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On large deviations and intersection of random interlacements

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We investigate random interlacements on $\mathbb{Z}^d$ with $d \geq 3$, and derive the large deviation rate for the probability that the capacity of the interlacement set in a macroscopic box is much smaller than that of the box. As an application, we obtain the large deviation rate for the probability that two independent interlacements have empty intersections in a macroscopic box. Additionally, we prove that conditioning on this event, one of the interlacements will be sparse in terms of capacity within the box. This result is an example of the entropic repulsion phenomenon for random interlacements.

Keywords: Entropic repulsion; large deviations; random interlacements

1. Introduction

Random interlacements, introduced in Sznitman (2010), serve as a model for understanding the trace of simple random walks and percolation with long-range correlations. While its percolative properties have been extensively studied (see, in particular, Sidoravicius and Sznitman (2009);Sznitman (2010)), the large deviation properties of this model, as well as the trace of simple random walks, are also a central object of study. Recent progress in this direction can be found in Chiarini and Nitzschner (2020a);Li (2017);Li and Sznitman (2014);Nitzschner and Sznitman (2020);Sznitman (2017, 2019a,b, 2021a,b, 2023).

In this article, we focus on the intersection of two independent interlacements. The percolative properties of the intersection set and its complement are studied by the second author in Zhuang (2021). To better understand the intersection set, we calculate the asymptotic probability that it leaves a macroscopic hole in space. In doing so, we show that the optimal strategy for two interlacements to avoid each other is to force the interlacement with smaller intensity to be "almost empty" in the box, while the other one behaves as usual. When the two interlacements have the same intensity, one of them will be forced to exhibit this behavior with asymptotically equal probability. This phenomenon falls within the scope of the entropic repulsion phenomenon for interlacements. (This phenomenon has been studied in the context of Gaussian free fields, with results such as Bolthausen, Deuschel and Zeitouni (1995);Deuschel and Giacomin (1999) for the hard-wall condition, and Chiarini and Nitzschner (2020b) for the disconnection condition. More recently, the authors in Chiarini and Nitzschner (2020a) investigated the disconnection condition for random interlacements.) An important step in the proof is to compute the large deviation rate for the probability that the capacity of one interlacement set in a macroscopic box is much smaller than that of the box. This rate function is determined by a constraint problem on capacity, which is of independent interest.

We now describe our results in more detail. See Section 2 for more details. For $d \geq 3$, consider two independent interlacements $\mathcal{I}_1, \mathcal{I}_2$ on $\mathbb{Z}^d$ with intensity parameters $u_1, u_2 \geq 0$ respectively. We will write $\mathcal{I}^u$ when only one of them is being considered. We use $\mathbb{P}$ for the probability measure governing these objects and $\mathbb{E}$ for the corresponding expectation. Let $B(x,r)$ (resp. $\widetilde{B}(x,r)$) denote the closed l^∞-norm box in $\mathbb{Z}^d$ (resp. $\mathbb{R}^d$) centered at x with radius r. We denote by $\mathrm{cap}(A)$ the discrete capacity for

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$A \subset\subset \mathbb{Z}^d$, and $\widetilde{\text{cap}}(A)$ the Brownian capacity for bounded $A \in \mathcal{B}$, where $\mathcal{B}$ is the collection of F_σ sets in $\mathbb{R}^d$ (see Section 2, in particular above (2.1), for precise definitions). A set $A \subset \mathbb{R}^d$ is called nice if it is the union of a finite number of closed boxes. For $\lambda \geq 0$, we define $f(\lambda)$ through the following constraint problem:

$$f(\lambda) = \inf_{A \text{ nice}, \widetilde{\text{cap}}(A) \leq \lambda} \widetilde{\text{cap}}(\widetilde{B}(0,1) \backslash A). \tag{1.1}$$

When $\lambda \geq \widetilde{\text{cap}}(\widetilde{B}(0,1))$, this function $f(\lambda)$ is trivially zero. See Proposition 4.1 for properties of f.

Our first result concerns the large deviation rate for the probability that the interlacement set in a macroscopic box has a small capacity.

Theorem 1.1. *For any $u > 0$ and $0 < \lambda < \frac{1}{d}\widetilde{\text{cap}}(\widetilde{B}(0,1))$, we have*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\text{cap}\left(B(0,N) \cap \mathcal{I}^u\right) < \lambda N^{d-2}\right] = -\frac{u}{d} f(d\lambda). \tag{1.2}$$

Here, the factor N^{d-2} appears twice because it corresponds to the order of the discrete capacity of $B(0,N)$. This factor is also present in related problems involving Gaussian free fields, such as in Bolthausen and Deuschel (1993);Chiarini and Nitzschner (2020b);Sznitman (2015). The dimension d appears in the rate function because the Brownian capacity is approximately d times the discrete capacity. For more precise statements, we refer to Lemmas 2.2 and 2.3.

Our next result gives the large deviation rate for the probability that two independent interlacements have no intersections in a macroscopic box, and shows that conditioned on this event, the one with the smaller intensity parameter will be negligible in terms of capacity.

Theorem 1.2. *Consider two independent random interlacements $\mathcal{I}_1$ and $\mathcal{I}_2$ with intensities u_1 and u_2 respectively.*

(1). *For any $u_1, u_2 > 0$,*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] = -\frac{\min\{u_1,u_2\}}{d} \widetilde{\text{cap}}(\widetilde{B}(0,1)). \tag{1.3}$$

(2). *For any $u_1 > u_2 > 0$ and $\epsilon > 0$,*

$$\lim_{N\to\infty} \mathbb{P}\left[\text{cap}\left(B(0,N) \cap \mathcal{I}_1\right) > (1-\epsilon)\text{cap}\left(B(0,N)\right) \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] = 1\,; \tag{1.4}$$

$$\lim_{N\to\infty} \mathbb{P}\left[\text{cap}\left(B(0,N) \cap \mathcal{I}_2\right) < \epsilon N^{d-2} \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] = 1. \tag{1.5}$$

In Theorems 1.1 and 1.2, we present the result for the case of $B(0,N)$, the blow-up of $\widetilde{B}(0,1)$, for brevity. However, analogous results should also hold for more general sets, such as regular sets. Additionally, we can show that the interlacement set with the smaller intensity parameter is negligible in the box in terms of *occupancy*. We refer to Section 6 for relevant discussions. In the case of $u_1 = u_2$, Claim (2) in Theorem 1.2 no longer holds, and further discussions can be found in Remark 5.2. However, we can establish that, with asymptotically probability a half, one of the interlacement sets is negligible in terms of occupancy. The justification for this requires a different approach, which we briefly outline in Section 6.

Theorem 1.2 is an example of the entropic repulsion phenomenon for random interlacements. This phenomenon suggests that conditional on certain rare results (such as disconnecting a box from far away

Chiarini and Nitzschner (2020a);Nitzschner and Sznitman (2020);Sznitman (2017), exceeding expected values in a box Sznitman (2019b, 2021a, 2023), or having macroscopic holes Sznitman (2019a)), random interlacements tend to favor configurations with the lowest energy, where "energy" refers to the rate function in the large deviation type estimates. The optimal strategy is achieved by so-called "tilted interlacements" which have a non-homogeneous density and were initially introduced in Li and Sznitman (2014). These tilted interlacements play a significant role in the aforementioned work and Li (2017) for providing plausibly tight asymptotic lower bounds for disconnection probability. It is conjectured that a similar phenomenon exists for simple random walks, which can be heuristically seen as interlacements with intensity zero. However, only a few rigorous results have been established for random interlacements or simple random walks so far, as shown in Chiarini and Nitzschner (2020a);Sznitman (2019a), and many open problems remain. In contrast, similar phenomena for the Gaussian free field are better understood due to the presence of a nice domain Markov property, which is absent in interlacements. Another important issue is that upper and lower bounds involving percolative properties of interlacements often rely on different definitions of critical thresholds, while the sharpness of the phase transition for interlacements is still an open question (though this issue has been resolved for the Gaussian free field in Duminil-Copin et al. (2023)).

Next, we provide a brief overview of the proof strategy for Theorems 1.1 and 1.2.

The lower bound of Theorem 1.1 directly follows from the definition of the function f. The idea is to constrain the interlacements to stay within the blow-up of the set which almost solves the constraint problem. By doing so, the lower bound of the rate function is given by the Brownian capacity of this set.

We now focus on the upper bound. Heuristically, similar to the decomposition of Gaussian free fields (see e.g. Sznitman (2015)), we can decompose interlacements into a local part and a global part. The local part describes the behavior of interlacements inside each mesoscopic box, taking into account the intensity profile such as the number of interlacement paths entering a slightly larger box or the average local time. On the other hand, the global part contains information about these intensity profiles and captures the global behavior of interlacements.

Since the local part is approximately independent between distantly located boxes, it is more difficult to tilt compared to the global part, which exhibits long-range correlations. Thus, when considering a rare event, we can assume that the local part remains untilted, in the sense that it behaves consistently according to the intensity profile, while only the global part is tilted. In other words, the rare event is realized through a certain tilting of the global part. Since the global part has an integrable structure, we can calculate the cost of this tilting, which yields an upper bound on the large deviation rate function.

To rigorously establish this intuition, we employ the coarse-graining procedure introduced in Sznitman (2015, 2017). Specifically, we partition the macroscopic box into mesoscopic boxes with appropriately chosen side lengths. By using the soft local time technique developed in Comets et al. (2013);Popov and Teixeira (2015), we can show that, with a super-exponentially small probability, interlacements will exhibit regular behavior in most boxes. For example, the fraction of occupied points will be consistent with the local-time profile, which may not necessarily be equal to the intensity parameter. This aligns with the intuition that the local part is much more resistant to being tilted. In our case, we call the interlacement set in a mesoscopic box *good* if it either has a small average local time or is "visible" for simple random walks, meaning its capacity is comparable to that of the box, see (3.1) and (3.2).

Under the constraint that most boxes are good, we can replace the event of "the interlacement set in the macroscopic box having small capacity" with a collection of events, each of which corresponds to the interlacement set having a small average local time in many mesoscopic boxes and yet the total capacity of these boxes exceeds a certain threshold. The probability of each of these events can be bounded above using the Laplace transform of local times (as shown in Proposition 2.4). By summing up these inequalities, we obtain an upper bound for the large deviation probability that matches the lower bound in the principal order.

Theorem 1.2 is a direct corollary of Theorem 1.1 by observing the following identity:

$$\mathbb{P}\Big[\mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\Big] = \mathbb{E}\Big[e^{-u_2 \cdot \mathrm{cap}(\mathcal{I}_1 \cap B(0,N))}\Big] = \mathbb{E}\Big[e^{-u_1 \cdot \mathrm{cap}(\mathcal{I}_2 \cap B(0,N))}\Big]. \tag{1.6}$$

In what follows, we highlight some of the open problems and challenges related to the entropic repulsion phenomenon for interlacements and related models.

One fundamental problem is to provide a proper characterization of "the convergence of a random set to tilted interlacements". It is crucial to clarify the metric space in which this convergence occurs. In this paper, we establish convergence in terms of capacity and occupancy. However, these notions alone may not be sufficient to uniquely determine tilted interlacements. Therefore, it is necessary to develop a characterization construction for tilted interlacements. This entails proposing conditions that are sufficient to characterize the model as tilted interlacements. The main difficulty arises from the fact that the characterization depends on both the scale and the tilted density profile.

Another challenging problem is the tilting of interlacements in the downward direction. As mentioned in Remark 5.5 of Sznitman (2019b), when investigating large deviation problems, difficulties arise in estimating the rate function if the optimal strategy involves downward tilting, particularly when the optimal strategy cannot be explicitly solved. In our specific case, this challenge does not arise since the optimal strategy is relatively simple. However, to address the general situation, one must roll up sleeves and overcome this difficulty.

We now discuss whether independence between different interlacements is preserved when conditioning on rare events. We speculate that for fairly general events, these interlacements should still behave independently, at least at the macroscopic level. Our Theorem 1.2 implies that in an appropriate sense, one of them will behave normally while the other will avoid the box, which partially confirms this speculation. However, to provide a precise statement, it is necessary to clarify in what sense these discrete objects are approximately independent.

Finally, let us delve into the entropic repulsion phenomena for the trace of simple random walks. To the best of our knowledge, few rigorous results in this direction have been established, and even the rate functions are unknown in many cases. A classical question regarding the large deviations of simple random walks is what happens when the range of a simple random walk is much smaller than its expectation over a certain time interval. The large deviation rate function is obtained in Phetpradap (2011), extending a similar result for Brownian motions in van den Berg, Bolthausen and den Hollander (2001). The optimal strategy, referred to as the "Swiss cheese" in van den Berg, Bolthausen and den Hollander (2001), depicts the random walk's range resembling Swiss cheese (or more precisely, Emmentaler cheese), covering a positive proportion of points in space but not all. We speculate that this "Swiss cheese" strategy actually forces simple random walks to behave similarly to tilted interlacements locally around some point; see Asselah and Schapira (2017, 2020) for recent developments in this area.

This work is organized as follows. In Section 2, we introduce our notation and setup, and recall a few useful results. Section 3 is dedicated to the coarse-graining strategy, which is a necessary step in obtaining the upper bound in Theorem 1.1. We discuss the constraint problem and Brownian capacities in Section 4. In Section 5, we conclude the proof of both theorems. In Section 6, we provide a sketch of an alternative approach (which can apply to the $u_1 = u_2$ case).

Finally, let us explain our convention concerning constants. Constants like c, c', C, C' may change from place to place, while constants with subscripts like c_1, C_1 remain fixed throughout the article. All constants may implicitly depend on the dimension d. The dependence on additional variables will be indicated at the first occurrence of each constant.

2. Notation and some useful results

In this section, we review the definitions of simple random walks, Brownian motions, and random interlacements, and collect some useful results about capacities and local times.

We begin with some notation. We consider $\mathbb{Z}^d$ and $\mathbb{R}^d$ with $d \geq 3$. For a real value a, we define $a_+ = \max\{a, 0\}$ and use $\lfloor a \rfloor$ to denote the largest integer not greater than a. Let $|\cdot|_1$ (resp. $|\cdot|_\infty$) denote the l^1-norms (resp. l^∞-norms) in both $\mathbb{Z}^d$ and $\mathbb{R}^d$. In what follows, we will add a tilde "$\widetilde{\ }$" above objects in the continuum to distinguish them from their discrete counterparts. We write $B(x,r) = \{y \in \mathbb{Z}^d : |x-y|_\infty \leq r\}$ for the closed l^∞-ball in $\mathbb{Z}^d$ centered at x and of radius r. Given $A \subset \mathbb{Z}^d$, we write $\partial_i A = \{x \in A : \exists y \in \mathbb{Z}^d \backslash A, |x-y|_1 = 1\}$ for its inner boundary and $\partial A = \partial_i(\mathbb{Z}^d \backslash A)$ for its outer boundary. Let $\widetilde{B}(x,r) = \{y \in \mathbb{R}^d : |x-y|_\infty \leq r\}$ denote the closed l^∞-ball in $\mathbb{R}^d$ centered at x and of radius r. Let $\widetilde{A}(x,r,R) = \{y \in \mathbb{R}^d : r < |x-y|_\infty \leq R\}$ denote the l^∞-annulus in $\mathbb{R}^d$ centered at x, with inner radius r and outer radius R. Given $A \subset \mathbb{R}^d$, we write ∂A for its boundary. Here we slightly abuse the notation $|\cdot|_1, |\cdot|_\infty$ and ∂ in $\mathbb{Z}^d$ and $\mathbb{R}^d$ (and hopefully they will be clear in the context). For $A \subset \mathbb{Z}^d$, let $\widetilde{A}$ denote its $\mathbb{R}^d$*-filling* (which contains all the l^∞-balls centered at the vertices in A with radius $1/2$ in $\mathbb{R}^d$). For a set B in $\mathbb{R}^d$ and any integer $N \geq 1$, let B_N stand for the *blow-up* of B in the discrete: $B_N = \{x \in \mathbb{Z}^d ; \inf_{y \in NB} |x-y|_\infty \leq 1\}$.

We turn to random walks on $\mathbb{Z}^d$. Let P_x denote the law of a continuous-time simple random walk $\{X_t\}_{t \geq 0}$ on $\mathbb{Z}^d$ with jump rate 1 started at a vertex x. Given $A \subset\subset \mathbb{Z}^d$, let $H_A = \inf\{t \geq 0 : X_t \in A\}$ denote the first *entrance* time and $T_A = \inf\{t \geq 0 : X_t \notin A\}$ denote the first *exit* time. Let ξ_1 be the first jumping time and $\widehat{H}_A = \inf\{t > \xi_1 : X_t \in A\}$ be the first *hitting* time. We write $e_A(x) = P_x[\widehat{H}_A = \infty]\mathbb{1}\{x \in K\}$ for the equilibrium measure of A, $\mathrm{cap}(A) = \sum_x e_A(x)$ for the discrete capacity of A, and $\bar{e}_A(x) = e_A(x)/\mathrm{cap}(A)$ for the normalized equilibrium measure. We write $g(x,y) = E_x \int_0^\infty \mathbb{1}\{X_t = y\} dt$ for the Green's function of the simple random walk.

We now briefly review Brownian motions on $\mathbb{R}^d$ and introduce relevant notation. Let W_x denote the law of a standard Brownian motion $\{W_t\}_{t \geq 0}$ in $\mathbb{R}^d$ started at a point x. Let $\mathcal{B}$ denote the collection of all F_σ sets in $\mathbb{R}^d$, i.e., the set that can be written as the increasing limit of compact sets. It is known that $\mathcal{B}$ contains all open sets and closed sets. Moreover, $\mathcal{B}$ is closed under countable unions and finite intersections. In this paper, all the sets we consider in $\mathbb{R}^d$ are *bounded* F_σ *sets*. We use $|\cdot|$ to denote the volume (d-dimensional Lebesgue measure) of these sets. Given $A \in \mathcal{B}$, let $\widetilde{H}_A = \inf\{t \geq 0 : W_t \in A\}$ denote the first entrance time of A and $\widetilde{T}_A = \inf\{t \geq 0 : W_t \notin A\}$ denote the first exit time from A. We write $\widetilde{e}_A$ for the equilibrium measure of A and $\widetilde{\mathrm{cap}}(A)$ for the Brownian capacity of A. We write $\widetilde{g}(x,y)$ for the Green's function of the standard Brownian motion. One can refer to Section 3.1 of Port and Stone (1978) for their precise definitions and basic properties. We call a set *regular* if its closure and interior have the same Brownian capacity. We call a set $A \subset \mathbb{R}^d$ *nice* if it is the union of a finite number of closed boxes.

From now on, we will use $N > 100^d$ to denote the side length of the macroscopic box, and $L = L(N)$ to denote the size of the mesoscopic boxes, also known as L-boxes, which will be defined below. For a given large N, we choose L as

$$L = \left\lfloor N^{2/d} (\log N)^{1/d} \right\rfloor . \tag{2.1}$$

Let

$$0 < \delta < 0.1 \tag{2.2}$$

be a constant that governs all the errors and approaches zero towards the end. We also select a large integer $K = K(\delta) > 100$ (the choice of K is provided in (5.1)).

An l^∞-ball $B(x,r)$ in $\mathbb{Z}^d$ is called an L*-box* if $r = L$, $x \in (2K+1)L\mathbb{Z}^d$ and $B(x,r) \subset B(0,N)$. For $x \in (2K+1)L\mathbb{Z}^d$, we use B_x to denote $B(x,L)$ and D_x to denote $B(x,KL)$. The sets D_x are pairwise disjoint. We denote the union of a collection of L-boxes by C and use $\mathrm{Card}(C)$ to denote the number of L-boxes it contains (note that this differs from the common usage of $\mathrm{Card}(\cdot)$). We consider C as a subset of $\mathbb{Z}^d$ and denote its $\mathbb{R}^d$-filling by $\widetilde{C}$.

The following lemma, collected from Proposition 2.5 in Sznitman (2017), states that when K is large (equivalently, when L-boxes are far apart from each other), the relative equilibrium measure defined on C is close to the equilibrium measure defined on each L-box. The proof is omitted.

Lemma 2.1. *If $L \geq 1$ and $K \geq c_1(\delta)$, then for any C which is the union of a collection of L-boxes, any L-box B contained in C, and any $x \in B$,*

$$(1-\delta)\bar{e}_B(x) \leq \frac{e_C(x)}{e_C(B)} \leq (1+\delta)\bar{e}_B(x),$$

where $e_C(B) = \sum_{y \in B} e_C(y)$.

The next lemma, collected from Proposition A.1 in Nitzschner and Sznitman (2020), states that the discrete capacity of C is close to d fraction of the Brownian capacity of its $\mathbb{R}^d$-filling $\widetilde{C}$ when both L (equivalently, N) and K are large. The proof is also omitted.

Lemma 2.2. *If $L \geq c_2(\delta)$ and $K \geq c_3(\delta)$, then for any C which is the union of a collection of L-boxes and its $\mathbb{R}^d$-filling $\widetilde{C}$*

$$(1-\delta)\widetilde{\mathrm{cap}}(\widetilde{C}) \leq d \cdot \mathrm{cap}(C) \leq (1+\delta)\widetilde{\mathrm{cap}}(\widetilde{C}).$$

The next lemma is a classical result that relates the Brownian capacity and the discrete capacity of its blow-up.

Lemma 2.3. *Suppose A is a bounded regular set in $\mathbb{R}^d$. Then, we have*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \mathrm{cap}(A_N) = \frac{1}{d}\widetilde{\mathrm{cap}}(A).$$

Proof. The lemma can be derived for nice sets from variational characterizations of both types of capacity. The details can be found in Bolthausen and Deuschel (1993), specifically in Equation (2.4) and Lemma 2.1.

To extend this result to arbitrary regular sets, we can express the interior of any set A as an increasing limit of nice sets and the closure of A as a decreasing limit of nice sets. By using Proposition 1.13 in Chapter 3 of Port and Stone (1978) and the fact that A is regular, this approximation yields the value of $\widetilde{\mathrm{cap}}(A)$. □

In the remainder of this section, we provide a brief introduction to the (continuous-time) random interlacements on $\mathbb{Z}^d$. We refer the reader to Drewitz, Ráth and Sapozhnikov (2014) for a detailed description of the random interlacements model, and to e.g. Li and Sznitman (2014) for the construction of the continuous-time interlacements.

Let W denote the space of continuous-time doubly-infinite l^1-neighbor paths in $\mathbb{Z}^d$, and let W^* denote the quotient space of W modulo time shift. We use π to denote the quotient map from W to W^*. We now define a Poisson point measure μ with intensity u on W^*, governed by the probability measure

$\mathbb{P}$, characterized by the following property. Given $A \subset\subset \mathbb{Z}^d$, let W_A^* denote the paths in W^* that pass through A, and let μ_A denote the restriction of μ to W_A^*. Then, μ_A has the same law as

$$\mu_A = \sum_{i=1}^{N_A} \delta_{\pi(X^i)}. \tag{2.3}$$

Here, $N_A \sim \text{Poisson}(u \cdot \text{cap}(A))$ is a Poisson random variable with mean $u \cdot \text{cap}(A)$. Conditional on N_A, N_A doubly-infinite paths $\{X_t^i\}_{t\in\mathbb{R}}$ on $\mathbb{Z}^d$ are sampled independently with the same law. Specifically, for each $1 \le i \le N_A$, X_0^i is a random point in A sampled according to the normalized equilibrium measure $\bar{e}_A$. Given X_0^i, the process $\{X_t^i\}_{t\ge 0}$ evolves as a continuous-time simple random walk starting from X_0^i, and the process $\{X_t^i\}_{t<0}$ is another independent continuous-time simple random walk starting from X_0^i conditioned on $\xi_1 = 0$ and $\widehat{H}_A = \infty$.

The interlacement set at level u, denoted by $\mathcal{I}^u$, is defined as the set of vertices occupied by at least one of these paths. Equation (2.3) gives the law of $\mathcal{I}^u \cap A$ for all $A \subset\subset \mathbb{Z}^d$, and from this, the law of $\mathcal{I}^u$ can be obtained. An important property is that

$$\mathbb{P}[\mathcal{I}^u \cap A = \varnothing] = \mathbb{P}[N_A = 0] = \exp(-u \cdot \text{cap}(A)). \tag{2.4}$$

For $x \in \mathbb{Z}^d$, let $L^u(x)$ denote the local time of $\mathcal{I}^u$ at the vertex x. Precise definitions can be found in Equation (2.15) of Chiarini and Nitzschner (2020b). Given two square integrable functions $f, h : \mathbb{Z}^d \to \mathbb{R}$, we use $\langle f, h\rangle = \sum_{x\in\mathbb{Z}^d} f(x)h(x)$ to denote their inner product. The following lemma provides the Laplace transform of $\langle L^u, e_A\rangle$, which is useful for bounding the local time of interlacements.

Lemma 2.4. *For $A \subset\subset \mathbb{Z}^d$ and $s < 1$,*

$$\mathbb{E}\left[e^{s\langle L^u, e_A\rangle}\right] = \exp\left(\frac{us \cdot \text{cap}(A)}{1-s}\right).$$

Here $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$.

Proof. For $|s| < 1$, this formula can be obtained by setting $V(x) = s \cdot e_A(x)$ in Equation (2.40) of Sznitman (2017). The result can be extended to $s \in (-\infty, -1]$ through analytic continuation. □

3. The coarse-graining procedure

In this section, we introduce the coarse-graining procedure introduced in Sznitman (2017) as well as the coupling of excursions introduced in Comets et al. (2013);Popov and Teixeira (2015).

First, we define good boxes which are important objects in the coarse-graining arguments. Recall that δ, as defined in (2.2), is a global constant that eventually goes to zero. Fixing random interlacements $\mathcal{I}^u$, we will call an L-box (defined below (2.2)) B_x *δ-good* if either of the following two conditions holds. In particular, we will call this box *Type-I δ-good* if

$$\max_{z\in\partial_i B_x} P_z\left[H_{\mathcal{I}^u\cap B_x} = \infty\right] < \delta, \tag{3.1}$$

Type-II δ-good if

$$\langle L^u, \bar{e}_{B_x}\rangle < \delta u, \tag{3.2}$$

and *δ-bad* otherwise. (Here, we assume that $H_\varnothing = \infty$.) The definition is similar in spirit to (3.11)-(3.13) in Sznitman (2017). Roughly speaking, a δ-good box is a box in which the behavior of random interlacements matches the density profile (which is not necessarily equal to the intensity parameter). For an L-box, if the average value of L^u is not too small, then the random interlacements will occupy a positive fraction of points, and thus (3.1) holds with high probability; otherwise, if the average value of L^u in the box is small, then (3.2) holds with high probability instead.

For $\rho > 0$, we define the event that most L-boxes are δ-good as

$$\mathcal{A} := \{\text{there are at most } \rho(N/L)^d \ \delta\text{-bad boxes in } B(0,N)\}. \tag{3.3}$$

In the next proposition, we will prove that for any $\rho > 0$, with overwhelmingly high probability, event $\mathcal{A}$ occurs.

Proposition 3.1. *For all $\delta, \rho > 0$ and $K \geq 100$, we have:*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}[\mathcal{A}^c] = -\infty. \tag{3.4}$$

Before giving the proof, we need to make some preparations. We introduce the notion of excursions and provide a way to couple excursions of interlacements in different boxes with independent random walk excursions. This coupling is based on the "soft local times" technique first introduced in Popov and Teixeira (2015), and the version we will use is from Section 2 of Comets et al. (2013).

Given an L-box centered at $x \in (2K+1)L\mathbb{Z}^d$, we decompose the paths of interlacements into excursions from $\partial_i B_x$ to ∂D_x by considering the hitting time of B_x and the exit time of D_x. One path may produce multiple excursions. We denote the total number of excursions from $\partial_i B_x$ to ∂D_x in $\mathcal{I}^u$ as N_u, and we denote these excursions as $Y^1, \ldots, Y^{N_u}$. See, for example, (2.29), (2.30), (2.41), and (2.42) in Sznitman (2017) for precise definitions. These excursions depend on the vertex x, but we omit it for simplicity. A priori, these excursions $Y^1, \ldots, Y^{N_u}$ and N_u are not independent for different x's. Therefore, we cannot directly use them to estimate $\mathbb{P}[\mathcal{A}^c]$. However, we can couple them with independent excursions and use these independent excursions to estimate $\mathbb{P}[\mathcal{A}^c]$.

We define Θ to be

$$\Theta = \Theta(L) = \mathrm{cap}(B_x). \tag{3.5}$$

Let $\mathbb{Q}$ denote the law of a family of Poisson point processes indexed by $x \in (2K+1)L\mathbb{Z}^d$ whose marked points are excursions in $\mathbb{Z}^d$. These processes are independent for different x's. Given $x \in (2K+1)L\mathbb{Z}^d$, let $(X^i)_{i\geq 1}$ be i.i.d. excursions from $\partial_i B_x$ to ∂D_x with the law of a continuous-time simple random walk started from a point in $\partial_i B_x$ sampled according to $\bar{e}_{B_x}$, and stopped upon leaving the box D_x. (These excursions and the counters $n(t)$ defined below all depend on x, but we omit it in the notation for simplicity.) We let the Poisson process have intensity Θ. Let $n(t)$ denote the number of points (i.e., excursions) of the Poisson process that appear in the time interval $[0,t]$. Then, for any $0 < a < b$, $n(b) - n(a)$ follows a Poisson distribution with mean $(b-a)\Theta$. An important feature of this family is that $(X^i)_{i\geq 1}$ and $n(t)$ are independent for different x's.

We will use the coupling described in Section 2 of Comets et al. (2013). We can construct both the law of $\mathbb{P}$ and $\mathbb{Q}$ from a single realization of the Poisson point process, as done in Comets et al. (2013). With abuse of notation, we will use $\mathbb{Q}$ to govern the coupling of $\mathbb{P}$ and $\mathbb{Q}$. Now we explain how we modify their setup. We set $A_j = B_x$ and $A'_j = D_x \cup \partial D_x$. The law of $\mathbb{P}$ will be sampled in the same way as their first process $(Z_1, Z_2, \ldots)$, and the law of $\mathbb{Q}$ will be sampled in the same way as their second process $(\tilde{Z}_1^{(j)}, \tilde{Z}_2^{(j)}, \ldots)$. We take $\tilde{H}_j(z) = \bar{e}_{B_x}(z)$ for $z \in \partial_i B_x$. In their construction, they choose the intensity to be 1, but we multiply the intensity by a factor of Θ. The following lemma will correspond to their Equation (2.5):

Lemma 3.2. *For $L \geq 1$ and $K \geq 100$, there exists $a > 1$ such that for all $z \in \partial_i B_x$ and $y \in \cup_x \partial D_x$*

$$\frac{1}{\sqrt{a}} < \frac{P_y\left[H_{B_x} = H_z | H_{\cup_x B_x} = H_{B_x} < \infty\right]}{\bar{e}_{B_x}(z)} < \sqrt{a}, \tag{3.6}$$

where $\cup_x$ denoted the union over $x \in (2K+1)L\mathbb{Z}^d$.

Proof. This statement follows from (2.25) in Proposition 2.5 of Sznitman (2017) by taking $A = B_x$ and $B = \cup_x B_x$ in the proposition. The original argument works for K sufficiently large, but one can extend it to all $K > 100$ by enlarging the constant a. □

We have the following property for the coupling. The proof follows verbatim that of Lemma 2.1 in Comets et al. (2013), and thus we omit here. The only difference is that we need to replace $(1+v)$ in their Equation (2.7) with a.

Lemma 3.3. *We have the following two properties for the coupling of $\mathbb{P}$ and $\mathbb{Q}$:*

- *If $n\left(\frac{\delta u}{a^2}\right) > \frac{\delta u}{a^3}\Theta$ and $N_u \leq \frac{\delta u}{a^3}\Theta$, then we have*

$$\{Y^1, \ldots, Y^{N_u}\} \subset \{X^1, \ldots, X^{n(\delta u/a)}\}. \tag{3.7}$$

- *If $n\left(\frac{\delta u}{a^4}\right) < \frac{\delta u}{a^3}\Theta$ and $N_u \geq \frac{\delta u}{a^3}\Theta$, then we have*

$$\{Y^1, \ldots, Y^{N_u}\} \supset \{X^1, \ldots, X^{n(\delta u/a^5)}\}. \tag{3.8}$$

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. For each L-box B_x, let $\mathcal{A}_x^1$ denote the event that all of the following inequalities hold:

$$n\left(\frac{\delta u}{a}\right) < \frac{\delta u \Theta}{\sqrt{a}}; \quad n\left(\frac{\delta u}{a^2}\right) > \frac{\delta u}{a^3}\Theta; \quad n\left(\frac{\delta u}{a^4}\right) < \frac{\delta u}{a^3}\Theta; \quad n\left(\frac{\delta u}{a^5}\right) > \frac{\delta u}{a^6}\Theta. \tag{3.9}$$

Since $\mathbb{E}[n(t)] = \Theta t$, we can apply standard tail bounds on Poisson random variables and use the fact that $\Theta \geq cL^{d-2}$ to obtain

$$\mathbb{Q}\left[(\mathcal{A}_x^1)^c\right] \leq \exp\left(-cL^{d-2}\right). \tag{3.10}$$

If the event $\mathcal{A}_x^1$ occurs, we can deduce from (3.7) and (3.8) that when $N_u \leq \frac{\delta u}{a^3}\Theta$,

$$\{Y^1, Y^2, \ldots, Y^{N_u}\} \subset \{X^1, X^2, \ldots, X^{\delta u/a}\} \subset \{X^1, X^2, \ldots, X^R\}, \tag{3.11}$$

while when $N_u > \frac{\delta u}{a^3}\Theta$,

$$\{Y^1, Y^2, \ldots, Y^{N_u}\} \supset \{X^1, X^2, \ldots, X^{n(\delta u/a^5)}\} \supset \{X^1, X^2, \ldots, X^M\}, \tag{3.12}$$

where we denote $R = \lfloor \delta u \Theta / \sqrt{a} \rfloor$ and $M = \lfloor \delta u \Theta / a^6 \rfloor$. Let $E = (X^1 \cup X^2 \cup \ldots \cup X^M) \cap B_x$ represent the vertices occupied by the excursions in B_x. We define $\mathcal{A}_x^2$ as the event

$$\max_{z \in \partial_i B_x} P_z[H_E = \infty] < \delta.$$

Then, we have:

$$\begin{aligned}\mathbb{Q}\big[(\mathcal{A}_x^2)^c\big] &\le \sum_{z\in\partial_i B_x} \mathbb{Q}\big[P_z[H_E=\infty]\ge\delta\big] \le \sum_{z\in\partial_i B_x} \frac{1}{\delta}\mathbb{Q}\otimes P_z[H_E=\infty] \\ &= \sum_{z\in\partial_i B_x} \frac{1}{\delta}\mathbb{Q}\otimes P_z[Z\cap E=\varnothing] = \sum_{z\in\partial_i B_x} \frac{1}{\delta} P_z\big[\mathbb{Q}[Z\cap X^1\cap B_x=\varnothing]^M\big] \\ &\le CL^{d-1}\max_{z\in\partial_i B_x} P_z\big[\mathbb{Q}[Z\cap X^1\cap B_x=\varnothing]^{cL^{d-2}}\big].\end{aligned}$$

Here, $\mathbb{Q}\otimes P_z$ represents the product measure of $\mathbb{Q}$ and P_z, and Z denotes the trajectory of the simple random walk starting from z. The second inequality follows by using Markov's inequality. The fourth step is by the independence of $X^1,\dots,X^M$. Finally, the last inequality holds because $|\partial_i B_x|\le CL^{d-1}$ and $M\ge cL^{d-2}$. The trajectory Z connects $\partial_i B_x$ and ∂D_x and thus has a capacity of at least $cL/\log L$ if $d=3$, and cL if $d\ge 4$. Hence, for given any Z, we have $\mathbb{Q}[Z\cap X^1\cap B_x=\varnothing]\le 1-c/\log L$ if $d=3$, and $\le 1-cL^{3-d}$ if $d\ge 4$. (These bounds follow easily from basic estimates on capacity through Green's function; see e.g. (1.3.6), (1.3.12) and (1.3.13) of Drewitz, Ráth and Sapozhnikov (2014).) Therefore,

$$\lim_{L\to\infty}\frac{1}{\log(L)}\log\mathbb{Q}\big[(\mathcal{A}_x^2)^c\big]=-\infty. \tag{3.13}$$

Recall that $R=\lfloor\delta u\Theta\rfloor$. Now we consider first R independent simple random walk excursions $X^1,\dots,X^R$ from $\partial_i B_x$ to ∂D_x under the law $\mathbb{Q}$ and their local times $L^1(\cdot),\dots,L^R(\cdot)$. Let $\mathcal{A}_x^3$ denote the event that

$$\sum_{i=1}^R\langle L^i,\bar{e}_{B_x}\rangle<\delta u.$$

By Lemma 2.1 in Sznitman (2017), $\langle L^i,e_{B_x}\rangle$ is dominated by an exponential random variable with mean smaller than one. Therefore, using a standard large deviation estimate and the fact that $R\le\delta u\Theta/\sqrt{a}$, we have $\mathbb{Q}\big[(\mathcal{A}_x^3)^c\big]\le Ce^{-cR}$. Consequently,

$$\lim_{L\to\infty}\frac{1}{\log(L)}\log\mathbb{Q}\big[(\mathcal{A}_x^3)^c\big]=-\infty. \tag{3.14}$$

We observe that

$$B_x \text{ is } \delta\text{-good on the event } \mathcal{A}_x^1\cap\mathcal{A}_x^2\cap\mathcal{A}_x^3. \tag{3.15}$$

We can prove this by separating into two cases:

- If $N_u\le\frac{\delta u}{a^3}\Theta$, then by (3.11),

$$\mathcal{I}^u\cap B_x\subset\Big(Y^1\cup\cdots\cup Y^{N_u}\Big)\cap B_x\subset\Big(X^1\cup\cdots\cup X^R\Big)\cap B_x.$$

  Together with the definition of $\mathcal{A}_x^3$, we know that in this case $\langle L^u,\bar{e}_{B_x}\rangle<\delta u$, and so B_x is Type-II δ-good.
- If $N_u>\frac{\delta u}{a^3}\Theta$, then by (3.12),

$$\mathcal{I}^u\cap B_x\supset\Big(Y^1\cup\cdots\cup Y^{N_u}\Big)\cap B_x\supset\Big(X^1\cup\cdots\cup X^M\Big)\cap B_x=E.$$

Together with the definition of $\mathcal{A}_x^2$, we know that in this case $\max_{z\in\partial_i B_x} P_z[H_{\mathcal{I}^u\cap B_x}=\infty]<\delta$, and so B_x is Type-I δ-good..

Proposition 3.1 follows from the property (3.15) and the estimates (3.10),(3.13),(3.14). Let U be the number of L-boxes, and let $X_1,\dots,X_U$ be the indicator functions for whether these boxes satisfy $\mathcal{A}_x^1\cap\mathcal{A}_x^2\cap\mathcal{A}_x^3$. Since $\mathcal{A}_x^1$, $\mathcal{A}_x^2$, and $\mathcal{A}_x^3$ are independent among disjoint boxes under the law $\mathbb{Q}$, the random variables X_i are i.i.d. Fix a large constant A. From (3.10), (3.13), and (3.14), we have that for L large enough and any $1\le i\le U$,

$$\mathbb{Q}[X_i=1]=\mathbb{Q}[\mathcal{A}_x^1\cap\mathcal{A}_x^2\cap\mathcal{A}_x^3]\ge 1-L^{-A}. \tag{3.16}$$

Now, fixing $t>0$, we have

$$\begin{aligned}\mathbb{Q}[\mathcal{A}^c]&\le\mathbb{Q}\Big[\sum_{i=1}^{U}(1-X_i)\ge\rho(N/L)^d\Big]\le e^{-\rho t(N/L)^d}\Big(\mathbb{E}[e^{t(1-X_1)}]\Big)^U\\&\le e^{-\rho t(N/L)^d}\Big(1+L^{-A}(e^t-1)\Big)^{C(N/L)^d}.\end{aligned}$$

The first inequality is due to (3.15), and the second inequality follows from Markov's inequality and independence. In the last step, we use (3.16) and the fact that $U\le C(N/L)^d$. For any $B>0$, we choose $t=B\log N$. By choosing a sufficiently large A, we can ensure that $(N/L)^d\log(1+L^{-A}(e^t-1))=o(1)$. Combined with (2.1), we obtain

$$\limsup_{N\to\infty}\frac{1}{N^{d-2}}\log\mathbb{Q}[\mathcal{A}^c]\le-\rho B.$$

Since this holds for arbitrary $B>0$, we can send the right-hand side to $-\infty$ and thus finish the proof of Proposition 3.1. □

4. Basic properties of f and Brownian capacity

In this section, we present two results. First, in Proposition 4.1, we analyze the properties of the function f introduced in the constraint problem (see (1.1) for its definition). Second, we prove Proposition 4.3, which relates the Brownian capacity of coarse-grained sets to their discrete capacity. This connection plays a crucial role in the proof of Theorem 1.1.

In the following proposition, we establish several properties of the function f.

Proposition 4.1. *The function f defined in (1.1) has the following properties:*

(1). *$f(\lambda)$ is decreasing.*
(2). *$f(0)=\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))$, and $f(\lambda)=0$ for all $\lambda\ge\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))$.*
(3). *$f(\lambda)>\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))-\lambda$, for all $0<\lambda<\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))$.*
(4). *$f(\lambda)$ is continuous.*

Proof. Claims (1) and (2) follow directly from the definition. We now prove Claim (3). Let $\lambda>0$ be fixed. For any nice set A with $\widetilde{\mathrm{cap}}(A)\le\lambda$, let $B=\widetilde{B}(0,1)\backslash A$. We will show that there exists $c=c(\lambda)>0$ such that

$$\widetilde{\mathrm{cap}}(B)\ge\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))-\lambda+c. \tag{4.1}$$

If $\widetilde{\mathrm{cap}}(A) \le \frac{1}{2}\lambda$, then by the subadditivity property of Brownian capacity (see e.g. Proposition 1.12 in Chapter 3 of Port and Stone (1978)), we have

$$\widetilde{\mathrm{cap}}(B) \ge \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) - \widetilde{\mathrm{cap}}(A) \ge \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) - \frac{1}{2}\lambda. \tag{4.2}$$

Now, suppose $\widetilde{\mathrm{cap}}(A) > \frac{1}{2}\lambda$. By subadditivity and the fact that $\widetilde{\mathrm{cap}}(A) \le \lambda$, we have

$$\widetilde{\mathrm{cap}}(B) \ge \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) - \widetilde{\mathrm{cap}}(A) \ge \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) - \lambda.$$

By Theorem 1.10 in Chapter 3 of Port and Stone (1978), we have

$$\begin{aligned}\widetilde{\mathrm{cap}}(A) + \widetilde{\mathrm{cap}}(B) &= \int_{x\in\partial\widetilde{B}(0,2)} \big(W_x[\widetilde{H}_A < \infty] + W_x[\widetilde{H}_B < \infty]\big)\,\widetilde{e}_{\widetilde{B}(0,2)}(dx) \\ &= \int_{x\in\partial\widetilde{B}(0,2)} \big(W_x[\widetilde{H}_{A\cup B} < \infty] + W_x[\widetilde{H}_A < \infty, \widetilde{H}_B < \infty]\big)\,\widetilde{e}_{\widetilde{B}(0,2)}(dx) \\ &\ge \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) + c.\end{aligned} \tag{4.3}$$

In the second step, we use the fact that $A \cap B = \varnothing$. In the last step, we use Theorem 1.10 in Chapter 3 of Port and Stone (1978) again and the fact that for any $x \in \widetilde{B}(0,2)$, we have $W_x[\widetilde{H}_A < \infty, \widetilde{H}_B < \infty] > c$ for some $c = c(\lambda) > 0$. This is because $W_x[\widetilde{H}_A < \infty] = \int \widetilde{g}(x,y)d\widetilde{e}_A(dy) \ge \int c d\widetilde{e}_A(dy) = c\widetilde{\mathrm{cap}}(A) \ge c$ (see e.g., Theorem 2.1 in Chapter 3 of Port and Stone (1978)), and similarly $W_x[\widetilde{H}_B < \infty] \ge c$. The fact $W_x[\widetilde{H}_A < \infty, \widetilde{H}_B < \infty]$ follows from these two inequalities and the strong Markov property. The desired inequality (4.1) follows from (4.2) and (4.3), which implies Claim (3).

Next, we will prove Claim (4). Since f is decreasing, it suffices to show that for any $\Delta > 0$, there exists $\epsilon = \epsilon(\Delta) > 0$ such that for all $\lambda \ge 0$, we have

$$f(\lambda) \ge f(\lambda - \epsilon) - \Delta. \tag{4.4}$$

Here we let $f(\lambda) = \widetilde{\mathrm{cap}}(\widetilde{B}(0,1))$ when $\lambda < 0$. Fix $\Delta > 0$ and choose a constant $\epsilon > 0$ to be determined later. For any $\lambda \ge 0$, according to the definition of f, there exists a nice set A such that

$$\widetilde{\mathrm{cap}}(A) \le \lambda \quad \text{and} \quad \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)\backslash A) \le f(\lambda) + \frac{\Delta}{2}.$$

In order to prove (4.4), it suffices to find a nice set $A' \subset A$ such that

$$\widetilde{\mathrm{cap}}(A') \le (\lambda - \epsilon)_+ \quad \text{and} \quad \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)\backslash A') \le f(\lambda) + \Delta. \tag{4.5}$$

If $f(\lambda) + \Delta \ge \widetilde{\mathrm{cap}}(\widetilde{B}(0,1))$, we can take $A' = \varnothing$ and then (4.5) holds. From now on, we assume that $f(\lambda) + \Delta < \widetilde{\mathrm{cap}}(\widetilde{B}(0,1))$. There exists a large integer $N = N(\Delta)$ such that

(a). For all l^∞-norm boxes E with side length $\frac{1}{N}$, we have $\widetilde{\mathrm{cap}}(E) \le \frac{\Delta}{2}$.

(b). Suppose a set $K \subset \mathbb{R}^d$ satisfies $|K \cap E| \ge \frac{1}{2}|E|$ for all l^∞-norm boxes E with side length $\frac{1}{N}$ on the boundary of $\widetilde{B}(0,1)$ (i.e. $E \subset \widetilde{B}(0,1)$ and $E \cap \partial\widetilde{B}(0,1) \neq \varnothing$). Then, we have $\widetilde{\mathrm{cap}}(K) \ge \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) - \frac{\Delta}{2}$.

Claim (a) follows form the scaling property of $\widetilde{\mathrm{cap}}(\cdot)$. We will now prove Claim (b). Consider a set K that satisfies the condition stated in Claim (b). By Theorem 1.10 of Chapter 3 in Port and Stone (1978), we have

$$\widetilde{\mathrm{cap}}(K) = \int_{x\in\partial\widetilde{B}(0,1)} W_x[\widetilde{H}_K < \infty]\widetilde{e}_{\widetilde{B}(0,1)}(dx) \quad \text{and} \quad \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) = \int_{x\in\partial\widetilde{B}(0,1)} \widetilde{e}_{\widetilde{B}(0,1)}(dx).$$

Therefore, it suffices to show that for N large enough, we have

$$\inf_{x\in\partial\widetilde{B}(0,1)} W_x[H_K < \infty] \geq 1 - \frac{\Delta}{2\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))}. \tag{4.6}$$

This follows from similar arguments to Wiener's test, as described in Theorem 3.3 of Chapter 3 in Port and Stone (1978). Recall that $\widetilde{A}(x,r,R)$ is the l^∞-annulus in $\mathbb{R}^d$ centered at x and with inner radius r and outer radius R, as defined in Section 2. Fix $x\in\partial\widetilde{B}(0,1)$. For $\{W_t\}_{t\geq 0}$ sampled from W_x and $1\leq i\leq \lfloor \log_2 N\rfloor$, we define $\sigma_i := \inf\{t>0 : W_t \in \partial\widetilde{B}(x, \frac{3}{2}\cdot\frac{2^{i-1}}{N})\}$, and $\tau_i := \inf\{t>\sigma_i : W_t \notin \partial\widetilde{A}(x,\frac{2^{i-1}}{N},\frac{2^i}{N})\}$. Then, we have:

$$W_x[H_K = \infty] \leq W_x[W_t \notin K \quad \forall \sigma_i \leq t\leq \tau_i, 1\leq i\leq \lfloor \log_2 N\rfloor].$$

By definition, the intervals $[\sigma_i,\tau_i]$ are disjoint from each other. By applying the strong Markov property subsequently to the stopping times σ_i for $1\leq i\leq\lfloor\log_2 N\rfloor$, we obtain:

$$W_x[H_K=\infty] \leq \prod_{i=1}^{\lfloor \log_2 N\rfloor} \sup_{z\in\partial\widetilde{B}(x,\frac{3}{2}\cdot\frac{2^{i-1}}{N})\}} W_z[W_t\notin K \quad \forall 0<t\leq \tau_i'].$$

where τ_i' is the first leaving time of $\widetilde{A}(x,\frac{2^{i-1}}{N},\frac{2^i}{N})$. Each term in the product can be bounded by $1-c$ for some $c>0$ independent of i and x. This is because the Brownian motion will hit $E_i := \widetilde{A}(x,\frac{2^{i-1}}{N},\frac{2^i}{N}) \cap \{w\in\widetilde{B}(0,1) : \exists y\in\partial\widetilde{B}(0,1), |w-y|_\infty \leq \frac{1}{N}\}$ with positive probability (since heuristically E_i is a $(d-1)$-dimensional object), and K occupies at least half of the volume of E_i by condition (b). Therefore,

$$W_x[H_K=\infty] \leq (1-c)^{\lfloor \log_2 N\rfloor}.$$

By enlarging the valued of N, we obtain the inequality (4.6).

We choose a sufficiently large N such that Claims (a) and (b) hold. Since $\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)\backslash A) \leq f(\lambda) + \frac{\Delta}{2} < \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) - \frac{\Delta}{2}$, according to Claim (b), there exists a box E on the boundary of $\widetilde{B}(0,1)$ with side length $\frac{1}{N}$ such that

$$|(\widetilde{B}(0,1)\backslash A)\cap E| < \frac{1}{2}|E|,$$

and therefore, $|A\cap E| > \frac{1}{2}|E|$. Let A' be the closure of $A\backslash E$, which remains a nice set. Then, using Claim (a), we have

$$\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)\backslash A') \leq \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)\backslash A) + \widetilde{\mathrm{cap}}(E) \leq f(\lambda)+\Delta$$

and there exists $\epsilon = \epsilon(\Delta) > 0$ such that

$$\begin{aligned}\widetilde{\mathrm{cap}}(A) &= \int_{x\in\partial\widetilde{B}(0,2)} W_x[\widetilde{H}_A<\infty]\widetilde{e}_{\widetilde{B}(0,2)}(dx) \\ &\geq \int_{x\in\partial\widetilde{B}(0,2)} \Big(W_x[\widetilde{H}_{A'}<\infty] + W_x[\widetilde{H}_{A\cap E}<\infty, \widetilde{H}_{A'}=\infty]\Big)\, \widetilde{e}_{\widetilde{B}(0,2)}(dx) \geq \widetilde{\mathrm{cap}}(A')+\epsilon.\end{aligned}$$

The first equation is from Theorem 1.10 in the Chapter 3 of Port and Stone (1978). The last inequality holds because the Brownian motion started from $\partial\widetilde{B}(0,2)$ has a (uniform) positive probability of entering $\widetilde{B}(0,1)$ via $\partial E \cap \widetilde{B}(0,1)$, then hitting the set $A \cap E$ (since $|A \cap E \ \geq \frac{1}{2}|E|$), and finally escaping to infinity without ever hitting A'. In other words, there exists a constant $c(\Delta) > 0$ such that

$$\inf_{x\in\partial\widetilde{B}(0,2)} W_x[\widetilde{H}_{A\cap E} < \infty, \widetilde{H}_{A'} = \infty] > c(\Delta),$$

allowing us to choose ϵ as $c(\Delta) \cdot \widetilde{\mathrm{cap}}(\widetilde{B}(0,1))$. Therefore, A' satisfies (4.5), and Claim (4) holds. □

Remark 4.2. By the continuity of f, the infimum in (1.1) can also be taken over all sets in $\mathbb{R}^d$ with C^1 boundaries. Finding the minimizer set A (if it exists, which we believe to be the case) in (1.1) is an interesting question in itself. To the best of the authors' knowledge, there is neither a quick "back-of-the-envelope" solution nor a readily available answer in the literature. We hope that experts in variational analysis could provide insights on this matter. It is equally interesting to consider more general sets rather than just $\widetilde{B}(0,1)$. It is worth noting that, except in very special cases (e.g., the union of two touching balls with a specifically chosen λ), there seems to be no trivial solution either.

We now turn to the Brownian capacity of the coarse-grained sets. The following proposition provides a comparison between the Brownian capacity and the discrete capacity of these sets.

Proposition 4.3. *Given $\delta > 0$ and $K \geq 1$, there exist constants $c_4(\delta,K)$ and $c_5(\delta,K)$ such that if $L > c_4(\delta,K)$, $\rho < c_5(\delta,K)$, and $t > 0$, then for any C_1 and C_2, which are two disjoint collections of L-boxes in $B(0,N)$ satisfying* $\mathrm{Card}(C_1) + \mathrm{Card}(C_2) \geq (1-\rho)\big(\frac{2N}{(2K+1)L}\big)^d$ *and* $\widetilde{\mathrm{cap}}(\widetilde{C}_1) \leq tN^{d-2}$, *we have the following inequality:*

$$\widetilde{\mathrm{cap}}(\widetilde{C}_2) \geq (f(t) - \delta)N^{d-2}. \tag{4.7}$$

Proof. Let $\epsilon = \epsilon(\delta,K) > 0$ be a constant to be chosen. Similar to (4.6), we can choose $N(\epsilon)$ and $\rho(\epsilon)$ such that for all $N \geq N(\epsilon)$ and $\rho \leq \rho(\epsilon)$, the following holds:

$$\text{For all } x \in \widetilde{B}(0,N), \text{ either } W_x[\widetilde{H}_{\widetilde{C}_1} = \infty] < \epsilon \text{ or } W_x[\widetilde{H}_{\widetilde{C}_2} = \infty] < \epsilon. \tag{4.8}$$

(Fix $x \in \widetilde{B}(0,N)$. The idea is to consider $r = \lfloor \log_2(\frac{N}{2KL}) \rfloor$ layers of sets around x. At each layer, either $\widetilde{C}_1$ or $\widetilde{C}_2$ will occupy a positive fraction of space, making it visible to a simple random walk. By combining this observation with the strong Markov property, we can derive the desired result.)

Now we define two sets $\widetilde{C}_1'$ and $\widetilde{C}_2'$ as follows:

$$\widetilde{C}_1' := \widetilde{C}_1 \cup \{x \in \widetilde{B}(0,N) : W_x[\widetilde{H}_{\widetilde{C}_1} = \infty] < \epsilon\}; \quad \widetilde{C}_2' := \widetilde{C}_2 \cup \{x \in \widetilde{B}(0,N) : W_x[\widetilde{H}_{\widetilde{C}_2} = \infty] < \epsilon\}.$$

By (4.8), we have $\widetilde{C}_1' \cup \widetilde{C}_2' = \widetilde{B}(0,N)$. According to Theorem 1.10 in Chapter 3 of Port and Stone (1978),

$$\begin{aligned}
\widetilde{\mathrm{cap}}(\widetilde{C}_1') &= \int_{x\in\partial\widetilde{B}(0,N)} W_x[\widetilde{H}_{\widetilde{C}_1'}<\infty]\widetilde{e}_{\widetilde{B}(0,N)}(dx)\\
&= \int_{x\in\partial\widetilde{B}(0,N)} \left(W_x[\widetilde{H}_{\widetilde{C}_1}<\infty]+W_x[\widetilde{H}_{\widetilde{C}_1'}<\infty,\widetilde{H}_{\widetilde{C}_1}=\infty]\right)\widetilde{e}_{\widetilde{B}(0,N)}(dx)\\
&\le \int_{x\in\partial\widetilde{B}(0,N)} \left(W_x[\widetilde{H}_{\widetilde{C}_1}<\infty]+W_x[\widetilde{H}_{\widetilde{C}_1'}<\infty]\sup_{y\in\widetilde{C}_1'} W_y[\widetilde{H}_{\widetilde{C}_1}=\infty]\right)\widetilde{e}_{\widetilde{B}(0,N)}(dx)\\
&\le \widetilde{\mathrm{cap}}(\widetilde{C}_1)+\epsilon\cdot\widetilde{\mathrm{cap}}(\widetilde{C}_1')\le tN^{d-2}+\epsilon\cdot\widetilde{\mathrm{cap}}(\widetilde{B}(0,N)).
\end{aligned}$$

The second equation follows from the fact that $\widetilde{C}_1\subset\widetilde{C}_1'$. The third step is a consequence of the strong Markov property. Finally, the last two steps follow from the construction of $\widetilde{C}_1'$ and the assumption $\widetilde{\mathrm{cap}}(\widetilde{C}_1)\le tN^{d-2}$. Similarly, we can show that

$$\widetilde{\mathrm{cap}}(\widetilde{C}_2')\le\widetilde{\mathrm{cap}}(\widetilde{C}_2)+\epsilon\cdot\widetilde{\mathrm{cap}}(\widetilde{B}(0,N)).$$

By choosing a sufficiently small ϵ, Equation (4.7) follows from the above two inequalities and the continuity of f as shown in Proposition 4.1. □

5. Proof of Theorems 1.1 and 1.2

In this section, we will prove Theorems 1.1 and 1.2. The lower bound of Theorem 1.1 can be derived from the definition of f because we can force the interlacements to stay within the blow-up of the minimizer of the constraint problem in (1.1), and the probability matches the large deviation rate. For the upper bound, we need to use the coarse-graining procedure introduced in Section 3 and enumerate all possible collections of type-II δ-good boxes. We will consider a quantity H defined in (5.2) which encapsulates the information of the local time in type-II δ-good boxes. The upper bound then follows from controls on H, particularly the Laplace transform in Lemma 2.4. Theorem 1.2 follows as a direct corollary of Theorem 1.1.

We begin with the lower bound of Theorem 1.1.

Proof of the lower bound in Theorem 1.1. Fix $\lambda>0$ and $\alpha>f(d\lambda)$. By Claim (4) in Proposition 4.1, we can choose a nice set A such that

$$\widetilde{\mathrm{cap}}(A)<d\lambda\quad\text{and}\quad\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)\backslash A)<\alpha.$$

Let $B=\widetilde{B}(0,1)\backslash A$. Recall that A_N and B_N stand for the blow-up of A and B, respectively. Then, $A_N\cup B_N\supset B(0,N)$. By Lemma 2.3, we have

$$\lim_{N\to\infty}\frac{1}{N^{d-2}}\mathrm{cap}(A_N)=\frac{1}{d}\widetilde{\mathrm{cap}}(A)<\lambda\quad\text{and}\quad\lim_{N\to\infty}\frac{1}{N^{d-2}}\mathrm{cap}(B_N)=\frac{1}{d}\widetilde{\mathrm{cap}}(B)<\frac{\alpha}{d}.$$

So,

$$\begin{aligned}
&\liminf_{N\to\infty}\frac{1}{N^{d-2}}\log\mathbb{P}[\mathrm{cap}(B(0,N)\cap\mathcal{I}^u)<\lambda N^{d-2}]\ge\liminf_{N\to\infty}\frac{1}{N^{d-2}}\log\mathbb{P}[\mathcal{I}^u\cap B(0,N)\subset A_N]\\
&\ge\liminf_{N\to\infty}\frac{1}{N^{d-2}}\log\mathbb{P}[\mathcal{I}^u\cap B_N=\varnothing]=\liminf_{N\to\infty}\frac{1}{N^{d-2}}\log[\exp(-u\cdot\mathrm{cap}(B_N))]>-\frac{u\alpha}{d}.
\end{aligned}$$

The third step follows from (2.4). This holds for arbitrary $\alpha > f(d\lambda)$, and thus

$$\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}[\mathrm{cap}(B(0,N)\cap \mathcal{I}^u) < \lambda N^{d-2}] \geq -\frac{u}{d} f(d\lambda).$$

This verifies the lower bound of Theorem 1.1. □

We now turn to the upper bound. Fix $0<\delta<1$. Pick a sufficiently large N (and recall (2.1) for its relation to L), K and some ρ such that

$$K > \max\{100, c_1(\delta), c_3(\delta)\}, \quad L > \max\{100^d, c_2(\delta), c_4(\delta,K)\}, \quad \text{and} \quad 0<\rho<c_5(\delta,K). \tag{5.1}$$

Let $\mathcal{A}'$ denote the event that there are at most $\rho(2N/(2K+1)/L)^d$ δ-bad boxes in $B(0,N)$. Then, by applying Proposition 3.1 with $\rho=\rho(2/(2K+1))^d$, the event $\mathcal{A}'$ happens with overwhelmingly high probability. We write

$$\mathcal{D} = \left\{\mathrm{cap}(\mathcal{I}^u \cap B(0,N)) < \lambda N^{d-2}\right\}.$$

We now define a quantity that is abnormally small under the event $\mathcal{A}'\cap\mathcal{D}$. Let C_1 denote the union of Type-I δ-good boxes, and C_2 denote the union of Type-II δ-good boxes (recall (3.1) and (3.2) for the definition). We define H as

$$H = \langle L^u, e_{C_2}\rangle. \tag{5.2}$$

Lemma 5.1. *On the event $\mathcal{A}'\cap\mathcal{D}$, we have*

$$\mathrm{cap}(C_2) \geq \frac{1-\delta}{d}\left(f\left(\frac{d\lambda}{1-2\delta}\right)-\delta\right)N^{d-2} \quad \textit{and} \quad H < 2u\delta\cdot \mathrm{cap}(C_2). \tag{5.3}$$

Proof. Suppose that the events $\mathcal{A}'$ and $\mathcal{D}$ both happen. Define a set S as

$$S = \cup_{B_x\in C_1}(\mathcal{I}^u\cap B_x).$$

Then, we have

$$\begin{aligned}
\mathrm{cap}(C_1) &\overset{*}{=} \sum_{x\in\partial_i B(0,N)} e_{B(0,N)}(x) P_x[H_{C_1}<\infty] \\
&\leq \sum_{x\in\partial_i B(0,N)} e_{B(0,N)}(x) \frac{P_x[H_S<\infty]}{\min_{B_y\in C_1, z\in\partial_i B_y} P_z[H_S<\infty]} \quad \text{(by strong Markov property)} \\
&\overset{(3.1)}{\leq} \sum_{x\in\partial_i B(0,N)} e_{B(0,N)}(x)\frac{1}{1-\delta}P_x[H_S<\infty] = \frac{1}{1-\delta}\mathrm{cap}(S) \overset{**}{\leq} \frac{\lambda}{1-\delta}N^{d-2}.
\end{aligned}$$

where we use Proposition 5.5 in Drewitz, Ráth and Sapozhnikov (2014) for the equality marked with $*$ and the definition of the event $\mathcal{D}$ for the inequality marked with $**$. We denote $\widetilde{C_1}$ and $\widetilde{C_2}$ as $\mathbb{R}^d$-fillings of C_1 and C_2, respectively. By Lemma 2.2 and the fact that $L>c_2(\delta)$, $K>c_3(\delta)$, we have

$$\widetilde{\mathrm{cap}}(\widetilde{C_1}) \leq \frac{d}{1-\delta}\mathrm{cap}(C_1) \leq \frac{d\lambda}{1-2\delta}N^{d-2}.$$

By $\mathcal{A}'$, we have that $\mathrm{Card}(C_1)+\mathrm{Card}(C_2)\geq(1-\rho)\big(\frac{2N}{(2K+1)L}\big)^d$. Hence, by Proposition 4.3 and the fact that $L>c_4(\delta,K),\rho<c_5(\delta,K)$, we have

$$\widetilde{\mathrm{cap}}(\widetilde{C}_2)\geq\left(f\left(\frac{d\lambda}{1-2\delta}\right)-\delta\right)N^{d-2}.$$

Thus, by Lemma 2.2 again,

$$\mathrm{cap}(C_2)\geq\frac{1-\delta}{d}\widetilde{\mathrm{cap}}(\widetilde{C}_2)\geq\frac{1-\delta}{d}\left(f\left(\frac{d\lambda}{1-2\delta}\right)-\delta\right)N^{d-2}. \tag{5.4}$$

We now turn to the upper bound of H. By Lemma 2.1 and $K\geq c_1(\delta)$, we have (here we let B denote the unique L-box that contains x)

$$\begin{aligned}H &= \sum_{x\in C_2}L^u(x)e_{C_2}(x)\leq\sum_{x\in C_2}L^u(x)(1+\delta)\bar{e}_B(x)e_{C_2}(B)\\ &= (1+\delta)\sum_{B\in C_2}e_{C_2}(B)\sum_{x\in B}L^u(x)\bar{e}_B(x)\\ &\overset{(3.2)}{<} (1+\delta)\sum_{B\in C_2}e_{C_2}(B)\delta u\leq 2\delta u\cdot\mathrm{cap}(C_2).\end{aligned}$$

This finishes the proof. □

We now prove the upper bound of Theorem 1.1 and thus complete its proof.

Proof of the upper bound in Theorem 1.1. By Lemma 2.4, for each choice of C_2,

$$\mathbb{P}[H<2\delta u\mathrm{cap}(C_2)]\leq e^{2u\sqrt{\delta}\mathrm{cap}(C_2)}\mathbb{E}\left[e^{-\frac{1}{\sqrt{\delta}}H}\right]=\exp\left(u\left[2\sqrt{\delta}-\frac{1}{1+\sqrt{\delta}}\right]\mathrm{cap}(C_2)\right). \tag{5.5}$$

This together with (5.3) gives

$$\begin{aligned}\mathbb{P}[\mathcal{A}'\cap\mathcal{D}] &\leq \sum_E\mathbb{P}\left[\mathcal{A}'\cap\mathcal{D}\cap\{C_2=E\}\right]\\ &\leq 2^{\left(\frac{2N}{(2K+1)L}\right)^d}\exp\left(u\left[2\sqrt{\delta}-\frac{1}{1+\sqrt{\delta}}\right]\frac{1-\delta}{d}\left(f\left(\frac{d\lambda}{1-2\delta}\right)-\delta\right)N^{d-2}\right).\end{aligned}$$

where in the first inequality, we sum over all possible choices of C_2. We know from (2.1) that $(N/L)^d=o(N^{d-2})$. This, combined with Proposition 3.1, shows that

$$\limsup_{N\to\infty}\frac{1}{N^{d-2}}\log\mathbb{P}[\mathcal{D}]\leq u\left[2\sqrt{\delta}-\frac{1}{1+\sqrt{\delta}}\right]\frac{1-\delta}{d}\left(f\left(\frac{d\lambda}{1-2\delta}\right)-\delta\right).$$

Let δ tend to zero. By Claim (4) in Proposition 4.1, i.e., the continuity of f, we have

$$\limsup_{N\to\infty}\frac{1}{N^{d-2}}\log\mathbb{P}[\mathcal{D}]\leq-\frac{u}{d}f(d\lambda).$$

Together with the lower bound, this completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We first prove (1.3). Suppose that $u_1 \geq u_2$. By using (2.4) and Lemma 2.2, we have

$$\begin{aligned}&\liminf_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] \\ &\geq \liminf_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_2 \cap B(0,N) = \varnothing\right] = -\frac{u_2}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)).\end{aligned}$$

Using the observation (1.6), we have

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] = \limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{E}\left[e^{-u_1 \mathrm{cap}(\mathcal{I}_2 \cap B(0,N))}\right] \\ &\leq \sup_{\lambda\geq 0} \limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{E}\left[e^{-u_1 \lambda N^{d-2}} \cdot \mathbb{1}\{\mathrm{cap}(\mathcal{I}_2 \cap B(0,N)) < \lambda N^{d-2}\}\right] \\ &= \sup_{\lambda\geq 0}\left[-u_1\lambda - \frac{u_2}{d} f(d\lambda)\right] = -\frac{u_2}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)).\end{aligned}$$

In the second line, we decompose the possible values of $\mathrm{cap}(\mathcal{I}_2 \cap B(0,N))$ into intervals of width of order N^{d-2} and apply Theorem 1.1 and the continuity of f (as shown in Claim (4) of Proposition 4.1). The last equation is by Claim (3) in Proposition 4.1. By combining the above two inequalities, we prove (1.3).

Now we suppose that $u_1 > u_2$. Fix $\epsilon > 0$. Equation (1.3) follows from the following inequality:

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing, \mathrm{cap}(B(0,N)\cap \mathcal{I}_1) \leq (1-\epsilon)\mathrm{cap}(B(0,N))\right] \\ &\overset{(1.6)}{\leq} \limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{E}\left[e^{-u_2 \mathrm{cap}(\mathcal{I}_1 \cap B(0,N))} \cdot \mathbb{1}\{\mathrm{cap}(\mathcal{I}_1 \cap B(0,N)) \leq (1-\epsilon)\mathrm{cap}(B(0,N)\}\right] \\ &\leq \sup_{0\leq\lambda\leq\frac{1-\epsilon}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))} \limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{E}\left[e^{-u_2 \lambda N^{d-2}} \cdot \mathbb{1}\{\mathrm{cap}(\mathcal{I}_1 \cap B(0,N)) < \lambda N^{d-2}\}\right] \\ &= \sup_{0\leq\lambda\leq\frac{1-\epsilon}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))} \left[-u_2\lambda - \frac{u_1}{d} f(d\lambda)\right] \overset{*}{<} -\frac{u_2}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)).\end{aligned} \tag{5.6}$$

In the third line, we decompose the possible values of $\mathrm{cap}(\mathcal{I}_1 \cap B(0,N))$ into intervals of width of order N^{d-2} and apply Theorem 1.1, Lemma 2.3 and the continuity of f. The inequality marked with $*$ is derived from Claim (3) in Proposition 4.1 and the fact that $u_1 > u_2$. By combining this with (1.3), we establish (1.4).

To prove (1.5), it is sufficient to show that

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing, \mathrm{cap}(B(0,N)\cap \mathcal{I}_2) \leq \epsilon\,\mathrm{cap}(B(0,N))\right] \\ &\overset{*}{<} -\frac{u_2}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)).\end{aligned} \tag{5.7}$$

We can prove it in a similar fashion as the previous two inequalities, and hence we omit it here. □

Remark 5.2. We now discuss the case where $u_1 = u_2$. The proof of Claim (1) in Theorem 1.2 still holds in this case. However, Claim (2) cannot be proved in the same fashion since the strict inequalities marked with $*$ in (5.6) and (5.7) no longer hold. We expect that when the intensities are the same, conditional on the non-intersection event, one of the interlacements will be sparse in the box while the other behaves as usual (and by symmetry, each of them has a half chance to be sparse). However, we are uncertain if such sparsity also encompass that the capacity of the interlacement set in the box will be significantly smaller than that of the box, as a very sparse set in terms of volume can still have a capacity comparable to the box. Nonetheless, we conjecture that the sparse interlacement set will have a small capacity in the interior of the box. Hence, we propose the following conjecture:

Conjecture 5.3. For $u_1 = u_2$ and any $\epsilon, \delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left[\mathrm{cap}(\mathcal{I}_1 \cap B(0,(1-\delta)N)) < \epsilon N^{d-2} \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] = \frac{1}{2}.$$

In the next section, we will discuss an alternative approach to the large deviation bound which is capable of dealing with the case of $u_1 = u_2$ in establishing the entropic repulsion result. In this approach, we can completely avoid the issue discussed above by characterizing sparsity in terms of occupancy instead of capacity.

6. An alternative approach and entropic repulsion regarding occupancy

In this section, we outline an alternative approach to proving (1.3) in Theorem 1.2. This approach is then applied to establish an entropic repulsion result regarding occupancy when $u_1 = u_2$. Finally, we mention its application to the case where $u_1 > u_2$.

Recall that we consider two independent interlacements $\mathcal{I}_1$ and $\mathcal{I}_2$ with intensities u_1 and u_2 respectively and that for $x \in \mathbb{Z}^d$, we let L_x^1 (resp. L_x^2) denote the local time of interlacements $\mathcal{I}_1$ (resp. $\mathcal{I}_2$) at the vertex x.

We start with the upper bound of (1.3) in Theorem 1.2. Note that the lower bound is completely straightforward, as shown in Section 5.

An alternative proof to the upper bound of (1.3). For an integer N, we choose L as defined in (2.1). We also select $\delta \in (0, 0.1)$ as a constant that governs all the errors and approaches zero towards the end, and a large constant $K = K(\delta)$. We require the following conditions to hold:

$$K \geq \max\{100, c_1(\delta), c_3(\delta)\}, L \geq \max\{100^d, c_2(\delta)\}.$$

We will use the definitions of boxes B_x and D_x as defined in Section 2.

The proof still follows from a coarse-graining procedure, but we will use different definitions of δ-good boxes. We will call an L-box B_x:

1. *Type-I δ-good* if $\langle L^1, \bar{e}_{B_x}\rangle < \delta u_1$,
2. *Type-II δ-good* if $\langle L^2, \bar{e}_{B_x}\rangle < \delta u_2$,
3. *Type-III good* if $\mathcal{I}_1 \cap \mathcal{I}_2 \cap B_x \neq \varnothing$,

and *δ-bad* otherwise. Note that these three conditions may not be mutually exclusive. For $\rho > 0$, we define the event $\overline{\mathcal{A}}$ as

$$\overline{\mathcal{A}} := \{\text{there are at most } \rho(N/L)^d \ \delta\text{-bad boxes in } B(0,N)\}.$$

We can establish the following lemma, which serves as an analogue to Proposition 3.1:

Lemma 6.1. *For all $\delta,\rho > 0$ and $K \geq 100$, we have:*

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\overline{\mathcal{A}}^c\right] = -\infty. \tag{6.1}$$

We can apply the excursion coupling technique independently to both interlacements, as described in Section 3. The remainder of the proof follows the same structure as that of Proposition 3.1. Note that an analogous result to (3.13) is easier in this case thanks to the presence of cL^{d-2} excursions instead of just one. Therefore, we omit the proof here.

We now proceed with the proof of the upper bound. Let

$$\mathcal{E} := \{\mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\}.$$

We define C_3 as the union of Type-I δ-good boxes and C_4 as the union of Type-II δ-good boxes. We also define H_1 and H_2 as follows:

$$H_1 = \langle L^1, \bar{e}_{C_3}\rangle \quad \text{and} \quad H_2 = \langle L^2, \bar{e}_{C_4}\rangle.$$

On the event $\overline{\mathcal{A}} \cap \mathcal{E}$, similar to Lemma 5.1, we can show that

$$H_1 < 2u_1\delta \cdot \mathrm{cap}(C_3) \quad \text{and} \quad H_2 < 2u_2\delta \cdot \mathrm{cap}(C_4). \tag{6.2}$$

On the event $\overline{\mathcal{A}} \cap \mathcal{E}$, where there are no Type-III good boxes, we have

$$\mathrm{Card}(C_3) + \mathrm{Card}(C_4) \geq \left(\frac{2N}{(2K+1)L}\right)^d - \rho\left(\frac{N}{L}\right)^d.$$

Using this fact and similar arguments to Proposition 4.3, we can show that for $L > C(\delta,K)$ and $\rho < c(\delta,K)$

$$\widetilde{\mathrm{cap}}(\widetilde{C}_3) + \widetilde{\mathrm{cap}}(\widetilde{C}_4) \geq \inf_{t\geq 0}\left(t + f(t) - \delta\right) N^{d-2}. \tag{6.3}$$

The infimum form is obtained by decomposing the choice of t into intervals of width of order 1 and applying the continuity of f in Claim (4) of Proposition 4.1. Combining this with Claim (3) in Proposition 4.1 and Lemma 2.3, we can obtain:

$$\mathrm{cap}(C_3) + \mathrm{cap}(C_4) \geq (1 - o_\delta(1))\frac{\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))}{d} N^{d-2}. \tag{6.4}$$

With (6.2) and (6.4) at hand, we can prove the upper bound in Claim (1) of Theorem 1.2 in a similar manner to the proof of the upper bound in Theorem 1.1. Similar to (5.5), we know that for $i = 1,2$

$$\mathbb{P}[H_i < 2\delta u_i \mathrm{cap}(C_{i+2})] \leq \exp\Big(- (u_1 + o_\delta(1))\mathrm{cap}(C_{i+2})\Big).$$

Therefore, we have

$$\begin{aligned}\mathbb{P}[\overline{\mathcal{A}} \cap \mathcal{E}] &\leq \sum_{E_1,E_2} \mathbb{P}[\overline{\mathcal{A}} \cap \mathcal{E} \cap \{C_3 = E_1\} \cap \{C_4 = E_2\}] \\ &\leq \exp\Big(-(u_1 + o_\delta(1))\mathrm{cap}(C_3) - (u_2 + o_\delta(1))\mathrm{cap}(C_4) + o_N(N^{d-2})\Big).\end{aligned}$$

Here, we sum over all possible choices of C_3 and C_4. In the second line, we used the independence of $\mathcal{I}_1$ and $\mathcal{I}_2$. Combining this inequality with (6.4), Lemma 6.1, and then sending δ to zero, we conclude that:

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}[\mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing] \le -\frac{\min\{u_1,u_2\}}{d} \widetilde{\mathrm{cap}}(\widetilde{B}(0,1)).$$

This finishes the proof. □

We now explain the application of this approach to the case where $u_1 = u_2$. As explained in Section 1, in this case we can show that each of the interlacement set will be sparse in the box in terms of occupancy with probability tending to $1/2$, conditional on the non-intersection event.

We start with the notation. Write

$$\overline{L}^1 = \frac{1}{(2N+1)^d} \sum_{x\in B(0,N)} \mathbb{1}\{L_x^1 > 0\} \quad \text{and} \quad \overline{L}^2 = \frac{1}{(2N+1)^d} \sum_{x\in B(0,N)} \mathbb{1}\{L_x^2 > 0\} \tag{6.5}$$

for the average "*occupancy*" of I_1 and I_2 in the box $B(0,N)$ resp. Then, we have:

Theorem 6.2. *For $u_1 = u_2 = u$, there exists $\epsilon_0 = \epsilon_0(u) > 0$ such that for any $0 < \epsilon < \epsilon_0$*

$$\lim_{N\to\infty} \mathbb{P}\left[\overline{L}^i < \epsilon \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] = \frac{1}{2}, \qquad i = 1,2. \tag{6.6}$$

Proof. We need to first modify the condition of δ-good boxes. Instead, we call an L-box:

1. *Type-I δ-good* if $\langle L^1, \bar{e}_{B_x}\rangle < \delta u$ and $\frac{1}{L^d}\sum_{y\in B_x} \mathbb{1}\{L_y^1 > 0\} < \delta$,
2. *Type-II δ-good* if $\langle L^2, \bar{e}_{B_x}\rangle < \delta u$ and $\frac{1}{L^d}\sum_{y\in B_x} \mathbb{1}\{L_y^2 > 0\} < \delta$,
3. *Type-III good* if $\mathcal{I}_1 \cap \mathcal{I}_2 \cap B_x \ne \varnothing$,

We refer to the previous proof for the choice of L, δ, K, ρ and the event $\overline{\mathcal{A}}$. Similarly to Proposition 3.1, we can prove that for $\delta,\rho > 0$ and $K \ge 100$

$$\lim_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\overline{\mathcal{A}}^c\right] = -\infty.$$

We will now use this new definition of good boxes and coarse-graining procedure to show that for any $\epsilon > 0$

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}\left[\overline{L}^1 \ge \epsilon, \overline{L}^2 \ge \epsilon, \mathcal{E}\right] < -\frac{u}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) \tag{6.7}$$

(recall that $\mathcal{E} = \{\mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\}$). This together with (1.3) implies that conditional on the event $\mathcal{E}$, the event inside the bracket will not happen with asymptotically probability one. Fix $\epsilon > 0$. Let C be the union of all L-boxes contained in $B(0,N)$. We define the event $\mathcal{E}'$ as follows:

$$\mathcal{E}' := \mathcal{E}_1' \cap \mathcal{E}_2' \cap \mathcal{E}, \quad \text{where } \mathcal{E}_i' := \left\{\frac{1}{(2N+1)^d} \sum_{x\in C} \mathbb{1}\{L_x^i > 0\} \ge \frac{\epsilon}{(2K+1)^d},\right\}, i = 1,2.$$

We now claim that

$$\limsup_{N\to\infty} \frac{1}{N^{d-2}} \log \mathbb{P}[\mathcal{E}'] < -\frac{u}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)). \tag{6.8}$$

Equation (6.7) directly follows from this estimate, as we can consider $(2K+1)^d$ copies of the event $\mathcal{E}'$, where the center of L-boxes are shifted by v for $v \in L\mathbb{Z}^d \cap [0,2KL]^d$. Similar estimates can be established for all these copies. The event in (6.7) implies that at least one of these events occurs, and therefore (6.7) is a direct consequence of (6.8).

We now prove (6.8). We define C_3 as the union of Type-I δ-good boxes and C_4 as the union of Type-II δ-good boxes. Then, similar to the previous proof we can show that

$$\mathbb{P}[\overline{\mathcal{A}} \cap \mathcal{E}'] \le \exp\Big(-(u+o_\delta(1))\big(\mathrm{cap}(C_3)+\mathrm{cap}(C_4)\big) + o_N(N^{d-2})\Big). \tag{6.9}$$

Next, we will provide a lower bound of $\mathrm{cap}(C_3)+\mathrm{cap}(C_4)$. We assume that the event $\overline{\mathcal{A}} \cap \mathcal{E}'$ happens. On the event $\overline{\mathcal{A}} \cap \mathcal{E}$, where there are no Type-III good boxes, we have

$$\mathrm{Card}(C_3)+\mathrm{Card}(C_4) \ge \left(\frac{2N}{(2K+1)L}\right)^d - \rho\left(\frac{N}{L}\right)^d.$$

On the event $\mathcal{E}'$, we know that for $i = 1,2$,

$$\begin{aligned}\frac{\epsilon}{(2K+1)^d} &\le \frac{1}{(2N+1)^d}\sum_{x\in C}\mathbb{1}\{L_x^i>0\} = \frac{1}{(2N+1)^d}\left[\sum_{x\in C_{i+2}}\mathbb{1}\{L_x^i>0\} + \sum_{x\in C\setminus C_{i+2}}\mathbb{1}\{L_x^i>0\}\right]\\ &\le \frac{1}{(2N+1)^d}\left[\delta L^d\mathrm{Card}(C_{i+2}) + (2L)^d(\mathrm{Card}(C)-\mathrm{Card}(C_{i+2}))\right],\end{aligned}$$

and thus

$$\mathrm{Card}(C_{i+2}) \le \frac{\mathrm{Card}(C) - \frac{\epsilon(2N+1)^d}{((4K+2)L)^d}}{1-\frac{\delta}{2^d}} \le \left(1+\frac{2\delta}{2^d}\right)\left(\mathrm{Card}(C) - \frac{\epsilon(2N+1)^d}{((4K+2)L)^d}\right).$$

Therefore, we can choose $\delta < c(\epsilon,K)$ and $\rho < c(\epsilon,K)$ such that

$$\mathrm{Card}(C_3) \ge c(\epsilon,K)(N/L)^d \quad \text{and} \quad \mathrm{Card}(C_4) \ge c(\epsilon,K)(N/L)^d,$$

and thus

$$\widetilde{\mathrm{cap}}(\widetilde{C}_3) \ge c'(\epsilon,K)N^{d-2} \quad \text{and} \quad \widetilde{\mathrm{cap}}(\widetilde{C}_4) \ge c'(\epsilon,K)N^{d-2}.$$

With these two inequalities, we can establish the following inequality similar to (6.3): for $L > C(\delta,K)$ and $\rho < c(\delta,K)$

$$\widetilde{\mathrm{cap}}(\widetilde{C}_3)+\widetilde{\mathrm{cap}}(\widetilde{C}_4) \ge \inf_{t\ge c'}\big(t+\max\{f(t),c'\}-\delta\big)N^{d-2} \ge (\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))+c'')N^{d-2}.$$

Note that c' does not depend on δ, so we can choose a small δ such that the last inequality holds. Combining this with Lemma 2.3 and (6.4), we complete the proof of (6.8) and thus (6.7).

Next, we will show that there exists some $\epsilon_0 = \epsilon_0(u) > 0$ such that for any $\epsilon < \epsilon_0$, we have

$$\limsup_{N\to\infty}\frac{1}{N^{d-2}}\log\mathbb{P}\left[\overline{L}^1 < \epsilon, \overline{L}^2 < \epsilon\right] < -\frac{u}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1)) \tag{6.10}$$

This result can be derived in a similar fashion as (6.7) via coarse-graining (with some modification on the definition of good boxes). In this case, as ϵ tends to zero, the left-hand side decreases to $-\frac{2u}{d}\widetilde{\mathrm{cap}}(\widetilde{B}(0,1))$.

Now, by combining (6.7) and (6.10), we can complete the proof of Theorem 6.2. We consider $\epsilon < \epsilon_0$. Conditional on the event $\mathcal{E}$, the following four events happen disjointly:

$$\{\overline{L}^1 \geq \epsilon, \overline{L}^2 \geq \epsilon\}; \quad \{\overline{L}^1 \geq \epsilon, \overline{L}^2 < \epsilon\}; \quad \{\overline{L}^1 < \epsilon, \overline{L}^2 \geq \epsilon\}; \quad \{\overline{L}^1 < \epsilon, \overline{L}^2 < \epsilon\}.$$

Based on (6.7) and (6.10), we observe that the first and fourth events occur with asymptotically zero probability, whereas the second and third events are symmetric. As a result, each of these events happens with asymptotically probability $1/2$, which implies Theorem 6.2. □

We end this section with several remarks:

Remark 6.3.

1. We expect that the optimal value of ϵ_0 is

$$\epsilon_0(u) = 1 - \exp(-u/g(0,0)) = \mathbb{E}[\mathbb{1}\{L_x^1 > 0\}].$$

2. One can also establish a more general entropic repulsion result by replacing the indicator function $\mathbb{1}\{L_x^i > 0\}$ in the definition of the average occupancy in (6.5) by $f(L_x^i)$, where $f : [0,\infty) \to [0,\infty)$ is any bounded function that remains uniformly positive when away from zero. An example is $f(x) = x \wedge 10$.
3. In the case where $u_1 > u_2$, we can prove similarly that for any $\epsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left[\overline{L}^2 < \epsilon \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] = 1. \tag{6.11}$$

In this case, we also expect a global characterization of local times under this conditioning. Define the local time profile $\mathscr{L}_N^1, \mathscr{L}_N^2$ by

$$\mathscr{L}_N^1 = \frac{1}{(2N+1)^d} \sum_{x\in\mathbb{Z}^d} L_x^1 \delta_{\frac{x}{N}}; \qquad \mathscr{L}_N^2 = \frac{1}{(2N+1)^d} \sum_{x\in\mathbb{Z}^d} L_x^2 \delta_{\frac{x}{N}}.$$

For $u_1 > u_2 > 0$ and any $R \geq 1$, we conjecture that

$$\begin{aligned} &\lim_{N\to\infty} \mathbb{E}\left[d_R\left(\mathscr{L}_N^1, u_1 dx\right) \wedge 1 \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] = 0, \text{ and} \\ &\lim_{N\to\infty} \mathbb{E}\left[d_R\left(\mathscr{L}_N^2, u_2 \widetilde{g}_{\widetilde{B}(0,1)}(x)^2 dx\right) \wedge 1 \middle| \mathcal{I}_1 \cap \mathcal{I}_2 \cap B(0,N) = \varnothing\right] = 0. \end{aligned} \tag{6.12}$$

Here, d_R denotes the 1-Wasserstein distance restricted to the box $\widetilde{B}(0,R)$, and for $z \in \mathbb{R}^d$, the function $\widetilde{g}_{\widetilde{B}(0,1)}(z) := W_z[\widetilde{H}_{\widetilde{B}(0,1)} = \infty]$ denotes (the opposite of) the harmonic potential of $\widetilde{B}(0,1)$.

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