ON GLOBAL $W^{2,\delta}$ ESTIMATES FOR THE MONGE-AMPÈRE EQUATION ON GENERAL BOUNDED CONVEX DOMAINS

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Dedicated to Professor Dang Duc Trong on the occasion of his 60th birthday

Abstract. We establish global $W^{2,\delta}$ estimates, for all $\delta < \frac{1}{n-1}$, for convex solutions to the Monge-Ampère equation with positive $C^{2,\beta}$ right-hand side and zero boundary values on general bounded convex domains in $\mathbb{R}^n$ $(n \geq 2)$. We exhibit examples showing that global $W^{2,\frac{n}{2(n-1)}}$ estimates fail in all dimensions, so the range of δ is sharp in two dimensions.

1. Introduction and statement of the main result

This note is concerned with global second derivative estimates for the convex Aleksandrov solution to the Monge-Ampère equation

$$\begin{cases} \det D^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

on general bounded convex domains $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$, where f is bounded between two positive constants $\lambda \leq \Lambda$, that is,

$$0 < \lambda \leq f \leq \Lambda. \tag{1.2}$$

Regarding interior second-order Sobolev estimates, building on the work of De Philippis–Figalli [DPF], De Philippis–Figalli–Savin [DPFS] and Schmidt [Sc], independently, show that $D^2u \in L^{1+\varepsilon}_{\text{loc}}(\Omega)$ for some constant $\varepsilon = \varepsilon(n, \lambda, \Lambda) > 0$. If f is assumed additionally to be continuous, then Caffarelli [C2] shows that $u \in W^{2,p}_{\text{loc}}(\Omega)$ for all $p \in (1, \infty)$.

Regarding global second-order Sobolev estimates, when Ω is uniformly convex with C^3 boundary, Savin [S2] extends the above estimates all the way to the boundary by showing respectively that $D^2u \in L^{1+\varepsilon}(\Omega)$, and $D^2u \in L^p(\Omega)$ when $f \in C(\overline{\Omega})$. The techniques in [S2] are based on the Boundary Localization Theorem established in [S1]. In general, for the Monge-Ampère equation with possibly nonzero boundary values, the uniform convexity of the boundary and the C^3 regularity of the boundary and boundary data are crucial for global $W^{2,p}$ estimates. In [W], Wang constructs explicit examples showing the failure of global $W^{2,3}$ estimates for the Monge–Ampère equation in two dimensions with positive

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constant right-hand side f when either the boundary data or the domain boundary failing to be C^3.

A natural question is to determine the optimal global integrability of the second derivatives for the solution u to (1.1)–(1.2) when Ω is a general bounded convex domain. To the best of our knowledge, this issue has not been studied before. On the other hand, thanks to Caffarelli [C1], $|u|$ is known to grow at most like $[\text{dist}(\cdot,\partial\Omega)]^{2/n}$ away from the boundary. Therefore, by the convexity of u, $|Du|$ grows like $[\text{dist}(\cdot,\partial\Omega)]^{2/n-1}$ away from $\partial\Omega$. These growths are shown to be optimal in the author's work [L3] for domains with portions of $(n-1)$-dimensional hyperplanes on their boundaries. Given these optimal growths, it is reasonable to expect that $\|D^2u\|$ grows like $[\text{dist}(\cdot,\partial\Omega)]^{2/n-2}$ away from the boundary. This, in turn, indicates that the optimal global integrability for D^2u should be $L^\mu(\Omega)$ for all $\mu < \frac{n}{2(n-1)}$. We are able to confirm this expectation in two dimensions. For higher dimensions, there is still a gap between our integrability result where $D^2u \in L^\delta(\Omega)$ for all $\delta < \frac{1}{n-1}$, and the non-integrability examples for the threshold exponent $\frac{n}{2(n-1)}$. This is due to our method of proving the $W^{2,\delta}$ estimates; see Remark 2.4 and Lemma 2.5.

Our main result states as follows.

Theorem 1.1. *Let $u \in C(\overline{\Omega})$ be the convex Aleksandrov solution to the Monge-Ampère equation* (1.1) *where Ω is a bounded convex domain in $\mathbb{R}^n$ ($n \geq 2$), and $f \in C^{2,\beta}(\overline{\Omega})$ satisfies* (1.2) *where $\beta \in (0,1)$. Then the following statements hold.*

(i) *For all $0 < \delta < \frac{1}{n-1}$, we have $D^2u \in L^\delta(\Omega)$ with estimate*

$$\int_\Omega \|D^2u\|^\delta \, dx \leq C(n,\Omega,\delta,\lambda,\Lambda,\|\log f\|_{C^2(\overline{\Omega})}).$$

(ii) *If, in addition, Ω is a rectangular box, then $D^2u \notin L^{\frac{n}{2(n-1)}}(\Omega)$.*

In the proof of Theorem 1.1(i), we use Pogorelov-type estimates which require u to be C^4. Therefore, it is natural to assume $f \in C^{2,\beta}(\overline{\Omega})$. It would be interesting to reduce the regularity of f in Theorem 1.1(i), and to improve the range of δ when $n \geq 3$.

The rest of this note is devoted to the proof of Theorem 1.1 and pertaining remarks.

2. Proof of Theorem 1.1

Let u be as in Theorem 1.1. Then u is strictly convex; see Caffarelli [C1] and also Figalli [F, Corollary 4.11]. Moreover, $u \in C^{4,\beta}(\Omega)$; see [F, Theorem 3.10].

2.1. Global $W^{2,\delta}$ estimates.

We will establish the following pointwise Hessian estimates.

Lemma 2.1. *Let Ω, u, and f be as in Theorem 1.1(i). Let $\gamma \in (1,2)$. Then, in Ω, we have*

$$\|D^2u(x)\| \leq \begin{cases} C(n,\gamma,\Omega,\lambda,\Lambda,\|\log f\|_{C^2(\overline{\Omega})})[\text{dist}(\mathrm{x},\partial\Omega)]^{-\gamma} & \text{when } n=2,\\ C(n,\Omega,\lambda,\Lambda,\|\log f\|_{C^2(\overline{\Omega})})[\text{dist}(\mathrm{x},\partial\Omega)]^{1-\mathrm{n}} & \text{when } n\geq 3.\end{cases}$$

Remark 2.2. *Lemma 2.1 improves upon Theorem 3.9 in Figalli* [F] *and Theorem 4.1 in Shi–Jiang* [SJ], *where the exponent in the Hessian estimate $\|D^2u(x)\| \leq C[dist(x,\partial\Omega)]^{-\kappa}$ was, respectively, $-(3n+2)$ and $-(2n+\tau)$ where $\tau \in (1,2)$, instead of* $\min\{-\gamma, 1-n\}$.

Clearly, the global $W^{2,\delta}$ estimates in Theorem 1.1(i) are a consequence of Lemma 2.1.

It remains to prove Lemma 2.1. One of our key tools is the following Pogorelov estimate, due to Trudinger and Wang [TW, Lemma 3.6].

Lemma 2.3. *Let $v \in C^4(\overline{\Omega})$ be the convex solution to the Monge-Ampère equation*

$$\begin{cases} \det D^2 v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded convex domain in $\mathbb{R}^n$ ($n \geq 2$), and $f \in C^2(\overline{\Omega})$ with $f > 0$ in $\overline{\Omega}$. Then

$$|v(x)|\|D^2 v(x)\| \leq C(n, \|v\|_{L^\infty(\Omega)}, \|\log f\|_{C^2(\overline{\Omega})})\big(1 + \|Dv\|^2_{L^\infty(\Omega)}\big) \quad \text{in } \Omega. \tag{2.1}$$

Proof of Lemma 2.1. We start with some general estimates for u. For the uniform estimate, we have (see [LMT, Theorem 3.42])

$$c(\Lambda, n)\|u\|^{n/2}_{L^\infty(\Omega)} \leq |\Omega| \leq C(\lambda, n)\|u\|^{n/2}_{L^\infty(\Omega)},$$

where $c(\Lambda, n) > 0$ and $C(\lambda, n) > 0$, so

$$0 < M_1(n, |\Omega|, \lambda) \leq \|u\|_{L^\infty(\Omega)} \leq M_2(n, |\Omega|, \Lambda). \tag{2.2}$$

Since u is convex and $u = 0$ on $\partial\Omega$, there holds

$$|u(x)| \geq \frac{\operatorname{dist}(x, \partial\Omega)}{\operatorname{diam}(\Omega)}\|u\|_{L^\infty(\Omega)} \quad \text{for all } x \in \Omega, \tag{2.3}$$

and

$$|Du(x)| \leq \frac{|u(x)|}{\operatorname{dist}(x, \Omega)} \quad \text{for all } x \in \Omega. \tag{2.4}$$

We recall the following Hölder estimate, due to Caffarelli [C1, Lemma 1],

$$|u(x)| \leq C_1(n, \alpha, \operatorname{diam}(\Omega), \Lambda)[\operatorname{dist}(x, \partial\Omega)]^\alpha \quad \text{for all } x \in \Omega, \tag{2.5}$$

where

$$\alpha := \begin{cases} \frac{2}{1+\gamma} \in (0,1) & \text{when } n = 2, \\ \frac{2}{n} & \text{when } n \geq 3. \end{cases} \tag{2.6}$$

For $h > 0$ small, let

$$\Omega_h := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > h\} \subset\subset \Omega,$$

and

$$A_h := \{x \in \Omega : u(x) < -h\} \subset\subset \Omega.$$

From (2.3), we deduce that

$$A_h \supset \Omega_{\operatorname{diam}(\Omega)h/M_1}. \tag{2.7}$$

Let $v := u + h$. Then, $v \in C^4(\overline{A}_h)$, $v < 0$ in A_h, and $v = 0$ on ∂A_h. Applying (2.1) to v in A_h, and recalling (2.2), we find that

$$\sup_{A_h}\big(|u + h|\|D^2 u\|\big) \leq C(n, |\Omega|, \Lambda, \|\log f\|_{C^2(\overline{\Omega})})(1 + \|Du\|^2_{L^\infty(A_h)}). \tag{2.8}$$

If $x \in A_h$, then $|u(x)| \geq h$, and (2.5) gives

$$\mathrm{dist}(x, \partial\Omega) \geq c_1 h^{\frac{1}{\alpha}}, \quad c_1 = c_1(n, \alpha, \mathrm{diam}\,(\Omega), \Lambda) > 0. \tag{2.9}$$

Combining (2.4) and (2.5) with the above estimate, we obtain

$$|Du(x)| \leq C_1[\mathrm{dist}(x, \partial\Omega)]^{\alpha-1} \leq C_2 h^{1-\frac{1}{\alpha}} \quad \text{in } A_h. \tag{2.10}$$

Thus, in A_{2h} where h is small, (2.8) and (2.10) imply that

$$\|D^2 u\| \leq C(1 + \|Du\|^2_{L^\infty(A_h)})h^{-1} \leq C(n, |\Omega|, \Lambda, \|\log f\|_{C^2(\overline{\Omega})})h^{1-\frac{2}{\alpha}}. \tag{2.11}$$

It follows from (2.7) that

$$\|D^2 u\| \leq \bar{C}(n, |\Omega|, \Lambda, \|\log f\|_{C^2(\overline{\Omega})})h^{1-\frac{2}{\alpha}} \quad \text{in } \Omega_{2\mathrm{diam}\,(\Omega)h/M_1}. \tag{2.12}$$

In view of (2.6), this easily concludes the proof of the lemma. □

Remark 2.4. *In the proof of Lemma 2.1, we use both estimates* (2.3) *and* (2.5)*. When $n = 2$, by choosing γ close to 1, we see that the lower bound and the upper bound for $|u(x)|$ are almost of the same order in dist$(x, \partial\Omega)$. This is responsible for the sharp range of δ in Theorem 1.1(i). However, for $n \geq 3$, the lower bound and the upper bound for $|u(x)|$ in* (2.3) *and* (2.5) *are not of the same order. Thus, to obtain an improved range for δ when $n \geq 3$ without further assumptions on the geometry of Ω, one needs completely different arguments.*

We note that for $n \geq 3$, local improvements on the range of δ are possible when the boundary has flat portions. Due to Theorem 1.1 (ii), the exponent $\frac{n}{2(n-1)}$ in the next lemma is sharp.

Lemma 2.5. *Let $u \in C(\overline{\Omega})$ be the convex Aleksandrov solution to* (1.1) *where $\Omega \supset (-2, 2)^{n-1} \times (0, 2)$ is a bounded convex domain in $\mathbb{R}^n$ ($n \geq 3$) with $(-2, 2)^{n-1} \times \{0\} \subset \partial\Omega$, and $f \in C^{2,\beta}(\overline{\Omega})$ satisfies* (1.2) *where $\beta \in (0, 1)$. Then for $K := (-1, 1)^{n-1} \times (0, c) \subset \Omega$ where $c = c(n, \lambda, \Omega) \in (0, 1/4)$ is small, we have $D^2 u \in L^\mu(K)$ for all $\mu \in (0, \frac{n}{2(n-1)})$ with estimate*

$$\|D^2 u\|_{L^\mu(K)} \leq C(n, \Omega, \lambda, \Lambda, \mu, \|\log f\|_{C^2(\overline{\Omega})}).$$

Proof. We use the same notation as in the proof of Lemma 2.1. Our proof consists of improving (2.7) and (2.12).

By [L3, Lemma 4.3], there exists $c_0 = c_0(n, \lambda, \Omega) \in (0, 1/4)$ such that for $K_0 := (-1, 1)^{n-1} \times (0, c_0)$, we have

$$|u(x)| \geq c_0[\mathrm{dist}(x, \partial\Omega)]^{\frac{2}{n}} \quad \text{if} \quad x \in K_0.$$

Therefore, for $0 < h \leq c_0^2$, we obtain the following local improvement of (2.7):

$$A_h \cap K_0 \supset \Omega_{c_0^{-n/2} h^{n/2}} \cap K_0. \tag{2.13}$$

Using (2.11), (2.6), and (2.13), we find

$$\|D^2 u\| \leq \bar{C}(n, |\Omega|, \Lambda, \|\log f\|_{C^2(\overline{\Omega})})h^{1-n} \quad \text{in } \Omega_{2c_0^{-n/2} h^{n/2}} \cap K_0.$$

Consequently,

$$\|D^2u(x)\| \leq C(n,\Omega,\Lambda,\lambda,\|\log f\|_{C^2(\overline{\Omega})})[\mathrm{dist}(x,\partial\Omega)]^{\frac{2}{n}-2} \quad \text{in} \quad K, \tag{2.14}$$

for $K := (-1,1)^{n-1}\times(0,c_1)\subset\Omega$ where $c_1 = c_1(n,\lambda,\Omega)\in(0,1/4)$ is small. This gives the conclusion of the lemma. □

2.2. **The rectangular box domain.** In this section, we prove Theorem 1.1(ii) where Ω is a rectangular box. By the affine invariance of the Monge-Ampère equation, we can assume, without loss of generality, that

$$\Omega = (-1,1)^{n-1}\times(0,2).$$

Our main estimate, inspired by Wang [W], shows that for a fixed positive fraction of

$$x'\in Q_n := [-1/2,1/2]^{n-1},$$

$D_{nn}u(x',x_n)$ blows up like $x_n^{\frac{2}{n}-2}$ when x_n is small. This is the expected rate discussed in Section 1.

For $x\in\mathbb{R}^n$, we write $x=(x_1,\dots,x_n)=(x',x_n)$ where $x'\in\mathbb{R}^{n-1}$. Denote $D_i=\frac{\partial}{\partial x_i}$, and $D_{ij}=\frac{\partial^2}{\partial x_i\partial x_j}$. Let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure. Below is our main measure-theoretic estimate.

Lemma 2.6. *Let $\Omega, u,$ and f be as in Theorem 1.1(ii). Then, for each $0<x_n<1/2$, there exists an $\mathcal{H}^{n-1}$ measurable subset $E_{x_n}\subset Q_n$ such that the following statements hold.*

(i) $\mathcal{H}^{n-1}(E_{x_n})\geq 1/2$.

(ii) *There exists a constant $c=c(n,\lambda,\Lambda)>0$ such that for all $x'\in E_{x_n}$, we have*

$$D_{nn}u(x',x_n)\geq \begin{cases} c(x_n|\log x_n|)^{-1} & \text{when } n=2,\\ c x_n^{\frac{2}{n}-2} & \text{when } n\geq 3.\end{cases} \tag{2.15}$$

Proof. We fix $x_n\in(0,1/2)$ in this proof.

In view of the Hadamard determinant inequality (see (2.22)), to obtain (2.15), it suffices to show that all the second pure derivatives $D_{ii}u(x',x_n)$ $(i=1,\dots,n-1)$ are bounded from above by $Cx_n^{2/n}$ when $n\geq 3$, and by $Cx_n|\log x_n|$ when $n=2$. We will establish these bounds using one-dimensional slicing arguments.

When $n=2$, we can strengthen the Hölder estimate (2.5) to the following global log-Lipschitz estimate (see [L3, Proposition 1.4])

$$|u(x)|\leq C(\mathrm{diam}\,(\Omega),\Lambda)\mathrm{dist}(x,\partial\Omega)(1+|\log \mathrm{dist}(x,\partial\Omega)|) \quad \text{for all } x\in\Omega\subset\mathbb{R}^2. \tag{2.16}$$

Now, if $x'\in Q_n$, then $\mathrm{dist}((x',x_n),\partial\Omega)=x_n$, and thus (2.5) and (2.16) give

$$|u(x',x_n)|\leq \begin{cases} C_0(n,\Lambda)x_n|\log x_n| & \text{when } n=2,\\ C_0(n,\Lambda)x_n^{\frac{2}{n}} & \text{when } n\geq 3.\end{cases} \tag{2.17}$$

Let

$$\alpha := \frac{2}{n}, \quad a := \frac{1}{n-1}(\frac{1}{2}+n-2).$$

Fix

$$\tilde{x} = (x_2, \dots, x_{n-1}, x_n) \quad \text{where } -\frac{1}{2} \leq x_i \leq \frac{1}{2} \quad \text{for } i = 2, \dots, n-2.$$

We show that there exists a set $S_{\tilde{x}} \subset (-1, 1)$ with $\mathcal{H}^1(S_{\tilde{x}}) \geq a$ for which $D_{11}u(x_1, \tilde{x})$, where $x_1 \in S_{\tilde{x}}$, is bounded from above by $Cx_n^{2/n}$ when $n \geq 3$, and by $Cx_n|\log x_n|$ when $n = 2$.

Indeed, by the convexity of u and $u = 0$ on $\partial\Omega$, we have

$$0 = u(1, \tilde{x}) \geq u(1/2, \tilde{x}) + D_1u(1/2, \tilde{x})(1/2).$$

Hence,

$$D_1u(1/2, \tilde{x}) \leq -2u(1/2, \tilde{x}) = 2|u(1/2, \tilde{x})|.$$

Similarly,

$$-D_1u(-1/2, \tilde{x}) \leq 2|u(-1/2, \tilde{x})|.$$

Therefore, invoking (2.17), we obtain a positive constant $C_1 = 4C_0(n, \Lambda)$ such that

$$D_1u(1/2, \tilde{x}) - D_1u(-1/2, \tilde{x}) \leq \begin{cases} C_1(n, \Lambda)x_n|\log x_n| & \text{when } n = 2, \\ C_1(n, \Lambda)x_n^\alpha & \text{when } n \geq 3. \end{cases} \tag{2.18}$$

We first consider the case $n \geq 3$. Let

$$S_{\tilde{x}} := \Big\{x_1 \in (-1/2, 1/2) : D_{11}u(x_1, \tilde{x}) < \frac{C_1x_n^\alpha}{1-a}\Big\},$$

and

$$L_{\tilde{x}} := (-1/2, 1/2) \setminus S_{\tilde{x}}.$$

Then

$$D_{11}u(x_1, \tilde{x}) \geq \frac{C_1x_n^\alpha}{1-a} \quad \text{for } x_1 \in L_{\tilde{x}}.$$

Consequently, (2.18) implies

$$\begin{aligned} C_1x_n^\alpha \geq D_1u(1/2, \tilde{x}) - D_1u(-1/2, \tilde{x}) &= \int_{-1/2}^{1/2} D_{11}u(x_1, \tilde{x})\, dx_1 \\ &\geq \int_{L_{\tilde{x}}} D_{11}u(x_1, \tilde{x})\, dx_1 \\ &\geq \frac{C_1x_n^\alpha}{1-a}\mathcal{H}^1(L_{\tilde{x}}). \end{aligned}$$

It follows that

$$\mathcal{H}^1(L_{\tilde{x}}) \leq 1 - a,$$

and hence

$$\mathcal{H}^1(S_{\tilde{x}}) \geq a \quad \text{for each } \tilde{x} = (x_2, \dots, x_{n-1}, x_n) \quad \text{where } |x_i| \leq \frac{1}{2}(i = 2, \dots, n-2). \tag{2.19}$$

Let

$$E_{i,x_n} := \Big\{x' \in Q_n : D_{ii}u(x', x_n) < \frac{C_1x_n^\alpha}{1-a}\Big\},$$

and

$$E_{x_n} = \bigcap_{i=1}^{n-1} E_{i,x_n}.$$

Then, by (2.19) and the Fubini Theorem, we have

$$\mathcal{H}^{n-1}(E_{i,x_n}) \geq a. \tag{2.20}$$

Note that if A and B are two $\mathcal{H}^{n-1}$ measurable subsets of Q_n, then

$$\mathcal{H}^{n-1}(A \cap B) = \mathcal{H}^{n-1}(A) + \mathcal{H}^{n-1}(B) - \mathcal{H}^{n-1}(A \cup B) \geq \mathcal{H}^{n-1}(A) + \mathcal{H}^{n-1}(B) - 1.$$

By induction, we then obtain from (2.20) that

$$\mathcal{H}^{n-1}(E_{x_n}) \geq \sum_{i=1}^{n-1} \mathcal{H}^{n-1}(E_{i,x_n}) - (n-2) \geq (n-1)a - (n-2) \geq 1/2. \tag{2.21}$$

For $x' \in E_{x_n}$, we have

$$D_{ii}u(x', x_n) \leq \frac{C_1 x_n^\alpha}{1-a} \quad \text{for all } i = 1, \ldots, n-1.$$

Thus, using the Hadamard determinant inequality

$$\det D^2 u(x', x_n) \leq \prod_{i=1}^{n} D_{ii}u(x', x_n), \tag{2.22}$$

together with $\det D^2 u(x', x_n) \geq \lambda$, we obtain

$$D_{nn}u(x', x_n) \geq \lambda(1-a)^{n-1} C_1^{1-n} x_n^{-(n-1)\alpha} = \lambda(1-a)^{n-1} C_1^{1-n} x_n^{\frac{2}{n}-2} \quad \text{for } x' \in E_{x_n}. \tag{2.23}$$

Due to (2.21) and (2.23), the set E_{x_n} satisfies the requirements of the lemma with $c = \lambda(1-a)^{n-1} C_1^{1-n}$.

Finally, we consider the case $n = 2$. Then $a = 1/2$. As above, it suffices to choose

$$E_{x_2} := \{x_1 \in (-1/2, 1/2) : D_{11}u(x_1, x_2) < 2C_1 x_2 |\log x_2|\}.$$

The lemma is proved. □

Completion of the proof of Theorem 1.1(ii). We can assume $\Omega = (-1,1)^{n-1} \times (0,2)$. Let $p > 0$. Then, Lemma 2.6 tells us that

$$\begin{aligned}
\int_\Omega \|D^2 u\|^p \, dx &\geq \int_0^{1/2} \int_{E_{x_n}} [D_{nn}u(x', x_n)]^p \, dx' dx_n \\
&\geq \begin{cases} \dfrac{1}{2} \displaystyle\int_0^{1/2} (c(x_n |\log x_n|)^{-1})^p \, dx_n & \text{when } n = 2, \\ \dfrac{1}{2} \displaystyle\int_0^{1/2} (c x_n^{\frac{2}{n}-2})^p \, dx_n & \text{when } n \geq 3 \end{cases} = +\infty,
\end{aligned}$$

if $p \geq \frac{n}{2(n-1)}$. This proves Theorem 1.1(ii), and completes the proof of Theorem 1.1. □

3. Further remarks

The method of the proof of Theorem 1.1(ii) can be extended to singular and degenerate Monge-Ampère equations. The following proposition is a representative.

Proposition 3.1. *Let Ω is a rectangular box in $\mathbb{R}^n$ $(n \geq 2)$. Let $f \in C^{2,\beta}(\overline{\Omega})$ be such that $0 < \lambda \leq f \leq \Lambda$ where $\beta \in (0,1)$. Let $s \in (-\infty, n-2)$. Let $u \in C(\overline{\Omega})$ be the nonzero convex Aleksandrov solution to the Monge-Ampère equation*

$$\begin{cases} \det D^2 u &= f|u|^s \quad \text{in } \Omega, \\ u = 0 & \quad\text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Then $D^2 u \notin L^{\frac{n-s}{2(n-s)-2}}(\Omega)$ if $s \leq 0$, and $D^2 u \notin L^{\frac{n-s}{2(n-s)-2}+\varepsilon}(\Omega)$ for any $\varepsilon > 0$ if $s > 0$.

Proof. Following the proof of Proposition 2.8 in [L1], we have $u \in C^{4,\beta}(\Omega)$. The case $s = 0$ follows from Theorem 1.1(ii) so we only consider $s \neq 0$. We assume that $\Omega = (-1,1)^{n-1} \times (0,2)$, and use the same notation as in Section 2.2. In particular, $x_n \in (0, 1/2)$. We consider two separate cases.

Case 1. We first consider the case $s < 0$. In Lemma 2.6, we replace (2.15) by

$$D_{nn} u(x', x_n) \geq c x_n^{\frac{2-2(n-s)}{n-s}} \tag{3.2}$$

where $c = c(n, \lambda, \Lambda, s) > 0$, from which it follows that $D^2 u \notin L^{\frac{n-s}{2(n-s)-2}}(\Omega)$.

To prove (3.2), we make the following changes in the proof of Theorem 1.1(ii). Due to [L2, Theorem 1.1 (i)], we can replace (2.17) by

$$|u(x', x_n)| \leq C_0(n, \Lambda, s) x_n^{\frac{2}{n-s}}. \tag{3.3}$$

We replace α by

$$\alpha_s := \frac{2}{n-s}.$$

From (2.22) and

$$\det D^2 u(x', x_n) \geq \lambda |u(x', x_n)|^s \geq \lambda (C_0 x_n^{\alpha_s})^s,$$

we have, instead of (2.23),

$$D_{nn} u(x', x_n) \geq \lambda (C_0 x_n^{\alpha_s})^s C_1^{1-n} (1-a)^{n-1} x_n^{-(n-1)\alpha_s} = c x_n^{\frac{2-2(n-s)}{n-s}},$$

which is (3.2) where $c = \lambda C_0^s C_1^{1-n} (1-a)^{n-1} > 0$.

Case 2. We next consider the case $0 < s < n-2$. Let

$$0 < \mu_1 < \frac{2}{n-s} < \mu_2 < 1.$$

In Lemma 2.6, we replace (2.15) by

$$D_{nn} u(x', x_n) \geq c x_n^{s\mu_2 - (n-1)\mu_1} \tag{3.4}$$

where $c = c(n, \lambda, \Lambda, s, \mu_1, \mu_2) > 0$.

Thus, given any $\varepsilon > 0$, we can choose μ_1 and μ_2 close to $\frac{2}{n-2}$ so that

$$(s\mu_2 - (n-1)\mu_1)(\frac{n-s}{2(n-s)-2} + \varepsilon) \leq -1,$$

which shows that $D^2u \notin L^{\frac{n-s}{2(n-s)-2}+\varepsilon}(\Omega)$.

To prove (3.4), we make the following changes in the proof of Theorem 1.1(ii). Due to [L2, Proposition 1], we can replace (2.17) by

$$|u(x', x_n)| \leq C_0(n, \Lambda, s, \mu_1)x_n^{\mu_1}. \tag{3.5}$$

We replace α by

$$\alpha_s := \mu_1.$$

By [L3, Theorem 1.1], we have

$$|u(x', x_n)| \geq c_1(n, s, \mu_2, \lambda)x_n^{\mu_2}.$$

From (2.22) and

$$\det D^2u(x', x_n) \geq \lambda|u(x', x_n)|^s \geq \lambda(c_1 x_n^{\mu_2})^s,$$

we have, instead of (2.23),

$$D_{nn}u(x', x_n) \geq \lambda(c_1 x_n^{\mu_2})^s C_1^{1-n}(1-a)^{n-1}x_n^{-(n-1)\mu_1} = cx_n^{s\mu_2-(n-1)\mu_1},$$

which is (3.4) where $c = \lambda c_1^s C_1^{1-n}(1-a)^{n-1} > 0$.

We have completed the proof of the proposition. □

Remark 3.2. *It would be interesting to establish an analogue of Theorem 1.1(i) for* (3.1) *when* $s \neq 0$. *If we apply* (2.1) *as in the proof of Lemma 2.1, then in* (2.8)*, the quantity* $\|\log f\|_{C^2(\overline{\Omega})}$ *has to be replaced by* $\|\log(f|u|^s)\|_{C^2(\overline{\Omega})}$ *which we do not have a priori control.*

4. Declarations

Conflict of Interest Statement: There is no conflict of interest.

Data Availability Statement: All data generated or analyzed during this study are included in this article.

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