_a : [0, \infty) \rightarrow [0, \infty)$, which gives travel time as a function of traffic flow. The

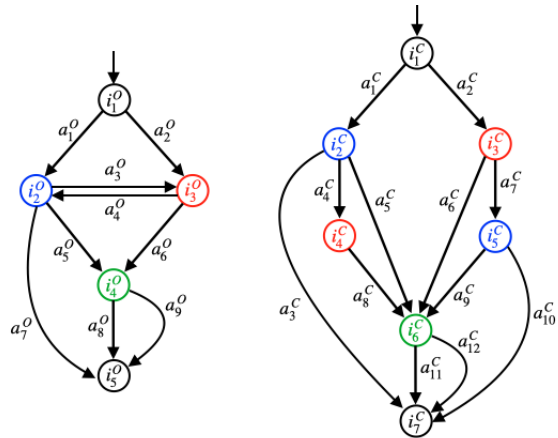


Fig. 1: Example of a single-origin single-destination original network G_O (top left, with superscript O), and its corresponding condensed DAG, or CoDAG, representation G (top right, with superscript C). Arc correspondences between the two networks are given by Table I, while node correspondences are indicated by color.

cost on arc $a \in A_O$ is then given by:

$$c_a(w_a, p_a) = s_a(w_a) + p_a.$$

Finally, let the demand of (infinitesimal) travelers entering from origin node o be denoted by g_o .

Note that sequential arc selection on networks with bidirectional arcs can result in a cyclic route. For example, a traveler navigating the left traffic network in Figure 1 using sequential arc selection may cycle between nodes i_2^O and i_3^O . To resolve this issue, we consider arc selection on the *condensed DAG (CoDAG) representation* of the original network G_O , a directed acyclic graph (DAG) representation, as proposed in [10]. The Condensed DAG representation preserves all acyclic routes from origin o to destination d in G_O , but precludes cyclic routes by design. Details regarding the construction and properties of CoDAG representations are provided in [10], Section II.

TABLE I: Arc correspondences between the graphs in Figure 1: The original network (top left) and the CoDAG (top right).

Original	a_1^O	a_2^O	a_3^O	a_4^O	a_5^O	a_6^O	a_7^O	a_8^O	a_9^O
CoDAG	a_1^T	a_2^T	a_4^T	a_7^T	a_5^T	a_6^T	a_3^T	a_{11}^T	a_{12}^T
					a_9^T	a_8^T	a_{10}^T		

We define $[\cdot] : A \rightarrow A_O$ to be a map from each CoDAG arc $a \in A$ to the corresponding arc in the original graph, $[a] \in A_O$ (as shown in Table I). For each arc $a \in A$, let i_a and j_a denote the start and terminal nodes, and for each node $i \in I$, let $A_i^-, A_i^+ \subset A$ denote the set of incoming and outgoing arcs.

A. Cost Model

Below, we assume that every traveler has access to G_O , and to the same CoDAG representation $G = (I, A)$ of G_O ; in particular, G is used to perform sequential arc selection to generate acyclic routes. The travelers' aggregative arc

selections generate network congestion. Specifically, for each $a \in A$, let the *flow* or *congestion level* on arc a be denoted by w_a , and let the total flow on the corresponding arc in the original network be denoted, with a slight abuse of notation, by $w_{[a]} := \sum_{a' \in [a]} w_{a'}$. Travelers perceive the cost on each arc $a \in A$ as:

$$\begin{aligned} \tilde{c}_{[a]}(w_{[a]}, p_{[a]}) &:= c_{[a]}(w_{[a]}, p_{[a]}) + \nu_a \\ &= s_{[a]}(w_{[a]}) + p_{[a]} + \nu_a, \end{aligned}$$

where ν_a is a zero-mean random variable. At each non-destination node $i \in I \setminus \{d\}$, travelers select among outgoing nodes $a \in A_i^+$ by comparing their perceived cost-to-go $\tilde{z}_a : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_O|} \rightarrow \mathbb{R}$, given recursively by:

$$\tilde{z}_a(w, p) := \tilde{s}_{[a]}(w_{[a]}) + p_{[a]} + \min_{a' \in A_{j_a}^+} \tilde{z}_{a'}(w, p), \quad j_a \neq d, \quad (1)$$

$$\tilde{z}_a(w, p) := \tilde{s}_{[a]}(w_{[a]}) + p_{[a]}, \quad j_a = d.$$

Consequently, the fraction of travelers who arrives at $i \in I \setminus \{d\}$ and choose arc $a \in A_i^+$ is given by:

$$P_{ij_a} := \mathbb{P}(\tilde{z}_a \leq \tilde{z}_{a'}, \forall a' \in A_i^+). \quad (2)$$

An explicit formula for the probabilities $\{P_{ij_a} : a \in A_i^+\}$, in terms of the statistics of \tilde{z}_a , is provided by the discrete-choice theory [20]. In particular, define $z_a(w) := \mathbb{E}[\tilde{z}_a(w)]$ and $\epsilon_a := \tilde{z}_a(w) - z_a(w)$, and define the latency-to-go at each node by:

$$\varphi_i(\{z_{a'}(w, p) : a' \in A_i^+\}) = \mathbb{E} \left[\min_{a' \in A_i^+} \tilde{z}_{a'}(w, p) \right]. \quad (3)$$

Then, from discrete-choice theory [20]:

$$P_{ij_a} = \frac{\partial \varphi_i}{\partial z_a}(z), \quad i \in I \setminus \{d\}, a \in A_i^+, \quad (4)$$

where, with a slight abuse of notation, we write $\varphi_i(z)$ for $\varphi_i(\{z_{a'} : a' \in A_i^+\})$.

To obtain a closed-form expression of φ , we employ the *logit Markovian model* [6, 7], under which the noise terms ϵ_a are described by the Gumbel distribution with scale parameter β . As a result, the expected minimum cost-to-go $z_a : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_O|} \rightarrow \mathbb{R}$, associated with traveling on each arc $a \in A$, assumes the following form:

$$\begin{aligned} z_a(w, p) & \\ &= s_{[a]} \left(\sum_{\bar{a} \in [a]} w_{\bar{a}} \right) + p_{[a]} - \frac{1}{\beta} \ln \left(\sum_{a' \in A_{j_a}^+} e^{-\beta z_{a'}(w, p)} \right). \end{aligned} \quad (5)$$

Note that (5) is well-posed, as z_a can be recursively computed from the destination back to the origin ([10], Section III).

¹Unlike existing TAMs, in our model, the latency of arcs in G can be coupled, since multiple copies of the same arc in G_O may exist in G .

B. CoDAG Equilibrium

Here, we define the *condensed DAG (CoDAG) equilibrium* (Definition 1), based on the CoDAG representation of the original traffic network. Specifically, we show that the CoDAG equilibrium exists, is unique, and solves a strictly convex optimization problem (Theorem 1).

Definition 1 (Condensed DAG Equilibrium): Fix a toll vector $p \in \mathbb{R}^{|A_O|}$, and fix $\beta > 0$. We call an arc-flow vector $\bar{w}^\beta(p) \in \mathbb{R}^{|A|}$ a *Condensed DAG (CoDAG) equilibrium* at p if, for each $i \in I \setminus \{d\}$, $a \in A_i^+$:

$$\begin{aligned} &\bar{w}_a^\beta(p) \\ &= \left(g_i + \sum_{a' \in A_i^+} \bar{w}_{a'}^\beta(p) \right) \frac{\exp(-\beta z_a(\bar{w}^\beta(p), p))}{\sum_{a' \in A_i^+} \exp(-\beta z_{a'}(\bar{w}^\beta(p), p))}, \end{aligned} \quad (6)$$

where $g_i = g_0 \cdot \mathbf{1}(i = o)$, and $w \in \mathcal{W}$, where:

$$\mathcal{W} := \left\{ w \in \mathbb{R}^{|A|} : \sum_{a \in A_i^+} w_a = \sum_{a \in A_i^-} w_a, \forall i \neq o, d, \quad (8) \right. \\ \left. \sum_{a \in A_o^+} w_a = g_o, w_a \geq 0, \forall a \in A \right\}$$

characterizes the conservation of flow in the CoDAG G . Note that \mathcal{W} is convex and compact.

At a CoDAG equilibrium $\bar{w}^\beta(p)$, the fraction of travelers at any intermediate node $i \in I \setminus \{d\}$ who selects an arc $a \in A_i^+$ is given by $\bar{\xi}_a^\beta(p)$, as defined below:

$$\bar{\xi}_a^\beta(p) := \frac{\bar{w}_a^\beta(p)}{\sum_{a' \in A_i^+} \bar{w}_{a'}^\beta(p)}.$$

The CoDAG equilibrium bears some resemblance to the Markovian Traffic Equilibrium (MTE) introduced in Baillon and Cominetti [7]. However, the CoDAG formulation by design precludes the possibility of assigning cyclic routes, and is capable of capturing couplings between arcs in the CoDAG G that correspond to the same arc in the original network G_O (see [10], Remark 6).

Below, we show that, given any CoDAG representation G of an original network G_O and any fixed toll vector $p \in \mathbb{R}^{|A_O|}$, the CoDAG equilibrium exists and is unique. Specifically, the CoDAG equilibrium is the unique minimizer of a strictly convex optimization problem over a compact set. This characterization provides powerful insight into the mathematical properties of the CoDAG equilibrium flow, and its dependence on the toll vector. These properties will be used in our work to establish the existence of an optimal toll (Theorem 2) and the convergence of our discrete-time toll dynamics to the optimal toll (Theorem 3).

For each $[a] \in A_O$, define $F : \mathcal{W} \times \mathbb{R}^{|A_O|} \rightarrow \mathbb{R}$ by:

$$\begin{aligned} &F(w, p) \\ &= \sum_{[a] \in A_O} \int_0^{w_{[a]}} [s_{[a]}(u) + p_{[a]}] du \end{aligned}$$

$$+ \frac{1}{\beta} \sum_{i \neq d} \left[\sum_{a \in A_i^+} w_a \ln w_a - \left(\sum_{a \in A_i^+} w_a \right) \ln \left(\sum_{a \in A_i^+} w_a \right) \right]. \quad (9)$$

Theorem 1: For each fixed toll vector $p \in \mathbb{R}^{|A_O|}$, the corresponding CoDAG equilibrium $\bar{w}^\beta(p) \in \mathcal{W}$ exists, is unique, and is the unique minimizer of $F(\cdot, p)$ over \mathcal{W} .

Proof: (Proof Sketch) The proof parallels that of [10], Theorem 1 and Lemma 1. For details, please see [10], Section III and Appendix B. ■

C. Social Optimality

We now describe the socially optimal flow which would lead to the most efficient use of the transportation network. More specifically, we define below the notion of *perturbed social optimality* considered in our work.

Definition 2 (Perturbed Socially Optimal Flow): We define a *perturbed socially optimal flow* with regularization parameter $\beta > 0$ to be a minimizer of the following convex optimization problem:

$$\begin{aligned} & \min_{w \in \mathcal{W}} \sum_{[a] \in A_O} w_{[a]} \cdot s_{[a]}(w_{[a]}) \\ & + \frac{1}{\beta} \sum_{i \neq d} \left[\sum_{a \in A_i^+} w_a \ln w_a - \left(\sum_{a \in A_i^+} w_a \right) \ln \left(\sum_{a \in A_i^+} w_a \right) \right], \end{aligned}$$

with \mathcal{W} given by (8), and $w_{[a]} := \sum_{a' \in [a]} w_{a'}$, as defined above.

In words, perturbed social optimality is characterized as the total latency experienced by travelers on each arc of the CoDAG G , augmented by an entropy term with regularization parameter β which captures stochasticity in the travelers' arc selections.

III. OPTIMAL TOLL: EXISTENCE AND UNIQUENESS

Below, we characterize the *optimal toll* $\bar{p} \in \mathbb{R}^{|A_O|}$ for which the corresponding CoDAG equilibrium $\bar{w}^\beta(\bar{p})$ is perturbed socially optimal (see Definition 2). Throughout the rest of the paper, we call \bar{p} the *optimal toll*.

Theorem 2: There exists a unique toll vector $\bar{p} \in \mathbb{R}^{|A_O|}$ that satisfies the following fixed-point equation:

$$\bar{p}_{[a]} = \bar{w}_{[a]}^\beta(\bar{p}) \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})), \quad \forall a \in A. \quad (10)$$

Moreover, $\bar{w}^\beta(\bar{p})$, the CoDAG equilibrium flow distribution corresponding to \bar{p} , is the perturbed socially optimal flow with regularization β .

To prove Theorem 2, we first show that $\bar{w}^\beta(p)$ is continuous and monotonic in the toll p (Lemmas 1 and 2). Then, we use these properties to establish the existence and uniqueness of a toll vector $\bar{p} \in \mathbb{R}^{|A_O|}$ satisfying the fixed-point equation (10) (Lemma 3). Finally, we prove that the CoDAG equilibrium flow allocation $\bar{w}^\beta(\bar{p})$ corresponding to \bar{p} is perturbed socially optimal (Lemma 4).

Below, we begin by establishing that the CoDAG equilibrium $\bar{w}^\beta(p)$ is a continuously differentiable and monotonic function of the toll $p \in \mathbb{R}^{|A_O|}$.

Lemma 1: $\bar{w}^\beta(p)$ is continuously differentiable in p .

Proof: (Proof Sketch) For each fixed toll vector $p \in \mathbb{R}^{|A_O|}$, the corresponding CoDAG equilibrium $\bar{w}^\beta(p)$ uniquely solves the KKT conditions of the optimization problem of minimizing $F(\cdot, p)$ over \mathcal{W} (Theorem 1). We write these KKT conditions as an *implicit function* $J : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_O|} \rightarrow \mathbb{R}^{|A|}$ of the flow and tolls (w, p) :

$$J(w, p) = \mathbf{0},$$

where $\mathbf{0}$ denotes the $|A|$ -dimensional zero vector. We can then derive an explicit expression for $\frac{d\bar{w}^\beta}{dp}(p)$ at each $p \in \mathbb{R}^{|A_O|}$ by proving that:

$$\frac{\partial J}{\partial w}(\bar{w}^\beta(p), p) \in \mathbb{R}^{|A| \times |A|}$$

is non-singular for each fixed p , and invoking the Implicit Function Theorem. For details, please see Appendix A.1 [21]. ■

Lemma 2: For any $p, p' \in \mathbb{R}^{|A_O|}$:

$$\sum_{a \in A} \left(\bar{w}_a^\beta(p') - \bar{w}_a^\beta(p) \right) (p'_{[a]} - p_{[a]}) \leq 0.$$

Proof: (Proof Sketch) By Theorem 1, the CoDAG equilibrium $\bar{w}^\beta(p)$ is the unique minimizer of the strictly convex function $F(\cdot, p) : \mathcal{W} \rightarrow \mathbb{R}$ defined by (9). Thus, $\bar{w}^\beta(p)$ can be characterized by the first-order optimality conditions of this optimization problem. This in turn allows us to establish monotonicity. For details, please see Appendix A.2 [21]. ■

We then use the above lemmas to prove that the fixed-point equation (10) yields a unique solution.

Lemma 3: There exists a unique $\bar{p} \in \mathbb{R}^{|A_O|}$ satisfying (10):

$$\bar{p}_{[a]} = \bar{w}_{[a]}^\beta(\bar{p}) \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})), \quad \forall [a] \in A_O.$$

Proof: (Proof Sketch) Existence follows from the Brouwer fixed point theorem, since $\bar{w}^\beta(p)$ is continuous in p (Lemma 1). Uniqueness follows via a contradiction argument; we show that the existence of two distinct fixed points of (10) would violate the monotonicity established by Lemma 2. For details, please see Appendix A.3 [21]. ■

Finally, we prove that the CoDAG equilibrium flow corresponding to $\bar{p} \in \mathbb{R}^{|A_O|}$ is perturbed socially optimal.

Lemma 4: $\bar{w}^\beta(\bar{p})$ is perturbed socially optimal.

Proof: (Proof Sketch) This follows by comparing the KKT conditions satisfied by $\bar{w}^\beta(\bar{p})$ (Theorem 1) with the KKT conditions of the optimization problem that defines the perturbed socially optimal flow in Definition 2. For details, please see Appendix A.4 [21]. ■

Together, Lemmas 1, 2, 3, and 4 prove Theorem 2.

IV. DYNAMICS AND CONVERGENCE

A. Discrete-time Dynamics

Here, we present discrete-time stochastic dynamics that describes the evolution of the traffic flow and tolls on the network. Formally, g_o units of traveler flow enter the network

at the origin node o at each time step $n \geq 0$. At each non-destination node $i \in I \setminus \{d\}$, a $\xi_a[n]$ fraction of travelers chooses an outgoing arc $a \in A_i^+$. We shall refer to $\xi_a[n]$ as the *aggregate arc selection probability*. Consequently, the flow induced on any arc $a \in A$ satisfies:

$$W_a[n] = \left(g_{i_a} + \sum_{a' \in A_i^+} W_{a'}[n] \right) \cdot \xi_a[n]. \quad (11)$$

At the conclusion of every time step n , travelers reach the destination node d and observe a noisy estimate of the cost-to-go values and tolls on all arcs in the network (including arcs not traversed during that time step). Let $K_i > 0$ denote node-dependent constants, and let $\{\eta_i[n+1] \in \mathbb{R} : i \in I, n \geq 0\}$ be independent bounded random variables² in $[\underline{\mu}, \bar{\mu}]$, with $0 < \underline{\mu} < \mu < \bar{\mu} < 1/\max\{K_i : i \in I \setminus \{d\}\}$ and $\mathbb{E}[\eta_{i_a}[n+1]] = \mu$ at each node $i \in I$ and discrete time index $n \geq 0$. At the next time $n+1$ and non-destination node $i \in I \setminus \{d\}$, a $\eta_i[n+1] \cdot K_i$ fraction of travelers at node $i \in I$ observes the latencies on each arc, and decides to switch to the outgoing arc that minimizes the (stochastic) observed cost-to-go. Meanwhile, $1 - \eta_i[n+1] \cdot K_i$ fraction of travelers selects the same arc they used at time step n . Thus, the arc selection probabilities evolve according to the following *perturbed best-response dynamics*:

$$\begin{aligned} \xi_a[n+1] & \\ = \xi_a[n] + \eta_{i_a}[n+1] \cdot K_{i_a} & \\ \cdot \left(-\xi_a[n] + \frac{\exp(-\beta[z_a(W[n], P[n])])}{\sum_{a' \in A_i^+} \exp(-\beta[z_{a'}(W[n], P[n])])} \right). & \end{aligned} \quad (12)$$

We assume that $\xi_a[0] > 0$ for each $a \in A$, i.e., each arc has some strictly positive initial traffic flow. This captures the stochasticity in travelers' perception of network congestion that causes each arc to be assigned a nonzero probability of being selected.

At each time step $n+1 \geq 0$, the tolls $P_{[a]}[n] \in \mathbb{R}^{|A_O|}$ on each arc $[a] \in A_O$ are updated by interpolating between the tolls implemented at time step n , and the marginal latency of that arc given the flow at time step n . That is:

$$\begin{aligned} P_{[a]}[n+1] & \\ = P_{[a]}[n] + \gamma \left(-P_{[a]}[n] + W_{[a]}[n] \cdot \frac{ds_{[a]}(W_{[a]}[n])}{dw} \right), & \end{aligned} \quad (13)$$

with $\gamma \in (0, 1)^3$, where with a slight abuse of notation, we denote $W_{[a]} := \sum_{a' \in [a]} W_{a'}$. Note that the update (13) is distributed, i.e., for each arc in the original network, the updated toll depends only on the flow of that arc, and not on the flow of any other arc. Moreover, we assume that $\gamma \ll \mu$, i.e., the toll updates (13) occur at a slower timescale compared to the arc selection probability updates (12).

To simplify our study of the convergence of the dynamics (12) and (13), we assume that the arc latency functions are affine in the congestion on the link.

²The random variables $\{\eta_a[n] : a \in A, n \geq 0\}$ are assumed to be independent of travelers' perception uncertainties.

³Our result also holds if γ is a random variable with bounded support.

Assumption 1: Each arc latency function $s_{[a]}$ is affine, i.e.,:

$$s_{[a]}(w_{[a]}) = \theta_{\bar{a},1} w_{[a]} + \theta_{[a],0}, \quad (14)$$

for some $\theta_{[a],1}, \theta_{[a],0} > 0$.

Under Assumption 1, the toll dynamics (13) can be alternatively written as follows

$$P_{[a]}[n+1] = P_{[a]}[n] + \gamma \left(-P_{[a]}[n] + W_{[a]}[n] \cdot \theta_{[a],1} \right). \quad (15)$$

B. Convergence Results

In this subsection, we show that the arc selection probability and toll updates (12)-(15) converge in the neighborhood of the socially optimal flow $\bar{w}^\beta(\bar{p})$ and the corresponding toll \bar{p} respectively.

Theorem 3: The joint evolution of arc selection probability and toll updates (12)-(15) satisfies

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{E}[\|\xi[n] - \bar{\xi}^\beta(\bar{p})\|_2^2 + \|P[n] - \bar{p}\|_2^2] & \\ = O\left(\mu + \frac{\gamma}{\mu}\right). & \end{aligned}$$

Consequently, for each $\delta > 0$:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}(\|\xi[n] - \bar{\xi}^\beta(\bar{p})\|_2^2 + \|P[n] - \bar{p}\|_2^2 \geq \delta) & \\ = O\left(\frac{\mu}{\delta} + \frac{\gamma}{\delta\mu}\right). & \end{aligned}$$

To prove Theorem 3, we employ the theory of two-timescale stochastic approximation [22]. Consequently, the asymptotic behavior of (12)-(15) can be characterized by studying the convergence properties of the corresponding continuous-time dynamical system. Since the tolls are updated at a slower rate compared to the traffic flows ($\gamma \ll \mu$), we consider the evolution of continuous-time flows $w(t)$ under a fixed toll $p \in \mathbb{R}^{|A_O|}$, and continuous-time tolls $p(t)$ with flow converged at the corresponding CoDAG equilibrium $\bar{w}^\beta(p(t))$ at each time. Specifically, for any fixed toll $p \in \mathbb{R}^{|A_O|}$, on each arc $a \in A$, the arc selection probabilities evolve as follows:

$$\begin{aligned} w_a(t) &= \xi_a(t) \cdot \left(g_{i_a} + \sum_{a' \in A_i^+} w_{a'}(t) \right), & (16) \\ \dot{\xi}_a(t) &= K_{i_a} \cdot \left(-\xi_a(t) + \frac{\exp(-\beta \cdot z_a(w(t), p))}{\sum_{a' \in A_i^+} \exp(-\beta \cdot z_{a'}(w(t), p))} \right). & (17) \end{aligned}$$

Meanwhile, on each arc $[a] \in A_O$ in the original network, we consider the following continuous-time toll dynamics:

$$\dot{p}_{[a]}(t) = -p_{[a]}(t) + \bar{w}_{[a]}^\beta(p(t)) \cdot \theta_{[a],1}. \quad (18)$$

We prove that, for each fixed toll $p \in \mathbb{R}^{|A_O|}$, the corresponding continuous-time ξ -dynamics (17) globally asymptotically converges to the corresponding CoDAG equilibrium $\bar{w}^\beta(p) \in \mathbb{R}^{|A|}$. Moreover, the continuous-time toll dynamics (18) globally converges to the optimal toll $\bar{p} \in \mathbb{R}^{|A_O|}$.

Lemma 5 (Informal): Suppose $w(0) \in \mathcal{W}$, i.e., the initial flow satisfies flow continuity. Under the continuous-time flow

dynamics (17) and (16), if $K_i \ll K_{i'}$ whenever $\ell_i < \ell_{i'}$, the continuous-time traffic allocation $w(t)$ globally asymptotically converges to the corresponding CoDAG equilibrium $\bar{w}^\beta(p)$.

Proof: (Proof Sketch) The following proof sketch parallels that of [10], Lemma 2, and is included for completeness. Recall that Theorem 1 establishes $\bar{w}^\beta(p)$ as the unique minimizer of the map $F(\cdot, p) : \mathcal{W} \rightarrow \mathbb{R}$, defined by (9). We show that $F(\cdot, p)$ is a Lyapunov function for the continuous-time flow dynamics induced by (17). To this end, we first unroll the dynamics (17) using (16), as follows:

$$\begin{aligned} \dot{w}_a(t) &= -K_{i_a} \cdot \left(1 - \frac{1}{K_{i_a}} \cdot \frac{\sum_{a' \in A_{i_a}^-} \dot{w}_{a'}(t)}{\sum_{\hat{a} \in A_{i_a}^+} w_{\hat{a}}(t)} \right) w_a(t) \\ &\quad + K_{i_a} \cdot \sum_{a' \in A_{i_a}^-} w_{a'}(t) \cdot \frac{\exp(-\beta z_a(w(t), p))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta z_{a'}(w(t), p))}. \end{aligned}$$

Next, we establish that if $w(0) \in \mathcal{W}$, then for each $t \geq 0$:

$$\dot{F}(t) = \dot{w}(t)^\top \nabla_w F(w(t)) \leq 0.$$

The proof then follows from LaSalle's Theorem (see [23, Proposition 5.22]). For a precise statement of Lemma 5, please see Appendix B.1 [21]; for the proof of the analogous theorem in [10], please see [10] Appendix C.1. ■

Lemma 6: The continuous-time toll dynamics (18) globally exponentially converges to the CoDAG equilibrium $\bar{w}^\beta(\bar{p})$ corresponding to the optimal toll \bar{p} .

Proof: Define $D \in \mathbb{R}^{|A_O| \times |A_O|}$ to be the diagonal and symmetric positive definite matrix whose $[a]$ -th diagonal element is given by:

$$\frac{ds_{[a]}(\bar{w}_{[a]}^\beta(\bar{p}))}{dw} = \theta_{[a],1} > 0,$$

for each $[a] \in A_O$. Note that D is independent of the toll p . Now, consider the Lyapunov function $V : \mathbb{R}^{|A_O|} \rightarrow \mathbb{R}$, defined by:

$$V(p) := \frac{1}{2} (p - \bar{p})^\top D^{-1} (p - \bar{p}).$$

The trajectory of the continuous-time toll dynamics (18), starting at $p(0)$, satisfies:

$$\begin{aligned} \dot{V}(p(t)) &= (p(t) - \bar{p})^\top D^{-1} \dot{p}(t) \\ &= \sum_{[a] \in A_O} \frac{(p_{[a]}(t) - \bar{p}_{[a]})}{\theta_{[a],1}} \cdot \left(-p_{[a]}(t) + \theta_{[a],1} \bar{w}_{[a]}^\beta(p(t)) \right) \\ &= \sum_{[a] \in A_O} \frac{(p_{[a]}(t) - \bar{p}_{[a]})}{\theta_{[a],1}} \\ &\quad \cdot \left(-p_{[a]}(t) + \bar{p}_{[a]} - \bar{p}_{[a]} + \theta_{[a],1} \bar{w}_{[a]}^\beta(p(t)) \right) \\ &= -2V(p(t)) \\ &\quad + \sum_{[a] \in A_O} (p_{[a]}(t) - \bar{p}_{[a]}) \left(\bar{w}_{[a]}^\beta(p(t)) - \bar{w}_{[a]}^\beta(\bar{p}) \right) \\ &\leq -2V(p(t)), \end{aligned}$$

where the final inequality follows due to the monotonicity of the map $\bar{w}^\beta(\cdot)$ (Lemma 2). ■

To conclude the proof of Theorem 3, it remains to check that the discrete-time dynamics (12)-(15), and the continuous-time dynamics (17)-(18), satisfy the technical conditions in Lemmas 7 and 8. In particular, Lemma 7 establishes that flows and tolls are uniformly bounded across the arc and time indices, while Lemma 8 asserts that the continuous-time flow and toll dynamics maps are Lipschitz continuous.

Lemma 7: The continuous-time flow and toll dynamics induced by (12)-(15) satisfy:

- 1) For each $a \in A$: $\{M_a[n+1] : n \geq 0\}$ is a martingale difference sequence with respect to the filtration $\mathcal{F}_n := \sigma(\cup_{a \in A} (W_a[1], \xi[1], p[1], \dots, W_a[n], \xi[n], p[n]))$.
- 2) There exist $C_w, C_m, C_p > 0$, independent of the node-dependent values $\{K_i : i \in I\}$, such that, for each $a \in A$ and each $n \geq 0$, we have $W_a[n] \in [C_w, g_o]$, $P_a[n] \in [0, C_p]$, and $|M_a[n]| \leq C_m$.

Likewise, the continuous-time flow and toll dynamics induced by (17) and (18) satisfy:

- 3) For each $a \in A$, $t \geq 0$, we have $w_a(t) \in [C_w, g_o]$ and $p_a(t) \in [0, C_p]$.

Proof: Please see Appendix B.2 [21]. ■

Lemma 8: The continuous-time flow dynamics (16) and toll dynamics (18) satisfy:

- 1) The map $\bar{\xi}^\beta : \mathbb{R}^{|A_O|} \rightarrow \mathbb{R}^{|A|}$ is Lipschitz continuous.
- 2) For each $a \in A$, the restriction of the cost-to-go map $z_a : \mathcal{W} \times \mathbb{R}^{|A_O|} \rightarrow \mathbb{R}$ to the set of realizable flows and tolls, i.e., $\mathcal{W}' \times [0, C_p]^{|A_O|}$, is Lipschitz continuous.
- 3) The map from the probability transitions $\xi \in \prod_{i \in I \setminus \{a\}} \Delta(A_i^+)$ and the traffic flows $w \in \mathcal{W}$ is Lipschitz continuous.
- 4) For each $a \in A$, the restriction of the continuous dynamics transition map $\rho_a : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_O|} \rightarrow \mathbb{R}^{|A|}$, defined recursively as follows for each $a \in A$:

$$\rho_a(\xi, p) := -\xi_a + \frac{\exp(-\beta z_a(w, p))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta z_{a'}(w, p))}$$

to the set of realizable flows and tolls, i.e., $\mathcal{W}' \times [0, C_p]^{|A_O|}$, is Lipschitz continuous.

- 5) For each $a \in A$, the map $r_{[a]} : \mathbb{R}^{|A_O|} \times \mathbb{R}^{|A_O|}$, defined as follows for each $a \in A$:

$$r_{[a]}(p) := -p_{[a]} + \bar{w}_{[a]}^\beta(p) \cdot \frac{ds_{[a]}(\bar{w}_{[a]}^\beta(p))}{dw},$$

is Lipschitz continuous.

Proof: Please see Appendix B.3 [21]. ■

V. EXPERIMENT RESULTS

This section presents experiments that validate the theoretical convergence results of Section IV. We present simulation results illustrating that, under (12)-(15), the traffic flows and tolls converge to a neighborhood of the socially optimal values, as claimed by Theorem 3.

Consider the network presented in Figure 1, following affine latency functions (14) with parameters given in Table

TABLE II: Parameters for simulation.

Notation	Default value
$\theta_{\bar{a},0}$	0, 1, 0, 1, 1, 0, 1, 1, 1 (ordered by edge index)
$\theta_{\bar{a},1}$	2, 1, 1, 1, 1, 1, 2, 2, 2 (ordered by edge index)
g_1	1
β	10
γ	0.02
$\eta_{i_a}[n]$	Uniform(0, 0.1), $\forall a \in A, i \in I \setminus \{d\}$

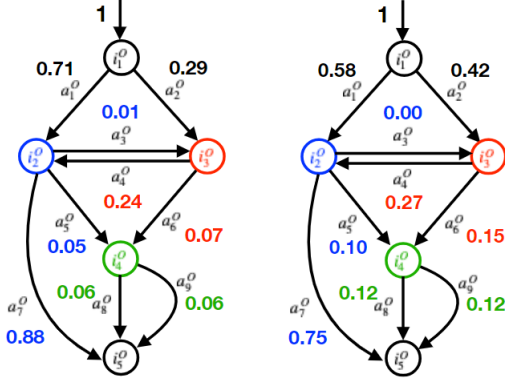


Fig. 2: Steady state traffic flow on each arc for the original network before (left) and after (right) tolls. Flows on arcs emerging from the same node are represented in the same color.

II. To validate Theorem 3, we evaluate and plot the traffic flow values $W_a[n]$ and toll values $P_a[n]$ on each arc $a \in A$ with respect to discrete time index $n \geq 0$. Figure 2 presents traffic flow values at the condensed DAG equilibrium (i.e., w^β) for the original network before and after tolls. Meanwhile, Figure 3 and 4 illustrate that w and p converge to the condensed DAG equilibrium in approximately 300 iterations. As in [10], flow convergence to the optimal allocation occurs even when the constants $\{K_i : i \in I\}$ are simply all set to 1. While the original traffic distribution is more concentrated on a few routes, tolls can distribute the traffic more evenly. This shows that tolls can improve overall social welfare by reducing congestion in over-utilized routes.

VI. CONCLUSION AND FUTURE WORK

This work introduces a discrete-time adaptive tolling scheme to minimize the total travel latency in a general traffic network with bidirectional edges. Our model assumes that, at each time, players near-instantaneously react via perturbed best response to the announced tolls. Accordingly, we formulate a two-timescale stochastic dynamical system that describes the joint evolution of traffic flow and tolls. We prove that the fixed point of these dynamics is unique and corresponds to the optimal traffic flow allocation from the perspective of minimizing the total travel time. Moreover, we prove that the stochastic dynamics converges to a neighborhood of the unique fixed point with high probability. Finally, we present simulation results that corroborate our theoretical findings.

Interesting avenues of future research include: (1) Extending our theoretical analysis to the setting where the latency function of each arc is not necessarily affine, (2) Developing

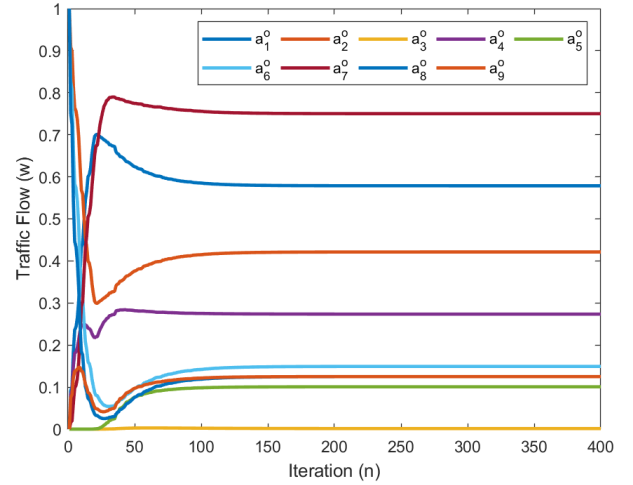


Fig. 3: Traffic flow W vs. time index n for the condensed DAG in Figure 1.

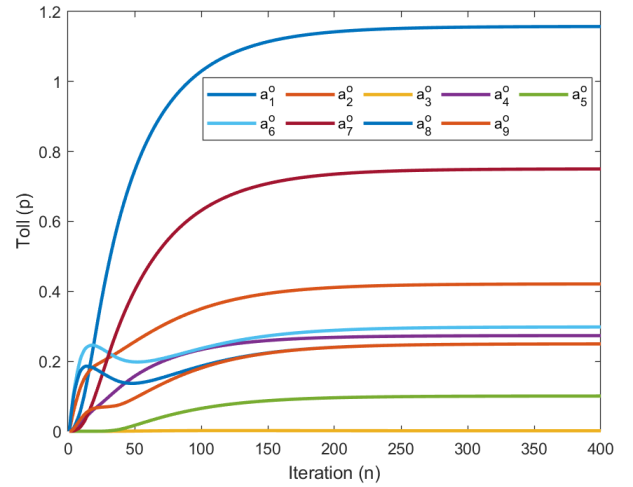


Fig. 4: Toll P vs. time index n for the condensed DAG in Figure 1.

tolling dynamics for the setting in which the central authority must learn the network latency functions and entropy regularization parameter $\beta > 0$ while simultaneously implementing an adaptive tolling scheme that converges to the optimal toll, and (3) Designing robust tolls for traffic networks in which some fraction of the population behaves unexpectedly or adversarially.

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APPENDIX

Please use the following link to access the ArXiv version with the appendix [21] (<https://arxiv.org/pdf/2307.05466.pdf>). The authors will make certain that this link stays active.

Below, we present proofs omitted in the main paper due to space limitations.

A. Proofs for Section III

Here, we provide the proofs of Lemmas 1, 2, 3, and 4.

1) *Proof of Lemma 1:* Define $F : \mathcal{W} \times \mathbb{R}^{|A_o|} \rightarrow \mathbb{R}$ by:

$$\begin{aligned} F(w, p) &:= \sum_{[a] \in A_o} \int_0^{w_{[a]}} [s_{[a]}(z) + p_{[a]}] dz \\ &+ \frac{1}{\beta} \sum_{i \neq d} \left[\sum_{a \in A_i^+} w_a \ln w_a - \left(\sum_{a \in A_i^+} w_a \right) \ln \left(\sum_{a \in A_i^+} w_a \right) \right]. \end{aligned} \quad (19)$$

The theory of constrained optimization implies that, for each p , the unique minimizer of $F(\cdot, p) : \mathcal{W} \rightarrow \mathbb{R}$ is completely characterized via a set of equality constraints, which we describe below. First, recall that since \mathcal{W} is a subset of an affine subspace of $\mathbb{R}^{|A|}$ characterized by $|I \setminus \{d\}|$ equality constraints, there exist $M \in \mathbb{R}^{|A| \times |I \setminus \{d\}|}$, of full column rank, and $b \in \mathbb{R}^{|I \setminus \{d\}|}$ such that:

$$\mathcal{W} = \{w \in \mathbb{R}^{|A|} : M^\top w + b = 0, w_a \geq 0, \forall a \in A\}.$$

Moreover, by using QR decomposition, we can assume that the columns of M are orthonormal. Next, let $B \in \mathbb{R}^{|A| \times (|A| - |I \setminus \{d\}|)}$ be given such that the columns of B have unit norm, are pair-wise orthogonal, and are each orthogonal to the subspace of $\mathbb{R}^{|A|}$ spanned by the columns of M , i.e., B^\top maps each vector in $\mathbb{R}^{|A|}$ to the coefficients of its projection onto the linear subspace orthogonal to \mathcal{W} , with respect to an ordered, orthonormal basis of that subspace. Then the theory of constrained optimization, and the strict convexity of $F(\cdot, p)$, imply that $\bar{w}^\beta(p)$, the unique minimizer of $F(\cdot, p)$, is completely characterized by the equations:

$$\begin{aligned} M^\top w + b &= 0, \\ B^\top \nabla_w F(w, p) &= 0. \end{aligned}$$

To this end, define $J : \mathbb{R}^{|A_o|} \times \mathbb{R}^{|A|} \rightarrow \mathbb{R}^{|A|}$ by:

$$J(w, p) := \begin{bmatrix} M^\top w + b \\ B^\top \nabla_w F(w, p) \end{bmatrix}.$$

Note that J is continuously differentiable almost everywhere, with:

$$\frac{\partial J}{\partial w}(w, p) = \begin{bmatrix} M^\top \\ B^\top \nabla_w^2 F(w, p) \end{bmatrix} \in \mathbb{R}^{|A| \times |A|}.$$

Suppose by contradiction that $\frac{\partial J}{\partial w}(w, p) \in \mathbb{R}^{|A| \times |A|}$ is singular at some (w, p) . Then $\frac{\partial J}{\partial w}(w, p)^\top \in \mathbb{R}^{|A| \times |A|}$ lacks full column rank, i.e.:

$$\begin{aligned} \dim(R(M) + R(\nabla_w^2 F(w, p)B)) &= \text{rank} \left(\begin{bmatrix} M & \nabla_w^2 F(w, p)B \end{bmatrix} \right) \\ &\leq |A| - 1. \end{aligned}$$

By the Boolean formula for sums of vector spaces:

$$\dim(R(M) \cap R(\nabla_w^2 F(w, p)B))$$

$$\begin{aligned} &= \dim(R(M)) + \dim(R(\nabla_w^2 F(w, p)B)) \\ &\quad - \dim(R(M) + R(\nabla_w^2 F(w, p)B)) \\ &= \dim(R(M)) + \dim(R(B)) \\ &\quad - \dim(R(M) + R(\nabla_w^2 F(w, p)B)) \\ &\geq |A| - (|A| - 1) \\ &= 1. \end{aligned}$$

Thus, there exists some nonzero vector $v \in R(M) \cap R(\nabla_w^2 F(w, p)B)$. Since $v \in R(M)$, and the columns of B are orthogonal to $R(M)$, we have $B^\top v = 0$. Meanwhile, since $v \in R(\nabla_w^2 F(w, p)B)$, there exists some nonzero $w \in \mathbb{R}^{|A| - d}$ such that $u = \nabla_w^2 F(w, p)Bw$. Thus, we have:

$$0 = B^\top u = B^\top \nabla_w^2 F(w, p)Bu,$$

a contradiction, since the fact that B^\top has full row rank and $\nabla_w^2 F(w, p)$ is symmetric positive definite implies that $B^\top \nabla_w^2 F(w, p)B$ is symmetric positive definite, and $u \neq 0$ by construction. This establishes that $\frac{\partial J}{\partial w}(w, p) \in \mathbb{R}^{|A| \times |A|}$ is non-singular at each $(w, p) \in \mathbb{R}^{|A_o|} \times \mathbb{R}^{|A|}$. The existence and continuity of $\frac{d\bar{w}^\beta}{dp}(p)$ at each $p \in \mathbb{R}^{|A_o|}$ now follows from the Implicit Function Theorem.

2) *Proof of Lemma 2:* In this subsection, we show that for any $p, p' \in \mathbb{R}^{|A_o|}$:

$$\sum_{a \in A} \left(\bar{w}_a^\beta(p') - \bar{w}_a^\beta(p) \right) (p'_{[a]} - p_{[a]}) \leq 0.$$

By Theorem 1, $\bar{w}^\beta(p)$ is the unique minimizer, in \mathcal{W} , of the following strictly convex function of w :

$$\begin{aligned} &\sum_{[a] \in A_o} \int_0^{w_{[a]}} [s_{[a]}(z) + p_{[a]}] dz \\ &+ \frac{1}{\beta} \sum_{i \neq d} \left[\sum_{a \in A_i^+} w_a \ln w_a - \left(\sum_{a \in A_i^+} w_a \right) \ln \left(\sum_{a \in A_i^+} w_a \right) \right]. \end{aligned}$$

Applying first-order conditions for optimality in constrained convex optimization, we obtain that, for each $w^1 \in \mathcal{W}$:

$$\begin{aligned} &\sum_{a \in A} \left[s_{[a]}(\bar{w}_{[a]}^\beta(p)) + p_{[a]} + \frac{1}{\beta} \ln \left(\frac{\bar{w}_a^\beta(p)}{\sum_{a' \in A_{i_a}^+} \bar{w}_{a'}^\beta(p)} \right) \right] \\ &\cdot (w_a^1 - \bar{w}_a^\beta(p)) \geq 0. \end{aligned}$$

Similarly, for $\bar{w}_{[a]}(p')$, we obtain that for each $w^2 \in \mathcal{W}$:

$$\begin{aligned} &\sum_{a \in A} \left[s_{[a]}(\bar{w}_{[a]}^\beta(p')) + p'_{[a]} + \frac{1}{\beta} \ln \left(\frac{\bar{w}_a^\beta(p')}{\sum_{a' \in A_{i_a}^+} \bar{w}_{a'}^\beta(p')} \right) \right] \\ &\cdot (w_a^2 - \bar{w}_a^\beta(p')) \geq 0. \end{aligned}$$

Taking $w^1 := \bar{w}_a^\beta(p')$, $w^2 := \bar{w}_a^\beta(p)$, and adding the above two inequalities, we have:

$$\begin{aligned} 0 &\leq \sum_{a \in A} (\bar{w}_a^\beta(p') - \bar{w}_a^\beta(p)) \\ &\cdot \left[s_{[a]}(\bar{w}_{[a]}^\beta(p)) - s_{[a]}(\bar{w}_{[a]}^\beta(p')) + p_{[a]} - p'_{[a]} \right] \end{aligned}$$

$$+ \frac{1}{\beta} \ln \left(\frac{\bar{w}_a^\beta(p)}{\sum_{a' \in A_i^+} \bar{w}_{a'}^\beta(p)} \right) - \frac{1}{\beta} \ln \left(\frac{\bar{w}_a^\beta(p')}{\sum_{a' \in A_i^+} \bar{w}_{a'}^\beta(p')} \right) \Big].$$

Since the maps $w_a \mapsto s_{[a]}(w_{[a]})$ and $w_a \mapsto \ln(w_a / \sum_{a' \in A_i^+} w_{a'})$ are non-decreasing, by rearranging terms, we obtain:

$$\sum_{a \in A} \left(\bar{w}_a^\beta(p') - \bar{w}_a^\beta(p) \right) (p'_{[a]} - p_{[a]}) \leq 0,$$

as desired. Additionally, it also holds that

$$\sum_{[a] \in A_O} \left(\bar{w}_{[a]}^\beta(p') - \bar{w}_{[a]}^\beta(p) \right) (p_{[a]} - p'_{[a]}) \leq 0.$$

3) *Proof of Lemma 3:* In this subsection, we show that there exists a unique $\bar{p} \in \mathbb{R}^{|A_O|}$ satisfying (10):

$$\bar{p}_{[a]} = \bar{w}_{[a]}^\beta(\bar{p}) \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})), \quad \forall [a] \in A_O.$$

Define $\psi: \mathbb{R}^{|A_O|} \rightarrow \mathbb{R}$ as:

$$\psi_{[a]}(p) := w_{[a]}(p) \cdot \frac{ds_{[a]}}{dw}(w_{[a]}(p)), \quad \forall [a] \in A_O.$$

Since $w_{[a]}(\cdot)$ is continuous (Lemma 1), and $s_{[a]}$ is continuously differentiable, the map ψ is continuous. Define the set:

$$K := \left\{ y \in \mathbb{R}^{|A_O|} : y \succeq 0, \|y\|_1 \leq |A_O| g_o \max_{[a] \in A_O} \frac{ds_{[a]}}{dw}(g_o) \right\}.$$

Observe that K is a compact and convex subset of $\mathbb{R}^{|A_O|}$, and ψ maps K to K , since for any $p \in K$, we have $\psi_a(p) \geq 0$ for each $a \in A$, and:

$$\begin{aligned} \|\psi(p)\|_1 &= \sum_{a \in A_O} \psi_a(p) \\ &= \sum_{a \in A_O} \bar{w}_{[a]}(p) \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}(p)) \\ &\leq \max_{a \in A_O} \frac{ds_{[a]}}{dw}(g_o) \cdot \sum_{a \in A_O} \bar{w}_{[a]}(p) \\ &\leq |A_O| g_o \cdot \max_{a \in A_O} \frac{ds_{[a]}}{dw}(g_o). \end{aligned}$$

. Thus, by the Brouwer's fixed point theorem, there exists a fixed point $\bar{p} \in K \subset \mathbb{R}^{|A_O|}$ of ψ , i.e., there exists $\bar{p} \in \mathbb{R}^{|A_O|}$ satisfying (10), i.e.,:

$$\bar{p}_{[a]} = \bar{w}_{[a]}^\beta(\bar{p}) \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})), \quad \forall [a] \in A_O.$$

Next, we show that \bar{p} is unique up to Markovian Traffic Equilibrium on the original traffic network, i.e., any $p' \in \mathbb{R}^{|A_O|}$ satisfies (10) if and only if $\bar{w}_{[a]}^\beta(p') = \bar{w}_{[a]}^\beta(\bar{p})$ for each $a \in A$. To show this, suppose by contradiction that there exists some $p' \in \mathbb{R}^{|A_O|}$ satisfying (10), such that $\bar{w}_{[a]}^\beta(p') \neq \bar{w}_{[a]}^\beta(\bar{p})$ for some $[a] \in A_O$. Then:

$$\begin{aligned} \bar{p}_{[a]} - p'_{[a]} &= \bar{w}_{[a]}^\beta(\bar{p}) \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})) - \bar{w}_{[a]}^\beta(p') \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(p')) \end{aligned}$$

$$\begin{aligned} &= \left[\bar{w}_{[a]}^\beta(\bar{p}) - \bar{w}_{[a]}^\beta(p') \right] \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})) \\ &\quad + \bar{w}_{[a]}^\beta(p') \cdot \left[\frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})) - \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(p')) \right]. \end{aligned}$$

Rearranging terms, and invoking the strict convexity and increasing nature of each $s_{[a]}$, and the fact that $\bar{w}_{[a]}^\beta(\bar{p}) \neq \bar{w}_{[a]}^\beta(p')$ for some $[a] \in A_O$, we obtain:

$$\begin{aligned} &\sum_{a \in A} \left[\bar{w}_a^\beta(\bar{p}) - \bar{w}_a^\beta(p') \right] (\bar{p}_{[a]} - p'_{[a]}) \\ &= \sum_{[a] \in A_O} \left[\bar{w}_{[a]}^\beta(\bar{p}) - \bar{w}_{[a]}^\beta(p') \right] (\bar{p}_{[a]} - p'_{[a]}) \\ &= \sum_{[a] \in A_O} \left[\bar{w}_{[a]}^\beta(\bar{p}) - \bar{w}_{[a]}^\beta(p') \right]^2 \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})) \\ &\quad + \sum_{[a] \in A_O} \bar{w}_{[a]}^\beta(p') \left[\bar{w}_{[a]}^\beta(\bar{p}) - \bar{w}_{[a]}^\beta(p') \right]^2 \\ &\quad \cdot \left[\frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})) - \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(p')) \right] \\ &> 0, \end{aligned}$$

which contradicts Theorem 1 .

The above arguments establish that if $p' \in \mathbb{R}^{|A_O|}$ satisfies (10), then $\bar{w}_{[a]}^\beta(p') = \bar{w}_{[a]}^\beta(\bar{p})$ for each $[a] \in A_O$. Through (10), we then have, for each $[a] \in A_O$:

$$\begin{aligned} \bar{p}_{[a]} &= \bar{w}_{[a]}^\beta(\bar{p}) \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})) \\ &= \bar{w}_{[a]}^\beta(p') \cdot \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(p')) \\ &= p'_{[a]}, \end{aligned}$$

so $p' = \bar{p}$. This concludes the proof.

4) *Proof of Lemma 4:* In this subsection, we show that $\bar{w}^\beta(\bar{p})$ is perturbed socially optimal. Let $w^* \in \mathbb{R}^{|A|}$ denote the perturbed socially optimal load. Recall that, by Theorem 1 and the definition of the perturbed socially optimal load:

$$\begin{aligned} &\bar{w}^\beta(\bar{p}) \\ &= \arg \min_{w \in \mathcal{W}} \left\{ \sum_{[a] \in A_O} \int_0^{w_{[a]}} [s_{[a]}(z) + p_{[a]}] dz \right. \\ &\quad \left. + \frac{1}{\beta} \sum_{i \neq d} \left[\sum_{a \in A_i^+} w_a \ln w_a - \left(\sum_{a \in A_i^+} w_a \right) \ln \left(\sum_{a \in A_i^+} w_a \right) \right] \right\}, \\ &w^* \\ &= \arg \min_{w \in \mathcal{W}} \left\{ \sum_{[a] \in A_O} w_{[a]} \ln w_{[a]} \right. \\ &\quad \left. + \frac{1}{\beta} \sum_{i \neq d} \left[\sum_{a \in A_i^+} w_a \ln w_a - \left(\sum_{a \in A_i^+} w_a \right) \ln \left(\sum_{a \in A_i^+} w_a \right) \right] \right\}. \end{aligned}$$

The proof follows by verifying that the variational inequalities corresponding to the above two optimization problems are the same. These two variational inequalities in question are respectively given by:

$$\sum_{[a] \in A_O} \left[s_{[a]}(\bar{w}_{[a]}^\beta(\bar{p})) + \bar{w}_{[a]}^\beta(\bar{p}) \frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(\bar{p})) \right]$$

$$+ \frac{1}{\beta} \ln \left(\frac{\bar{w}_{[a]}^\beta(\bar{p})}{\sum_{a' \in A_{i_a}^+} \bar{w}_{[a']}^\beta(\bar{p})} \right) \left(w_a - \bar{w}_{[a]}^\beta(\bar{p}) \right) > 0,$$

$$\forall w \in \mathcal{W}, w \neq \bar{w}_{[a]}^\beta(\bar{p}),$$

$$\sum_{[a] \in A_o} \left[s_{[a]}(w_{[a]}^*) + w_{[a]}^* \frac{ds_{[a]}}{dw}(w_{[a]}^*) \right. \\ \left. + \frac{1}{\beta} \ln \left(\frac{\bar{w}_{[a]}^*}{\sum_{a' \in A_{i_a}^+} w_{[a']}^*} \right) \right] (w_a - w_a^*) > 0,$$

$$\forall w \in \mathcal{W}, w \neq w^*,$$

and are thus, indeed, identical. This confirms that $\bar{w}^\beta(\bar{p}) = w^*$, and concludes the proof.

B. Proofs for Section IV

1) *Statement of Lemma 5:* The complete, rigorous statement of Lemma 2, is as follows—Suppose $w(0) \in \mathcal{W}$, i.e., the initial flow satisfies flow continuity, and:

$$K_i > \frac{g_o}{C_w} \max\{K_{i_a} : \hat{a} \in A_i^-\}$$

for each $i \in I \setminus \{d\}$, with C_w given by Lemma 7. Then under the continuous-time flow dynamics (16) and (17), the continuous-time traffic allocation $w(t)$ globally asymptotically converges to the corresponding CoDAG equilibrium $\bar{w}^\beta(p)$.

The proof of Lemma 5 follows by applying the proof of the analogous theorem in [10] (Appendix C.1), and replacing the latencies $s_{[a]}(w_{[a]})$ with the total cost $s_{[a]}(w_{[a]}) + p_{[a]}$.

2) *Proof of Lemma 7:* First, we rewrite the discrete ξ -dynamics (12) as a Markov process with a martingale difference term:

$$\xi_a[n+1] = \xi_a[n] + \mu(\rho_a(\xi[n], P[n]) + M_a[n+1]),$$

where $\rho_a : \mathbb{R}^{|A|} \times \mathbb{R}^{|A_o|} \rightarrow \mathbb{R}^{|A|}$ is given by:

$$\rho_a(\xi, p) := K_{i_a} \left(-\xi_a + \frac{\exp(-\beta \cdot z_a(w, p))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta \cdot z_{a'}(w, p))} \right), \quad (21)$$

with $w \in \mathbb{R}^{|A|}$ defined arc-wise by $w_a = (g_{i_a} + \sum_{\hat{a} \in A_{i_a}^-} w_{a'}) \cdot \xi_a$, and:

$$M_a[n+1] := \left(\frac{1}{\mu} \eta_{i_a}[n+1] - 1 \right) \cdot \rho_a(\xi[n], P[n]). \quad (22)$$

Here, $W_a[n] = (g_{i_a} + \sum_{a' \in A_{i_a}^-} W_{a'}[n])$, as given by (11).

Below, we state and prove Lemma 7.

Proof:

1) We have:

$$\mathbb{E}[M_a[n+1] | \mathcal{F}_n] \\ = \left(\frac{1}{\mu} \mathbb{E}[\eta_{i_a}[n+1]] - 1 \right) \cdot K_{i_a} \\ \cdot \left(-\xi_a[n] + \frac{\exp(-\beta[z_a(W[n], P[n])])}{\sum_{a' \in A_{i_a}^+} \exp(-\beta[z_{a'}(W[n], P[n])])} \right) \\ = 0.$$

2) We separate the proof of this part of the lemma into the following steps.

- First, we show that for each $a \in A$, $n \geq 0$, we have $\xi_a[n] \in (0, 1]$.

Fix $a \in A$ arbitrarily. Then $\xi_a[0] \in (0, 1]$ by assumption, and for each $n \geq 0$:

$$\frac{\exp(-\beta[z_a(W[n], P[n])])}{\sum_{a' \in A_{i_a}^+} \exp(-\beta[z_{a'}(W[n], P[n])])} \in (0, 1],$$

since the exponential function takes values in $(0, \infty)$. Thus, by Lemma 7, we have $\xi_a[n] \in (0, 1]$ for each $n \geq 0$.

- Second, we show that for each $a \in A$, $n \geq 0$, we have $W_a[n] \in (0, g_o)$.

Note that (11), together with the assumption that $W[0] \in \mathcal{W}$, implies that $W[n] \in \mathcal{W}$ for each $n \geq 0$. Now, fix $a \in A$, $n \geq 0$ arbitrarily. Let $\mathbf{R}(a) \subseteq \mathbf{R}$ denote the set of all routes passing through a , and for each $r \in \mathbf{R}(a)$, let $a_{r,k}$ denote the k -th arc in r . Then, by the conservation of flow encoded in R :

$$W_a[n] = g_o \cdot \sum_{r \in \mathbf{R}(a)} \prod_{k=1}^{|r|} \xi_{a_{r,k}} \\ \leq g_o \cdot \sum_{r \in \mathbf{R}} \prod_{k=1}^{|r|} \xi_{a_{r,k}} \\ = g_o.$$

Similarly, since $\xi_a[n] \in (0, 1]$ for each $a \in A$, $n \geq 0$, we have:

$$W_a[n] = g_o \cdot \sum_{r \in \mathbf{R}(a)} \prod_{k=1}^{|r|} \xi_{a_{r,k}} > 0.$$

- Third, we show that there exists $C_p > 0$ such that $P_a[n] \in [0, C_p]$ for each $a \in A$, $n \geq 0$.

Above, we have established that $W_a[n] \in (0, g_o)$ for each $a \in A$, $n \geq 0$. Moreover, by assumption, $s_{[a]}(\cdot)$ is non-negative, continuously differentiable, strictly increasing, and strictly convex. Thus, taking $C_{ds} := (ds_{[a]}/dw)(g_o)$, we obtain:

$$\frac{ds_{[a]}}{dw}(W_{[a]}[n]) \in [0, C_{ds}].$$

Now, take $C_p := \max\{\max_{a \in A} P_a[0], g_o C_{ds}\}$. By the definition of C_p , we have $P_a[0] \in [0, C_p]$ for each $a \in A$. Moreover, for each $n \geq 0$:

$$W_{[a]}[n] \cdot \frac{ds_{[a]}}{dw}(W_{[a]}[n]) \in [0, g_o C_{ds}] \subseteq [0, C_p].$$

Thus, by Lemma 7, we conclude that $P_a[n] \in [0, C_p]$ for each $n \geq 0$.

- Fourth, we show that there exists $C_z > 0$ such that $|z_a(W[n], P[n])| \leq C_z$ for each $a \in A$, $n \geq 0$. Fix $a \in A_d^- = \{a \in A : m_a = 1\}$ arbitrarily. Then, from (5):

$$z_a(w, p) = s_{[a]}(w_{[a]}) + p_{[a]} \in [0, s_{[a]}(g_o) + C_p],$$

$$\Rightarrow |z_a(w, p)| \leq s_{[a]}(g_o) + C_p := C_{z,1}.$$

Now, suppose that at some height $k \in [m(G) - 1]$, there exists some $C_{z,k} > 0$ such that, for each $n \geq 0$, and each $a \in A$ satisfying $m_a \leq k$ and each $n \geq 0$, we have $|z_a(w, p)| \leq C_{z,k}$. Then, for each $n \geq 0$, and each $a \in A$ satisfying $m_a = k + 1$ (at least one such $a \in A$ must exist, by [10], Proposition 2):

$$\begin{aligned} & z_a(w, p) \\ &= s_{[a]}(w_{[a]}) + p_{[a]} - \frac{1}{\beta} \ln \left(\sum_{a' \in A_{j_a}^+} e^{-\beta \cdot z_{a'}(w, p)} \right) \\ &\leq s_{[a]}(g_o) + C_p - \frac{1}{\beta} \ln (|A_{j_a}^+| e^{-\beta \cdot C_z}) \\ &= s_{[a]}(g_o) + C_p + C_z, \end{aligned}$$

and:

$$\begin{aligned} & z_a(w, p) \\ &= s_{[a]}(w_{[a]}) + p_{[a]} - \frac{1}{\beta} \ln \left(\sum_{a' \in A_{j_a}^+} e^{-\beta \cdot z_{a'}(w, p)} \right) \\ &\geq 0 + 0 - \frac{1}{\beta} \ln (|A_{j_a}^+| e^{\beta \cdot C_z}) \\ &= -\frac{1}{\beta} \ln |A| - C_z, \end{aligned}$$

from which we conclude that:

$$\begin{aligned} & |z_a(w, p)| \\ &\leq \max \left\{ s_{[a]}(g_o) + C_p + C_z, \frac{1}{\beta} \ln |A| + C_z \right\} \\ &:= C_{z,k+1}, \end{aligned}$$

with $C_{z+1} \geq C_z$. This completes the induction step, and the proof is completed by taking $C_z := C_{z,m(G)}$.

- Fifth, we show that there exists some $C_\xi > 0$ such that $\xi_a[n] \geq C_\xi$ for each $a \in A$, $n \geq 0$.

Define:

$$C_\xi := \min \left\{ \min \{ \xi_{a'}[0] : a' \in A \}, \frac{1}{|A|} e^{-2\beta C_z} \right\} > 0.$$

By definition of C_ξ , we have $\xi_a[0] \geq C_\xi$. Moreover, for each $n \geq 0$, we have:

$$\begin{aligned} & \frac{\exp(-\beta [z_a(W[n], P[n])])}{\sum_{a' \in A_{i_a}^+} \exp(-\beta [z_{a'}(W[n], P[n])])} \\ &\geq \frac{e^{-\beta C_z}}{|A_{i_a}^+| \cdot e^{\beta C_z}} \\ &\geq \frac{1}{|A|} e^{-2\beta C_z} \\ &\geq C_\xi. \end{aligned}$$

Thus, by Lemma 7, we have $\xi_a[n] \geq C_\xi$ for each $n \geq 0$.

- Sixth, we show that there exists $C_w > 0$ such that, for each $a \in A$, $n \geq 0$, we have $W_a[n] \geq C_w$.

Fix $a \in A$, $n \geq 0$. Let $r \in \mathbf{R}$ be any route in the corresponding DAG containing $a \in A$. By unwinding the recursive definition of $W_a[n]$ from the flow allocation probability values $\{\xi_a[n] : a \in A, n \geq 0\}$, we have:

$$\begin{aligned} W_a[n] &= g_o \cdot \sum_{\substack{r' \in \mathbf{R} \\ a \in r'}} \prod_{a' \in r'} \xi_{a'}[n] \\ &\geq g_o \cdot \prod_{a' \in r} \xi_{a'}[n] \\ &\geq g_o \cdot (C_\xi)^{|r|} \\ &\geq g_o \cdot (C_\xi)^{\ell(G)} \\ &:= C_w. \end{aligned}$$

- Seventh, we show that there exists $C_m > 0$ such that, for each $a \in A$, $n \geq 0$, we have $M_a[n] \geq C_m$.

Define, for convenience, $C_\mu := \max\{\bar{\mu} - \mu, \mu - \underline{\mu}\}$. Since $\eta_{i_a}[n] \in [\underline{\mu}, \bar{\mu}]$, we have from (22) that for each $a \in A$, $n \geq 0$:

$$\begin{aligned} & M_a[n+1] \\ &= \left(\frac{1}{\mu} \eta_{i_a}[n+1] - 1 \right) \cdot K_{i_a} \\ &\quad \cdot \left(-\xi_a[n] + \frac{\exp(-\beta [z_a(W[n], P[n])])}{\sum_{a' \in A_{i_a}^+} \exp(-\beta [z_{a'}(W[n], P[n])])} \right). \end{aligned}$$

Applying the triangle inequality, we obtain:

$$\begin{aligned} |M_a[n+1]| &\leq \frac{1}{\mu} K_{i_a} C_\mu \cdot (1+1) \\ &= \frac{2}{\mu} C_\mu \cdot \max_{i \in I \setminus \{d\}} K_i \\ &:= C_m. \end{aligned}$$

- 3) We separate the proof of this part of the lemma into the following steps.

- First, we show that for each $a \in A$, $t \geq 0$, we have $\xi_a(t) \in (0, 1]$.

Fix $a \in A$. By assumption, $\xi_a(0) \in (0, 1]$, and at each $t \geq 0$:

$$\frac{\exp(-\beta z_a(w, p))}{\sum_{a' \in A_{i_a}^+} \exp(-\beta z_{a'}(w, p))} \in (0, 1].$$

Thus, by Lemma 7, we conclude that $\xi_a(t) \in (0, 1]$ for each $t \geq 0$.

- Second, we show that $w_a(t) \in [0, g_o]$ for each $t \geq 0$.

The proof here is nearly identical to the proof that $W_a[n] \in (0, g_o)$ in the second bullet point of the second part of this Proposition, and is omitted for brevity.

- Third, we show that $p_a(t) \in [0, C_p]$ for each $t \geq 0$.

Above, we have established that $w_a(t) \in (0, g_o)$ for each $a \in A$, $t \geq 0$. Let $C_p > 0$ be as defined in the third bullet point of the second part of this Proposition, i.e., $C_p = \max \left\{ \max_{a \in A} P_a[0], g_o \cdot \frac{ds_{[a]}}{dw}(g_o) \right\}$. Then, for each $a \in A$, $t \geq 0$, we have:

$$w_a(t) \cdot \frac{ds_{[a]}}{dw}(w_a(t)) \in [0, C_p].$$

Note also that $P_a[0] \leq C_p$ for each $a \in A$, by definition of C_p . Thus, Lemma 7 implies that $p_a(t) \in [0, C_p]$ for each $t \geq 0$.

- Fourth, we show that $|z_a(w_a(t), p_a(t))| \leq C_z$ for each $t \geq 0$.

The proof here is nearly identical to the proof that $|z_a(W_a[n], P_a[n])| \leq C_z$ in the fourth bullet point of the second part of this Proposition, and is omitted for brevity.

- Fifth, we show that there exists some $C_\xi > 0$ such that $\xi_a(t) \geq C_\xi$ for each $a \in A$, $t \geq 0$.

Define:

$$C_\xi := \min \left\{ \min\{\xi_{a'}(0) : a' \in A\}, \frac{1}{|A|} e^{-2\beta C_z} \right\} > 0.$$

By definition of C_ξ , we have $\xi_a(0) \geq C_\xi$. Moreover, for each $n \geq 0$, we have:

$$\begin{aligned} & \frac{\exp(-\beta[z_a(W[n], P[n])])}{\sum_{a' \in A_{i_a}^+} \exp(-\beta[z_{a'}(W[n], P[n])])} \\ & \geq \frac{e^{-\beta C_z}}{|A_{i_a}^+| \cdot e^{\beta C_z}} \\ & \geq \frac{1}{|A|} e^{-2\beta C_z} \\ & \geq C_\xi. \end{aligned}$$

Thus, by Lemma 7, we have $\xi_a(t) \geq C_\xi$ for each $t \geq 0$.

- Sixth, we show that there exists $C_w > 0$ such that, for each $a \in A$, $t \geq 0$, we have $w_a(t) \geq C_w$.

The proof here is nearly identical to the proof that $W_a[n] \geq C_w$ in the fourth bullet point of the second part of this Proposition, and is omitted for brevity. \blacksquare

Below, we state and prove Lemma 8, which together with Lemma A.1 supplies all the technical conditions necessary for Borkar's stochastic approximation theory to be applied.

3) Proof of Lemma 8:

Proof:

- 1) Since $\bar{\xi}^\beta(p)$ can be derived component-wise from \bar{w}^β , we first show that $\bar{w}^\beta : \mathbb{R}^{|A \circ|} \rightarrow \mathbb{R}^{|A|}$ is Lipschitz continuous. We do so by showing that w^β is continuously differentiable with bounded derivative. To this end, recall from the proofs of Lemma 1 and Lemma 6, the matrix $M \in \mathbb{R}^{|A| \times d}$, $b \in \mathbb{R}^d$, with $d \in [|A|]$ describing the dimension of \mathcal{W} (as a manifold with boundary), and the matrices $B \in \mathbb{R}^{|A| \times (|A| - d)}$, $C \in \mathbb{R}^{|A| \times |A \circ|}$.

As established in the proof of Proposition 1, there exists a continuously differentiable function $J : \mathbb{R}^{|A \circ|} \times \mathbb{R}^{|A|} \rightarrow \mathbb{R}^{|A|}$, and matrices $M \in \mathbb{R}^{|A| \times d}$ and $B \in \mathbb{R}^{|A| \times (|A| - d)}$, such that $J(p, w^\beta(p)) = 0$ for each $p \in \mathbb{R}^{|A \circ|}$, the columns of B and the columns of M are orthonormal, $R(M)$ and $R(B)$ are orthogonal subspaces

whose direct sum is $\mathbb{R}^{|A|}$, and:

$$\frac{\partial J}{\partial w}(w, p) = \begin{bmatrix} M^\top \\ B^\top \nabla_w^2 F(w, p) \end{bmatrix} \in \mathbb{R}^{|A| \times |A|},$$

where, as in the proof of Proposition 1, $F : \mathcal{W} \times \mathbb{R}^{|A \circ|} \rightarrow \mathbb{R}$ is given by (9), reproduced below:

$$\begin{aligned} & F(w, p) \\ & := \sum_{[a] \in A_0} \int_0^{w_{[a]}} [s_{[a]}(z) + p_{[a]}] dz \\ & + \frac{1}{\beta} \sum_{i \neq d} \left[\sum_{a \in A_i^+} w_a \ln w_a - \left(\sum_{a \in A_i^+} w_a \right) \ln \left(\sum_{a \in A_i^+} w_a \right) \right]. \end{aligned}$$

Thus, the Implicit Function Theorem implies that:

$$\begin{aligned} & \frac{d\bar{w}^\beta}{dp}(p) \\ & = - \left[\frac{\partial J}{\partial w}(\bar{w}^\beta(p), p) \right]^{-1} \frac{\partial J}{\partial p}(\bar{w}^\beta(p), p) \\ & = - \begin{bmatrix} M^\top \\ B^\top \nabla_w^2 F(\bar{w}^\beta(p), p) \end{bmatrix}^{-1} \begin{bmatrix} O \\ B^\top \frac{d}{dp} \nabla_w F(\bar{w}^\beta(p), p) \end{bmatrix}, \end{aligned} \quad (23)$$

where $\nabla_w F(w, p) \in \mathbb{R}^{|A|}$, and $\frac{\partial}{\partial p} \nabla_w F(w, p) \in \mathbb{R}^{|A| \times |A \circ|}$. To study (23) further, we wish to rewrite the $B^\top \nabla_w^2 F(w, p)$ term. To this end, note that since $\begin{bmatrix} M & B \end{bmatrix} \in \mathbb{R}^{|A| \times |A|}$ is an orthogonal matrix, and $\nabla_w^2 F(w, p)$ is symmetric positive definite (since $F(p, \cdot)$ is strictly convex for each $p \in \mathbb{R}^{|A \circ|}$), the matrix:

$$Q := \begin{bmatrix} M^\top \\ B^\top \end{bmatrix} \nabla_w^2 F(\bar{w}^\beta(p), p) \begin{bmatrix} M & B \end{bmatrix} \in \mathbb{R}^{|A| \times |A|}$$

is symmetric positive definite as well. Now, let $Q_{11} := M^\top \nabla_w^2 F(w, p) M \in \mathbb{R}^{d \times d}$, $Q_{12} := M^\top \nabla_w^2 F(w, p) B \in \mathbb{R}^{d \times (|A| - d)}$, and $Q_{22} := B^\top \nabla_w^2 F(w, p) B \in \mathbb{R}^{(|A| - d) \times (|A| - d)}$ denote the various block matrices of Q , as shown below:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix}.$$

We then have:

$$\begin{aligned} B^\top \nabla_w^2 F(w, p) & = \begin{bmatrix} O & I \end{bmatrix} \begin{bmatrix} M^\top \\ B^\top \end{bmatrix} \nabla_w^2 F(w, p) \\ & = \begin{bmatrix} O & I \end{bmatrix} Q \begin{bmatrix} M^\top \\ B^\top \end{bmatrix} \\ & = Q_{12}^\top M^\top + Q_{22} B^\top, \end{aligned}$$

where the matrices O and $i \in I$ above are the zero matrix of dimension $(|A| - d) \times d$ and identity matrix of

dimension $(|A|-d) \times (|A|-d)$, respectively. Substituting back into (23), we obtain:

$$\begin{aligned}
& \frac{d\bar{w}^\beta}{dp}(p) \\
&= - \begin{bmatrix} M^\top \\ B^\top \nabla_w^2 F(w, p) \end{bmatrix}^{-1} \begin{bmatrix} O \\ B^\top \frac{d}{dp} \nabla_w F(w, p) \end{bmatrix} \quad (24) \\
&= - \begin{bmatrix} M^\top \\ Q_{12}^\top M^\top + Q_{22} B^\top \end{bmatrix}^{-1} \begin{bmatrix} O \\ B^\top \frac{d}{dp} \nabla_w F(w, p) \end{bmatrix} \\
&= - \left(\begin{bmatrix} I & O \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} M^\top \\ B^\top \end{bmatrix} \right)^{-1} \begin{bmatrix} O \\ B^\top \end{bmatrix} \frac{d}{dp} \nabla_w F(w, p) \\
&= - [M \ B] \begin{bmatrix} I & O \\ -Q_{22}^{-1} Q_{12}^\top & Q_{22}^{-1} \end{bmatrix} \begin{bmatrix} O \\ B^\top \end{bmatrix} \frac{d}{dp} \nabla_w F(w, p) \quad (25)
\end{aligned}$$

$$\begin{aligned}
&= -BQ_{22}^{-1}B^\top C \quad (26) \\
&= -B(B^\top \nabla_w^2 F(w, p)B)^{-1}B^\top C.
\end{aligned}$$

Below, to establish the Lipschitz continuity of $\bar{w}^\beta(\cdot)$, we provide a uniform bound for $\frac{d\bar{w}^\beta}{dp}(p)$ over all values of $p \in \mathbb{R}^{|A_O|}$, by providing a uniform upper bound for the minimum eigenvalue of $\nabla_w^2 F(\bar{w}^\beta(p), p)$ over all values of $p \in \mathbb{R}^{|A_O|}$.

From Lemma 7, $\bar{w}_a^\beta(p) \in [C_w, g_o]$ for each $p \in \mathbb{R}^{|A_O|}$ and $a \in A$. Thus, all the second partial derivatives of F , as given by:

$$\begin{aligned}
& \frac{\partial^2}{\partial w_a \partial w_{a'}} F(\bar{w}^\beta(p), p) \\
&= \left(\frac{ds_{[a]}}{dw}(\bar{w}^\beta(p)) + \frac{1}{\bar{w}_a^\beta(p)} \right) \cdot \mathbf{1}\{a = a'\} \\
& \quad - \frac{1}{\sum_{a' \in A_{i_a}^+} \bar{w}_{a'}^\beta(p)} \cdot \mathbf{1}\{i_{a'} = i_a\}
\end{aligned}$$

are well-defined and continuous. Next, consider the continuous map from each $p \in \mathbb{R}^{|A_O|}$ to the minimum eigenvalue of $\nabla_w^2 F(\bar{w}^\beta(p), p)$, officially stated as:

$$p \mapsto \min_{\substack{v \in \mathbb{R}^{|A|} \\ \|v\|_2=1}} v^\top \nabla_w^2 F(\bar{w}^\beta(p), p)v.$$

Since F is strictly convex, $\nabla_w^2 F(\bar{w}^\beta(p), p)$ is symmetric positive definite for each $p \in \mathbb{R}^{|A_O|}$, meaning that the output of the above map is strictly positive for each $p \in \mathbb{R}^{|A_O|}$. Moreover, note that the entries of $F(\bar{w}^\beta(p), p)$ only depend on p through the value of $\bar{w}^\beta(p)$, which is bounded in $[C_w, g_o]^{|A|}$. Thus, for each $p \in \mathbb{R}^{|A_O|}$:

$$\begin{aligned}
& \min_{\substack{v \in \mathbb{R}^{|A|} \\ \|v\|_2=1}} v^\top \nabla_w^2 F(\bar{w}^\beta(p), p)v \\
& \geq \min_{w \in [C_w, g_o]^{|A|}} v^\top \nabla_w^2 F(w, p)v \\
& := C_F > 0,
\end{aligned}$$

where $C_F := \min_{w \in [C_w, g_o]^{|A|}} v^\top \nabla_w^2 F(w, p)v$ is independent of p , and is strictly positive, since the minimum of a strictly positive-valued function over a compact set is strictly positive. We thus have a uniform bound on the derivative of \bar{w}^β over all values of $p \in \mathbb{R}^{|A_O|}$ at which it is evaluated:

$$\begin{aligned}
& \left\| \frac{d\bar{w}^\beta}{dp}(p) \right\|_2 \\
& \leq \|B\|_2^2 \|C\|_2 \cdot \|(B^\top \nabla_w^2 F(\bar{w}^\beta(p), p)B)^{-1}\|_2 \\
& = \|B\|_2^2 \|C\|_2 \cdot \frac{1}{\min_{\substack{\hat{v} \in \mathbb{R}^{|A|-d} \\ \|\hat{v}\|_2=1}} \hat{v}^\top B^\top \nabla_w^2 F(\bar{w}^\beta(p), p)B \hat{v}} \\
& \leq \|B\|_2^2 \|C\|_2 \cdot \frac{1}{\min_{\substack{v \in \mathbb{R}^{|A|} \\ \|v\|_2=1}} v^\top \nabla_w^2 F(\bar{w}^\beta(p), p)v} \\
& \leq \|B\|_2^2 \|C\|_2 \cdot \frac{1}{C_F}.
\end{aligned}$$

- 2) We shall establish the Lipschitz continuity of (the restriction of) z_a , for each $a \in A$, by providing uniform bounds on its partial derivatives across all values of its arguments $(w, p) \in \mathcal{W} \times [0, C_p]^{|A_O|}$.

The proof follows by induction on the height index $k \in [m(G)]$. For each $a \in A$, let $\tilde{z}_a : \mathbb{R}^{|A|} \rightarrow \mathbb{R}$ be the continuous extension of $z_a : \mathcal{W} \rightarrow \mathbb{R}$ to the Euclidean space $\mathbb{R}^{|A|}$ containing \mathcal{W} . By definition of Lipschitz continuity, if \tilde{z}_a is Lipschitz for some $a \in A$, then so is z_a . For each $a \in A_d^- = \{a \in A : m_a = 1\}$ and any $w \in \mathbb{R}^{|A|}$:

$$\tilde{z}_a(w) = s_{[a]}(w_{[a]}) + p_{[a]}.$$

Thus, for any $\hat{a} \in A$, and any $w \in \mathbb{R}^{|A|}$, $p \in \mathbb{R}^{|A_O|}$:

$$\begin{aligned}
& \frac{\partial \tilde{z}_a}{\partial w_{\hat{a}}}(w, p) = \frac{ds_{[a]}}{dw}(w_{[a]}) \cdot \mathbf{1}\{\hat{a} \in [a]\} \in [0, C_{ds}], \\
& \frac{\partial \tilde{z}_a}{\partial p_{[\hat{a}]}}(w, p) = \mathbf{1}\{\hat{a} \in [a]\} \in [0, 1].
\end{aligned}$$

We set $C_{z,1} := \max\{C_{ds}, 1\}$.

Now, suppose that there exists some depth $k \in [m(G) - 1]$ and some constant $C_{z,k} > 0$ such that, for any $a \in A$ satisfying $m_a \leq k$, and any $w \in \mathcal{W}$, $n \geq 0$, the map $\tilde{z}_a : \mathbb{R}^{|A|} \rightarrow \mathbb{R}$ is continuously differentiable, with:

$$\left| \frac{\partial \tilde{z}_a}{\partial w_{\hat{a}}}(w) \right| \leq C_{z,k}, \quad \left| \frac{\partial \tilde{z}_a}{\partial p_{[\hat{a}]}}(w) \right| \leq C_{z,k}.$$

Continuing with the induction step, fix $a \in A$ such that $m_a = k + 1$ (there exists at least one such link, by [10], Proposition 1, Part 4). From [10], Proposition 1, Part 2, we have $m_{a'} \leq k$ for each $a' \in A_{i_a}^+$. Thus, the induction hypothesis implies that, for any $\hat{a} \in A$:

$$\tilde{z}_a(w, p) = s_{[a]}(w_{[a]}) + p_{[a]} - \frac{1}{\beta} \sum_{a' \in A_{i_a}^+} e^{-\beta z_{a'}(w, p)}.$$

Computing partial derivatives with respect to each component of w , we obtain:

$$\begin{aligned} \frac{\partial \tilde{z}_a}{\partial w_{\hat{a}}}(w, p) &= \frac{ds_{[a]}}{dw}(w_{[a]}) \cdot \mathbf{1}\{\hat{a} \in [a]\} \\ &\quad + \sum_{a' \in A_{j_a}^+} e^{-\beta \tilde{z}_{a'}(w, p)} \cdot \frac{\partial \tilde{z}_{a'}}{\partial w_{\hat{a}}}(w, p), \\ \Rightarrow \left| \frac{\partial \tilde{z}_a}{\partial w_{\hat{a}}}(w) \right| &\leq C_{ds} + |A| \cdot C_{z, k}. \end{aligned}$$

Computing partial derivatives with respect to each component of p , we obtain:

$$\begin{aligned} \frac{\partial \tilde{z}_a}{\partial p_{[\hat{a}]}}(w, p) &= \mathbf{1}\{\hat{a} \in [a]\} \\ &\quad + \sum_{a' \in A_{j_a}^+} e^{-\beta \tilde{z}_{a'}(w, p)} \cdot \frac{\partial \tilde{z}_{a'}}{\partial p_{\hat{a}}}(w, p), \\ \Rightarrow \left| \frac{\partial \tilde{z}_a}{\partial w_{\hat{a}}}(w) \right| &\leq 1 + |A| \cdot C_{z, k}. \end{aligned}$$

We can complete the induction step by taking $C_{z, k+1} := \max\{C_{ds}, 1\} + |A| \cdot C_{z, k}$.

This establishes that, for each $a \in A$, the map z_a is continuously differentiable, with partial derivatives uniformly bounded by a uniform constant, $C_z := C_{z, m(G)}$. This establishes the Lipschitz continuity of the map z_a for each $a \in A$, and thus proves this part of the proposition.

- 3) Recall that the map from traffic allocation probabilities (ξ) to traffic flows (w) is given as follows, for each $a \in A$:

$$w_a = \left(g_{i_a} + \sum_{\hat{a} \in A_i^-} w_{\hat{a}} \right) \cdot \xi_a = g_o \cdot \sum_{r \in \mathbf{R}} \prod_{k=1}^{|r|} \xi_{a_{r, k}},$$

where $a_{r, k}$ denotes the k -th arc along a given route $r \in \mathbf{R}$, for each $k \in |r|$. It is there clear that the map from ξ to w is continuously differentiable. Moreover, the domain of this map is compact; indeed, for each $a \in A$, we have $\xi_a \in [0, 1]$, and for each non-destination node $i \neq d$, we have $\sum_{a \in A_i^+} \xi_a = 1$. Thus, the map $\xi \mapsto w$ has continuously differentiable derivatives with magnitude bounded above by some constant uniform in the compact set of realizable probability allocations ξ . This is equivalent to stating that the map $\xi \mapsto w$ is Lipschitz continuous.

- 4) Above, we have established that the maps z_a and $\xi \mapsto w$ are Lipschitz continuous. Since the addition and composition of Lipschitz maps is Lipschitz, it suffices to verify that the map $\hat{\rho} : \mathbb{R}^{|A|} \rightarrow \mathbb{R}^{|A|}$, defined element-wise by:

$$\hat{\rho}_a(z) := \frac{e^{-\beta z_a}}{\sum_{a' \in A_{i_a}^+} e^{-\beta z_{a'}}}, \quad \forall a \in A$$

is Lipschitz continuous. We do so below by computing, and establishing a uniform bound for, its partial deriva-

tives. For each $\hat{a} \in A$:

$$\begin{aligned} \frac{\partial \hat{\rho}_a}{\partial z_{\hat{a}}} &= \frac{1}{\left(\sum_{a' \in A_{i_a}^+} e^{-\beta z_{a'}} \right)^2} \\ &\quad \cdot \left(\sum_{a' \in A_{i_a}^+} e^{-\beta z_{a'}} \cdot (-\beta) e^{-\beta z_a} \cdot \frac{\partial z_a}{\partial z_{\hat{a}}} \right. \\ &\quad \left. - e^{-\beta z_a} \cdot \sum_{a' \in A_{i_a}^+} (-\beta) e^{-\beta z_{a'}} \frac{\partial z_{a'}}{\partial z_{\hat{a}}} \right), \\ &= \frac{e^{-\beta z_a}}{\sum_{a' \in A_{i_a}^+} e^{-\beta z_{a'}}} \cdot \beta \cdot \frac{\partial z_a}{\partial z_{\hat{a}}} \\ &\quad + \frac{\beta e^{-\beta z_a}}{\left(\sum_{a' \in A_{i_a}^+} e^{-\beta z_{a'}} \right)^2} \cdot \sum_{a' \in A_{i_a}^+} e^{-\beta z_{a'}} \frac{\partial z_{a'}}{\partial z_{\hat{a}}}, \end{aligned}$$

where we have used the fact that:

$$\begin{aligned} \sum_{a' \in A_{i_a}^+} e^{-\beta z_{a'}} \frac{\partial z_{a'}}{\partial z_{\hat{a}}} &= \sum_{a' \in A_{i_a}^+} e^{-\beta z_{a'}} \cdot \mathbf{1}\{a' = \hat{a}\} \\ &\leq \max_{a' \in A_{i_a}^+} e^{-\beta z_{a'}}. \end{aligned}$$

Thus, applying the triangle inequality, we obtain:

$$\left| \frac{\partial \hat{\rho}_a}{\partial z_{\hat{a}}} \right| = \beta + \beta = 2\beta.$$

This concludes the proof for this part of the proposition.

- 5) For each $a, a' \in A$:

$$\begin{aligned} &\frac{r_{[a]}}{p_{[a']}}(p) \\ &= - \frac{\partial p_{[a]}}{\partial p_{[a']}} + \left(\frac{ds_{[a]}}{dw}(\bar{w}_{[a]}^\beta(p)) + \bar{w}_{[a]}^\beta(p) \frac{d^2 s_{[a]}}{dw^2}(\bar{w}_{[a]}^\beta(p)) \right) \\ &\quad \cdot \frac{\partial \bar{w}_{[a]}^\beta}{\partial p_{[a']}}(p). \end{aligned}$$

Define:

$$C_{dds} := \max_{x \in [0, g_o]} \left\{ \frac{d^2 s_{[a]}(x)}{dw^2} \right\}.$$

Meanwhile, by the first part of this proposition, and the Cauchy-Schwarz inequality:

$$\begin{aligned} \left| \frac{\partial \bar{w}_{[a]}^\beta}{\partial p_{[a']}}(p) \right| &= \left| e_{[a]}^\top \frac{d\bar{w}^\beta}{dp}(p) e_{[a']} \right| \\ &\leq \left\| \frac{d\bar{w}^\beta}{dp}(p) \right\|_2 \\ &\leq \|B\|_2^2 \|C\|_2^2 \cdot \frac{1}{C_F}. \end{aligned}$$

Thus, we have:

$$\begin{aligned} \left| \frac{r_{[a]}}{p_{[a']}}(p) \right| &\leq 1 + (C_{ds} + g_o C_{dds}) \cdot \|B\|_2^2 \|C\|_2^2 \cdot \frac{1}{C_F} \\ &:= C_r. \end{aligned}$$

Thus, $C_r > 0$ uniformly upper bounds the partial derivatives of $r_{[a]}$ over all of its components $p_{[a']}$ and all arguments $p \in \mathbb{R}^{|A \circ|}$. This establishes the Lipschitz continuity of each $r_{[a]}$, and thus concludes the proof. \blacksquare

4) *Proof of Theorem 3:* We complete the proof of Theorem 3, restated below: There exists some $\epsilon > 0$ such that, if $\|P[0] - \bar{p}\|_2 \leq \epsilon$, then (a):

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{E}[\|\xi[n] - \bar{\xi}^\beta(\bar{p})\|_2^2 + \|P[n] - \bar{p}\|_2^2] \\ &= O\left(\mu + \frac{a}{\mu}\right), \end{aligned}$$

and (b) for each $\delta > 0$:

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \mathbb{P}[\|\xi[n] - \bar{\xi}^\beta(\bar{p})\|_2^2 + \|P[n] - \bar{p}\|_2^2 \geq \delta] \\ &= O\left(\frac{\mu}{\delta} + \frac{a}{\delta\mu}\right). \end{aligned}$$

The result follows by applying the global convergence of the continuous-time toll dynamics 18 under the affine latency assumption, as provided by Lemma 6. Theorem 3 now follows by applying the two-timescale stochastic approximation results in Borkar [22], Chapters 2 and 9.