



Sharp Asymptotics for Arm Probabilities in Critical Planar Percolation

Hang Du¹, Yifan Gao², Xinyi Li³, Zijie Zhuang⁴

¹ Massachusetts Institute of Technology, Cambridge, USA. E-mail: hangdu@mit.edu

² City University of Hong Kong, Kowloon, Hong Kong. E-mail: yifangao@cityu.edu.hk

³ Peking University, Beijing, China. E-mail: xinyili@bicmr.pku.edu.cn

⁴ University of Pennsylvania, Philadelphia, USA. E-mail: zijie123@wharton.upenn.edu

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Abstract: In this work, we consider critical planar site percolation on the triangular lattice and derive sharp estimates on the asymptotics of the probability of half-plane j -arm events for $j \geq 1$ and planar (polychromatic) j -arm events for $j > 1$, building upon a recent, not yet peer-reviewed result of Binder and Richards (Convergence rates of random discrete model curves approaching sle curves in the scaling limit. Preprint, 2020). These estimates greatly improve previous results and in particular answer (a large part of) a question of Schramm (*ICM Proc.*, 2006).

1. Introduction

Percolation is without doubt one of the most studied statistical mechanics models in probability. As an ideal playground for the study of phase transitions and criticality, it has received considerable attention from probabilists and statistic physicists in the past 60+ years. Despite its simple setup, it is the source of many fascinating yet difficult mathematical problems, with some already well answered and many more still very far from being solved. Starting from the beginning of 21st century, there have been a lot of breakthroughs in the study of a particular case of this model, namely the critical planar percolation on the triangular lattice. In the ground-breaking work by Smirnov [31], it is shown that the both crossing probabilities and the exploration process have conformally invariant scaling limits that can be given through Cardy's formula and described as a (chordal) Schramm-Loewner evolution with parameter 6 (SLE₆) respectively. Later, in [6] Camia and Newman give the characterization of the full scaling limit, via the collection of non-self-crossing loops that is known now as the conformal loop ensemble (CLE). To keep the introduction concise, we refer readers to the classical book [12] and a recent survey [9] for more on the history and some recent progresses of this model.

Critical exponents are central notions in understanding the behavior of critical models. In the case of percolation, many of them can be derived from the so-called arm exponents, i.e. exponents in the power-law decay of the probability of arm events (as the mesh

size tends to 0), which we will introduce shortly. The value of various arm exponents have been calculated unrigorously in the physics literature a long time ago, see e.g. [1, 7, 8, 10, 27].

In mathematics, for critical planar percolation on the triangular lattice, Smirnov and Werner rigorously computed precise values of the half-plane arm exponents and (polychromatic) j -arm exponents with $j > 1$ in [32], using Kesten's scaling relation from [16], Cardy's formula as well as relations between interfaces and SLE_6 . Roughly at the same time, Lawler, Schramm and Werner obtained the one-arm exponent in the plane in [21], also via SLE-related calculations. We will briefly recall their results in Claim (3) of Lemma 2.6 and (1.10) respectively.

However, these asymptotics (with an $o(1)$ in the exponent) are not completely satisfactory. One particular reason is that in dealing with scaling limits involving microscopic quantities from critical planar percolation, such as the pivotal, cluster or interface measures from [11] (see in particular Theorems 4.3, 5.1 and 5.5 of the said reference) and the natural parametrization of the interface (see Theorem 1.4 in [13]), instead of renormalizing with an explicit factor η^α where η is the mesh size and α the corresponding exponent, one normalizes by something implicit, namely the asymptotic arm probability (which corresponds to a_j in this paper); see also Remark 4.10 in [11] for more on this issue.

Thus, it is natural to ask if there are precise estimates of the arm events, just as what have been obtained for other critical models, e.g., sharp asymptotics of the one-point function of the loop-erased random walk (LERW); see [14, 19, 22] for the two-, three- and four-dimensional cases respectively.

In fact, much before the scaling limit results mentioned in the paragraph above were obtained, Schramm already asked the question in the proceedings of ICM 2006 if it is possible to improve the estimates. In Problem 3.1 of [29], he pointed out that "it would be especially nice to obtain estimates that are sharp up to multiplicative constants".

In a few special cases (where the exponent is an integer, namely the half-plane 2- and 3-arm exponents and planar 5-arm exponent), both the value and up-to-constants estimates can be obtained without relating to SLE's; see Lemma 5 of [17], Theorem 24 of [24] or the first exercise sheet of [35]. Improvements for other arm asymptotics are much more difficult. Mendelson, Nachmias and Watson (see [23]) and Binder, Chayes and Lei (see [4]) obtained power-law convergence rate for the Cardy's formula which yields a slight improvement of half-plane one-arm asymptotics (see (1.2) below), although still not as strong as up-to-constants estimates.

In this paper, we answer a large part (except the planar one-arm case; see Remark 1.6 for more detail) of Schramm's question, building upon a recent result of Binder and Richards [5] (not yet peer-reviewed, only available as the PhD thesis of Richards [26]) which proved power-law convergence rate of the exploration path of critical planar percolation (which also answered a related question by Schramm; see the beginning of Sect. 1.2 for more discussions). More precisely, we derive sharp estimates for the probability of half-plane arm events and planar j -arm events with $j > 1$, with power-law error bounds in most cases, which gives much more than what he asked for.

1.1. Main results. We start with necessary notation. Let \mathbb{T} denote the triangular lattice where each face is an equilateral triangle and \mathbb{T}^* denote the dual graph. We consider the critical Bernoulli percolation on \mathbb{T}^* in which each hexagon is colored red (=open) or blue (=closed) independently with equal probability. For $j \geq 1$, let $\mathcal{B}_j(r, R)$ denote the half-plane j -arm event that there exist j disjoint crossings with alternating colors

of the semi-annulus $A^+(r, R)$ in the upper half-plane with inner and outer radii r and R respectively (see Sect. 2.4 for the precise definitions of arm events and Sect. 2.2 for the discretization of domains). We will also consider a variant of half-plane arm events, denoted by $\mathcal{H}_j(r, R)$, which corresponds to j -alternating arms from the segment $[-r, r]$ to the semi-circle C_R^+ within the half-disk B_R^+ . Write $b_j(r, R) := \mathbb{P}[\mathcal{B}_j(r, R)]$ and define h_j similarly. In the seminal work [32], it is showed that for any $j \geq 1$, the sequence $b_j(r, R)$ has a power-law decay in R with exponent

$$\beta_j = j(j+1)/6, \quad \text{such that as } R \rightarrow \infty, \quad b_j(r, R) = R^{-\beta_j + o(1)}. \quad (1.1)$$

In [23] the following improvement of b_1 is obtained:

$$b_1(1, n) = e^{O(\sqrt{\log \log n})} n^{-1/3} = (\log n)^{O(1/\sqrt{\log \log n})} n^{-1/3}. \quad (1.2)$$

We are now ready to state our main results for half-plane arm probabilities, which give sharp asymptotics with power-law error bounds for both b_j and h_j .

Theorem 1.1. *For any $j \geq 1$, there exist constants $c(j, r) > 0$ and $C_b(j, r)$, $C_h(j, r) \geq 0$ such that for any real $n > r$,*

$$b_j(r, n) = C_b n^{-\beta_j} \left(1 + O(n^{-c})\right); \quad h_j(r, n) = C_h n^{-\beta_j} \left(1 + O(n^{-c})\right). \quad (1.3)$$

Note that in the cases of $j = 1, 2, 3$, $C_b(j, 1)$, $C_h(j, 1) > 0$, hence it makes sense to pick $r = 1$, and our theorem applies to classical half-plane arm events out of one hexagon. The same asymptotics also hold for any fixed “inner initial configuration”, i.e., an outer face which we will define in Sect. 2.5. In this case, the constants depend on j and the initial configuration.

We now turn to (polychromatic) planar j -arm events. Again, see Sects. 2.2 and 2.4 for conventions and precise definitions. We start with the classical j -arm event $\mathcal{P}_j(r, R)$, which is defined as the event that there exist j disjoint crossings of the annulus $A(r, R)$ and not all of the same color (except in the case $j = 1$). We will also consider $\mathcal{A}_j(r, R)$, $\mathcal{X}_j(r, R)$, $\mathcal{Y}_j(r, R)$ and $\mathcal{Z}_j(r, R)$, which are variants satisfying the requirements of color sequences, some with additional location constraints and some with additional connection constraints; see (2.4) and (2.5) below for precise definitions.

Write $p_j(r, R) = \mathbb{P}[\mathcal{P}_j(r, R)]$ and define a_j , x_j , y_j and z_j similarly. In [32], it is showed that for any $j \geq 2$, the sequence $p_j(r, R)$ has a power-law decay in R with critical exponent

$$\alpha_j = (j^2 - 1)/12, \quad \text{such that as } R \rightarrow \infty, \quad p_j(r, R) = R^{-\alpha_j + o(1)}. \quad (1.4)$$

We now state our main results for planar arm probabilities which give sharp asymptotics with power-law bounds for variants x_j and y_j and as a consequence sharp asymptotics without explicit error bounds for a_j and up-to-constants estimates for p_j .

Theorem 1.2. *For any $j \geq 2$, there exist constants $c(j, r)$ and $C_x(j, r)$, $C_y(j, r) \geq 0$ such that for all real $n > r$,*

$$x_j(r, n) = C_x n^{-\alpha_j} \left(1 + O(n^{-c})\right); \quad y_j(r, n) = C_y n^{-\alpha_j} \left(1 + O(n^{-c})\right). \quad (1.5)$$

Note that when $j = 2, 3, 4, 5$, $C_x(j, 1), C_y(j, 1) > 0$, and also when $j = 6$, $C_x(6, 1) > 0$. Hence in these cases (note that special care is needed for $x_6(1, n)$; see Remark 4.9 for discussions) our results hold for arm events out of a single hexagon. As an immediate consequence of Theorem 1.2 and the up-to-constants asymptotic equivalence of different arm events (see Claim (2) of Lemma 2.6),

$$p_j(r, n) \asymp a_j(r, n) \asymp n^{-\alpha_j}. \quad (1.6)$$

Moreover, we are able to obtain sharper estimates (albeit without a power-law error bound) than (1.6) for the alternating arm probability $a_j(r, n)$. For simplicity, we only give them for arm events out of one hexagon for $j \leq 5$ (but similar asymptotics hold for arm events with other inner initial configurations, including the case of $a_6(1, n)$). See Remark 4.9 for more discussions.

Theorem 1.3. *For $j = 2, 3, 4, 5$, there exists some $C_a(j) > 0$ such that for all real $n > 1$,*

$$a_j(1, n) = C_a n^{-\alpha_j} (1 + o(1)). \quad (1.7)$$

A direct application of Theorem 1.3 for $j = 2, 4$ is that we can replace renormalizing factors in the scaling limit of pivotal and interface measures by precise powers of the mesh size in Theorems 4.3 and 5.5 of [11] and that in the natural parametrization in Theorem 1.4 of [13]. Another application is the improvement for the asymptotics of the correlation (or characteristic) length and sensitivity length (see (2) in [33] for definition), which is also a central object in the study of near-critical and dynamical percolation. In particular, our asymptotics on planar 4-arm events imply that various versions of the correlation length for planar critical percolation on triangular lattice are up-to-constants equivalent to $|p - 1/2|^{-4/3}$. For a more thorough account on the correlation length, see e.g., Section 7 of [24].

1.2. Comments. In this subsection, we briefly comment on the proof and discuss possible directions for generalization.

We start with the strategy of the proof. As Schramm has already pointed out immediately below his question in [29], the crux of the matter lies in “the passage between the discrete and continuous setting” and he posed a related question on obtaining “reasonable estimates for the speed of convergence of the discrete processes”.

This question was solved recently by Binder and Richards in [5], which constitutes the PhD thesis [26] of Richards; see also [3] for an extended abstract. More precisely, they verify that the framework developed by Viklund in [34] for the power-law convergence rate of random discrete models towards SLE indeed works for percolation. In particular, they obtain a power-law convergence rate for the exploration process of planar critical percolation up to some stopping time. See Sect. 2.8 for detailed discussions of their results.

We now explain how we derive sharp asymptotics for arm events out of the power-law convergence. For better illustration we will discuss the classical half-plane arm-events (i.e., Theorem 1.1 for b_j) in more detail and only briefly point out necessary modifications in other cases.

At first glance, sharp asymptotics for arm events should follow naturally once we relate the discrete exploration path to SLE_6 . However, a few obstacles prevent us from applying the power-law convergence rate directly.

The first one is that such results only give information down to the mesoscopic level (in terms of the mesh size), while in this work we endeavor to reach the microscopic level. To overcome this difficulty, we resort to coupling techniques developed for critical planar percolation, which allow us to “decouple” the microscopic “initial configurations” (if we figuratively regard arm events as the consequence of outward explorations) with the macroscopic boundary conditions. Inspired by the arguments of Theorem 1.2 in [22] for the one-point function of 3D LERW, the ideas above can be crystallized in the following estimates, from which Theorem 1.1 for b_j can be directly derived.

Denote $r_\star(j) = \min\{r \in \mathbb{N} : \star_j(r, n) > 0 \text{ for all sufficiently large } n\}$ for $\star \in \{b, h, p, a, x, y, z\}$. It is clear that when dealing with some arm probability $\star_j(r, n)$, it suffice to focus on the case $r \geq r_\star(j)$. Note that $r_b(j) \leq r_h(j)$ and $r_a(j) = r_x(j) \leq r_y(j) = r_z(j)$; in addition, $r_h(j) = 1$ for $j \leq 3$ and $r_y(j) = 1$ for $j \leq 5$.

Proposition 1.4. *Given $j \geq 1$, for any $r \geq r_b(j)$ and $m \in (1.1, 10)$,*

$$\frac{b_j(r, m^2n)}{b_j(r, mn)} = \frac{b_j(r, mn)}{b_j(r, n)} \left(1 + O(n^{-c})\right), \quad (1.8)$$

where $O(n^{-c})$ is independent of the choice of m .

The second obstacle is that the classical arm event \mathcal{B}_j is not the ideal choice to be described by the exploration path in a discretized domain. More precisely, working with \mathcal{B}_j directly involves dealing with a sequence of semi-annuli with shrinking inner radii. Although in principle one should be able to produce similar power-law convergence in these varying domains as a perturbation of the half-disk, a more sensible choice is to make use of couplings of percolation configurations conditioned on arm events (see the discussion on the proof of Proposition 1.4 below), and work with the variant \mathcal{H}_j instead; see Sects. 3.1 and 4.1 for more detail.

The third obstacle is that the result of Binder and Richards provides convergence rate only for the exploration up to some stopping time, not for the whole path (although it is believed to hold for the latter as well). To overcome this difficulty, we enlarge the domain in which we compare percolation explorations and SLE₆, so that the segment before the stopping time already suffices to provide comparison of arm probabilities; see Sect. 4.1 for more detail.

We now discuss the proof of Proposition 1.4 in more detail. The main ingredients are Propositions 3.4 and 3.5, in which we derive comparisons of conditioned arm probabilities, which allow us to compare arm probabilities between different boundary conditions and also to compare different types of arm events. These estimates are established through coupling results established in Propositions 3.1 and 3.3 for $j = 1$ and $j > 1$ respectively as well as the power-law convergence of the exploration process in the form of Proposition 4.1.

A crucial ingredient for these couplings is the separation lemma, which, first developed by Kesten in [16], plays a key role in establishing quasi-multiplicativity properties of arm probabilities. We will discuss it (and several variants) in detail in Sects. 2.6 and 2.7.

We now turn to the plane case. In a fashion similar to Theorem 1.1, Theorem 1.2 for y_j can be derived from the following estimates.

Proposition 1.5. *Given $j \geq 2$, for any $r \geq r_y(j)$ and $m \in (1.1, 10)$,*

$$\frac{y_j(r, m^2n)}{y_j(r, mn)} = \frac{y_j(r, mn)}{y_j(r, n)} \left(1 + O(n^{-c})\right), \quad (1.9)$$

where $O(n^{-c})$ is independent of the choice of m .

This proposition follows from the same argument as that of Proposition 1.4 as the particular definition of \mathcal{Y}_j allows us to relate it to the exploration process. Note that in order to overcome the third obstacle as in the half-plane case, we will also enlarge the domain accordingly; see Sect. 4.3 for more detail. The claim for x_j follows from a coupling result that relates y_j and x_j ; see Proposition 3.10.

Theorem 1.3, whose proof is inspired by Proposition 4.9 of [11], is a corollary of Theorem 1.2, Proposition 3.9 (which relates the conditional arm probabilities for \mathcal{X}_j and \mathcal{A}_j), and Claim (4) of Lemma 2.6 (which gives the convergence of macroscopic arm probabilities without a speed).

Remark 1.6. We now briefly mention some open questions and discuss possible generalizations.

(1) Planar 1-arm events are crucial objects to understand critical percolation clusters. In [21] it was shown that the 1-arm probabilities satisfy

$$p_1(r, R) = R^{-5/48 + o_R(1)} \quad \text{as } R \rightarrow \infty. \quad (1.10)$$

It is then a natural question to wonder if our asymptotics also hold in this case. In fact, the key difficulty in extending our arguments to the one-arm case lies in the fact that planar one-arm events cannot be described by a single chordal exploration path. Instead, they can be described either by the collection of all interface loops or by the so-called radial exploration process, which correspond to CLE_6 or the radial SLE_6 respectively in the scaling limit. While we believe power-law convergence analogous to Theorem 4.1.11 in [26] should also hold for these objects, solid arguments leading to such results still seem rather out of reach for the moment.

(2) Recently, in [25], the exact value of the monochromatic two-arm exponent has been calculated. We expect that the couplings of conditional monochromatic two-arm events can be proved similarly to the half-plane one-arm case; see Proposition 3.1 and [11, Proposition 5.2]. The monochromatic two-arm event can also be explored using a radial exploration process, and the main difficulty in extending this work to that case is the lack of power-law convergence of this process. For the monochromatic j -arm event with $j > 2$, the existence of their exponents was established in [2] using quasi-multiplicativity arguments, but their explicit values are not known yet. To establish sharp asymptotics for these events, additional difficulties arise in the coupling step which require some non-trivial ideas. Essentially, one needs to show that conditional on the monochromatic j -arm event, with positive probability these j arms will not be jammed together. Another missing input is the power-law convergence of the radial exploration process or full scaling limits.

(3) In fact, power-law convergence for the harmonic explorer (by Lan, Ma and Zhou [18] as well as Binder and Richards [5]) and FK-Ising model (by Binder and Richards [5]) has also been established. We wonder if it is possible to obtain sharp asymptotics also for the FK-Ising arm events and the one-point function of the harmonic explorer (as well as the convergence in natural parametrization to SLE_4). We plan to investigate them in future works.

Finally, we explain how this article is organized. In Sect. 2, we settle the setup and introduce various key notions and preliminary results, including arm events, faces, separation lemmas, and the power-law convergence rate of the exploration paths. We give, without proof, various coupling results concerning arm events in Sect. 3, which are crucial to the main arguments and derive from these couplings some relations on conditioned arm probabilities. Section 4 is dedicated to the proof of the main theorems in which we

also include the proofs of Propositions 1.4 and 1.5. In Sect. 5, we unfold the proofs for various coupling results postponed from Sect. 3. A few technical proofs for preliminary results in Sect. 2 are given in the appendices.

2. Notation and Preparatory Results

This section is dedicated to setup and preparatory results. In Sect. 2.1 we introduce overall notation. We then fix the setup and recall some basic tools for percolation in Sects. 2.2–2.3 resp. In Sect. 2.4, we give the definition for various arm events, state some existing results their its asymptotics and give a “functional equation” type result on sequences that will be useful for obtaining sharp asymptotics. In Sects. 2.6 and 2.7, we state and prove the separation lemmas in the half-plane and plane resp., which are crucial to the coupling argument in this work. Finally in Sect. 2.8, we recall the power-law convergence rate of the rescaled exploration path towards SLE₆ from [26] and prove a variant tailored for this work.

2.1. Notation and conventions. Let \mathbb{C} stand for the complex plane and $\mathbb{H} = \{x + iy : y > 0\}$ stand for the upper half-plane. For convenience of notation we regard them as \mathbb{R}^2 and a subset thereof and use both sets of notation interchangeably. Let $B(x, R) = \{z : |z - x| < R\}$ denote the ball of radius R around x and $C(x, R) = \partial B(x, R)$. For $0 < r < R$, we write $A(x, r, R) := \{z : r < |z - x| < R\}$ for the annulus of radii $r < R$ around x . We will omit “ x ” when $x = 0$, i.e., abbreviate $B(0, R)$, $C(0, R)$, $A(0, r, R)$ as B_R , C_R , $A(r, R)$. We will add $+$ in the superscript to indicate the quantities are defined in the upper half-plane \mathbb{H} . For example, if $A \subseteq \mathbb{C}$, write A^+ for the set $A \cap \mathbb{H}$. Then, $B^+(x, R)$, $C^+(x, R)$, $A^+(x, r, R)$, B_R^+ , C_R^+ , $A^+(r, R)$ are the corresponding subsets of \mathbb{H} . Without further specification, all the sequences we consider are indexed by real numbers, not only integers.

We write $a_n = n^{\alpha+o(1)}$ if for all $\epsilon > 0$, $n^{\alpha-\epsilon} \leq a_n \leq n^{\alpha+\epsilon}$ for all n large enough. We write $a_n = O(n^\alpha)$ if there exists $C > 0$ such that $|a_n| \leq Cn^\alpha$ for all n , and write $a_n \asymp b_n$ if $a_n, b_n > 0$ for all n large enough, and a_n/b_n is bounded both from above and below. We always write j for the number of arms and η for the scale of lattice. Without further specification, all the arcs are written counterclockwise. We write $d(x, y)$ for the Euclid distance of $x, y \in \mathbb{R}^2$. Given a real number x , we write $\lfloor x \rfloor := \max_{z \in \mathbb{Z}} \{z \leq x\}$ for the integer part of x .

Finally, let us explain our convention concerning constants. Constants like $\varepsilon, \delta, c, c'$, C or C' may change from line to line while those with a subscript like c_1 are kept fixed throughout the paper. All constants are universal except otherwise stated.

2.2. Setting for percolation. Let \mathbb{T} be the triangular lattice embedded in the complex plane \mathbb{C} where each face is an equilateral triangle of side length 1. More precisely, the set of sites (vertices) in \mathbb{T} is given by $T := \mathbb{Z} + e^{i\pi/3}\mathbb{Z}$, and two sites are neighbors if they are at distance 1.

We consider the critical site percolation on \mathbb{T} , defined through declaring each site $v \in T$ as open or closed equally with probability $\frac{1}{2}$, independently of the other sites. More precisely, let $\{0, 1\}^T$ be the sample space of configurations $(\omega_v)_{v \in T}$, where $\omega_v = 1$ if v is open, and $\omega_v = 0$ if v is closed. Write \mathbb{P} for the product measure with parameter $\frac{1}{2}$ on T . For better illustration, a site will be colored red (resp. blue) if it is open (resp.

closed). In studying scaling limits we will also be interested in percolation on the rescaled lattice. For $\eta > 0$, let $\eta\mathbb{T}$ be \mathbb{T} rescaled by η . With slight abuse of notation, we will also use \mathbb{P} to denote the product measure with parameter $\frac{1}{2}$ on $\eta\mathbb{T}$. However, we will fix $\eta = 1$ in the majority of this work and emphasize the mesh size only when we do rescale the lattice, namely, in Sects. 2.8, 4.1 and 4.3.

We also view the percolation on the triangular lattice \mathbb{T} as a random coloring of **hexagons** (i.e. faces) on \mathbb{T}^* , which is dual to \mathbb{T} such that each vertex (resp. edge) on \mathbb{T} corresponds to a hexagonal face (resp. an edge) on \mathbb{T}^* . We say two hexagons are **neighbors** (or adjacent) if they have a common edge. A **path** on \mathbb{T}^* is a sequence of neighboring hexagons and they are all distinct, and we call it a **loop** if the two endpoints of the path are also neighbors. We will also consider paths on \mathbb{T}^* as a graph, which we refer to as **b-paths** (“boundary paths”; we will mostly consider those as part of the boundary of a cluster or a domain) to distinguish them from paths of hexagons we have just defined. In the language of this work, arms (to be defined in the next paragraph) are paths while interfaces are b-paths. We say two interfaces are **adjacent** if there is a hexagon on \mathbb{T}^* touched by both of them.

Let ∂_1 and ∂_2 be two disjoint parts of the boundary of a hexagonal domain D (in many cases two opposite sides of a topological quadrangle or inner and outer boundaries of a topological annulus). A **crossing** from ∂_1 to ∂_2 in D is a monochromatic path of hexagons in $\eta\mathbb{T}^*$ such that the two endpoints intersect ∂_1 and ∂_2 respectively, and all other hexagons are inside D . A crossing will also be called an **arm** when we consider arm events in what follows.

We now turn to the issues related to discretization of domains. By default, we use the conventions in [6] and refer readers thereto for more details. However, to avoid inconsistency for the discretization of (half)-disks, (semi-)annuli and (semi-)circles that are ubiquitous in this work, we give a special rule for the discretization of these objects; see the last paragraph of this subsection for details. We also remark that as will be discussed in Remark 2.19, our results are indeed quite stable against different choices of discretization. Hence our convention here is in fact merely an inessential technical matter.

We start with the default convention. A set of hexagons E is called connected or simply connected if so is the set obtained by embedding E in \mathbb{C} . If E is a simply connected set of hexagons, we write ΔE for the outer boundary of E , i.e., the set of hexagons that are not in E but adjacent to hexagons in E , and we write ∂E for the topological boundary of E by viewing E as a domain in \mathbb{C} . The bounded simply connected set E of hexagons is called a **Jordan set** if ΔE is a loop.

Given a Jordan domain D in \mathbb{C} , we write $D_{[\eta]}$ for the η -approximation of D , i.e., the largest Jordan set of $\eta\mathbb{T}^*$ that is contained in D (if there is a tie, choose one arbitrarily). For $a \in \partial D$, we write a_η for the vertex of $\eta\mathbb{T}^*$ in $\partial D_{[\eta]}$ such that it is closest to a and it is on the common edge shared by two adjacent hexagons in $\Delta D_{[\eta]}$ (if there is a tie, choose one arbitrarily). When $\eta = 1$, we omit η or $[\eta]$ from the subscript unless we wish to emphasize that this is a discretized object.

We now turn to special discretization rules for points and intervals on the real axis, circles, disks and annuli centered at the origin. Note that if one applies the rules in the paragraph above to e.g., B_r^+ and $A(r, R)$ which are two touching objects in the continuum, the resulting hexagonal domains will have non-empty “cracks” in-between. Hence, we apply the following special rule for them.

We first designate, in an arbitrary way, the discrete circles C_R for each $R \geq 1$ as a loop of b-paths on \mathbb{T}^* that disconnects the origin from infinity which is at most $O(1)$ -

away from C_R , such that C_1 is the boundary of the hexagon containing the origin and there do not exist $R_1 < R_2$ such that C_{R_1} crosses C_{R_2} . With slight abuse of notation we still denote by C_R the discretized object; same for other objects in this paragraph. Then, for each $1 \leq r < R$ we regard B_R as the hexagonal domain encircled by C_R and $A(r, R)$ as $B_R \setminus B_r$ respectively. Similarly, we also designate for each $R \geq 1$ the discrete semi-circle C_R^+ as a b-path that disconnect 0_1 from infinity in $\mathbb{H}_{[1]}$ with the special rule that C_1^+ is the upper three edges of the hexagonal face in \mathbb{T}^* containing 0_1 . We then define half-disks B_R^+ and semi-annuli $A^+(r, R)$ in a similar fashion. With slight abuse of notation, we write $[-R, R]$ for the b-path that forms the part of the boundary of $\mathbb{H}_{[1]}$ encircled by C_R^+ and define other intervals accordingly. When working on the discretization with respect to the rescaled lattice $\eta\mathbb{T}$, we first counter-rescale to $\eta = 1$, apply the rules above and then rescale everything back to $\eta\mathbb{T}$.

2.3. Some classical tools. The aim of this subsection is to review some classical tools in percolation theory, namely, the Harris-FKG inequality, the BK-Reimer inequality and the Russo–Seymour–Welsh theory. These tools will be extensively used throughout this paper. We refer readers to [12] for more details.

Definition 2.1 (Monotone events). There is a natural partial order on the set of configurations $\{0, 1\}^T$ given by $\omega_1 \leq \omega_2$ if and only if $\omega_1(v) \leq \omega_2(v)$ for all $v \in T$. An event A is called increasing if $1_A(\omega_1) \leq 1_A(\omega_2)$ whenever $\omega_1(v) \leq \omega_2(v)$, where 1_A is the indicator function of A . An event A is called decreasing if its complement A^c is increasing.

Lemma 2.2 (Harris-FKG inequality). *If A and B are both increasing or decreasing events, then*

$$\mathbb{P}[A \cap B] \geq \mathbb{P}[A]\mathbb{P}[B].$$

Lemma 2.3 (BK-Reimer inequality). *Let $A \square B$ denotes the disjoint occurrence of the events A and B , meaning that $\omega \in A \square B$ if there exist disjoint sets of sites E and F (possibly depending on ω) with the property that one can verify that $\omega \in A$ (resp. $\omega \in B$) by looking at sites in E (resp. F) only. Then, we have*

$$\mathbb{P}[A \square B] \leq \mathbb{P}[A]\mathbb{P}[B].$$

Lemma 2.4 (Russo–Seymour–Welsh (RSW) Theorem). *For any topological quadrangle (i.e., a Jordan domain with four marked points on the boundary) D in \mathbb{C} with two opposite sides ∂_1 and ∂_2 , there exist constants $\eta_0 > 0$ and $c = c(D) > 0$ such that for all $\eta \leq \eta_0$, with probability at least c , there is an open crossing in $\eta\mathbb{T}^*$ from ∂_1 to ∂_2 in $D_{[\eta]}$ (which we call a quad crossing).*

The combination of the Harris-FKG inequality and RSW theorem allows us to “glue” arms of the same color to create paths that have macroscopic geometric restrictions with uniformly positive probability. E.g., a loop that disconnects the annulus $A(R, 2R)$ can be constructed through gluing of quad crossings. We will colloquially refer to these arguments as “RSW-FKG gluing”.

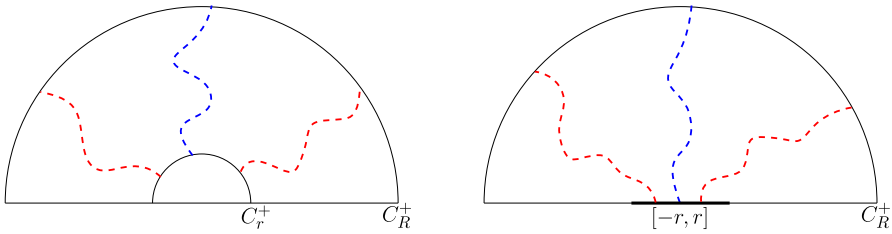


Fig. 1. Left: The event $\mathcal{B}_3(r, R)$. Right: The event $\mathcal{H}_3(r, R)$. The semi-annulus $A^+(r, R)$ and the line segment $[-r, r]$ (in bold black) should be viewed as b-paths discretized according to the convention given at the end of Sect. 2.2. Dashed red and blue curves represent arms (composed of hexagons) with respective colors

2.4. Arm events and asymptotics. In this subsection, we briefly review the definition of classical arm events in planar percolation and introduce variants that we will specifically use in this work. We then review existing asymptotics of the arm probabilities and finally lay out a “functional equation” lemma on sequences that is tailored for the derivation of sharp asymptotics in our work.

We first introduce two types of half-plane arm events. We fix $\eta = 1$ throughout these definitions. In the following, the symbols B_R^+ , C_R^+ , $A^+(r, R)$, $[-r, r]$, C_r and $A(r, R)$ refer to discrete objects introduced at the end of Sect. 2.2. For $j \geq 1$, let $\mathcal{B}_j(r, R)$ and $\mathcal{H}_j(r, R)$ stand for the half-plane j -arm events from scale r to scale R in the semi-annulus and half-disk respectively, defined as

$$\mathcal{B}_j(r, R) = \{\text{There are } j \text{ disjoint arms from } C_r^+ \text{ to } C_R^+ \text{ in } A^+(r, R) \text{ of alternating colors}\}; \quad (2.1)$$

$$\mathcal{H}_j(r, R) = \{\text{There are } j \text{ disjoint arms from } [-r, r] \text{ to } C_R^+ \text{ in } B_R^+ \text{ of alternating colors}\}, \quad (2.2)$$

where **alternating colors** means that the color pattern is red, blue, red, ..., in counter-clockwise order. See Fig. 1 for an illustration.

Remark 2.5. We now briefly comment on the definitions above.

(1) Among the two definitions above, $\mathcal{B}_j(r, R)$ is the classical half-plane arm event in the literature, however it is not the convenient setup for us to relate to the scaling limit of the exploration process. To overcome this difficulty, we introduce $\mathcal{H}_j(r, R)$, which perfectly solves this problem. See Sect. 4.1, in particular Lemma 4.3 for how the definition of $\mathcal{H}_j(r, R)$ comes into play.

(2) As discussed in Remark 2 in [32], in the half-plane case it is not restrictive to study arm events of alternating colors thanks to a “color-switching” trick. In fact, the probability of $\mathcal{B}_j(r, R)$ (or $\mathcal{H}_j(r, R)$) remains the same for any color sequence.

We now introduce the (polychromatic) arm events in the plane. For $j \geq 2$, the classical j -arm event from scale r to R in the plane is given by

$$\mathcal{P}_j(r, R) = \{\text{There are } j \text{ disjoint arms from } C_r \text{ to } C_R \text{ in } A(r, R), \text{ not all of the same color}\}. \quad (2.3)$$

However, in this work we are going to mainly work with the following variants of arm events, which allow us to link the exploration process under these arm events to specific

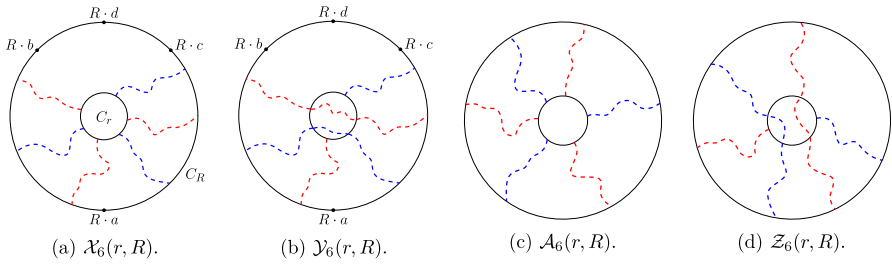


Fig. 2. Illustration of the events $\mathcal{X}_j(r, R)$, $\mathcal{Y}_j(r, R)$, $\mathcal{A}_j(r, R)$ and $\mathcal{Z}_j(r, R)$ when $j = 6$. The annuli should be viewed as hexagonal domains on \mathbb{T}^* under the convention given at the end of Sect. 2.2. The dashed red and blue curves represent arms of the respective colors

behaviors of the SLE_6 process; see also Remark 2.5 for relevant discussions in the half-plane case. We refer readers to Fig. 2 for illustrations. Fix four points $a = (0, -1)$, $b = (-1/2, \sqrt{3}/2)$, $c = (1/2, \sqrt{3}/2)$ and $d = (0, 1)$ on C_1 . Let

$$l(j) = \lfloor j/4 \rfloor + \lfloor (j+1)/4 \rfloor + 1; \quad r(j) = \lfloor (j+2)/4 \rfloor + \lfloor (j+3)/4 \rfloor - 1.$$

(Note that $l(j) + r(j) = j$.) We call j disjoint arms from C_r to C_R in $A(r, R)$ are “with the prescribed pattern” if there are $l(j)$ of them from C_r to $R \cdot \widehat{ba}$ (the counterclockwise arc from $R \cdot b$ to $R \cdot a$) with color pattern red, blue, red ... (counted clockwise from the point $R \cdot a$); and the remaining $r(j)$ ones from C_r to $R \cdot \widehat{ac}$ with color pattern blue, red, blue ... (counted counterclockwise from $R \cdot a$). Then, we define the following variants of planar j -arm events $\mathcal{X}_j, \mathcal{Y}_j$ as

$$\mathcal{X}_j(r, R) = \{\text{There exist } j \text{ disjoint arms from } C_r \text{ to } C_R \text{ in } A(r, R) \text{ with the prescribed pattern}\}, \quad (2.4)$$

$$\begin{aligned} \mathcal{Y}_j(r, R) = \{ & \text{There exist } j \text{ disjoint arms from } C_r \text{ to } C_R \text{ in} \\ & A(r, R) \text{ with the prescribed pattern} \\ & \text{and furthermore the left } (k+1)\text{-th arm and the right} \\ & k\text{-th arm are connected} \\ & \text{in } B_R \text{ by a path of their (common)} \\ & \text{color for all } 1 \leq k \leq (l(j) - 1) \wedge r(j)\}. \end{aligned} \quad (2.5)$$

We will also consider two other variants \mathcal{A}_j and \mathcal{Z}_j , which correspond to the events \mathcal{X}_j and \mathcal{Y}_j respectively but with no constraints on the endpoints on C_R . When j is even, \mathcal{A}_j is the arm event with an alternating color sequence, referred to in the literature as the **alternating arm events**. Note that \mathcal{A}_j is different from \mathcal{P}_j except for $j = 2$, although they are comparable; See Claim (2) of Lemma 2.6.

We now turn to arm probabilities. We write

$$b_j(r, R) = \mathbb{P}[\mathcal{B}_j(r, R)], \text{ and similarly } h_j, p_j, x_j, y_j \text{ and } a_j \text{ for arm probabilities.} \quad (2.6)$$

We now collect several quick facts and review some classical estimates on the asymptotics of arm probabilities. For simplicity, we will fix $r \geq r_h(j)$ and write $h_j(n)$ as a shorthand for $h_j(r, n)$ (and resp. for other arm probabilities) throughout this subsection.

We now collect several classical facts on the asymptotics of arm probabilities. In the following lemma, we show that in both half-plane and plane cases, the probabilities of variants arm events are up-to-constants equal to each other.

Lemma 2.6. *The half-plane arm probabilities are comparable for $j \geq 1$ and so is the case for planar arm events and $j \geq 2$. Namely*

- (1). $h_j(n) \asymp b_j(n)$;
- (2). $p_j(n) \asymp x_j(n) \asymp a_j(n)$ and for $r > r_0(j)$ such that $y_j(n) > 0$ for all large n , we further have $a_j(n) \asymp y_j(n)$.

Moreover, with the arm exponents defined in (1.1) and (1.4) resp.,

- (3). $b_j(R_1, R_2) = (R_1/R_2)^{-\beta_j+o(1)}$ as $R_1 \geq r_b(j)$ and $R_1/R_2 \rightarrow 0$. Also for $j \geq 2$, $p_j(R_1, R_2) = (R_1/R_2)^{-\alpha_j+o(1)}$ as $R_1 \geq r_p(j)$ and $R_1/R_2 \rightarrow 0$. The same holds for h_j, a_j, x_j, y_j .

Finally, the macroscopic arm probabilities have the following asymptotics:

- (4). There exist $f_j, g_j : (0, 1) \rightarrow \mathbb{R}^+$ such that for $0 < \epsilon < 1$,

$$\lim_{n \rightarrow \infty} x_j(\epsilon n, n) = f_j(\epsilon) \quad \text{and} \quad \lim_{n \rightarrow \infty} a_j(\epsilon n, n) = g_j(\epsilon).$$

The proofs for Claims (1) and (2) are classical applications of separation lemmas (which we will introduce shortly in this section) and will be postponed to Appendix A. Claim (3) is the main result in [32]. Claim (4) of a_j for $j \in \{2, 3, 4, 6\}$ is Lemma 2.9 in [11], while the case of other a_j and x_j follows from essentially the same arguments.

Remark 2.7. The power of (3) lies in the fact that R_1 can also go to infinity with R_2 as long as R_1/R_2 goes to zero. In the context, we will use two particular forms frequently. One is that for fixed $\alpha \in (0, 1)$

$$\begin{aligned} h_j(n^\alpha, n) &\asymp b_j(n^\alpha, n) = n^{-\beta_j(1-\alpha)+o(1)} \quad \text{and} \\ x_j(n^\alpha, n) &\asymp y_j(n^\alpha, n) \asymp p_j(n^\alpha, n) = n^{-\alpha_j(1-\alpha)+o(1)}; \end{aligned} \quad (2.7)$$

The other is that there exists $c > 0$ such that when $R_1/R_2 \rightarrow \infty$

$$b_3(R_1, R_2) \leq (R_1/R_2)^{1+c} \quad \text{and} \quad p_6(R_1, R_2) \leq (R_1/R_2)^{2+c},$$

because $\beta_3 = 2 > 1$ and $\alpha_6 = 35/12 > 2$.

In the following lemma, we show that the arm probabilities are “almost” monotone. We also postpone its proof to Appendix A.

Lemma 2.8. *For any $j \geq 1$*

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \inf_{n \leq s \leq t \leq (1+\epsilon)n} \frac{h_j(t)}{h_j(s)} \geq 1.$$

The same inequalities also hold for b_j and x_j, y_j, p_j and a_j (for $j \geq 2$).

The following lemma is a “functional equation” type result on sequences and it shows that we can obtain Theorem 1.1 from Proposition 1.4 and Theorem 1.2 from Proposition 1.5 together with the preliminary estimates Lemmas 2.6 and 2.8 (which imply that the sequence of various types of arm probabilities satisfy Assumption (2) in the following lemma). Its proof can be found in Appendix B.

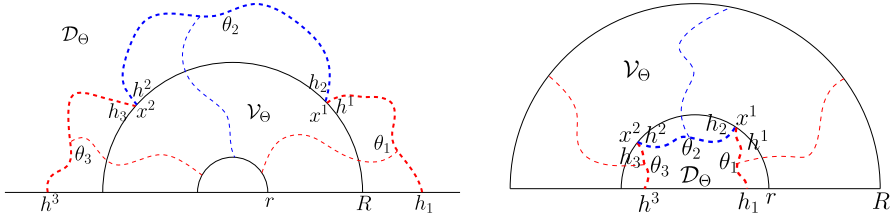


Fig. 3. Left: A configuration of outer faces around C_R^+ . Right: A configuration of inner faces around C_r^+ . In both cases, $j = 3$ and the configuration of faces $\Theta = \{\theta_1, \theta_2, \theta_3\}$ is depicted in heavy dashed curves with corresponding colors. We have indicated the relative position of \mathcal{D}_Θ and \mathcal{V}_Θ . The events $\mathcal{B}_j^\Theta(r, R)$ occur when the arms indicated by normal dashed curves with colors exist. Moreover, h_i and h^i are the end-hexagons of θ_i , and x^1 and x^2 are the endpoints of Θ

Lemma 2.9. Consider a set of positive real numbers $\{a_n : n \in [1, \infty)\}$ such that:

- (1) $\frac{a_{m^2n}}{a_{mn}} = \frac{a_m}{a_n} \left(1 + O(n^{-c})\right)$ for all $m \in (1.1, 10)$, where the constants in $O(n^{-c})$ is independent of m in the given range.
- (2) $\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \inf_{n \leq s \leq (1+\epsilon)n} \frac{a_t}{a_s} \geq 1$.

Then, there exist $0 < C < \infty$ and $-\infty < \alpha < \infty$ such that $a_n = Cn^\alpha \left(1 + O(n^{-c})\right)$.

2.5. Faces. When working on couplings of arm events, a crucial concept we will need throughout the arguments is **faces**, which forms a special example of stopping sets. The notion of stopping set is in some sense a two-dimensional version of the stopping time, whose meaning will be given at the end of this subsection.

We begin with faces in the half-plane. Recall the conventions given at the end of Sect. 2.2 for discretization. We call a set of paths $\Theta = \{\theta_1, \dots, \theta_j\}$ a configuration of **outer faces** around C_r^+ if the following conditions are satisfied:

- For $1 \leq i \leq j$, if i is odd (resp. even), then θ_i is a red (resp. blue) path from h_i to h^i such that $\theta_i \subset \mathbb{H} \setminus B_r^+$.
- For $1 \leq i \leq j$, the end-hexagons h_i 's and h^i 's satisfy the condition that h_1, h^j touch $\partial\mathbb{H}_\eta$, and $h^1, h_2, \dots, h^{j-1}, h_j$ touch C_r^+ . Furthermore, $(h_1, h^1, \dots, h_j, h^j)$ are listed in counterclockwise order, and h^i is adjacent to h_{i+1} for $1 \leq i \leq j-1$.

See Fig. 3 for an illustration. The paths $\theta_1, \dots, \theta_j$ are called the **outer faces** of Θ . The set of points $x^i := h^i \cap h_{i+1} \cap C_r^+$ for all $1 \leq i \leq j-1$ are called the **endpoints** of Θ . The concepts of configurations of **inner faces** around C_r are defined similarly except that we require instead that $\theta_i \subset B_r^+$ for all i . In both cases, each path will be called a face. We will also use Θ to denote the union of all hexagons in θ_i , $1 \leq i \leq j$ with a slight abuse of notation. For Θ a configurations of outer faces, we write \mathcal{D}_Θ and \mathcal{V}_Θ for the connected components of $\mathbb{H}_\eta \setminus \Theta$ whose boundary contains ∞ and 0 respectively. In these definitions, the initials D and V stand for “discovered” and “vacant”, signifying the status of these regions when we explore inwards; same for inner faces. See also the definition for stopping sets. When Θ represents a configuration of inner faces, we exchange the role of ∞ and 0 in the definition accordingly. See Fig. 3 for an illustration.

Suppose $r < R$. If Θ is a configuration of outer faces around C_R^+ , then we write $\mathcal{B}_j^\Theta(r, R)$ (resp. $\mathcal{H}_j^\Theta(r, R)$) for the event that each outer face is connected to C_r^+ (resp.

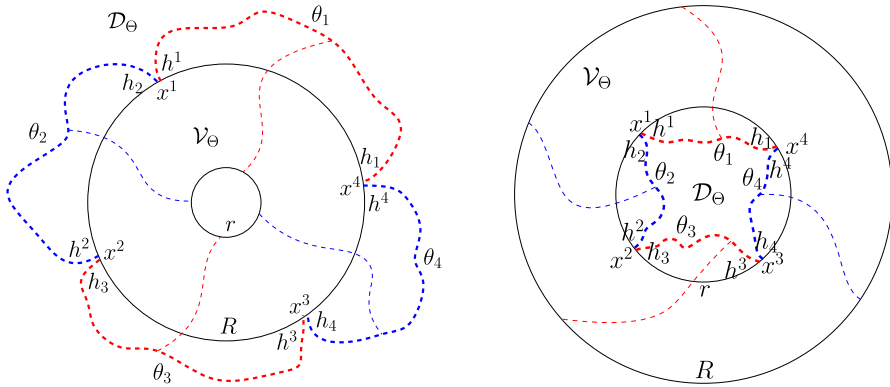


Fig. 4. Left: A configuration of outer faces around C_R . Right: A configuration of inner faces around C_r . In both cases, $j = 4$ and the configuration of faces $\Theta = \{\theta_1, \dots, \theta_4\}$ is depicted in heavy dashed curves with corresponding colors. We have indicated the relative position of \mathcal{D}_Θ and \mathcal{V}_Θ . The events $\mathcal{A}_j^\Theta(r, R)$ occur when the arms indicated by normal dashed curves with colors exist. Moreover, h_i and h^i are the end-hexagons of θ_i , and x^1, \dots, x^4 are the endpoints of Θ

$[-r, r]$) by an arm in \mathcal{V}_Θ of the same color. The same definition applies to a configuration of inner faces Θ around C_r^+ , with each face connected to C_R^+ by an arm also in \mathcal{V}_Θ . When Θ is specified, there is no confusion in whether $\mathcal{B}_j^\Theta(r, R)$ refers to a configuration of inner or outer faces.

In a similar fashion, in the plane, we define the configuration of outer or inner faces around a circle as a circular chain of j (an even integer) paths of alternating colors. More precisely, we call the set of paths $\Theta = \{\theta_1, \dots, \theta_j\}$ a configuration of outer (resp. inner) faces around C_r if

- $\theta_1, \dots, \theta_j$ have alternating colors and they are contained in $\mathbb{C} \setminus B_r$ (resp. B_r).
- The end-hexagons of these j paths, $(h_1, h^1, \dots, h_j, h^j)$, touching C_r , are listed in counterclockwise order (still, h_i and h^i are the two ends of θ_i). Moreover, h^i is adjacent to h_{i+1} for all $1 \leq i \leq j$ (where we set $h_{j+1} := h_1$).

See Fig. 4 for an illustration. Again, the paths θ_i 's are called the outer (resp. inner) faces of Θ , and we can define the end-points x^1, \dots, x^j analogously (by setting $x^i := h^i \cap h_{i+1} \cap C_r$). Similar to the half-plane case, for any configuration of outer faces Θ , let \mathcal{D}_Θ and \mathcal{V}_Θ denote the connected components of $\mathbb{C} \setminus \Theta$ which contains ∞ and 0 respectively. If Θ stand for a configuration of inner faces instead, we then exchange the role of 0 and ∞ in the definition accordingly. For $0 < r < R$, if Θ is a configuration of outer faces around C_R , we write $\mathcal{A}_j^\Theta(r, R)$ for the event that each outer face is connected to C_r by an arm of the corresponding color in \mathcal{D}_Θ . The corresponding events for inner faces are defined in a similar way.

By a **stopping set**, we mean a random configuration of inner or outer faces Θ which can be determined by a “one-sided” exploration process. That is to say, there exists an exploration process such that when the exploration stops, we are able to determine Θ from the configuration in \mathcal{D}_Θ only, and all the hexagons in \mathcal{V}_Θ are left unexplored.

2.6. Separation lemmas in the half-plane. In this subsection, we give different versions of the separation lemma in the half-plane. Their counterparts in the plane will be given in the next subsection separately.

The separation lemma for planar percolation first appears in Kesten's work [16]. This is the key tool that allows one to deal with conditional arm probabilities and obtain quasi-multiplicativity for arm events. It is worth mentioning that its counterpart in the setup of Brownian motion established by Lawler in [20] also has multiple implications in the study of special points of Brownian motion.

We will state the separation lemma in terms of interfaces. By an **interface** we mean a b -path that separates clusters of different colors in some domain.

We now introduce the notion of “quality” to measure how separated the interfaces are. Suppose $u < v$. Let Γ be a set of interfaces crossing $A^+(u, v)$ from C_u^+ to C_v^+ . Suppose that Γ contains $j \geq 1$ interfaces and let $\{x^1, \dots, x^j\}$ be the collection of endpoints of these interfaces on C_v^+ listed in counterclockwise order. Then Γ has an **exterior quality** defined as

$$Q_{\text{ex}}(\Gamma) := \frac{1}{v} d(v, x^1) \wedge d(x^1, x^2) \wedge \dots \wedge d(x^j, -v). \quad (2.8)$$

In an analogous way, we define the **interior quality** for interfaces and denote it by Q_{in} . We can also define the quality Q for a configuration of faces Θ in a similar fashion. Note that there is no need to add subscripts here since in this case the type of quality can be read from the type of faces. We call Θ to be **well-separated** if $Q(\Theta) > j^{-1}$.

The separation lemma roughly shows that conditioned on the interfaces reach a long distance without intersecting each other, then their endpoints will separate at some macroscopic distance with a universal positive probability. We now give the first version of separation lemma, which involves no initial configuration of faces, and is rather standard. This version can be viewed as the half-plane counterpart of [30, Lemma A.4] and its proof uses a similar argument, so we omit it.

Lemma 2.10 (The Standard Separation lemma). *For any $j \geq 2$, there exists $c(j) > 0$ such that for any $100j \leq 2r \leq R$, the following hold.*

- Let Γ be the set of interfaces crossing $A^+(r, R)$ connecting C_r^+ and C_R^+ , then,

$$\mathbb{P}[Q_{\text{in}}(\Gamma) \wedge Q_{\text{ex}}(\Gamma) > j^{-1} \mid \mathcal{B}_j(r, R)] > c. \quad (2.9)$$

- Let Γ be the set of interfaces in B_R^+ connecting $[-r, r]$ and C_R^+ , then

$$\mathbb{P}[Q_{\text{ex}}(\Gamma) > j^{-1} \mid \mathcal{H}_j(r, R)] > c. \quad (2.10)$$

The following strong separation lemma is the half-plane version of [11, Proposition A.1]. In fact, Proposition 2.11 is called “strong” because it holds for any given initial configuration of faces which might be of very bad quality, not because it implies Lemma 2.10 (in fact it does NOT—note that in this strong version we do require the initial configuration to be j faces, which does not appear in the conditioning in Lemma 2.10). For the sake of completeness, we will prove it in Appendix C.

Proposition 2.11 (Strong separation lemma). *For any $j \geq 2$, there exists $c(j) > 0$ such that for any $100j\eta \leq 2r \leq R$, the following hold.*

- (Outward) For any configuration of inner faces $\Theta = \{\theta_1, \dots, \theta_j\}$ with endpoints $\{x^1, \dots, x^{j-1}\}$ around C_r^+ , let Γ be the $(j-1)$ -tuple of interfaces in \mathcal{V}_Θ which start from x^1, \dots, x^{j-1} respectively until they reach C_R^+ . Then

$$\mathbb{P}[Q_{\text{ex}}(\Gamma) > j^{-1} \mid \mathcal{B}_j^\Theta(r, R)] > c. \quad (2.11)$$

- (Inward) In a similar fashion, the same claim also holds for any configuration of outer faces Θ with C_r^+ , C_R^+ and Q_{ex} replaced by C_R^+ , C_r^+ and Q_{in} respectively.

In this work, we will also need a slightly strengthened version of the strong separation lemma stated below. Compared with Proposition 2.11, it lifts the restriction of requiring exactly j faces as the initial configuration, although it still requires an a priori upper bound on the number of faces. We say that a color configuration on \mathbb{T}^* coincides with a configuration of (inner or outer) faces if they have the same color on the common hexagons. Moreover, for a color configuration ω_0 on \mathbb{T}^* (or only on a subset D of \mathbb{T}^*), we denote by $\{\omega_D = \omega_0\}$ the event that the color configuration inside D coincides with ω_0 .

Proposition 2.12 (Slightly stronger separation lemma). *For any $K \geq j \geq 2$, there exists $c = c(j, K) > 0$ such that for any $10j \cdot 2^K \leq 2^K r \leq 2^K R_1 \leq R_2 \leq R$, letting Γ be the set of interfaces crossing $A^+(R_1, R_2)$, the following holds:*

- (Outward) For any configuration of inner faces Θ around $C_{R_1}^+$ with no more than K faces, and any color configuration ω_0 that coincides with Θ and satisfies $\mathbb{P}[\mathcal{B}_j(r, R) \mid \omega_{\mathcal{D}_\Theta} = \omega_0] > 0$,

$$\mathbb{P}[|\Gamma| = j - 1 \text{ and } Q_{\text{ex}}(\Gamma) > j^{-1} \mid \mathcal{B}_j(r, R), \omega_{\mathcal{D}_\Theta} = \omega_0] > c. \quad (2.12)$$

- (Inward) In a similar fashion, for any configuration of outer faces Θ around $C_{R_2}^+$ with no more than K faces, and any suitable color configuration ω_0 ,

$$\text{The same inequality holds with } Q_{\text{ex}} \text{ replaced by } Q_{\text{in}}. \quad (2.13)$$

The claims (2.12) and (2.13) above also hold if $\mathcal{B}_j(r, R)$ is replaced by $\mathcal{H}_j(r, R)$.

Proof. We focus on (2.12) as all other cases are almost the same. Given any configuration of inner faces Θ around $C_{R_1}^+$ with k ($j \leq k \leq K$) faces, and any configuration ω_0 that coincides with Θ and satisfies $\mathbb{P}[\mathcal{B}_j(r, R) \mid \omega_{\mathcal{D}_\Theta} = \omega_0] > 0$, denote

$$\mathcal{B} = \mathcal{B}_j(r, R), \quad \mathcal{Q} = \{\omega_{\mathcal{D}_\Theta} = \omega_0\} \text{ and } \mathcal{U} = \{|\Gamma| = j - 1 \text{ and } Q_{\text{ex}}(\Gamma) > j^{-1}\}$$

for conciseness. Our aim is to show $\mathbb{P}[\mathcal{U} \mid \mathcal{B} \cap \mathcal{Q}] > c(j, K)$ for some constant $c(j, K) > 0$.

We first deal with the case $R_2 = R$ and we proceed by induction on K . The case $K = j$ reduces to (2.11) since when Θ has exactly j faces, $\mathbb{P}[\cdot \mid \mathcal{B} \cap \mathcal{Q}]$ is identical to $\mathbb{P}[\cdot \mid \mathcal{B}_j^\Theta(r, R)]$ in $\mathcal{V}(\Theta)$. Now, for some $K > j$, assume the result (when $R_2 = R$) is true for $(K - 1)$ and Θ has exactly K faces.

Consider the event

$$\mathcal{B}_K^\Theta(R_1, 2R_1) = \{\text{all of these } (K - 1) \text{ interfaces reach } C_{2R_1}^+\}.$$

Assume that $\mathcal{B}_K^\Theta(R_1, 2R_1)$ fails, then once we explore the $(K - 1)$ interfaces starting from the $(K - 1)$ endpoints and stop exploring immediately upon reaching $\mathbb{R} \cup C_{2R_1}^+$,

we will see that some of them merge together or hit the real line before reaching $C_{2R_1}^+$. Thus the exploration process induces a configuration of inner faces Θ' around $C_{2R_1}^+$ with no more than $(K - 1)$ faces. Hence, the induction hypothesis gives

$$\mathbb{P}\left[\mathcal{U} \mid \mathcal{B}, \mathcal{Q}, (\mathcal{B}_K^\Theta(R_1, 2R_1))^c\right] > c(j, K - 1). \quad (2.14)$$

On the other hand, assume that $\mathcal{B}_K^\Theta(R_1, 2R_1)$ happens. Clearly

$$\mathcal{B} \cap \mathcal{B}_K^\Theta(R_1, 2R_1) \subset \mathcal{B}_K^\Theta(R_1, 2R_1) \cap \mathcal{B}_j(4R_1, R),$$

and also $\mathcal{B}_j(4R_1, R)$, $\mathcal{B}_K^\Theta(R_1, 2R_1)$ and \mathcal{Q} are mutually independent, so

$$\mathbb{P}\left[\mathcal{B}, \mathcal{B}_K^\Theta(R_1, 2R_1) \mid \mathcal{Q}\right] \leq \mathbb{P}\left[\mathcal{B}_K^\Theta(R_1, 2R_1)\right] \cdot \mathbb{P}\left[\mathcal{B}_j(4R_1, R)\right]. \quad (2.15)$$

Furthermore, let Γ_1, Γ_2 be the set of interfaces crossing $A^+(R_1, 2R_1)$ and $A^+(4R_1, R)$, respectively. Then, by RSW-FKG gluing, (2.9) and (2.11) (taking j therein as K), we see there exist $c_1 = c_1(j, K) > 0$, $c_2 = c_2(j) > 0$ and $c_3 = c_3(K) > 0$ such that

$$\begin{aligned} \mathbb{P}\left[\mathcal{U}, \mathcal{B}, \mathcal{B}_K^\Theta(R_1, 2R_1) \mid \mathcal{Q}\right] &> c_1 \cdot \mathbb{P}\left[\mathcal{B}_j(4R_1, R), Q_{\text{in}}(\Gamma_2) > j^{-1}\right] \mathbb{P}\left[\mathcal{B}_K^\Theta(R_1, 2R_1), \right. \\ &\quad \left. Q_{\text{ex}}(\Gamma_1) > K^{-1}\right] \\ &> c_1 \cdot c_2 \mathbb{P}[\mathcal{B}_j(4R_1, R)] \cdot c_3 \mathbb{P}[\mathcal{B}_K^\Theta(R_1, 2R_1)] \\ &\geq c_1 c_2 c_3 \mathbb{P}[\mathcal{B}, \mathcal{B}_K^\Theta(R_1, 2R_1)], \end{aligned}$$

where the last inequality follows from (2.15). This implies

$$\mathbb{P}\left[\mathcal{U} \mid \mathcal{B}, \mathcal{Q}, \mathcal{B}_K^\Theta(R_1, 2R_1)\right] = \frac{\mathbb{P}[\mathcal{V}, \mathcal{B}, \mathcal{B}_K^\Theta(R_1, 2R_1) \mid \mathcal{Q}]}{\mathbb{P}[\mathcal{B}, \mathcal{B}_K^\Theta(R_1, 2R_1) \mid \mathcal{Q}]} > c_1 c_2 c_3 = c_4(j, K). \quad (2.16)$$

Now we pick $c(j, K) = c(j, K - 1) \wedge c_4(j, K)$, then (2.14) and (2.16) together with the total probability formula yields $\mathbb{P}[\mathcal{U} \mid \mathcal{B}, \mathcal{Q}] > c(j, K)$. This proves (2.12) for the case $R_2 = R$.

For the general case of (2.12), by the quasi-multiplicativity (see [30, (3.14)]) and what we just proved, we have for some constants $c_5, c_6, c_7 > 0$ depending only on j, K ,

$$\begin{aligned} \mathbb{P}[\mathcal{U}, \mathcal{B} \mid \mathcal{Q}] &> c_5 \mathbb{P}[\mathcal{U}, \mathcal{B}_j(r, R_2) \mid \mathcal{Q}] \mathbb{P}[\mathcal{B}_j(R_2, R)] > c_6 \mathbb{P}[\mathcal{B}_j(r, R_2) \mid \mathcal{Q}] \\ &\quad \mathbb{P}[\mathcal{B}_j(R_2, R)] > c_7 \mathbb{P}[\mathcal{B} \mid \mathcal{Q}]. \end{aligned}$$

This shows $\mathbb{P}[\mathcal{U} \mid \mathcal{B}, \mathcal{Q}] > c_7(j, K)$, as required. \square

Remark 2.13. The constant 2 in the condition $2r \leq R$ in the statement of Lemma 2.10 and Proposition 2.11 is not essential. Indeed, similar results still hold for any constant greater than 1 when r is large enough, and the proof is just a verbal change of the proof given in Appendix C. Then it is clear from the above proof that Proposition 2.12 still holds for large r when the constant 2^K is replaced by any constant greater than 1.

2.7. Separation lemmas in the plane. This subsection is devoted to separation lemmas in the plane. As the arguments are similar to the half-plane case, we only give the statements in Proposition 2.14 and omit the proofs.

To start, we give the definition of the quality of interfaces, just as in the half-plane case. For $u < v$, let Γ be a set of percolation interfaces that start from C_u and end at the first visit of C_v . Suppose that Γ contains j interfaces with endpoints x^1, \dots, x^j on C_v , then we define the **exterior quality** of Γ as

$$Q_{\text{ex}}(\Gamma) := \frac{1}{v} d(x^1, x^2) \wedge d(x^2, x^3) \wedge \dots \wedge d(x^j, x^1). \quad (2.17)$$

Again, we define the **interior quality** for interfaces in the same fashion and denote it by Q_{in} as well as the quality for a configuration of faces Θ and denote it by $Q(\Theta)$. If Θ satisfies $Q(\Theta) > j^{-1}$, we call it **well-separated**. Recall the notation \mathcal{V}_Θ and \mathcal{D}_Θ from Sect. 2.5.

There are counterparts of Lemma 2.10 and Proposition 2.11 in case of plane. However, some additional complicity arises from parity: since the number of interfaces crossing an annulus must be even, we cannot naively argue something like “the j interfaces are well-separated” for an arbitrary j . Lemma A.4 and Proposition A.1 in [11] only consider the case when j is even, but the same argument with little change yields similar results for odd j . More precisely, for any $j \geq 2$, we set $J = 2\lfloor j/2 \rfloor$. For any $10j \leq 2r \leq R$, and any configuration of outer faces $\Theta = \{\theta_1, \dots, \theta_J\}$ around C_R , let Γ be the set of interfaces crossing $A(r, R)$. Then, there exists $c = c(j) > 0$ such that

$$\mathbb{P}[Q_{\text{in}}(\Gamma) > J^{-1} \mid \mathcal{A}_J^\Theta(r, R)] > c. \quad (2.18)$$

Similar result holds for any configuration of inner faces with J faces. From this we can prove (by almost identical arguments) the counterpart of Proposition 2.12 in the plane case.

Proposition 2.14. *For any $K \geq j \geq 2$, there exists $c = c(j, K) > 0$ such that for any $10j \cdot 2^K \leq 2^K R_1 \leq R_2 \leq R$, let Γ be the set of interfaces crossing $A(R_1, R_2)$, then the following holds. For any configuration of outer faces Θ around C_{R_2} with no more than K faces, any color configuration ω_0 which coincides with Θ and satisfies $\mathbb{P}[\mathcal{X}_j(r, R) \mid \omega_{\mathcal{D}_\Theta} = \omega_0] > 0$,*

$$\mathbb{P}[|\Gamma| = J \text{ and } Q_{\text{ex}}(\Gamma) > J^{-1} \mid \mathcal{X}_j(r, R), \omega_{\mathcal{D}_\Theta} = \omega_0] > c. \quad (2.19)$$

Moreover, the same claim also holds for $\mathcal{Y}_j(r, R)$.

The first claim (2.19) can be deduced similarly as before. For the second claim, we note that although conditioning on $\mathcal{Y}_j(r, R)$ involves more requirements on connectedness, it could be resolved by a similar RSW-FKG gluing technique as in the proof of Lemma 3.3 in [11]. We omit the details.

2.8. Percolation exploration process and scaling limits. In this subsection, we define the percolation exploration process and discuss its power-law convergence towards SLE₆ obtained in an unpublished work by Binder and Richards. In particular, we give in Proposition 2.16 a variant of their results tailored for our work. We will provide some estimates about the exploration process in Lemma 2.17 and show that it is stable under perturbations of the boundary in Proposition 2.18.

Given a Jordan domain Ω and two boundary points a, b , for any $0 < \eta < 1$, recall from Sect. 2.2 that $\Omega_{[\eta]}$ is a connected subgraph of $\eta\mathbb{T}$ (where we assume that η is small enough) and a_η and b_η are two vertices on the boundary of $\Omega_{[\eta]}$. In addition, a_η and b_η divide the boundary of $\Omega_{[\eta]}$ into two connected parts, and we will assign red and blue colors to the external hexagons adjacent to these two parts, respectively, according to the context. Define the exploration process γ_η as a directed path from a_η to b_η on $\eta\mathbb{T}^*$ which is the interface between the two red and blue clusters containing the boundary.

The Schramm-Loewner evolution (SLE), first introduced by Schramm in [28], is a family of conformally invariant random curves that serves as the canonical candidate for the scaling limit of curves, in particular interfaces of various critical models in 2D. In the case of critical planar percolation, Smirnov proved in [31] that the scaling limit of the exploration process γ_η is indeed the chordal SLE₆ path from a to b in Ω , which we denote by γ below.

As discussed in Sect. 1.2, Binder and Richards improve the above convergence by giving a power-law rate for curves up to some stopping time. We now describe their results in more detail. Given $U \subset \Omega$, let T_η be the first time that γ_η enters $U_{[\eta]}$ and T be the first time that γ enters U . Define the distance d between two paths $l_1 : [0, t_1] \rightarrow \mathbb{R}^2$ and $l_2 : [0, t_2] \rightarrow \mathbb{R}^2$ as

$$d(l_1, l_2) = \inf_{\theta} \sup_{0 \leq s \leq t_1} |l_2(\theta(s)) - l_1(s)|,$$

where θ runs over all increasing homeomorphisms from $[0, t_1]$ to $[0, t_2]$. We call Ω a **nice domain** if Ω is a domain with a piecewise smooth boundary. Note that the domains we consider in Sects. 4.1 and 4.3 (both denoted by Ω) are nice domains.

In Theorem 4.1.11 of Richards' thesis [26], it is proved that under some coupling of γ and γ_η , they are close upon specific stopping times with high probability. We collect their results here.

Theorem 2.15 ([26], not yet peer-reviewed). *When Ω is a nice domain and $U = B(b, \epsilon) \cap \Omega$ for some $\epsilon > 0$, there exists $u = u(\Omega, U) > 0$ such that under some coupling of γ and γ_η*

$$\mathbb{P}[d(\gamma_\eta|_{[0, T_\eta]}, \gamma|_{[0, T]}) > \eta^u] = O(\eta^u) \text{ as } \eta \rightarrow 0.$$

Combined with estimates on critical percolation, we can prove the following proposition which states that the time can be taken as the hitting time of a nice domain containing the endpoint in our setup.

Proposition 2.16. *For a nice domain Ω and some nice domain $U \subset \Omega$ that contains a neighborhood of b , there exists $u = u(\Omega, U) > 0$ such that under some coupling of γ and γ_η (which we will later call **good coupling**),*

$$\mathbb{P}[d(\gamma_\eta|_{[0, T_\eta]}, \gamma|_{[0, T]}) > \eta^u] < O(\eta^u) \text{ as } \eta \rightarrow 0. \quad (2.20)$$

We will also use a variant of this proposition, i.e., Proposition 4.1, in the proof of Proposition 1.4.

Proof. Fix $\epsilon > 0$ such that $U \cap \Omega \supset B(b, \epsilon) \cap \Omega$ and let $V = B(b, \epsilon) \cap \Omega$. We write T'_η for the first time that γ_η enters $V_{[\eta]}$ and T' for the first time that γ enters V . Note that

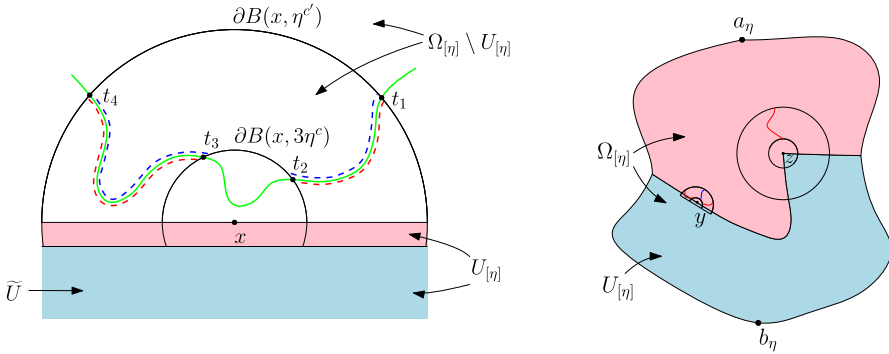


Fig. 5. Left: Illustration of the event \mathcal{E}^c . On this event, there exist three arms of alternating colors from $\partial B(x, 3\eta^c)$ to $\partial B(x, \eta^c)$ in $B(x, \eta^c) \setminus \tilde{U}$. Right: For a point y away from the non-smooth point of ∂U , we can find a $(\pi + 0.01)$ -cone that contains $B(y, \eta^c/2) \setminus \tilde{U}$. If there exists $x \in B(y, \eta^c)$ such that the event \mathcal{E}^c occurs for this point, then there exist three arms of alternating colors from inner radius $5\eta^c$ to outer radius $\eta^c/3$ in this cone. For a point z on the non-smooth point of ∂U , if the event \mathcal{E}^c occurs for some point x contained in $B(z, \eta^{c'})$, we have at least one arm from $B(z, 2\eta^{c'})$ to distance $\frac{1}{O(1)}$ away from z

the stopping times in Theorem 2.15 are T'_η and T' by taking the same Ω and ϵ . Hence, there exists $c > 0$ such that under some coupling \mathbb{P} of γ_η and γ ,

$$\mathbb{P} \left[d \left(\gamma_\eta|_{[0, T'_\eta]}, \gamma|_{[0, T']} \right) > \eta^c \right] < O(\eta^c). \quad (2.21)$$

We now show that \mathbb{P} is indeed a good coupling which satisfies (2.20). Since $U_{[\eta]} \supset V_{[\eta]}$ and $U \supset V$, we have $T_\eta \leq T'_\eta$ and $T \leq T'$. Now, it is sufficient to prove that γ_η and γ respectively hit $U_{[\eta]}$ and U almost at the same time and place with high probability. Let $\tilde{U} = \{z \in U_{[\eta]} : d(z, \Omega_{[\eta]} \setminus U_{[\eta]}) \geq \eta^c\}$; see Fig. 5 (left). Write

$$\mathcal{E} = \left\{ \text{there does not exist } x \in \partial U_{[\eta]} \text{ and } 0 < t_1 < t_2 < t_3 < t_4 \text{ such that } \gamma_\eta(t_1), \gamma_\eta(t_4) \in \partial B(x, \eta^c), \right. \\ \left. \gamma_\eta(t_2), \gamma_\eta(t_3) \in \partial B(x, 3\eta^c), \text{ and moreover, } \gamma_\eta|_{[t_1, t_4]} \text{ does not intersect with } \tilde{U} \right\}, \quad (2.22)$$

where c' is a constant to be chosen and $c > c' > 0$.

Next, we bound $\mathbb{P}[\mathcal{E}^c]$ from above using Claim (2) of Lemma 2.6. If \mathcal{E}^c happens, then we have three arms of alternating colors from $B(x, 3\eta^c)$ to $B(x, \eta^c)$ in $B(x, \eta^c) \setminus \tilde{U}$; see Fig. 5 (left). Specifically, there are four arms along $\gamma_\eta|_{[t_1, t_2]}$ and $\gamma_\eta|_{[t_3, t_4]}$. However, the arms in the region bounded by $\gamma_\eta|_{[t_1, t_2]}$ and $\gamma_\eta|_{[t_3, t_4]}$ may intersect.

Since Ω and U are piecewise smooth, we can find a net \mathcal{N} with $O(\eta^{-c})$ points on $\partial U_{[\eta]}$ such that each point x on $\partial U_{[\eta]}$ is η^c -close to a point y in \mathcal{N} . Then we have $B(x, 3\eta^c) \subset B(y, 4\eta^c)$ and $B(x, \eta^c) \supset B(y, \eta^c/2)$. Hence, the event \mathcal{E}^c implies that for some point $y \in \mathcal{N}$, there are three arms of alternating colors from $B(y, 4\eta^c)$ to $B(y, \eta^c/2)$ in $B(y, \eta^c/2) \setminus \tilde{U}$. For the points in \mathcal{N} which are away from the non-smooth point of ∂U , say a large constant (depending on Ω, U) times η^c , the domain $B(y, \eta^c/2) \setminus \tilde{U}$ is contained in a $(\pi + 0.01)$ -cone centered at a point in \tilde{U} that is η^c -close to y . In such cases, if the preceding event occurs for y , then there is a three-arm event for this cone from inner radius $5\eta^c$ to outer radius $\eta^c/3$; see Fig. 5 (right). Therefore, using

Claim (2) of Lemma 2.6 with b_3 , we can bound the probability of this event from above by $O(\eta^{(1+\delta)(c-c')})$ for some universal constant $\delta > 0$ (we can take δ to be close to 1 here, however, any $\delta > 0$ is sufficient for our purpose). Note that Lemma 2.6 is stated for the half-plane case, however, it is relatively easy to extend to the $(\pi + 0.01)$ -cone case by the conformal invariance of critical percolation. For points in \mathcal{N} that are $O(\eta^{c'})$ -close to the non-smooth point of ∂U , the domain $B(y, \eta^{c'}/2) \setminus \tilde{U}$ may be much larger than a half space. In this case, we can cover all these points by $O(1)$ number of balls, each with radius $\eta^{c'}$. If the event \mathcal{E}^c occurs for a point x contained in these balls, then there exists at least one monochromatic arm from these balls to distance $\frac{1}{O(1)}$ since the curve γ_η must reach within distance $\eta^{c'}$ from one of these balls; see Fig. 5 (right). Here $\frac{1}{O(1)}$ means that the sequence is larger than some positive constant depending on Ω and U for all sufficiently small η . Using Claim (2) of Lemma 2.6 with p_1 , we can bound this probability from above by $O(1) \times O(\eta^{\delta c'})$ where we decrease the value of δ if necessary. Therefore,

$$\mathbb{P}[\mathcal{E}^c] \leq O(\eta^{-c}) \times O(\eta^{(1+\delta)(c-c')}) + O(\eta^{\delta c'}).$$

Hence, we can choose c' small such that $\mathbb{P}[\mathcal{E}^c] \leq O(\eta^{c''})$.

Next, we show that

$$\mathcal{E} \cap \left\{ d(\gamma_\eta|_{[0, T'_\eta]}, \gamma|_{[0, T']}) \leq \eta^c \right\} \subset \left\{ d(\gamma_\eta|_{[0, T_\eta]}, \gamma|_{[0, T]}) \leq \eta^{c'} \right\}.$$

Assume that $d(\gamma_\eta|_{[0, T'_\eta]}, \gamma|_{[0, T']}) \leq \eta^c$ and the event \mathcal{E} occurs. When γ_η hits $U_{[\eta]}$ for the first time, by the event \mathcal{E} , we see that it also hits \tilde{U} before traveling a distance of $\eta^{c'}$. Specifically, we can take z be the hit point of γ_η on $\partial U_{[\eta]}$ and let the time be t_0 . Let t_2 be the last hit time of $\partial B(z, 3\eta^c)$ before t_0 , and t_1 be the last hit time of $\partial B(z, \eta^{c'})$ before t_2 . Similarly, define t_3 be the first hit time of $\partial B(z, 3\eta^c)$ after t_0 , and t_4 be the first hit time of $\partial B(z, \eta^{c'})$ after t_1 . Then, by the event \mathcal{E} , the path $\gamma_\eta|_{[t_1, t_4]}$ must intersect \tilde{U} . Therefore, γ_η must hit \tilde{U} before traveling $\eta^{c'}$ which implies that the process γ must also hit U around z . Similarly, when the process γ first hits U , γ_η must get η^c close to some point on $\partial U_{[\eta]}$ and by the event \mathcal{E} it must hit U before traveling a distance of $\eta^{c'}$. So, γ_η and γ hit respectively $U_{[\eta]}$ and U almost at the same time and place, and their d -distance is smaller than $\eta^{c'}$. This completes the proof of the above relationship. Therefore, under the coupling of γ_η and γ in (2.21),

$$\begin{aligned} & \mathbb{P} \left[d(\gamma_\eta|_{[0, T_\eta]}, \gamma|_{[0, T]}) > \eta^{c'} \right] \\ & \leq \mathbb{P} \left[d(\gamma_\eta|_{[0, T'_\eta]}, \gamma|_{[0, T']}) > \eta^c \right] + \mathbb{P}[\mathcal{E}^c] \leq O(\eta^c) + O(\eta^{c''}). \end{aligned}$$

Letting $u = \min\{c, c', c''\}$ yields (2.20) as desired. \square

We will also use the following estimate frequently in Sect. 4. The proof follows verbatim from the same argument as in the previous proposition, using Claim (3) in Lemma 2.6 with b_3 and p_1 , so we omit it here.

Lemma 2.17. *For any $u > 0$, there exists a constant $v \in (0, u)$ depending on u such that the following holds. For a nice domain Ω and two boundary points a, b , let P be a*

piecewise smooth path contained in $\overline{\Omega}$ that has a positive distance away from a, b and P_η be its discretization; see Sect. 2.2. Define the event

$$\mathcal{E} = \left\{ \text{there does not exist } x \in P_\eta \text{ and } 0 < t_1 < t_2 < t_3 < t_4 \text{ such that } \gamma_\eta(t_1), \gamma_\eta(t_4) \in \partial B(x, \eta^v), \right. \\ \left. \gamma_\eta(t_2), \gamma_\eta(t_3) \in \partial B(x, \eta^u), \text{ and moreover, } \gamma_\eta|_{[t_1, t_4]} \text{ does not intersect } P_\eta \right\}. \quad (2.23)$$

Then, we have

$$\mathbb{P}[\mathcal{E}] \geq 1 - O(\eta^c),$$

where c is a constant depending only on u, v and the $O(\eta^c)$ term depends on Ω and P . Furthermore, on the event \mathcal{E} , whenever γ_η reaches within distance η^u from P_η , it hits P_η before travelling a distance of η^v

In applications, we can also take P_η as part of the discretization of a smooth path with a bounded number of connected components. This lemma also extends to the case where P_η is n^{-v} -close to a, b as long as Ω is locally close to the half-plane at these parts of P_η . For instance, when $\Omega = B^+(0, 1)$, $a = (1, 0)$, $b = (-1, 0)$, we can take $P_\eta = C_1^+ \setminus (B(a, n^{-v}) \cup B(b, n^{-v}))$ where C_1^+ is the discretization of C_1^+ with lattice scale η .

The next proposition states that the exploration process is insensitive to the change of boundaries. More precisely, when the boundary changes by η^δ , then the exploration process changes by at most $\eta^{c(\delta)}$ with probability at least $(1 - \eta^{c(\delta)})$.

Proposition 2.18. *Suppose Ω is a nice domain and $\Omega_{[\eta]}$ be the discretization of Ω defined in Sect. 2.2. Let Ω'_η be a Jordan set of hexagons on $\eta\mathbb{T}^*$, i.e., Ω'_η contains a set of hexagons on $\eta\mathbb{T}^*$ and this set is simply connected when embedded in \mathbb{C} . Let a'_η, b'_η be two boundary points of Ω'_η . If for some $\delta > 0$,*

$$d(\partial\Omega_{[\eta]}, \partial\Omega'_\eta) < \eta^\delta, \quad d(a_\eta, a'_\eta) < \eta^\delta, \quad d(b_\eta, b'_\eta) < \eta^\delta,$$

then there exists a coupling \mathbb{P} of γ_η and γ'_η and a constant $c(\delta) > 0$ such that

$$\mathbb{P}\left[d\left(\gamma_\eta, \gamma'_\eta\right) > \eta^c\right] < O(\eta^c).$$

Proof. Without loss of generality, we assume that $\Omega'_\eta \supset \Omega_{[\eta]}$ (otherwise we can compare $\Omega_{[\eta]}$ and Ω'_η both with Ω''_η , where $\Omega''_\eta = \{x \in \Omega_{[\eta]} : d(x, \partial\Omega_{[\eta]}) > \eta^\delta\}$). We can couple the critical percolation on $\Omega_{[\eta]}$ and Ω'_η such that the two configurations have the same color on $\Omega_{[\eta]}$. Under this coupling, the exploration processes in these two domains can only be different after one of them hits $\partial\Omega_{[\eta]}$ (since they have the same path in the interior of $\Omega_{[\eta]}$). However, we can find some $c \in (0, \delta)$ such that with more than $(1 - \eta^c)$ probability, each time they hit $\partial\Omega_{[\eta]}$, they will remain the same after travelling a distance of η^c (otherwise we either have a half plane 3-arm event from an η^δ ball on the boundary to distance η^c , or, not a non-smooth point, a whole plane 1-arm event from distance η^c to $\frac{1}{O(1)}$). This happens with probability less than $\eta^{c'}$ for some $c' > 0$ when c is close enough to 0). This completes the proof. \square

Remark 2.19. With similar arguments, we can show that the sharp asymptotics we obtain in this paper for arm events under the specific discretization, also hold for other discretizations, as they differ by at most a power of the mesh size. In other words, arm events are insensitive to the way we discretize domains. In particular, the good coupling we construct in Proposition 2.16 (as well as Proposition 4.1) also works for the discretized domains under the convention given at the end of Sect. 2.2.

3. Couplings and Conditional Arm Probabilities

In this section, we will give various couplings concerning arm events in the half-plane and the plane (see Propositions 3.1, 3.3 and 3.6) and deduce useful estimates on the conditional arm probabilities from these couplings (see Propositions 3.4, 3.5, 3.8, 3.10). In [11], the authors have established the coupling results concerning one-arm ([11, Proposition 5.2]) and four-arm ([11, Proposition 3.1]) events in the plane, respectively. Although many techniques developed in [11] can be adapted for variants of arm events we are considering here (e.g., $\mathcal{H}_j(r, R)$, $\mathcal{X}_j(r, R)$ and $\mathcal{Y}_j(r, R)$), certain intricacies arise from our choice of such events, causing extra difficulties. We postpone the proofs to Sect. 5 as they are rather technical and independent of the main story. To simplify the statements of the results, we assume the mesh size $\eta = 1$ throughout this section.

3.1. The half-plane case. In this subsection, we deal the coupling results in the half-plane. For simplicity of presentation, the $j = 1$ case and the $j \geq 2$ cases are stated separately.

3.1.1. One-arm In this subsection, we will state the coupling result for the one-arm event in the half-plane. We say that a b-path in the half-plane is a **circuit** if it connects the discretized negative real line to the discretized positive real line. If $1 < r < u$, let $\Gamma_{\text{out}}(r, u)$ (resp. $\Gamma_{\text{in}}(r, u)$) be the outermost (resp. innermost) red circuit in the semi-annulus $A^+(r, u)$, i.e., the red circuit which is closest to C_u^+ (resp. C_r^+). If such circuits do not exist, then set $\Gamma(r, u) = \emptyset$.

Proposition 3.1. *There exists $\delta > 0$ such that for all $100 < 10r < R$ and $m \in (1.1, 10)$, denoting $u := \sqrt{rR}$, then the following hold:*

- (Inward coupling) *There is a coupling \mathbb{Q} of the conditional laws $\mathbb{P}[\cdot \mid \mathcal{H}_1(r, R)]$ and $\mathbb{P}[\cdot \mid \mathcal{H}_1(r, mR)]$ such that if we sample $(\omega_1, \omega_2) \sim \mathbb{Q}$, then with probability at least $(1 - (r/R)^\delta)$, $\Gamma_{\text{out}}(r, u)$ for both ω_1 and ω_2 are non-empty and identical. Furthermore, the percolation configurations in the connected component of $\mathbb{H} \setminus \Gamma_{\text{out}}(r, u)$ whose boundary contains 0 are also identical.*
- (Outward coupling) *A similar coupling exists for the conditional laws $\mathbb{P}[\cdot \mid \mathcal{H}_1(r, R)]$ and $\mathbb{P}[\cdot \mid \mathcal{B}_1(r, R)]$ with Γ_{out} and 0 replaced by Γ_{in} and ∞ respectively.*

Remark 3.2. In the statement above, we take $u = \sqrt{rR}$ merely for convenience. In fact, it can be taken as $r^d R^{1-d}$ for any $0 < d < 1$ (with the corresponding $\delta(d)$). The same applies to all couplings in this section.

The proof of the above proposition is essentially the same as that of [11, Proposition 5.2], which deals with one-arm events in the plane (the proof in fact dates back to Kesten's work on the IIC [15]) in which the key tools are the RSW theory and FKG inequality. Therefore, we omit the proof and refer the readers to the said paper for details.

3.1.2. j -arms with $j \geq 2$ In this subsection, we concentrate on the j -arm events with $j \geq 2$ in the half-plane. We shall present couplings in both the inward and outward directions simultaneously since they share the same virtue. Note that for $j \geq 2$, $\mathcal{H}_j(r, R)$ is equivalent to the event that there are at least $(j - 1)$ interfaces starting from C_R^+ and end at their first hitting of $[-r, r]$; similarly, $\mathcal{B}_j(r, R)$ is equivalent to the event that there are at least $(j - 1)$ interfaces crossing the annulus $A^+(r, R)$. In both cases, we write

$\Gamma = \{\gamma_1, \dots, \gamma_n\}$ (where $n \geq j - 1$) for the set of interfaces crossing the whole region, and order them counterclockwise. We say two interfaces in Γ are **adjacent** if there is a hexagon on \mathbb{T}^* touched by both of them.

Proposition 3.3. *For any $j \geq 2$, there exists $\delta(j) > 0$ such that for any $100j < 10r < R$ and $m \in (1.1, 10)$, denoting $u = \sqrt{rR}$, the following hold:*

- (Inward coupling) *There is a coupling \mathbb{Q} of $\mathbb{P}[\cdot \mid \mathcal{H}_j(r, R)]$ and $\mathbb{P}[\cdot \mid \mathcal{H}_j(r, mR)]$ such that, if we sample $(\omega_1, \omega_2) \sim \mathbb{Q}$, then with probability at least $(1 - (r/R)^\delta)$, there exists a common configuration of outer faces Θ^* with j faces around C_u^+ in both ω_1 and ω_2 , and ω_1 coincides with ω_2 in \mathcal{V}_{Θ^*} . Furthermore, when this is the case, Θ^* is a stopping set (recall the definition in Sect. 2.5) and for any $1 < r' < r$, we have*

$$\mathbb{P}[\mathcal{H}_j(r', R) \mid \mathcal{H}_j(r, R), \Theta^*] = \mathbb{P}[\mathcal{H}_j(r', mR) \mid \mathcal{H}_j(r, mR), \Theta^*]. \quad (3.1)$$

- (Outward coupling) *A similar coupling exists for $\mathbb{P}[\cdot \mid \mathcal{H}_j(r, R)]$ and $\mathbb{P}[\cdot \mid \mathcal{B}_j(r, R)]$ with outer faces replaced by inner faces. For any $R' > R$, when the coupling succeeds (so that the common inner faces Θ^* exists), we have*

$$\mathbb{P}[\mathcal{H}_j(r, R') \mid \mathcal{H}_j(r, R), \Theta^*] = \mathbb{P}[\mathcal{B}_j(r, R') \mid \mathcal{B}_j(r, R), \Theta^*]. \quad (3.2)$$

As a consequence of these couplings in Proposition 3.1 (for the $j = 1$ case) and Proposition 3.3 (for the $j \geq 2$ cases), we know that the law $\mathbb{P}[\cdot \mid \mathcal{H}_j(n^\alpha, n)]$ will coincide with $\mathbb{P}[\cdot \mid \mathcal{H}_j(n^\alpha, mn)]$ at scale n^α with high probability for $\alpha \in (0, 1)$ (not to be confused with the arm exponents α_j). From the (slightly stronger) separation lemma Proposition 2.12, we can show that the failure event will not contribute much to $\mathbb{P}[\mathcal{H}_j(r, n) \mid \mathcal{H}_j(n^\alpha, n)]$. Thus, we obtain the following comparison of conditional arm probabilities.

Proposition 3.4. *For any $r \geq r_h(j)$, $m \in (1.1, 10)$ and $\alpha \in (0, 1)$,*

$$\mathbb{P}[\mathcal{H}_j(r, n) \mid \mathcal{H}_j(n^\alpha, n)] = \mathbb{P}[\mathcal{H}_j(r, mn) \mid \mathcal{H}_j(n^\alpha, mn)] \left(1 + O(n^{-c})\right). \quad (3.3)$$

Here, $O(n^{-c})$ may depend on r and α , but not m .

Proof. We only give the proof for $j \geq 2$, since the case $j = 1$ is similar (and easier indeed). Denote $\beta = (1 + \alpha)/2$, and choose a large integer $K_0 = K_0(j, \alpha)$ such that

$$\mathbb{P}[\mathcal{B}_{K_0}(n^\alpha, n^\beta)] \leq n^{-\beta j - 1}. \quad (3.4)$$

Consider probability measures $\mathbb{P}_1 = \mathbb{P}[\cdot \mid \mathcal{H}_j(n^\alpha, n)]$ and $\mathbb{P}_2 = \mathbb{P}[\cdot \mid \mathcal{H}_j(n^\alpha, mn)]$ with sample spaces Ω_1 and Ω_2 . For a pair $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, we say $\omega_1 =_{n^\beta} \omega_2$ if there exists a common configuration Θ^* of outer faces around $C_{n^\beta}^+$, such that ω_1 and ω_2 are identical in \mathcal{V}_{Θ^*} . Divide $\Omega_1 \times \Omega_2$ into the union of $\mathcal{K}_1 = \{(\omega_1, \omega_2) : \omega_1 =_{n^\beta} \omega_2\}$, $\mathcal{K}_2 = \{(\omega_1, \omega_2) : \omega_1 \in \mathcal{B}_{K_0}(n^\alpha, n^\beta) \text{ or } \omega_2 \in \mathcal{B}_{K_0}(n^\alpha, n^\beta)\} \setminus \mathcal{K}_1$ and $\mathcal{K}_3 = (\mathcal{K}_1 \cup \mathcal{K}_2)^c$. By Proposition 3.3, we can couple $\mathbb{P}_1, \mathbb{P}_2$ by \mathbb{Q} in a way that

$$\mathbb{Q}[\mathcal{K}_1] \geq 1 - n^{-C} \text{ for some } C(\alpha, j) > 0. \quad (3.5)$$

We define a pair of random stopping sets (Θ_1, Θ_2) under \mathbb{Q} as follow. For $(\omega_1, \omega_2) \sim \mathbb{Q}$, since Θ^* is a stopping set by Proposition 3.3, we can perform an exploration process outside $C_{n^\beta}^+$ for both configurations to check whether there exists a common configuration of outer faces Θ^* with j faces for both $\omega_i, i = 1, 2$ and without exploring any hexagon in \mathcal{V}_{Θ^*} when this is the case. If so (see the left part of Fig. 6), we then further explore

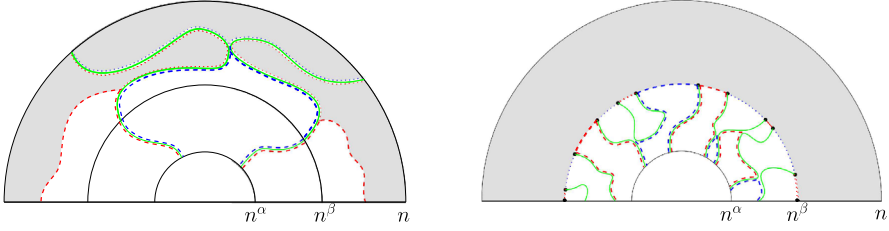


Fig. 6. Proof of Proposition 3.4. Sketch for $\mathbb{P}[\cdot \mid \mathcal{H}_3(n^\alpha, n)]$. Left: The case when the coupling succeeds: there is a common configuration of outer faces Θ^* with 3 faces around $C_{n^\beta}^+$ for both $\omega_1 \sim \mathbb{P}[\cdot \mid \mathcal{H}_j(n^\alpha, n)]$ and $\omega_2 \sim \mathbb{P}[\cdot \mid \mathcal{H}_j(n^\alpha, mn)]$. The two interfaces that cross the annulus $A^+(n^\alpha, n)$ are in green. Faces Θ^* together with hexagons that are adjacent to these two interfaces further create a common configuration of outer faces around $C_{n^\alpha}^+$, which is sketched in dashed red/blue curves. The region that has been explored (\mathcal{V}_{Θ^*}) is in gray. Right: The case when the coupling fails. In this case, we reveal the color of all hexagons in $A^+(n^\beta, n)$, which is illustrated in gray. Then, the color of all hexagons neighboring $C_{n^\beta}^+$ are revealed, which is sketched in dotted/dashed red/blue arcs on $C_{n^\beta}^+$. Running all of the exploration processes (in green) from $C_{n^\beta}^+$ towards $C_{n^\alpha}^+$, we obtain a configuration of outer faces around $C_{n^\alpha}^+$, which is sketched in dashed red/blue curves. In this case, the number of faces could be larger than j

the $(j - 1)$ interfaces starting from the endpoints of Θ^* and end upon reaching $C_{n^\alpha}^+$ and hence obtain a pair of configurations of outer faces $(\Theta_1(\omega_1), \Theta_2(\omega_2))$ around $C_{n^\alpha}^+$. Otherwise (see the right part of Fig. 6), we further explore all the hexagons outside $C_{n^\beta}^+$ together with all the interfaces in $A^+(n^\alpha, n^\beta)$ from $C_{n^\beta}^+$ to $C_{n^\alpha}^+$ for both $\omega_i, i = 1, 2$. This will also yield a pair of configurations of outer faces $(\Theta_1(\omega_1), \Theta_2(\omega_2))$ around $C_{n^\alpha}^+$. Note that in either case, all hexagons in $\mathcal{V}_{\Theta_i(\omega_i)}, i = 1, 2$ are left unexplored, hence we get a pair of stopping sets $(\Theta_1, \Theta_2)(\omega_1, \omega_2) = (\Theta_1(\omega_1), \Theta_2(\omega_2))$.

For $(\omega_1, \omega_2) \in \mathcal{K}_1$, $\Theta_1(\omega_1)$ and $\Theta_2(\omega_2)$ are identical with j faces, and we deduce from (3.1) that

$$\mathbb{P}[\mathcal{H}_j(r, n) \mid \mathcal{H}_j(n^\alpha, n), \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1] = \mathbb{P}[\mathcal{H}_j(r, mn) \mid \mathcal{H}_j(n^\alpha, mn), \omega_{\mathcal{D}_{\Theta_2(\omega_2)}} = \omega_2]. \quad (3.6)$$

For $(\omega_1, \omega_2) \in \mathcal{K}_3$, we claim it holds uniformly that

$$\mathbb{P}[\mathcal{H}_j(r, n) \mid \mathcal{H}_j(n^\alpha, n), \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1] \leq n^{-\beta_j \alpha + o(1)}, \quad (3.7)$$

$$\mathbb{P}[\mathcal{H}_j(r, mn) \mid \mathcal{H}_j(n^\alpha, mn), \omega_{\mathcal{D}_{\Theta_2(\omega_2)}} = \omega_2] \leq n^{-\beta_j \alpha + o(1)}. \quad (3.8)$$

Since $\mathcal{H}_j(r, n) \subset \mathcal{B}_j(r, 2^{-K_0} n^\alpha) \cap \mathcal{B}_j(2^{-K_0} n^\alpha, n)$ and $\mathcal{B}_j(r, 2^{-K_0} n^\alpha)$ is independent of $\mathcal{B}_j(2^{-K_0} n^\alpha, n)$ and $\{\omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1\}$, the left hand side of (3.7) is uniformly bounded by

$$\frac{\mathbb{P}[\mathcal{B}_j(r, 2^{-K_0} n^\alpha)] \mathbb{P}[\mathcal{B}_j(2^{-K_0} n^\alpha, n) \mid \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1]}{\mathbb{P}[\mathcal{H}_j(n^\alpha, n) \mid \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1]} \leq n^{-\beta_j \alpha + o(1)} \cdot \frac{\mathbb{P}[\mathcal{B}_j(2^{-K_0} n^\alpha, n) \mid \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1]}{\mathbb{P}[\mathcal{H}_j(n^\alpha, n) \mid \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1]}.$$

Noting that $\Theta_1(\omega_1)$ has no more than K_0 faces for any $(\omega_1, \omega_2) \in \mathcal{K}_3$, and letting \mathcal{U} be the event that the interfaces crossing $A^+(2^{-K_0} n^\alpha, n^\alpha)$ are well-separated on $C_{2^{-K_0} n^\alpha}^+$, it follows that

$$\begin{aligned} \mathbb{P}[\mathcal{H}_j(n^\alpha, n) \mid \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1] &\stackrel{\text{RSW}}{\geq} c(j, K_0) \mathbb{P}[\mathcal{U} \cap \mathcal{B}_j(2^{-K_0} n^\alpha, n) \mid \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1] \\ &\stackrel{(2.13)}{\geq} c'(j, K_0) \mathbb{P}[\mathcal{B}_j(2^{-K_0} n^\alpha, n) \mid \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1]. \end{aligned}$$

This proves (3.7), and (3.8) follows in the same fashion.

Now, writing

$$\begin{aligned} F &:= \mathbb{P}[\mathcal{H}_j(r, n) \mid \mathcal{H}_j(n^\alpha, n), \omega_{\mathcal{D}_{\Theta_1(\omega_1)}} = \omega_1], \\ G &:= \mathbb{P}[\mathcal{H}_j(r, mn) \mid \mathcal{H}_j(n^\alpha, mn), \omega_{\mathcal{D}_{\Theta_2(\omega_2)}} = \omega_2] \end{aligned}$$

for short, by the total probability formula we have

$$\left| \mathbb{P}_1[\mathcal{H}_j(r, n)] - \mathbb{P}_2[\mathcal{H}_j(r, mn)] \right| = \left| \sum_{(\Theta_1, \Theta_2)} (F - G) \mathbb{Q}[(\Theta_1, \Theta_2) = (\Theta_1, \Theta_2)] \right|. \quad (3.9)$$

The right-hand side of (3.9) can be bounded by

$$\begin{aligned} & \left| \sum_{(\Theta_1, \Theta_2)} (F - G) \mathbb{Q}[(\Theta_1, \Theta_2) = (\Theta_1, \Theta_2), \mathcal{K}_1] \right| \\ & + \sum_{i=2}^3 \sum_{(\Theta_1, \Theta_2)} (F + G) \mathbb{Q}[(\Theta_1, \Theta_2) = (\Theta_1, \Theta_2), \mathcal{K}_i]. \end{aligned}$$

The first term equals 0 by (3.6), the second term (that corresponds to \mathcal{K}_2 in the sum) is bounded by $2\mathbb{Q}[\mathcal{K}_2] \leq 4n^{-\beta_j-1}$ from (3.4), and the last term is bounded by $n^{-\beta_j\alpha-C+o(1)}$ from (3.5), (3.7) and (3.8). As a result we see $|\mathbb{P}_1[\mathcal{H}_j(r, n)] - \mathbb{P}_2[\mathcal{H}_j(r, mn)]| \leq n^{-\beta_j\alpha-C+o(1)}$ for some $C > 0$. In addition, by Lemma 2.6 and (2.7) in Remark 2.7, $\mathbb{P}[\mathcal{H}_j(r, n) \mid \mathcal{H}_j(n^\alpha, n)] = n^{-\beta_j\alpha+o(1)}$. The claim (3.3) hence follows. \square

Proposition 3.5. *For any $r \geq r_h(j) (\geq r_b(j))$, $m \in (1.1, 10)$ and $\alpha \in (0, 1)$,*

$$\mathbb{P}[\mathcal{B}_j(r, n) \mid \mathcal{B}_j(r, n^\alpha)] = \mathbb{P}[\mathcal{H}_j(r, n) \mid \mathcal{H}_j(r, n^\alpha)] \left(1 + O(n^{-c})\right). \quad (3.10)$$

Proof. In contrast to Proposition 3.4, this is an outward coupling, in which case we do not need to spare an additional scale n^β to use Proposition 2.12, and thus much easier to deal with. Concretely, we first examine whether $\mathbb{P}[\cdot \mid \mathcal{B}_j(r, n^\alpha)]$ and $\mathbb{P}[\cdot \mid \mathcal{H}_j(r, n^\alpha)]$ induce the same configuration of inner faces around $C_{n^\alpha}^+$, which will happen with probability larger than $(1 - n^{-c})$ by (3.2). If it fails, for both events $\mathcal{B}_j(r, n)$ and $\mathcal{H}_j(r, n)$, we further need to fulfill an arm event $\mathcal{B}_j(n^\alpha, n)$ which has probability bounded by $n^{-\beta_j\alpha+o(1)}$ by (2.7). The result follows immediately. \square

3.2. The planar cases. In this subsection, we provide the coupling results in the plane, which are analogous to those in the half-plane in the previous subsection. The one-arm and four-arm cases have already been developed in [11], but as new difficulties arise for the variants we consider in this work and for general j , we decide to include a proof, which we postpone to the last section. Recall the definitions of $\mathcal{X}_j, \mathcal{Y}_j, \mathcal{A}_j$ for $j \geq 2$ in (2.5) and (2.4) and below (2.5) resp. We let

$$J = 2\lfloor j/2 \rfloor.$$

In the next proposition, we couple together conditional laws in the planar case. For discussions of these couplings, see Remark 3.7 below.

Proposition 3.6. *For any $j \geq 2$, there exists $\delta(j) > 0$ such that for all $100j \leq 10r \leq R$ and $m \in (1.1, 10)$, denoting $u = \sqrt{rR}$, then the following hold.*

- (Inward coupling) *There is a coupling \mathbb{Q} of the conditional laws $\mathbb{P}[\cdot \mid \mathcal{X}_j(r, R)]$ and $\mathbb{P}[\cdot \mid \mathcal{A}_j(r, R)]$ such that if we sample $(\omega_1, \omega_2) \sim \mathbb{Q}$, then with probability at least $(1 - (r/R)^\delta)$, there exists a common configuration of outer faces Θ^* with J faces around C_u in both ω_1 and ω_2 , and ω_1 coincides with ω_2 in \mathcal{V}_{Θ^*} . Furthermore, when this is the case, Θ^* is a stopping set and for any $1 \leq r' \leq r$,*

$$\mathbb{P}[\mathcal{X}_j(r', R) \mid \mathcal{X}_j(r, R), \omega_{\mathcal{D}_{\Theta^*}} = \omega_1] = \mathbb{P}[\mathcal{A}_j(r', R) \mid \mathcal{A}_j(r, R), \omega_{\mathcal{D}_{\Theta^*}} = \omega_2]. \quad (3.11)$$

- (Another inward coupling) *A similar coupling exists for the conditional laws $\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, R)]$ and $\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, mR)]$. In this case, if the coupling succeeds (i.e., Θ^* exists),*

$$\mathbb{P}[\mathcal{Y}_j(r', R) \mid \mathcal{Y}_j(r, R), \omega_{\mathcal{D}_{\Theta^*}} = \omega_1] = \mathbb{P}[\mathcal{Y}_j(r', mR) \mid \mathcal{Y}_j(r, mR), \omega_{\mathcal{D}_{\Theta^*}} = \omega_2]. \quad (3.12)$$

Remark 3.7. The major difference of these couplings with Proposition 3.3 lies in (3.11) and (3.12). In the half-plane case, the “domain Markov property” can be easily applied, and thus (3.1) and (3.2) follows naturally from the existence of common faces for both percolation configurations. In the planar case, however, due to the parity issue and the complicity of $\mathcal{Y}_j(r, R)$, (3.11) and (3.12) do not hold trivially. We tackle this kind of difficulty in Sect. 5 by coupling together extra structures rather than just configurations of faces.

As a result of Proposition 3.6, we get the following estimates on conditional arm probabilities.

Proposition 3.8. *For any $r \geq r_y(j)$, $m \in (1.1, 10)$ and $\alpha \in (0, 1)$, we have*

$$\mathbb{P}[\mathcal{Y}_j(r, n) \mid \mathcal{Y}_j(n^\alpha, n)] = \mathbb{P}[\mathcal{Y}_j(r, mn) \mid \mathcal{Y}_j(n^\alpha, mn)] \left(1 + O(n^{-c})\right), \quad (3.13)$$

where $O(n^{-c})$ may depend on r and α , but not m .

Proposition 3.9. *For any $r \geq r_x(j)(= r_a(j))$ and $\varepsilon > 0$, we have*

$$\mathbb{P}[\mathcal{X}_j(r, n) \mid \mathcal{X}_j(\varepsilon n, n)] = \mathbb{P}[\mathcal{A}_j(r, n) \mid \mathcal{A}_j(\varepsilon n, n)] \left(1 + O(\varepsilon^c)\right). \quad (3.14)$$

The proof of Proposition 3.8 is almost identical to that of Proposition 3.4, and by similar arguments and replacing n^α with εn , we obtain a similar proof of Proposition 3.9. We omit the proofs.

We now consider outward coupling in the plane. Recall the definition of $\mathcal{Z}_j(r, R)$. There are similar couplings of $\mathbb{P}[\cdot \mid \mathcal{A}_j(r, n^\alpha)]$ and $\mathbb{P}[\cdot \mid \mathcal{Z}_j(r, n^\alpha)]$ as in the half-plane case, from which we can deduce a comparison of the probability of \mathcal{X}_j conditioned on \mathcal{A}_j and \mathcal{Y}_j on \mathcal{Z}_j , which is summarized in the following proposition. As the key idea is essentially same as before, for brevity we choose not to state the precise coupling results, and only give a sketch of the proof for the comparison result.

Proposition 3.10. *For any $r \geq r_z(j)(\geq r_a(j))$, $m \in (1.1, 10)$ and $\alpha \in (0, 1)$, we have*

$$\mathbb{P}[\mathcal{X}_j(r, n) \mid \mathcal{A}_j(r, n^\alpha)] = \mathbb{P}[\mathcal{Y}_j(r, n) \mid \mathcal{Z}_j(r, n^\alpha)] \left(D(j) + O(n^{-c})\right) \quad (3.15)$$

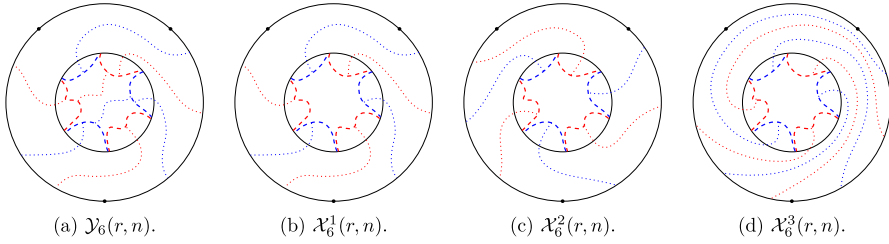


Fig. 7. An illustration of the events $\mathcal{Y}_6(r, n)$ and $\mathcal{X}_6(r, n)$ of three different types of connecting patterns. All four annuli are $A(n^\alpha, n)$ with the same configuration of inner faces Θ^* in dashed red/blue curves. The three marked points on C_n are $n \cdot a$, $n \cdot b$ and $n \cdot c$, respectively (from bottom, counterclockwise). The outer connecting patterns for Θ^* are sketched in dotted red/blue curves, and there is an additional inner connecting pattern in the first picture which results in the only possibility for the outer connection for $\mathcal{Y}_6(r, n)$

where $D(j)$ is a constant given by

$$D(j) = \begin{cases} 1, & j \equiv 1 \pmod{2} \\ \frac{j}{2}, & j \equiv 2 \pmod{4} \\ \frac{j}{4}, & j \equiv 0 \pmod{4} \end{cases} \quad (3.16)$$

and $O(n^{-c})$ may depend on r , α and j , but not m .

We first explain why the constant $D(j)$ in (3.16) appears. As in Proposition 3.6, we can couple $\mathbb{P}[\cdot \mid \mathcal{A}_j(r, n^\alpha)]$ and $\mathbb{P}[\cdot \mid \mathcal{Z}_j(r, n^\alpha)]$ in a way such that if we sample (ω_1, ω_2) according to this coupling, ω_1 and ω_2 will coincide on a configuration of inner faces Θ^* around C_{n^α} with high probability. We now consider the events $\mathcal{X}_j(r, n)$ and $\mathcal{Y}_j(r, n)$ conditioned on the color configuration in \mathcal{D}_{Θ^*} . When j is odd, conditioned on \mathcal{D}_{Θ^*} , $\mathcal{X}_j(r, n)$ is equivalent to $\mathcal{Y}_j(r, n)$ and the proof is almost identical with the previous cases. However, when j is even, these two events are distinct: there are $j/2$ kinds of connecting patterns for $\mathcal{X}_j(r, n)$, since the information inside Θ^* cannot tell us which faces should be connected to the corresponding segments on C_n ; but for $\mathcal{Y}_j(r, n)$, there are only 1 (if $j \equiv 2 \pmod{4}$) or 2 (if $j \equiv 0 \pmod{4}$) possible connecting patterns, because the information inside Θ^* already determines the order of faces in Θ^* . See Fig. 7 for an illustration of the case when $j = 6$. If we realize that all possible connecting patterns have nearly the same probability, then the definition of $D(j)$ in (3.16) makes sense. For this equality fact, we draw an analogy with the evolution of a Markov chain at a heuristic level. Similar to how states in a Markov chain lose memory of the initial values and converge to a stationary distribution, the interfaces tend to “forget” their initial positions and “mix” into a common distribution after traversing a long distance. On a more technical level, this equality can be established by comparing each pattern to a specific reference pattern via coupling arguments, as shown in (3.18) below.

Sketch of Proof for Proposition 3.10 when j is even. Denote \mathcal{G} for the event that there are exactly j interfaces crossing $A(r, n)$, then conditioned on the event $\mathcal{X}_j(r, n)$ or $\mathcal{Y}_j(r, n)$, \mathcal{G} has negligible conditional probability by BK-Reimer’s inequality. We partition $\mathcal{X}_j(r, n) \cap \mathcal{G}$ into the disjoint union of $D(j)$ events according to the locations of interfaces. More precisely, let $\hat{\Gamma}$ be the set of the $\frac{j}{2}$ interfaces with red on their left (seen from outside to inside). We label each $\gamma \in \hat{\Gamma}$ in the following two ways: label γ by γ_i (resp. γ^i) with $1 \leq i \leq \frac{j}{2}$, if γ is the i -th element in $\hat{\Gamma}$ when counting around the circle

C_r (resp. C_n) counterclockwise starting from $(0, -r)$ (resp. $(0, -n)$). For $1 \leq i \leq D(j)$, define $\mathcal{X}_j^i(r, n)$ as

$$\mathcal{X}_j^i(r, n) = \begin{cases} \mathcal{X}_j(r, n) \cap \mathcal{G} \cap \{\gamma_1 = \gamma^i\} & j \equiv 2 \pmod{4}, \\ \mathcal{X}_j(r, n) \cap \mathcal{G} \cap \{\gamma_1 = \gamma^i \text{ or } \gamma^{i+\frac{j}{4}}\} & j \equiv 0 \pmod{4}. \end{cases} \quad (3.17)$$

See Fig. 7 for illustration of the events $\mathcal{X}_j^i(r, n)$'s. Then $\mathcal{X}_j(r, n) \cap \mathcal{G}$ is the disjoint union of $\mathcal{X}_j^i(r, n)$, $1 \leq i \leq D(j)$. We construct a coupling of $\mathbb{P}[\cdot \mid \mathcal{A}(r, n^\alpha)]$ and $\mathbb{P}[\cdot \mid \mathcal{Z}_j(r, n^\alpha)]$ for each $1 \leq i \leq D(j)$. In such couplings, we keep a record of the relative location of interfaces, so once we successfully couple a face, the conditional version of $\mathcal{X}_j^i(r, n)$ and $\mathcal{Y}_j(r, n) \cap \mathcal{G}$ are identical. This allows us to deduce that

$$\mathbb{P}[\mathcal{X}_j^i(r, n) \mid \mathcal{A}_j(r, n^\alpha)] = \mathbb{P}[\mathcal{Y}_j(r, n) \cap \mathcal{G} \mid \mathcal{Z}_j(r, n^\alpha)] \left(1 + O(n^{-c})\right) \quad (3.18)$$

holds for each $1 \leq i \leq D(j)$, and the desired result follows readily. \square

4. Proof of Main Theorems

In this section, we will combine the coupling results from the previous section and the power-law convergence of the exploration process discussed in Sect. 2.8 to prove the main results of this work, namely Theorems 1.1, 1.2 and 1.3. In Sects. 4.1 and 4.2, we will prove Theorem 1.1, which is derived directly from Proposition 1.4. In order to prove Proposition 1.4, we use Propositions 3.4, 3.5 and 4.2 as inputs. In Sects. 4.3 and 4.4, we will prove Theorems 1.2 and 1.3. The former is derived directly from Proposition 1.5, while the latter theorem is a quick corollary of the former. In order to prove Proposition 1.5, we use Propositions 3.8 and 4.6 (the latter is an analogue of Proposition 4.2 in the plane), as inputs.

Note that in Sects. 4.1 and 4.3, we will consider the rescaled lattice and the discretization scheme in Sect. 2.2 comes into play.

4.1. Comparison estimates in the half-plane. In this subsection, our main goal is Proposition 4.2 in which we compare the probabilities of \mathcal{H}_j , the modified half-plane arm events (recall (2.2) for definitions), at different scales.

As discussed in Sect. 1.2, the key ingredient is the convergence rate of percolation exploration process towards SLE₆ upon some stopping time as shown in Theorem 4.1.10 in [26] and discussed in Sect. 2.8. Instead of citing Proposition 2.16 directly, we will use a modified version stated as follows.

Recall the notation from Sect. 2.8. We consider the nice domain (in the sense of Sect. 2.2)

$$\Omega = B^+(0, 1) \cup B\left((-3/4, 0), 1/4\right), \quad a = (1, 0) \text{ and } b = (-3/4, -1/4);$$

see Fig. 8 for an illustration. We assign red color to the counterclockwise arc \widehat{ab} and blue color to the counterclockwise arc \widehat{ba} . Given a mesh size $\eta > 0$, the percolation exploration process starts from a_η and ends at b_η . For $\alpha \in (0, 1)$ and $m > 0$, let $\tilde{T}_{1/n}$ (resp. $\tilde{T}_{1/(mn)}$) denote the first time that $\gamma_{1/n}$ (resp. $\gamma_{1/(mn)}$) hits $[-1, -n^{\alpha-1}]$. Then we have $\tilde{T}_{1/n}, \tilde{T}_{1/(mn)} < \infty$ for all sufficiently large n . Sometimes we may use $\tilde{T}_{1/mn}$ and $\gamma_{1/mn}$ to simplify the notation.

Recall (2.2) that $\mathcal{H}_j(r, R)$ is the event that in B_R^+ with the lattice scale $\eta = 1$ there exist j disjoint arms connecting $[-r, r]$ to C_R^+ with alternating colors. Rescaling the lattice by $\eta = 1/R$ (in fact we will consider the cases $R = n$ and $R = mn$ in this and the next subsection) we get that $\mathcal{H}_j(r, R)$ is equivalent to the event that in B_1^+ with the lattice scale η , there exist j such arms connecting $[-r/R, r/R]$ to C_1^+ . From here to the end of this subsection, with slight abuse of notation, we will tacitly assume the equivalence between events on the rescaled and those on the unrescaled lattices.

Proposition 4.1. *There exists $u(\Omega) > 0$ such that for any $\alpha \in (0, 1)$ and $m \in (1.1, 10)$ there exists a coupling of $\gamma_{1/n}$ and $\gamma_{1/(mn)}$ (which we will refer to as the **good coupling**) satisfying*

$$\mathbb{P} \left[d \left(\gamma_{1/n} \big|_{[0, \tilde{T}_{1/n}]}, \gamma_{1/(mn)} \big|_{[0, \tilde{T}_{1/(mn)}]} \right) > n^{-u} \right] = O(n^{-u}).$$

Moreover, for any $\beta > 0$, $O(n^{-u})$ depends only on β for all $\alpha \in (\beta, 1)$ and $m \in (1.1, 10)$.

Proof. Let $U = B((-3/4, 0), 1/4) \setminus \overline{B(0, 1)}$ where $\overline{B(0, 1)}$ is the closure of $B(0, 1)$. Applying Proposition 2.16 with Ω and U , we obtain that there exists a coupling of $\gamma_{1/n}$ and $\gamma_{1/(mn)}$ such that

$$\mathbb{P} \left[d \left(\gamma_{1/n} \big|_{[0, T_{1/n}]}, \gamma_{1/(mn)} \big|_{[0, T_{1/(mn)}]} \right) > n^{-c} \right] = O(n^{-c}). \quad (4.1)$$

where $T_{1/n}$ is the first time that $\gamma_{1/n}$ enters $U_{[1/n]}$, $T_{1/(mn)}$ is the first time that $\gamma_{1/(mn)}$ enters $U_{[1/(mn)]}$, and c is the constant in Proposition 2.16 which depends only on U and Ω .

The remaining part is similar to Proposition 2.16 except that here we let \mathcal{E} be the event that there does not exist $x \in [-1, -n^{\alpha-1})$ such that $\gamma_{1/n}$ enters $B(x, n^{-c})$ but does not hit $[-1, -n^{\alpha-1})$ between the last entrance of $B(x, n^{-c'})$ before hitting $B(x, n^{-c})$ and the first exit time of $B(x, n^{-c'})$ after hitting $B(x, n^{-c})$, and the same condition holds for $\gamma_{1/(mn)}$. We refer to Lemma 2.17 for a quantitative version. Then, by Lemma 2.17, we can choose $c' \in (0, c)$ such that $\mathbb{P}[\mathcal{E}^c] \leq O(n^{-c''})$. Similar to Proposition 2.16, we have

$$\begin{aligned} & \mathcal{E} \cap \left\{ d \left(\gamma_{1/n} \big|_{[0, T_{1/n}]}, \gamma_{1/(mn)} \big|_{[0, T_{1/(mn)}]} \right) \leq n^{-c} \right\} \\ & \subset \left\{ d \left(\gamma_{1/n} \big|_{[0, \tilde{T}_{1/n}]}, \gamma_{1/(mn)} \big|_{[0, \tilde{T}_{1/(mn)}]} \right) \leq n^{-c'} \right\}. \end{aligned}$$

Hence, under the coupling of $\gamma_{1/n}$ and $\gamma_{1/(mn)}$ in (4.1),

$$\begin{aligned} & \mathbb{P} \left[d \left(\gamma_{1/n} \big|_{[0, \tilde{T}_{1/n}]}, \gamma_{1/(mn)} \big|_{[0, \tilde{T}_{1/(mn)}]} \right) > n^{-c'} \right] \\ & \leq \mathbb{P} \left[d \left(\gamma_{1/n} \big|_{[0, T_{1/n}]}, \gamma_{1/(mn)} \big|_{[0, T_{1/(mn)}]} \right) > n^{-c} \right] + \mathbb{P}[\mathcal{E}^c] \leq O(n^{-c}) + O(n^{-c''}). \end{aligned}$$

Taking $u = \min\{c, c', c''\}$, we obtain the proposition. \square

Recall the definition of $h_j(r, R)$ in (2.6). The following proposition states that h_j are almost the same at different scales within a power-law error at different scales when r is not too small compared to R .

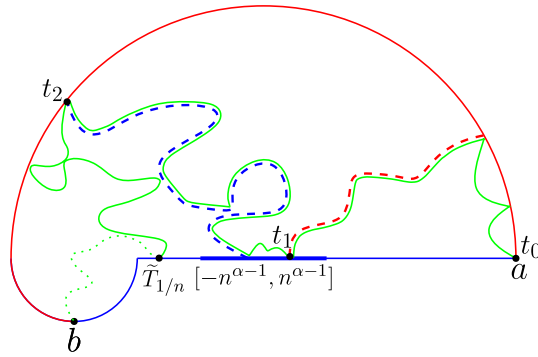


Fig. 8. Sketch of the event $\mathcal{H}_j(n^\alpha, n)$ when $j = 2$. The whole domain is $\Omega_{[1/n]}$. The exploration process $\gamma_{1/n}$ from $a_{1/n}$ to $b_{1/n}$ on $\Omega_{[1/n]}$ is in green. $\tilde{T}_{1/n}$ is the first time that $\gamma_{1/n}$ hits $[-1, -n^{\alpha-1}]$, after which time $\gamma_{1/n}$ is sketched in dotted. The segment $[-n^{\alpha-1}, n^{\alpha-1}]$ is in bold blue line. Two arms, neighboring the interface $\gamma_{1/n}$, from this segment to the top boundary C_1^+ are in dashed red and blue respectively

Proposition 4.2. *There exist $c_8, c > 0$ such that for all $\alpha \in (1 - c_8, 1)$ and $m \in (1.1, 10)$,*

$$h_j(n^\alpha, n) = h_j(mn^\alpha, mn) \left(1 + O(n^{-c})\right).$$

Importantly, here $O(n^{-c})$ is independent of m and α .

To prove it, we need Lemmas 4.3 and 4.4. In Lemma 4.3, we will show that $\mathcal{H}_j(n^\alpha, n)$ can be determined by the exploration process $\gamma_{1/n}$ upon time $\tilde{T}_{1/n}$ and similarly $\mathcal{H}_j(mn^\alpha, mn)$ can be determined by $\gamma_{1/mn}$ upon $\tilde{T}_{1/mn}$. In Lemma 4.4, we will show that the indicator functions of corresponding events of the exploration process are identical under three constraints. Finally, we prove that all of these constraints happen with high probability under the good coupling of $\gamma_{1/n}$ and $\gamma_{1/mn}$. Therefore, the probabilities of $\mathcal{H}_j(n^\alpha, n)$ and $\mathcal{H}_j(mn^\alpha, mn)$ are close.

We say that $\gamma_{1/n}$ travels j times between C_1^+ and $[-n^{\alpha-1}, n^{\alpha-1}]$ before $\tilde{T}_{1/n}$ if there exist $0 = t_0 < t_1 < t_2 < \dots < t_j \leq \tilde{T}_{1/n}$ such that $\gamma_{1/n}(t_0) \in C_1^+$, $\gamma_{1/n}(t_1) \in [-n^{\alpha-1}, n^{\alpha-1}]$, $\gamma_{1/n}(t_2) \in C_1^+$, etc.; the same for (4.3). Following from simple geometrical arguments, we arrive at the following lemma. See also Fig. 8 for an illustration.

Lemma 4.3. *One has*

$$\mathcal{H}_j(n^\alpha, n) = \{\gamma_{1/n} \text{ travels } j \text{ times between } C_1^+ \text{ and } [-n^{\alpha-1}, n^{\alpha-1}] \text{ before } \tilde{T}_{1/n}\} \quad (4.2)$$

and

$$\mathcal{H}_j(mn^\alpha, mn) = \{\gamma_{1/mn} \text{ travels } j \text{ times between } C_1^+ \text{ and } [-n^{\alpha-1}, n^{\alpha-1}] \text{ before } \tilde{T}_{1/mn}\}. \quad (4.3)$$

Note that if we work with $\mathcal{B}_j(n^\alpha, n)$ instead, we need to deal with a varying domain which is a less-than-ideal strategy. This is the main reason that we introduce and work with $\mathcal{H}_j(n^\alpha, n)$. We refer readers to Remark 2.5 and the paragraph below Proposition 1.4 for more discussions.

Before stating the second lemma, we introduce three events which all happen with high probability under the good coupling of $\gamma_{1/n}$ and $\gamma_{1/mn}$ in Proposition 4.1. Let

$u = u(\Omega)$ be the constant from Proposition 4.1. We will call $C_1^+ \cup [-n^{\alpha-1}, n^{\alpha-1}]$ the **designated boundary** and note that $(-1, 0)$, $(1, 0)$, $(-n^{\alpha-1}, 0)$ and $(n^{\alpha-1}, 0)$ are the extremal points of the designated boundary. Define $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ as

$$\begin{aligned}\mathcal{J}_1 &= \left\{ d\left(\gamma_{1/n}|_{[0, \tilde{T}_{1/n}]}, \gamma_{1/mn}|_{[0, \tilde{T}_{1/mn}]}\right) \leq n^{-u} \right\}; \\ \mathcal{J}_2 &= \left\{ \text{there does not exist } x \text{ on the boundary such that it is } n^{-v} \text{ away from the extremal} \right. \\ &\quad \text{points, and } \gamma_{1/n} \text{ enters } B(x, n^{-u}) \text{ but does not hit the designated boundary between} \\ &\quad \text{the last entrance and first exit times of } B(x, n^{-v}), \text{ and the same for } \gamma_{1/mn} \left. \right\}; \\ \mathcal{J}_3 &= \left\{ \gamma_{1/n} \text{ and } \gamma_{1/mn} \text{ do not enter } B((-1, 0), 2n^{-v}), B((-n^{\alpha-1}, 0), 2n^{-v}) \right. \\ &\quad \left. \text{or } B((n^{\alpha-1}, 0), 2n^{-v}) \text{ and do not reenter } B((1, 0), 2n^{-v}) \text{ after leaving } B((1, 0), 1/4) \right\}.\end{aligned}$$

In the event \mathcal{J}_2 , we refer to the last entrance of $B(x, n^{-v})$ before hitting $B(x, n^{-u})$ and the first exit time of $B(x, n^{-v})$ after hitting; see Lemma 2.17 for a quantitative version. We will let v be the constant in Lemma 2.17 applied with u . If \mathcal{J}_2 happens, then each time $\gamma_{1/n}$ (or $\gamma_{1/mn}$) enters $B(x, n^{-u})$, it will hit the designated boundary between the last entrance and first exit times of $B(x, n^{-v})$ for all points x on the designated boundary that are n^{-v} away from the extremal points.

We now give the second lemma which states that the indicator functions of $\mathcal{H}_j(n^\alpha, n)$ and $\mathcal{H}_j(mn^\alpha, mn)$ are identical under three constraints. In the proof, we will use the equivalence established in Lemma 4.3.

Lemma 4.4. *One has*

$$\mathbb{1}_{\mathcal{H}_j(n^\alpha, n)} = \mathbb{1}_{\mathcal{H}_j(mn^\alpha, mn)} \text{ on } \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{J}_3. \quad (4.4)$$

Proof. Assume that $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ and $\mathcal{H}_j(n^\alpha, n)$ happen so that $\gamma_{1/n}$ fulfills the right-hand side of (4.2). Suppose that $\gamma_{1/n}$ hits $[-n^{\alpha-1}, n^{\alpha-1}]$ at the point x . From \mathcal{J}_3 we know that x is n^{-v} away from the extremal points. By \mathcal{J}_1 and \mathcal{J}_2 , the process $\gamma_{1/mn}$ enters $B(x, n^{-u})$ and so must hit the designated boundary between the last entrance and first exit times of $B(x, n^{-v})$, therefore also before $\tilde{T}_{1/mn}$. Thus, $\gamma_{1/mn}$ must hit $[-n^{\alpha-1}, n^{\alpha-1}]$ nearly at the same place as $\gamma_{1/n}$ before $\tilde{T}_{1/mn}$. Similarly, when $\gamma_{1/n}$ hits a point x in C_1^+ before $\tilde{T}_{1/n}$ and after hitting $[-n^{\alpha-1}, n^{\alpha-1}]$ (this together with \mathcal{J}_3 ensures that the hitting point is $2n^{-v}$ away from the point $(1, 0)$), the process $\gamma_{1/mn}$ must hit C_1^+ nearly at the same place as $\gamma_{1/n}$ before $\tilde{T}_{1/mn}$. Hence, $\gamma_{1/mn}$ fulfilling the right-hand side of (4.3) and so $\mathcal{H}_j(mn^\alpha, mn)$ happens. Therefore,

$$\begin{aligned}\mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{J}_3 \cap \mathcal{H}_j(n^\alpha, n) &\subset \mathcal{H}_j(mn^\alpha, mn) \text{ and similarly} \\ \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{J}_3 \cap \mathcal{H}_j(mn^\alpha, mn) &\subset \mathcal{H}_j(n^\alpha, n).\end{aligned}$$

This completes the proof of (4.4). \square

Proof of Proposition 4.2. First, we give upper bounds to $\mathbb{P}[\mathcal{J}_1^c]$, $\mathbb{P}[\mathcal{J}_2^c]$ and $\mathbb{P}[\mathcal{J}_3^c]$. By Proposition 4.1, under the good coupling of $\gamma_{1/n}$ and $\gamma_{1/mn}$, $\mathbb{P}[\mathcal{J}_1^c] \leq O(n^{-u})$. Taking v as the constant in Lemma 2.17 applied with the same u , we obtain that $\mathbb{P}[\mathcal{J}_2^c] \leq O(n^{-c})$ for some constant $c > 0$ depending only on u, v . In our case, the designated boundary is not a discretization of a smooth path, but Lemma 2.17 can still be applied to this case, as Ω is locally contained in a half-space at every point on the designated boundary. If \mathcal{J}_3^c happens, we have a planar 1-arm event from a $(2n^{-v})$ -ball centered at one of the

extremal points to distance $1/4$ since we can find an arm along the exploration process that is $2n^{-v}$ -close to that extremal point. So,

$$\mathbb{P}[\mathcal{J}_3^c] \leq O(1) \times O(n^{-cv})$$

(where we reduce the value of c if necessary).

Therefore, under the good coupling

$$\begin{aligned} |h_j(n^\alpha, n) - h_j(mn^\alpha, mn)| &\stackrel{(4.4)}{\leq} 1 - \mathbb{P}[\mathcal{J}_1^c \cup \mathcal{J}_2^c \cup \mathcal{J}_3^c] \leq \mathbb{P}(\mathcal{J}_1^c) + \mathbb{P}(\mathcal{J}_2^c) + \mathbb{P}(\mathcal{J}_3^c) \\ &= O(n^{-u}) + O(n^{-c}) + O(n^{-cv}). \end{aligned}$$

By applications of the RSW theory, $h_j(mn^\alpha, mn) \geq n^{C(\alpha-1)}$. We can take c_8 such that

$$-Cc_8 > \max\{-u, -c, -cv\}.$$

Then, for all $\alpha \in (1 - c_8, 1)$, $|h_j(n^\alpha, n) - h_j(mn^\alpha, mn)| = O(n^{-c'})h_j(mn^\alpha, mn)$ as desired. \square

Remark 4.5. When $j = 1$, the event $\mathcal{H}_j(r, R)$ is simply equivalent to the event that there is a red crossing between $[-r/R, r/R]$ and C_1^+ in B_1^+ with the lattice scale $\eta = 1/R$. This special case of Proposition 4.2 has been proved in Proposition 5.6 of [23] and the main theorem of [4].

4.2. Proof of Theorem 1.1. In this subsection, we will prove Proposition 1.4 and complete the proof of Theorem 1.1. We will first prove a version of (1.8) for the case of h_j with Propositions 4.2 and 3.4 as inputs, whose proof is simpler than that of (1.8) for b_j yet already contains the main idea. Then, we will turn back to (1.8) for b_j with Proposition 3.5 as an additional input.

Proof of Proposition 1.4. As discussed above, we start with the case of h_j . Fix $r \geq r_h(j)$. Let $m \in (1.1, 10)$ and $\alpha \in (1 - c_8, 1)$ (c_8 is the constant defined in Proposition 4.2). In this proof we write $f(n) \simeq g(n)$ as the shorthand of $f(n) = g(n)(1 + O(n^{-c}))$ where $O(n^{-c})$ may depend on j and on the choice of α but are independent of the choice of m .

We first transform $\frac{h_j(r, mn)}{h_j(r, n)}$ into comparisons of mesoscopic arm probabilities thanks to Proposition 3.4. Since $\mathcal{H}_j(r, n) \subset \mathcal{H}_j(n^\alpha, n)$ and $\mathcal{H}_j(r, mn) \subset \mathcal{H}_j(n^\alpha, mn)$,

$$\frac{h_j(r, mn)}{h_j(r, n)} = \frac{\mathbb{P}[\mathcal{H}_j(r, mn) | \mathcal{H}_j(n^\alpha, mn)] \cdot \mathbb{P}[\mathcal{H}_j(n^\alpha, mn)]}{\mathbb{P}[\mathcal{H}_j(r, n) | \mathcal{H}_j(n^\alpha, n)] \cdot \mathbb{P}[\mathcal{H}_j(n^\alpha, n)]} \stackrel{(3.3)}{\simeq} \frac{\mathbb{P}[\mathcal{H}_j(n^\alpha, mn)]}{\mathbb{P}[\mathcal{H}_j(n^\alpha, n)]}. \quad (4.5)$$

We then use Proposition 4.2 to pass from scales mn, n to scales m^2n, mn :

$$\frac{\mathbb{P}[\mathcal{H}_j(n^\alpha, mn)]}{\mathbb{P}[\mathcal{H}_j(n^\alpha, n)]} \simeq \frac{\mathbb{P}[\mathcal{H}_j(mn^\alpha, m^2n)]}{\mathbb{P}[\mathcal{H}_j(mn^\alpha, mn)]}. \quad (4.6)$$

Next, similar to the reverse of the first step, we transform from mesoscopic comparison to the ratio $\frac{h_j(r, m^2n)}{h_j(r, mn)}$. By Proposition 3.4, we have

$$\frac{\mathbb{P}[\mathcal{H}_j(mn^\alpha, m^2n)]}{\mathbb{P}[\mathcal{H}_j(mn^\alpha, mn)]} \stackrel{(3.3)}{\simeq} \frac{\mathbb{P}[\mathcal{H}_j(mn^\alpha, m^2n)]}{\mathbb{P}[\mathcal{H}_j(mn^\alpha, mn)]} \cdot \frac{\mathbb{P}[\mathcal{H}_j(r, m^2n) | \mathcal{H}_j(mn^\alpha, m^2n)]}{\mathbb{P}[\mathcal{H}_j(r, mn) | \mathcal{H}_j(mn^\alpha, mn)]} = \frac{h_j(r, m^2n)}{h_j(r, mn)}. \quad (4.7)$$

The combination of (4.5), (4.6) and (4.7) gives

$$\frac{h_j(r, mn)}{h_j(r, n)} \stackrel{(4.5)}{\simeq} \frac{\mathbb{P}[\mathcal{H}_j(n^\alpha, mn)]}{\mathbb{P}[\mathcal{H}_j(n^\alpha, n)]} \stackrel{(4.6)}{\simeq} \frac{\mathbb{P}[\mathcal{H}_j(mn^\alpha, m^2n)]}{\mathbb{P}[\mathcal{H}_j(mn^\alpha, mn)]} \stackrel{(4.7)}{\simeq} \frac{h_j(r, m^2n)}{h_j(r, mn)}. \quad (4.8)$$

We now turn back to b_j . First, by Proposition 3.5,

$$\frac{b_j(r, mn)}{b_j(r, n)} = \frac{\mathbb{P}[\mathcal{B}_j(r, mn) | \mathcal{B}_j(r, n^\alpha)]}{\mathbb{P}[\mathcal{B}_j(r, n) | \mathcal{B}_j(r, n^\alpha)]} \stackrel{(3.10)}{\simeq} \frac{\mathbb{P}[\mathcal{H}_j(r, mn) | \mathcal{H}_j(r, n^\alpha)]}{\mathbb{P}[\mathcal{H}_j(r, n) | \mathcal{H}_j(r, n^\alpha)]} \stackrel{(4.5)}{\simeq} \frac{\mathbb{P}[\mathcal{H}_j(n^\alpha, mn)]}{\mathbb{P}[\mathcal{H}_j(n^\alpha, n)]}. \quad (4.9)$$

Similarly, we can use Proposition 3.5 and (4.7) instead of (4.5) to obtain

$$\frac{\mathbb{P}[\mathcal{H}_j(mn^\alpha, m^2n)]}{\mathbb{P}[\mathcal{H}_j(mn^\alpha, mn)]} \simeq \frac{b_j(r, m^2n)}{b_j(r, mn)}. \quad (4.10)$$

The combination of the above two estimates as well as (4.6) gives

$$\frac{b_j(r, mn)}{b_j(r, n)} \stackrel{(4.9)}{\simeq} \frac{\mathbb{P}[\mathcal{H}_j(n^\alpha, mn)]}{\mathbb{P}[\mathcal{H}_j(n^\alpha, n)]} \stackrel{(4.6)}{\simeq} \frac{\mathbb{P}[\mathcal{H}_j(mn^\alpha, m^2n)]}{\mathbb{P}[\mathcal{H}_j(mn^\alpha, mn)]} \stackrel{(4.10)}{\simeq} \frac{b_j(r, m^2n)}{b_j(r, mn)}. \quad (4.11)$$

This finishes the proof of Proposition 1.4. \square

Proof of Theorem 1.1. For a given j , it suffices to consider only $r \geq r_b(j)$. It follows from Proposition 1.4 and Lemmas 2.8 and 2.9 that there exist $0 < C < \infty$ and $-\infty < \alpha < \infty$ such that $b_j(r, n) = Cn^\alpha(1 + O(n^{-c}))$. By Lemma 2.6, $\alpha = -\beta_j$ and so

$$b_j(r, n) = Cn^{-\beta_j}(1 + O(n^{-c})).$$

And the same holds for h_j . This finishes the proof of Theorem 1.1. \square

4.3. Comparison estimates in the plane. Our goal of this subsection is Proposition 4.6 in which we compare the probabilities of \mathcal{Y}_j at different scales. The proof is similar to that of the half-plane case in spirit, in which we relate \mathcal{Y}_j to the exploration process and then make use of the power-law rate of convergence from [26]. However when the number of arms is odd, there are inevitably two neighboring ones of the same color that are not separated by an interface. This issue poses extra difficulties. See Lemma 4.7 and the paragraph right above it for more details.

Fix in this subsection the nice domain (in the sense of Sect. 2.2) and four points on its boundary

$$\Omega = B_1 \cup B\left((0, 1), 2 \sin(\pi/12)\right), \quad a = (0, -1), \quad b = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \\ c = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \text{and } d = (0, 1).$$

See Fig. 9 for an illustration. We also let

$$e = (0, 1 + 2 \sin(\pi/12)) \text{ and } U = B(d, 2 \sin(\pi/2)) \setminus \overline{B_1} \subset \Omega \\ \text{which contains a neighborhood of } e.$$

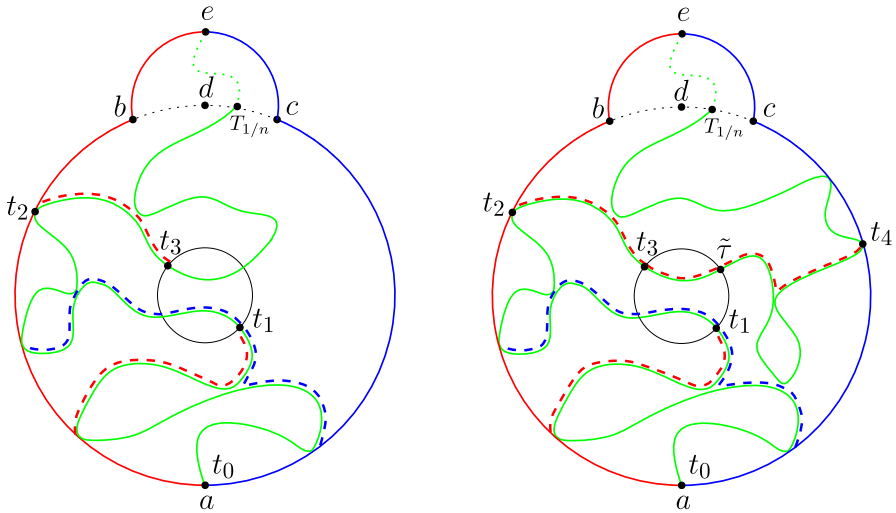


Fig. 9. Sketch of events $\mathcal{Y}_4(n^\alpha, n)$ on the left and $\mathcal{Y}_5(n^\alpha, n)$ on the right. The whole domain is $\Omega_{[1/n]}$. The small circle inside this domain is $C_{n^{\alpha-1}}$. The exploration process $\gamma_{1/n}$ from $a_{1/n}$ to $e_{1/n}$ on $\Omega_{[1/n]}$ is in green. $T_{1/n}$ is the first time that $\gamma_{1/n}$ enters $U_{[1/n]}$, after which time $\gamma_{1/n}$ is sketched in dotted curves. The arms, neighboring the interface $\gamma_{1/n}$, to enforce the occurrence of arm events are in dashed red and blue. Note that on the right the “disjointness condition” holds: $\gamma_{1/n}(t_2, t_4)$ does not hit \widehat{cdb} , $\tilde{\sigma} = t_3$, and the left boundaries of $\gamma_{1/n}(t_2, \tilde{\sigma})$ and $\gamma_{1/n}(\tilde{\tau}, t_4)$ are disjoint

Note that the value $2 \sin(\pi/12)$ equals to the length of \widehat{bc} (and \widehat{cd}). Given a mesh size η , the percolation exploration process starts from a_η and ends at e_η . We assign red color to the counterclockwise arc \widehat{eba} and blue color to the counterclockwise arc \widehat{ace} . See Fig. 9 for an illustration of the setup. Recall from Sect. 2.8 that $T_{1/n}$ is the first time that $\gamma_{1/n}$ enters $U_{1/n}$. The stopping times $T_{1/mn}$ and T are defined similarly.

Recall (2.5) that $\mathcal{Y}_j(r, R)$ is the event that in C_R with mesh size $\eta = 1$ there exist j disjoint arms connecting C_r with C_R with the prescribed color pattern and some additional constraints. Rescaling the lattice by $1/R$, we get that $\mathcal{Y}_j(r, R)$ is equivalent to the event there exist j such arms connecting $C_{r/R}$ to C_1 in B_1 with mesh size $\eta = 1/R$. Similar to the half-plane case in Sect. 4.1, we will consider two cases $R = n$ and mn in this subsection and we will tacitly assume the equivalence between events on the rescaled and those on the unrescaled lattices.

Recall the definition of $y_j(r, R)$ in (2.6). The following proposition plays a similar role to that of Proposition 4.2 in the half-plane case. It states that y_j are almost the same at different scales within a power-law error when r is not too small compared to R .

Proposition 4.6. *There exists $c_9 > 0$ such that for all $\alpha \in (1 - c_9, 1)$ and $m \in (1.1, 10)$,*

$$y_j(n^\alpha, n) = y_j(mn^\alpha, mn)(1 + O(n^{-c})).$$

Here, $O(n^{-c})$ is independent of m and α .

We postpone the proof until the end of this subsection and turn to Lemmas 4.7 and 4.8 which play same roles as Lemmas 4.3 and 4.4 in the half-plane case.

We say that $\gamma_{1/n}$ hits $C_{n^{\alpha-1}}$, \widehat{ba} , $C_{n^{\alpha-1}}$, \widehat{ac} , \dots , $(j-1)$ times before $T_{1/n}$ if there exists a sequence of times $0 = t_0 < t_1 < t_2 < \dots < t_{j-1} \leq T_{1/n}$ such that

$$\gamma_{1/n}(t_1) \in C_{n^{\alpha-1}}, \gamma_{1/n}(t_2) \in \widehat{ba}, \gamma_{1/n}(t_3) \in C_{n^{\alpha-1}}, \gamma_{1/n}(t_4) \in \widehat{ac}, \text{ etc.} \quad (4.12)$$

In particular, $\gamma_{1/n}$ is allowed to hit other arcs in the middle of two consecutive hitting times. For instance, it can hit \widehat{ac} between the times hitting $C_{n^{\alpha-1}}$ and \widehat{ba} . We say that this time sequence satisfies the “disjointness condition” if

$$\begin{aligned} & \text{left boundaries of } \gamma_{1/n}[t_{j-3}, \tilde{\sigma}] \text{ and } \gamma_{1/n}[\tilde{\tau}, t_{j-1}] \\ & \text{(which are the last two red arms) are disjoint,} \end{aligned} \quad (4.13)$$

where $\tilde{\sigma}$ denotes the first hitting time of $C_{n^{\alpha-1}}$ after t_{j-3} and $\tilde{\tau}$ denotes the last exit time from $C_{n^{\alpha-1}}$ before t_{j-1} . Following from simple geometrical arguments, we arrive at the following lemma. See also Fig. 9 for an illustration.

Lemma 4.7. *For even $j \geq 2$, one has*

$$\mathcal{Y}_j(n^\alpha, n) = \{\gamma_{1/n} \text{ hits } C_{n^{\alpha-1}}, \widehat{ba}, C_{n^{\alpha-1}}, \widehat{ac}, \dots, (j-1) \text{ times before } T_{1/n}\}, \quad (4.14)$$

and for odd $j \geq 3$, one has

$$\mathcal{Y}_j(n^\alpha, n) = \left\{ \gamma_{1/n} \text{ hits } C_{n^{\alpha-1}}, \widehat{ba}, C_{n^{\alpha-1}}, \widehat{ac}, \dots, (j-1) \text{ times before } T_{1/n} \text{ and the last two crossings satisfy the “disjointness condition” defined in (4.13)} \right\}. \quad (4.15)$$

The situation here is more complicated, since $\mathcal{Y}_j(n^\alpha, n)$ corresponds to different events according to whether j is even or odd. When j is odd, in the event $\mathcal{Y}_j(n^\alpha, n)$ there are two neighboring arms of the same color which lead to the introduction of the “disjointness condition”.

Similarly, we can show that for even $j \geq 2$,

$$\mathcal{Y}_j(mn^\alpha, mn) = \{\gamma_{1/mn} \text{ hits } C_{n^{\alpha-1}}, \widehat{ba}, C_{n^{\alpha-1}}, \widehat{ac}, \dots, (j-1) \text{ times before } T_{1/mn}\}, \quad (4.16)$$

and the same for odd $j \geq 3$ with the extra “disjointness condition”.

Before stating the second lemma, we introduce several events $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ (and \mathcal{K}_4 when j is odd) which all happen with high probability under the good coupling from Proposition 2.16 of $\gamma_{1/n}, \gamma_{1/mn}$ and γ , whose joint law we also denote by \mathbb{P} . Let u be the constant from Proposition 2.16. We will call $\widehat{ba} \cup \widehat{ac} \cup \widehat{C}_{n^{\alpha-1}}$ the **designated boundary** and note that b, a, d are the extremal points of the designated boundary. Define $\mathcal{K}_1, \mathcal{K}_2$ and \mathcal{K}_3 as

$$\begin{aligned} \mathcal{K}_1 &= \left\{ d(\gamma_{1/n}|_{[0, T_{1/n}]}, \gamma_{1/mn}|_{[0, T_{1/mn}]}) > 2n^{-u} \right\}; \\ \mathcal{K}_2 &= \left\{ \text{there does not exist any point } x \text{ on the designated boundary such that it is } n^{-v} \text{ away} \right. \\ & \quad \text{from the extremal points, and } \gamma_{1/n} \text{ enters } B(x, 2n^{-u}) \text{ but does not hit the designated} \\ & \quad \text{hit the designated boundary between the last entrance and first exit times of } B(x, n^{-v}), \\ & \quad \left. \text{and the same for } \gamma_{1/mn} \right\}; \\ \mathcal{K}_3 &= \left\{ \gamma_{1/n} \text{ and } \gamma_{1/mn} \text{ do not enter } B(b, 2n^{-v}), \text{ or } B(d, 2n^{-v}), \text{ and do not reenter } B(a, 2n^{-v}) \right. \\ & \quad \left. \text{after leaving } B(a, 1/4) \right\}, \end{aligned}$$

and for odd j 's, define \mathcal{K}_4 as

$$\mathcal{K}_4 = \left\{ \begin{array}{l} \text{there does not exist a time sequence that satisfies (4.12) and (4.13)} \\ \text{but the last two arms are } (2n^{-u})\text{-close for neither } \gamma_{1/n} \text{ nor } \gamma_{1/mn} \end{array} \right\}.$$

Here, $v \in (0, u)$ is a constant to be chosen. If \mathcal{K}_2 happens, then each time $\gamma_{1/n}$ (or $\gamma_{1/mn}$) enters $B(x, 2n^{-u})$, it will hit the designated boundary between the last entrance and first exit times of $B(x, n^{-v})$ for all points x on the designated boundary that are n^{-v} away from the extremal points.

We now give the second lemma which states that the indicator functions of $\mathcal{Y}_j(n^\alpha, n)$ and $\mathcal{Y}_j(mn^\alpha, mn)$ are identical under $\cap_i \mathcal{K}_i$. In the proof, we will use the equivalence between arm events and behavior of exploration processes, established in Lemma 4.7 and (4.16).

Lemma 4.8. *One has*

$$\mathbb{1}_{\mathcal{Y}_j(n^\alpha, n)} = \mathbb{1}_{\mathcal{Y}_j(mn^\alpha, mn)} \text{ on } \cap_{i=1}^3 \mathcal{K}_i \text{ (for } j \text{ even) or } \cap_{i=1}^4 \mathcal{K}_i \text{ (for } j \text{ odd).} \quad (4.17)$$

Proof. We begin with the case where j is even. Assume that $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ and $\mathcal{Y}_j(n^\alpha, n)$ happen and so $\gamma_{1/n}$ fulfills the right-hand side of (4.14). Suppose that $\gamma_{1/n}$ hits $C_{n^{\alpha-1}}$ at the point x . By \mathcal{K}_1 and \mathcal{K}_2 , the process $\gamma_{1/mn}$ enters $B(x, 2n^{-u})$ and so must hit $C_{n^{\alpha-1}}$ between the last entrance and first exit times of $B(x, n^{-v})$. Thus, $\gamma_{1/mn}$ hits $C_{n^{\alpha-1}}$ nearly at the same place. Suppose that $\gamma_{1/n}$ hits \widehat{ba} at the point x . From \mathcal{K}_3 we know that x is n^{-v} away from the extremal points. By \mathcal{K}_1 and \mathcal{K}_2 , the process $\gamma_{1/mn}$ enters $B(x, 2n^{-u})$ and so hits \widehat{ba} between the last entrance and first exit times of $B(x, n^{-v})$, therefore also before $T_{1/mn}$. The same holds for \widehat{ac} . Therefore, $\gamma_{1/mn}$ hits each side nearly at the same place as $\gamma_{1/n}$ before $T_{1/mn}$ and so $\gamma_{1/mn}$ fulfills the right-hand side of (4.16) which implies that $\mathcal{Y}_j(mn^\alpha, mn)$ happens. Therefore,

$$\mathcal{K}_1 \cap \mathcal{K}_2 \cap \mathcal{K}_3 \cap \mathcal{Y}_j(n^\alpha, n) \subset \mathcal{Y}_j(mn^\alpha, mn).$$

In the same way, we can show that

$$\mathcal{K}_1 \cap \mathcal{K}_2 \cap \mathcal{K}_3 \cap \mathcal{Y}_j(mn^\alpha, mn) \subset \mathcal{Y}_j(n^\alpha, n).$$

This completes the proof of (4.17).

For odd j , we add one more constraint \mathcal{K}_4 to ensure that we can still find a time sequence that satisfies the “disjointness condition” even after minor perturbation of the process. Then, it is easy to verify that $\mathbb{1}_{\mathcal{Y}_j(n^\alpha, n)} = \mathbb{1}_{\mathcal{Y}_j(mn^\alpha, mn)}$ on $\mathcal{K}_1 \cap \mathcal{K}_2 \cap \mathcal{K}_3 \cap \mathcal{K}_4$, since after a minor perturbation of $2n^{-u}$, the same time sequence (which may differ by a distance of n^{-v}) still satisfies (4.12) and the “disjointness condition”. \square

We are now ready to prove the main result of this subsection.

Proof of Proposition 4.6. First, we give upper bounds to $\mathbb{P}[\mathcal{K}_1^c]$, $\mathbb{P}[\mathcal{K}_2^c]$, $\mathbb{P}[\mathcal{K}_3^c]$ and $\mathbb{P}[\mathcal{K}_4^c]$. By Proposition 2.16, there exists $u > 0$ such that under the good coupling of $\gamma_{1/n}$, $\gamma_{1/mn}$ and γ

$$\mathbb{P}[\mathcal{K}_1^c] = \mathbb{P}[d(\gamma_{1/n}|_{[0, T_{1/n}]}, \gamma_{1/mn}|_{[0, T_{1/mn}]}) > 2n^{-u}] = O(n^{-u}). \quad (4.18)$$

Taking v as the constant in Lemma 2.17 applied with the same u , we obtain that $\mathbb{P}[\mathcal{K}_2^c] < O(n^{-c})$ for some $c > 0$ depending only on u, v . Similar to the estimates of $\mathbb{P}[\mathcal{J}_2^c]$

and $\mathbb{P}[\mathcal{J}_3^c]$ in the proof of Proposition 4.2, we can choose some $v_0 \in (0, u)$ such that $\mathbb{P}[\mathcal{K}_2^c] \leq O(n^{-c})$ for all $v \in (0, v_0)$ where c is a constant depending only on u (note that \mathcal{K}_2^c is an increasing event in v). In addition, for all $v > 0$, $\mathbb{P}[\mathcal{K}_3^c] \leq O(n^{-cv})$ where we reduce c if necessary.

Therefore, for even j and $v = v_0$, under the good coupling

$$\begin{aligned} & |y_j(n^\alpha, n) - y_j(mn^\alpha, mn)| \\ & \stackrel{(4.17)}{\leq} \mathbb{P}[\mathcal{K}_1^c] + \mathbb{P}[\mathcal{K}_2^c] + \mathbb{P}[\mathcal{K}_3^c] \leq O(n^{-u}) + O(n^{-c}) + O(n^{-cv_0}). \end{aligned} \quad (4.19)$$

Now, we give an upper bound to $\mathbb{P}[\mathcal{K}_4^c]$. We will call $C_1 \cup C_{n^{\alpha-1}}$ the extended boundary (which is different from the designated boundary we defined before). We will argue that if \mathcal{K}_4^c happens, we have a whole-plane 6-arm event from a $(2n^{-u})$ -ball in B_1 to distance n^{-v} , or on the extended boundary a half-plane 4-arm event from distance $3n^{-v}$ to $1/4$. We assume that \mathcal{K}_4^c happens for $\gamma_{1/n}$ and $j \equiv 1 \pmod{4}$. The cases for $\gamma_{1/mn}$ or $j \equiv 3 \pmod{4}$ can be treated similarly.

If \mathcal{K}_4^c happens, then the last two red arms, i.e., the left boundaries of $\gamma_{1/n}[t_{j-3}, \tilde{\sigma}]$ and $\gamma_{1/n}[\tilde{\tau}, t_{j-1}]$ are both $(2n^{-u})$ -close to a point x . If x is n^{-v} away from the extended boundary, we write σ^1 (resp. σ^3) for the first hitting time of $C(x, 2n^{-u})$ after t_{j-3} (resp. $\tilde{\tau}$) and τ^1 (resp. τ^3) for the last exit time of $C(x, n^{-v})$ before σ^1 (resp. σ^3). Write σ^2 (resp. σ^4) for the first exit time of $C(x, n^{-v})$ after σ^1 (resp. σ^3) and τ^2 (resp. τ^4) for the last exit time of $C(x, 2n^{-u})$ before σ^2 (resp. σ^4). Then,

$$\tau^1 < \sigma^1 \leq \tau^2 < \sigma^2 < \tau^3 < \sigma^3 \leq \tau^4 < \sigma^4.$$

Furthermore, the left boundaries of $\gamma_{1/n}[\tau^1, \sigma^1]$, $\gamma_{1/n}[\tau^2, \sigma^2]$, $\gamma_{1/n}[\tau^3, \sigma^3]$, $\gamma_{1/n}[\tau^4, \sigma^4]$ are four disjoint red arms from $B(x, 2n^{-u})$ to distance n^{-v} , and the right boundaries of $\gamma_{1/n}[\tau^1, \sigma^1]$, $\gamma_{1/n}[\tau^3, \sigma^3]$ are two disjoint blue arms from $B(x, 2n^{-u})$ to distance n^{-v} . So, there is a whole-plane 6-arm event from a $(2n^{-u})$ -ball in B_1 to distance n^{-v} . If x is n^{-v} -close to a point y on the extended boundary, then the left and right boundaries of $\gamma_{1/n}[\tau^2, \sigma^2]$, $\gamma_{1/n}[\tau^3, \sigma^3]$ (where we replace $2n^{-u}$ in the definition to $2n^{-v}$ and n^{-v} to $1/4$) are four disjoint crossings from $B(y, 3n^{-v})$ to distance $1/4$. We can cover B_1 by a $(2n^{-u})$ -net with $O(n^{2u})$ elements and cover the extended boundary by a $(2n^{-v})$ -net with $O(n^v)$ elements. Thus,

$$\mathbb{P}[\mathcal{K}_4^c] = O(n^{-cv}) + O\left(n^{2u+(2+c)(v-u)}\right), \text{ where we reduce } c \text{ if necessary.}$$

Therefore, for odd $j \geq 3$ and $v \leq v_0$, under the good coupling

$$\begin{aligned} & |y_j(n^\alpha, n) - y_j(mn^\alpha, mn)| \leq \mathbb{P}[\mathcal{K}_1^c] + \mathbb{P}[\mathcal{K}_2^c] + \mathbb{P}[\mathcal{K}_3^c] + \mathbb{P}[\mathcal{K}_4^c] \\ & \leq O(n^{-u}) + O(n^{-c}) + O(n^{-cv}) + O(n^{-cv}) + O\left(n^{2u+(2+c)(v-u)}\right). \end{aligned}$$

Take $v \leq v_0$ such that

$$2u + (2+c)(v-u) < 0. \quad (4.20)$$

By applications of RSW theory, $y_j(mn^\alpha, mn) > n^{C(\alpha-1)}$. We can take c_9 such that

$$-C c_9 > \max\{-u, -c, -cv, 2u + (2+c)(v-u)\}.$$

Then, for all $\alpha \in (1 - c_9, 1)$, $|y_j(n^\alpha, n) - y_j(mn^\alpha, mn)| = O(n^{-c'}) y_j(mn^\alpha, mn)$ as desired. \square

4.4. Proof of Theorems 1.2 and 1.3. In this subsection, we will complete the proof of Theorems 1.2 and 1.3.

The proof of Proposition 1.5 is the same as that of Proposition 1.4 except that we will replace Proposition 3.4 with Proposition 3.8 and Proposition 4.2 with Proposition 4.6, and is therefore omitted. We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We start with y_j . For a given j , it suffices to consider only $r \geq r_y(j)$. Combining with Proposition 1.5 and Lemmas 2.8 and 2.9, there exist $0 < C < \infty$ and α such that $y_j(r, n) = Cn^\alpha(1 + O(n^{-c}))$. By Lemma 2.6, $\alpha = -\alpha_j$.

We now turn to x_j . By applying Proposition 3.10 we obtain that

$$\frac{x_j(r, mn)}{x_j(r, n)} = \frac{y_j(r, mn)}{y_j(r, n)} \left(1 + O(n^{-c})\right) \quad (4.21)$$

for r, m satisfying the requirement of Proposition 1.5. We thus establish an estimate similar to (1.9) for x_j , which yields (1.5) for x_j . \square

We now turn to Theorem 1.3.

Proof of Theorem 1.3. By Proposition 3.9,

$$\frac{a_j(1, n)}{x_j(1, n)} \stackrel{(3.14)}{=} \frac{a_j(\epsilon n, n)}{x_j(\epsilon n, n)} \left(1 + O(\epsilon^c)\right). \quad (4.22)$$

Thanks to Claim (4) of Lemma 2.6, the first term of the right-hand side of (4.22) converges to $g_j(\epsilon)/f_j(\epsilon)$ as $n \rightarrow \infty$. Hence the LHS of (4.22) must also converge to a constant as $n \rightarrow \infty$. Combined with (1.5) for x_j , this finishes the proof. \square

Remark 4.9. Note that although $a_6(1, n) > 0$, we did not include the case $j = 6$ in the statement of Theorem 1.3, because $y_6(1, n) = 0$ by definition and hence (4.21) no longer holds. In this case (plus some other inner initial configurations), obtaining sharp asymptotics for $x_6(1, n)$ as well as for $a_6(1, n)$ requires some extra coupling argument, which we omit for brevity.

5. Proof of Coupling Results

In this section, we give the proof of the couplings in the half-plane and plane cases stated in Propositions 3.3 and 3.6 respectively. As the idea for all these couplings are quite similar, we will give a detailed proof for Proposition 3.3 in Sect. 5.1 but only sketch the differences in the details of the proof for Proposition 3.6 in Sect. 5.2.

Before going into details for each specific setup, we briefly explain here the core idea of these couplings. In essence, one divides the domain into exponential scales (“layers” in the text) and couples configurations of each layer step by step. In each step, when passing from one scale to the next, one can show by the separation lemma and RSW-FKG gluing technique that (although under different types of conditioning) the laws of nice configurations in previous scales are absolutely continuous with respect to each other with bounded Radon-Nykodym derivative, and hence different conditional laws can be coupled with positive probability at each scale. Thus the coupling will succeed in the end with high probability. It can also be proved that with high probability, some good events will happen which encompass all the dependence of the past and future configurations under the conditional laws. The result we want follows from the combination of these two facts.

5.1. $j \geq 2$ arms in the half-plane. In this subsection, we give the proof of Proposition 3.3. We divide the (discretized) half-plane \mathbb{H} into the disjoint union of dyadic semi-annuli $A_i = B_{r_{i+1}}^+ \setminus B_{r_i}^+$ (for simplicity in this subsection we do not add the $+$ in the superscript), where $r_i = 2^i$ for $i = 0, 1, 2, \dots$. Call the A_i 's the **layers** in \mathbb{H} . We will do the coupling layer by layer.

In the following, we fix some annulus region $A_i \cup A_{i+1}$ between C_r^+ and C_R^+ and consider the percolation configuration inside this annulus. We start by defining the **good event** in $A_i \cup A_{i+1}$.

Definition 5.1 (*Good event*). Define the good event $\mathcal{G}_j^{(i)}$ associated with the percolation configuration inside $A_i \cup A_{i+1}$ as follows:

- There are exactly $(j - 1)$ interfaces crossing the annulus $A_i \cup A_{i+1}$. Denote them by $\gamma_1, \dots, \gamma_{j-1}$ in counterclockwise order, then γ_k is adjacent to γ_{k+1} for any $1 \leq k \leq j - 2$. In addition, $\gamma_1, \dots, \gamma_{j-1}$ are well-separated on both ends.
- There is a red path connecting $[r_i, 2r_i]$ to γ_1 in A_i and a red path connecting $[2r_i, 4r_i]$ to γ_1 in A_{i+1} .
- There are two red (if j is odd) or blue (if j is even) paths which connect $[-2r_i, -r_i]$ and $[-4r_i, -2r_i]$ to γ_{j-1} in A_i and A_{i+1} , respectively.

We will need that the good event $\mathcal{G}_j^{(i)}$ in $A_i \cup A_{i+1}$ happens with at least some positive probability depending only on j .

Lemma 5.2. *There is a constant $c(j) > 0$ such that for any $r_i \geq 10j$, we have $\mathbb{P}[\mathcal{G}_j^{(i)}] \geq c$.*

This lemma could be proved in a way similar to the second proof of [11, Lemma 3.4] by constructing pivotal points through the exploration process (to ensure the interface conditions) together with RSW-FKG gluing (to ensure the well-separateness conditions). Here we present the following alternative proof which relies on a combination of RSW-FKG gluing and BK-Reimer's inequality.

Proof. Define a **quasi-good** event \mathcal{U} in $A_i \cup A_{i+1}$ as follows:

- There are two configurations of well-separated inner and outer faces, say $\Theta = \{\theta_1, \dots, \theta_j\}$ and $\Theta' = \{\theta'_1, \dots, \theta'_j\}$, which are around $C_{4r_i}^+$ and $C_{r_i}^+$ resp., and lying completely in $\mathbb{H} \setminus B_{3r_i}^+$ and in $B_{2r_i}^+$ resp.
- For each $1 \leq k \leq j$, both θ_k and θ'_k are red (if k is odd) or blue (if k is even). Furthermore, there is a path connecting θ_k, θ'_k together with the aforementioned color.

From RSW-FKG gluing technique we see $\mathbb{P}[\mathcal{U}] \geq c_0$ for some positive constant $c_0(j)$. If \mathcal{U} happens, there will be exactly $(j - 1)$ interfaces crossing $A_i \cup A_{i+1}$ which end at the $(2j - 2)$ endpoints of Θ, Θ' , respectively. Write

$$\mathcal{W} := \left\{ \text{each interface is adjacent to both of its neighbors} \right\}.$$

By the observation that $\mathcal{U} \cap \mathcal{W} \subset \mathcal{G}_j^{(i)}$ and $\mathcal{U} \cap \mathcal{W}^c \subset \mathcal{U} \cap \mathcal{B}_1(2r_i, 3r_i)$ (recall the notation \square from Sect. 2.3), we conclude that (by applying BK-Reimer's inequality (Lemma 2.3) in the last step)

$$\mathbb{P}[\mathcal{G}_j^{(i)}] \geq \mathbb{P}[\mathcal{U} \cap \mathcal{W}] \geq \mathbb{P}[\mathcal{U}] - \mathbb{P}[\mathcal{U} \cap \mathcal{B}_1(2r_i, 3r_i)] \geq \mathbb{P}[\mathcal{U}](1 - \mathbb{P}[\mathcal{B}_1(2r_i, 3r_i)]) \quad (5.1)$$

which is bounded below by some $c(j) > 0$. \square

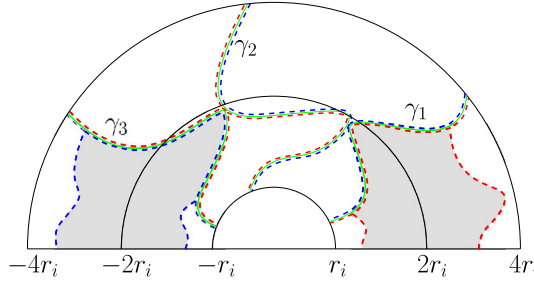


Fig. 10. The good set \mathcal{S} when $\mathcal{G}_j^{(i)}$ holds with $j = 4$. The three interfaces γ_i , $i = 1, 2, 3$ crossing $A^+(r_i, 4r_i)$ are in green. All the hexagons that are adjacent to these interfaces are sketched in dashed red or blue curves, and they make up part of the good set \mathcal{S} . The gray regions on both sides are bounded by the interfaces γ_1 and γ_3 with some innermost paths (crossings) in A_{i+1} and outermost ones in A_i . The (colored) hexagons in these gray regions make up the rest part of \mathcal{S}

Definition 5.3 (*Good set*). If $\mathcal{G}_j^{(i)}$ holds, let \mathcal{S} be the union of the following colored hexagons (we keep the notation used in the definition of $\mathcal{G}_j^{(i)}$):

- The hexagons that touch at least one of the $(j - 1)$ interfaces;
- The hexagons in the region enclosed by four curves, namely the innermost red path in A_{i+1} from $[2r_i, 4r_i]$ to γ_1 , the outermost red path in A_i from $[r_i, 2r_i]$ to γ_1 , γ_1 , and the real axis;
- The hexagons in the region enclosed by four curves, namely the innermost red (if j is odd) or blue (if j is even) path in A_{i+1} from $[-4r_i, -2r_i]$ to γ_j , the outermost red (if j is odd) or blue (if j is even) path in A_i from $[-2r_i, -r_i]$ to γ_{j-1} , γ_{j-1} , and the real axis.

If $\mathcal{G}_j^{(i)}$ fails, set $\mathcal{S} = \emptyset$. For a nonempty set \mathcal{S} of hexagons in $A_i \cup A_{i+1}$, we say that \mathcal{S} is a **good set** if \mathcal{S} is a possible realization for \mathcal{S} such that $\mathcal{G}_j^{(i)}$ holds. See Fig. 10 for an illustration.

Fix a large integer $K_1(j)$ such that

$$p(K_1) := \sup_{i \in \mathbb{N}} \mathbb{P}[\mathcal{B}_{K_1}(r_i, r_{i+1})] < 2^{-10-8\beta_j}. \quad (5.2)$$

Note that this is possible since $\beta_j \asymp j^2$; same for the definition of K_2 and K_4 later in this section, noting that $a_j \asymp j^2$ too. The following proposition is a generalization of (3.1) and (3.2) of [11].

Lemma 5.4. *There are constants $C, C' > 0$ depending only on j , such that for any $2^{K_1+1}r \leq r_i < 4r_i \leq 2^{-K_1-1}R$, the following holds: for any good set $\mathcal{S} \subset A_i \cup A_{i+1}$, any configuration of outer faces Θ around $C_{r_i+K_1+3}^+$ with no more than K_1 faces, and any color configuration ω_0 coincides with Θ and satisfies $\mathbb{P}[\mathcal{H}_j(r, R) \mid \omega_{\mathcal{D}_\Theta} = \omega_0] > 0$,*

$$C^{-1}2^{-|S|} \leq \mathbb{P}[\mathcal{S} = S \mid \mathcal{H}_j(r, R), \omega_{\mathcal{D}_\Theta} = \omega_0] \leq C2^{-|S|}. \quad (5.3)$$

Furthermore,

$$\mathbb{P}[\mathcal{G}_j^{(i)} \mid \mathcal{H}_j(r, R), \omega_{\mathcal{D}_\Theta} = \omega_0] \geq C'. \quad (5.4)$$

And similar bounds also hold if we replace $\mathcal{H}_j(r, R)$ by $\mathcal{B}_j(r, R)$. In a similar fashion, same results also hold for inner faces Θ around $C_{r_i-K_1-1}^+$ satisfying the same requirements.

Proof. We give the proof of (5.3) and (5.4), and the rest are similar. Rewrite the probability in (5.3) by Bayes' theorem as

$$\mathbb{P}[S = S \mid \mathcal{H}_j(r, R), \omega_{\mathcal{D}_\Theta} = \omega_0] = 2^{-|S|} \times \frac{\mathbb{P}[\mathcal{H}_j(r, R) \mid S = S, \omega_{\mathcal{D}_\Theta} = \omega_0]}{\mathbb{P}[\mathcal{H}_j(r, R) \mid \omega_{\mathcal{D}_\Theta} = \omega_0]}. \quad (5.5)$$

Let Θ_1 (resp. Θ_2) be the configuration of inner faces around $C_{4r_i}^+$ (resp. the configuration of outer faces around $C_{r_i}^+$) induced by S . Denote \mathcal{R} for the event that there are j arms connecting C_R^+ to each face of Θ_1 , and \mathcal{W} for the event that there are j arms connecting $[-r, r]$ to each face of Θ_2 (each arm has the same color with the face it connects to). Then we see that the event in numerator of (5.5) is equivalent to $\mathcal{R} \cap \mathcal{W}$ conditioned on $\{\omega_{\mathcal{D}_\Theta} = \omega_0\}$. By independence it has probability $\mathbb{P}[\mathcal{R} \mid \omega_{\mathcal{D}_\Theta} = \omega_0] \cdot \mathbb{P}[\mathcal{W}]$.

Since Θ_2 is well-separated, by the separation lemma (Lemma 2.10) and RSW-FKG gluing,

$$\mathbb{P}[\mathcal{W}] \asymp \mathbb{P}[\mathcal{H}_j(r, r_{i-1})]. \quad (5.6)$$

In addition, by (2.13) in Proposition 2.12 and RSW-FKG gluing, we also have

$$\mathbb{P}[\mathcal{R} \mid \omega_{\mathcal{D}_\Theta} = \omega_0] \asymp \mathbb{P}[\mathcal{B}_j(r_{i+3}, R) \mid \omega_{\mathcal{D}_\Theta} = \omega_0]. \quad (5.7)$$

Finally, similar to the proof of Proposition 2.12, we can show the following quasimultiplicativity

$$\mathbb{P}[\mathcal{B}_j(r_{i+3}, R) \mid \omega_{\mathcal{D}_\Theta} = \omega_0] \cdot \mathbb{P}[\mathcal{H}_j(r, r_{i-1})] \asymp \mathbb{P}[\mathcal{H}_j(r, R) \mid \omega_{\mathcal{D}_\Theta} = \omega_0]. \quad (5.8)$$

Combining (5.5) and (5.6), (5.7) and (5.8) we see that (5.3) holds. Summing over all good set $S \subset A_i \cup A_{i+1}$, and using Lemma 5.2, we get (5.4) as desired. \square

Proof of Proposition 3.3. We focus on the first claim in Proposition 3.3. Recall that $u = \sqrt{rR}$. Let $i_0 < i_N$ be the integers such that $r_{i_0-1} < u \leq r_{i_0} < r_{i_N} \leq R < r_{i_N+1}$. We inductively couple the conditional laws $\mathbb{P}_1 = \mathbb{P}[\cdot \mid \mathcal{H}_j(r, R)]$ and $\mathbb{P}_2 = \mathbb{P}[\cdot \mid \mathcal{H}_j(r, mR)]$ from outside to inside layer by layer. The virtue of the coupling is same as the maximal coupling for a Markov chain on a finite state space.

For initialization, sample the color configuration ω_1, ω_2 outside $C_{i_N}^+$ according to \mathbb{P}_1 and \mathbb{P}_2 independently, and set an index $I = 0$. Assume now we have sampled two color configurations ω_1 and ω_2 outside $C_{r_i+K_1+1}^+$ for some $i \geq i_0$ according to some coupling of these two conditional distribution, we proceed as follow:

If $I \geq 0$, then for both configurations, we independently explore all the interfaces crossing $A^+(r_{i+K_1}, r_{i+K_1+1})$ from outside to inside and stop exploring when reaching $\mathbb{R} \cup C_{r_{i+K_1}}^+$. This exploration process will induce two configurations of outer faces around $C_{r_i+K_1}^+$ for both $\omega_i, i = 1, 2$, denoted by $\hat{\Theta}_i, i = 1, 2$. Also note that $\mathcal{V}_{\hat{\Theta}_1}$ and $\mathcal{V}_{\hat{\Theta}_2}$ are left unexplored. If either $\hat{\Theta}_1$ or $\hat{\Theta}_2$ has more than K_1 faces, we just keep I unchanged, explore all the hexagons outside $C_{r_i+K_1}^+$ and proceed the previous procedure on the scale $C_{r_i+K_1}^+$. Otherwise, we increase I by 1, and by applying Lemma 5.4 for both $\mathcal{H}_j(r, R)$

and $\mathcal{H}_j(r, mR)$, we can construct a coupling \mathbb{Q}_i of the laws $\mathbb{P}_1[\cdot \mid \omega_{\mathcal{D}_{\hat{\Theta}_1}} = \omega_1]$ and $\mathbb{P}_2[\cdot \mid \omega_{\mathcal{D}_{\hat{\Theta}_2}} = \omega_2]$ with the following property:

$$\mathbb{Q}_i[\mathcal{S}(\omega_1) = \mathcal{S}(\omega_2) \neq \emptyset] > C'', \text{ for some } C''(j) > 0, \quad (5.9)$$

where $\mathcal{S}(\omega_1), \mathcal{S}(\omega_2)$ are the sets induced by ω_1, ω_2 in $A_i \cup A_{i+1}$, respectively.

We sample the pair of color configurations (ω_1, ω_2) outside $C_{r_i}^+$ from \mathbb{Q}_i , and perform the following exploration process to detect whether $\{\mathcal{S}(\omega_1) = \mathcal{S}(\omega_2) \neq \emptyset\}$ holds:

- Step 1** For each of the endpoints x of $\hat{\Theta}_1$, we explore the interface of ω_1 inside $B_{r_i+K_1}^+$ starting from x , and stop exploring until it hits $\mathbb{R} \cup C_{r_i}^+$. Denote Γ for the subset of the aforementioned interfaces reaching $C_{r_i}^+$. Then we can check whether $|\Gamma| = j - 1$ and each interface in Γ is adjacent to its neighbors in $A^+(r_i, 4r_i)$.
- Step 2** If the condition in **Step 1** holds, denote γ_1 and γ_{j-1} for the leftmost and rightmost interface in Γ . In the quad Q_1 enclosed by $C_{4r_i}^+, \gamma_1, C_{r_i}^+$ and \mathbb{R} , we start from $Q_1 \cap C_{2r_i}^+$ and explore in ω_1 to find the innermost red path in A_{i+1} connecting $[2r_i, 4r_i]$ to γ_1 , and the outermost red path in A_i connecting $[r_i, 2r_i]$ to γ_1 . Our exploration process stops whenever we find such paths, or if we cannot find them, we stop exploring until reaching $C_{r_i}^+$ or $C_{4r_i}^+$. Perform the same exploration on the left side of $A_i \cup A_{i+1}$. Up to now, we already have enough information to fix $\mathcal{S}(\omega_1)$ and to assert whether the good event $\mathcal{G}_j^{(i)}$ happens or not for ω_1 .
- Step 3** Run the same exploration process for ω_2 . Now we can check whether $\mathcal{G}_j^{(i)}$ happens for both ω_1 and ω_2 and $\mathcal{S}(\omega_1) = \mathcal{S}(\omega_2)$. If so, we set $I = -1$; otherwise we keep I unchanged.

If I is still non-negative, we explore the entire color configuration ω_1, ω_2 outside $C_{r_i}^+$, and proceed the procedure above on the scale r_i . Otherwise $I = -1$, then the identical nonempty good set $\mathcal{S}(\omega_1) = \mathcal{S}(\omega_2)$ induces a configuration of outer faces Θ^* with j faces around $C_{r_i}^+$, which is common for both $\omega_i, i = 1, 2$. Note that our exploration process ensures that none of the hexagons in \mathcal{V}_{Θ^*} are explored, i.e. the law in \mathcal{V}_{Θ^*} is still the critical Bernoulli percolation, and thus by the domain Markov property, $\mathbb{P}[\cdot \mid \mathcal{H}_j(r, R), \omega_{\mathcal{D}_{\Theta^*}} = \omega_1]$ and $\mathbb{P}[\cdot \mid \mathcal{V}_j(r, mR), \omega_{\mathcal{D}_{\Theta^*}} = \omega_2]$ are precisely equal. Hence, we can couple the color configuration in \mathcal{V}_{Θ^*} of these two distributions identically. In this case we have (ω_1, ω_2) satisfies the requirement in proposition 3.3, and Θ^* is a stopping set as desired.

Finally, we control the probability that our coupling remains unsuccessful until scale r_{i_0} . Write $N_1 = \frac{N-2K_1}{2K_1+2}$, then $N_1 \geq \frac{1}{4K_1+4} \log_2 \frac{R}{r} - 2$ since $N \geq \frac{1}{2} \log_2 \frac{R}{r} - 1$. On the one hand, every time I increases by 1, we always have a positive probability C'' to couple successfully, no matter how things went in previous layers. Thus

$$\mathbb{P}[I \geq N_1] \leq (1 - C'')^{N_1} \leq (r/R)^\delta \text{ for some } \delta(j) > 0.$$

On the other hand, $I < N_1$ implies there are more than $N - (K_1 + 1)N_1 - K_1 = N/2$ integers $i \in [i_0, i_N - 1]$ satisfying $\mathbf{1}_{\mathcal{B}_{K_1}(r_i, r_{i+1})}(\omega_1) + \mathbf{1}_{\mathcal{B}_{K_1}(r_i, r_{i+1})}(\omega_2) \geq 1$, so that

$$\mathcal{T} := \left\{ \sum_{i=i_0}^{i_N-1} \mathbf{1}_{\mathcal{B}_{K_1}(r_i, r_{i+1})} \geq N/4 \right\}$$

happens for either ω_1 or ω_2 . Combining

$$\mathbb{P}_1[T] \leq 2^N p(K_1)^{N/4} / \mathbb{P}[\mathcal{H}_j(r, R)], \quad \mathbb{P}_2[T] \leq 2^N p(K_1)^{N/4} / \mathbb{P}[\mathcal{H}_j(r, mR)],$$

with the a priori estimates of arm probabilities, we obtain

$$\mathbb{P}[0 \leq I \leq N_1] \leq 2^N p(K_1)^{N/4} \left[(R/r)^{-\beta_j+o(1)} + (mR/r)^{-\beta_j+o(1)} \right],$$

which is bounded by $(r/R)^\delta$ for some $\delta(j) > 0$ from the choice of K_1 in (5.2). Altogether we conclude $\mathbb{P}[I \geq 0] \leq (r/R)^\delta$ for some $\delta(j) > 0$, as desired.

With these tools in hand, we can rerun the coupling from inside to outside to conclude the proof of the second claim of Proposition 3.3. \square

5.2. Coupling arm events in the plane. In this subsection, we prove Proposition 3.6. Recall the definitions of events $\mathcal{X}_j(r, R)$, $\mathcal{Y}_j(r, R)$, and $\mathcal{A}_j(r, R)$. We shall couple $\mathbb{P}[\cdot \mid \mathcal{X}_j(r, R)]$ with $\mathbb{P}[\cdot \mid \mathcal{A}_j(r, R)]$ and $\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, R)]$ with $\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, mR)]$. Just as in the half-plane case, it is quite feasible to couple these conditional laws so that ω_1 and ω_2 coincide on some outer face Θ^* around C_u ($u = \sqrt{rR}$) with high probability. However, as hinted earlier in Remark 3.7, the sole existence of an outer face is insufficient for decoupling the configurations on the inner and outer sides of Θ^* and thus we cannot establish (3.11), (3.12) immediately. To tackle this issue, we aim to couple the configurations on some “good set” which not only induces an outer face, but also encodes enough geometric information that decouples the configurations on both sides of the face. We will present the precise definitions for the whole-plane good sets later and obtain very similar estimates as before. As discussed at the beginning of this section, we will just sketch the proofs and note necessary modifications of arguments from Sect. 5.1. For clarification, the cases of odd j ’s and even j ’s are dealt with separately.

As before, we divide the plane into the union of dyadic annuli $A_i = B_{r_{i+1}} \setminus B_{r_i}$, where $r_i = 2^i$ for $i = 0, 1, 2, \dots$. Call them the **layers** in the plane.

5.2.1. Inward coupling: the odd j case We begin with the case for odd j . For each $r_i > 10j$, define good event in $A_i \cup A_{i+1}$ as follows:

- There are exactly $(j-1)$ interfaces crossing the annulus, say $\gamma_1, \dots, \gamma_{j-1}$, in counterclockwise order and they are well-separated on both sides of the annulus.
- for $1 \leq k \leq j-2$, γ_k is adjacent to γ_{k+1} , while γ_1 is *not* adjacent to γ_{j-1} .
- The hexagons between γ_1 and γ_{j-1} that touch $\gamma_1 \cup \gamma_{j-1}$ are all red (if $j \equiv 1 \pmod{4}$) or blue (if $j \equiv 3 \pmod{4}$). Further, there are two paths with the aforementioned color connecting γ_1 and γ_{j-1} , which lie in A_i and A_{i+1} , respectively.

And if $\mathcal{G}_j^{(i)}$ holds, let \mathcal{S} be the union of all hexagons which touch at least one of the interfaces γ_k , $1 \leq k \leq j-1$, or lie in the quad enclosed by γ_1, γ_{j-1} , the outermost and innermost paths connecting γ_1 and γ_{j-1} in A_i and A_{i+1} , respectively. We set $\mathcal{S} = \emptyset$ if $\mathcal{G}_j^{(i)}$ fails, and for a nonempty set of colored hexagons in $A_i \cup A_{i+1}$, we say that $\mathcal{S} \neq \emptyset$ is a **good set** if \mathcal{S} is a possible realization of \mathcal{S} . The key feature regarding to the good set is the following: for any good set $S \subset A_i \cup A_{i+1}$, denoting Θ_S the outer face around C_{r_i} induced by S , then for any ω_1, ω_2 such that $\mathcal{S}(\omega_1) = \mathcal{S}(\omega_2) = S$,

$$\mathbb{P}[\cdot \mid \mathcal{X}_j(r, R), \omega_{\mathcal{D}_{\Theta_S}} = \omega_1] = \mathbb{P}[\cdot \mid \mathcal{A}_j(r, R), \omega_{\mathcal{D}_{\Theta_S}} = \omega_2], \quad (5.10)$$

and

$$\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, R), \omega_{\mathcal{D}_{\Theta_S}} = \omega_1] = \mathbb{P}[\cdot \mid \mathcal{Y}_j(r, mR), \omega_{\mathcal{D}_{\Theta_S}} = \omega_2]. \quad (5.11)$$

We have the following estimates similarly as before:

- There exists $c(j) > 0$, such that for any $r_i > 10j$, we have $\mathbb{P}[\mathcal{G}_j^{(i)}] > c$.
- For any fixed large integer K_2 , there exists $C = C(j, K_2) > 0$, such that for any good set S in $A_i \cup A_{i+1}$ with $2^{K_2+1}r \leq r_i < r_{i+2} \leq 2^{-K_2-1}R$, any configuration of outer faces Θ around $C_{r_i+K_2+3}^+$ with no more than K_2 faces and any color configuration ω_0 which satisfies $\mathbb{P}[\mathcal{A}_j(r, R) \mid \omega_{\mathcal{D}_{\Theta}} = \omega_0] > 0$,

$$C^{-1}2^{-|S|} \leq \mathbb{P}[S = S \mid \mathcal{A}_j(r, R), \omega_{\mathcal{D}_{\Theta}} = \omega_0] \leq C2^{-|S|}. \quad (5.12)$$

As a result, $\mathbb{P}[\mathcal{G}_j^{(i)} \mid \mathcal{A}_j(r, R), \omega_{\mathcal{D}_{\Theta}} = \omega_0] \geq cC^{-1}$.

- In addition, similar results also hold for $\mathcal{A}_j(r, R)$ replaced by $\mathcal{X}_j(r, R)$ or $\mathcal{Y}_j(r, R)$.

Sketch of Proof for Proposition 3.6, odd j case. Fix a large integer $K_2 = K_2(j)$ such that

$$p(K_2) := \sup_{i \in \mathbb{N}} \mathbb{P}[\mathcal{A}_{K_2}(r_i, r_{i+1})] < 2^{-10-8\alpha_j}.$$

The first two items can be established similarly as in Lemmas 5.2 and 5.4, results for $\mathcal{X}_j(r, R)$ and $\mathcal{Y}_j(r, R)$ can also be obtained from the same manner. From these estimates, following the framework of the proof of Proposition 3.3, we can construct a coupling of $\mathbb{P}[\cdot \mid \mathcal{A}_j(r, R)]$ (resp. $\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, R)]$) and $\mathbb{P}[\cdot \mid \mathcal{X}_j(r, R)]$ (resp. $\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, mR)]$) such that if we sample (ω_1, ω_2) according to such coupling, then with probability at least $(1 - (r/R)^\delta)$ for some $\delta(j) > 0$, ω_1 and ω_2 are identical on some good set lying outside of B_u . Thanks to (5.10) (resp. (5.11)), we may further couple the configurations inside the face identically. Therefore, under such a coupling, with probability at least $(1 - (r/R)^\delta)$ there exists a common outer face Θ^* around C_u which is a stopping set for both ω_1, ω_2 , and (3.11) (resp. (3.12)) holds. This completes the proof. \square

5.2.2. Inward coupling: even j case Finally, we deal with the case for even j . This is essentially the case studied in [11]; we follow the idea therein and define good events $\mathcal{G}_j^{(i)}$ in $A_i \cup A_{i+1}$ as

- There are exactly j interfaces crossing the annulus $A_i \cup A_{i+1}$, and they are well-separated on both ends. In addition, each of the interfaces is adjacent with its two neighbors.

If $\mathcal{G}_j^{(i)}$ holds, let \mathcal{S} be the union of all colored hexagons in $A_i \cup A_{i+1}$ which touch at least one of the j interfaces, otherwise we set $\mathcal{S} = \emptyset$. For a nonempty set S of colored hexagons in $A_i \cup A_{i+1}$, we say that S is a **good set** if $S \neq \emptyset$ is a possible realization of \mathcal{S} .

The proof of the first item in Proposition 3.6 is essentially same as before. More precisely, for any good set S and ω_1, ω_2 such that $\mathcal{S}(\omega_1) = \mathcal{S}(\omega_2) = S$, it holds that

$$\mathbb{P}[\cdot \mid \mathcal{X}_j(r, R), \omega_{\mathcal{D}_{\Theta_S}} = \omega_1] = \mathbb{P}[\cdot \mid \mathcal{A}_j(r, R), \omega_{\mathcal{D}_{\Theta_S}} = \omega_2], \quad (5.13)$$

which implies (3.11) for $\Theta^* = \Theta_S$. However, for the second item, in order to guarantee (3.12), it is not enough to make ω_1 and ω_2 coincide on some good set. Indeed, we

need to further specify the xconnecting pattern between this good set and the outer boundary C_R (or C_{mR}). For a color configuration ω_0 sampled from $\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, R)]$ (resp. $\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, mR)]$), assume that under ω_0 , for some $u' \in [r, R]$ there is a configuration of outer faces Θ around C_u with j faces. Denote $\tilde{\theta}$ for the first red face in Θ when counting from $(0, -u')$ counterclockwise, and denote $\hat{\theta}$ for the red face in Θ which is connected to C_R (resp. C_{mR}) by the first red arm counting from $(0, -R)$ (resp. $(0, -mR)$) counterclockwise. Similar to (5.11), for any good set S and any ω_1, ω_2 such that $\mathcal{S}(\omega_i) = S$ and $\tilde{\theta}(\omega_i) = \hat{\theta}(\omega_i)$ for $i = 1, 2$, we have

$$\mathbb{P}[\cdot \mid \mathcal{Y}_j(r, R), \omega_{\mathcal{D}_{\Theta_S}} = \omega_1] = \mathbb{P}[\cdot \mid \mathcal{Y}_j(r, mR), \omega_{\mathcal{D}_{\Theta_S}} = \omega_2]. \quad (5.14)$$

Then we have the following estimates:

- There exists $c(j) > 0$, such that for any $r_i > 10j$, we have $\mathbb{P}[\mathcal{G}_j^{(i)}] > c$.
- Fix a large integer K_3 , there exists $C = C(j) > 0$, such that for any good set S in $A_i \cup A_{i+1}$ with $2^{K_3+1}r \leq r_i < r_{i+2} < 2^{-K_3-1}R$, any configuration of outer faces Θ around $C_{r_i+K_3+3}^+$ with no more than K_3 faces and any color configuration ω_0 satisfies $\mathbb{P}[\mathcal{Y}_j(r, R) \mid \omega_{\mathcal{D}_{\Theta}} = \omega_0] > 0$, denoting Θ_S for the configuration of outer faces around C_{r_i} induced by S , then it holds that

$$C^{-1}2^{-|S|} \leq \mathbb{P}[S = S, \tilde{\theta}_S = \hat{\theta}_S \mid \mathcal{Y}_j(r, R), \omega_{\mathcal{D}_{\Theta}} = \omega_0] \leq C2^{-|S|}. \quad (5.15)$$

As a result, $\mathbb{P}[\mathcal{G}_j^{(i)}, \tilde{\theta}_S = \hat{\theta}_S \mid \mathcal{Y}_j(r, R), \omega_{\mathcal{D}_{\Theta}} = \omega_0] \geq cC^{-1}$.

Sketch of Proof for Proposition 3.6, even j case. Fix a large integer $K_3(j)$ such that

$$p(K_3) := \sup_{i \in \mathbb{N}} \mathbb{P}[\mathcal{A}_{K_3}(r_i, r_{i+1})] \leq 2^{-10-8\alpha_j}.$$

The estimates above can be proved similarly, with the caveat that for (5.15), we construct some specific gluing to make sure that $\tilde{\theta}_S$ coincides with $\hat{\theta}_S$. Then, following the same framework as before, (5.13), (5.14) together with the estimates above yield the desired results. \square

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A. Proof of Lemmas 2.6 and 2.8

Proof of Lemma 2.6. We begin with the proof of $h_j(n) \asymp b_j(n)$. Since $\mathcal{H}_j(r, n) \subset \mathcal{B}_j(r, n)$, we have $h_j(n) \leq b_j(n)$. It suffices to prove that $h_j(n) \geq cb_j(n)$ for some constant $c > 0$ and all n large enough. Let $r' = 10j \vee r$ and Γ be the set of interfaces connecting C_n^+ and $C_{r'}^+$ in $A^+(r', n)$. Then,

$$\begin{aligned} h_j(r, n) &\geq \mathbb{P}[\mathcal{H}_j(r, n) | Q_{\text{in}}(\Gamma) > j^{-1}, \mathcal{B}_j(r', n)] \\ &\quad \times \mathbb{P}[Q_{\text{in}}(\Gamma) > j^{-1} | \mathcal{B}_j(r', n)] \times \mathbb{P}[\mathcal{B}_j(r', n)] \\ &\geq c \times c \times b_j(r', n) \geq cb_j(r, n). \end{aligned} \quad (\text{A.1})$$

(The second inequality follows from RSW theory and Lemma 2.10. The third inequality follows from the fact that $\mathcal{B}_j(r, n) \subset \mathcal{B}_j(r', n)$.) We can derive in the same way that $x_j(n) \asymp a_j(n) \asymp y_j(n)$ using Proposition 2.14 instead of Lemma 2.10. The relation $p_j(n) \asymp a_j(n)$ follows from a color-switching argument; see Remark 6 in [32] for more details.

Now, we consider Claim (3). We can show the up-to-constants equivalence between arm probabilities using similar arguments as in (A.1). The methods in [32] for calculating asymptotics of arm probabilities down to microscopic scales (i.e., $b_j(n)$ or $p_j(n)$ in our notation) can also be applied to derive mesoscopic asymptotics.

Claim (4) follows from [6, 31] and the estimates of the half-plane there arm event and whole-plane six arm event in critical percolation. One can refer to the proof of Lemma 2.9 in [11] for more details. \square

Proof of Lemma 2.8. We only prove the case of h_j . The cases for other arm probabilities can be proved in the same way. Let Γ be the set of interfaces connecting $[-r, r]$ and C_n^+ in B_n^+ . For $\epsilon > 0$, define $\mathcal{H}_j^\epsilon(r, n)$ by

$$\mathcal{H}_j^\epsilon(r, n) = \mathcal{H}_j(r, n) \cap \{Q_{\text{ex}}(\Gamma) \geq \epsilon\}.$$

Let \mathcal{R}^ϵ denote the event that there exists a point x on C_n^+ such that there are three disjoint arms (not all of the same color) connecting $C(x, \epsilon n)$ and $C(x, n/2)$. By Claim (3) in Lemma 2.6, $\mathbb{P}[\mathcal{R}^\epsilon] \leq O(1/\epsilon) \times O(\epsilon^{1+c}) = O(\epsilon^c)$. Thanks to the spatial independence of percolation

$$\begin{aligned} \mathbb{P}[\mathcal{H}_j(r, n) \setminus \mathcal{H}_j^\epsilon(r, n)] &\leq \mathbb{P}[\mathcal{H}_j(r, n/2)] \cdot \mathbb{P}[\mathcal{R}^\epsilon] \\ &\leq C \mathbb{P}[\mathcal{H}_j(r, n)] \times O(\epsilon^c) = O(\epsilon^c) \mathbb{P}[\mathcal{H}_j(r, n)]. \end{aligned}$$

Therefore,

$$\mathbb{P}[\mathcal{H}_j^\epsilon(r, n) | \mathcal{H}_j(r, n)] \geq 1 - O(\epsilon^c). \quad (\text{A.2})$$

For all $n \leq t \leq (1 + \epsilon/K)n$, $\mathbb{P}[\mathcal{H}_j(r, t) | \mathcal{H}_j^\epsilon(r, n)] \geq 1 - j(1 - c)^K$. This is because on the event $\mathcal{H}_j^\epsilon(r, n)$ each outer face has a length of at least ϵn which can be partitioned into at least K pieces with a length of at least $(1 + \epsilon/K)n$, and by applications of RSW theory, each piece has a probability at least c to be connected with $C_{(1+\epsilon/K)n}^+$, independently of others. Therefore, for all $n \leq t \leq (1 + \epsilon/K)n$,

$$\begin{aligned} h_j(t) &= \mathbb{P}[\mathcal{H}_j(r, t)] \geq \mathbb{P}[\mathcal{H}_j(r, t) | \mathcal{H}_j^\epsilon(r, n)] \times \mathbb{P}[\mathcal{H}_j^\epsilon(r, n) | \mathcal{H}_j(r, n)] \times \mathbb{P}[\mathcal{H}_j(r, n)] \\ &\geq (1 - j(1 - c)^K)(1 - O(\epsilon^c))h_j(r, n). \end{aligned}$$

Picking K large and ϵ small, we then complete the proof. \square

B. Proof of Lemma 2.9

Proof of Lemma 2.9. For $m \in (1.1, 10)$ and integer $k \geq 1$, let $b_k(m) = a_{m^{k+1}}/a_{m^k}$. We note that all constants in $O(\cdot)$'s in this proof are uniform w.r.t. $m \in (1.1, 10)$. By Assumption (1) in the statement, $b_{k+1}(m) = b_k(m)(1 + O(m^{-ck}))$. Then, it is easy to see that $\lim_{k \rightarrow \infty} b_k(m)$ exists (which will be denoted as $C(m)$) and is positive (since $b_k(m)$ is positive when k is sufficiently large). In addition,

$$b_k(m) = C(m) \left(1 + O(m^{-ck})\right), \quad \text{and equivalently} \quad \frac{a_{m^{k+1}}}{C(m)^{k+1}} = \frac{a_{m^k}}{C(m)^k} \left(1 + O(m^{-ck})\right).$$

Therefore, $\lim_{k \rightarrow \infty} \frac{a_{m^k}}{C(m)^k}$ exists (which will be denoted by $B(m)$) and is positive. In addition,

$$a_{m^k} = B(m)C(m)^k \left(1 + O(m^{-ck})\right). \quad (\text{B.1})$$

Next, we use Assumption (2) to prove that there exist $\alpha \in (-\infty, \infty)$ and $C \in (0, \infty)$ such that $C(m) = m^\alpha$ and $B(m) = C$ for all m . It suffices to show that

$$\begin{aligned} \log(C(m_1))/\log(m_1) &= \log(C(m_2))/\log(m_2) \text{ and} \\ B(m_1) &= B(m_2) \text{ for all } m_1, m_2 \in (1.1, 10). \end{aligned}$$

Fix $m_1, m_2 \in (1.1, 10)$. Without loss of generality, we can assume that $\log_{m_2}(m_1)$ is irrational, because we can always find another m_3 such that $\log_{m_3}(m_1)$ and $\log_{m_3}(m_2)$ are both irrational and the case of m_1, m_2 follows from those of m_1, m_3 and m_2, m_3 . Let $\delta, \epsilon > 0$ denote two small constants to be chosen later. Since $\log_{m_2}(m_1)$ is irrational, we can find a sequence of increasing integers $\{p_j\}_{j \geq 1}$ and $\{q_j = \lfloor p_j \log_{m_2}(m_1) \rfloor\}_{j \geq 1}$ such that for all $j \geq 1$

$$q_j \leq p_j \log_{m_2}(m_1) \leq q_j + \epsilon. \quad (\text{B.2})$$

By Assumption (2), there exists a constant $c = c(\delta) > 0$ such that for all $\epsilon < c$

$$\liminf_{n \rightarrow \infty} \inf_{10^{-\epsilon} n \leq t \leq n} \frac{a_t}{a_s} > 1 - \delta.$$

By (B.2) and the fact that $m_2 < 10$, we have $10^{-\epsilon} m_1^{p_j} \leq m_2^{-\epsilon} m_1^{p_j} \leq m_2^{q_j} \leq m_1^{p_j}$ and so for j large enough

$$(1 - \delta) a_{m_1^{p_j}} \leq a_{m_2^{q_j}} \leq \frac{1}{1 - \delta} a_{m_1^{p_j}}.$$

Together with (B.1)

$$(1 - \delta) \left(1 + O(m_1^{-cp_j}) + O(m_2^{-cq_j})\right) \leq \frac{B(m_2)C(m_2)^{q_j}}{B(m_1)C(m_1)^{p_j}} \leq \frac{1}{1 - \delta} \left(1 + O(m_1^{-cp_j}) + O(m_2^{-cq_j})\right).$$

Let j tend to infinity. Since $\lim_{j \rightarrow \infty} \frac{q_j}{p_j} = \log_{m_2}(m_1)$, we have

$$\alpha := \log(C(m_1))/\log(m_1) = \log(C(m_2))/\log(m_2).$$

Note that when $\alpha \geq 0$, $10^{-\alpha\epsilon} m_1^{\alpha p_j} \leq m_2^{\alpha q_j} \leq m_1^{\alpha p_j}$; when $\alpha < 0$, $10^{-\alpha\epsilon} m_1^{\alpha p_j} \geq m_2^{\alpha q_j} \geq m_1^{\alpha p_j}$. Therefore,

$$(1 - \delta) 10^{-\epsilon(\alpha \wedge 0)} B(m_1) \left(1 + O(m_1^{-cp_j})\right) \leq B(m_2) \leq \frac{1}{1 - \delta} B(m_1) 10^{\epsilon(\alpha \vee 0)} \left(1 + O(m_1^{-cp_j})\right).$$

Let j tend to infinity, then ϵ to zero and finally δ to zero. We have $B(m_1) = B(m_2)$. This completes the proof. \square

C. Proof of the Strong Separation Lemma

Proof of Proposition 2.11. The proof we employ here is a combination of techniques used in [11,30]. We only prove the first bullet point (2.11). Write $s = \text{dist}(r, x_2, \dots, x_j, -r)$. Then $Q(\Theta) = s/r$. Let $M = \lfloor \log_2(r/s) \rfloor \vee 0$ and $L = \lfloor \log_2(R/r) \rfloor$. Set

$$r_i = \begin{cases} r, & i = 0; \\ r + 2^{i-1}s, & 1 \leq i \leq M; \\ 2^{i-M}r, & M+1 \leq i \leq M+L. \end{cases}$$

Let Γ_i be the collection of interfaces in Γ truncated at their first hitting on $C_{r_i}^+$. Recall that $Q_{\text{ex}}(\Gamma_i)$ is the exterior quality of Γ_i on $C_{r_i}^+$ defined in (2.8). Let $d_i := r_i Q_{\text{ex}}(\Gamma_i)$ be the minimal distance between $-r_i$, r_i and the endpoints of Γ_i . Define the **relative qualities**

$$Q^*(i) := \begin{cases} d_i/(2^i s), & 0 \leq i \leq M; \\ Q_{\text{ex}}(\Gamma_i), & M+1 \leq i \leq M+L. \end{cases}$$

We set $Q^*(i) := 0$ if not all j interfaces manage to reach $C_{r_i}^+$. Furthermore, set

$$f_i := \mathbb{P}[Q^*(i) > 0], \quad g_i(\rho) := \mathbb{P}[Q^*(i) > \rho] \text{ for } \rho > 0.$$

With the above definitions, it suffices to show

$$g_{M+L}(j^{-1}) \geq c(j) f_{M+L} \quad (\text{C.1})$$

for some $c(j) > 0$ that only depends on j . To this end, we need the following facts about the quality, which can be obtained by using RSW-FKG gluing techniques, similar to the appendices of [11,30] for the plane case. We omit the details.

- By application of the RSW theory, there is $c_{10}(j) > 0$ such that

$$f_1 \geq c_{10}. \quad (\text{C.2})$$

- For any $\delta > 0$, there exists $\rho_0(\delta, j)$ such that for all $\rho \leq \rho_0$ and $i \geq 0$,

$$f_{i+1} - g_{i+1}(\rho) \leq \delta f_i. \quad (\text{C.3})$$

- For any $\rho > 0$, there exists $R(\rho, j) > 0$ such that for all $i \geq 0$,

$$g_{i+1}(j^{-1}) \geq R g_i(\rho). \quad (\text{C.4})$$

Let $K = K(\rho)$ be the smallest integer in the range $1 \leq K \leq M+L$ such that $g_i(\rho) \leq f_i/2$ for all $K < i \leq M+L$, where we set $K = M+L$ if $g_{M+L}(\rho) > f_{M+L}/2$. We claim that there exists $\rho_1 > 0$ such that for all $\rho \leq \rho_1$,

$$f_K \leq 2g_K(\rho). \quad (\text{C.5})$$

This follows by definition if $K \geq 2$. If $K = 1$, we set $\delta_0 := c_{10}/2$ and $\rho_1 := \rho_0(\delta_0, j)$ with c_{10} and ρ_0 from the first and second bullet points above respectively. By (C.3), for all $\rho \leq \rho_1$, we have $f_1 - g_1(\rho) \leq \delta_0$. This combined with (C.2) gives $f_1 - g_1(\rho) \leq f_1/2$ for all $\rho \leq \rho_1$, which implies (C.5).

In the following, for any $\delta > 0$, we let $\rho \leq \rho_0(\delta, j) \wedge \rho_1$, then both (C.3) and (C.5) hold. Furthermore, by (C.3), $f_i \leq 2\delta f_{i-1}$ for all $K < i \leq M + L$. Iterating this, we get

$$f_{M+L} \leq (2\delta)^{M+L-K} f_K \leq (2\delta)^{M+L-K} 2g_K(\rho), \quad (\text{C.6})$$

where in the last inequality we have used (C.5). Repeated application of (C.4) gives that

$$g_K(\rho) \leq R(\rho)^{-1} R(j^{-1})^{K+1-M-L} g_{M+L}(j^{-1}). \quad (\text{C.7})$$

Combining (C.6) with (C.7), we have

$$f_{M+L} \leq 2R(\rho)^{-1} R(j^{-1}) (2\delta/R(j^{-1}))^{M+L-K} g_{M+L}(j^{-1}).$$

Letting δ be sufficiently small such that $2\delta/R(j^{-1}) \leq 1$, we obtain (C.1) by setting $c(j)^{-1} = 2R(\rho)^{-1} R(j^{-1})$. This completes the proof. \square

References

1. Aizenman, M., Duplantier, B., Aharony, A.: Path-crossing exponents and the external perimeter in 2d percolation. *Phys. Rev. Lett.* **83**(7), 1359 (1999)
2. Beffara, V., Nolin, P.: On monochromatic arm exponents for 2D critical percolation. *Ann. Probab.* **39**(4), 1286–1304 (2011)
3. Binder, I.: Rate of convergence of critical interfaces to SLE curves. In *Extended Abstracts Fall 2019*, pp. 43–50. Birkhäuser (2021)
4. Binder, I., Chayes, L., Lei, H.K.: On the rate of convergence for critical crossing probabilities. *Ann. Inst. Henri Poincaré Probab. Stat.* **51**(2), 672–715 (2015)
5. Binder, I., Richards, L.: Convergence rates of random discrete model curves approaching sle curves in the scaling limit. Preprint, (2020)
6. Camia, F., Newman, C.M.: Two-dimensional critical percolation: the full scaling limit. *Commun. Math. Phys.* **268**(1), 1–38 (2006)
7. Den Nijs, M.: A relation between the temperature exponents of the eight-vertex and q-state potts model. *J. Phys. A Math. Gen.* **12**(10), 1857 (1979)
8. den Nijs, M.: Extended scaling relations for the magnetic critical exponents of the potts model. *Phys. Rev. B* **27**(3), 1674 (1983)
9. Duminił-Copin, H.: Sixty years of percolation. In: *Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018*, pp. 2829–2856. World Scientific (2018)
10. Duplantier, B.: Harmonic measure exponents for two-dimensional percolation. *Phys. Rev. Lett.* **82**(20), 3940 (1999)
11. Garban, C., Pete, G., Schramm, O.: Pivotal, cluster, and interface measures for critical planar percolation. *J. Am. Math. Soc.* **26**(4), 939–1024 (2013)
12. Grimmett, G.: *Percolation*. Grundlehren der Mathematischen Wissenschaften, vol. 321. Springer, Berlin (1999)
13. Holden, N., Li, X., Sun, X.: Natural parametrization of percolation interface and pivotal points. *Ann. Inst. Henri Poincaré Probab. Stat.* **58**(1), 7–25 (2022)
14. Kenyon, R.: The asymptotic determinant of the discrete Laplacian. *Acta Math.* **185**(2), 239–286 (2000)
15. Kesten, H.: The incipient infinite cluster in two-dimensional percolation. *Probab. Theory Relat. Fields* **73**, 369–394 (1986)
16. Kesten, H.: Scaling relations for 2D-percolation. *Commun. Math. Phys.* **109**(1), 109–156 (1987)
17. Kesten, H., Sidoravicius, V., Zhang, Y.: Almost all words are seen in critical site percolation on the triangular lattice. *Electron. J. Probab.* **3**(10), 1–75 (1998)
18. Lan, S.-Y., Ma, J., Zhou, W.: The rate of convergence of harmonic explorer to SLE4. *arXiv preprint arXiv:2003.01949* (2020)
19. Lawler, G., Sun, X., Wu, W.: Four-dimensional loop-erased random walk. *Ann. Probab.* **47**(6), 3866–3910 (2019)
20. Lawler, G.F.: The dimension of the frontier of planar Brownian motion. *Electron. Commun. Probab.* **1**(5), 29–47 (1996)
21. Lawler, G.F., Schramm, O., Werner, W.: One-arm exponent for critical 2D percolation. *Electron. J. Probab.* **7**(3), 1–13 (2002)

22. Li, X., Shiraishi, D.: One-point function estimates for loop-erased random walk in three dimensions. *Electron. J. Probab.* **24**(111), 1–46 (2019)
23. Mendelson, D., Nachmias, A., Watson, S.S.: Rate of convergence for Cardy's formula. *Commun. Math. Phys.* **329**(1), 29–56 (2014)
24. Nolin, P.: Near-critical percolation in two dimensions. *Electron. J. Probab.* **13**, 1562–1623 (2008)
25. Nolin, P., Qian, W., Sun, X., Zhuang, Z.: Backbone exponent for two-dimensional percolation. *arXiv preprint arXiv:2309.05050* (2023)
26. Richards, L.: Convergence rates of random discrete model curves approaching SLE curves in the scaling limit. PhD thesis, University of Toronto (Canada), (2021). Available for download at <https://hdl.handle.net/1807/106368>
27. Saleur, H., Duplantier, B.: Exact determination of the percolation hull exponent in two dimensions. *Phys. Rev. Lett.* **58**(22), 2325 (1987)
28. Schramm, O.: Scaling limits of loop-erased random walks and uniform spanning trees. *Isr. J. Math.* **118**, 221–288 (2000)
29. Schramm, O.: Conformally invariant scaling limits: an overview and a collection of problems. In: *International Congress of Mathematicians. Vol. I*, pp. 513–543. European Mathematical Society (2007)
30. Schramm, O., Steif, J.E.: Quantitative noise sensitivity and exceptional times for percolation. *Ann. Math.* **171**(2), 619–672 (2010)
31. Smirnov, S.: Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.* **333**(3), 239–244 (2001)
32. Smirnov, S., Werner, W.: Critical exponents for two-dimensional percolation. *Math. Res. Lett.* **8**(5–6), 729–744 (2001)
33. Tassion, V., Vanneuville, H.: Noise sensitivity of percolation via differential inequalities. *Proc. Lond. Math. Soc.* **126**(4), 1063–1091 (2023)
34. Viklund, F.J.: Convergence rates for loop-erased random walk and other Loewner curves. *Ann. Probab.* **43**(1), 119–165 (2015)
35. Werner, W.: Lectures on two-dimensional critical percolation. In: *IAS Park City Mathematics Series*, vol. 16, pp. 297–360. American Mathematical Society (2009)

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