LINKING STRUCTURES ACROSS LOGIC AND SPACE: THE ROLE OF EULER DIAGRAMS

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When a parent says to their child, "If you finish your homework, then you can play video games", they make a statement of the form 'if p, then q'. Statements of this kind are often called logical implications (LIs, or conditional statements), where p and q represent the hypothesis and conclusion of the LI, respectively. However, the parent likely intends for this statement to be interpreted *bidirectionally* (p if and only if q); otherwise, they allow for the possibility that their child does not finish their homework but plays video games anyway. Such presumed bidirectionality of LIs, commonly found among college mathematics students, may be a consequence of assumptions people make within the underlying language (say, English) in which we embed logic.

To help students sort out logical statements, instructors of introduction-to-proofs courses sometimes encourage students to rely upon spatial representations, particularly Euler diagrams. The *truth set* of a statement is the set of elements of the universal set, U, that make the statement true; an *Euler diagram* uses topological relationships to represent logical relationships between truth sets associated with an LI's hypothesis and conclusion. Conventionally, the truth set of the hypothesis is depicted as contained within the conclusion's truth set (*i.e.*, $p \rightarrow q$ corresponds to $P \subseteq Q$), with both contained within U (see Figure 1).

The purpose of this article is to introduce and exemplify *logico-spatial linked structuring* (LSLS) as a theoretical perspective on students' linking between spatial and non-spatial representations when dealing with logical statements. Although we conceptualize LSLS broadly, this article focuses on the specific case of Euler diagrams as spatial representations of LIs. We argue, in line with Dawkins and Roh (2022), that Euler diagrams can be a constructive resource for students, particularly as a medium through which to visualize logical relationships and subsequently carry out mental actions to establish the validity of given statements. Euler diagrams may be especially useful to students who treat LIs as actions but not yet objects (see our Theoretical Perspective section). However, as our examples illustrate, reasoning via Euler diagrams can be challenging. Because Euler diagrams are polysemous, there is rich activity required to reason with them in ways productive for logic.

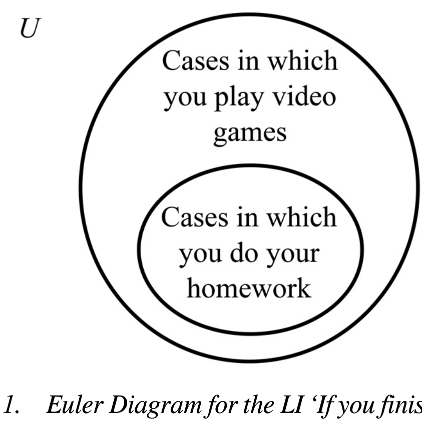

Figure 1. Euler Diagram for the LI 'If you finish your homework, then you can play video games'.

We therefore begin with a discussion of potential affordances and challenges that students may experience when using Euler diagrams. We then discuss the broader theoretical perspective in which LSLS is grounded, followed by an introduction to LSLS itself. Finally, we illustrate the utility of LSLS by drawing on examples of student reasoning.

Affordances and challenges of Euler diagrams

When reasoning with an LI, students can represent the implication using any of the representational tools at their disposal. Dawkins has promoted an instructional approach in which students use familiar mathematical contexts to make sense of the hypothesis, p, and conclusion, q, and relationships between them (Dawkins & Norton, 2022; Dawkins & Roh, 2022). Students make sense of LIs by *populating* truth sets P and Q, associated with p and q (respectively), with familiar examples or with 'carriers' (objects acting as arbitrary elements of truth sets). Their approach suggests Euler diagrams can help students not only represent LIs, but also reason about the corresponding converse, inverse, and contrapositive statements (Hub & Dawkins, 2018). Our work builds on that of Dawkins and colleagues by exemplifying and characterizing potential challenges that can emerge in these spatial representations—challenges that have not yet been problematized in the literature—and by identifying spatial mental actions that students use when reasoning with Euler diagrams.

One affordance of Euler diagrams is their potential to help students identify discrepancies between a statement's intended meaning and its logical interpretation. For instance, in Figure 1's representation of the video-game LI, the existence of space inside the outer region, but outside of the inner region, suggests a case in which one could play video

For the Learning of Mathematics **44**, 2 (July, 2024)
FLM Publishing Association, New Westminster, BC, Canada

games but *not* do their homework (in contrast to the statement's presumed meaning).

One challenge identified in our work, and illustrated later in the article, comes from how students might link the statement 'if *p*, then *q*'—in particular, 'if you *have p*, then you *have q*'—to a spatial representation. This statement could be reasonably interpreted as suggesting that *p* and *q* are objects that one can 'have'—and moreover, that statement *p* (or space *P*) includes, or carries with it, statement *q* (or space *Q*). Spatially, then, it makes sense that a student might think about *Q* as contained within *P*.

Additionally, an important consideration when dealing with LIs is the universal set from which the truth set of the hypothesis (*P*) is populated. Conventionally, the universe is represented by a rectangle drawn around all other components of the Euler diagram. In the video-game example, the universal set might be the set of all *possible cases* in which one does their homework, plays video games, neither, or both. As mathematics educators, however, we should call into question potential challenges of mathematical conventions. In particular, why represent the universal set as a rectangle—or, for that matter, represent Euler diagrams in the plane? Of course, there is a certain practicality to the latter; most human experience in writing/drawing occurs on a flat surface. However, planar representations are perhaps not the best at illustrating the logical equivalence between an LI and its contrapositive.

C. S. Peirce (2020/1903) was aware of this planar issue, and he suggested an alternative: "But there is [...] no particular appropriateness in drawing the diagrams on a plane surface rather than on a sphere" (p. 234). Indeed, as illustrated in Figure 2, an Euler diagram showing $P \subseteq Q$ on a sphere might better represent the contrapositive than the equivalent diagram in the plane. The boundaries of *P* and *Q* may be stretched (i.e., those regions may be enlarged) over the surface of the sphere (while maintaining $P \subseteq Q$). Then, rotating the sphere on its axis so that the opposite side is visible, the boundary of *P* is shown to be exterior to the boundary of *Q*, which may make the relationship $Q^c \subseteq P^c$, and thus the implication $\sim q \rightarrow \sim p$, more visually salient.

Euler diagrams are further complicated by the issue of quantification. Even within symbolic representations, quantifiers are often hidden to students (Shipman, 2016). Euler diagrams may exacerbate this challenge in that they represent only *qualitative* relationships between sets. Conventionally, the size of an enclosed space within an Euler diagram is not correlated with the cardinality of the associated truth set; in fact, an enclosed space could represent an *empty* set (*e.g.,* when $p \rightarrow q$ is vacuously true, or when *p* iff *q* is true; see Figure 3). This is consistent with Dawkins's (2019) claim that "non-quantified logic presents many pragmatic barriers to students' learning of mathematical logic" (p. 20).

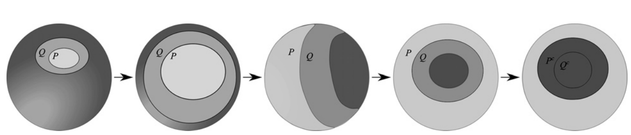

Figure 2. Transforming an LI into its contrapositive on a sphere.

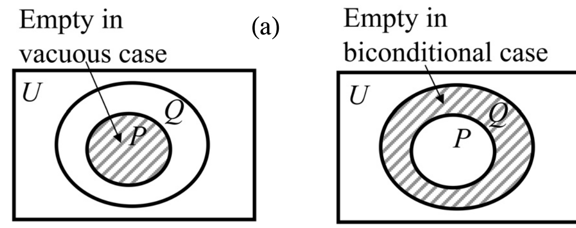

Figure 3. Potentially empty sets in Euler diagrams.

Theoretical Perspective

Our research is theoretically grounded in Piaget's (1970) genetic epistemology wherein logico-mathematical objects arise as products of psychology; they are constructed by coordinating mental actions within structures for reversing and composing them. Within our Piagetian framing, we understand challenges in reasoning with LIs as consequences of students' ways of operating, and in particular, their means for transforming and quantifying LIs.

Following his own Piagetian approach, action-process-object-schema (APOS) theory, Dubinsky (1991) made a distinction between operating with LIs as *actions* and operating on them as *objects*. As actions, LIs consist of three parts: a hypothesis (*p*), a conclusion (*q*), and the implication connecting them. Treating LIs as actions, a student can reason that, if *p* is given, then *q* follows (*modus ponens*). However, they would be unlikely to conclude that if *q* is false, then *p* is false (*modus tollens*, which relies on the contrapositive). Indeed, prior research indicates the latter form of reasoning is much more challenging (*e.g.,* Yopp, 2020). As objects, LIs can be taken as wholes, be quantified, and be transformed into their converse, contrapositive, and negation. Thus, students who treat LIs as objects can act on them in ways that would be quite challenging for students who treat LIs as actions. We argue that Euler diagrams can serve as a tool, especially for students who treat LIs as actions, for visualizing relationships between truth sets and applying spatial actions to carry out corresponding logical actions.

In other mathematical domains, teacher-researchers have supported students' constructions of mathematical objects by engaging them in tasks that involve coordinating relevant mental actions and reflecting on that activity (Wheatley, 1992). Likewise, we might support students' constructions of LIs as objects by engaging students in tasks that involve coordinating and reflecting on relevant mental actions. Dawkins and Roh (2022) offer such an approach. Dawkins and Norton (2022) have specified related mental actions such as *populating* sets and *negating* (the act of "constructing an opposite property or predicate to a given one", p. 5).

In interpreting or creating an Euler diagram, how one mentally organizes and interprets its constituent *spatial* components determines the logico-mathematical meaning that they read from the diagram. Conversely, one's initial assimilation of a logical statement, such as an LI, influences one's structuring of the corresponding Euler diagram. *Spatial structuring*, the act of constructing an organization or form for one or more spatial objects (Battista, Frazee & Winer 2018), is a critical part of this process. Spatial objects, including Euler diagrams, must be mentally constructed through an active process of spatial organizing and interrelating. In its most basic form, an Euler diagram contains one or more simple closed loops, each of which partitions space into two regions (an inside and an outside). To make meaning of an Euler diagram, students need to engage in mental operations by which they organize its constituent parts and establish logical meaning for those parts in a coherent mental model. This establishment of logical meaning, and the corresponding linking between spatial and non-spatial representations, constitutes the major impetus of our proposed LSLS framework.

Logico-Spatial Linked Structuring

The main contribution of this theoretical article is the introduction of the LSLS framework. LSLS rests upon the hypothesis that the use of Euler diagrams to reason about LIs involves a conceptual linking between two distinct representational forms: logical statements (which may be abstract/symbolic or contextualized) and spatial representations. For us, *logical structuring* is a term that encompasses any act of organizing, interrelating, or transforming structures involved in logical statements, including quantifiers (*e.g.,* 'for all x in the set of squares') and open statements (*e.g.,* 'x has the properties of a rhombus'). *Spatial structuring* is defined above.

We define LSLS to be conceptual *linking* between logical structuring and spatial structuring. That is, LSLS entails establishing logical meaning for a spatial representation, building a spatial representation for one or more logical statements, and ideally being able to flexibly switch between representational forms. For us, linking two representational forms means not just association, but also coherence between the two representations, according to a student's conceptions of how to structure objects within each mode of representation.

LSLS is an extension of a related construct from research on students' geometric measurement reasoning. Battista, Frazee, and Winer (2018) introduced *spatial-numerical linked structuring* as a form of reasoning that links spatial structuring to numerical structuring (the latter being the mental process of organizing a set of numerical or algebraic symbols). For example, they report on a Grade 6 student's reasoning when asked to find the area of the polygon in Figure 4a. The student spatially structured the polygon by decomposing it into two right triangles and a rectangle (Figure 4b); their numerical structuring, $(5 \times 3/2) + (5 \times 4) + (3 \times 4/2)$, was linked to their spatial structuring. In our analysis of student reasoning about LIs and Euler diagrams, we found a similar linking in students' reasoning—not between spatial and numerical structures, but between spatial and *logical* structures.

LSLS helps us understand how students establish logico-spatial meaning, but it also helps us understand why Euler diagrams may be helpful tools for many students. Following Dubinsky (1991), we hypothesize that students with an object conception of LI might directly transform an implication symbolically (*e.g.,* using a normative system of symbolic logic) and be able to make sense of that transformation, without necessarily appealing to spatial representations (Figure 5, top arrow). However, students with an *action* conception of LI may need to work out the meaning and structure of the transformation (*e.g.,* negation), possibly by appealing to Euler diagrams. We argue that LSLS provides researchers with a way to describe and explain students' shifts between, and links across, representations.

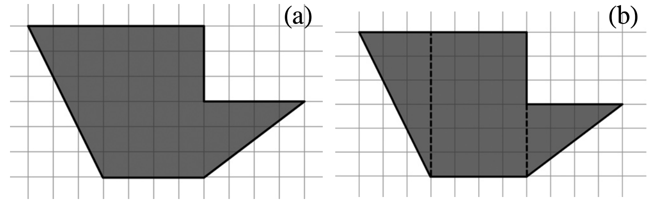

Figure 4. Example of area-measurement SNLS.

When students shift from the realm of logical statements to spatial representations, they engage in what we call a *logico-spatial mapping* (left arrow). Logico-spatial mappings occur when students construct an initial spatial structuring using their understanding of how Euler diagrams are constructed. Importantly, however, this spatial structuring is also linked to their understanding of the logical situation under consideration—hence, *logico-spatial* mapping. Once students shift to the domain of spatial representations, their focus shifts to spatial operations for interpreting and, if necessary, transforming spatial information (bottom arrow). Our examples of student reasoning suggest four specific spatial operations (see Figure 5), defined below.

- Spatial locating: drawing or imagining a point, boundary, or region
- Engulfing: taking a larger space as possessing all of the characteristics (*e.g.,* properties or elements) of a space that it contains
- Boundary crossing: coordinating spatial locating and negating, focusing on a point
- Spatial inversion: coordinating spatial locating and negating, focusing on a region

Lastly, students use their spatial representations and reasoning to make a logical conclusion, a transition that we characterize as *spatial-logical mapping* (right arrow).

Examples of LSLS in student reasoning

Our work focuses on the experiences and reasoning of undergraduate students enrolled in an introduction-to-proofs course—students with no prior experience of advanced,

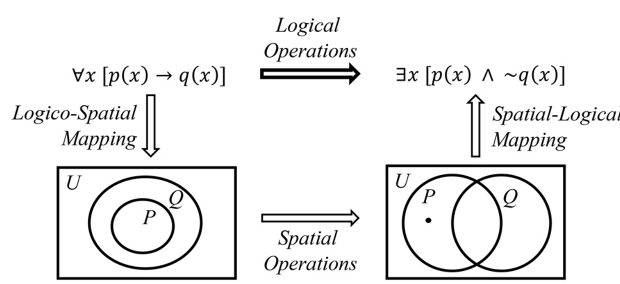

Figure 5. LSLS for transforming an LI into its negation

proofs-based mathematics. The excerpts provided come from clinical interviews that occurred during the third week of the semester with two students, Carmen and Kai. Prior to these interviews, the class had focused discussions around the topics of mathematical statements, quantification, and LIs, including representing statements and LIs using Euler diagrams. These examples are intended to illustrate (1) differences in how Carmen and Kai spatially represented LIs using Euler diagrams, (2) the spatial operations that each student performed in their problem-solving, and (3) the interaction between their spatial and logical reasoning.

To prompt students to attend to and reason about the logical structure itself without relying on domain-specific mathematical knowledge, our clinical interview tasks purposefully utilized some contexts unfamiliar to students (*e.g.,* algebraic topology). Our examples draw on two students' responses to the task below.

Suppose that the following statement is true: *If two topological spaces are homeomorphic, then their homology groups are isomorphic.* Based on this fact alone, decide if the following statements are true, false, or uncertain.

a. If two topological spaces have isomorphic homology groups, then the spaces are homeomorphic. [Converse]

b. If the homology groups of two topological spaces are not all isomorphic, then the spaces are not homeomorphic. [Contrapositive]

c. There is a pair of homeomorphic topological spaces whose homology groups are not all isomorphic. [Negation]

As a point of comparison, conventionally, how might a student solve this task using Euler diagrams? With the given LI, they would likely draw a diagram depicting boundary P within another boundary Q. They might also consider other Euler diagrams satisfying the LI—*e.g.,* a single boundary labeled P and Q (biconditional case), or only labeled Q (vacuous case where P is empty). Regarding statement (a), then, they might use spatial locating to identify the regions associated with the premise (Q) and the conclusion (P) and notice that points within the former could, but *need not*, be contained in the latter (likely envisioning boundary crossing)—concluding an uncertain truth value. For (b), they might engage in spatial inversion to visualize the complement of each truth set, then engage in spatial locating and boundary crossing to reason that since $P \subseteq Q$, any point in Q^c is also in P^c, concluding (b) is true. For (c), they might spatially locate a point contained within P, think about the spatial analog to the statement 'whose homology groups are not all isomorphic' (Q^c), and reason that since $P \subseteq Q$, (c) must be false.

Example 1

Carmen was a senior aerospace engineering student minoring in mathematics. Carmen's interviews taught us important lessons about how students might reason about Euler diagrams—in particular, the conception of one set 'engulfing' another, which we see in her response to statement (b), the contrapositive. She correctly concluded (b) would be true, without appealing to Euler diagrams, and her LSLS (coupled with presumed bidirectionality) gave her further confidence in her conclusion.

Carmen We're saying that the two spaces are *not* isomorphic. So that's kind of the negative of the part from *[the original statement]*. So, then we can say, in the second part, it would have to negate the second part *[of the original]* as well. So, if we have like p and q, we would have to negate both of these to get not p and not q.

The interviewer asked Carmen whether she could represent her reasoning using an Euler diagram. She drew a small circle labeled P, and a larger circle, Q, containing P.

Carmen So, if I have a circle within a circle. So, this is my circle P, and this is my circle Q. *[…]* Maybe I should have flipped it around, to go with that *[reverses the labeling of the two circles, so that $p \rightarrow q$ was represented as $P \supseteq Q$; see Figure 6a]*. So, if p, then q. The P engulfs the entire space of Q, so that if I have the statement p, then I'm definitely going to have the statement q. *[…]* But if I don't have p *[draws a rectangle around the Euler diagram to represent the universe]*, I'm outside of this circle in this region *[points to the region outside of the larger circle],* um, then I definitely do not have q.

Using the LSLS framework, we infer that Carmen completed a logico-spatial mapping directly, without first engaging in the activity of populating sets (which is understandable since these are unfamiliar mathematical objects). She seemed to interpret circles P and Q as representing *statements* p and q rather than truth sets associated with those statements. Working within an Euler diagram, Carmen seemed uncertain about how to spatially structure circles P and Q to represent the logical relationship $p \rightarrow q$. We hypothesize from Carmen's statements "P engulfs […] Q" and "If I have the statement p, then I'm definitely going to have the statement q" that she interpreted the space P (and statement p) as containing, or possibly carrying, the space Q. We infer Carmen *spatially inverted* P by *negating* p and *locating* the region outside of P ("then I'm outside of this circle in this region"; see Figure 6b), which necessarily

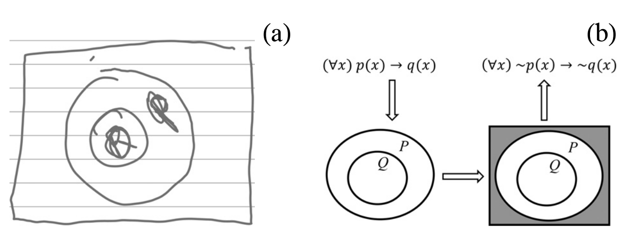

Figure 6. (a) Carmen's Euler diagram representation and (b) LSLS diagram of Carmen's logico-spatial reasoning about statement (b).

implies moving outside of Q ("then I definitely do not have q"). From her spatial operations, she thus concluded statement (b) is true—a reasonable conclusion, given her spatial structuring of P "engulfing" the space Q.

Example 2

Kai was a sophomore computational mathematics major. In contrast to Carmen whose initial focus was on *statements p* and q, Kai's spatial structuring seemed to refer to their corresponding *truth sets*. He handled this spatial structuring by considering two cases, $P \subset Q$ and $P = Q$, correctly determining that the truth value of statement (a) is "uncertain".

Kai — It could be true, if these two statements were equivalent. So, if they were the same space, like truth set, then the converse would hold for both implications. But, because I don't know what it is, um, it's uncertain. *[…]* In this case *[the latter]*, it would be true, and in this case, it would be false *[draws Euler diagrams shown in Figure 7]*. But because both of these are possible, it's uncertain.

As illustrated in Figure 8, Kai seemed to engage in LSLS by constructing a mapping from the logical form of statement (a) to two potential Euler diagrams. Given that $p \rightarrow q$, he knew that either $P \subset Q$ or $P = Q$. We infer that, in both cases, Kai's attention was on the boundary of P in relation to the boundary of Q. When $P = Q$, Kai recognized, through an act of *boundary crossing*, that the boundary of P coincides with the boundary of Q (shown in Figure 8a), leaving no space for a point x outside of P but within Q (making the set of counterexamples to $q \rightarrow p$ empty: "in this case, it would be true"). When $P \subset Q$, Kai structured P as being *engulfed* (to use Carmen's language) within Q. He assumed that the nonempty region between P's boundary and Q's boundary implied the existence of a nonempty set of counterexamples (suggesting *spatial inversion*), and thus $q \rightarrow p$ would be false (see lower right-hand diagram in Figure 8b). Thus, Kai correctly concluded that statement (a) is "uncertain", making a logical conclusion from his spatial reasoning.

Later in the interview, Kai was asked a follow-up question regarding statement (a). Kai seemed to assimilate the question as asking, "Is there any way to form an Euler diagram that shows $p \rightarrow q$ and that shows $q \rightarrow p$, without the circles P and Q entirely coinciding?"

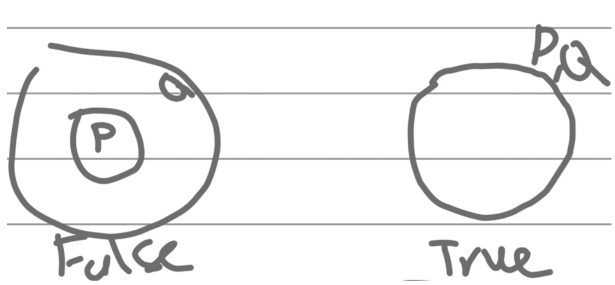

Figure 7. Kai's Euler diagrams in reasoning about statement (a).

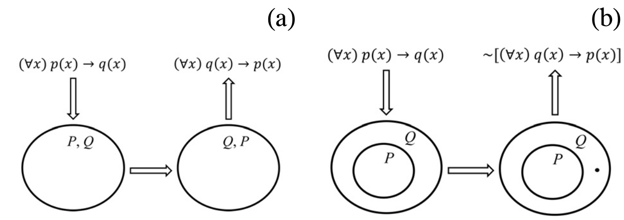

Figure 8. LSLS diagrams illustrating Kai's logico-spatial reasoning.

Researcher — Could it be the case that there's a pair that makes this true, and yet P and Q aren't the same?

Kai — Right now, I'm thinking 'a pair of topological homeomorphic' *[statement p]* is like the inner circle, and the outer circle is 'their homology groups are isomorphic' *[statement q]*. *[…]* So, the other condition would be if P is outside, in any way, it could be like P and then Q on the inside *[draws Figure 9a]*. And P and Q overlap *[draws Figure 9b]*. Or even P and Q, like, no relationship *[draws Figure 9c]*, because we know if two of them are homeomorphic, then it has to be isomorphic. So, there's no way that they don't have a relationship *[rules out Figure 9c]*. Oh! Can it be on the outside? Because it could, if the two spaces are homeomorphic *[pause]* then their homology groups are isomorphic. But it could also be like other things. *[…]* Well, it could be other conditions of the homology groups. But isomorphic would always be true. So, this would be, on the outside here, it would be, in this circle, it would be that the topological spaces are homeomorphic and their homology groups are not isomorphic. Which is not possible, because this *[original statement]* is true. So, it cannot be bigger either *[rules out Figure 9a]*. And, with regard to the, this one, which is the intersection. It's also outside. So, anything—Yeah, yeah. This cannot be true either *[rules out Figure 9b]* because, once again, it cannot be homeomorphic and non-isomorphic homology groups. So, I would rule that out as well.

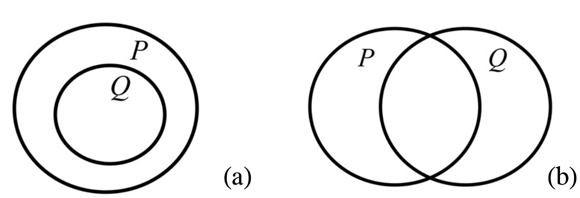

Figure 9. Recreation of Kai's Euler diagrams

Having ruled out these three possibilities, Kai felt confident that, if $p \rightarrow q$ is true, then there are only two spatial possibilities: P is contained properly within Q, or P and Q coincide exactly.

Kai seemed initially unsure whether the relationship $p \rightarrow q$ could be represented spatially where part of P is exterior to Q. Similar to the previous excerpt, in his initial logico-spatial mapping, Kai engaged in case-based reasoning (see Figure 9): "it could be like P and then Q on the inside", "P and Q overlap", or "P and Q, like, no relationship". He then switched between logical reasoning and his spatial structuring to rule out all alternative possibilities.

When reasoning about Figure 9a and Figure 9b, Kai seemed to engage in a spatial locating action. Consistent with his reasoning in the previous excerpt, he seemed to interpret closed regions as necessarily indicating the existence of at least one point. For instance, when reasoning about Figure 9a, he said, "It would be that the topological spaces are homeomorphic and their homology groups are not isomorphic", which he used to rule out this diagram. When reasoning about Figure 9b, he said, "This cannot be true either. Because, once again, it cannot be homeomorphic and non-isomorphic homology groups". He seemed to assume the existence of a point inside P but outside of Q. Kai's overall LSLS is depicted in Figure 10: establish a spatial relationship given $p \rightarrow q$, reason about whether P can be exterior to Q in any way (by engaging in *point-locating*), interpret this spatial relationship logically, and conclude that this logical statement is in contradiction with the assumed LI.

Discussion

As Hub and Dawkins (2018) and Dawkins and Roh (2022) have suggested, Euler diagrams can be a useful tool for supporting student reasoning about LIs. We hypothesize they may be especially useful for students who treat LIs as actions, as they provide students with a medium through which logical transformations may be spatially visualized. The LSLS framework provides researchers with a way to capture how students link spatial and non-spatial representations in reasoning about logical statements, with this article focusing specifically on Euler diagrams and LIs. This work fits within the research literature by characterizing and illustrating potential challenges that may emerge in students' transitions to spatial representations, including the assumptions they might make and the mental actions they might use—contributing to Dawkins and Norton's (2022) prior research on students' mental actions when reasoning about LIs.

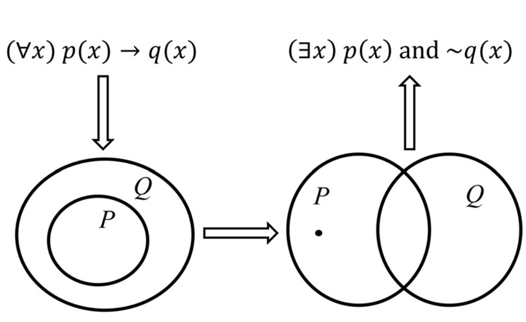

Figure 10. Kai's transformation of an LI into its negation via LSLS.

The examples presented teach us that students, when given freedom to explore their own ways of making sense of Euler diagrams, employ different spatial structurings linked to their logical interpretations of LIs. Both Carmen and Kai engaged in acts of spatial locating, engulfing, boundary crossing, and spatial inversion; however, there were important differences in their meanings of these actions. For the act of engulfing in particular, Carmen interpreted $p \rightarrow q$ as saying, "if I have the statement p, then I have the statement q", which she represented spatially by engulfing Q within P. For her, the truth of p necessitates the truth of q, and so the space P carries with it the space Q. Kai's acts of engulfing, by contrast, seemed to focus more on *points within regions* rather than regions as wholes. He seemed to reason that if point x lies within P, then it must also lie within Q, meaning Q engulfs P. Thus, Carmen and Kai established qualitatively distinct spatial structurings of Euler diagrams. Carmen seemed to conceptualize P and Q as singular objects, whereas Kai seemed to conceptualize P and Q as collections of points where statements p and q are true, respectively.

Furthermore, as seen in Euler diagrams drawn by both Carmen and Kai, students may not always (or not immediately) include a representation of the universe in their Euler diagrams. Carmen added a rectangle to represent the universe in her Euler diagram in Figure 6a only after she engaged in spatial inversion to visualize the set Q^c. Likewise, Kai did not represent the universe in any of his three Euler diagrams in Figures 7 or 9. Future research might investigate how students engage in Euler diagrams drawn on the sphere, as discussed in the 'Affordances and challenges' section. We hypothesize that, with the universe represented as a sphere, by stretching boundaries of P and Q to the back of the sphere, the spatial inversion from $P \subseteq Q$ to $Q^c \subseteq P^c$ may be more visually salient for students—though this representation would likely introduce new spatial operations, like rotating and stretching.

Additionally, toward identifying potential challenges associated with Euler diagrams, we aimed to examine what students might assume about LIs and their transformations from the underlying *space* in which Euler diagrams are constructed—especially with regard to quantification. The main assumption from the examples considered in this article was the following: If an Euler diagram contains a closed region, then that region represents a nonempty set. This was most apparent in Kai's reasoning that $p \rightarrow q$ can be represented spatially in two different ways, shown in Figure 8. Kai assumed that, since the diagram in Figure 8b shows a closed region between the boundaries of P and Q, that region must indicate the existence of at least one point.

Prior work has documented students' tendency to interpret LIs as biconditional (*i.e.,* assuming its converse is also true). Some of this research attributes the assumption to colloquial uses of LI (*e.g.,* Epp, 2003), such as our opening video-game example. We might refer to such examples as logical assumptions inherited from the underlying language. Representing LIs and their transformations with Euler diagrams introduces the possibility of similar assumptions, drawn from the underlying space in which the diagrams are embedded. In Kai's example, we see an assumption about quantification (existence) drawn from that underlying space.

In summary, although Euler diagrams provide students with a tool for representing LIs, and a visual medium for working out the logical validity of statements given certain assumptions, Euler diagrams may introduce new challenges. To reason about Euler diagrams as spatial representations, students need to engage in spatial reasoning and operations, and—critically—they need to link these meanings and representations to logical statements. Through LSLS, researchers have a way of theoretically framing students' logical-spatial meaning-making and students' links between spatial and non-spatial representations.

Acknowledgments

We thank Paul Dawkins and our anonymous reviewers for their comments and suggestions. This work was supported by the National Science Foundation under Grant DUE-2141626.

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How human is mathematics? One issue that keeps surfacing in this discussion is the different value systems, the different "aesthetics", of the two communities: math educators and professional mathematicians. At the level of communities (neglecting small minorities within each community), these value systems are best studied not by what people say, but by how they behave professionally. Consider for example the expected writing style in research journals or the procedures for academic promotion. At this level, the two value systems are not only different—they are almost orthogonal. For in education we are always stressing the human side of mathematics, whereas official pure mathematics has always evolved in the direction of more objectivity, formality, "purity"—in short, away from the human, personal perspective. Building bridges between these two cultures seems to me a task that is both important and difficult. I believe topics like the relationship of proof to mental images and "generic" objects (or generic arguments), and the gradual refinement of intuitive argument to formal proof, are good starting points for such bridges. Unfortunately, because of the atrocities performed in the name of mathematical formalism on people of all ages, math educators have largely tended to avoid this topic. I hope we see more "human approach to formalism" in the future.

— Uri Leron (1986) Reply to a letter from Dick Tahta, **6**(2), 41.

Editor's Note: Uri Leron died in late 2023. His contributions to mathematics education and to the *FLM* community were immense. The conversations I had with him early in my career fundamentally changed how I thought about my research on proof.
