PROBABILITY and MATHEMATICAL PHYSICS

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5:1
2024

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PROBABILITY and
MATHEMATICAL
PHYSICS
Vol. 5, No. 1, 2024
https://doi.org/10.2140/pmp.2024.5.1

REGULARITY AND CONFLUENCE OF GEODESICS FOR THE SUPERCRITICAL LIOUVILLE QUANTUM GRAVITY METRIC

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Let h be the planar Gaussian free field and let D_h be a supercritical Liouville quantum gravity (LQG) metric associated with h. Such metrics arise as subsequential scaling limits of supercritical Liouville first passage percolation (Ding and Gwynne, 2020) and correspond to values of the matter central charge $\mathbf{c}_{\mathrm{M}} \in (1, 25)$. We show that a.s. the boundary of each complementary connected component of a D_h-metric ball is a Jordan curve and is compact and finite-dimensional with respect to D_h. This is in contrast to the *whole* boundary of the D_h-metric ball, which is noncompact and infinite-dimensional with respect to D_h (Pfeffer, 2021). Using our regularity results for boundaries of complementary connected components of D_h-metric balls, we extend the confluence of geodesics results of Gwynne and Miller (2019) to the case of supercritical Liouville quantum gravity. These results show that two D_h-geodesics with the same starting point and different target points coincide for a nontrivial initial time interval.

1. Introduction

1.1. ***Overview.*** *Liouville quantum gravity* (LQG) is a family of models of random surfaces originating in the physics literature in the 1980s [4; 11; 35]. One way to define LQG surfaces is in terms of the *matter central charge*, a parameter $\mathbf{c}_{\mathrm{M}} \in (-\infty, 25)$. Let $U \subset \mathbb{C}$ be open. For a Riemannian metric tensor g on U, let Δ_g be the associated Laplace–Beltrami operator and let $\det \Delta_g$ denote its determinant. Heuristically speaking, an LQG surface parametrized by U is the random two-dimensional Riemannian manifold (U, g), where g is sampled from the "uniform measure on Riemannian metric tensors on U weighted by $(\det \Delta_g)^{-\mathbf{c}_{\mathrm{M}}/2}$". We refer to the case when $\mathbf{c}_{\mathrm{M}} < 1$ as the *subcritical case* and the case when $\mathbf{c}_{\mathrm{M}} \in (1, 25)$ as the *supercritical case*.

The above definition of LQG is very far from rigorous, but it is nevertheless possible to define LQG surfaces rigorously. One way to do this is via the *David–Distler–Kawai* (*DDK*) *ansatz* [4; 11], which

MSC2020: 60D05, 60G60.
Keywords: Liouville quantum gravity, Gaussian free field, Liouville quantum gravity metric, supercritical Liouville quantum gravity, confluence of geodesics, Liouville first passage percolation.

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says that, at least in the subcritical phase, the Riemannian metric tensor associated with an LQG surface can be expressed in terms of the exponential of a variant of the Gaussian free field (GFF) h on U. We refer to [3; 39; 40] for background on the GFF. The GFF h is a random distribution, not a function, so its exponential is not well-defined. But, one can construct objects associated with the exponential of h by replacing h by a family of continuous functions $\{h_\varepsilon\}_{\varepsilon>0}$ which approximate h, then taking a limit as $\varepsilon \to 0$.

In the subcritical and critical cases, i.e., when $\boldsymbol{c}_{\mathrm{M}} \le 1$, this approach has been used to construct the *Liouville quantum gravity area measure* (i.e., the volume form) as a limit of regularized versions of $e^{\gamma h}$ integrated against Lebesgue measure [13; 14; 15; 25; 37], where $\gamma \in (0, 2]$ is related to the central charge by

$$\boldsymbol{c}_{\mathrm{M}} = 25 - 6Q^2, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{1-1}$$

Most mathematical works on LQG consider only the case when $\boldsymbol{c}_{\mathrm{M}} \le 1$ and use γ, rather than $\boldsymbol{c}_{\mathrm{M}}$, as the parameter for the model.

The focus of the present paper is the *metric* (Riemannian distance function) associated with an LQG surface, which can be defined for all $\boldsymbol{c}_{\mathrm{M}} \in (-\infty, 25)$. Let us explain the construction of this metric for the GFF on the whole plane. For $t > 0$ and $z \in \mathbb{C}$, we define the heat kernel $p_t(z) := \frac{1}{2\pi t} e^{-|z|^2/2t}$ and we denote its convolution with the whole-plane GFF[1] h by

$$h_\varepsilon^*(z) := (h * p_{\varepsilon^2/2})(z) = \int_{\mathbb{C}} h(w) p_{\varepsilon^2/2}(z-w)\, dw^2 \quad \text{for all } z \in \mathbb{C} \tag{1-2}$$

where the integral is interpreted in the sense of distributional pairing.

For a parameter $\xi > 0$, we define the *ε-Liouville first passage percolation* (LFPP) metric associated with h, with parameter ξ, by

$$D_h^\varepsilon(z, w) := \inf_P \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)|\, dt \quad \text{for all } z, w \in \mathbb{C} \tag{1-3}$$

where the infimum is over all piecewise continuously differentiable paths $P : [0, 1] \to \mathbb{C}$ from z to w.

To extract a nontrivial limit of the metrics D_h^ε, we need to renormalize. We define our renormalizing factor by

$$\mathfrak{a}_\varepsilon := \text{median of } \inf\left\{ \int_0^1 e^{\xi h_\varepsilon^*(P(t))} |P'(t)|\, dt : P \text{ is a left-right crossing of } [0,1]^2 \right\}, \tag{1-4}$$

where a left-right crossing of $[0, 1]^2$ is a piecewise continuously differentiable path $P : [0, 1] \to [0, 1]^2$ joining the left and right boundaries of $[0, 1]^2$. We emphasize that $\mathfrak{a}_\varepsilon$ is the median of a random variable (the inf of the lengths of the left-right crossings) so is deterministic.

It was shown in [8, Proposition 1.1] that for each $\xi > 0$, there exists $Q = Q(\xi) > 0$ such that

$$\mathfrak{a}_\varepsilon = \varepsilon^{1-\xi Q + o_\varepsilon(1)}, \quad \text{as } \varepsilon \to 0. \tag{1-5}$$

Furthermore, Q is a nonincreasing function of ξ and satisfies $\lim_{\xi\to 0} Q(\xi) = \infty$ and $\lim_{\xi\to\infty} Q(\xi) = 0$.

[1] The whole-plane GFF h is only defined modulo additive constant. Throughout the paper, we assume that the additive constant is chosen so that the average of h over the unit circle is zero unless otherwise stated.

As explained in [8] (see also [17]), the parameter ξ is (heuristically) related to the matter central charge by

$$\mathbf{c}_{\mathrm{M}} = 25 - 6Q(\xi)^2. \tag{1-6}$$

The dependence of Q on ξ, or equivalently the dependence of ξ on $\mathbf{c}_{\mathrm{M}}$, is not known explicitly except that $Q(1/\sqrt{6}) = 5/\sqrt{6}$, which corresponds to $\mathbf{c}_{\mathrm{M}} = 0$ [6]. Define

$$\xi_{\mathrm{crit}} := \inf\{\xi > 0 : Q(\xi) = 2\}. \tag{1-7}$$

We do not know ξ_{crit} explicitly, but the bounds from [22, Theorem 2.3] give the reasonably good approximation $\xi_{\mathrm{crit}} \in [0.4135, 0.4189]$. By (1-6) and the properties of $Q(\xi)$ from [8, Proposition 1.1], we have

$$\mathbf{c}_{\mathrm{M}} < 1 \quad \Leftrightarrow \quad \xi < \xi_{\mathrm{crit}} \qquad \text{and} \qquad \mathbf{c}_{\mathrm{M}} \in (1, 25) \quad \Leftrightarrow \quad \xi > \xi_{\mathrm{crit}}. \tag{1-8}$$

In the subcritical case, it was shown in [5] that for $\xi < \xi_{\mathrm{crit}}$, the rescaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ admit nontrivial subsequential scaling limits with respect to the topology of uniform convergence on compact subsets of $\mathbb{C} \times \mathbb{C}$. Subsequently, it was shown in [20] that the subsequential limit is unique and is characterized by a certain list of natural axioms. The limit D_h of $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ is called the *LQG metric* with parameter ξ.

The LQG metric in the subcritical case induces the same topology as the Euclidean metric, but its geometric properties are very different. For example, the Hausdorff dimension of the metric space $(\mathbb{C}, D_h)$ is $\gamma/\xi > 2$ [23]. Another important property of D_h is *confluence of geodesics*, which states that two D_h-geodesics (i.e., paths of minimal D_h-length) with the same starting point and different target points typically coincide for a nontrivial initial time interval. Note that this is not true for geodesics for a smooth Riemannian metric. Confluence of geodesics for the subcritical LQG metric was first established in [18] and played a key role in the uniqueness proof in [20]. See also [16; 24] for extensions of the confluence property for subcritical LQG, [26] for an earlier proof of confluence of geodesics for the Brownian map (which is equivalent to LQG with $\mathbf{c}_{\mathrm{M}} = 0$ [30; 32]), and [1; 27; 29] for stronger confluence results in the Brownian map setting.

In this paper, we will mainly be interested in the supercritical and critical cases, i.e., $\xi \geq \xi_{\mathrm{crit}}$. It was shown in [8] that for this range of parameter values, the rescaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ are tight with respect to the topology on lower semicontinuous functions on $\mathbb{C} \times \mathbb{C}$ introduced by Beer [2] (see Definition 2.1). Later, after this paper appeared on the arXiv, it was shown in [9] that the subsequential limit is unique. The proof in [9] uses some of the results in this paper (in particular, those in Section 3.2), so throughout this paper we will work with subsequential limits.

If D_h is a subsequential limit of LFPP for $\xi > \xi_{\mathrm{crit}}$, then D_h is a metric on $\mathbb{C}$ which is allowed to take on infinite values. This metric does not induce the Euclidean topology: rather, there is an uncountable, Euclidean-dense set of *singular points* $z \in \mathbb{C}$ such that

$$D_h(z, w) = \infty \quad \text{for all } w \in \mathbb{C} \setminus \{z\}. \tag{1-9}$$

On the other hand, for two fixed points $z, w \in \mathbb{C}$, a.s. $D_h(z, w) < \infty$, and the restriction of D_h to the complement of the set of singular points defines a complete metric [34]. Roughly speaking, singular points for D_h correspond to α-thick points of h for $\alpha > Q$, i.e., points $z \in \mathbb{C}$ for which $h_\varepsilon^*(z)$ behaves like $\alpha \log \varepsilon^{-1}$ as $\varepsilon \to 0$ [8; 34]. It was shown in [7] that the metric D_h induces the Euclidean topology on $\mathbb{C}$ for $\xi = \xi_{\text{crit}}$. In particular, there are no singular points in this case.

Due to the existence of singular points, D_h-metric balls in the supercritical case are highly irregular objects. A D_h-ball has empty Euclidean interior (since the singular points are Euclidean dense). Moreover, the D_h-boundary of a D_h-metric ball is not D_h-compact and has infinite Hausdorff dimension with respect to D_h [34] (see Theorem 1.2). See Figure 1 for a simulation of a supercritical LQG metric ball.

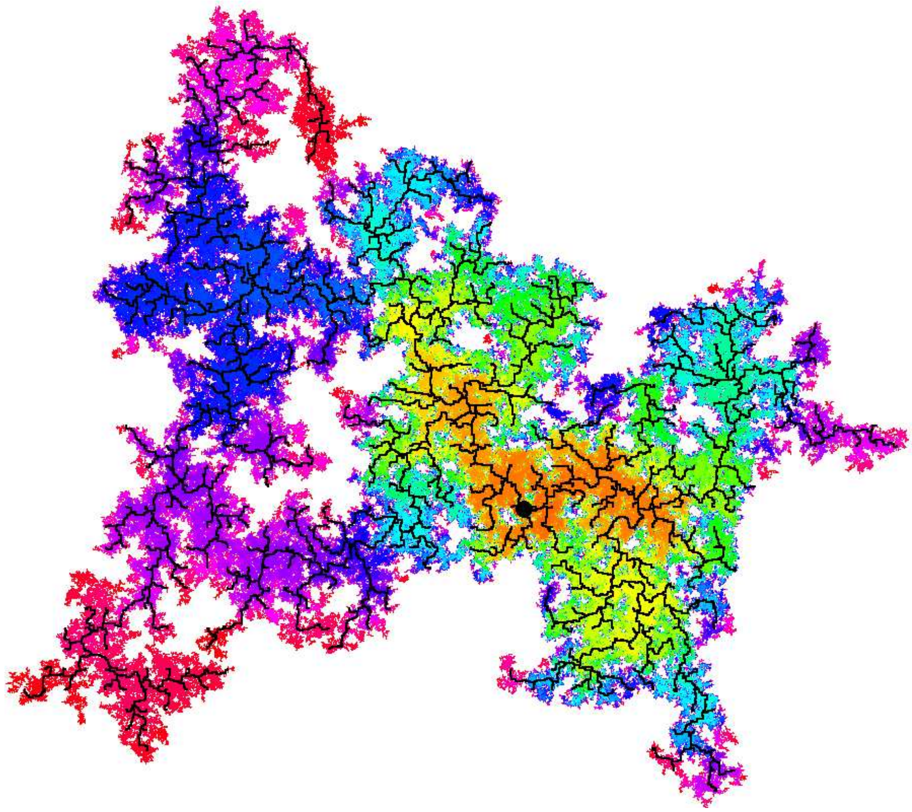

Figure 1. Simulation of an LFPP metric ball for $\xi = 1.6 > \xi_{\text{crit}}$. The colors indicate distance to the center point (marked with a black dot) and the black curves are geodesics from the center point to other points in the ball. These geodesics have a tree-like structure, which is consistent with our confluence of geodesics results. We also note that there are many "holes" corresponding to complementary connected components of the ball. The boundary of each of these holes is of the form $\partial \mathcal{B}_s^{y,\bullet}$ for some $y \in \mathbb{C}$. The simulation was produced using LFPP with respect to a discrete GFF on a 1024×1024 subset of $\mathbb{Z}^2$. It is believed that this variant of LFPP falls into the same universality class as the variant in (1-2). The geodesics go from the center of the metric ball to points in the intersection of the metric ball with the grid $20\mathbb{Z}^2$. The code for the simulation was provided by J. Miller.

In contrast, we will show that the boundary of a *filled* D_h-metric ball (i.e., the union of the ball and the points which it disconnects from some specified target point) is a Jordan curve and is compact and finite-dimensional with respect to D_h (Theorem 1.4).

Using our regularity results for outer boundaries of D_h-metric balls, we will then extend the confluence of geodesic results from [18] to the critical and supercritical cases (Theorems 1.6 and 1.7). Unlike in the subcritical case [20], these confluence results are not needed for the proof of the uniqueness of the critical and supercritical LQG metrics in [9]. However, they are of independent interest.

An important tool in our work is [34], which shows that subsequential limits of supercritical LFPP satisfy a list of axioms similar to the axioms for a weak LQG metric from [12] (see Definition 2.3), and establishes a number of estimates for any metric satisfying these axioms. All of the results in this paper are valid for any metric satisfying the axioms from [34].

1.2. *Ordinary and filled LQG metric balls.* Throughout the paper, we let h be a whole-plane GFF, we fix $\xi > 0$, and we let D_h be a weak LQG metric associated with h with parameter ξ. For now, the reader can think of D_h as a subsequential limit of the rescaled LFPP metrics $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$, but we emphasize that all of our results also hold for any metric satisfying the axioms stated in Definition 2.3 below. Also, most of our results are stated for $\xi > 0$ (not just $\xi \geq \xi_{\text{crit}}$), but many of the statements are either obvious or already proven elsewhere when $\xi \in (0, \xi_{\text{crit}})$. For $\xi > \xi_{\text{crit}}$, the metric D_h does not induce the Euclidean topology. We therefore make the following notational convention.

Notation 1.1. Throughout, topological concepts such as "open", "closed", "boundary", etc., are always defined with respect to the Euclidean topology unless otherwise stated. Similarly, for a set $A \subset \mathbb{C}$, ∂A denotes its boundary with respect to the Euclidean topology and $\overline{A}$ denotes its closure with respect to the Euclidean topology. Moreover, $z_n \to z$ always refers to convergence with respect to the Euclidean topology, unless otherwise stated.

For any set A, the boundary of A with respect to D_h is contained in the Euclidean boundary ∂A: this is because the Euclidean metric is continuous with respect to D_h. The reverse inclusion does not necessarily hold. For example, a D_h-metric ball is Euclidean-closed (Lemma 3.1) and has empty Euclidean interior (since the set of singular points is dense), so the Euclidean boundary of such a ball is equal to the whole ball. On the other hand, the D_h-distance from each point in the D_h-boundary to the center point of the ball is equal to the radius of the ball. Hence, any D_h-metric ball with the same center point and a strictly smaller radius is disjoint from the D_h-boundary of the ball, so in particular the D_h-boundary of the ball is not equal to the whole ball.

We also briefly recall the definition of Hausdorff dimension. For $\Delta > 0$, the Δ-Hausdorff content of a metric space (X, d) is

$$\inf\left\{ \sum_{j=1}^{\infty} r_j^\Delta : \text{there is a covering of } X \text{ by } d\text{-metric balls with radii } \{r_j\}_{j\in\mathbb{N}} \right\}$$

and the *Hausdorff dimension* of (X, d) is the infimum of the values of Δ for which the Δ-Hausdorff content is zero.

For $x \in \mathbb{C}$ and $s > 0$, we write

$$\mathcal{B}_s(x) := \{z \in \mathbb{C} : D_h(x, z) \le s\} \quad \text{and} \quad \mathcal{B}_s := \mathcal{B}_s(0) \tag{1-10}$$

for the closed D_h-metric ball of radius s. Recall that a *singular point* for D_h is a point which lies at infinite distance from every other point. A *nonsingular point* is a point which is not a singular point (i.e., a point which lies at finite distance from some other point). Using the fact that the singular points for D_h are Euclidean-dense, Pfeffer [34, Proposition 1.14] established the following.

Theorem 1.2 [34]. *Assume that $\xi > \xi_{\text{crit}}$. Almost surely, for each nonsingular point $x \in \mathbb{C}$ and each $s > t > 0$, the D_h-boundary (hence also the Euclidean boundary) of the D_h-metric ball $\mathcal{B}_s(x)$ cannot be covered by finitely many D_h-metric balls of radius t. Furthermore, $\partial\mathcal{B}_s(x) = \mathcal{B}_s(x)$ is not D_h-compact and has infinite Hausdorff dimension with respect to D_h.*

The reason why $\partial\mathcal{B}_s(x) = \mathcal{B}_s(x)$ is that, as noted above, the fact that the set of singular points for D_h is Euclidean dense implies that $\mathcal{B}_s(x)$ has empty Euclidean interior. Theorem 1.2 tells us that the boundaries of D_h-metric balls are in some sense highly irregular. One of the main contributions of this paper is to show that, in contrast, the boundaries of *filled* D_h-metric balls are well-behaved.

Definition 1.3. Let $x \in \mathbb{C}$ and $y \in \mathbb{C} \cup \{\infty\}$. For $s \ge 0$, we define the *filled D_h-metric ball centered at x and targeted at y* with radius $s > 0$ by

$$\mathcal{B}_s^{y,\bullet}(x) := \begin{cases} \text{the union of the closed metric ball } \mathcal{B}_s(x) \text{ and the set} & \\ \text{of points which this metric ball disconnects from } y & \text{for } s < D_h(x, y), \\ \mathbb{C} & \text{for } s \ge D_h(x, y). \end{cases}$$

We will most often work with filled metric balls centered at zero and filled metric balls targeted at infinity, so to lighten notation, we abbreviate

$$\mathcal{B}_s^\bullet(x) := \mathcal{B}_s^{\infty,\bullet}(x), \quad \mathcal{B}_s^{y,\bullet} := \mathcal{B}_s^{y,\bullet}(0) \quad \text{and} \quad \mathcal{B}_s^\bullet := \mathcal{B}_s^\bullet(0). \tag{1-11}$$

We note that filled D_h-metric balls differ from ordinary D_h-metric balls since the complement of an ordinary D_h-metric ball is typically not connected (see Figure 1). In fact, a.s. each such complement has infinitely many connected components; see [34, Proposition 1.14]. The following theorem summarizes our main results concerning the boundaries of filled D_h-metric balls.

Theorem 1.4. *Almost surely, for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$, the filled metric ball boundary $\partial\mathcal{B}_s^{y,\bullet}(x)$ is a Jordan curve. Moreover, this boundary is D_h-compact and its Hausdorff dimension is bounded above by a finite constant which depends only on the law of D_h.*

We emphasize that the statement of Theorem 1.4 holds a.s. for all choices of x, y, s simultaneously. We show in Lemma 3.4 below that the boundaries of $\mathcal{B}_s^{y,\bullet}(x)$ with respect to the Euclidean metric and D_h coincide, so Theorem 1.4 also applies to the D_h-boundary of $\mathcal{B}_s^{y,\bullet}(x)$.

In the subcritical case $\xi < \xi_{\text{crit}}$, Theorem 1.4 follows from the fact that D_h induces the Euclidean topology and the Hausdorff dimension of $(\mathbb{C}, D_h)$ is finite. See [31, Proposition 2.1] for a proof that the boundary of a filled metric ball is a Jordan curve for any geodesic metric on $\mathbb{C}$ which induces the Euclidean topology. For $\xi \geq \xi_{\text{crit}}$, however, the proof of Theorem 1.4 requires nontrivial ideas. In particular, we first establish a general criterion for the boundary of an open domain to be a (not necessarily simple) curve (Proposition 4.3), which is a variant of the well-known fact that if the boundary of a simply connected domain in $\mathbb{C}$ is locally connected, then it is a curve (see, e.g., [36, Section 2.2]). We then use some geometric estimates for supercritical LQG to check this criterion for the boundary of a filled supercritical LQG metric ball (see Section 4.2), which shows that the filled metric ball boundary is a curve. Finally, we use some fairly straightforward topological considerations to show that the boundary of a filled metric ball does not have cut points, so is in fact a *simple* curve (see Lemma 4.10). The basic idea of our proof is similar to the proof of [31, Proposition 2.1], which proceeds by checking that the boundary of a filled metric ball for a geodesic metric which induces the Euclidean topology is locally connected and has no cut points. However, our proof is much more involved since our metric does not induce the Euclidean topology.

Theorem 1.4 implies that for $x \in \mathbb{C}$ and $s > 0$, the boundaries of the connected components of $\mathbb{C} \setminus \mathcal{B}_s(x)$ have finite D_h-Hausdorff dimension. Since $\partial\mathcal{B}_s(x)$ itself has infinite D_h-Hausdorff dimension (Theorem 1.2), we get that "most" points of $\partial\mathcal{B}_s(x)$ do not lie on the boundary of any connected component of $\mathbb{C}\setminus\mathcal{B}_s(x)$. Points of this type can arise as accumulation points of arbitrarily small connected components of $\mathbb{C} \setminus \mathcal{B}_s(x)$. See [24, Theorem 1.14] for an analogous result in the subcritical case.

In fact, we will prove a slightly stronger Hausdorff dimension statement than the one in Theorem 1.4. For $x \in \mathbb{C}$ and $y \in \mathbb{C} \cup \{\infty\}$, we define the *metric net*

$$\mathcal{N}_s^y(x) := \bigcup_{t \in [0,s]} \partial\mathcal{B}_t^{y,\bullet}(x). \tag{1-12}$$

Theorem 1.5. *There is a deterministic constant $\Delta \in (0, \infty)$ (depending on the law of D_h) such that a.s. for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s > 0$ the Hausdorff dimension of $\mathcal{N}_s^y(x)$ with respect to D_h is at most Δ.*

The Hausdorff dimension of the metric net with respect to D_h or with respect to the Euclidean metric is not known, even heuristically, for any $\xi > 0$, with one exception: when $\boldsymbol{c}_{\mathrm{M}} = 0$ ($\xi = 1/\sqrt{6}$), we expect that the Hausdorff dimension with respect to D_h is 3 (this is consistent with scaling relations for quantum Loewner evolution in [30; 32; 33]). It was shown in [18, Theorem 1.11] that in the subcritical case, the dimensions of the metric net with respect to the Euclidean and LQG metrics are each a.s. equal to deterministic constants. We expect that the same is true in the supercritical case.

1.3. ***Confluence of geodesics.*** Theorem 1.4 (and the estimates which go into its proof) can be used to extend the confluence of geodesic results from [18] to the critical and supercritical cases. In particular, we obtain the following theorem for all $\xi > 0$.

Theorem 1.6 (confluence of geodesics at a point). *Almost surely, for each radius $s > 0$ there exists a radius $t \in (0, s)$ such that any two D_h-geodesics from 0 to points outside of the filled D_h-metric ball $\mathcal{B}_s^\bullet = \mathcal{B}_s^\bullet(0)$ coincide on the time interval $[0, t]$.*

Another form of confluence concerns geodesics across an annulus between two filled D_h-metric balls (Definition 1.3). Let us first note that every D_h-geodesic from 0 to a point $z \in \partial\mathcal{B}_s^\bullet$ stays in $\mathcal{B}_s^\bullet$. For some points z there might be many such D_h-geodesics, but there is always a distinguished D_h-geodesic from 0 to z, called the *leftmost geodesic*, which lies (weakly) to the left of every other D_h-geodesic from 0 to z if we stand at z and look outward from $\mathcal{B}_s^\bullet$ (see Lemma 5.4).

Theorem 1.7 (confluence of geodesics across a metric annulus). *Almost surely, for each $0 < t < s < \infty$ there is a finite set of D_h-geodesics from 0 to $\partial\mathcal{B}_t^\bullet$ such that every leftmost D_h-geodesic from 0 to a point of $\partial\mathcal{B}_s^\bullet$ coincides with one of these D_h-geodesics on the time interval $[0, t]$. In particular, there are a.s. only finitely many points of $\partial\mathcal{B}_t^\bullet$ which are hit by leftmost D_h-geodesics from 0 to points of $\partial\mathcal{B}_s^\bullet$.*

Theorems 1.6 and 1.7 are identical to [18, Theorems 1.3 and 1.4], except that they apply for all $\xi > 0$ rather than just $\xi < \xi_{\text{crit}}$. The proofs of Theorems 1.6 and 1.7 are given in Section 5. Many of the proofs in [18] carry over verbatim to the critical and supercritical cases, but other parts require nontrivial adaptations. To avoid unnecessary repetition, we will only explain the parts of the proofs which are different in the critical and supercritical cases.

1.4. *Outline.* The rest of this paper is structured as follows. In Section 2 we review the axioms for a weak LQG metric from [34], then restate some results from the existing literature (mostly from [34]) which we will need for our proofs. In Section 3, we prove a number of regularity estimates for the boundaries of filled D_h-metric balls, which enable us to prove Theorem 1.5 as well as all of Theorem 1.4 except for the statement that $\partial\mathcal{B}_s^{y,\bullet}$ is a Jordan curve. In Section 4, we prove that $\partial\mathcal{B}_s^{y,\bullet}$ is a Jordan curve, which completes the proof of Theorem 1.4. To do this, we first prove a general criterion for the boundary of a simply connected domain to be a curve, then check this criterion for $\partial\mathcal{B}_s^{y,\bullet}$ using the estimates from Section 3. In Section 5 we explain how to prove our confluence of geodesic results, Theorems 1.6 and 1.7, by adapting the arguments of [18] and applying the estimates of Section 3.

2. Preliminaries

2.1. *Notational conventions.* We write $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $a < b$, we define the discrete interval $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$.

If $f : (0, \infty) \to \mathbb{R}$ and $g : (0, \infty) \to (0, \infty)$, we say that $f(\varepsilon) = O_\varepsilon(g(\varepsilon))$ (resp. $f(\varepsilon) = o_\varepsilon(g(\varepsilon))$) as $\varepsilon \to 0$ if $f(\varepsilon)/g(\varepsilon)$ remains bounded (resp. tends to zero) as $\varepsilon \to 0$. We similarly define $O(\cdot)$ and $o(\cdot)$ errors as a parameter goes to infinity.

Let $\{E^\varepsilon\}_{\varepsilon>0}$ be a one-parameter family of events. We say that E^ε occurs with

- *polynomially high probability* as $\varepsilon \to 0$ if there is a $p > 0$ (independent from ε and possibly from other parameters of interest) such that $\mathbb{P}[E^\varepsilon] \geq 1 - O_\varepsilon(\varepsilon^p)$;
- *superpolynomially high probability* as $\varepsilon \to 0$ if $\mathbb{P}[E^\varepsilon] \geq 1 - O_\varepsilon(\varepsilon^p)$ for every $p > 0$.

We similarly define events which occur with polynomially or superpolynomially high probability as a parameter tends to ∞.

For $z \in \mathbb{C}$ and $r > 0$, we write $B_r(z)$ for the open Euclidean ball of radius r centered at z. More generally, for $X \subset \mathbb{C}$ we write $B_r(X) = \bigcup_{z\in X} B_r(z)$. We also define the open annulus

$$\mathbb{A}_{r_1,r_2}(z) := B_{r_2}(z) \setminus \overline{B_{r_1}(z)} \quad \text{for all } 0 < r_r < r_2 < \infty. \tag{2-1}$$

For a region $A \subset \mathbb{C}$ with the topology of a Euclidean annulus, we write $D_h(\text{across } A)$ for the D_h-distances between the inner and outer boundaries of A and $D_h(\text{around } A)$ for the infimum of the D_h-lengths of paths in A which disconnect the inner and outer boundaries of A.

2.2. *Weak LQG metrics.* In this subsection, we will state the axiomatic definition of a weak LQG metric from [34]. We first define the topology on the space of metrics that we will work with.

Definition 2.1. Let $X \subset \mathbb{C}$. A function $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *lower semicontinuous* if whenever $(z_n, w_n) \in X \times X$ with $(z_n, w_n) \to (z, w)$, we have $f(z, w) \le \liminf_{n\to\infty} f(z_n, w_n)$. The *topology on lower semicontinuous functions* is the topology whereby a sequence of such functions $\{f_n\}_{n\in\mathbb{N}}$ converges to another such function f if and only if

(i) whenever $(z_n, w_n) \in X \times X$ with $(z_n, w_n) \to (z, w)$, we have $f(z, w) \le \liminf_{n\to\infty} f_n(z_n, w_n)$;

(ii) for each $(z, w) \in X \times X$, there exists a sequence $(z_n, w_n) \to (z, w)$ such that $f_n(z_n, w_n) \to f(z, w)$.

It follows from [2, Lemma 1.5] that the topology of Definition 2.1 is metrizable (see [8, Section 1.2]). Furthermore, [2, Theorem 1(a)] shows that this metric can be taken to be separable.

Definition 2.2. Let (X, d) be a metric space, with d allowed to take on infinite values.

- For a curve $P : [a, b] \to X$, the *d-length* of P is defined by

$$\operatorname{len}(P; d) := \sup_T \sum_{i=1}^{\#T} d(P(t_i), P(t_{i-1}))$$

  where the supremum is over all partitions $T : a = t_0 < \cdots < t_{\#T} = b$ of $[a, b]$. Note that the d-length of a curve may be infinite.

- We say that (X, d) is a *length space* if for each $x, y \in X$ and each $\varepsilon > 0$, there exists a curve of d-length at most $d(x, y) + \varepsilon$ from x to y. A curve from x to y of d-length *exactly* $d(x, y)$ is called a *geodesic*.

- For $Y \subset X$, the *internal metric of d on Y* is defined by

$$d(x, y; Y) := \inf_{P\subset Y} \operatorname{len}(P; d) \quad \text{for all } x, y \in Y \tag{2-2}$$

  where the infimum is over all paths P in Y from x to y. Note that $d(\cdot, \cdot; Y)$ is a metric on Y, except that it is allowed to take infinite values.

- If $X \subset \mathbb{C}$, we say that d is a *lower semicontinuous metric* if the function $(x, y) \to d(x, y)$ is lower semicontinuous with respect to the Euclidean topology. We equip the set of lower semicontinuous metrics on X with the topology on lower semicontinuous functions on $X \times X$, as in Definition 2.1, and the associated Borel σ-algebra.

The following is a restatement of [34, Definition 1.6].

Definition 2.3 (weak LQG metric). Let $\mathcal{D}'$ be the space of distributions (generalized functions) on $\mathbb{C}$, equipped with the usual weak topology. For $\xi > 0$, a *weak LQG metric with parameter* ξ is a measurable functions $h \mapsto D_h$ from $\mathcal{D}'$ to the space of lower semicontinuous metrics on $\mathbb{C}$ with the following properties. Let h be a *GFF plus a continuous function* on $\mathbb{C}$: i.e., h is a random distribution on $\mathbb{C}$ which can be coupled with a random continuous function f in such a way that $h-f$ has the law of the whole-plane GFF. Then the associated metric D_h satisfies the following axioms.

I. **Length space.** Almost surely, $(\mathbb{C}, D_h)$ is a length space.

II. **Locality.** Let $U \subset \mathbb{C}$ be a deterministic open set. The D_h-internal metric $D_h(\cdot,\cdot;U)$ is a.s. given by a measurable function of $h|_U$.

III. **Weyl scaling.** For a continuous function $f : \mathbb{C} \to \mathbb{R}$, define

$$(e^{\xi f} \cdot D_h)(z,w) := \inf_{P : z \to w} \int_0^{\operatorname{len}(P;D_h)} e^{\xi f(P(t))}\, dt \quad \text{for all } z, w \in \mathbb{C}, \tag{2-3}$$

where the infimum is over all D_h-continuous paths from z to w in $\mathbb{C}$ parametrized by D_h-length. Then a.s. $e^{\xi f} \cdot D_h = D_{h+f}$ for every continuous function $f : \mathbb{C} \to \mathbb{R}$.

IV. **Translation invariance.** For each deterministic point $z \in \mathbb{C}$, a.s. $D_{h(\cdot+z)} = D_h(\cdot+z, \cdot+z)$.

V. **Tightness across scales.** Suppose that h is a whole-plane GFF and let $\{h_r(z)\}_{r>0, z\in\mathbb{C}}$ be its circle average process. There are constants $\{\mathfrak{c}_r\}_{r>0}$ such that the following is true. Let $A \subset \mathbb{C}$ be a deterministic Euclidean annulus. In the notation defined at the end of Section 2.1, the random variables

$$\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(\text{across } rA) \quad \text{and} \quad \mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(\text{around } rA)$$

and the reciprocals of these random variables for $r > 0$ are tight. Finally, there exists $\Lambda > 1$ such that for each $\delta \in (0,1)$,

$$\Lambda^{-1}\delta^{\Lambda} \le \frac{\mathfrak{c}_{\delta r}}{\mathfrak{c}_r} \le \Lambda \delta^{-\Lambda} \quad \text{for all } r > 0. \tag{2-4}$$

The axioms of Definition 2.3 are the same as the axioms which define a weak LQG metric in [12, Section 1.2], with two exceptions: one works with lower semicontinuous metrics instead of continuous metrics, and the tightness across scales axiom (Axiom V) is formulated differently: we require tightness for rescaled distances around and across Euclidean annuli, rather than requiring tightness of the rescaled metrics themselves.

It was shown in [34] that if h is a GFF plus a continuous function and D is a weak LQG metric, then a.s. the Euclidean metric is D_h-continuous (see Proposition 2.11 below for a quantitative version of this). In particular, a.s. every D_h-continuous path (e.g., a D_h-geodesic) is also Euclidean continuous.

Axiom V allows us to get bounds for D_h-distances which are uniform across different Euclidean scales. This axiom serves as a substitute for exact scale invariance (i.e., the LQG coordinate change formula), which is difficult to prove for subsequential limits of LFPP before we know that the subsequential limit is unique. See [12; 20; 34] for further discussion of this point.

The following theorem is proven as [34, Theorem 1.7], building on the tightness result from [8].

Theorem 2.4 [34]. *Let $\xi > 0$. For every sequence of ε's tending to zero, there is a weak LQG metric D with parameter ξ and a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ for which the following is true. Let h be a whole-plane GFF, or more generally a whole-plane GFF plus a bounded continuous function. Then the rescaled LFPP metrics $\mathfrak{a}_{\varepsilon_n}^{-1} D_h^{\varepsilon_n}$, as defined in* (1-3) *and* (1-4), *converge in probability to D_h with respect to the metric on lower semicontinuous functions on $\mathbb{C} \times \mathbb{C}$.*

Theorem 2.4 implies in particular that for each $\xi > 0$, there exists a weak LQG metric with parameter ξ.

Remark 2.5. It was shown in [9], subsequently to this paper, that the axioms in Definition 2.3 uniquely characterize D_h, up to multiplication by a deterministic positive constant. This implies that one has actual convergence (not just subsequential convergence) in Theorem 2.4 and that Axiom V can be improved to the LQG coordinate change formula for spatial scaling. Some of the results of this paper (in particular, those in Section 3.2) are used in [9].

2.3. *Results from prior work.* Throughout the rest of the paper, we fix $\xi > 0$ and a weak LQG metric $D : h \mapsto D_h$ with parameter ξ. We will not make the dependence on the parameter ξ or the particular choice of metric D explicit in our estimates. We also let h be a whole-plane GFF and we let $\{h_r(z) : r > 0, z \in \mathbb{C}\}$ be its circle average process (as in Axiom V).

Many of the quantitative estimates in this paper involve a parameter $\mathbb{r} > 0$, which represents the "Euclidean scale". The estimates are required to be uniform in the choice of $\mathbb{r}$. The reason for including $\mathbb{r}$ is the same as in other papers concerning weak LQG metrics, such as [12; 18; 20; 34]: we only have tightness across scales (Axiom V), rather than exact scale invariance, so it is not possible to directly transfer estimates from one Euclidean scale to another.

In this subsection, we state some previously known results for the GFF and the LQG metric (mostly from [34]) which we will cite regularly. We start with the fact that D_h-geodesics exist [34, Proposition 1.12], which is not immediate from the axioms since Axiom I only shows that $D_h(z, w)$ is the infimum of the D_h-lengths of paths joining z and w, not that a length-minimizing path exists.

Lemma 2.6 [34]. *Almost surely, for any two nonsingular points $z, w \in \mathbb{C}$, there exists an LQG geodesic P joining z and w.*

We will frequently use without comment the following fact, which implies in particular that every D_h-bounded set is Euclidean bounded. See [34, Lemma 3.12] for a proof.

Lemma 2.7 [34]. *Almost surely, for every Euclidean-compact set $K \subset \mathbb{C}$,*

$$\lim_{R\to\infty} D_h(K, \partial B_R(0)) = \infty.$$

It was shown in [34, Lemma 3.1] that one has the following stronger version of Axiom V.

Lemma 2.8 [34]. *Let $U \subset \mathbb{C}$ be open and let $K_1, K_2 \subset U$ be two disjoint, deterministic compact sets (allowed to be singletons). The rescaled internal distances $\mathfrak{c}_r^{-1} e^{-\xi h_r(0)} D_h(rK_1, rK_2; rU)$ and their reciprocals are tight.*

The following proposition, which is [34, Proposition 1.8], is a more quantitative version of Lemma 2.8 in the case when K_1, K_2 are connected and are not singletons. It will be our most important estimate for D_h-distances.

Proposition 2.9 [34]. *Let $U \subset \mathbb{C}$ be an open set (possibly all of $\mathbb{C}$) and let $K_1, K_2 \subset U$ be connected, disjoint compact sets which are not singletons. Also let $\{\mathfrak{c}_r\}_{r>0}$ be the scaling constants from Axiom V. For each $\mathbb{r} > 0$, it holds with superpolynomially high probability as $A \to \infty$, at a rate which is uniform in the choice of $\mathbb{r}$, that*

$$A^{-1}\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \le D_h(\mathbb{r}K_1, \mathbb{r}K_2; \mathbb{r}U) \le A\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}. \tag{2-5}$$

Recall the notation for D_h-distance across and around Euclidean annuli from Section 2.1. We will most frequently use Proposition 2.9 to lower-bound $D_h(\text{across } \mathbb{A}_{a\mathbb{r},b\mathbb{r}}(z))$ and upper-bound $D_h(\text{around } \mathbb{A}_{a\mathbb{r},b\mathbb{r}}(z))$ where $b > a > 0$ are fixed. To do this, we first note that due to Axiom IV we can assume without loss of generality that $z = 0$. To lower-bound $D_h(\text{across } \mathbb{A}_{a\mathbb{r},b\mathbb{r}}(z))$ we apply Proposition 2.9 with $K_1 = \partial B_a(0)$, $K_2 = \partial B_b(0)$, and $U = \mathbb{C}$. To upper-bound $D_h(\text{around } \mathbb{A}_{a\mathbb{r},b\mathbb{r}}(z))$, we apply Proposition 2.9 twice, with the sets K_1, K_2, U and K_1', K_2', U' chosen so that the union of any path from K_1 to K_2 in U and any path from K_1' to K_2' in U' is contained in $\mathbb{A}_{a,b}(0)$ and disconnects the inner and outer boundaries of $\mathbb{A}_{a,b}(0)$.

Axiom V only gives polynomial upper and lower bounds for the ratios of the scaling constants $\mathfrak{c}_r$. The following proposition, which is [34, Proposition 1.9], gives much more precise bounds for these scaling constants and relates them to LFPP.

Proposition 2.10 [34]. *With Q as in (1-5), the scaling constants from Axiom V satisfy $\mathfrak{c}_r = r^{\xi Q + o_r(1)}$ as $r \to 0$ or $r \to \infty$.*

We also have a Hölder continuity condition for the Euclidean metric with respect to D_h. See [34, Proposition 3.8].

Proposition 2.11 [34]. *Let $\chi \in (0, (\xi(Q+2))^{-1})$ and let $U \subset \mathbb{C}$ be a Euclidean-bounded open set. For each $\mathbb{r} > 0$, it holds with polynomially high probability as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$, that*

$$|z - w| \le D_h(z, w)^\chi \quad \text{for all } z, w \in \mathbb{r}U \text{ with } |z - w| \le \varepsilon \mathbb{r}. \tag{2-6}$$

In particular, the identity mapping from $(\mathbb{C}, D_h)$ to $\mathbb{C}$, equipped with the Euclidean metric, is χ-Hölder continuous when restricted to any Euclidean-compact set.

We note that in contrast to the subcritical case (see [18, Theorem 1.7]), the Hölder continuity in Proposition 2.11 only goes in one direction.

Finally, we state an estimate which is a consequence of the fact that the restrictions of the GFF h to disjoint concentric annuli are nearly independent. See [19, Lemma 3.1] for a proof of a slightly more general result.

Lemma 2.12 [19]. *Fix $0 < \mu_1 < \mu_2 < 1$. Let $\{r_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $r_{k+1}/r_k \le \mu_1$ for each $k \in \mathbb{N}$ and let $\{E_{r_k}\}_{k\in\mathbb{N}}$ be events such that $E_{r_k} \in \sigma\big((h - h_{r_k}(0))|_{\mathbb{A}_{\mu_1 r_k, \mu_2 r_k}(0)}\big)$ for each $k \in \mathbb{N}$ (here we use the notation for Euclidean annuli from Section 2.1). For $K \in \mathbb{N}$, let $N(K)$*

be the number of $k \in [1, K]_{\mathbb{Z}}$ for which E_{r_k} occurs. For each $a > 0$ and each $b \in (0, 1)$, there exists $p = p(a, b, \mu_1, \mu_2) \in (0, 1)$ and $c = c(a, b, \mu_1, \mu_2) > 0$ such that if

$$\mathbb{P}[E_{r_k}] \geq p \quad \textit{for all } k \in \mathbb{N}, \tag{2-7}$$

then

$$\mathbb{P}[N(K) < bK] \leq ce^{-aK} \quad \textit{for all } K \in \mathbb{N}. \tag{2-8}$$

3. Estimates for the outer boundary of an LQG metric ball

We continue to assume that $\xi > 0$, h is a whole-plane GFF, and D_h is a weak LQG metric with parameter ξ. In this section, we will prove a variety of estimates for D_h-distance which will eventually lead to proofs of Theorem 1.5 and the compactness and finite-dimensionality parts of Theorem 1.4. We start out in Section 3.1 by proving some basic facts about D_h which are relatively straightforward consequences of existing results, e.g., the fact that D_h-metric balls are Euclidean closed and every filled D_h-metric ball contains a Euclidean ball with the same center point. In Section 3.2, we will prove a technical lemma which will be a key tool in our proofs: basically, it says that points on the boundary of a filled D_h-metric ball can be surrounded by paths with small D_h-lengths (Lemma 3.6). Using this lemma, in Section 3.3 we will prove a lower bound for the Euclidean distance between the boundaries of two filled metric balls with the same center point. Finally, in Section 3.4 we will prove Theorem 1.5 and part of Theorem 1.4.

3.1. ***Basic facts about the LQG metric.*** Before proving our main results for LQG metric ball boundaries, we will record some facts about D_h which are easy consequences of the axioms from Definition 2.3 and the estimates from Section 2.3. For our first statement, we recall that ∂ always denotes the boundary with respect to the *Euclidean* topology.

Lemma 3.1. *Almost surely, for each $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$, the ordinary metric ball $\mathcal{B}_s(x)$ and the filled metric ball $\mathcal{B}_s^{y,\bullet}(x)$ are both Euclidean-closed and $\partial \mathcal{B}_s^{y,\bullet}(x) \subset \mathcal{B}_s(x)$.*

Proof. The function $z \mapsto D_h(x, z)$ is lower semicontinuous, so if z_n is a sequence of points in $\mathcal{B}_s(x)$ with $|z_n - z| \to 0$, then $D_h(x, z) \leq \liminf_{n\to\infty} D_h(x, z_n) \leq s$, so $z \in \mathcal{B}_s(x)$. Hence $\mathcal{B}_s(x)$ is Euclidean-closed. Consequently, each connected component of $\mathbb{C} \setminus \mathcal{B}_s(x)$ is Euclidean-open. In particular, the connected component of $\mathbb{C} \setminus \mathcal{B}_s(x)$ containing y, namely $\mathbb{C} \setminus \mathcal{B}_s^{y,\bullet}(x)$, is Euclidean-open, so $\mathcal{B}_s^{y,\bullet}(x)$ is Euclidean-closed. Since $\mathcal{B}_s(x)$ is Euclidean-closed, it contains the boundary of each of its complementary connected components. In particular, $\partial \mathcal{B}_s^{y,\bullet}(x) \subset \mathcal{B}_s(x)$. □

Our next several lemmas are based on the following straightforward consequence of Lemma 2.12; see [34, Proposition 1.13] for a proof.

Lemma 3.2 [34]. *Almost surely, for each nonsingular point $z \in \mathbb{C}$ there is a sequence of disjoint D_h-continuous loops $\{\pi_n\}_{n\in\mathbb{N}}$, each of which separates a neighborhood of z from ∞, such that the Euclidean radius of π_n, the D_h-length of π_n, and the D_h-distance from z to π_n each tend to zero as $n \to \infty$.*

Since the set of singular points is a.s. Euclidean-dense, a.s. every D_h-metric ball has empty Euclidean interior. In contrast, the following lemma tells us that a filled D_h-metric ball a.s. contains a Euclidean ball with the same center.

Lemma 3.3. *Almost surely, for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$, the filled D_h-metric ball $\mathcal{B}_s^{y,\bullet}(x)$ contains a Euclidean ball centered at x with positive radius.*

Proof. Let $\{\pi_n\}_{n\in\mathbb{N}}$ be a sequence of loops surrounding x as in Lemma 3.2. Let P be a D_h-geodesic from x to y. The Euclidean radii and the D_h-lengths of the π_n's shrink to zero as $n \to \infty$ and P is Euclidean continuous. Hence a.s. for each sufficiently large $n \in \mathbb{N}$, the loop π_n disconnects x from y, the D_h-length of π_n is less than $s/2$, and P hits π_n before time $s/2$. This shows that π_n is contained in $\mathcal{B}_s(x)$, so $\pi_n \subset \mathcal{B}_s^{y,\bullet}(x)$. Since π_n disconnects a Euclidean ball of positive radius centered at x from y, this gives the lemma statement. □

For our next lemma, we recall that ∂ always denotes the boundary with respect to the *Euclidean* topology.

Lemma 3.4. *Almost surely, for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$,*

$$D_h(x, z) = s \quad \textit{for all } z \in \partial\mathcal{B}_s^{y,\bullet}(x). \tag{3-1}$$

Furthermore, the Euclidean boundary $\partial\mathcal{B}_s^{y,\bullet}(x)$ is equal to the D_h-boundary of $\mathcal{B}_s^{y,\bullet}(x)$.

Proof. By Lemma 3.1, a.s. for each x, y, s as in the lemma statement we have $\partial\mathcal{B}_s^{y,\bullet}(x) \subset \mathcal{B}_s(x)$, so $D_h(x, z) \le s$ for each $z \in \partial\mathcal{B}_s^{y,\bullet}(x)$. We need to prove the reverse inequality. To this end, we fix x, y, s as in the lemma statement. All statements are required to hold for all choices of x, y, s simultaneously.

Let $z \in \partial\mathcal{B}_s(x)$. Then $D_h(x, z) \le s < \infty$ so z is not a singular point. Let $\{\pi_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint D_h-continuous loops surrounding z as in Lemma 3.2. Since $D_h(x, w) \le s < D_h(x, y)$ for each $w \in \partial\mathcal{B}_s^{y,\bullet}(x)$ and $\partial\mathcal{B}_s^{y,\bullet}(x)$ is Euclidean-closed, $\partial\mathcal{B}_s^{y,\bullet}(x)$ lies at positive Euclidean distance from y. The Euclidean radius of π_n tends to zero as $n \to \infty$ and each π_n disconnects a neighborhood of z from ∞. Hence for each large enough $n \in \mathbb{N}$, y lies in the unbounded complementary connected component of π_n, and hence π_n disconnects a neighborhood of z from y.

If $D_h(x, z) < s$, then since the D_h-length of π_n and the D_h-distance from z to π_n both tend to zero as $n \to \infty$, the triangle inequality shows that $\pi_n \subset \mathcal{B}_s(x)$ for each large enough $n \in \mathbb{N}$. But, π_n disconnects a neighborhood of z from y for each large enough n, so if $D_h(x, z) < s$ then z must be in the interior of $\mathcal{B}_s^{y,\bullet}(x)$, not in $\partial\mathcal{B}_s^{y,\bullet}(x)$. We thus obtain (3-1).

Since $\partial\mathcal{B}_s^{y,\bullet}(x) \subset \mathcal{B}_s(x) \subset \mathcal{B}_s^{y,\bullet}(x)$ and the D_h-boundary of any set is contained in its Euclidean boundary, to get the last statement of the lemma, we need to show that each point $z \in \partial\mathcal{B}_s^{y,\bullet}(x)$ is a D_h-accumulation point of $\mathbb{C} \setminus \mathcal{B}_s^{y,\bullet}(x)$. Since the loop π_n disconnects z from y for each large enough n, it follows that π_n disconnects $\mathbb{C} \setminus \mathcal{B}_s^{y,\bullet}(x)$ into at least two connected components for each large enough n. Since $\mathbb{C} \setminus \mathcal{B}_s^{y,\bullet}(x)$ is connected, it follows that π_n contains a point $z_n \in \mathbb{C} \setminus \mathcal{B}_s^{y,\bullet}(x)$ for each large enough n. Since the D_h-distance from z to π_n and the D_h-length of π_n each tend to zero as $n \to \infty$, we infer that z is a D_h-accumulation point of $\mathbb{C} \setminus \mathcal{B}_s^{y,\bullet}(x)$, as required. □

Finally, we record a more quantitative version of Lemma 3.3 which applies when the center point of the filled metric ball is fixed. In the lemma statement and later in the paper, we will use the notation

$$\tau_r = D_h(0, \partial B_r(0)) = \inf\{t > 0 : \mathcal{B}_t \not\subset B_r(0)\} \quad \text{for all } r > 0. \tag{3-2}$$

Lemma 3.5. *Let $\mathbb{r} > 0$ and let $\tau_{\mathbb{r}}$ be as in* (3-2). *It holds with polynomially high probability as $\varepsilon \to 0$, uniformly over the choice of $\mathbb{r}$, that $B_{\varepsilon\mathbb{r}}(0) \subset \mathcal{B}^\bullet_{\tau_{\mathbb{r}}}$.*

Proof. Let $\zeta \in (0, \xi Q/100)$ be a small exponent. By Proposition 2.9, it holds with superpolynomially high probability as $\varepsilon \to 0$, uniformly in $\mathbb{r}$, that

$$D_h(\text{across } \mathbb{A}_{\mathbb{r}/2,\mathbb{r}}(0)) \geq \varepsilon^\zeta \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \quad \text{and} \quad D_h(\text{around } \mathbb{A}_{\varepsilon\mathbb{r},2\varepsilon\mathbb{r}}(0)) \leq \varepsilon^{-\zeta} \mathfrak{c}_{\varepsilon\mathbb{r}} e^{\xi h_{\varepsilon\mathbb{r}}(0)}. \tag{3-3}$$

Here we use the notation for D_h-distances across and around Euclidean annuli as explained in Section 2.1.

By Proposition 2.10, we have $\mathfrak{c}_{\varepsilon\mathbb{r}}/\mathfrak{c}_{\mathbb{r}} = \varepsilon^{\xi Q + o_\varepsilon(1)}$, with the rate of convergence of the $o_\varepsilon(1)$ uniform in $\mathbb{r}$, so with superpolynomially high probability as $\varepsilon \to 0$,

$$\frac{D_h(\text{around } \mathbb{A}_{\varepsilon\mathbb{r},2\varepsilon\mathbb{r}}(0))}{D_h(\text{across } \mathbb{A}_{\mathbb{r}/2,\mathbb{r}}(0))} \leq \varepsilon^{\xi Q - 2\zeta + o_\varepsilon(1)} e^{\xi(h_{\varepsilon\mathbb{r}}(0) - h_{\mathbb{r}}(0))}. \tag{3-4}$$

The random variable $h_{\varepsilon\mathbb{r}}(0) - h_{\mathbb{r}}(0)$ is centered Gaussian with variance $\log \varepsilon^{-1}$, so by the Gaussian tail bound it holds with polynomially high probability as $\varepsilon \to 0$ that $e^{\xi(h_{\varepsilon\mathbb{r}}(0) - h_{\mathbb{r}}(0))} \leq \varepsilon^{-(\xi Q - 3\zeta)}$. By (3-4), it therefore holds with polynomially high probability as $\varepsilon \to 0$ that

$$D_h(\text{around } \mathbb{A}_{\varepsilon\mathbb{r},2\varepsilon\mathbb{r}}(0)) < D_h(\text{across } \mathbb{A}_{\mathbb{r}/2,\mathbb{r}}(0)). \tag{3-5}$$

Suppose that (3-5) holds. We claim that $B_{\varepsilon\mathbb{r}}(0) \subset \mathcal{B}^\bullet_{\tau_{\mathbb{r}}}$. Let π be a path in $\mathbb{A}_{\varepsilon\mathbb{r},2\varepsilon\mathbb{r}}(0)$ which disconnects the inner and outer boundaries of $\mathbb{A}_{\varepsilon\mathbb{r},2\varepsilon\mathbb{r}}(0)$ and has D_h-length less than $D_h(\text{across } \mathbb{A}_{\mathbb{r}/2,\mathbb{r}}(0))$. Also let P be a D_h-geodesic from 0 to a point of $\partial\mathcal{B}^\bullet_{\tau_{\mathbb{r}}} \cap \partial B_{\mathbb{r}}(0)$. Then P hits π before leaving $B_{\mathbb{r}/2}(0)$ and the segment of P after it leaves $B_{\mathbb{r}/2}(0)$ has D_h-length at least $D_h(\text{across } \mathbb{A}_{\mathbb{r}/2,\mathbb{r}}(0))$. Since the D_h-length of π is smaller than $D_h(\text{across } \mathbb{A}_{\mathbb{r}/2,\mathbb{r}}(0))$, we get that $\pi \subset \mathcal{B}_{\tau_{\mathbb{r}}}$. Since π disconnects $B_{\varepsilon\mathbb{r}}(0)$ from ∞, it follows that $B_{\varepsilon\mathbb{r}}(0) \subset \mathcal{B}^\bullet_{\tau_{\mathbb{r}}}$. □

3.2. ***Regularity of distances on outer boundaries of metric balls.*** A key ingredient for many of the proofs in this paper is the following lemma, which implies every point on the boundary of a filled D_h-metric ball can be surrounded by a path of small D_h-length, in a sense which is uniform over all points in any Euclidean-bounded open set (this is in contrast to Lemma 3.2, which does not give any uniform control on the rate of convergence). A closely related lemma for LQG geodesics is proven in [34, Section 2.4]. Note that we include a Euclidean scale parameter $\mathbb{r}$ in the estimates of this subsection since we will need them to be uniform across Euclidean scales.

Lemma 3.6. *For each $\alpha \in (0, 1)$, there exists $\beta = \beta(\alpha) > 0$ such that for each Euclidean-bounded open set $U \subset \mathbb{C}$ and each $\mathbb{r} > 0$, it holds with polynomially high probability as $\varepsilon \to 0$, uniformly over the choice of $\mathbb{r}$, that following is true. Suppose $z \in \mathbb{r}U$, $x, y \in \mathbb{C} \setminus B_{\varepsilon^\alpha\mathbb{r}}(z)$, and $s > 0$ such that the boundary of the filled metric ball $\partial\mathcal{B}^{y,\bullet}_s(x)$ intersects $B_{\varepsilon\mathbb{r}}(z)$. Then*

$$D_h(\textit{around } \mathbb{A}_{\varepsilon\mathbb{r},\varepsilon^\alpha\mathbb{r}}(z)) \leq \varepsilon^\beta D_h(\textit{across } \mathbb{A}_{\varepsilon\mathbb{r},\varepsilon^\alpha\mathbb{r}}(z)). \tag{3-6}$$

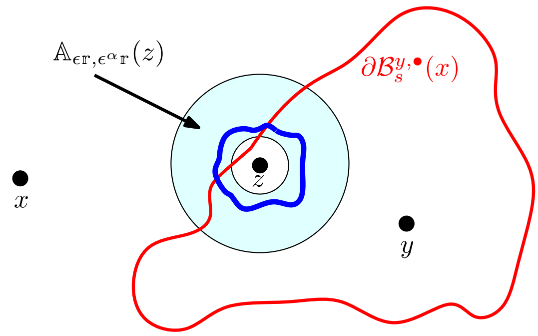

Figure 2. Illustration of the statement of Lemma 3.6. The lemma asserts that if $z \in \mathbb{r}U$, $x, y \in \mathbb{C} \setminus B_{\varepsilon^\alpha \mathbb{r}}(z)$, and $s > 0$ such that the filled metric ball boundary $\partial\mathcal{B}_s^{y,\bullet}(x)$ (red) intersects $B_{\varepsilon\mathbb{r}}(z)$, then we can find a path (blue) which disconnects the inner and outer boundaries of the annulus $\mathbb{A}_{\varepsilon\mathbb{r},\varepsilon^\alpha\mathbb{r}}(z)$ which has short D_h-length, in the sense of (3-6).

See Figure 2 for an illustration of the statement of Lemma 3.6. We will most often use the following slightly weaker estimate, which is an immediate consequence of Lemma 3.6.

Corollary 3.7. *Suppose we are in the setting of Lemma 3.6. On the polynomially high probability event of that lemma, the following is true. Suppose* $z \in \mathbb{r}U$, $x, y \in \mathbb{C} \setminus B_{\varepsilon^\alpha\mathbb{r}}(z)$, *and* $s > 0$ *such that either* $\partial\mathcal{B}_s^{y,\bullet}(x) \cap B_{\varepsilon\mathbb{r}}(z) \neq \emptyset$ *or there is a* D_h*-geodesic* P *from* x *to* y *with* $P(s) \in B_{\varepsilon\mathbb{r}}(z)$. *Then*

$$D_h(\text{around } \mathbb{A}_{\varepsilon\mathbb{r},\varepsilon^\alpha\mathbb{r}}(z)) \leq \varepsilon^\beta s. \tag{3-7}$$

Proof. If P is a D_h-geodesic from x to y, then necessarily $P(s) \in \partial\mathcal{B}_s^{y,\bullet}(x)$, so if $P(s) \in B_{\varepsilon\mathbb{r}}(z)$ then $\partial\mathcal{B}_s^{y,\bullet}(x) \cap B_{\varepsilon\mathbb{r}}(z) \neq \emptyset$. Hence, Lemma 3.6 implies that for z, x, y, s as in the lemma statement the bound (3-6) is satisfied. Since any D_h-geodesic from x to a point of $\partial\mathcal{B}_s^{\bullet,y} \cap B_{\varepsilon\mathbb{r}}(z)$ has length s and must cross between the inner and outer boundaries of $B_{\varepsilon^\alpha\mathbb{r}}(z) \setminus B_{\varepsilon\mathbb{r}}(z)$, we see that (3-6) implies (3-7). □

Intuitively, the reason why Lemma 3.6 and Corollary 3.7 are true is that points on the boundary of a filled D_h-metric ball or on a D_h-geodesic should be in some sense far from being singular points (since they are at finite distance from at least one point). Hence it should be possible to find short paths which disconnect small Euclidean neighborhoods of such points from ∞ (roughly speaking, this is a quantitative version of Lemma 3.2).

Corollary 3.7 can be thought of as a substitute for the fact that for supercritical LQG (unlike in the subcritical case) we do not know that the identity mapping $(\mathbb{C}, |\cdot|) \to (\mathbb{C}, D_h)$ is Hölder continuous. To be more precise, the corollary tells us that points on the outer boundary of a filled D_h-metric ball or on a D_h-geodesic can be surrounded by paths of small Euclidean size whose D_h-length is small. By forcing these paths to cross other paths, we will be able to establish upper bounds for the D_h-distance between points near filled metric ball boundaries or geodesics in terms of their Euclidean distance. Since many estimates for the LQG metric only require us to work with points near filled metric ball boundaries or geodesics, this will be a suitable substitute for Hölder continuity.

The idea of the proof of Lemma 3.6 is to surround the Euclidean ball $B_{\varepsilon\mathbb{r}}(z)$ by logarithmically many disjoint concentric Euclidean annuli contained in $\mathbb{A}_{\varepsilon\mathbb{r},\varepsilon^\alpha\mathbb{r}}(z)$ with the property that the D_h-distances around and across each of the annuli are comparable. This will be done using Lemma 2.12. We will order these annuli from outside to inside. Using a deterministic lemma (see Lemma 3.9), we will argue that in order for a filled metric ball boundary to intersect $B_\varepsilon(z)$, there must be at least one annulus such that the D_h-distance around this annulus is smaller than a positive power of ε times the sum of the D_h-distances across the subsequent annuli. This latter sum provides a lower bound for $D_h(\text{across } \mathbb{A}_{\varepsilon\mathbb{r},\varepsilon^\alpha\mathbb{r}}(z))$.

Let us now construct the concentric annuli that we will work with. An *annulus with aspect ratio* 2 is an open annulus of the form $A = \mathbb{A}_{r,2r}(z)$ for some $z \in \mathbb{C}$ and $r > 0$. For an annulus A with aspect ratio 2 and a number $c > 0$, we define

$$E_c(A) := \{D_h(\text{around } A) \le (1/c) D_h(\text{across } A)\}. \tag{3-8}$$

Lemma 3.8. *For each $\alpha \in (0,1)$, there exists $\eta \in (0, 1-\alpha)$ and $c \in (0,1)$ such that for each Euclidean-bounded open set $U \subset \mathbb{C}$ and each $\mathbb{r} > 0$, it holds with polynomially high probability as $\varepsilon \to 0$, uniformly over the choice of $\mathbb{r}$, that the following is true. For each $z \in \mathbb{r}U$, there exist $N := \lceil \eta \log \varepsilon^{-1} \rceil$ disjoint concentric annuli $A_1, \dots, A_N \subset \mathbb{A}_{\varepsilon\mathbb{r},\varepsilon^\alpha\mathbb{r}}(z)$ which each disconnects $\partial B_{\varepsilon\mathbb{r}}(z)$ from $\partial B_{\varepsilon^\alpha\mathbb{r}}(z)$ such that $E_c(A_n)$ occurs for each $n = 1, \dots, N$.*

Proof. This is a straightforward consequence of the near-independence of the restriction of the GFF to disjoint concentric annuli (Lemma 2.12) together with a union bound over points in an fine mesh of $\mathbb{r}U$. Let us now give the details.

For $z \in \mathbb{C}$ and $k \in \mathbb{N}$, let $A_{k,\varepsilon}(z) := \mathbb{A}_{2^{2k}\varepsilon\mathbb{r},2^{2k+1}\varepsilon\mathbb{r}}(z)$. Note that the annuli $A_{k,\varepsilon}(z)$ for different values of k are disjoint and for each k, the region between the annuli $A_{k,\varepsilon}(z)$ and $A_{k+1,\varepsilon}(z)$ is the annulus $\mathbb{A}_{2^{2k+1}\varepsilon\mathbb{r},2^{2k+2}\varepsilon\mathbb{r}}(z)$. Furthermore, if we set $K_\varepsilon := \lfloor \frac{1}{3} \log_2 \varepsilon^{-(1-\alpha)} \rfloor - 1$, then

$$A_{k,\varepsilon}(z) \subset \mathbb{A}_{2\varepsilon\mathbb{r},\varepsilon^\alpha\mathbb{r}/2}(z) \quad \text{for all } k \in [1, K_\varepsilon]_{\mathbb{Z}}. \tag{3-9}$$

The reason why we want $2\varepsilon\mathbb{r}$ and $\varepsilon^\alpha\mathbb{r}/2$ instead of just $\varepsilon\mathbb{r}$ and $\varepsilon^\alpha\mathbb{r}$ in (3-9) is that we will need to slightly adjust the radii of our annuli when we pass from a statement for points in a fine mesh to a statement for all points simultaneously.

By the definition (3-8) of $E_c(A_{k,\varepsilon}(z))$, this event is a.s. determined by the internal metric of D_h on $A_{k,\varepsilon}(z)$. By the locality and Weyl scaling properties of D_h (Axioms II and III), each of the events $E_c(A_{k,\varepsilon}(z))$ is a.s. determined by the restriction of h to $A_{k,\varepsilon}(z)$, viewed modulo additive constant. By the translation invariance and tightness across scales properties of D_h (Axioms IV and V), for any $p \in (0,1)$ we can find $c = c(p) \in (0,1)$ such that $\mathbb{P}[E_c(A_{k,\varepsilon}(z))] \ge p$ for each $z \in \mathbb{C}$, $k \in \mathbb{N}$, and $\varepsilon > 0$.

We may therefore apply Lemma 2.12 to find $c \in (0,1)$ and $\eta \in (0, 1-\alpha)$ such that for each $z \in \mathbb{C}$, it holds with probability at least $1 - O_\varepsilon(\varepsilon^3)$ that there are at least $\eta \log \varepsilon^{-1}$ values of $k \in [1, K_\varepsilon]_{\mathbb{Z}}$ for which $E_c(A_{\varepsilon,k}(z))$ occurs. By a union bound, it holds with polynomially high probability as $\varepsilon \to 0$ that for each $z \in \left(\frac{1}{4}\varepsilon\mathbb{r}\mathbb{Z}^2\right) \cap B_{\varepsilon\mathbb{r}}(\mathbb{r}U)$, there are at least $\eta \log \varepsilon^{-1}$ values of $k \in [1, K_\varepsilon]_{\mathbb{Z}}$ for which $E_c(A_{\varepsilon,k}(z))$ occurs. Henceforth assume that this is the case.

Let $z \in U$. We can find $z' \in \left(\frac{1}{4}\varepsilon\mathbb{r}\mathbb{Z}^2\right) \cap B_{\varepsilon\mathbb{r}}(U)$ such that $z \in B_{\varepsilon\mathbb{r}/2}(z')$. Then $B_{\varepsilon\mathbb{r}}(z) \subset B_{2\varepsilon\mathbb{r}}(z')$ and $B_{\varepsilon^\alpha\mathbb{r}/2}(z') \subset B_{\varepsilon^\alpha\mathbb{r}}(z)$. By (3-9), the conditions in the lemma statement hold with $A_1, \dots, A_N$ chosen to be $N = \lceil \eta \log \varepsilon^{-1} \rceil$ of the annuli $A_{k,\varepsilon}(z')$ for $k \in [1, K_\varepsilon]_{\mathbb{Z}}$ for which $E_c(A_{k,\varepsilon}(z'))$ occurs. □

The following deterministic lemma will allow us to choose one of the annuli A_n from Lemma 3.8 in such a way that $D_h(\text{around } A_n)$ is much smaller than $\sum_{j=n+1}^N D_h(\text{across } A_j)$. See [34, Lemma 2.20] for a proof.

Lemma 3.9. *Let $X_1, \dots, X_N$ be nonnegative real numbers. For each $c > 0$,*

$$\#\left\{n \in [1, N]_{\mathbb{Z}} : X_n \geq c \sum_{j=n+1}^N X_j\right\} \leq \max\left\{1, \frac{\log\left(\frac{1}{X_N} \max_{n \in [1,N]_{\mathbb{Z}}} X_n\right)}{\log(c+1)} - \frac{\log c}{\log(c+1)} + 2\right\}. \tag{3-10}$$

Proof of Lemma 3.6. Let $\alpha > 0$ and let η and c be chosen as in Lemma 3.8. Also fix a Euclidean-bounded open set $U \subset \mathbb{C}$ and a number $\mathbb{r} > 0$. Throughout the proof, we work on the polynomially high probability event of Lemma 3.8.

Let $z \in \mathbb{r}U$ and let $A_1, \dots, A_N \subset \mathbb{A}_{\varepsilon\mathbb{r}, \varepsilon^\alpha\mathbb{r}}(z)$ be the disjoint concentric annuli from Lemma 3.8, numbered from outside in. For $n \in [1, N]_{\mathbb{Z}}$, define

$$X_n := D_h(\text{around } A_n), \tag{3-11}$$

so that by the definition of $E_c(A_n)$,

$$X_n \leq (1/c) D_h(\text{across } A_n). \tag{3-12}$$

Suppose that there exists $x, y \in \mathbb{C} \setminus B_{\varepsilon^\alpha\mathbb{r}}(z)$ and $s > 0$ such that $\partial\mathcal{B}_s^{y,\bullet}(x) \cap B_{\varepsilon\mathbb{r}}(z) \neq \varnothing$. We need to show that (3-6) holds for an appropriate choice of β. With a view toward applying Lemma 3.9, we claim that

$$X_n \geq c \sum_{j=n+1}^N X_j \quad \text{for all } n \in [1, N-1]_{\mathbb{Z}}. \tag{3-13}$$

Indeed, suppose by way of contradiction that (3-13) does not hold for some $n \in [1, N-1]_{\mathbb{Z}}$, i.e., $X_n < c \sum_{j=n+1}^N X_j$. By (3-12), for this choice of n,

$$D_h(\text{around } A_n) < \sum_{j=n+1}^N D_h(\text{across } A_j) \leq D_h(A_n, B_{\varepsilon\mathbb{r}}(z)), \tag{3-14}$$

where the last inequality follows since the A_j's are disjoint, numbered from outside in, and surround $B_{\varepsilon\mathbb{r}}(z)$. Let π be a path in A_n which disconnects the inner and outer boundaries of A_n and has D_h-length strictly less than $D_h(A_n, B_{\varepsilon\mathbb{r}}(z))$.

Let $w \in \partial\mathcal{B}_s^{y,\bullet}(x) \cap B_{\varepsilon\mathbb{r}}(z)$. By Lemma 3.4, $D_h(x, w) = s$. There is a D_h-geodesic $P : [0, s] \to \mathbb{C}$ from x to w. Let τ be the first time that P hits π. Since P is a geodesic, the D_h-distance from x to each point of π is at most $\tau + (D_h\text{-length of } \pi)$, which by the preceding paragraph is less than $\tau + D_h(A_n, B_{\varepsilon\mathbb{r}}(z))$.

On the other hand, P must travel from A_n to $B_{\varepsilon\mathbb{r}}(z)$ after time τ, so $s \geq \tau + D_h(A_n, B_{\varepsilon\mathbb{r}}(z))$. Therefore, each point of π lies at D_h-distance less than s from x, so $\pi \subset \mathcal{B}_s(x)$. Since $x, y \notin B_{\varepsilon^\alpha\mathbb{r}}(z)$ and π is contained in $B_{\varepsilon^\alpha \mathbb{r}}(z)$ and disconnects $B_{\varepsilon \mathbb{r}}(z)$ from $\partial B_{\varepsilon^\alpha \mathbb{r}}(z)$, we get that $\mathcal{B}_s(x)$ disconnects $B_{\varepsilon \mathbb{r}}(z)$ from x and y. Therefore, $B_{\varepsilon \mathbb{r}}(z) \cap \mathcal{B}_s^{y,\bullet}(x) = \varnothing$, which is our desired contradiction. Hence (3-13) holds.

By (3-13), there are N values of $n \in [1, N]_{\mathbb{Z}}$ for which $X_n \geq c \sum_{j=n+1}^N X_j$. Therefore, Lemma 3.9 gives

$$\frac{\log\left(\frac{1}{X_N} \max_{n \in [1,N]_{\mathbb{Z}}} X_n\right)}{\log(c+1)} \geq N - O(1), \tag{3-15}$$

where the $O(1)$ denotes a constant depending only on c (not on ε). Therefore,

$$\begin{aligned} D_h(\text{around } B_{\varepsilon^\alpha \mathbb{r}}(z) \setminus B_{\varepsilon \mathbb{r}}(z)) &\leq X_N \quad (\text{by the definition (3-11) of } X_n) \\ &\leq (c+1)^{-N+O(1)} \max_{n \in [1,N]_{\mathbb{Z}}} X_n \quad (\text{by rearranging (3-15)}) \\ &\leq O(1)(c+1)^{-N} D_h(\text{across } \mathbb{A}_{\varepsilon \mathbb{r}, \varepsilon^\alpha \mathbb{r}}(z)) \quad (\text{by (3-12)}). \end{aligned} \tag{3-16}$$

Since $N = \lceil \eta \log \varepsilon^{-1} \rceil$, for small enough ε the quantity $O(1)(c+1)^{-N}$ is bounded above by ε^β for an appropriate choice of $\beta > 0$. This gives (3-6). □

3.3. *Lower bound for D_h-distances across LQG annuli.* An easy consequence of Lemma 3.6 is the following lemma, which gives a polynomial lower bound for the Euclidean distance between the outer boundaries of concentric filled D_h-metric balls. This lemma will play an important role in the proof of Theorem 1.5 and in the proof of confluence of geodesics.

Lemma 3.10. *There exists $\beta > 0$ such that the following is true. Fix $b > 1$ and for $\mathbb{r} > 0$ let $\tau_{\mathbb{r}} = D_h(0, \partial B_{\mathbb{r}}(0))$ be as in* (3-2). *It holds with probability tending to* 1 *as $\delta \to 0$, uniformly in the choice of $\mathbb{r}$, that for each $s, t \in [\tau_{\mathbb{r}}, \tau_{b\mathbb{r}}]$ with $|s-t| \leq \delta \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$,*

$$\operatorname{dist}(\partial \mathcal{B}_s^\bullet, \partial \mathcal{B}_t^\bullet) \geq \left(\frac{|s-t|}{\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}} \right)^{1/\beta} \mathbb{r}, \tag{3-17}$$

where dist *denotes Euclidean distance.*

Note that in the subcritical case, (3-17) is immediate from the local Hölder continuity of D_h with respect to the Euclidean metric [12, Theorem 1.7], so the lemma has nontrivial content only in the case when $\xi \geq \xi_{\text{crit}}$. We will deduce Lemma 3.10 from the following more quantitative statement, which allows for a general choice of starting points and target points for the filled metric balls. For the statement, we recall the notation $B_r(X) = \bigcup_{z \in X} B_r(z)$ for the Euclidean r-neighborhood of a set $X \subset \mathbb{C}$.

Lemma 3.11. *For each $\alpha \in (0, 1)$, there exists $\beta = \beta(\alpha) > 0$ such that the following is true. Let $U \subset \mathbb{C}$ be a Euclidean-bounded open set and let $\mathbb{r} > 0$. With polynomially high probability as $\varepsilon \to 0$, uniformly over the choice of $\mathbb{r}$, it holds for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in [\varepsilon \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}, D_h(x, y)]$ that*

$$\operatorname{dist}\left(\partial \mathcal{B}_s^{y,\bullet}(x) \cap (\mathbb{r} U \setminus B_{\varepsilon^{\alpha/\beta} \mathbb{r}}(\{x, y\})), \partial \mathcal{B}_{(1-\varepsilon)s}^{y,\bullet}(x)\right) \geq \varepsilon^{1/\beta} \mathbb{r}, \tag{3-18}$$

where dist *denotes Euclidean distance, we define $B_{\varepsilon^{\alpha/\beta}}(\{x, \infty\}) = B_{\varepsilon^{\alpha/\beta}}(x)$, and we make the convention that the distance from any set to the empty set is ∞ (which is consistent with the convention that the infimum of the empty set is ∞).*

Proof. Let $\tilde{\beta} = \tilde{\beta}(\alpha) > 0$ be the parameter β from Corollary 3.7 and let $\beta = \tilde{\beta}/2$. By Corollary 3.7 (applied with $\varepsilon^{1/\beta}$ instead of ε), it holds with polynomially high probability as $\varepsilon \to 0$ that for each $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, each $s \in [0, D_h(x, y)]$, and each $z \in \partial \mathcal{B}_s^{y,\bullet}(x) \cap (\mathbb{r} U \setminus B_{\varepsilon^{\alpha/\beta}\mathbb{r}}(\{x, y\}))$,

$$D_h(\text{around } \mathbb{A}_{\varepsilon^{1/\beta}\mathbb{r}, \varepsilon^{\alpha/\beta}\mathbb{r}}(z)) \le \varepsilon^2 s. \tag{3-19}$$

We henceforth work on the polynomially high probability event that this is the case.

Let x, y, s, z be as above with x nonsingular and let π be a path in $\mathbb{A}_{\varepsilon^{1/\beta}\mathbb{r}, \varepsilon^{\alpha/\beta}\mathbb{r}}(z)$ which disconnects the inner and outer boundaries of this annulus and has D_h-length at most $(\varepsilon/2)s$. If $\partial \mathcal{B}_{(1-\varepsilon)s}^{y,\bullet}(x) \cap B_{\varepsilon^{1/\beta}\mathbb{r}}(z) \neq \varnothing$, then since $x, y \notin B_{\varepsilon^{\alpha/\beta}\mathbb{r}}(z)$ it must be the case that each of $\partial \mathcal{B}_{(1-\varepsilon)s}^{y,\bullet}(x)$ and $\partial \mathcal{B}_s^{y,\bullet}(x)$ intersects π. This implies that the D_h-distance between $\partial \mathcal{B}_{(1-\varepsilon)s}^{y,\bullet}(x)$ and $\partial \mathcal{B}_s^{y,\bullet}(x)$ is at most $(\varepsilon/2)s$. This cannot be the case since Lemma 3.4 implies the D_h-distance between $\partial \mathcal{B}_{(1-\varepsilon)s}^{y,\bullet}(x)$ and $\partial \mathcal{B}_s^{y,\bullet}(x)$ is εs. Therefore, $\partial \mathcal{B}_{(1-\varepsilon)s}^{y,\bullet}(x) \cap B_{\varepsilon^{1/\beta}\mathbb{r}}(z) = \varnothing$, so (3-18) holds. ☐

Proof of Lemma 3.10. Let β be the parameter from Lemma 3.11 with $\alpha = 1/2$. By Lemma 3.5, it holds with probability tending to 1 as $\delta \to 0$ that $B_{\delta^{1/(2\beta)}\mathbb{r}}(0) \subset \mathcal{B}_{\tau_{\mathbb{r}}}^{\bullet}$, which means that also $B_{\delta^{1/(2\beta)}\mathbb{r}}(0) \subset \mathcal{B}_s^{\bullet}$ for each $s \ge \tau_{\mathbb{r}}$. Furthermore, by tightness across scales (Axiom V) it holds with probability tending to 1 as $\delta \to 0$ that $\tau_{\mathbb{r}} \ge \delta \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$. Hence with probability tending to 1 as $\delta \to 0$, we have

$$\tau_{\mathbb{r}} \ge \delta \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \quad \text{and} \quad \partial \mathcal{B}_s^{\bullet} \cap (B_{b\mathbb{r}}(0) \setminus B_{\delta^{1/(2\beta)}\mathbb{r}}(0)) = \partial \mathcal{B}_s^{\bullet}$$

for each $s \in [\tau_{\mathbb{r}}, \tau_{b\mathbb{r}}]$.

We now apply Lemma 3.11 (with $U = B_b(0)$) and a union bound over dyadic values of ε, followed by the estimate of the preceding paragraph, to get that with probability tending to 1 as $\delta \to 0$, the following is true. For $\varepsilon \in (0, \delta) \cap \{2^{-k}\}_{k \in \mathbb{N}}$ and each $s \in [\tau_{\mathbb{r}}, \tau_{b\mathbb{r}}]$,

$$\operatorname{dist}(\partial \mathcal{B}_s^{\bullet}, \partial \mathcal{B}_{(1-\varepsilon)s}^{\bullet}) \ge \varepsilon^{1/\beta} \mathbb{r}. \tag{3-20}$$

By Lemma 2.8, for any $p \in (0, 1)$ we can find $C = C(p, b) > 1$ such that for each $\mathbb{r} > 0$,

$$\mathbb{P}\big[C^{-1} \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \le \tau_{\mathbb{r}} \le \tau_{b\mathbb{r}} \le C \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}\big] \ge p. \tag{3-21}$$

Now suppose that the event in (3-21) holds and the event in (3-20) holds with $C\delta$ in place of δ, which happens with probability at least $p - o_\delta(1)$. By (3-21), for any $s, t \in [\tau_{\mathbb{r}}, \tau_{b\mathbb{r}}]$ with $s - \delta \mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)} \le t \le s$, we have $t \le (1 - \varepsilon)s$ for some dyadic $\varepsilon \in (0, C\delta)$ which satisfies

$$\varepsilon \ge \frac{s - t}{2s} \ge \frac{1}{2C} \frac{s - t}{\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}}. \tag{3-22}$$

We conclude by combining (3-22) with (3-20), replacing β by a slightly smaller number to absorb the factor of $1/(2C)$ into a small power of ε, and noting that the parameter p from (3-21) can be made arbitrarily close to 1. ☐

3.4. ***The metric net is finite-dimensional.*** We will now use Lemma 3.11 to prove Theorem 1.5. Since we are proving an a.s. statement, we no longer need to include the Euclidean scale parameter $\mathbb{r}$.

Proof of Theorem 1.5. We write $\dim_h$ for Hausdorff dimension with respect to D_h. Fix a Euclidean-bounded open set $U \subset \mathbb{C}$, a number $r > 0$, and numbers $s_2 > s_1 > 0$. By the countable stability of Hausdorff dimension, it suffices to show that there exists $\Delta \in (0, \infty)$ (not depending on U, r, s_1, s_2) such that a.s. for each nonsingular point $x \in U$ and each $y \in \mathbb{C} \cup \{\infty\}$,

$$\dim_h\big((\mathcal{N}^y_{s_2}(x) \setminus \mathcal{N}^y_{s_1}(x)) \cap (U \setminus B_r(\{x, y\}))\big) \le \Delta. \tag{3-23}$$

See Figure 3 for an illustration of the proof. The idea is as follows. We consider the set of $\varepsilon^{1/\beta} \times \varepsilon^{1/\beta}$ squares with corners in $\varepsilon^{1/\beta}\mathbb{Z}^2$ which intersect a neighborhood of U. By Proposition 2.9 and an estimate for the maximum of the circle average process $h_{\varepsilon^{1/\beta}}$, each of these squares can be surrounded by a path π_S of Euclidean diameter comparable to $\varepsilon^{1/\beta}$ whose D_h-length is at most a negative power of ε. The number of D_h-balls of radius ε needed to cover each of these paths is at most a negative power of ε. Using Lemma 3.11, we show that for each $s \in [s_1, s_2]$, each D_h-geodesic from a point of $\partial\mathcal{B}^{y,\bullet}_s(x)$ to x must hit π_S for one of the $\varepsilon^{1/\beta} \times \varepsilon^{1/\beta}$ squares S which intersects $\partial\mathcal{B}^{y,\bullet}_{s-\varepsilon}(x)$, and it must do so before time ε. This shows that the set in (3-23) is contained in the union of a polynomial (in ε) number of D_h-balls of radius 2ε.

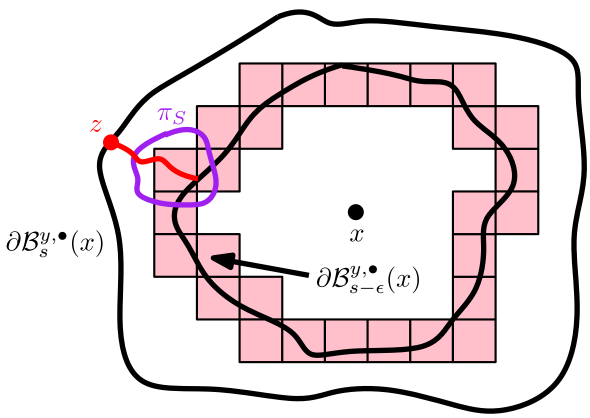

Figure 3. Illustration of the proof of Theorem 1.5. The figure shows a point $z \in \partial\mathcal{B}^{y,\bullet}_s(x)$ for some $s \in [s_1, s_2]$ (red) and the subset of $\mathcal{S}_\varepsilon$ consisting of squares S which intersect $\partial\mathcal{B}^{y,\bullet}_{s-\varepsilon}(x)$ (pink). Thanks to Lemma 3.11, we can arrange that each of these squares lies at Euclidean distance at least $\varepsilon^{1/\beta}$ from $\partial\mathcal{B}^{y,\bullet}_s$. Moreover, using basic estimates for D_h we can arrange that each square S is surrounded by a path $\pi_S \subset B_{\varepsilon^{1/\beta}}(S) \setminus S$ whose D_h-length is bounded above by a negative power of ε (one such path is shown in purple). Hence the number of D_h-balls needed to cover π_S is at most a negative power of ε. If P is a D_h-geodesic from 0 to z, then $P([s-\varepsilon, s])$ (red) must intersect π_S for some $S \in \mathcal{S}_\varepsilon$, so z is contained in the ε-neighborhood of one of the D_h-metric balls in our covering of π_S. This leads to an upper bound for the number of D_h-balls of radius 2ε needed to cover the metric net.

Step 1: *regularity events.* Let $\tilde\beta > 0$ be the parameter β from Lemma 3.11 with $\alpha = 1/2$, say, and let $\beta = \tilde\beta/2$. By Lemma 3.11, it holds with probability tending to 1 as $\varepsilon \to 0$ that for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in [s_1, s_2 \wedge D_h(x, y)]$,

$$\operatorname{dist}\big(\partial\mathcal{B}_s^{y,\bullet}(x) \cap (U \setminus B_r(\{x, y\})), \partial\mathcal{B}_{s-\varepsilon}^{y,\bullet}(x)\big) \geq 4\varepsilon^{1/\beta}. \tag{3-24}$$

Note that here we have used that $s \in [s_1, s_2]$ to absorb an s-dependent constant factor into a power of ε.

Let $\mathcal{S}_\varepsilon$ be the set of $\varepsilon^{1/\beta} \times \varepsilon^{1/\beta}$ squares with corners in $\varepsilon^{1/\beta}\mathbb{Z}^2$ which intersect the LQG s_2-neighborhood $\mathcal{B}_{s_2}(U)$. For each $S \in \mathcal{S}_\varepsilon$, we define the annular region

$$A_S := B_{\varepsilon^{1/\beta}}(S) \setminus S. \tag{3-25}$$

Since $\mathcal{B}_{s_2}(U)$ is a.s. contained in some Euclidean-bounded open set, we can apply Proposition 2.9 and a union bound over $S \in \mathcal{S}_\varepsilon$ to get that with probability tending to 1 as $\varepsilon \to 0$,

$$D_h(\text{around } A_S) \leq \varepsilon^{o_\varepsilon(1)} \mathfrak{c}_{\varepsilon^{1/\beta}} e^{\xi h_{\varepsilon^{1/\beta}}(v_S)} \quad \text{for all } S \in \mathcal{S}_\varepsilon, \tag{3-26}$$

where v_S is the center of S and the rate of convergence of the $o_\varepsilon(1)$ is deterministic and uniform over all $S \in \mathcal{S}_\varepsilon$.

The random variables $e^{\xi h_{\varepsilon^{1/\beta}}(v_S)}$ for $S \in \mathcal{S}_\varepsilon$ are centered Gaussian with variances $\log \varepsilon^{-1/\beta} + O_\varepsilon(1)$. By the Gaussian tail bound and a union bound over $O_\varepsilon(\varepsilon^{-2/\beta})$ squares, we get that with probability tending to 1 as $\varepsilon \to 0$, we have $h_{\varepsilon^{1/\beta}}(v_S) \leq (2/\beta + o_\varepsilon(1)) \log \varepsilon^{-1}$ for each $S \in \mathcal{S}_\varepsilon$. By Proposition 2.10, we also have $\mathfrak{c}_{\varepsilon^{1/\beta}} = \varepsilon^{(1/\beta)\xi Q + o_\varepsilon(1)}$. By plugging these estimates into (3-26), we get that with probability tending to 1 as $\varepsilon \to 0$,

$$D_h(\text{around } A_S) \leq \varepsilon^{-(1/\beta)(2-Q)+o_\varepsilon(1)} \quad \text{for all } S \in \mathcal{S}_\varepsilon, \tag{3-27}$$

where the rate of convergence of the $o_\varepsilon(1)$ is deterministic and uniform over all $S \in \mathcal{S}_\varepsilon$.

Henceforth assume that (3-24) and (3-27) both hold, which happens with probability tending to 1 as $\varepsilon \to 0$. We will prove an upper bound for the number of D_h-balls of radius ε needed to cover the set on the left side of (3-23).

Step 2: *defining a collection of D_h-metric balls.* For $S \in \mathcal{S}_\varepsilon$, let π_S be a path in A_S which separates the inner and outer boundaries of A_S and which has D_h-length at most $\varepsilon^{-(1/\beta)(2-Q)+o_\varepsilon(1)}$ (such a path exists by (3-27)). There is a set $\mathcal{M}_S$ of $\#\mathcal{M}_S \leq \varepsilon^{-(1/\beta)(2-Q)-1+o_\varepsilon(1)}$ D_h-metric balls of radius ε whose union contains π_S. Since $\#\mathcal{S}_\varepsilon = O_\varepsilon(\varepsilon^{-2/\beta})$, we have

$$\#\Big(\bigcup_{S \in \mathcal{S}_\varepsilon} \mathcal{M}_S\Big) \leq \varepsilon^{-\Delta + o_\varepsilon(1)} \quad \text{for } \Delta = \frac{1}{\beta}(4 - Q) + 1. \tag{3-28}$$

By the definition of Hausdorff dimension, to prove (3-23) with Δ as in (3-28), it suffices to show (continuing to assume (3-24) and (3-27)), that for each nonsingular point $x \in \mathbb{C}$ and $y \in \mathbb{C} \cup \{\infty\}$,

$$(\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)) \cap (U \setminus B_r(\{x, y\})) \subset \bigcup_{S \in \mathcal{S}_\varepsilon} \bigcup_{\mathcal{B} \in \mathcal{M}_S} \mathcal{B}', \tag{3-29}$$

where $\mathcal{B}'$ denotes the D_h-ball with the same center as $\mathcal{B}$ and twice the radius.

Step 3: *covering the metric net.* To prove (3-29), let $x \in \mathbb{C}$, $y \in \mathbb{C} \cup \{\infty\}$, $s \in [s_1, D_h(x, y) \wedge s_2]$, and $z \in \partial \mathcal{B}_s^{y,\bullet}(x) \cap (U \setminus B_r(\{x, y\}))$. We need to show that $z \in \mathcal{B}'$ for some $\mathcal{B} \in \bigcup_{S \in \mathcal{S}_\varepsilon} \mathcal{M}_S$. Let P be a D_h-geodesic from x to z. By Lemma 3.4, $D_h(x, z) = s$ so $P : [0, s] \to \mathbb{C}$. We have $P(s - \varepsilon) \in \partial \mathcal{B}_{s-\varepsilon}^{y,\bullet}(x)$. Furthermore, since $x \in U$ and $s \leq s_2$, we have $P(s - \varepsilon) \in \mathcal{B}_{s_2}(U)$, so there exists $S \in \mathcal{S}_\varepsilon$ such that $P(s - \varepsilon) \in S$. By (3-24), $P(s - \varepsilon)$ lies at Euclidean distance at least $4\varepsilon^{1/\beta}$ from $z = P(s)$, so $z \notin B_{\varepsilon^{1/\beta}}(S)$. Therefore, the path π_S disconnects $P(s - \varepsilon)$ from z, so P must cross π_S between time $s - \varepsilon$ and time s. This implies that there is a point of π_S which lies at D_h-distance at most ε from z. This point is contained in $\mathcal{B}$ for one of the ε-balls $\mathcal{B} \in \mathcal{M}_S$. Therefore, $z \in \mathcal{B}'$, as required. □

Our proof of Theorem 1.5 also yields the following proposition, which is a slightly stronger version of the compactness statement from Theorem 1.4.

Proposition 3.12. *Almost surely, for each* $x \in \mathbb{C}$, *each* $y \in \mathbb{C} \cup \{\infty\}$, *and each* $0 < s_1 < s_2 < D_h(x, y)$, *the set* $\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)$ *is precompact with respect to* D_h (*i.e., its* D_h*-closure is compact*).

Proof. Let $s_2 > s_1 > 0$, let $U \subset \mathbb{C}$ be a Euclidean-bounded open set, and let $r > 0$. The proof of Theorem 1.5 shows there exists $\Delta > 0$ such that with probability tending to 1 as $\varepsilon \to 0$, it holds for each $x \in U$ and each $y \in \mathbb{C} \cup \{\infty\}$ that the set $(\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)) \cap (U \setminus B_r(\{x, y\}))$ can be covered by $\varepsilon^{-\Delta + o_\varepsilon(1)}$ D_h-balls of radius ε. Hence, a.s. there is a sequence $\varepsilon_k \to 0$ (depending on U, r) such that for each x, y as above and each $k \in \mathbb{N}$, this set can be covered by $\varepsilon_k^{-2\Delta}$ D_h-balls of radius ε_k.

Let $\{U_n\}_{n \in \mathbb{N}}$ be an increasing sequence of Euclidean-bounded open sets whose union is all of $\mathbb{C}$ and let $\{r_n\}_{n \in \mathbb{N}}$ be a sequence of positive radii tending to zero. By the conclusion of the preceding paragraph, a.s. for each $n \in \mathbb{N}$ there exists a sequence $\varepsilon_{n,k} \to 0$ such that for each $k \in \mathbb{N}$, each $x \in U_n$, and each $y \in \mathbb{C} \cup \{\infty\}$, the set $(\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)) \cap (U_n \setminus B_{r_n}(\{x, y\}))$ can be covered by $\varepsilon_{n,k}^{-2\Delta}$ D_h-balls of radius $\varepsilon_{n,k}$.

If x is a singular point or $y = x$, then $\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x) = \varnothing$, so we can assume without loss of generality that x is nonsingular and $y \neq x$. For each nonsingular $x \in \mathbb{C}$ and each $y \in \mathbb{C} \cup \{\infty\}$ with $y \neq x$, the set $\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)$ is Euclidean-bounded, so both x and this set are contained in U_n for each large enough $n \in \mathbb{N}$. By Lemma 3.3, $\mathcal{B}_{s_1}^{y,\bullet}(x)$ contains $B_{r_n}(x)$ for each large enough $n \in \mathbb{N}$, which implies that $\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)$ is disjoint from $B_{r_n}(x)$ for each large enough $n \in \mathbb{N}$. Furthermore, if $s_2 < D_h(x, y)$ then since $\mathcal{B}_{s_2}^{y,\bullet}(x) \supset \mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)$ is Euclidean-closed (Lemma 3.1) and does not contain y, this set lies at positive Euclidean distance from y.

Hence, a.s. for each nonsingular $x \in \mathbb{C}$ and each $y \in \mathbb{C} \cup \{\infty\}$ with $0 < s_2 < D_h(x, y)$, the set $\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)$ is contained in $U_n \setminus B_{r_n}(\{x, y\})$ for each large enough $n \in \mathbb{N}$. Therefore, as shown above, $\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)$ can be covered by finitely many D_h-balls of radius $\varepsilon_{n,k}$ for each large enough $n \in \mathbb{N}$ and each large enough $k \in \mathbb{N}$. Hence $\mathcal{N}_{s_2}^y(x) \setminus \mathcal{N}_{s_1}^y(x)$ is totally bounded with respect to D_h, hence precompact with respect to D_h.

This proves the proposition for a deterministic choice of s_1 and s_2. Every interval $[s_1, s_2] \subset (0, \infty)$ is contained in $[s_1', s_2']$ for some $s_1', s_2' \in \mathbb{Q} \cap (0, \infty)$ with s_1', s_2' arbitrarily close to s_1, s_2. This gives the proposition statement in general. □

4. Outer boundaries of LQG metric balls are Jordan curves

The goal of this section is to prove the following proposition, which is the missing ingredient needed to prove Theorem 1.4.

Proposition 4.1. *Almost surely, for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$, the set $\partial \mathcal{B}_s^{y,\bullet}(x)$ is a Jordan curve.*

4.1. ***A criterion for a domain boundary to be a curve.*** In this subsection we will prove a general criterion for the boundary of a planar domain to be a curve. Our criterion will be stated in terms of disconnecting sets, defined as follows.

Definition 4.2. Let $X, Y \subset \mathbb{C}$ and $A_1, A_2 \subset X$. We say that Y *disconnects* A_2 from A_1 in X if the following is true: A_1 is disjoint from Y; and any two points $x \in A_1$ and $y \in A_2 \setminus Y$ lie in different connected components of $X \setminus Y$.

We note that by definition Y disconnects any subset of $Y \cap X$ from any subset of $X \setminus Y$.

Proposition 4.3. *Let $U \subset \mathbb{C}$ be a domain containing 0, not all of $\mathbb{C}$, such that ∂U is compact. We assume that either U is bounded and simply connected; or U is unbounded and $U \cup \{\infty\}$ is a simply connected subset of the Riemann sphere. Suppose that for each $\varepsilon > 0$, there exists $\delta > 0$ such that each subset of U which can be disconnected from 0 in U by a set Y of Euclidean diameter at most δ with $Y \cap \partial U \neq \emptyset$ has Euclidean diameter at most ε. Then ∂U is the image of a (not necessarily simple) curve.*

The criterion of Proposition 4.3 is similar in spirit to the concept of ∂U being locally connected (see, e.g., [36, Section 2.2]), which is a different condition that implies that ∂U is a curve. The reason why we require that $Y \cap \partial U \neq \emptyset$ is to rule out the possibility that Y is a small loop surrounding 0, in which case Y would disconnect most of U from 0.

For the proof of Proposition 4.3, we first need to recall some standard definitions from complex analysis. See, e.g., [36] for more detail on these concepts. A *crosscut* of a domain $U \subset \mathbb{C}$ is a simple curve $C : [0, 1] \to \overline{U}$ such that $C(0), C(1) \in \partial U$ and $C((0, 1)) \subset U$. If ∂U is bounded, we define a *null chain* in U to be a sequence of crosscuts $\{C_n\}_{n \in \mathbb{N}}$ with the following properties.

(i) $C_n \cap C_{n+1} = \emptyset$ for each $n \in \mathbb{N}$.

(ii) C_n disconnects C_{n+1} from C_1 in U for each $n \in \mathbb{N}$.

(iii) As $n \to \infty$, the Euclidean diameter of C_n converges to zero.

If $\{C_n\}$ and $\{C_n'\}$ are two null chains, we say that $\{C_n\}$ and $\{C_n'\}$ are *equivalent* if for each large enough $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ for which C_m' disconnects C_n from C_1 in U and C_m disconnects C_n' from C_1' in U. A *prime end* of ∂U is an equivalence class of null chains.

For a prime end p represented by a null chain $\{C_n\}$, we define A_p to be the intersection over all $n \in \mathbb{N}$ of the closure of the set of points in U which are disconnected from C_1 by C_n in U. Then $A_p \subset \partial U$. We call A_p the *set of points corresponding to p*.

In the next two lemmas, we assume that U is a domain satisfying the hypotheses of Proposition 4.3.

Lemma 4.4. *Each prime end of U corresponds to a single point of ∂U.*

Proof. Let p be a prime end for U and let $\{C_n\}_{n\in\mathbb{N}}$ be a null chain corresponding to p. By possibly removing finitely many of the C_n's, we can assume without loss of generality that C_1 disconnects 0 from C_n for each $n \geq 2$. Since the diameter of C_n tends to zero as $n \to \infty$, our assumption on U implies that the diameter of the set of points in U which are disconnected from 0 in U by C_n, hence also its closure, tends to zero as $n \to \infty$. Hence the (decreasing) intersection of the closures of these sets has diameter zero, so is a single point. □

In what follows, if U is unbounded we view ∞ as a point of U, so that by the Riemann mapping theorem there exists a conformal map from the open unit disk $\mathbb{D}$ to U.

Lemma 4.5. *Every conformal map $\phi : \mathbb{D} \to U$ extends to a continuous map $\overline{\mathbb{D}} \to \overline{U}$.*

Proof. By [36, Theorem 2.15] there is a bijection $\widehat{\phi}$ from $\partial\mathbb{D}$ to prime ends of U such that for each $u \in \partial\mathbb{D}$ and each null chain $\{C_n\}_{n\in\mathbb{N}}$ for the prime end $\widehat{\phi}(u)$, $\{f^{-1}(C_n)\}_{n\in\mathbb{N}}$ is a null chain for u. By Lemma 4.4, for each $u \in \partial\mathbb{D}$ the prime end $\widehat{\phi}(u)$ corresponds to a single point of ∂U. Let $\phi(u)$ be this point. We need to show that ϕ, thus extended, is continuous.

Obviously, ϕ is continuous at each point of $\mathbb{D}$, so consider a point $u \in \partial\mathbb{D}$ and a sequence $\{z_k\}_{k\in\mathbb{N}}$ in $\overline{\mathbb{D}}$ which converges to u. We will show that $\phi(z_k) \to \phi(u)$.

For this purpose let $\varepsilon > 0$ and let $\{C_n\}_{n\in\mathbb{N}}$ be a null chain for the prime end $\widehat{\phi}(u)$, as above. By possibly removing one of the C_n's, we can assume without loss of generality that $0 \notin C_n$ for each n. By [36, Proposition 2.12], each of the cross cuts C_n separates U into exactly two connected components. Let G_n be the one of these connected components which does not contain 0. Then $\phi(u) \in \partial G_n$. Since the Euclidean diameter of C_n tends to 0 as $n \to \infty$, our hypothesis on U implies that the Euclidean diameter of G_n, and hence also the Euclidean diameter of $\overline{G}_n$, tends to 0 as $n \to \infty$. Hence there is some $n_* \in \mathbb{N}$ such that for $n \geq n_*$, each point of $\overline{G}_n$ lies at Euclidean distance at most ε from $\phi(u)$.

By the defining property of $\widehat{\phi}$, the sets $\phi^{-1}(C_n)$ are a null chain for the prime end $u \in \partial\mathbb{D}$. In particular, each $\phi^{-1}(C_n)$ separates $\mathbb{D}$ into two connected components, namely $\phi^{-1}(G_n)$ and $\phi^{-1}(U \setminus \overline{G}_n)$. Each prime end of U which does not correspond to a point of ∂G_n corresponds to a point which lies at positive distance from G_n, so can be represented by a null chain whose cross cuts (except for their endpoints) are contained in $U \setminus \overline{G}_n$.

We claim that $\phi(\overline{\phi^{-1}(G_n)}) \subset \overline{G}_n$. Since $\phi|_{\mathbb{D}}$ is a homeomorphism from $\mathbb{D}$ to U, we have that $\phi(\overline{\phi^{-1}(G_n)} \cap \mathbb{D}) \subset \overline{G}_n$. Now let $w \in \overline{\phi^{-1}(G_n)} \cap \partial\mathbb{D}$ and suppose by way of contradiction that $\phi(w) \notin \overline{G}_n$. By the preceding paragraph there is a null chain $\{\widetilde{C}_n\}_{n\in\mathbb{N}}$ for $\phi(w)$ whose cross cuts (except for their endpoints) are contained in $U \setminus \overline{G}_n$. But, then $\{\phi^{-1}(\widetilde{C}_n)\}_{n\in\mathbb{N}}$ is a null chain for w whose cross cuts (except for their endpoints) are contained in $\phi^{-1}(U \setminus \overline{G}_n)$, hence lie at positive distance from $\phi^{-1}(G_n)$. This contradicts the fact that $w \in \overline{\phi^{-1}(G_n)}$, as desired. Therefore, $\phi(\overline{\phi^{-1}(G_n)}) \subset \overline{G}_n$.

Recall the sequence $z_k \to z$ from above. For each large enough k, z_k is disconnected from $\phi^{-1}(0)$ in $\mathbb{D}$ by $\phi^{-1}(C_n)$, so $z_k \in \overline{\phi^{-1}(G_n)}$. It therefore follows from the conclusion of the preceding paragraph that for each such k, we have $\phi(z_k) \in \overline{G}_n$ and hence $|\phi(z_k) - \phi(u)| < \varepsilon$. Since ε is arbitrary, this gives the continuity of ϕ. □

Proof of Proposition 4.3. Lemma 4.5 implies that ∂U is a curve, since it is the continuous image of $\partial\mathbb{D}$ under ϕ (in fact, the statement of Lemma 4.5 is equivalent to the statement that ∂U is a curve; see [36, Theorem 2.1]). □

4.2. *Proof of Proposition 4.1.* In this subsection we will use Proposition 4.3 to prove Proposition 4.1. Let us first introduce the domain U that we will work with. For each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$, we let

$$U_s^y(x) := (\text{connected component of the interior of } \mathcal{B}_s^{y,\bullet}(x) \text{ which contains } x). \tag{4-1}$$

By Lemma 3.3, a.s. x lies in the interior of $\mathcal{B}_s^{y,\bullet}(x)$ for every x, y, s as above. Hence, almost surely $U_s^y(x)$ is well-defined for every such x, y, s.

Once we show that $\partial\mathcal{B}_s^{y,\bullet}(x)$ is a Jordan curve, we will get that $U_s^y(x)$ is in fact the only connected component of the interior of $\mathcal{B}_s^{y,\bullet}(x)$. However, we do not rule out the possibility that the interior of $\mathcal{B}_s^{y,\bullet}(x)$ is not connected *a priori*. The following lemma will allow us to work with $U_s^y(x)$ instead of $\mathcal{B}_s^{y,\bullet}(x)$ throughout the proof of Proposition 4.1.

Lemma 4.6. *Almost surely, for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$, the following is true, with $U_s^y(x)$ as in* (4-1). *We have $\partial U_s^y(x) = \partial \overline{U}_s^y(x) = \partial\mathcal{B}_s^{y,\bullet}(x)$ and $U_s^y(x)$ is simply connected. Furthermore, each D_h-geodesic from x to a point of $\partial\mathcal{B}_s^{y,\bullet}(x)$ is contained in $U_s^y(x)$ except for its terminal endpoint.*

Proof. All of the statements in the proof are required to hold a.s. for each x, y, s as in the lemma statement. To prove that $\partial U_s^y(x) = \partial\mathcal{B}_s^{y,\bullet}(x)$, we first argue that $\partial U_s^y(x) \subset \partial\mathcal{B}_s^{y,\bullet}(x)$. Indeed, each $z \in \partial U_s^y(x)$ is an accumulation point of $U_s^y(x) \subset \mathcal{B}_s^{y,\bullet}(x)$, so in particular $\partial U_s^y(x) \subset \mathcal{B}_s^{y,\bullet}(x)$. Hence it suffices to show that if z is in the interior of $\mathcal{B}_s^{y,\bullet}(x)$, then $z \notin \partial U_s^y(x)$. Indeed, for such a z either $z \in U_s^y(x)$ or z belongs to a connected component of the interior of $\mathcal{B}_s^{y,\bullet}(x)$ other than $U_s^y(x)$. In the former case, $z \notin \partial U_s^y(x)$ since $U_s^y(x)$ is open. In the latter case, $z \notin U_s^y(x)$ since the other connected components of the interior of $\mathcal{B}_s^{y,\bullet}(x)$ are open sets disjoint from $U_s^y(x)$, so they are also disjoint from $\partial U_s^y(x)$.

To prove that $\partial\mathcal{B}_s^{y,\bullet}(x) \subset \partial U_s^y(x)$, let $z \in \partial\mathcal{B}_s^{y,\bullet}(x)$. By Lemma 3.4, $D_h(x, z) = s$. Let $P : [0, s] \to \mathbb{C}$ be a D_h-geodesic from x to z. Then $P \subset \mathcal{B}_s^{y,\bullet}(x)$. Furthermore, for $t < s$ we have $D_h(x, P(t)) = t$, so Lemma 3.4 implies that $P(t) \notin \partial\mathcal{B}_s^{y,\bullet}(x)$. Therefore, $P([0, s))$ is contained in the interior of $\mathcal{B}_s^{y,\bullet}(x)$ and hence $P([0, s)) \subset U_s^y(x)$. This shows that z is an accumulation point of $U_s^y(x)$, so $z \in \partial U_s^y(x)$.

We have shown that $\partial\mathcal{B}_s^{y,\bullet}(x) = \partial U_s^y(x)$. Since $U_s^y(x) \subset \overline{U}_s^y(x) \subset \mathcal{B}_s^{y,\bullet}(x)$, we have $\partial\overline{U}_s^y(x) = \partial\mathcal{B}_s^{y,\bullet}(x)$.

The argument in the second paragraph of the proof shows that each D_h-geodesic from x to a point of $\partial\mathcal{B}_s^{y,\bullet}(x) = \partial U_s^y(x)$ is contained in $U_s^y(x)$ except for its terminal endpoint.

Since $U_s^y(x)$ is connected, to show that $U_s^y(x)$ is simply connected, it suffices to show that $\mathbb{C} \setminus U_s^y(x)$ is connected. Let $\mathcal{V}$ be the set of connected components of the interior of $\mathcal{B}_s^{y,\bullet}(x)$ other than $U_s^y(x)$ (we will eventually show that $\mathcal{V} = \emptyset$, but we do not know this yet). We can write $\mathbb{C} \setminus U_s^y(x)$ as the union of $\overline{\mathbb{C} \setminus \mathcal{B}_s^{y,\bullet}(x)}$ and the sets $\overline{V}$ for $V \in \mathcal{V}$. Each of the sets $\overline{\mathbb{C} \setminus \mathcal{B}_s^{y,\bullet}(x)}$ and $\overline{V}$ is the closure of a connected set, so is connected. Furthermore, each ∂V for $V \subset \mathcal{V}$ is contained in $\partial\mathcal{B}_s^{y,\bullet}(x)$ (by the same argument that we used for $U_s^y(x)$ above), which in turn is contained in $\overline{\mathbb{C} \setminus \mathcal{B}_s^{y,\bullet}(x)}$. Hence $\mathbb{C} \setminus U_s^y(x)$ is the union of connected sets which all intersect a common connected set, so is connected. □

The set $U_s^y(x)$ contains points z with $D_h(x, z) > s$. For such points z, it is possible that a D_h-geodesic from x to z intersects $\partial U_s^y(x)$. Since we will be interested in sets which are disconnected from x in $U_s^y(x)$ (see Proposition 4.3), it is important for us to work with paths which are contained in $U_s^y(x)$. The following lemma will allow us to do so.

Lemma 4.7. *Almost surely, for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$, the following is true. For each $z \in U_s^y(x)$ and each $\delta > 0$, there is a path in $U_s^y(x)$ from x to z with D_h-length at most $D_h(x, z) + \delta$.*

Proof. See Figure 4 for an illustration of the proof. The statement is vacuous if z is a singular point (i.e., $D_h(x, z) = \infty$), so assume that z is not a singular point.

Let P be a D_h-geodesic from x to z. If $t := D_h(x, z) \leq s$, then $P \subset \mathcal{B}_s(x) \subset \mathcal{B}_s^{y,\bullet}(x)$. Furthermore, since $D_h(x, w) = s \geq t$ for each $w \in \partial \mathcal{B}_s^{y,\bullet}(x)$ (Lemma 3.4), $P([0, t))$ is contained in the interior of $\mathcal{B}_s^{y,\bullet}(x)$. Since $P(t) = z \in U_s^y(x)$, we get that P is contained in the interior of $\mathcal{B}_s^{y,\bullet}(x)$. By the definition (4-1) of $U_s^y(x)$, it follows that $P \subset U_s^y(x)$.

Hence we only need to treat the case when $D_h(x, z) > s$. By Lemma 3.4, the path P can hit $\partial \mathcal{B}_s^{y,\bullet}(x)$ at most once, namely at time s. Consequently, P cannot exit and subsequently reenter $\mathcal{B}_s^{y,\bullet}(x)$, so $P \subset \mathcal{B}_s^{y,\bullet}(x)$. Furthermore, $P(t)$ is contained in the interior of $\mathcal{B}_s^{y,\bullet}(x)$ for each $t \neq s$.

If $P(s) \notin \partial \mathcal{B}_s^{y,\bullet}(x)$, then we are done so we can assume without loss of generality that $P(s) \in \partial \mathcal{B}_s^{y,\bullet}(x)$. Since $U_s^y(x)$ is open and connected, hence path connected, we can find a simple path P' in $U_s^y(x)$ from x to z (we make no assumption on the D_h-length of P'). In fact, since $U_s^y(x)$ is homeomorphic to the disk and P is a simple path, we can arrange that P' does not intersect P except at x and z. Let V be the unique bounded complementary connected component of $\mathbb{C} \setminus (P \cup P')$. Then V is a Jordan domain and $P(s) \in \partial V$. Furthermore, $\partial V \subset \mathcal{B}_s^{y,\bullet}(x)$, so each point of V is disconnected from y by $\mathcal{B}_s^{y,\bullet}(x)$. Hence $\overline{V} \subset \mathcal{B}_s^{y,\bullet}(x)$. In fact, $\partial V \setminus \{P(s)\}$ is contained in the interior of $\mathcal{B}_s^{y,\bullet}(x)$, so it follows that $\overline{V} \setminus \{P(s)\}$ is contained in the interior of $\mathcal{B}_s^{y,\bullet}(x)$. Since $\overline{V} \setminus \{P(s)\}$ is connected and contains x it follows that $\overline{V} \setminus \{P(s)\} \subset U_s^y(x)$.

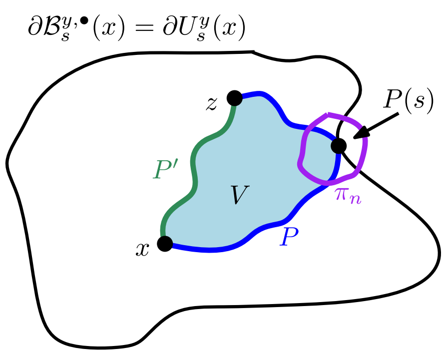

Figure 4. Illustration of the proof of Lemma 4.7. The path P is a D_h-geodesic from x to z. If $P(s) \in \partial U_s^y(x)$, we replace a segment of P by a segment of the small loop π to get a path from x to z which is contained in $U_s^y(x)$ and which is not much longer than P.

By Lemma 3.2, there is a sequence of disjoint D_h-continuous loops $\{\pi_n\}_{n\in\mathbb{N}}$, each of which separates a neighborhood of $P(s)$ from ∞, such that the Euclidean radius of π_n, the D_h-length of π_n, and the D_h-distance from z to π_n each tend to zero as $n \to \infty$. If $n \in \mathbb{N}$ is chosen to be sufficiently large, then π_n is disjoint from P', the D_h-length of π_n is at most δ, and there is a segment η_n of π_n which is a crosscut of V (i.e., it is contained in V except for its endpoints). The segment η_n joins $P(t_1)$ to $P(t_2)$ for some $t_1 < s < t_2$. Let $\widetilde{P}$ be the concatenation of $P|_{[0,t_1]}$, η_n, and $P|_{[t_2,D_h(x,z)]}$. Then $\widetilde{P}$ is a path in $\overline{V} \setminus \{P(s)\}$ from x to z with D_h-length at most $D_h(x,z)+\delta$. By the preceding paragraph, $\widetilde{P}$ is contained in $U_s^y(x)$. □

We will now check the criterion of Proposition 4.3 for the domain $U_s^y(x)$.

Lemma 4.8. *There exists $\theta > 1$ such that a.s. for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x,y))$, there exists a random $\bar{\varepsilon} = \bar{\varepsilon}(x,y,s) > 0$ with the following property. For each $\varepsilon \in (0, \bar{\varepsilon}]$, each set $A \subset U_s^y(x)$ which can be disconnected from x in $U_s^y(x)$ by a set Y of Euclidean diameter at most ε^θ which intersects $\partial U_s^y(x)$ has Euclidean diameter at most ε.*

Proof. See Figure 5 for an illustration of the proof.

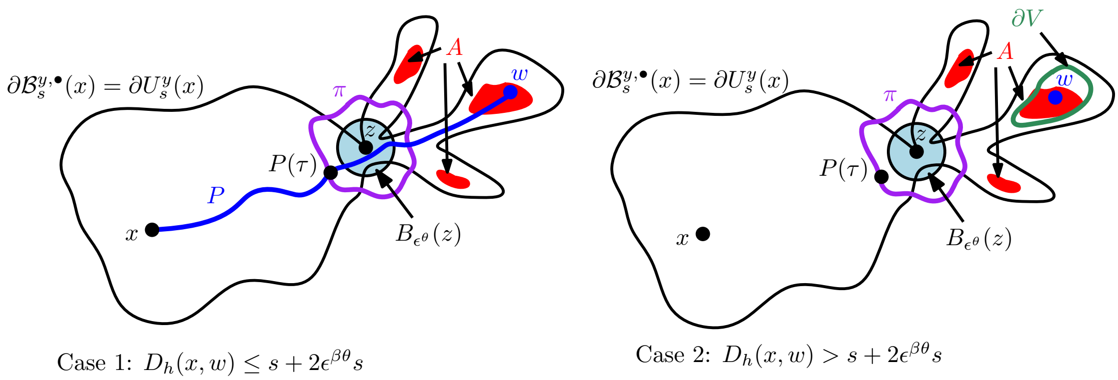

Figure 5. Illustration of the proof of Lemma 4.8. The red set $A \subset U_s^y(x)$ is disconnected from x in $U_s^y(x)$ by the Euclidean ball $\overline{B_{\varepsilon^\theta}(z)}$. Using Corollary 3.7, we produce a path π (purple) disconnecting the inner and outer boundaries of the annulus $\mathbb{A}_{\varepsilon^\theta,\varepsilon^{\theta/2}}(z)$ with D_h-length at most $\varepsilon^{\beta\theta} s$. We seek to bound the Euclidean distance from a point $w \in A$ to $\overline{B_{\varepsilon^\theta}(z)}$. The left panel shows the case when $D_h(x,w) \le s + 2\varepsilon^{\beta\theta} s$. In this case, Lemma 4.7 gives a path P from x to w whose D_h-length is at most $s + 3\varepsilon^{\beta\theta} s$. The path P must hit π, say at a time τ. Our upper bound for the D_h-length of π shows that $D_h(x,w) - \tau \le 4\varepsilon^{\beta\theta} s$. Using a Hölder continuity bound for the Euclidean metric with respect to D_h (Proposition 2.11), we then obtain an upper bound for the Euclidean diameter of $P([\tau, D_h(x,w)])$, which then gives an upper bound for the Euclidean distance from w to $B_{\varepsilon^\theta}(z)$. The right panel shows the case when $D_h(x,w) > s + 2\varepsilon^{\beta\theta} s$. In this case, we consider the complementary connected component V of the ball $\mathcal{B}_{s+2\varepsilon^{\beta\theta} s}(x)$ with $w \in V$. We bound the D_h-distance from x to $\overline{B_{\varepsilon^\theta}(z)}$ in terms of $\sup_{u\in\partial V} \operatorname{dist}(u, \overline{B_{\varepsilon^\theta}(z)})$, then bound this last quantity using the previous case.

Let $\theta > 1$ to be chosen later. We will first state some estimates which hold a.s. for each x, y, s as in the lemma statement and each small enough $\varepsilon > 0$ (depending on x, y, s). We will then truncate on the event that these estimates are satisfied and show that the conclusion of the lemma statement is satisfied.

Almost surely, for each $s > 0$ and each $x \in \mathbb{C}$, the D_h-ball $\mathcal{B}_s(x)$ is Euclidean-bounded, so its Euclidean 1-neighborhood $B_1(\mathcal{B}_s(x))$ is also Euclidean-bounded. We note that this latter set contains every point which lies at Euclidean distance less than 1 from $\partial \mathcal{B}_s^{y,\bullet}(x) = \partial U_s^y(x)$ (see Lemma 4.6) for each $y \in \mathbb{C} \cup \{\infty\}$.

If we let $\{V_n\}_{n\in\mathbb{N}}$ be an increasing sequence of Euclidean-bounded open sets whose union is all of $\mathbb{C}$, then a.s. for each $x \in \mathbb{C}$ and each $s > 0$ we have $B_1(\mathcal{B}_s(x)) \subset V_n$ for large enough $n \in \mathbb{N}$. Furthermore, if $y \in \mathbb{C} \cup \{\infty\}$ and $s \in (0, D_h(x, y))$, then x and y each lie at positive Euclidean distance from $\partial \mathcal{B}_s^{y,\bullet}(x)$ (see Lemma 3.3). We may therefore apply Corollary 3.7 (with $\alpha = 1/2$, ε^θ instead of ε, and $U = V_n$), then send $n \to \infty$, to get that there exists $\beta > 0$ such that a.s. for each x, y, s as in the lemma statement, it holds for small enough $\varepsilon > 0$ that

$$D_h(\text{around } \mathbb{A}_{\varepsilon^\theta, \varepsilon^{\theta/2}}(z)) \le \varepsilon^{\beta\theta} s \quad \text{for all } z \in \partial \mathcal{B}_s^{y,\bullet}(x). \tag{4-2}$$

By Proposition 2.11 (again applied to each of the sets V_n above), if $\chi \in (0, (\xi(Q+2))^{-1})$, then for each $x \in \mathbb{C}$ and each $s > 0$, it is a.s. the case that for each small enough $\varepsilon > 0$,

$$|z - w| \le \varepsilon^{\chi\beta\theta} s^\chi \quad \text{for all } z, w \in B_1(\mathcal{B}_s(x)) \text{ with } D_h(z, w) \le 4\varepsilon^{\beta\theta} s. \tag{4-3}$$

By Lemma 3.3, it is a.s. the case that for each x, y, s as in the lemma statement and each small enough $\varepsilon > 0$,

$$B_{4\varepsilon^{\theta/2}}(x) \subset \mathcal{B}_s^{y,\bullet}(x). \tag{4-4}$$

By the definition (4-1) of $U_s^y(x)$, we see that (4-4) implies that also $B_{4\varepsilon^{\theta/2}}(x) \subset U_s^y(x)$.

We henceforth work on the full-probability event that for each x, y, s as in the lemma statement, (4-2), (4-3), and (4-4) all hold each small enough $\varepsilon > 0$. We will show that the lemma statement holds provided $\theta > \max\{2, 1/(\beta\chi)\}$. To see this, let x, y, s be as in the lemma statement, assume that $\varepsilon > 0$ is small enough that the above three estimates hold. Let $A \subset U_s^y(x)$ be a set which can be disconnected from x in $U_s^y(x)$ by a set Y of Euclidean diameter at most ε^θ which intersects $\partial U_s^y(x)$. We claim that the Euclidean diameter of A is at most ε.

Choose $z \in Y \cap \partial U_s^y(x)$. Then $Y \subset \overline{B_{\varepsilon^\theta}(z)}$ so A is disconnected from x in $U_s^y(x)$ by $\overline{B_{\varepsilon^\theta}(z)}$. We can assume without loss of generality that $A \not\subset \overline{B_{\varepsilon^{\theta/2}}(z)}$ (otherwise, the Euclidean diameter of A is at most $\varepsilon^{\theta/2} < \varepsilon$). Furthermore, we have $z \in \partial U_s^y(x) \subset B_1(\mathcal{B}_s(x))$ and by (4-4), the Euclidean distance from x to $\partial U_s^y(x) = \partial \mathcal{B}_s^{y,\bullet}(x)$ is at least $4\varepsilon^{\theta/2}$, so $x \notin B_{\varepsilon^{\theta/2}}(z)$.

The estimate (4-2) implies that there is a path π in $\mathbb{A}_{\varepsilon^\theta, \varepsilon^{\theta/2}}(z)$ which disconnects the inner and outer boundaries of $\mathbb{A}_{\varepsilon^\theta, \varepsilon^{\theta/2}}(z)$ and satisfies

$$(D_h\text{-length of } \pi) \le \varepsilon^{\beta\theta} s. \tag{4-5}$$

Since $z \in \partial \mathcal{B}_s^{y,\bullet}(x)$, the path π intersects $\partial U_s^y(x) = \partial \mathcal{B}_s^{y,\bullet}(x)$. So,

$$D_h(u, \partial \mathcal{B}_s^{y,\bullet}(x)) \le \varepsilon^{\beta\theta} s \quad \text{for all } u \in \pi. \tag{4-6}$$

We claim that for each $w \in A \setminus \overline{B_{\varepsilon^{\theta/2}}(z)}$,

$$\operatorname{dist}(w, \overline{B_{\varepsilon^{\theta/2}}(z)}) \le \varepsilon/4, \tag{4-7}$$

where dist denotes Euclidean distance. Once (4-7) is established, we will obtain that the Euclidean diameter of A is at most $\varepsilon/2 + 2\varepsilon^{\theta/2} \le \varepsilon$, as desired. To prove (4-7), we treat two cases depending on the value of $D_h(x, w)$.

Case 1: $D_h(x, w) \le s + 2\varepsilon^{\beta\theta} s$. By Lemma 4.7, there is a path P from x to w in $U_s^y(x)$ with D_h-length $T \le D_h(x, w) + \varepsilon^{\beta\theta} s$. We take P to be parametrized by its D_h-length. By our choice of z, P passes through $\overline{B_{\varepsilon^\theta}(z)}$. Since $x, w \notin B_{\varepsilon^{\theta/2}}(z)$, P must hit the path π. Let τ be the first time that P hits π. By (4-6), $D_h(P(\tau), \partial\mathcal{B}_s^{y,\bullet}(x)) \le \varepsilon^{\beta\theta} s$.

Since each point of $\partial\mathcal{B}_s^{y,\bullet}(x)$ lies at D_h-distance s from x (Lemma 3.4), this implies that $\tau \ge s - \varepsilon^{\beta\theta} s$ and hence that

$$T - \tau \le T - s + \varepsilon^{\beta\theta} s \le D_h(x, w) - s + 2\varepsilon^{\beta\theta} s \le 4\varepsilon^{\beta\theta} s.$$

By (4-3), if we let

$$\sigma := s \wedge \inf\{t > \tau : P(t) \notin B_1(\partial\mathcal{B}_s^{y,\bullet}(x))\}$$

then the Euclidean diameter of $P([\tau, \sigma])$ is at most $\varepsilon^{\chi\beta\theta} s^\chi$, which by our choice of θ is at most $\varepsilon/4$ (provided ε is small enough).

Since $P(\tau) \in \pi$ and $\pi \subset B_{\varepsilon^{\theta/2}}(\partial\mathcal{B}_s^{y,\bullet}(x))$, each point of $P([\tau, \sigma])$ lies at Euclidean distance at most $\varepsilon^{\theta/2} + \varepsilon/4 < 1$ from $\partial\mathcal{B}_s^{y,\bullet}(x)$. Therefore, $\sigma = D_h(x, w)$ and $P(\sigma) = w$. Hence w lies at Euclidean distance at most $\varepsilon/4$ from $B_{\varepsilon^{\theta/2}}(z)$, as required.

Case 2: $D_h(x, w) > s + 2\varepsilon^{\beta\theta} s$. Let V be the connected component of $\mathbb{C} \setminus \mathcal{B}_{s+2\varepsilon^{\beta\theta}s}(x)$ which contains w. Then V is contained in $\mathbb{C} \setminus \mathcal{B}_s^{w,\bullet}(x)$, which is the connected component of $\mathbb{C} \setminus \mathcal{B}_s(x)$ which contains w. By Lemmas 3.1 and 3.4, $\partial V = \partial\mathcal{B}^{w,\bullet}_{s+2\varepsilon^{\beta\theta}s}(x)$ lies at positive Euclidean distance from $\partial\mathcal{B}_s^{w,\bullet}(x)$ and hence also from $\mathcal{B}_s(x)$. It follows that V lies at positive Euclidean distance from $\partial\mathcal{B}_s^{y,\bullet}(x)$, so $\overline{V}$ is contained in the interior of $\mathcal{B}_s^{y,\bullet}(x)$. Since $\overline{V}$ is connected and $w \in U_s^y(x)$, we have $\overline{V} \subset U_s^y(x)$.

We claim that $\overline{V}$ is disjoint from $\overline{B_{\varepsilon^\theta}(z)}$. Indeed, if $\overline{V}$ intersects $\overline{B_{\varepsilon^\theta}(z)}$, then since $w \in \overline{V} \setminus \overline{B_{\varepsilon^{\theta/2}}(z)}$ and $\overline{V}$ is connected, it must be the case that $\overline{V}$ intersects the inner and outer boundaries of the annulus $\mathbb{A}_{\varepsilon^\theta, \varepsilon^{\theta/2}}(z)$. Hence $\overline{V}$ intersects π. By (4-6), the D_h-distance from $\overline{V}$ to $\partial\mathcal{B}_s^{y,\bullet}(x)$ is at most $\varepsilon^{\beta\theta} s$, so the D_h-distance from $\overline{V}$ to x is at most $s + \varepsilon^{\beta\theta} s$. But, by Lemma 3.4 (applied to $\partial\mathcal{B}^{w,\bullet}_{s+2\varepsilon^{\beta\theta}s}(x)$), the D_h-distance from $\overline{V}$ to x is equal to $s + 2\varepsilon^{\beta\theta} s$, which is a contradiction.

Since $\overline{V}$ is connected, $w \in \overline{V}$, and w is disconnected from x by $\overline{B_{\varepsilon^\theta}(z)}$ in $U_s^y(x)$, we get that $\overline{V}$ is disconnected from x by $\overline{B_{\varepsilon^\theta}(z)}$ in $U_s^y(x)$.

Let $V_\infty = \mathbb{C} \setminus \mathcal{B}^\bullet_{s+2\varepsilon^{\beta\theta}s}(x)$ be the unbounded connected component of $\mathbb{C} \setminus \mathcal{B}_{s+2\varepsilon^{\beta\theta}s}(x)$. We will now reduce to the case when $V \neq V_\infty$. See Figure 6 for an illustration of this part of the argument. Obviously, if $V_\infty \not\subset U_s^y(x)$, then $V \neq V_\infty$, so we can assume that $V_\infty \subset U_s^y(x)$ (which implies that $U_s^y(x)$ is unbounded). Then $\mathbb{C} \setminus \mathcal{B}^\bullet_{2s}(x) \subset V_\infty \subset U_s^y(x)$. We can choose a path Π in $U_s^y(x)$ from x to a point of $\mathbb{C} \setminus \mathcal{B}^\bullet_{2s}(x)$ in a manner which depends only on $U_s^y(x)$ and $\mathbb{C} \setminus \mathcal{B}^\bullet_{2s}(x)$ (not on ε). Let ε_0 be the Euclidean distance from Π to $\partial U_s^y(x)$. Then ε_0 is a random number depending on x, y, s (not on ε). If $\varepsilon^\theta < \varepsilon_0$ then the path Π

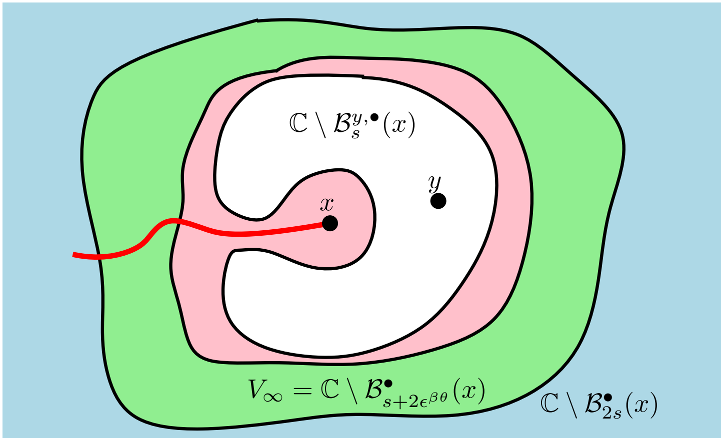

Figure 6. Illustration of how we reduce to the case when $V \neq V_\infty$ in the proof of Lemma 4.8. Here we have shown the case when $U_s^y(x)$ (the union of the pink, green, and blue regions) is unbounded. The set V_∞ is the union of the green and blue regions. We can choose a path Π (red) in $U_s^y(x)$ from x to the blue region $\mathbb{C} \setminus \mathcal{B}^\bullet_{2s}(x)$ in a manner which does not depend on ε. The Euclidean distance from Π to $\partial U_s^y(x)$ is a positive constant $\varepsilon_0 > 0$ which does not depend on ε. Hence, if $\varepsilon^\theta < \varepsilon_0$ then $\mathbb{C} \setminus \mathcal{B}^\bullet_{2s}(x)$ is not disconnected from x in $U_s^y(x)$ by $\overline{B_{\varepsilon^\theta}(z)}$. Since $\mathbb{C} \setminus \mathcal{B}^\bullet_{2s}(x) \subset V_\infty$, the same is true for V_∞. Since V is disconnected from x in $U_s^y(x)$ by $\overline{B_{\varepsilon^\theta}(z)}$, we infer that if $\varepsilon^\theta < \varepsilon_0$, then $V \neq V_\infty$.

cannot intersect $\overline{B_{\varepsilon^\theta}(z)}$. Hence if $\varepsilon^\theta < \varepsilon_0$, then $\mathbb{C} \setminus \mathcal{B}^\bullet_{2s}(x)$ is not disconnected from x in $U_s^y(x)$ by $\overline{B_{\varepsilon^\theta}(z)}$. Since $\mathbb{C} \setminus \mathcal{B}^\bullet_{2s}(x) \subset V_\infty$ and V is disconnected from x in $U_s^y(x)$ by $\overline{B_{\varepsilon^\theta}(z)}$ (as explained above), this implies that so long as $\varepsilon < \varepsilon_0$, we have $V \neq V_\infty$. We henceforth assume that $\varepsilon < \varepsilon_0$, so that $\overline{V}$ is compact.

The Euclidean-furthest point of $\overline{V}$ from $\overline{B_{\varepsilon^\theta}(z)}$ must lie on ∂V, so since $w \in \overline{V}$ we have

$$\operatorname{dist}(w, \overline{B_{\varepsilon^\theta}(z)}) \leq \sup_{u \in \partial V} \operatorname{dist}(u, \overline{B_{\varepsilon^\theta}(z)}).$$

Each point of ∂V lies at D_h-distance $s + 2\varepsilon^{\beta\theta} s$ from x, so we can apply Case 1 with ∂V in place of A to get that $\sup_{u \in \partial V} \operatorname{dist}(u, \overline{B_{\varepsilon^\theta}(z)}) \leq \varepsilon/4$. This yields (4-7). □

We can now apply Proposition 4.3 to get the following.

Lemma 4.9. *Almost surely, for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$ the set $\partial \mathcal{B}_s^{y,\bullet}(x)$ is the image of a (not necessarily simple) curve.*

Proof. By Lemma 4.6, $\partial \mathcal{B}_s^{y,\bullet}(x) = \partial U_s^y(x)$, so it suffices to show that $\partial U_s^y(x)$ is a curve. By Lemma 4.6, it is a.s. the case that for each x, y, s as in the lemma statement, the set $U_s^y(x)$ is simply connected. Furthermore, $\partial U_s^y(x) \subset \partial \mathcal{B}_s(x)$ is Euclidean-compact. By Proposition 4.3, to show that $\partial U_s^y(x)$ is a curve it therefore suffices to show that for each $\varepsilon > 0$, there exists $\delta > 0$ such that each set $A \subset U_s^y(x)$ which can be disconnected from x in $U_s^y(x)$ by a set of Euclidean diameter at most δ which intersects $\partial U_s^y(x)$ has Euclidean diameter at most ε. This follows from Lemma 4.8. □

To prove that $\partial\mathcal{B}_s^{y,\bullet}(x)$ is a *Jordan* curve, we need to prove that it can be represented by a curve with no double points. The following lemma will help us to do that.

Lemma 4.10. *Almost surely, for each nonsingular point $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$, the following is true. Let $\psi : \partial\mathbb{D} \to \partial\mathcal{B}_s^{y,\bullet}(x)$ be a continuous map (which exists by Lemma 4.9). Let $u, v \in \partial\mathbb{D}$ be distinct points such that $\psi(u) = \psi(v)$. Let I and J be the two closed arcs of $\partial\mathbb{D}$ between u and v. Then either $\psi(I) \subset \psi(J)$ or $\psi(J) \subset \psi(I)$.*

Proof. Since $\partial\mathcal{B}_s^{y,\bullet}(x)$ disconnects x from y, the homotopy class of the loop ψ in $(\mathbb{C} \cup \{\infty\}) \setminus \{x, y\}$ is nontrivial. Since $\psi(u) = \psi(v)$, each of $\psi|_I$ and $\psi|_J$ is a loop in $\mathbb{C}$, and $\psi|_{\partial\mathbb{D}}$ is the concatenation of these two loops. The concatenation of two homotopically trivial loops is also homotopically trivial. Therefore, one of $\psi|_I$ or $\psi|_J$ is not homotopic to a point in $(\mathbb{C} \cup \{\infty\}) \setminus \{x, y\}$. This implies that one of $\psi(I)$ or $\psi(J)$ disconnects x from y.

Assume without loss of generality that $\psi(I)$ disconnects x from y. Since $\psi(I), \psi(J) \subset \partial\mathcal{B}_s^{y,\bullet}(x)$, no point of $\psi(J) \setminus \psi(I)$ can be disconnected from y by $\psi(I)$. Hence, $\psi(J) \subset \overline{O}$, where O is the connected component of $\mathbb{C} \setminus \psi(I)$ which contains y. By assumption, $x \notin O$.

If $z \in \psi(J) \setminus \psi(I)$ then $z \in O \cap \partial\mathcal{B}_s^{y,\bullet}(x)$. By Lemma 3.4, we have $D_h(x, z) = s$. If $P : [0, s] \to \mathbb{C}$ is a D_h-geodesic from x to z, then since $z \in O$ there is a time $\tau < s$ such that $P(\tau) \in \psi(I)$. But $\psi(I) \subset \partial\mathcal{B}_s^{y,\bullet}(x)$, so by Lemma 3.4, $D_h(x, P(\tau)) = s$. This contradicts the fact that P is a D_h-geodesic. We conclude that $\psi(J) \subset \psi(I)$. □

Proof of Proposition 4.1. Throughout the proof, we fix a nonsingular point $x \in \mathbb{C}$, a point $y \in \mathbb{C} \cup \{\infty\}$, and $s \in (0, D_h(x, y))$. All statements are required to hold a.s. for all choices of x, y, s simultaneously.

Let $U_s^y(x)$ be as in (4-1) and let $\phi = \phi_s^y : \mathbb{D} \to U_s^y(x)$ be a conformal map (such a map exists since $U_s^y(x)$ is simply connected, see Lemma 4.6). Since $\partial U_s^y(x) = \partial\mathcal{B}_s^{y,\bullet}(x)$ (Lemma 4.6), it follows from Lemma 4.9 and [36, Theorem 2.1] (or just Lemma 4.5) that the map ϕ extends to a continuous map $\overline{\mathbb{D}} \to \overline{U}_s^y(x)$. We henceforth assume that ϕ has been replaced by such a continuous extension. We will show that ϕ, thus extended, is a homeomorphism.

We say that $z \in \partial\mathcal{B}_s^{y,\bullet}(x) = \partial U_s^y(x)$ is a *cut point* if $\partial U_s^y(x) \setminus \{z\}$ is not connected. By [36, Theorem 2.6], it suffices to show that $\partial U_s^y(x)$ has no cut points.

Assume by way of contradiction that $z \in \partial U_s^y(x)$ is a cut point. By [36, Proposition 2.5], $\#\phi^{-1}(z) \geq 2$ (in principle $\#\phi^{-1}(z)$ could be infinite, even uncountable). Furthermore, if $\mathcal{I}$ is the set of connected components of $\partial\mathbb{D} \setminus \phi^{-1}(z)$, then the set of connected components of $\partial U_s^y(x) \setminus \{z\}$ is $\{\phi(I) : I \in \mathcal{I}\}$.

Each $I \in \mathcal{I}$ is an open arc of $\partial\mathbb{D}$ whose endpoints are distinct points of $\phi^{-1}(z)$. Let $J = J(I) := \partial\mathbb{D} \setminus I$. By Lemma 4.10, either $\phi(\bar{I}) \subset \phi(J)$ or $\phi(J) \subset \phi(\bar{I})$. By the preceding paragraph, $\phi(\bar{I})$ is the union of $\{z\}$ and a connected component of $\partial U_s^y(x) \setminus \{z\}$; and $\phi(J)$ is the union of $\{z\}$ and the other connected components of $\partial U_s^y(x) \setminus \{z\}$. Therefore, $\phi(\bar{I}) \cap \phi(J) = \{z\}$. Hence one of $\phi(\bar{I})$ or $\phi(J)$ is equal to $\{z\}$. This means that $\partial U_s^y(x) \setminus \{z\}$ has only one connected component, so z was not a cut point after all. □

Proof of Theorem 1.4. Proposition 4.1 implies that a.s. $\partial\mathcal{B}_s^{y,\bullet}(x)$ is a Jordan curve for each nonsingular $x \in \mathbb{C}$, each $y \in \mathbb{C} \cup \{\infty\}$, and each $s \in (0, D_h(x, y))$. Theorem 1.5 implies that a.s. each of these filled metric ball boundaries has finite Hausdorff dimension with respect to D_h. Proposition 3.12 implies that a.s.

each of these filled metric ball boundaries is precompact with respect to D_h. Since each such boundary is Euclidean-closed, it is also D_h-closed, and hence D_h-compact. □

5. Confluence of geodesics

In this section we will explain how to adapt the proof of confluence of geodesics for subcritical LQG from [18] to the supercritical case. This will lead to proofs of Theorems 1.6 and 1.7. Many of the arguments of [18] carry over verbatim to the supercritical case, but in some places nontrivial modifications to the arguments, using results from Sections 3 and 4 of the present paper, are needed. As such, we will not repeat the full argument from [18]. Instead, we will only explain the parts of the argument which require modification. We aim to strike a balance between minimizing repetition of arguments from the subcritical case and making the paper readable without the reader having to frequently refer to [18].

The proof of confluence of geodesics for subcritical LQG has four steps.

1. Establish some preliminary facts about geodesics, such as uniqueness of geodesics between typical points and certain monotonicity properties for the cyclic ordering of geodesics from 0 to points of the boundary of the filled metric ball $\mathcal{B}_s^\bullet$ [18, Section 2.1].
2. Suppose we condition on $(\mathcal{B}_s^\bullet, h|_{\mathcal{B}_s^\bullet})$ and $I \subset \partial\mathcal{B}_s^\bullet$ is an arc chosen in a way which depends only on $(\mathcal{B}_s^\bullet, h|_{\mathcal{B}_s^\bullet})$. Show that if I can be disconnected from ∞ in $\mathbb{C} \setminus \mathcal{B}_s^\bullet$ by a set of small Euclidean diameter, then it holds with high conditional probability that there is a "shield" in $\mathbb{C} \setminus \mathcal{B}_s^\bullet$ which disconnects I from ∞ with the property that no D_h-geodesic started from 0 can cross this shield [18, Sections 3.2 and 3.3].
3. Start with a positive radius t and a collection of boundary arcs $\mathcal{I}_0$ of $\partial\mathcal{B}_t^\bullet$. Iteratively apply Step 2 for several successive radii $s_k > t$ to iteratively "kill off" all of the geodesics started from 0 which pass through $I \in \mathcal{I}_0$. Repeat until the number of remaining arcs in $\mathcal{I}_0$ which have not yet been killed off is at most a large deterministic constant (independent of the initial choice of $\mathcal{I}_0$). By sending the size of the arcs in $\mathcal{I}_0$ to zero (and the number of such arcs to ∞), conclude that for each fixed $s > t$, there are a.s. only finitely many points on $\partial\mathcal{B}_t^\bullet$ which are hit by D_h-geodesics from 0 to $\partial\mathcal{B}_s^\bullet$ [18, Section 3.4]. This yields Theorem 1.7.
4. Reduce from finitely many points on $\partial\mathcal{B}_t^\bullet$ to a single point by "killing off" the points one at a time [18, Section 4]. This yields Theorem 1.6.

See [18, Section 3.1] for a detailed overview of Steps 2 and 3.

Most of the arguments involved in Step 1 carry over verbatim to the supercritical case once we know that the boundary of a filled metric ball is a Jordan curve (Proposition 4.1). So, we will not repeat many of these arguments here. Rather, we will just state a few of the most important results; see Section 5.1.

Step 2 requires nontrivial modifications in the supercritical case. This is because the definition of the event used to build the "shield" in the subcritical case involves a bound for the LQG diameters of certain small squares, which are infinite in the supercritical case. So, we need to work with somewhat different events in the supercritical case. Because of this, we will give most of the details for Step 2 in this paper. This is done in Sections 5.2.1 and 5.2.2.

Step 3 requires only very minor modifications as compared to the subcritical case. In particular, in the subcritical case, the Hölder continuity of the LQG metric with respect to the Euclidean metric is used in one place. In our setting, we can replace this use of Hölder continuity by using Lemma 3.10, and then the argument goes through verbatim. As such, we will not give much detail about this step; see Section 5.2.3.

As in the case of Step 2, Step 4 requires nontrivial modifications in the supercritical case. Again, this is because the event used to "kill off" all but one of the geodesics in the subcritical case involves bounds for LQG diameters. We will provide most of the details for the parts of Step 4 which require modification; see Section 5.3.

5.1. *Preliminary results about LQG metric balls and geodesics.* We know that D_h-geodesics and outer boundaries of filled D_h-metric balls are simple, Euclidean-continuous curves (Lemma 2.6 and Proposition 4.1). Furthermore, we know that $D_h(0, z) = s$ for each $s > 0$ and each $z \in \partial \mathcal{B}_s^\bullet$ (Lemma 3.4). With these facts in hand, most of the results in [18, Section 2.1] and their proofs carry over verbatim to the supercritical case.

We first state a result to the effect that ordinary and filled LQG metric balls are local sets for h as defined in [38, Lemma 3.9]. Let us recall the definition. Suppose (h, A) is a coupling of h with a random set A. We say that a closed set $A \subset \mathbb{C}$ is a *local set* for h if for any open set $U \subset \mathbb{C}$, the event $\{A \cap U \neq \emptyset\}$ is conditionally independent from $h|_{\mathbb{C}\setminus U}$ given $h|_U$. If A is determined by h (which will be the case for all of the local sets we consider), this is equivalent to the statement that A is determined by $h|_U$ on the event $\{A \subset U\}$. For a local set A, we can condition on the pair $(A, h|_A)$: this is by definition the same as conditioning on the σ-algebra $\bigcap_{\varepsilon>0} \sigma(A, h|_{B_\varepsilon(A)})$. The conditional law of $h|_{\mathbb{C}\setminus A}$ given $(A, h|_A)$ is that of a zero-boundary GFF on $\mathbb{C} \setminus A$ plus a harmonic function on $\mathbb{C} \setminus A$ which is determined by $(A, h|_A)$.

Lemma 5.1. *Let $x \in \mathbb{C}$ and $y \in \mathbb{C} \cup \{\infty\}$ be deterministic. If τ is a stopping time for the filtration generated by $(\mathcal{B}_s(x), h|_{\mathcal{B}_s(x)})$, then $\mathcal{B}_\tau(x)$ is a local set for h. The same is true with $\mathcal{B}_s^{y,\bullet}(x)$ in place of $\mathcal{B}_s(x)$.*

Proof. Note that $\mathcal{B}_s(x)$ and $\mathcal{B}_s^{y,\bullet}(x)$ are Euclidean-closed (Lemma 3.1). In light of this, the lemma follows from exactly the same proof as [18, Lemma 2.1] (see also [19, Lemma 2.2]). □

Our next result gives the uniqueness of D_h-geodesics between typical points.

Lemma 5.2. *For each fixed $z, w \in \mathbb{C}$, a.s. there is a unique D_h-geodesic from z to w.*

Proof. We know that a.s. $D_h(z, w) < \infty$ and there is at least one D_h-geodesic from z to w (Lemma 2.6). The a.s. uniqueness of this geodesic follows from exactly the same argument as in the subcritical case; see [28, Theorem 1.2]. □

We emphasize that Lemma 5.2 only holds a.s. for a *fixed* choice of z and w. We expect that there are exceptional pairs of points z, w which are joined by multiple distinct D_h-geodesics (such points are known to exist in the subcritical case, see [1; 16; 29]). We also record the following analog of [18, Lemma 2.3].

Lemma 5.3. *For $q \in \mathbb{Q}^2$, let P_q be the a.s. unique D_h-geodesic from 0 to q. The following holds a.s. If $q \in \mathbb{Q}^2$, P' is a D_h-geodesic started from 0, and $u \in P_q \cap P'$, then there is a time $s \geq 0$ such that $P_q(s) = P'(s) = u$ and $P_q(t) = P'(t)$ for each $t \in [0, s]$.*

Proof. Lemma 5.2 implies that a.s. the D_h-geodesic from 0 to q is unique for each $q \in \mathbb{Q}^2$. The lemma now follows from exactly the same argument as in [18, Lemma 2.3]. □

The following result, which is the supercritical analog of [18, Lemma 2.4], tells us that for $z \in \partial \mathcal{B}_s^\bullet$, there are two distinguished D_h-geodesics from 0 to z.

Lemma 5.4. *Almost surely, for each $s > 0$ and each $z \in \partial \mathcal{B}_s^\bullet$, there exists a (necessarily unique)* ***leftmost (resp. rightmost) geodesic*** *P_z^- (resp. P_z^+) from 0 to z such that each D_h-geodesic from 0 to z lies (weakly) to the right (resp. left) of P_z^- (resp. P_z^+) if we stand at z and look outward from $\mathcal{B}_s^\bullet$. Moreover, there are sequences of points $q_n^-, q_n^+ \in \mathbb{Q}^2 \setminus \mathcal{B}_s^\bullet$ such that the D_h-geodesics from 0 to $q_n^\pm$ satisfy $P_{q_n^\pm} \to P_z^\pm$ uniformly with respect to the Euclidean topology.*

See Figure 7 for an illustration of the statement and proof of Lemma 5.4. The proof of [18, Lemma 2.4] uses the Arzelà–Ascoli theorem and the continuity of the subcritical LQG metric with respect to the Euclidean metric to take limits of D_h-geodesics. In order to do this in the supercritical case, we need the following lemma.

Lemma 5.5. *Almost surely, the following is true. Let $\{z_n\}_{n\in\mathbb{N}}$, $\{w_n\}_{n\in\mathbb{N}}$, z, and w be nonsingular points for D_h such that $z_n \to z$, $w_n \to w$, and $\limsup_{n\to\infty} D_h(z_n, w_n) < \infty$. Let $\{P_n\}_{n\in\mathbb{N}}$ be a sequence of D_h-rectifiable paths from z_n to w_n, each parametrized by D_h-length, such that $\operatorname{len}(P_n; D_h) - D_h(z_n, w_n) \to 0$ as $n \to \infty$, where $\operatorname{len}(P_n; D_h)$ denotes the D_h-length. There is a subsequence along which the paths P_n converge uniformly with respect to the Euclidean metric to a D_h-rectifiable path P from z to w. If $\lim_{n\to\infty} D_h(z_n, w_n) = D_h(z, w)$, then P is a D_h-geodesic.*

The statement of Lemma 5.5 allows for uniform convergence of paths which are defined on $[0, T_n]$ where T_n possibly depends on n. To make sense of uniform convergence under these circumstances, we view all of our paths as being defined on $[0, \infty)$ by extending them to be constant after time T_n.

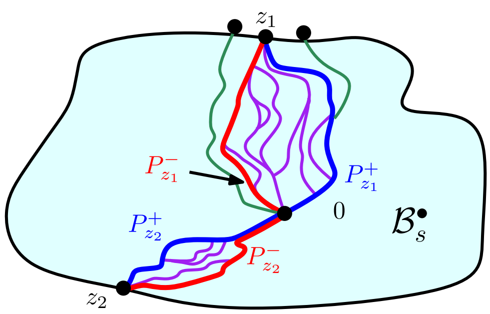

Figure 7. Two points $z_1, z_2 \in \partial\mathcal{B}_s^\bullet$ and their associated leftmost and rightmost D_h-geodesics (red and blue). Other D_h-geodesics from 0 to z_1 and z_2 are shown in purple. We have also shown two D_h-geodesics from 0 to points of $\mathbb{Q}^2 \setminus \mathcal{B}_s^\bullet$ (green) which approximate $P_{z_1}^-$ and $P_{z_1}^+$, respectively. Note that $P_{z_1}^-$ and $P_{z_1}^+$ intersect only at their endpoints, whereas $P_{z_2}^-$ and $P_{z_2}^+$ coincide for an initial time interval. Theorem 1.6 implies that a.s. the latter situation holds simultaneously for every $z \in \partial\mathcal{B}_s^\bullet$, but this has not been established yet. A similar figure and caption appeared in [18].

Proof of Lemma 5.5. Let $T_n := \mathrm{len}(P_n; D_h)$, so that $P_n : [0, T_n] \to \mathbb{C}$. Since P_n is parametrized by D_h-length, for $0 \le t \le s \le T_n$, we have that $D_h(P_n(s), P_n(t)) \le s - t$ and that $D_h(z_n, w_n) \le (T_n - s) + D_h(P_n(s), P_n(t)) + t$. Therefore,

$$T_n - D_h(z_n, w_n) \ge s - t - D_h(P_n(s), P_n(t)) \ge 0.$$

Since $T_n - D_h(z_n, w_n) \to 0$ as $n \to \infty$ by hypothesis,

$$\lim_{n\to\infty} \sup_{0\le t\le s\le T_n} |s - t - D_h(P_n(s), P_n(t))| = 0. \tag{5-1}$$

In particular, (5-1) implies that the P_n's are D_h-equicontinuous.

Since $\limsup_{n\to\infty} D_h(z_n, w_n) < \infty$ and D_h-metric balls are Euclidean-bounded, there is a bounded open subset of $\mathbb{C}$ which contains P_n for each $n \in \mathbb{N}$. Since the identity mapping $(\mathbb{C}, D_h) \to (\mathbb{C}, |\cdot|)$ is continuous and the P_n's are D_h-equicontinuous, it follows that the P_n's are Euclidean equicontinuous. Hence there is a sequence $\mathcal{N}$ of positive integers tending to ∞ and a Euclidean-continuous path $P : [0, T] \to \mathbb{C}$ from z to w such that $P_n \to P$ uniformly with respect to the Euclidean topology along $\mathcal{N}$.

Since D_h is lower semicontinuous with respect to the Euclidean metric, equation (5-1) implies that $D_h(P(s), P(t)) \le |s - t|$ for any two times $s, t \in [0, T]$. Consequently, P is D_h-rectifiable and for any $0 \le t \le s \le T$, the D_h-length of $P([t, s])$ is at most $s - t$. If $\lim_{n\to\infty} D_h(z_n, w_n) = D_h(z, w)$, then $T = D_h(z, w)$. Since the D_h-length of P is at most T, it follows that the D_h-length of P is exactly T and P is a D_h-geodesic. □

Proof of Lemma 5.4. The proof is essentially the same as [18, Lemma 2.4], but there are a couple of minor differences so we will give the details. Fix a point $w \in \partial\mathcal{B}_s^\bullet \setminus \{z\}$. Let A^- and A^+, respectively, be the clockwise and counterclockwise arcs of $\partial\mathcal{B}_s^\bullet$ from w to z, not including w and z themselves. Note that these arcs are well-defined since $\partial\mathcal{B}_s^\bullet$ is a Jordan curve (Proposition 4.1). We can choose sequences $z_n^- \in A^-$ (resp. $z_n^+ \in A^+$) which converge to z from the left (resp. right) with respect to the Euclidean topology (with "left" and "right" defined as in the lemma statement).

The set $\mathbb{Q}^2$ is a.s. D_h-dense in $\mathbb{C} \setminus \{\text{singular points}\}$ [34, Proposition 1.13] and the set $\mathcal{B}_\varepsilon(z_n^\pm) \setminus \mathcal{B}_s^\bullet$ contains a D_h-open set for each $\varepsilon > 0$. Applying this with ε equal to the minimum of $\frac{1}{n}$ and $\frac{1}{4} D_h(z_n^\pm, A^\mp)$, we see that for $n \in \mathbb{N}$ we can find $q_n^\pm \in \mathbb{Q}^2 \setminus \mathcal{B}_s^\bullet$ such that

$$D_h(q_n^\pm, z_n^\pm) \le \min\left\{\tfrac{1}{n}, \tfrac{1}{2} D_h(q_n^\pm, A^\mp)\right\}. \tag{5-2}$$

Let $P_{q_n^\pm}$ be the (a.s. unique, by Lemma 5.2) D_h-geodesic from 0 to $q_n^\pm$. Then $P_{q_n^\pm}(s) \in \partial\mathcal{B}_s^\bullet$. Since $D_h(0, z_n^\pm) = s$ (Lemma 3.4),

$$D_h(P_{q_n^\pm}(s), q_n^\pm) = D_h(0, q_n^\pm) - s \le D_h(q_n^\pm, z_n^\pm).$$

From this and (5-2),

$$D_h(P_{q_n^\pm}(s), A^\mp) \ge D_h(q_n^\pm, A^\mp) - D_h(P_{q_n^\pm}(s), q_n^\pm) \ge \tfrac{1}{2} D_h(q_n^\pm, A^\mp) > 0.$$

Hence $P_{q_n^\pm}(s) \notin A^\mp$. From (5-2) and since $z_n^- \to z$ from the left, we see that also $P_{q_n^-}(s) \to z$ from the left. The same is true for z_n^+, but with "right" in place of "left".

Since $D_h(0, z_n^\pm) = s$, we have $0 \le D_h(0, q_n^\pm) - s \le \frac{1}{n}$. We may therefore apply Lemma 5.5 to get that after possibly passing to a subsequence, we can arrange that the paths $P_{q_n^\pm}$ converge uniformly with respect to the Euclidean metric to D_h-geodesics $P_z^\pm$ from 0 to z. By Lemma 5.3, no D_h-geodesic from 0 to z can cross any of the geodesics $P_{q_n^\pm}$. If a geodesic from 0 to z does not lie in the closure of the open subset of $\mathcal{B}_s^\bullet$ lying to the right of P_z^- and to the left of P_z^+, then it must cross $P_{q_n^-}$ or $P_{q_n^+}$ for some n. Hence each geodesic from 0 to z lies to the right of P_z^- and to the left of P_z^+. □

Our next lemma is used in the iterative argument used to prove confluence of geodesics (see Step 3 of the outline at the beginning of this section).

Lemma 5.6. *Almost surely, the following is true for each $0 < s < s' < \infty$. Let $\mathcal{I}$ be a finite collection of disjoint arcs of $\partial\mathcal{B}_s^\bullet$. For each $I \in \mathcal{I}$, let I' be the set of $z \in \partial\mathcal{B}_{s'}^\bullet$ such that the leftmost D_h-geodesic from 0 to z passes through I. Then each I' is either empty or is a connected arc of $\partial\mathcal{B}_{s'}^\bullet$ and the arcs I' for different choices of $I \in \mathcal{I}$ are disjoint.*

Proof. Since we know that each $\partial\mathcal{B}_s^\bullet$ is a Jordan curve and $D_h(0, z) = s$ for each $z \in \partial\mathcal{B}_s^\bullet$, the proofs of [18, Lemmas 2.6 and 2.7] extend verbatim to the supercritical case (note that [18, Lemma 2.5] is a deterministic statement which can be reused in the supercritical case). In particular, [18, Lemma 2.7] gives precisely the statement of the present lemma. □

Finally, we record an FKG inequality for the LQG metric, which is proven in exactly the same way as [18, Proposition 2.8]. For the statement, we note that if D is a weak LQG metric with parameter ξ as in Definition 2.3, $U \subset \mathbb{C}$ is open, and $\mathring{h}$ is a zero-boundary GFF on U, then we can define $D_{\mathring{h}}$ as a random lower semicontinuous metric on U follows. Let h be a whole-plane GFF. We can write $h|_U = \mathring{h} + \mathfrak{h}$, where $\mathfrak{h}$ is a random harmonic function on U (see, e.g., [21, Lemma 2.2]). We then define $D_{\mathring{h}} = e^{-\xi\mathfrak{h}} \cdot D_{\mathring{h}}$, using the notation (2-3). As explained in [18, Remark 1.2], it is easily seen that $D_{\mathring{h}}$ is a measurable function $\mathring{h}$.

Proposition 5.7 (FKG for the LQG metric). *Let $\xi > 0$, let $U \subset \mathbb{C}$ be an open domain, let $\mathring{h}$ be a zero-boundary GFF on U, and let D be a weak LQG metric with parameter ξ. Let Φ and Ψ be bounded, real-valued measurable functions on the space of lower semicontinuous metrics on U which are nondecreasing in the sense that for any two such metrics D_1, D_2 with $D_1(z, w) \le D_2(z, w)$ for all $z, w \in U$, one has $\Phi(D_1) \le \Phi(D_2)$ and $\Psi(D_1) \le \Psi(D_2)$. Suppose further that Φ and Ψ are a.s. continuous at $D_{\mathring{h}}$ in the sense that for every (possibly random) sequence of continuous functions $\{f^n\}_{n\in\mathbb{N}}$ which converges to zero uniformly on U, one has $\Phi(e^{\xi f^n} \cdot D_{\mathring{h}}) \to \Phi(D_{\mathring{h}})$ and $\Psi(e^{\xi f^n} \cdot D_{\mathring{h}}) \to \Psi(D_{\mathring{h}})$. Then* $\mathrm{Cov}(\Phi(D_{\mathring{h}}), \Psi(D_{\mathring{h}})) \ge 0$.

Proof. This follows from Weyl scaling (Axiom III) together with the FKG inequality for the GFF given in [18, Lemma 2.10], via exactly the same argument as in the proof of [18, Proposition 2.8]. □

5.2. *Finitely many leftmost geodesics across an LQG annulus.* In this subsection we explain how to extend the core part of the argument in [18], corresponding to Steps 2 and 3 above, to the supercritical case. We start in Section 5.2.1 by defining an event for a Euclidean annulus which will be used to build "shields" which D_h-geodesics cannot cross. Then, in Section 5.2.2, we explain how to use this event to "kill off" all of the geodesics which pass through a given boundary arc of a filled D_h-metric ball. We will give most of the details of the arguments in these two subsections since nontrivial modifications are required as compared to the analogous arguments in [18]. In Section 5.2.3, we state a more quantitative version of Theorem 1.7 (Theorem 5.15) and explain why the theorem follows from the same proof as its subcritical analog from [18, Section 3.4], except for one trivial modification.

5.2.1. *Good annuli.* We now define an event for a Euclidean annulus which will eventually be used to build "shields" surrounding boundary arcs of a filled D_h-metric ball through which D_h-geodesics to 0 cannot pass. See Figure 8 for an illustration.

For $\varepsilon > 0$, $z \in \mathbb{C}$, and a set $V \subset \mathbb{C}$, we define the collection of Euclidean squares

$$\mathcal{S}_\varepsilon^z(V) := \big\{[x, x+\varepsilon] \times [y, y+\varepsilon] : (x, y) \in \varepsilon\mathbb{Z}^2 + z,\ ([x, x+\varepsilon] \times [y, y+\varepsilon]) \cap V \neq \varnothing\big\}. \tag{5-3}$$

Note that $\mathcal{S}_\varepsilon^z(V)$ depends only on the value of z modulo $\varepsilon\mathbb{Z}^2$ and that $\mathcal{S}_\varepsilon^z(V) - z = \mathcal{S}_\varepsilon^0(V - z)$.

For $z \in \mathbb{C}$, $r > 0$, and $\delta \in (0, 1)$, we define $\mathcal{U}_r(z) = \mathcal{U}_r(z; \delta)$ to be the (finite) set of open subsets U of the annulus $\mathbb{A}_{3r,4r}(z)$ such that $\mathbb{A}_{3r,4r}(z) \setminus U$ is a finite union of sets of the form $S \cap \mathbb{A}_{3r,4r}(z)$ for squares $S \in \mathcal{S}_{\delta r}^z(\mathbb{A}_{3r,4r}(z))$. For $U \in \mathcal{U}_r(z; \delta)$ and $\varepsilon > 0$, we define

$$U_\varepsilon := \{u \in U : \operatorname{dist}(z, \partial U) > \varepsilon\} \tag{5-4}$$

where dist denotes Euclidean distance.

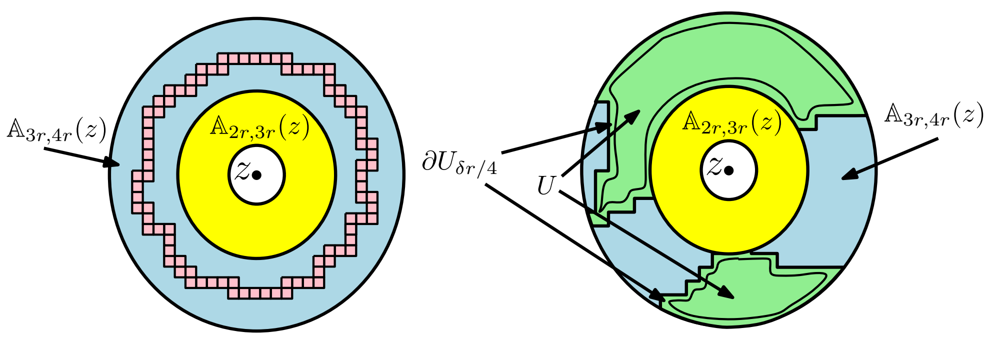

Figure 8. Illustration of the definitions in Section 5.2.1. The set $\mathcal{U}_r(z) = \mathcal{U}_r(z; \delta)$ consists of open subsets U of $\mathbb{A}_{3r,4r}(z)$ such that $\mathbb{A}_{3r,4r}(z) \setminus U$ is a finite union of sets of the form $S \cap \mathbb{A}_{3r,4r}(z)$ for $\delta r \times \delta r$ squares $S \in \mathcal{S}_{\delta r}(\mathbb{A}_{3r,4r}(z))$ (i.e., with corners in $\delta\mathbb{Z}^2$). One such set is shown in light green in the right panel. For each $U \in \mathcal{U}_r(z)$, $E_r^U(z)$ is the event that (1) the D_h-distance across the yellow annulus $\mathbb{A}_{2r,3r}(z)$ is bounded below, (2) there is a path of squares in $\mathbb{A}_{3r,4r}(z)$ which disconnects the inner and outer boundaries of this annulus, with the property that the D_h-distance around $B_{2\delta r}(S) \setminus B_{\delta r}(S)$ is small for each square S in the path (the squares are shown in pink in the left panel), and (3) the harmonic part of $h|_U$ is bounded above on the set $U_{\delta r/4} \subset U$ (outlined in black in right panel).

For $z \in \mathbb{C}$, $r > 0$, parameters $c, \delta \in (0,1)$ and $A > 0$, and $U \in \mathcal{U}_r(z;\delta)$, we let $E_r^U(z) = E_r^U(z; c, \delta, A)$ be the event that the following is true.

1. $D_h(\text{across } \mathbb{A}_{2r,3r}(z)) \geq c\,\mathfrak{c}_r e^{\xi h_r(z)}$.
2. There exists a collection of $\delta r \times \delta r$ squares $S_1, \dots, S_N \in \mathcal{S}_{\delta r}(\mathbb{A}_{3.1r,3.9r}(z))$ with the following properties.
   (a) The squares S_{j-1} and S_j share a side for each $j = 1, \dots, N$, where here we set $S_0 = S_N$.
   (b) The union of the squares $S_1, \dots, S_N$ contains a path which disconnects the inner and outer boundaries of $\mathbb{A}_{3.1r,3.9r}(z)$.
   (c) For each $j = 1, \dots, N$, we have $D_h(\text{around } B_{2\delta r}(S_j) \setminus B_{\delta r}(S_j)) \leq \frac{1}{100} c\,\mathfrak{c}_r e^{\xi h_r(z)}$.
3. Let $\mathfrak{h}^U$ be the harmonic part of $h|_U$. Then, in the notation (5-4),

$$\sup_{u \in U_{\delta r/4}} |\mathfrak{h}^U(u) - h_r(z)| \leq A. \tag{5-5}$$

We also define

$$E_r(z) = E_r(z; c, \delta, A) := \bigcap_{U \in \mathcal{U}_r(z;\delta)} E_r^U(z). \tag{5-6}$$

The first two conditions in the definition of $E_r^U(z)$ do not depend on U, so the only difference between $E_r(z)$ and $E_r^U(z)$ is that for the former event, Condition 3 is required to hold for all choices of U simultaneously.

The events $E_r(z)$ and $E_r^U(z)$ are defined in exactly the same manner as in [18, Section 3.2] except that in [18], Condition 2 is replaced by an upper bound for the D_h-diameters of the squares in $\mathcal{S}_{\delta r}(\mathbb{A}_{3r,4r}(z))$. Of course, such a diameter upper bound does not hold in the supercritical case, which is the reason for the modification.

The occurrence of $E_r^U(z)$ or $E_r(z)$ is unaffected by adding a constant to the field. By this and the locality of D_h (Axiom II), these events are determined by $h|_{\mathbb{A}_{2r,5r}(z)}$, viewed modulo additive constant.

We think of annuli $\mathbb{A}_{2r,5r}(z)$ for which $E_r(z)$ occurs as "good". We will show in Lemma 5.8 just below that $\mathbb{P}[E_r(z)]$ can be made close to 1 by choosing the parameters δ, c, A appropriately, in a manner which is uniform over the choices of r and z, The reason for separating $E_r(z)$ and $E_r^U(z)$ is that conditioning on $E_r^U(z)$ is easier than conditioning on $E_r(z)$ (see Lemma 5.10 just below).

We will eventually apply Condition 3 with U equal to $\mathbb{A}_{3r,4r}(z)$ minus the union of the set of squares in $\mathcal{S}_\varepsilon^z(\mathbb{A}_{3r,4r}(z))$ which intersect a filled D_h-metric ball $\mathcal{B}_\tau^\bullet$, for an appropriate stopping time τ. Condition 3 together with the Markov property of h allows us to show that with uniformly positive conditional probability given $h|_{\mathbb{C}\setminus U}$ and the event $E_r^U(z)$, the maximal D_h-distance between the centers of any two squares in $\mathcal{S}_{\delta r}^z(U)$ which are contained in the same connected component of U is small (see Lemma 5.10). This combined with Condition 2 will show that with uniformly positive conditional probability given $h|_{\mathbb{C}\setminus U}$ and $E_r^U(z)$, there is a collection of paths in $\mathbb{A}_{3r,4r}(z)$ which each have small D_h-length and whose union disconnects the inner and outer boundaries of $\mathbb{A}_{3r,4r}(z)$ in $\mathbb{C} \setminus \mathcal{B}_\tau^\bullet$ (see Lemma 5.13). Due to Condition 1, we can arrange that D_h-distance from each point of each of these paths to $\partial \mathcal{B}_\tau^\bullet$ will be smaller than $D_h(\text{across } \mathbb{A}_{2r,3r}(z))$. This will show that no D_h-geodesic from a point outside of $B_{4r}(z) \cup \mathcal{B}_\tau^\bullet$ can cross $\mathbb{A}_{2r,3r}(z)$ before entering $\mathcal{B}_\tau^\bullet$. See Figure 9 for an illustration of how the events $E_r^U(z)$ will eventually be used.

Lemma 5.8. *For each $p \in (0,1)$, we can find parameters $c, \delta \in (0,1)$ and $A > 0$ such that, in the notation* (5-6), *we have $\mathbb{P}[E_r(z)] \geq p$ for each $z \in \mathbb{C}$ and $r > 0$.*

In order to show that Condition 2 in the definition of $E_r^U(z)$ occurs with high probability, we will use the following lemma.

Lemma 5.9. *Fix $\zeta > 0$ and $0 < a < b < \infty$. For each $z \in \mathbb{C}$ and $r > 0$, it holds with superpolynomially high probability as $\delta \to 0$, uniformly over the choice of z and r, that there exists a collection of $\delta r \times \delta r$ squares $S_1, \dots, S_N \in \mathcal{S}_{\delta r}(\mathbb{A}_{ar,br}(z))$ with the following properties.*

1. *The squares S_{j-1} and S_j share a side for each $j = 1, \dots, N$, where here we set $S_0 = S_N$.*
2. *The union of the squares $S_1, \dots, S_N$ contains a path which disconnects the inner and outer boundaries of $\mathbb{A}_{ar,br}(z)$.*
3. *For each $j = 1, \dots, N$, we have $D_h(\text{around } B_{2\delta r}(S_j) \setminus B_{\delta r}(S_j)) \leq \delta^{\xi Q - \zeta} \mathfrak{c}_r e^{\xi h_r(z)}$.*

Proof. This can be proven using level sets of the GFF (see, e.g., the arguments in [10, Section 2] or [17, Section 5.1]), but we will give a different argument based on estimates for weak LQG metrics with parameter $\tilde{\xi}$, where $\tilde{\xi}$ is large.

Let $\tilde{\xi} > \xi$ to be chosen later, in a manner depending on ζ. Let $\widetilde{D}_h$ be a weak $\tilde{\xi}$-LQG metric with respect to h (e.g., a subsequential limit of LFPP with parameter $\tilde{\xi}$). We denote objects associated with $\tilde{\xi}$ and $\widetilde{D}_h$ with a tilde.

By Proposition 2.9 and a union bound, it holds with superpolynomially high probability as $\delta \to 0$, uniformly over the choices of z and r, that for each $S \in \mathcal{S}_{\delta r}(\mathbb{A}_{ar,br}(z))$,

$$D_h(\text{around } B_{2\delta r}(S) \setminus B_{\delta r}(S)) \leq \delta^{-\xi\zeta} \mathfrak{c}_{\delta r} e^{\xi h_{\delta r}(v_S)}, \quad \widetilde{D}_h(\text{across } B_{2\delta r}(S) \setminus B_{\delta r}(S)) \geq \delta^{\tilde{\xi}\zeta} \tilde{\mathfrak{c}}_{\delta r} e^{\tilde{\xi} h_{\delta r}(v_S)}.$$

where v_S is the center of S. Since $\mathfrak{c}_{\delta r} = \delta^{\xi Q + o_\delta(1)} \mathfrak{c}_r$ and similarly for $\tilde{\mathfrak{c}}_{\delta r}$, we can rewrite this as

$$\begin{aligned} D_h(\text{around } B_{2\delta r}(S) \setminus B_{\delta r}(S)) &\leq \delta^{\xi(Q-\zeta)+o_\delta(1)} \mathfrak{c}_r e^{\xi h_{\delta r}(v_S)}, \\ \widetilde{D}_h(\text{across } B_{2\delta r}(S) \setminus B_{\delta r}(S)) &\geq \delta^{\tilde{\xi}(\widetilde{Q}+\zeta)+o_\delta(1)} \tilde{\mathfrak{c}}_r e^{\tilde{\xi} h_{\delta r}(v_S)}. \end{aligned} \tag{5-7}$$

By another application of Proposition 2.9, it holds with superpolynomially high probability as $\delta \to 0$ that there is a path $\tilde{\pi}$ in $\mathbb{A}_{ar,br}(z)$ which disconnects the inner and outer boundaries of $\mathbb{A}_{ar,br}(z)$ and has $\widetilde{D}_h$-length at most $\delta^{-\tilde{\xi}\zeta} \tilde{\mathfrak{c}}_r e^{\tilde{\xi} h_r(z)}$. Let $S_1, \dots, S_N$ be the squares in $\mathcal{S}_{\delta r}(\mathbb{A}_{ar,br}(z))$ which are hit by $\tilde{\pi}$, listed in numerical order. Then $S_1, \dots, S_N$ satisfy Properties 1 and 2 in the lemma statement.

For each j, the path $\tilde{\pi}$ crosses between the inner and outer boundaries of $B_{2\delta r}(S_j) \setminus B_{\delta r}(S_j)$, so by (5-7),

$$\delta^{\tilde{\xi}(\widetilde{Q}+\zeta)+o_\delta(1)} \tilde{\mathfrak{c}}_r e^{\tilde{\xi} h_{\delta r}(v_{S_j})} \leq \widetilde{D}_h(\text{across } B_{2\delta r}(S_j) \setminus B_{\delta r}(S_j)) \leq (\widetilde{D}_h\text{-length of } \tilde{\pi}) \leq \delta^{-\tilde{\xi}\zeta} \tilde{\mathfrak{c}}_r e^{\tilde{\xi} h_r(z)}. \tag{5-8}$$

Rearranging this inequality, then taking the $1/\tilde{\xi}$ power of both sides, gives

$$e^{h_{\delta r}(v_{S_j}) - h_r(z)} \leq \delta^{-(\widetilde{Q}+2\zeta)}. \tag{5-9}$$

As $\tilde{\xi} \to \infty$, we have $\widetilde{Q} \to 0$ [8, Proposition 1.1]. Hence, if $\tilde{\xi}$ is chosen to be sufficiently large (depending on ζ) then we can arrange that $\widetilde{Q} < \zeta$, so $e^{h_{\delta r}(v_{S_j}) - h_r(z)} \le \delta^{-3\zeta}$. Plugging this into the first inequality in (5-7) shows that for each j,

$$D_h(\text{around } B_{2\delta r}(S_j) \setminus B_{\delta r}(S_j)) \le \delta^{\xi(Q-4\zeta)+o_\delta(1)} \mathfrak{c}_r e^{\xi h_r(z)}.$$

Since ζ is arbitrary, this implies that can arrange for the S_j's to satisfy Condition 3. □

Proof of Lemma 5.8. By translation invariance and tightness across scales (Axioms IV and V), the laws of the reciprocals of the scaled distances $\mathfrak{c}_r^{-1} e^{-\xi h_r(z)} D_h(\text{across } \mathbb{A}_{2r,3r}(z))$ for $z \in \mathbb{C}$ and $r > 0$ are tight. Therefore, we can find $c = c(p) > 0$ such that for each $z \in \mathbb{C}$ and $r > 0$, Condition 1 in the definition of $E_r^U(z)$ occurs with probability at least $1 - (1-p)/3$. By Lemma 5.9, we can find $\delta = \delta(p, c) \in (0, 1)$ such that Condition 2 in the definition of $E_r(z)$ occurs with probability at least $1 - (1-p)/3$. For a given choice of δ, the collection of open sets $\mathcal{U}_r(z; \delta)$ is finite, and is equal to $r\mathcal{U}_1(0; \delta) + z$ (here we use the translation by z in (5-3)). Since $\mathfrak{h}^U$ is continuous away from ∂U, for any fixed choice of $U \in \mathcal{U}_1(0; \delta)$, a.s. $\sup_{u \in U_{\delta/4}} |\mathfrak{h}^U(u)| < \infty$. By combining this with the translation and scale invariance of the law of h, modulo additive constant, we find that there exists $A > 0$ (depending on δ) such that with probability at least $1-(1-p)/3$, Condition 3 in the definition of $E_r^U(z)$ holds simultaneously for every $U \in \mathcal{U}_r(z; \delta)$. □

We now want to show that if we condition on $E_r^U(z)$, then with positive conditional probability the D_h-distances between certain points in U are very small. For $r > 0$, $z \in \mathbb{C}$, and $U \in \mathcal{U}_r(z)$, let $\mathcal{V}(U)$ be the set of connected components of U. Also let

$$\mathcal{Z}(U) := \{\text{center points of squares } S \in \mathcal{S}_{\delta r}^z(U) \text{ with } S \subset \overline{U}\} \tag{5-10}$$

be the set of centers of squares which are entirely contained in $\overline{U}$. We define the event

$$H_r^U(z) := \Big\{ \max_{V \in \mathcal{V}(U)} \sup_{u, v \in V \cap \mathcal{Z}(U)} D_h(u, v; V) \le \tfrac{1}{2} c \mathfrak{c}_r e^{\xi h_r(z)} \Big\}, \tag{5-11}$$

i.e., $H_r^U(z)$ is the event that for any $V \in \mathcal{V}(U)$, the D_h-internal distance in V between any two of the centers of the squares which are entirely contained in V is bounded above by $\frac{1}{2} c \mathfrak{c}_r e^{\xi h_r(z)}$ (this quantity is relevant due to Condition 1 in the definition of $E_r^U(z)$). We think of annuli $\mathbb{A}_{2r,5r}(z)$ for which $E_r^U(z) \cap H_r^U(z)$ occurs (for a suitable choice of U) as "very good".

We note that $H_r^U(z)$ does *not* include an upper bound for the D_h-distance between two arbitrary points of V. This is because there are a.s. singular points contained in V, but a.s. none of the (finitely many) points in $\mathcal{Z}(U)$ are singular points, so a.s. any two points in $\mathcal{Z}(U)$ lie at finite D_h-distance from each other.

The following is the analog of [18, Lemma 3.3] in our setting. It says that an annulus has positive conditional probability to be "very good" given that it is "good".

Lemma 5.10. *For any choice of parameters c, δ, A, there is a constant $\mathfrak{p} = \mathfrak{p}(c, \delta, A) > 0$ such that for each $r > 0$, each $z \in \mathbb{C}$, and each $U \in \mathcal{U}_r(z)$,*

$$\mathbb{P}\big[H_r^U(z) \,\big|\, h|_{\mathbb{C} \setminus U}, E_r^U(z)\big] \ge \mathfrak{p}. \tag{5-12}$$

Proof. This is proven via essentially the same argument as [18, Lemma 3.3]: we subtract a large bump function from $h|_U$ to get a lower bound for $\mathbb{P}[H_r^U(z) \mid h|_{\mathbb{C}\setminus U}]$, then we use the FKG inequality (Proposition 5.7) to add in the conditioning on $E_r^U(z)$ (we only need to use the FKG inequality for the second condition in the definition of $E_r^U(z)$ since the other two conditions are determined by $h|_{\mathbb{C}\setminus U}$). The proof is actually slightly simpler than that of [18, Lemma 3.3] since we are not trying to bound distances between points which are arbitrarily close to ∂V, so unlike in [18] we do not need to worry about the diameters of the squares in $\mathcal{S}_{\delta r}^z(V)$. □

5.2.2. *Cutting off geodesics from a boundary arc.* For $c, \delta \in (0,1)$ and $A > 0$, define $E_r(z) = E_r(z; c, \delta, A)$ as in (5-6). We will use the events $E_r(z)$ to build "shields" which prevent D_h-geodesics from hitting a given arc of a filled metric ball. For $z \in \mathbb{C}$ and $\mathbb{r} > 0$, let $\rho_{\mathbb{r}}^0(z) := \mathbb{r}$ and for $n \in \mathbb{N}$, inductively define

$$\rho_{\mathbb{r}}^n(z) := \inf\left\{r \geq 6\rho_{\mathbb{r}}^{n-1}(z) : r = 2^k \mathbb{r} \text{ for some } k \in \mathbb{Z},\ E_r(z) \text{ occurs}\right\}. \tag{5-13}$$

Since $E_r(z)$ is determined by $h|_{\mathbb{A}_{2r,5r}(z)}$, it follows that $\rho_{\mathbb{r}}^n(z)$ is a stopping time for the filtration generated by $h|_{B_{5r}(z)}$ for $r \geq \mathbb{r}$. The following lemma allows us to produce lots of annuli for which $E_r(z)$ occurs.

Lemma 5.11. *There exists a choice of parameters $c, \delta \in (0,1)$ and $A > 0$ and another parameter $\eta > 0$, depending only on the choice of metric D, such that the following is true. For each compact set $K \subset \mathbb{C}$, it holds with probability $1 - O_\varepsilon(\varepsilon^2)$ (at a rate depending on K) that*

$$\rho_{\varepsilon\mathbb{r}}^{\lfloor \eta \log \varepsilon^{-1} \rfloor}(z) \leq \varepsilon^{1/2}\mathbb{r} \quad \textit{for all } z \in \left(\tfrac{1}{4}\varepsilon\mathbb{r}\mathbb{Z}^2\right) \cap B_{\varepsilon\mathbb{r}}(\mathbb{r}K). \tag{5-14}$$

Proof. This follows from the variant of Lemma 2.12 where our radii are increasing rather than decreasing [18, Lemma 2.12] together with a union bound, exactly as in the proof of [18, Lemma 3.4]. □

We henceforth let c, δ, A, and η be as in Lemma 5.11. For $\varepsilon > 0$, $\mathbb{r} > 0$, and a compact set $K \subset \mathbb{C}$, let

$$R_{\mathbb{r}}^\varepsilon(K) := 6 \sup\left\{\rho_{\varepsilon\mathbb{r}}^{\lfloor \eta \log \varepsilon^{-1} \rfloor}(z) : z \in \left(\tfrac{1}{4}\varepsilon\mathbb{r}\mathbb{Z}^2\right) \cap B_{\varepsilon\mathbb{r}}(K)\right\} + \varepsilon\mathbb{r}, \tag{5-15}$$

so that each of the radii $\rho_{\varepsilon\mathbb{r}}^n(z)$ for $z \in \left(\frac{1}{4}\varepsilon\mathbb{r}\mathbb{Z}^2\right) \cap B_{\varepsilon\mathbb{r}}(K)$ and $n \in [1, \eta \log \varepsilon^{-1}]_{\mathbb{Z}}$ is determined by $R_{\mathbb{r}}^\varepsilon(K)$ and $h|_{B_{R_{\mathbb{r}}^\varepsilon(K)}(K)}$. Lemma 5.11 shows that for each fixed choice of K, $\mathbb{P}[R_{\mathbb{r}}^\varepsilon(\mathbb{r}K) \leq (6\varepsilon^{1/2} + \varepsilon)\mathbb{r}]$ tends to 1 as $\varepsilon \to 0$, at a rate which is uniform in $\mathbb{r}$.

For $s > 0$, define

$$\sigma_{s,\mathbb{r}}^\varepsilon := \inf\{s' > s : B_{R_{\mathbb{r}}^\varepsilon(\mathcal{B}_s^\bullet)}(\mathcal{B}_s^\bullet) \subset \mathcal{B}_{s'}^\bullet\}, \tag{5-16}$$

so that $\mathcal{B}^\bullet_{\sigma_{s,\mathbb{r}}^\varepsilon}$ contains $B_{6\rho_{\varepsilon\mathbb{r}}^{\lfloor \eta \log \varepsilon^{-1} \rfloor}(z)}(z)$ for each $z \in B_{\varepsilon\mathbb{r}}(\mathcal{B}_s^\bullet)$. Since each $\rho_{\varepsilon\mathbb{r}}^{\lfloor \eta \log \varepsilon^{-1} \rfloor}(z)$ is a stopping time for the filtration generated by $h|_{B_{5r}(z)}$ for $r \geq \varepsilon\mathbb{r}$, it follows that if τ is a stopping time for $\{(\mathcal{B}_t^\bullet, h|_{\mathcal{B}_t^\bullet})\}_{t \geq 0}$, then so is $\sigma_{\tau,\mathbb{r}}^\varepsilon$ (this would still be true if we replaced 6 by 5 in (5-16)). The following lemma, which is analogous to [18, Lemma 3.6], will be used to "kill off" the D_h-geodesics from 0 which hit a given boundary arc of a filled D_h-metric ball.

Lemma 5.12. *There exists $\alpha > 0$, depending only on the choice of metric, such that the following is true. Let $\mathbb{r} > 0$, let τ be a stopping time for the filtration generated by $\{(\mathcal{B}_s^\bullet, h|_{\mathcal{B}_s^\bullet})\}_{s \geq 0}$, and let $x \in \partial\mathcal{B}_\tau^\bullet$ and*

$\varepsilon \in (0,1)$ *be chosen in a manner depending only on* $(\mathcal{B}^\bullet_\tau, h|_{\mathcal{B}^\bullet_\tau})$. *There is an event* $G^\varepsilon_x \in \sigma(\mathcal{B}^\bullet_{\sigma^\varepsilon_{\tau,\mathfrak{r}}}, h|_{\mathcal{B}^\bullet_{\sigma^\varepsilon_{\tau,\mathfrak{r}}}})$ *with the following properties.*

A. *If, in the notation* (5-15), *we have* $R^\varepsilon_{\mathfrak{r}}(\mathcal{B}^\bullet_\tau) \le \operatorname{diam} \mathcal{B}^\bullet_\tau$ (*where* diam *denotes Euclidean diameter*) *and* G^ε_x *occurs, then no* D_h*-geodesic from* 0 *to a point in* $\mathbb{C} \setminus B_{R^\varepsilon_{\mathfrak{r}}(\mathcal{B}^\bullet_\tau)}(\mathcal{B}^\bullet_\tau)$ *can enter* $B_{\varepsilon\mathfrak{r}}(x) \setminus \mathcal{B}^\bullet_\tau$.

B. *There is a deterministic constant* $C_0 > 1$ *depending only on the choice of metric such that a.s.* $\mathbb{P}\big[G^\varepsilon_x \,\big|\, \mathcal{B}^\bullet_\tau, h|_{\mathcal{B}^\bullet_\tau}\big] \ge 1 - C_0 \varepsilon^\alpha$.

Proof. The proof is similar to that of the subcritical version [18, Lemma 3.6], but the geometric part of the argument (i.e., the verification of Property A) is slightly different due to the different way in which the events $E_r(z)$ are defined in the supercritical case. We will therefore repeat part of the proof in order to explain the details of this geometric argument. See Figure 9 for an illustration.

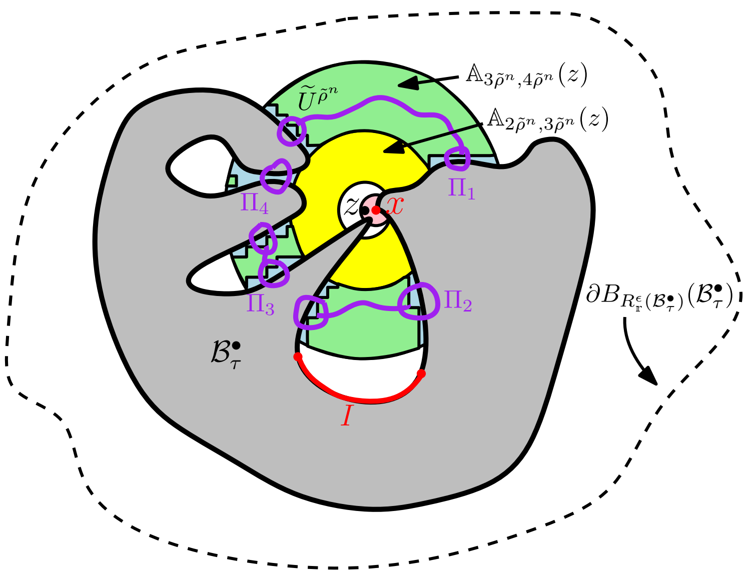

Figure 9. Illustration of the proof of Lemma 5.12. The point $z \in \frac{1}{4}\varepsilon\mathfrak{r}\mathbb{Z}^2$ is chosen so that $B_{\varepsilon\mathfrak{r}}(x) \subset B_{2\varepsilon\mathfrak{r}}(z)$. On the event G^ε_x defined in (5-20), there is some $n \in [1, \eta \log \varepsilon^{-1}]_{\mathbb{Z}}$ for which the event $H^{\widetilde{U}^{\tilde\rho^n}}_{\tilde\rho^n}(z)$ as defined in (5-11) occurs. For this choice of n, we can use the definition of $H^{\widetilde{U}^{\tilde\rho^n}}_{\tilde\rho^n}(z)$ together with Condition 2 in the definition of $E^{\widetilde{U}^{\tilde\rho^n}}_{\tilde\rho^n}(z)$ to build paths Π_k (purple) in the connected components of $\mathbb{A}_{3\tilde\rho^n,4\tilde\rho^n}(z) \setminus \mathcal{B}^\bullet_\tau$ which disconnect $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$ from ∞ in $\mathbb{C} \setminus \mathcal{B}^\bullet_\tau$ and whose D_h-lengths are each less than $c\mathfrak{c}_{\tilde\rho^n} e^{\xi h_{\tilde\rho^n}(z)}$. That is, each of the purple paths is D_h-shorter than the D_h-distance across $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$. In order for a path P from a point outside of $B_{R^\varepsilon_{\mathfrak{r}}(\mathcal{B}^\bullet_s)}(\mathcal{B}^\bullet_s)$ to 0 to enter $B_{\varepsilon\mathfrak{r}}(x) \setminus \mathcal{B}^\bullet_\tau$, it would first have to hit one of these purple paths, which would give us a path to 0 which is shorter than P. Hence such a path P cannot be a D_h-geodesic. The condition that $R^\varepsilon_{\mathfrak{r}}(\mathcal{B}^\bullet_\tau) \le \operatorname{diam} \mathcal{B}^\bullet_\tau$ ensures that $\mathbb{A}_{3\tilde\rho^n,4\tilde\rho^n}(z)$ intersects $\mathcal{B}^\bullet_\tau$. We can also prevent D_h-geodesics from hitting an arc I of $\partial\mathcal{B}^\bullet_\tau$ by choosing x so that $B_{\varepsilon\mathfrak{r}}(x)$ disconnects I from ∞ in $\mathbb{C}\setminus\mathcal{B}^\bullet_\tau$; see Lemma 5.13.

Step 1: *setup.* We can choose $z \in \left(\frac{1}{4}\varepsilon\mathbb{r}\mathbb{Z}^2\right) \cap B_{\varepsilon\mathbb{r}}(\mathcal{B}_\tau^\bullet)$ such that $B_{\varepsilon\mathbb{r}}(x) \subset B_{2\varepsilon\mathbb{r}}(z)$, in a manner depending only on $(\mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet})$. Recalling the set of squares $\mathcal{S}_{\delta r}^z(\cdot)$ from (5-3), for $r > 0$ we define

$$\widetilde{U}^r := \widetilde{U}^r(z) := \mathbb{A}_{3r,4r}(z) \setminus \bigcup\{S \in \mathcal{S}_{\delta r}^z(\mathbb{A}_{3r,4r}(z)) : S \cap \mathcal{B}_\tau^\bullet \neq \varnothing\}. \tag{5-17}$$

Note that $\widetilde{U}^r$ belongs to the set $\mathcal{U}_r(z)$ of Section 5.2.1 and $\widetilde{U}^r$ is determined by $(\mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet})$.

Let $\tilde{\rho}^0 := \varepsilon\mathbb{r}$ and for $n \in \mathbb{N}$, inductively define

$$\tilde{\rho}^n = \tilde{\rho}^n_{\varepsilon\mathbb{r}}(z) := \inf\left\{r \geq 6\tilde{\rho}^{n-1} : r = 2^k\mathbb{r} \text{ for some } k \in \mathbb{Z},\ E_r^{\widetilde{U}^r}(z) \text{ occurs}\right\}. \tag{5-18}$$

In other words, $\tilde{\rho}^n$ is defined in the same manner as $\rho^n_{\varepsilon\mathbb{r}}(z)$ from (5-13) (with $\varepsilon\mathbb{r}$ in place of $\mathbb{r}$) but with $E_r^{\widetilde{U}^r}(z)$ instead of $E_r(z)$. This means that $E_{\tilde{\rho}^n}^U(z)$ is only required to occur for $U = \widetilde{U}^{\tilde{\rho}^n}$ instead of for every $U \in \mathcal{U}_{\tilde{\rho}^n}(z)$. By this and the definition (5-15) of $R_{\mathbb{r}}^\varepsilon(\mathcal{B}_\tau^\bullet)$,

$$\tilde{\rho}^n \leq \rho^n_{\varepsilon\mathbb{r}}(z) \quad \text{for all } n \in \mathbb{N}_0 \qquad \text{and hence} \quad \tilde{\rho}^{\lfloor \eta \log \varepsilon^{-1} \rfloor} \leq \tfrac{1}{6} R_{\mathbb{r}}^\varepsilon(\mathcal{B}_\tau^\bullet). \tag{5-19}$$

The reason for considering $\tilde{\rho}^n$ instead of $\rho^n_{\varepsilon\mathbb{r}}(z)$ is because we can only condition on $E_r^U(z)$, not on $E_r(z)$, in Lemma 5.10.

Recalling that $\mathcal{V}(\widetilde{U}^{\tilde{\rho}^n})$ denotes the set of connected components of $\widetilde{U}^{\tilde{\rho}^n}$, we define

$$G_x^\varepsilon := \left\{\exists n \in [1, \eta \log \varepsilon^{-1}]_{\mathbb{Z}} \text{ such that } H_{\tilde{\rho}^n}^{\widetilde{U}^{\tilde{\rho}^n}}(z) \text{ occurs}\right\}, \tag{5-20}$$

where $H_{\tilde{\rho}^n}^{\widetilde{U}^{\tilde{\rho}^n}}(z)$ is the event of (5-11) with $U = \widetilde{U}^{\tilde{\rho}^n}$.

Since z and $\widetilde{U}^r$ for $r > 0$ are each determined by $(\mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet})$, it follows that each $E^{\widetilde{U}^r}(z)$ is determined by $(\mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet})$ and $h|_{\mathbb{A}_{2r,5r}(z)}$. Hence $\tilde{\rho}^n$ is a stopping time for the filtration generated by $h|_{B_{5r}(z)}$ for $r \geq \varepsilon\mathbb{r}$ and $(\mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet})$. By (5-19) and the definition (5-16) of $\sigma_{\tau,\mathbb{r}}^\varepsilon$, we have $B_{5\tilde{\rho}^n}(z) \subset \mathcal{B}^\bullet_{\sigma_{\tau,\mathbb{r}}^\varepsilon}$. By combining these statements with (5-20) and the locality of the metric (Axiom II), we get that $G_x^\varepsilon \in \sigma(\mathcal{B}^\bullet_{\sigma_{\tau,\mathbb{r}}^\varepsilon}, h|_{\mathcal{B}^\bullet_{\sigma_{\tau,\mathbb{r}}^\varepsilon}})$.

We need to check Properties A and B for the event G_x^ε.

Step 2: *proof that G_x^ε satisfies Property A.* Assume that $R_{\mathbb{r}}^\varepsilon(\mathcal{B}_\tau^\bullet) \leq \operatorname{diam} \mathcal{B}_\tau^\bullet$ and G_x^ε occurs. Choose $n \in [1, \eta \log \varepsilon^{-1}]_{\mathbb{Z}}$ as in the definition (5-20) of G_x^ε. Then

$$\varepsilon\mathbb{r} \leq \tilde{\rho}^n \leq \tfrac{1}{6} R_{\mathbb{r}}^\varepsilon(\mathcal{B}_\tau^\bullet) \leq \tfrac{1}{6} \operatorname{diam} \mathcal{B}_\tau^\bullet.$$

By our choice of z, this means that both the inner and outer boundaries of $\mathbb{A}_{3\tilde{\rho}^n,4\tilde{\rho}^n}(z)$ intersect $\mathcal{B}_\tau^\bullet$ and $\mathbb{A}_{2\tilde{\rho}^n,3\tilde{\rho}^n}(z)$ disconnects $B_{\varepsilon\mathbb{r}}(x)$ from ∞. We will argue that no D_h-geodesic from a point outside of $\mathbb{C} \setminus B_{R_{\mathbb{r}}^\varepsilon(\mathcal{B}_\tau^\bullet)}(\mathcal{B}_\tau^\bullet)$ to 0 can cross between the inner and outer boundaries of $\mathbb{A}_{2\tilde{\rho}^n,3\tilde{\rho}^n}(z)$ before hitting $\mathcal{B}_\tau^\bullet$, which implies that no such D_h-geodesic can hit $B_{\varepsilon\mathbb{r}}(x)$ before entering $\mathcal{B}_\tau^\bullet$. The idea of the proof is that the definition (5-11) of $H_{\tilde{\rho}^n}^{\widetilde{U}^{\tilde{\rho}^n}}(z)$ together with Condition 2 in the definition of $E_{\tilde{\rho}^n}^{\widetilde{U}^{\tilde{\rho}^n}}(z)$ allow us to build a collection of paths in $\mathbb{A}_{3\tilde{\rho}^n,4\tilde{\rho}^n}(z)$ which act as "shortcuts". Let us now explain the construction of these paths.

Step 2(*a*): *constructing paths in $\mathbb{A}_{3\tilde{\rho}^n,4\tilde{\rho}^n}(z)$.* Let $S_1, \dots, S_N$ be the path of squares in $\mathcal{S}_{\delta\tilde{\rho}^n}(\mathbb{A}_{3.1\tilde{\rho}^n,3.9\tilde{\rho}^n}(z))$ as in Condition 2 in the definition of $E_{\tilde{\rho}^n}^{\widetilde{U}^{\tilde{\rho}^n}}(z)$. Let K be the number of squares in $\{S_1, \dots, S_N\}$ which intersect $\mathcal{B}_\tau^\bullet$ (equivalently, the number of such squares which are not contained in $\widetilde{U}^{\tilde{\rho}^n}$). For $k \in [1, K]_{\mathbb{Z}}$, let j_k be the k-th smallest value of $j \in [1, N]_{\mathbb{Z}}$ for which S_j intersects $\mathcal{B}_\tau^\bullet$. Also set $S_{j_0} = S_{j_K}$.

For each $k \in [1, K]_{\mathbb{Z}}$ such that S_{j_k} intersects $\partial\mathcal{B}^\bullet_\tau$, we will define a path Π_k associated with S_{j_k} in such a way that the following properties are satisfied.

(i) Each Π_k has D_h-length strictly less than $c\mathfrak{c}_{\tilde\rho^n} e^{\xi h_{\tilde\rho^n}(z)}$ and intersects $\partial\mathcal{B}^\bullet_\tau$.

(ii) The union of the paths Π_k over all k such that $S_k \cap \partial\mathcal{B}^\bullet_\tau \neq \varnothing$ disconnects $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z) \setminus \mathcal{B}^\bullet_\tau$ from $\partial B_{4\tilde\rho^n}(z) \setminus \mathcal{B}^\bullet_\tau$ in $\mathbb{C} \setminus \mathcal{B}^\bullet_\tau$.

The paths Π_k are shown in purple in Figure 9.

To define these paths, let $k \in [1, K]_{\mathbb{Z}}$ such that $S_{j_k} \cap \partial\mathcal{B}^\bullet_\tau \neq \varnothing$. We consider two cases. If $j_{k-1}+1 = j_k$, we let Π_k be a path around $B_{2\delta\tilde\rho^n}(S_{j_k}) \setminus B_{\delta\tilde\rho^n}(S_{j_k})$ whose D_h-length is at most $\frac{1}{100} c\,\mathfrak{c}_{\tilde\rho^n} e^{\xi h_{\tilde\rho^n}(z)}$, as afforded by Condition 2 in the definition of $E^{\widetilde U^{\tilde\rho^n}}_{\tilde\rho^n}(z)$.

If $j_{k-1}+1 < j_k$, then there is a connected component V of $\widetilde U^{\tilde\rho^n}$ whose boundary intersects the boundaries of each of $S_{j_{k-1}}$ and S_{j_k} such that $S_{j_{k-1}+1}, \dots, S_{j_k-1} \subset \overline{V}$. Since the event $H^{\widetilde U^{\tilde\rho^n}}_{\tilde\rho^n}(z)$ of (5-11) occurs, there is a path π_V in V from the center point of $S_{j_{k-1}+1}$ to the center point of S_{j_k-1} whose D_h-length is at most $\frac{1}{2} c\,\mathfrak{c}_{\tilde\rho^n} e^{\xi h_{\tilde\rho^n}(z)}$. By the last part of Condition 2 in the definition of $E^{\widetilde U^{\tilde\rho^n}}_{\tilde\rho^n}(z)$, there are paths π_0 and π_1 in the annular regions $B_{2\delta\tilde\rho^n}(S_{j_{k-1}+1}) \setminus B_{\delta\tilde\rho^n}(S_{j_{k-1}+1})$ and $B_{2\delta\tilde\rho^n}(S_{j_k-1}) \setminus B_{\delta\tilde\rho^n}(S_{j_k-1})$, respectively, which disconnect the inner and outer boundaries of these annular regions and whose D_h-lengths are each at most $\frac{1}{100} c\,\mathfrak{c}_{\tilde\rho^n} e^{\xi h_{\tilde\rho^n}(z)}$. Note that the paths π_0 and π_1 necessarily intersect both π_V and $\partial\mathcal{B}^\bullet_\tau$. Let Π_k be a concatenation of π_0, π_V, π_1.

It is clear from the above definitions that our desired property (i) is satisfied. To check property (ii), consider a path $\mathfrak{P}$ from a point of $\partial B_{4\tilde\rho^n}(z) \setminus \mathcal{B}^\bullet_\tau$ to a point of $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z) \setminus \mathcal{B}^\bullet_\tau$ in $\mathbb{C} \setminus \mathcal{B}^\bullet_\tau$. There is a subpath $\mathfrak{P}'$ of $\mathfrak{P}$ which is contained in $\overline{\mathbb{A}_{3\tilde\rho^n,4\tilde\rho^n}(z)}$ and whose endpoints lie on the inner and outer boundaries of $\mathbb{A}_{3\tilde\rho^n,4\tilde\rho^n}(z)$, respectively. Since the union of the squares $S_1, \dots, S_N$ contains a path which disconnects the inner and outer boundaries of $\mathbb{A}_{3\tilde\rho^n,4\tilde\rho^n}(z)$ (Condition 2 in the definition of $E^{\widetilde U^{\tilde\rho^n}}_{\tilde\rho^n}(z)$), there must be some j such that $S_j \not\subset \mathcal{B}^\bullet_\tau$ and $\mathfrak{P}'$ intersects S_j. If $S_j \cap \partial\mathcal{B}^\bullet_\tau \neq \varnothing$, then $j = j_k$ for some k and $\mathfrak{P}'$ intersects the path Π_k, so we are done. Otherwise, there exists $k \in [1, K]_{\mathbb{Z}}$ for which $j \in [j_{k-1}+1, j_k-1]_{\mathbb{Z}}$. Let O be the connected component of $\mathbb{A}_{3\tilde\rho^n,4\tilde\rho^n}(z) \setminus \mathcal{B}^\bullet_\tau$ which contains $S_{j_{k-1}+1}, \dots, S_{j_k-1}$. Then $\mathfrak{P}' \subset O$. Furthermore, by construction, the path Π_k disconnects $\partial O \cap \partial B_{3\tilde\rho^n}(z)$ and $\partial O \cap \partial B_{4\tilde\rho^n}(z)$ in O. Therefore, $\mathfrak{P}'$ must intersect Π_k, as required.

Step 2(*b*): *preventing a D_h-geodesic from crossing* $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$. Due to Lemma 3.4, a D_h-geodesic from a point outside of $\mathcal{B}^\bullet_\tau$ to 0 hits $\partial\mathcal{B}^\bullet_\tau$ exactly once. So, if such a geodesic hits $B_{\varepsilon\mathfrak{r}}(x) \setminus \mathcal{B}^\bullet_\tau$, then it hits $B_{\varepsilon\mathfrak{r}}(x)$ *before* entering $\mathcal{B}^\bullet_\tau$. Therefore, to prove Property A, it suffices to consider a path P from a point outside of $\mathbb{C} \setminus B_{R^\varepsilon_{\mathfrak{r}}(\mathcal{B}^\bullet_\tau)}(\mathcal{B}^\bullet_\tau)$ to 0 which enters $B_{\varepsilon\mathfrak{r}}(x)$ before entering $\mathcal{B}^\bullet_\tau$ and show that P cannot be a D_h-geodesic.

Since $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$ disconnects $B_{\varepsilon\mathfrak{r}}(x)$ from ∞, the path P must cross from the outer boundary of $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$ to the inner boundary of $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$ before hitting $B_{\varepsilon\mathfrak{r}}(x)$, and hence also before hitting $\mathcal{B}^\bullet_\tau$. By Condition 1 in the definition of $E^{\widetilde U^{\tilde\rho^n}}_{\tilde\rho^n}(z)$, each path between the inner and outer boundaries of $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$ has D_h-length at least $c\mathfrak{c}_{\tilde\rho^n} e^{\xi h_{\tilde\rho^n}(z)}$. Hence, the D_h-length of the segment of P after the first time it enters $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$ must be at least $c\mathfrak{c}_{\tilde\rho^n} e^{\xi h_{\tilde\rho^n}(z)} + \tau$.

But, P must cross between the inner and outer boundaries of $\mathbb{A}_{3\tilde\rho^n,4\tilde\rho^n}(z)$ before entering $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$, so P must hit one of the paths Π_k above before entering $\mathbb{A}_{2\tilde\rho^n,3\tilde\rho^n}(z)$. Since Π_k intersects $\partial\mathcal{B}^\bullet_\tau$ and has

D_h-length strictly less than $c\mathfrak{c}_{\tilde\rho^n} e^{\xi h_{\tilde\rho^n}(z)}$, it follows that each point of Π_k lies at D_h-distance strictly less than $c\mathfrak{c}_{\tilde\rho^n} e^{\xi h_{\tilde\rho^n}(z)} + \tau$ from 0. Combining this with the conclusion of the preceding paragraph shows that P cannot be a D_h-geodesic to 0.

Step 3: *proof that G_x^ε satisfies Property B.* Recall the definition of G_x^ε from (5-20). From Lemma 5.10 and an elementary conditioning argument, exactly as in the proof of [18, Lemma 3.6, Step 3], we obtain that for every $n \in [1, \eta \log \varepsilon^{-1}]_{\mathbb{Z}}$, a.s.

$$\mathbb{P}\big[H_{\tilde\rho^n}^{\widetilde U^{\tilde\rho^n}}(z) \mid \widetilde U^{\tilde\rho^n}, h|_{\mathbb{C}\setminus \widetilde U^{\tilde\rho^n}}\big] \geq \mathfrak{p}, \tag{5-21}$$

where $\mathfrak{p} > 0$ is as in Lemma 5.10. Note that by the definition (5-18) of $\tilde\rho^n$, it is automatically the case that the event $E_r^{\widetilde U^{\tilde\rho^n}}(z)$ occurs. By the definition (5-11) and the locality property of D_h, the event $H_{\tilde\rho^n}^{\widetilde U^{\tilde\rho^n}}(z)$ is a.s. determined by $\widetilde U^{\tilde\rho^n}$ and the restriction of h to $\widetilde U^{\tilde\rho^n}$. Since the open sets $\widetilde U^{\tilde\rho^n}$ for different values of n are disjoint from each other and from $\mathcal{B}_\tau^\bullet$, we can apply (5-21) iteratively to get

$$\mathbb{P}[G_x^\varepsilon \mid \mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet}] \geq 1 - (1-\mathfrak{p})^{\lfloor \eta \log \varepsilon^{-1} \rfloor}.$$

See [18, Lemma 3.6, Step 3] for details. This last estimate gives Property B for an appropriate choice of C_0 and α. □

Analogously to [18, Lemma 3.7], we also have the following variant of Lemma 5.12 where we prevent D_h-geodesics from hitting a boundary arc rather than a neighborhood of a point.

Lemma 5.13. *Let α be as in Lemma 5.12. Let $\mathbb{r} > 0$, let τ be a stopping time for the filtration generated by $\{(\mathcal{B}_s^\bullet, h|_{\mathcal{B}_s^\bullet})\}_{s\geq 0}$. Also let $\varepsilon \in (0,1)$ and $I \subset \partial \mathcal{B}_\tau^\bullet$ be an arc, each chosen in a manner depending only on $(\mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet})$, such that I can be disconnected from ∞ in $\mathbb{C}\setminus\mathcal{B}_\tau^\bullet$ by a set of Euclidean diameter at most $\varepsilon\mathbb{r}$. There is an event $G_I \in \sigma(\mathcal{B}^\bullet_{\sigma^\varepsilon_{\tau,\mathbb{r}}}, h|_{\mathcal{B}^\bullet_{\sigma^\varepsilon_{\tau,\mathbb{r}}}})$ with the following properties.*

A. *If $R_{\mathbb{r}}^\varepsilon(\mathcal{B}_\tau^\bullet) \leq \operatorname{diam} \mathcal{B}_\tau^\bullet$ and G_I occurs, then no D_h-geodesic from 0 to a point in $\mathbb{C}\setminus \mathcal{B}^\bullet_{\sigma^\varepsilon_{\tau,\mathbb{r}}}$ can pass through I.*

B. *There is a deterministic constant $C_0 > 1$ depending only on the choice of metric such that a.s. $\mathbb{P}\big[G_I \,\big|\, \mathcal{B}_\tau^\bullet, h|_{\mathcal{B}_\tau^\bullet}\big] \geq 1 - C_0\varepsilon^\alpha$.*

Proof. This follows from Lemma 5.12 via exactly the same argument as in the proof of [18, Lemma 3.7]. □

5.2.3. *Proof of Theorem 1.7.* To prove Theorem 1.7, it remains to carry out Step 3 in the outline at the beginning of this subsection. For this step, the argument from [18] carries over almost verbatim so we will not give details.

We first define the regularity event that we will work on. Fix $\mathbb{r} > 0$ and define $\tau_{\mathbb{r}}$ as in (3-2). Also let $\beta > 0$ be the parameter from Lemma 3.10. For $a \in (0,1)$, we define $\mathcal{E}_{\mathbb{r}}(a)$ to be the event that the following is true.

1. $B_{a\mathbb{r}}(0) \subset \mathcal{B}^\bullet_{\tau_{\mathbb{r}}}$.
2. $\tau_{3\mathbb{r}} - \tau_{2\mathbb{r}} \geq a\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$.

3. For each $s, t \in [\tau_{2\mathbb{r}}, \tau_{3\mathbb{r}}]$ with $|s-t| \le a\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$, we have
$$\frac{1}{\mathbb{r}} \operatorname{dist}(\partial\mathcal{B}^\bullet_s, \partial\mathcal{B}^\bullet_t) \ge \left(\frac{|t-s|}{\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}} \right)^{1/\beta},$$
where dist denotes Euclidean distance.

4. In the notation (5-13), we have $\rho_{\varepsilon\mathbb{r}}^{\lfloor \eta \log \varepsilon^{-1} \rfloor}(z) \le \varepsilon^{1/2}\mathbb{r}$ for each $z \in \left(\frac{1}{4}\varepsilon\mathbb{r}\mathbb{Z}^2\right) \cap B_{4\mathbb{r}}(0)$ and each $\varepsilon \in (0, a] \cap \{2^{-k}\}_{k\in\mathbb{N}}$ (here η is as in Lemma 5.11).

Our above definition of $\mathcal{E}_{\mathbb{r}}(a)$ is identical to the analogous definition in [18, Section 3.4] except that in [18], Condition 3 is replaced by a Hölder continuity condition for D_h with respect to the Euclidean metric. This condition is of course not true in the supercritical case.

Lemma 5.14. *For each $p \in (0, 1)$, there exists $a = a(p) > 0$ such that $\mathbb{P}[\mathcal{E}_{\mathbb{r}}(a)] \ge p$ for every $\mathbb{r} > 0$.*

Proof. By Lemma 3.5, if a is chosen to be sufficiently small then the probability of Condition 1 is at least $1 - (1-p)/4$. By tightness across scales (Axiom V), after possibly decreasing a we can arrange that the probability of Condition 2 is also at least $1-(1-p)/4$. By Lemma 3.10, after possibly shrinking a we can arrange that the probability that Condition 1 holds is at least $1-(1-p)/4$. By Lemma 5.11 and a union bound over dyadic values of ε with $\varepsilon \in (0, a]$, the probability of Condition 4 is at least $1-(1-p)/4$. Combining these estimates shows that $\mathbb{P}[\mathcal{E}_{\mathbb{r}}(a)] \ge p$. ☐

The following is a more quantitative version of Theorem 1.7, analogous to [18, Theorem 3.9].

Theorem 5.15. *For each $a \in (0, 1)$, there is a constant $b_0 > 0$ depending only on a and constants $b_1, \alpha > 0$ depending only on the choice of metric D such that the following is true. For each $\mathbb{r} > 0$, each $N \in \mathbb{N}$, and each stopping time τ for $\{(\mathcal{B}^\bullet_s, h|_{\mathcal{B}^\bullet_s})\}_{s\ge 0}$ with $\tau \in [\tau_{\mathbb{r}}, \tau_{2\mathbb{r}}]$ a.s., the probability that $\mathcal{E}_{\mathbb{r}}(a)$ occurs and there are more than N points of $\partial\mathcal{B}^\bullet_\tau$ which are hit by leftmost D_h-geodesics from 0 to $\partial\mathcal{B}^\bullet_{\tau+N^{-\alpha}\mathfrak{c}_{\mathbb{r}}e^{\xi h_{\mathbb{r}}(0)}}$ is at most $b_0 e^{-b_1 N^\alpha}$.*

It is easy to see that Theorem 5.15 implies Theorem 1.7; see the beginning of [18, Section 3] for a proof of this in the subcritical case. The supercritical case is identical, with the caveat that we use Lemma 3.10 to show that $r \mapsto \tau_r$ is continuous and surjective.

The proof of Theorem 5.15 is identical to the proof of its subcritical analog, which is given in [18, Section 3.4], with one minor exception, which we discuss just below.

For the sake of completeness, we provide a short outline of the argument; see [18, Section 3.4] for details. We work on the event $\mathcal{E}_{\mathbb{r}}(a)$ defined at the beginning of this subsection. Start with an arbitrary initial collection $\mathcal{I}_0$ of arcs of $\partial\mathcal{B}^\bullet_\tau$ which cover $\partial\mathcal{B}^\bullet_\tau$ and intersect only at their endpoints. By a deterministic geometric lemma [18, Lemma 2.14], if $\#\mathcal{I}_0$ is large then at least half of the arcs in $\mathcal{I}_0$ can be disconnected from ∞ in $\mathbb{C} \setminus \mathcal{B}^\bullet_\tau$ by a set of Euclidean diameter at most a constant times $(\#\mathcal{I}_0)^{-1/2}$. We apply Lemma 5.12 (with $\varepsilon \asymp (\#\mathcal{I}_0)^{-1/2}$) to each of these arcs to get that with high probability, the following is true. For at least $1/4$ of the arcs $I \in \mathcal{I}_0$, there is no D_h-geodesic from 0 to a point outside of the Euclidean $r\mathbb{r}$-neighborhood of $\mathcal{B}^\bullet_\tau$ which passes through I; here $r > 0$ is related to the number $R^\varepsilon_{\mathbb{r}}(\mathcal{B}^\bullet_\tau)$ from Lemma 5.12 and can be bounded above by a negative power of $\#\mathcal{I}_0$.

We then choose a new radius $s_1 > \tau$ so that $B_{r\mathbb{r}}(\mathcal{B}^\bullet_\tau) \subset \mathcal{B}^\bullet_{s_1}$. By Condition 3 in the definition of $\mathcal{E}_\mathbb{r}(a)$, we have that $s_1 - \tau$ is small if r is small, and hence if $\#\mathcal{I}_0$ is large. We apply the same argument with τ replaced by s_1 and with $\mathcal{I}_0$ replaced by the set $\mathcal{I}_1$ of arcs of $\partial\mathcal{B}^\bullet_{s_1}$ defined so that for each $I \in \mathcal{I}_1$, all of the leftmost geodesics from 0 to points of I pass through the same arc in $\mathcal{I}_0$ (see Lemma 5.6). With high probability we have $\#\mathcal{I}_1 \le \frac{3}{4}\#\mathcal{I}_0$. We then iterate this procedure, defining radii $\tau < s_1 < s_2 \dots$. At each step we typically reduce the number of surviving arcs by a constant factor. Moreover, since the increase in the radius at each step is bounded above by a negative power of the number of surviving arcs, the total increase in the radius of the metric ball needed to get down to N surviving arcs can be bounded above independently of the choice of $\mathcal{I}_0$. To conclude, we apply this to a sequence of initial arc collections $\{\mathcal{I}_0^k\}_{k\in\mathbb{N}}$ such that $\#\mathcal{I}_0^k \to \infty$ and the maximal Euclidean diameter of the arcs in $\mathcal{I}_0^k$ tends to zero.

Roughly speaking, the reason why we get the quantitative estimate $b_0 e^{-b_1 N^\alpha}$ is that for each step of the iteration, if we condition on the previous steps, there is a positive conditional probability to kill off a positive fraction of the remaining arcs; and we only need to "succeed" for a positive fraction of the steps. See [18, Lemma 3.10].

The one minor point where the argument in the supercritical case differs from the argument of [18, Section 3.4] is as follows. In [18] Condition 3 in the definition of $\mathcal{E}_\mathbb{r}(a)$ is replaced by a Hölder continuity condition. However, this condition is only used once in [18], in the proof of [18, Lemma 3.11], in order to prove that a filled D_h-metric ball contains a small Euclidean neighborhood of a smaller filled D_h-metric ball. Condition 3 can be used in place of the Hölder continuity condition from [18] for this purpose.

We note that the proof of Theorem 5.15 uses Lemmas 5.6 and 5.13 and also reuses the deterministic estimate from [18, Lemma 2.15].

5.3. ***Reducing to a single geodesic.*** In this subsection we will explain how to deduce Theorem 1.6 from Theorem 1.7. That is, we will explain how to go from finitely many points on the boundary of a filled metric ball which are hit by D_h-geodesics, to just one such point. The main tool which allows us to do this is Lemma 5.16 just below, which says that for certain appropriately chosen arcs I of the boundary of a filled metric ball centered at 0, there is a positive chance that *every* D_h-geodesic from 0 to a point sufficiently far away from the filled metric ball passes through I.

To state this result precisely, we need to introduce a particular way of measuring (Euclidean) distances in a planar domain. Let $O \subset \mathbb{C}$ be a domain bounded by a Jordan curve. Following [18, equation (2.18)], for $z, w \in \overline{O}$, we define

$$d^O(z,w) = \inf\bigl\{\mathrm{diam}(X) : X \text{ is a connected subset of } O \text{ with } z, w \in \overline{X}\bigr\}, \tag{5-22}$$

where here diam denotes the Euclidean diameter. Then d^O is a metric on $\overline{O}$ which is bounded below by the Euclidean metric on $\mathbb{C}$ restricted to $\overline{O}$ and bounded above by the internal Euclidean metric on $\overline{O}$. Note that d^O is not a length metric.

Lemma 5.16. *For each $A > 1$, $\varepsilon \in (0, (A-1)/100)$, and $p \in (0,1)$, there exists $\mathfrak{p} = \mathfrak{p}(A, \varepsilon, p) > 0$ such that the following is true. Let $\mathbb{r} > 0$ and let $\tau_\mathbb{r} = D_h(0, \partial B_\mathbb{r}(0))$ be as in (3-2). Let $I \subset \partial\mathcal{B}^\bullet_{\tau_\mathbb{r}}$ be a closed boundary arc, chosen in a manner depending only on $(\mathcal{B}^\bullet_{\tau_\mathbb{r}}, h|_{\mathcal{B}^\bullet_{\tau_\mathbb{r}}})$, with the property that the*

$d^{\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}}$-neighborhood of radius $\varepsilon\mathbb{r}$ of $\partial\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}\setminus I$ (with $d^{\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}}$ defined as in (5-22)) does not disconnect I from ∞ in $\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}$. With probability at least p, it holds with conditional probability at least $\mathfrak{p}$ given $(\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}, h|_{\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}})$ that every D_h-geodesic from 0 to a point of $\mathbb{C}\setminus B_{A\mathbb{r}}(0)$ passes through I.

The proof of Lemma 5.16 is very similar to the proof of its subcritical analog, [18, Lemma 4.1]. However, just like in the setting of Sections 5.2.1 and 5.2.2, we need to make some nontrivial changes to the definitions of the events involved so we will explain most of the details of the proof.

The proof of Lemma 5.16 is similar to the proof of Lemma 5.13, but simpler since we only need something to happen with positive probability, not probability close to 1, and this probability is allowed to depend on the parameter ε. We will define "good" events $E^U_{\mathbb{r}}$ for certain domains U, which occur with high probability (see (5-27)). We will then argue that if $E^U_{\mathbb{r}}$ occurs, then there is a positive chance that the distances between certain points in U are very small (Lemma 5.17). We will then choose U in a manner which depends on $\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}$ and I. We will use Lemma 5.17 to argue that with positive conditional probability given $(\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}, h|_{\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}})$, there is a "shortcut" in U which prevents D_h-geodesics from 0 to points of $\mathbb{C}\setminus B_{A\mathbb{r}}(0)$ from hitting $\partial\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}\setminus I$.

To lighten notation, let

$$\mathcal{B}^* := \left\{z \in \overline{\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}} : d^{\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}}(z, \partial\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}\setminus I) < \varepsilon\mathbb{r}/4\right\} \tag{5-23}$$

be a slightly smaller $d^{\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}}$-neighborhood of $\partial\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}\setminus I$ than the one appearing in Lemma 5.16.

By the Hölder continuity of the Euclidean metric with respect to D_h (Proposition 2.11), we can find $c = c(A, \varepsilon, p) > 0$ such that with probability at least $1-(1-p)/3$, each subset of $B_{A\mathbb{r}}(0)$ with Euclidean diameter at least $\varepsilon\mathbb{r}/4$ has D_h-diameter at least $c\mathfrak{c}_{\mathbb{r}}e^{\xi h_{\mathbb{r}}(0)}$. By the definition (5-23) of $\mathcal{B}^*$, each path in $\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}$ from $\partial\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}\setminus I$ to a point of $\mathbb{C}\setminus(\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}\cup\mathcal{B}^*)$ has Euclidean diameter at least $\varepsilon\mathbb{r}/4$. Hence, with probability at least $1-(1-p)/3$,

$$D_h\left(\partial\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}\setminus I, \mathbb{C}\setminus(\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}\cup\mathcal{B}^*); \mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}\right) \ge c\mathfrak{c}_{\mathbb{r}}e^{\xi h_{\mathbb{r}}(0)}. \tag{5-24}$$

Define the collection of $\delta\mathbb{r}\times\delta\mathbb{r}$ squares $\mathcal{S}_{\delta\mathbb{r}}(B_{A\mathbb{r}}(0)) = \mathcal{S}^0_{\delta\mathbb{r}}(B_{A\mathbb{r}}(0))$ with corners in $\delta\mathbb{r}\mathbb{Z}^2$ as in (5-3) with $z=0$. By Lemma 2.8, the random variables $\mathfrak{c}_{\mathbb{r}}^{-1}e^{-\xi h_{\mathbb{r}}(0)}\tau_{\mathbb{r}}$ and their reciprocals are tight. By combining this with Corollary 3.7, we can find $\delta = \delta(c, A, \varepsilon) \in (0, \varepsilon^2/100)$ such that with probability at least $1-(1-p)/3$,

$$D_h(\text{around } B_{\delta^{1/2}\mathbb{r}}(S)\setminus B_{\delta\mathbb{r}}(S)) \le \tfrac{1}{100}c\,\mathfrak{c}_{\mathbb{r}}e^{\xi h_{\mathbb{r}}(0)} \quad \text{for all } S \in \mathcal{S}_{\delta\mathbb{r}}(\partial\mathcal{B}^\bullet_{\tau_{\mathbb{r}}}). \tag{5-25}$$

Let $\mathcal{U}_{\mathbb{r}}$ be the (finite) set of subdomains U of $B_{A\mathbb{r}}(0)$ such that $B_{A\mathbb{r}}(0)\setminus U$ is a finite union of sets of the form $S\cap B_{A\mathbb{r}}(0)$ for $S\in\mathcal{S}_{\delta\mathbb{r}}(B_{A\mathbb{r}}(0))$. For $U\in\mathcal{U}_r$, let $\mathfrak{h}^U$ be the harmonic part of $h|_U$. Also let $U_{\delta\mathbb{r}/4}$ be the set of points in U which lie at Euclidean distance at least $\delta\mathbb{r}/4$ from ∂U. Since there are only finitely many sets in $\mathcal{U}_{\mathbb{r}}$ and by the translation and scale invariance of the law of h, modulo additive constant, we can find $C = C(\delta, A, \varepsilon) > 0$ such that with probability at least $1-(1-p)/3$, it holds simultaneously for each $U\in\mathcal{U}_{\mathbb{r}}$ that

$$\sup_{u\in U_{\delta\mathbb{r}/4}} |\mathfrak{h}^U(u) - h_{\mathbb{r}}(0)| \le C. \tag{5-26}$$

For a given choice of $U \in \mathcal{U}_{\mathbb{r}}$, let $E_{\mathbb{r}}^U$ be the event that (5-24), (5-25), and (5-26) all hold, so that

$$\mathbb{P}\Big[\bigcap_{U \in \mathcal{U}_{\mathbb{r}}} E_{\mathbb{r}}^U\Big] \geq p. \tag{5-27}$$

The reason for considering $E_{\mathbb{r}}^U$ instead of $\bigcap_{U \in \mathcal{U}_{\mathbb{r}}} E_{\mathbb{r}}^U$ is the same as in Section 5.2.1: it is easier to condition on $E_{\mathbb{r}}^U$ than on $\bigcap_{U \in \mathcal{U}_{\mathbb{r}}} E_{\mathbb{r}}^U$ (see Lemma 5.17 just below).

We note that the definition of $E_{\mathbb{r}}^U$ given just above is identical to the definition of the analogous event in [18, Section 4.1], except that in [18] the condition (5-25) is replaced by an upper bound for the D_h-diameters of the squares $S \in \mathcal{S}_{\delta\mathbb{r}}(B_{A\mathbb{r}}(0))$. Such an upper bound does not hold in the supercritical case.

For $U \in \mathcal{U}_{\mathbb{r}}$, let

$$\mathcal{Z}(U) := \{\text{centers of squares } S \in \mathcal{S}_{\delta\mathbb{r}}(U) \text{ with } S \subset \overline{U}\} \tag{5-28}$$

and let $H_{\mathbb{r}}^U$ to be the event that for each $z \in \mathcal{Z}(U)$, there is a path $\Pi = \Pi_z$ in U which disconnects 0 from ∞ and hits z and which has D_h-length at most $\frac{c}{2}\mathfrak{c}_{\mathbb{r}} e^{\xi h_{\mathbb{r}}(0)}$. Note that for some choices of $U \in \mathcal{U}_{\mathbb{r}}$ and $z \in \mathcal{Z}(U)$, there is *no* path in U which disconnects 0 from ∞ and hits z. For such a choice of U we have $\mathbb{P}[H_{\mathbb{r}}^U] = 0$. The following lemma will play an analogous role to Lemma 5.10 from Section 5.2.1.

Lemma 5.17. *There is a constant $\mathfrak{p} = \mathfrak{p}(A, \varepsilon, p) > 0$ such that the following is true. Suppose $U \in \mathcal{U}_{\mathbb{r}}$ is connected and contains a path which disconnects 0 from ∞. On the event that $U \cap (\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet \cup \mathcal{B}^*) = \emptyset$, a.s.*

$$\mathbb{P}\big[H_{\mathbb{r}}^U \,\big|\, h|_{\mathbb{C}\setminus U}, E_{\mathbb{r}}^U\big] \geq \mathfrak{p}. \tag{5-29}$$

Proof. This follows from the Markov property of the GFF and the FKG inequality (Proposition 5.7), via exactly the same argument as in the proof of [18, Lemma 3.3 or Lemma 4.2]. □

Proof of Lemma 5.16. Most of the proof is exactly the same as the proof of [18, Lemma 4.1], but the geometric part of the argument is slightly different so we will repeat part of the argument to explain the differences. See Figure 10 for an illustration of the proof.

Step 1: *choosing a random domain U.* We first choose the domain U to which we will apply Lemma 5.17. The choice will depend on $\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet$ and I, which is why we need a lower bound for the probability of the intersection of all of the $E_{\mathbb{r}}^U$'s in (5-27).

Since $\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet \subset \overline{B_{\mathbb{r}}(0)}$ and $\varepsilon < (A-1)/100$, we have $\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet \cup \mathcal{B}^* \subset B_{(A+1)\mathbb{r}/2}(0)$. By hypothesis, the $d^{\mathbb{C}\setminus\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet}$-neighborhood of $\partial\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet \setminus I$ of radius $\varepsilon\mathbb{r}$ does not disconnect I from ∞ in $\mathbb{C}\setminus\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet$. Hence we can choose, in a manner depending only on $\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet$ and I, a path $\mathfrak{P}$ in $B_{A\mathbb{r}}(0) \setminus \mathcal{B}_{\tau_{\mathbb{r}}}^\bullet$ from a point of I to a point of $B_{(A+1)\mathbb{r}/2}(0)$ such that each point of $\mathfrak{P}$ lies at $d^{\mathbb{C}\setminus\mathcal{B}_{\tau_{\mathbb{R}}}^\bullet}$-distance at least $\varepsilon\mathbb{r}$ from $\partial\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet \setminus I$. By slightly perturbing $\mathfrak{P}$ if necessary, we can assume that $\mathfrak{P}$ does not hit any of the corners of any of the squares in $\mathcal{S}_{\delta\mathbb{r}}(B_{A\mathbb{r}}(0))$.

Let $\widetilde{U}$ be the interior of the union of all of the $\delta\mathbb{r} \times \delta\mathbb{r}$ squares $S \in \mathcal{S}_{\delta\mathbb{r}}(B_{A\mathbb{r}}(0))$ which intersect $\mathfrak{P} \cup \mathbb{A}_{(A+1)\mathbb{r}/2, A\mathbb{r}}(0)$ but do not intersect $\mathcal{B}_{\tau_{\mathbb{r}}}^\bullet$ or $\partial B_{A\mathbb{r}}(0)$. Let U be the connected component of $\widetilde{U}$ which intersects $\mathbb{A}_{(A+1)\mathbb{r}/2, A\mathbb{r}}(0)$. Then $U \in \mathcal{U}_{\mathbb{r}}$, as defined just above (5-26).

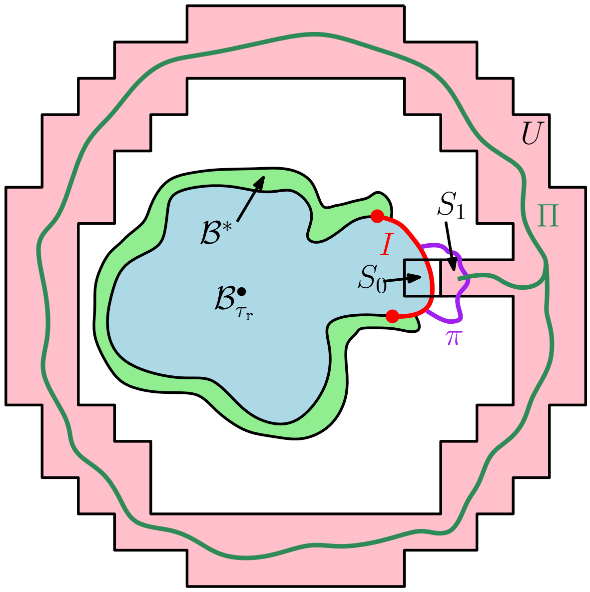

Figure 10. Illustration of the proof of Lemma 5.16. If $E^U_{\mathfrak{r}} \cap H^U_{\mathfrak{r}}$ occurs for the domain $U \in \mathcal{U}_{\mathfrak{r}}$ defined in the proof, then by the definition of $H^U_{\mathfrak{r}}$ we can find a path Π in U which contains the center of the square S_1, disconnects $\mathcal{B}^\bullet_{\tau_r} \cup \mathcal{B}^*$ from ∞, and whose D_h-length is small. Furthermore, we can find a path $\pi \subset (\mathbb{C} \setminus \mathcal{B}^\bullet_{\tau_{\mathfrak{r}}} \cup \mathcal{B}_*)$ — a segment of the path around $B_{2\delta\mathfrak{r}}(S_0) \setminus B_{\delta\mathfrak{r}}(S_0)$ given by (5-25) — which intersects both I and Π. By (5-24), the sum of the D_h-lengths of π and Π is smaller than the D_h-distance from $\mathbb{C} \setminus (\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}} \cup \mathcal{B}^*)$ to $\partial \mathcal{B}^\bullet_{\tau_{\mathfrak{r}}} \setminus I$ restricted to paths which do not enter $\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}}$. This prevents a D_h-geodesic from 0 to a point outside of $B_{A\mathfrak{r}}(0)$ from hitting $\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}} \setminus I$.

By definition, $U \cap \mathcal{B}^\bullet_{\tau_{\mathfrak{r}}} = \varnothing$. We claim that also $U \cap \mathcal{B}^* = \varnothing$. Indeed, each of the $\delta\mathfrak{r} \times \delta\mathfrak{r}$ squares S in the union defining U is contained in $\mathbb{C} \setminus \mathcal{B}^\bullet_{\tau_{\mathfrak{r}}}$ and has Euclidean diameter at most $\sqrt{2}\delta\mathfrak{r} < \varepsilon\mathfrak{r}/4$. If one of these squares intersected $\mathcal{B}^*$, then by the triangle inequality and the definition (5-22) of $d^{\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}}}$, the $d^{\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}}}$-distance from $\mathfrak{P}$ to $\partial\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}} \setminus I$ would be at most $\varepsilon\mathfrak{r}/2$, contrary to the definition of $\mathfrak{P}$.

Hence $U \cap (\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}} \cup \mathcal{B}^*) = \varnothing$. Since $\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}}$ is a local set for h (Lemma 5.1) and $\mathfrak{P}$ is determined by $(\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}}, h|_{\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}}})$, for each deterministic $\mathfrak{U} \in \mathcal{U}_{\mathfrak{r}}$, the event $\{U = \mathfrak{U}\}$ is determined by $h|_{\mathbb{C}\setminus\mathfrak{U}}$. Furthermore, by definition the set U is connected and contains a path which disconnects 0 from ∞. Therefore, the bound (5-29) of Lemma 5.17 holds a.s. for our (random) choice of U.

Step 2: *bounding conditional probabilities.* By (5-27) and Markov's inequality

$$\mathbb{P}\big[\mathbb{P}[E^U_{\mathfrak{r}} \,\big|\, \mathcal{B}^\bullet_{\tau_{\mathfrak{r}}}, h|_{\mathcal{B}^\bullet_{\tau_{\mathfrak{r}}}}] \geq 1-(1-p)^{1/2}\big] \geq 1-(1-p)^{1/2}. \tag{5-30}$$

By this together with the bound (5-29), and since p can be made arbitrarily close to 1, to conclude the proof of the lemma we only need to show that if $E^U_{\mathfrak{r}} \cap H^U_{\mathfrak{r}}$ occurs, then every D_h-geodesic from 0 to a point of $\mathbb{C} \setminus B_{A\mathfrak{r}}(0)$ passes through I. This will be accomplished via a similar argument to the proof of Lemma 5.12, as we now explain.

Step 3: *preventing D_h-geodesics from hitting $\partial\mathcal{B}^\bullet_{\tau_\mathfrak{r}} \setminus I$.* Let S_1 be the first square in $\mathcal{S}_{\delta\mathfrak{r}}(B_{A\mathfrak{r}}(0))$ hit by $\mathfrak{P}$ whose interior is contained in U. Since $\mathfrak{P}$ starts from a point of $I \subset \partial\mathcal{B}^\bullet_{\tau_\mathfrak{r}}$ and $U \cap \partial\mathcal{B}^\bullet_{\tau_\mathfrak{r}} = \varnothing$, S_1 is not the first square of $\mathcal{S}_{\delta\mathfrak{r}}(B_{A\mathfrak{r}}(0))$ hit by $\mathfrak{P}$. Hence, there is a square S_0 which is hit by $\mathfrak{P}$ prior to the first time $\mathfrak{P}$ hits S_1 such that S_0 and S_1 share a side (here we use that $\mathfrak{P}$ does not hit any of the four corners of S_1). By the definition of S_1, we have $S_0 \cap \partial\mathcal{B}^\bullet_{\tau_\mathfrak{r}} \neq \varnothing$.

By (5-25), there is a path $\widetilde{\pi}$ in the annular region $B_{\delta^{1/2}\mathfrak{r}}(S_0) \setminus B_{\delta\mathfrak{r}}(S_0)$ which disconnects the inner and outer boundaries of this annular region and has D_h-length at most $\frac{1}{100} c\,\mathfrak{c}_\mathfrak{r} e^{\xi h_\mathfrak{r}(0)}$. Since $S_1 \subset B_{\delta^{1/2}\mathfrak{r}}(S_0) \setminus B_{\delta\mathfrak{r}}(S_0)$ and S_1 is disjoint from $\mathcal{B}^\bullet_{\tau_\mathfrak{r}}$, there is a subpath π of $\widetilde{\pi}$ which is contained in $\overline{\mathbb{C} \setminus \mathcal{B}^\bullet_{\tau_\mathfrak{r}}}$ and which disconnects S_1 from ∞ in $\overline{\mathbb{C} \setminus \mathcal{B}^\bullet_{\tau_\mathfrak{r}}}$. Since S_0 intersects $\pi \cap I$, which is disjoint from $\mathcal{B}_*$, and $\delta^{1/2} \leq \varepsilon/10$, the definition (5-22) of $d^{\mathbb{C}\setminus\mathcal{B}^\bullet_{\tau_\mathfrak{r}}}$ shows that the path π cannot intersect $\partial\mathcal{B}^\bullet_{\tau_\mathfrak{r}} \setminus I$. Hence π must intersect I.

By the definition of $H^U_\mathfrak{r}$ (just below (5-28)), there is a path Π in U which contains a point of S_1 and which disconnects 0 from ∞ (and hence also $\mathcal{B}^\bullet_{\tau_\mathfrak{r}}$ from $\mathbb{C} \setminus B_{A\mathfrak{r}}(0)$). The union of π and Π is connected, has D_h-length strictly less than $c\mathfrak{c}_\mathfrak{r} e^{\xi h_\mathfrak{r}(0)}$, intersects I, and disconnects $\mathcal{B}^\bullet_{\tau_\mathfrak{r}}$ from $\mathbb{C} \setminus B_{A\mathfrak{r}}(0)$.

Any path P from a point of $B_{A\mathfrak{r}}(0)$ to 0 which first hits $\partial\mathcal{B}^\bullet_{\tau_\mathfrak{r}}$ at a point not in I must hit Π and then must subsequently cross from a point of $\mathbb{C} \setminus (\mathcal{B}_* \cup \mathcal{B}^\bullet_{\tau_\mathfrak{r}})$ to $\partial\mathcal{B}^\bullet_{\tau_\mathfrak{r}} \setminus I$. By the preceding paragraph, the D_h-distance from the first point of Π hit by P to 0 is strictly smaller than $\tau_\mathfrak{r} + c\mathfrak{c}_\mathfrak{r} e^{\xi h_\mathfrak{r}(0)}$. On the other hand, (5-24) shows that the D_h-length of the segment of P which crosses from $\mathbb{C} \setminus (\mathcal{B}_* \cup \mathcal{B}^\bullet_{\tau_\mathfrak{r}})$ to $\partial\mathcal{B}^\bullet_{\tau_\mathfrak{r}} \setminus I$ is at least $c\mathfrak{c}_\mathfrak{r} e^{\xi h_\mathfrak{r}(0)}$, so the D_h-length of the segment of P after it first hits U is at least $\tau_\mathfrak{r} + c\mathfrak{c}_\mathfrak{r} e^{\xi h_\mathfrak{r}(0)}$. Therefore, P cannot be a D_h-geodesic. □

Now that Lemma 5.16 is established, we can conclude the proof of Theorem 1.6 in exactly the same way as in the subcritical case (see [18, Section 4.2]). The proof of [18, Lemma 4.3] requires some containment relations between filled D_h-metric balls and Euclidean balls, but these are easily supplied by Lemma 3.5 and Proposition 2.11.

Acknowledgments

We thank the two referees for helpful comments on an earlier version of this article. We thank Jason Miller and Josh Pfeffer for helpful discussions. Ding was partially supported by NSF grants DMS-1757479 and DMS-1953848. Gwynne was partially supported by a Clay research fellowship.

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Received 2 Oct 2022. Revised 26 Jul 2023. Accepted 2 Sep 2023.

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Probability and Mathematical Physics (ISSN 2690-1005 electronic, 2690-0998 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online.

PMP peer review and production are managed by EditFlow® from MSP.

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