BPS INVARIANTS OF SYMPLECTIC LOG CALABI-YAU FOURFOLDS

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ABSTRACT. Using the Fredholm setup of Farajzadeh-Tehrani [Peking Math. J. (2023), https://doi.org/10.1007/s42543-023-00069-1], we study genus zero (and higher) relative Gromov-Witten invariants with maximum tangency of symplectic log Calabi-Yau fourfolds. In particular, we give a short proof of Gross [Duke Math. J. 153 (2010), pp. 297–362, Cnj. 6.2] that expresses these invariants in terms of certain integral invariants by considering generic almost complex structures to obtain a geometric count. We also revisit the localization calculation of the multiple-cover contributions in Gross [Prp. 6.1] and recalculate a few terms differently to provide more details and illustrate the computation of deformation/obstruction spaces for maps that have components in a destabilizing (or rubber) component of the target. Finally, we study a higher genus version of these invariants and explain a decomposition of genus one invariants into different contributions.

CONTENTS

1. INTRODUCTION

1.1. **Notation and setup.** Suppose (X, ω) is a smooth closed symplectic manifold, $D \subset X$ is a smooth symplectic divisor, and J is an ω-tame almost complex structure on X preserving TD (i.e., $JTD = TD$). Then, for every genus g J-holomorphic map $u \colon \Sigma \longrightarrow X$ with a smooth domain that is not mapped into D and represents the homology class $A \in H_2(X, \mathbb{Z})$, there is a finite (possibly empty) set of positive integers

$$\mathfrak{s} \equiv (s_1, \ldots, s_k) \in \mathbb{Z}_+^k, \qquad \sum_{i=1}^k s_i = A \cdot D \geq 0,$$

such that $u^{-1}(D) = \{z_1, \ldots, z_k\} \subset \Sigma$ and u has a well-defined tangency of order s_a at z_a with D as in the holomorphic case. We usually require z_a to be (ordered) marked points on Σ. Furthermore, we may consider extra classical marked points

z_a on Σ and define $s_a = 0$ for them. Then, $u^{-1}(D) \subset \{z_1, \ldots, z_k\}$ and $\mathfrak{s} \in \mathbb{N}^k$. This way, we do not need to distinguish between the classical and contact marked points as each non-negative integer s_a determines the type of the marked point z_a. Two marked J-holomorphic maps

$$\big(u, C \equiv (\Sigma, z_1, \ldots, z_k))\big) \quad \text{and} \quad \big(u', C' \equiv (\Sigma', z'_1, \ldots, z'_k)\big)$$

are equivalent if there exists a holomorphic identification $h \colon C \xrightarrow{\cong} C'$ of the marked domains C and C' such that $u = u' \circ h$. Following the terminology of [12], we denote the space of the equivalence classes of k-marked genus g J-holomorphic maps of contact type $\mathfrak{s}$ with D by $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$. *Note that $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ is simply the virtually-main open stratum of a suitable compactification that will be described in Section* 3.2. By [13, (1.7)], the real expected dimension of $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ is

$$(1.1) \qquad 2\Big(\langle c_1(TX(-\log D)), A \rangle + (1-g)(\dim_{\mathbb{C}} X - 3) + k \Big),$$

where $TX(-\log D)$ is the logarithmic tangent bundle constructed in [15]. In Section 2, we will review the construction and properties of the logarithmic tangent bundle $TX(-\log D)$.

This paper concerns the case where $c_1(TX(-\log D)) = 0$, $\dim_{\mathbb{C}} X = 2$, and $k = 1$. Consequently, by (1.1), $\dim_{\mathbb{C}} \mathcal{M}_{g,\mathfrak{s}}(X, D, A) = g$. The focal point of this paper, as articulated in Theorems 1.5 and 1.7, concerns genus zero curves where $\dim_{\mathbb{C}} \mathcal{M}_{0,\mathfrak{s}}(X, D, A) = 0$. Notably, integer invariants are derived by enumerating the elements of $\mathcal{M}_{0,\mathfrak{s}}(X, D, A)$ where $\mathfrak{s} = (s_1 = A \cdot D)$.

Remark 1.1. For any finite set S, with $\mathfrak{s} \in (\mathbb{N}^S)^k$ instead of $\mathbb{N}^k$, the definition of the moduli space $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ above and the formula (1.1) naturally extend to the case where $D = \bigcup_{i \in S} D_i$ is a simple normal crossings symplectic divisor in the sense of [14, Dfn. 2]; see [12]. However, the classical relative compactification does not generalize easily. There are different approaches in the algebraic and symplectic categories to construct a well-behaved compact moduli space; for instance see [8,9, 12,25,28,36,39].

Definition 1.2. We say

- $\big[u, \Sigma, (z_1, \ldots, z_k)\big] \in \mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ has **maximal tangency** with D if

$$s_1 = s_A \equiv A \cdot D > 0 \qquad (\text{and } s_i = 0 \quad \forall i > 1);$$

- $((X, \omega), D)$ is a **symplectic log Calabi-Yau pair** if

$$(1.2) \qquad c_1(TX(-\log D)) = c_1(TX) - \mathrm{PD}_X(D) = 0,$$

where $\mathrm{PD}_X(D) \in H^2(X, \mathbb{Z})$ is the Poincare dual of D in X;
- $((X, \omega), D)$ is a symplectic log Calabi-Yau (or **CY**) **fourfold** if $\dim_{\mathbb{R}} X = 4$.

Let $\mathrm{Symp}_{\log}(X, D)$ denote the space of symplectic structures ω on X such that D is a symplectic submanifold and $((X, \omega), D)$ is a symplectic log CY pair. To drop ω from the notation, we say (X, D) is a symplectic log CY pair if $\mathrm{Symp}_{\log}(X, D) \neq \emptyset$. It is worth noting that if (X, D) is a symplectic log CY fourfold, by (1.2) and the adjunction formula, D will be a symplectic 2-torus.

1.2. **Integrality conjecture.** As mentioned after (1.1), for the main result of this paper, we are interested in the case where (X, D) is a symplectic log Calabi-Yau fourfold, Σ has genus 0 (i.e., $\Sigma = \mathbb{P}^1$) and one marked point z_1, and u has maximal tangency of order s_A with D at z_1. In this case $\dim \mathcal{M}_{0,(s_A)}(X, D, A) = 0$ and we study the rational invariants $N_A \equiv N_{0,A}$ arising from the so-called relative compactification $\overline{\mathcal{M}}_{0,(s_A)}^{\mathrm{rel}}(X, D, A)$ of $\mathcal{M}_{0,(s_A)}(X, D, A)$. Furthermore, for $g > 0$, in Section 4, we study an invariant $N_{g,A}$ arising from the integration of the degree $2g$ "Hodge class" λ_g against the "virtual fundamental class" of $\overline{\mathcal{M}}_{g,(s_A)}^{\mathrm{rel}}(X, D, A)$. Under similar assumptions in the algebraic category, Conjecture 6.2 in [24] (which is equivalent to a conjecture of Kontsevich and Soibelman in [31]) makes the following predictions.

Conjecture 1.3. *Suppose (X, D) is a log Calabi-Yau surface.[1] Let N_A denote virtual count of the degree A rational curves in X that have maximal tangency with D at one point. Then genus zero invariants n_A formally derived from N_A by the formula*

$$(1.3) \qquad N_A = \sum_{\substack{B \in H_2(X, \mathbb{Z}) \\ dB = A, d > 0}} d^{-2} \binom{d(s_B - 1) - 1}{d - 1} n_B \qquad \forall A \in H_2(X, \mathbb{Z})$$

are integer.

Similarly to the BPS or Gopakumar-Vafa formula for the genus zero Gromov-Witten (or GW) invariants of symplectic CY sixfolds, this conjecture predicts that by taking into account the contribution of multiple-cover maps we obtain an integral invariant which we call the genus zero relative/log BPS invariants of (X, D). In this paper, we present a direct geometric proof utilizing J-holomorphic curve techniques that aligns with the Ionel-Parker's proof [30] of the integrality of BPS numbers in the context of compact symplectic Calabi-Yau sixfolds. More specifically, the last statement of Theorem 1.5 shows that these BPS numbers can be realized as a geometric count of finitely many "logarithmically-rigid" curves.

A proof of Conjecture 1.3 has been given by Bousseau-Brini-van Garrel [5], employing a somewhat indirect argument that leads to quiver Donaldson-Thomas invariants. Conjecture 4.1 in Section 4, due to P. Bousseau, extends the conjecture above to a higher-genus statement concerning the integrality of coefficients of a potential function. This function is defined in terms of $N_{g,A}$ and a quntum parameter q after undergoing some rearrangements, akin to (1.3). Conjecture 4.10 in Section 4 provides a direct generalization of Conjecture 1.3 to the case of genus one invariants.

1.3. **Compactification and transversality statements.** The moduli spaces

$$\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$$

are often not compact. We need more conditions on J along D to construct a compactification; see [12, Sec. 1]. For (ω, J) in a suitable space $\mathrm{AK}(X, D)$ of almost Kähler structures as in [12, Thm. 1.4 or Rmk. 1.5], we can use the relative compactification $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A)$ studied in [18,29,32,33] or the (analytical) log compactification $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{log}}(X, D, A)$ constructed in [12]. In general (assuming the smoothness

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[1]i.e., complex dimension 2.

of D), the relative compactification $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A)$ includes nodal maps with components either in X or in an expanded degeneration[2] of that so that the contact order $\mathfrak{s}$ remains well-defined. The irreducible components of a relative map are ordered such that the order 0 maps have their image in X (intersecting D discretely), while the higher-order components have their image in a rubber component-a fixed $\mathbb{P}^1$-bundle over D. The elements of $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A)$ will be called relative curves. Further details can be found in Section 3.2, expanded degenerations are defined in (3.4), relative maps are defined in Definition 3.2 and illustrated in Figure 1. The log compactification is essentially derived by considering a partial order in lieu of the total order. The elements of $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{log}}(X, D, A)$ will be called log curves. Consequently, the log compactification is slightly smaller than the relative compactification in the sense that there is a surjective map

$$\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A) \longrightarrow \overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{log}}(X, D, A);$$

see [12, Prp 4.5]. Nevertheless, the Fredholm theory developed in [13] applies to both $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A)$ and $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{log}}(X, D, A)$ with slight changes. Necessary details are provided in Section 2 and 3.

The log compactification $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{log}}(X, D, A)$ in [12] is expected to be related to the algebraic log compactification of Gross-Siebert and Abramovich-Chen [8, 9, 25]. Nevertheless, Conjecture 1.3 and this paper concern Gromov-Witten invariants arising from the relative compactification. An essential consideration to keep in mind is that, in Theorem 1.5 and various other instances, the specific choice of the compactification is not important. To be more specific, Theorem 1.5 is applicable to both choices of compactification, with the distinction only manifesting in the multiple cover moduli spaces of $\mathbb{P}^1$; see Remark 1.6. The latter are well-understood smooth orbifolds. Therefore, for the sake of simplicity in notation, in such cases, we omit the superscript and represent the compactified moduli space as $\overline{\mathcal{M}}_{0,\mathfrak{s}}(X, D, A)$.

Remark 1.4. In the symplectic category, there are diverse approaches to defining Gromov-Witten invariants from moduli spaces of pseudoholomorphic curves. Specifically, when certain positivity conditions hold for the symplectic manifold (X, ω), in the classical case (c.f. [34, Thm. 3.15, 6.6.1]), and for a pair $((X, \omega), D)$, in the relative context (c.f. [18, Dfn. 4.7]), one can define genus zero Gromov-Witten invariants as certain signed count of J-holomorphic curves. This count is computed over a generic selection of J, and the approach can be extended to "semi-positive" cases by perturbing the Cauchy-Riemann equation (2.1), particularly within the stable domain range where $2g + k \geq 3$; c.f. [41, Thm 3.16, Prp 3.21] and [13, Crl. 1.8]. However, in general, for cases beyond the aforementioned scenarios, one would need to provide a round about or construct a virtual fundamental class. The Gromov-Witten invariants are then defined through integration against or intersection with the virtual fundamental class. For relative moduli spaces $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A)$, such virtual treatments can be found in [11] and [37]. The former is in the more general case of pseudoholomorphic curves with boundary on a Lagrangian and the latter happens in the category of "exploded manifolds". The Calabi-Yau fourfolds considered in this paper are semi-positive. When $g = 0$,

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[2]This degeneration is a normal crossings variety composed of X and finite copies of the $\mathbb{P}^1$-bundle $\mathbb{P}_X D \equiv \mathbb{P}(\mathcal{N}_X D \oplus \mathbb{C})$ over D.

Theorems 1.5 and 1.7 show that Gromov-Witten invariants can be defined as a geometric count.

A stable 1-marked genus zero J-holomorphic map $(u, (\Sigma, z_1))$ with a genus zero nodal domain Σ is **simple**[3] if the restriction u_v of u to each irreducible component Σ_v of Σ is not multiply-covered (or equally it is somewhere injective) whenever u_v is not constant, and the images of two such components in X are distinct; see [34, Sec. 2.5]. An element of $\overline{\mathcal{M}}_{0,(s_A)}(X, D, A)$ is called simple if the underlying stable map in the classical genus zero 1-marked degree A stable map compactification $\overline{\mathcal{M}}_{0,1}(X, A)$ is simple. Let

$$\overline{\mathcal{M}}_{0,(s_A)}^{\star}(X, D, A) \subset \overline{\mathcal{M}}_{0,(s_A)}(X, D, A)$$

denote the subspace of simple log/relative curves. Since every log CY fourfold (X, D) is semi-positive in the sense of [18, Dfn. 4.7] or equally [13, Dfn. 1.6], Theorem 1.5 is mainly a corollary of [13, Thm. 1.5(2)], [13, Prp. 1.7(2)], Lemma 2.4, and some classical results. We will explain the proof with details in Section 2.

Theorem 1.5. *Suppose (X, D) is a symplectic log Calabi-Yau fourfold. Then, for every $E > 0$, there exists a Baire subset $\mathrm{AK}^{\mathrm{reg}}(X, D) \subset \mathrm{AK}(X, D)$ such that the forgetful map*

$$\mathrm{AK}^{\mathrm{reg}}(X, D) \longrightarrow \mathrm{Symp}_{\log}(X, D), \quad (\omega, J) \longrightarrow \omega,$$

has connected fibers and, for every $(\omega, J) \in \mathrm{AK}^{\mathrm{reg}}(X, D)$ and $A \in H_2(X, \mathbb{Z})$ with $\omega(A) < E$,

- *the moduli space $\mathcal{M}_{0,(s_A)}^{\star}(X, D, A)$ is cut transversely and consists of finitely many points[4];*
- *every log/relative curve in $\overline{\mathcal{M}}_{0,(s_A)}^{\star}(X, D, A)$ has a smooth domain, i.e.,*

(1.4) $$\overline{\mathcal{M}}_{0,(s_A)}^{\star}(X, D, A) = \mathcal{M}_{0,(s_A)}^{\star}(X, D, A);$$

- *and, each multiple-cover map in*

$$\mathcal{M}_{0,(s_A)}^{\mathrm{mc}}(X, D, A) \equiv \overline{\mathcal{M}}_{0,(s_A)}(X, D, A) - \overline{\mathcal{M}}_{0,(s_A)}^{\star}(X, D, A)$$

has an irreducible image in X. In particular, $\overline{\mathcal{M}}_{0,(s_A)}(X, D, A)$ is a disjoint union of closed components

(1.5) $$\overline{\mathcal{M}}_{0,(s_A)}(X, D, A) \cong \coprod_{\substack{B \in H_2(X, \mathbb{Z}) \\ dB = A, d \in \mathbb{Z}_+}} \overline{\mathcal{M}}_{0,(d)}\left(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\right)^* \times \mathcal{M}_{0,(s_B)}^{\star}(X, D, B).$$

We will explain the meaning of the superscript $*$ on $\overline{\mathcal{M}}_{0,(d)}\left(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\right)^*$ in Remark 1.6.

To be more precise, we show that for every arbitrary D-compatible almost Kähler structure (ω, J), each element of the moduli space $\mathcal{M}_{0,(s_A)}(X, D, A)$ is automatically **super-rigid** in the sense that $\mathcal{M}_{0,(s_A)}(X, D, A)$ does not have a sequence of maps with distinct images accumulating at a multiple-cover map; see [44, Dfn. 2.3]. This statement mainly follows from [26, Thm. 1'] that concerns the injectivity of all $\mathbb{R}$-linear $\bar{\partial}$-operator on a line bundle of negative degree. With some effort, Theorem 1.5 can also be extracted from the automatic transversality results of Wendl

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[3]Note that (Σ, z_1) is a unstable; c.f. [13, Dfn. 3.4] for the definition of simple maps in general.

[4]i.e., it has no accumulation point in $\overline{\mathcal{M}}_{0,(s_A)}^{\star}(X, D, A)$.

in [43], by viewing $X - D$ as a manifold with a cylindrical end and working with punctured surfaces. However, our logarithmic Fredholm setup has the advantage of working with closed surfaces and is more concrete for applications such as localization calculations. The super-rigidity statement above is similar but more straightforward than Wendl's super-rigidity theorem [44, Thm. A] for symplectic Calabi-Yau sixfolds that holds for some choices of J. In particular, the connectivity of $\mathrm{AK}^{\mathrm{reg}}(X, D)$ in Theorem 1.5 does not hold for symplectic Calabi-Yau sixfolds, resulting in a wall-crossing phenomenon where embedded J-holomorphic curves bifurcate when passing through certain real codimension 1 walls in the space of almost complex structures; see [1,44]. As we explain in Section 4, for dimensional reasons, the analogy above between the genus zero integral curve counts in compact symplectic Calabi-Yau sixfolds and log Calabi-Yau fourfolds does not naturally extend to higher genus.

Remark 1.6. As mentioned earlier, Theorem 1.5 is applicable to both log and relative compactifications, with the distinction only manifesting in the multiple cover moduli spaces of $\mathbb{P}^1$. The relative and log compactifications of $\mathcal{M}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)$ are slightly different and the difference can potentially[5] result in different Gromov-Witten invariants (i.e., different multiple-cover contributions). Additionally, the set of relative degree d multiple cover relative maps contributing to (1.5) can be topologically identified with $\overline{\mathcal{M}}^{\mathrm{rel}}_{0,(d)}(\mathbb{P}^1, \infty, [d\mathbb{P}^1])$; however, if $s_B > 1$, as mentioned in [24, p. 352], the algebraic moduli structure or the orbifold structure somewhat differs from the standard structure on $\overline{\mathcal{M}}^{\mathrm{rel}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)$ and depends on s_B; see the paragraph after (1.7) below. In [24], the correct moduli/orbifold structure is indicated by a superscript $*$; i.e., the moduli space is denoted by $\overline{\mathcal{M}}^{\mathrm{rel}}_{0,(d)}(\mathbb{P}^1, \infty, [d\mathbb{P}^1])^*$. We will explain these technical details in Section 3.

1.4. **Multiple cover contributions and the BPS invariants.** For every 1-marked curve $[u, \mathbb{P}^1, z_1] \in \mathcal{M}_{0,(s_A)}(X, D, A)$, after a reparametrization of the domain, we may assume that the contact point with D happens at $z_1 = \infty \in \mathbb{P}^1$. If u is a degree d multiple-cover map, we have

$$(1.6) \qquad u = \overline{u} \circ h \colon \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \longrightarrow X$$

such that $h(z)$ is a degree d polynomial in z, and $\overline{u} \colon \mathbb{P}^1 \longrightarrow X$ is a degree B somewhere-injective J-holomorphic map with

$$A = dB \quad \text{and} \quad \overline{u}^{-1}(D) = \infty \in \mathbb{P}^1, \qquad d \in \mathbb{Z}_+.$$

The tuple $\big(\overline{u}, (\mathbb{P}^1, \infty)\big)$ represents an element of $\mathcal{M}^\star_{0,(s_B)}(X, D, B)$ and the tuple $(h, (\mathbb{P}^1, \infty))$ represents an element of the open-dense main stratum

$$\mathcal{M}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)$$

of the relative/log moduli space $\overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^*$. In the limit, a sequence of degree d relative holomorphic maps in $\mathcal{M}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)$ may converge to a relative/log map h' with nodal domain. In the relative compactification, h' is a map

$$(1.7) \qquad h' \colon (\Sigma, z_1) \longrightarrow (\mathbb{P}^1[\ell], \infty) \equiv \underbrace{\mathbb{P}^1 {}_\infty\cup_0 \mathbb{P}^1 \cdots {}_\infty\cup_0 (\mathbb{P}^1, \infty)}_{\ell+1 \text{ copies of } \mathbb{P}^1}$$

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[5]The multiple-cover localization calculation is done using the relative compactification.

with image in a chain (or expanded degeneration) of rational curves such that the component containing the marked point z_1 is mapped to the last $\mathbb{P}^1$ and z_1 is mapped to ∞; see Figure 1. Nevertheless, the image of $u' = \overline{u} \circ h'$ will remain the same irreducible curve image($\overline{u}$).

To explain the second part of Remark 1.6, the set of relative maps $(h', (\Sigma, z_1))$ in (1.7) can be topologically identified with $\overline{\mathcal{M}}^{\text{rel}}_{0,(d)}(\mathbb{P}^1, \infty, [d\mathbb{P}^1])$. However, the s_B-tangency of $\overline{u}$ with D forces the tangency orders of maps to the rubber components of $\mathbb{P}^1[\ell]$ to all be divisible by s_B. This results in different maps to rubber components and modified obstruction bundles.

Theorem 1.7 confirms and strengthens Conjecture 1.3.

Theorem 1.7. *Suppose (X, D) is a symplectic log Calabi-Yau fourfold. The relative Gromov-Witten invariants*

$$(1.8) \qquad N_A \equiv \#[\overline{\mathcal{M}}^{\text{rel}}_{0,(s_A)}(X, D, A)]^{\text{vir}} \in \mathbb{Q} \qquad \forall A \in H_2(X, \mathbb{Z}), s_A > 0,$$

can be defined without using virtual techniques (see Remark 1.4) and only depend on the deformation equivalence class of $\omega \in \text{Symp}_{\log}(X, D)$. Furthermore,

$$(1.9) \qquad N_A = \sum_{\substack{B \in H_2(X,\mathbb{Z}) \\ dB = A, d \in \mathbb{Z}_+}} \text{mc}(d, s_B) n_B$$

such that $n_B \in \mathbb{Z}$ for all $0 \neq B \in H_2(X, \mathbb{Z})$ and

$$(1.10) \qquad \text{mc}(d, s_B) = d^{-2} \binom{d(s_B - 1) - 1}{d - 1}.$$

For generic J,

$$n_A \equiv \#\mathcal{M}^\star_{0,(s_A)}(X, D, A)$$

is a well-defined count of "logarithmically immersed" curves.

The notion of **logarithmically immersed** curve is defined in Section 2. If $s_B = 1$, note that

$$\text{mc}(d, 1) = d^{-2} \binom{-1}{d - 1} = \frac{(-1)^{d-1}}{d^2}.$$

In [24, Prp. 6.1], the coefficients $\text{mc}(d, s_B)$ are calculated using the relative localization. As mentioned in [24, p. 352], the moduli space $\overline{\mathcal{M}}^{\text{rel}}_{0,(d)}(\mathbb{P}^1, \infty, [d\mathbb{P}^1])^*$ is a nonsingular Deligne-Mumford stack or complex orbifold of complex dimension $d - 1$. The contribution $\text{mc}(d, s_B)$ of d-fold multiple-covers of a somewhere injective degree B map $\overline{u}$ to N_{dB} is

$$(1.11) \qquad \text{mc}(d, s_B) = \int_{\overline{\mathcal{M}}^{\text{rel}}_{0,(d)}(\mathbb{P}^1, \infty, [d\mathbb{P}^1])^*} c_{\text{top}}(\text{Ob}(d, s_B)),$$

where $\text{Ob}(d, s_B)$ is the obstruction bundle of rank $d - 1$. Recall from Remark 1.6 that the relative moduli space of multiple covers of $\mathbb{P}^1$ and thus the obstruction bundle depends on s_B. In order to compute (1.11), one needs to explicitly describe $\text{Ob}(d, s_B)$ and choose a suitable lift of the natural $\mathbb{C}^*$-action on $\mathbb{P}^1$ to the (logarithmic) normal bundle of $\overline{u}$. The latter determines the lift of the action to $\text{Ob}(d, s_B)$. When $s_B = 1$, this is done in the proof of [7, Thm. 5.1]. For $s_B > 1$, the proper choice of the lift is provided in the proof of [24, Prp. 6.1] and the justification is

mostly left to the reader. In the second half of Section 3, we review the efficient argument of [24] and describe an alternative approach that involves different weights[6] and is closer to the realm of [22]. While the latter approach is computationally cumbersome, it illustrates how the analytic description of the deformation-obstruction spaces in this paper works for relative maps that have components in the rubber components of the target.

1.5. **Concluding remarks and an example.** From the algebraic perspective, [24, Cnj. 6.2] is proved for toric del Pezzo surfaces in [21]. There are several other recent works such as [10] and [20] that address [24, Cnj. 6.2] or a variation of that from the algebraic or tropical perspectives. The (expected) relation of these invariants to certain count of J-holomorphic disks in $X - D$, mirror symmetry, loop quiver Donaldson-Thomas invariants, and local CY threefold invariants are explained in [10], [5], and other recent works. We do not know of any other work that approaches these invariants from a purely symplectic/analytic perspective.

In Section 4, following [4], we will discuss a natural higher genus version of the relative invariants (1.8) and provide an explicit decomposition of the genus one invariants into contributions of genus-zero curves and "conjecturally integral" "reduced" genus one invariants.

In the algebraic geometry literature, it is shown that the genus 0 relative GW invariants N_A and their higher genus variants are explicitly related to the GW invariants of the local CY threefold K_X, where K_X is the total space of the canonical bundle of X; see [6]. The integral invariants can then be defined using the Gopakumar-Vafa formula for CY threefolds. It is shown in [21, Lmm. 12] that these integral invariants are related by an integral matrix with an integral inverse that has a natural interpretation in terms of Donaldson-Thomas invariants of loop quivers. Therefore, the integrality of the two definitions of BPS invariants are equivalent.

Example 1.8. Let $X = \mathbb{P}^2$, D be a cubic curve, ℓ denote the line class in $H_2(\mathbb{P}^2, \mathbb{Z})$. The first row of the table below[7] contains the relative GW invariants $N_d \equiv N_{d\ell}$ for $1 \leq d \leq 6$ and the second row contains the BPS numbers n_d in the same degree range.

d	1	2	3	4	5	6
N_d	9	135/4	244	36999/16	635634/25	307095
n_d	9	27	234	2232	25380	305829

For instance, the number n_2 of maximally tangent conics can be calculated in the following way. Let C be a conic in $\mathbb{P}^2$ that is maximally tangent to the cubic curve D at a point $P \in D$. Then,

$$\mathcal{O}_D(6P) = \mathcal{O}(2)|_D,$$

where $\mathcal{O}_D(6P)$ is the degree 6 line bundle on the elliptic curve D corresponding to the divisor $6P$ and $\mathcal{O}(2)$ is the degree 2 line bundle on $\mathbb{P}^2$. Thus, the set of such points P is the fiber over $\mathcal{O}(2)|_D \in \mathrm{Pic}^6(D)$ of the map

$$D \cong \mathrm{Pic}^1(D) \longrightarrow \mathrm{Pic}^6(D), \qquad L \longrightarrow L^{\otimes 6},$$

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[6]It corresponds to infinitesimally extending the action to a neighborhood of the contributing curve in X.

[7]Borrowed from [23, Table 7.1-7.2].

between the degree 1 and 6 Picard groups. The latter is a 36:1 covering map, so there are 36 such points in the pre-image of $\mathcal{O}(2)|_D$. Moreover, for each of these points, the totally tangent conic is unique. However, for a similar reason, 9 of these 36 points correspond to double of maximally tangent lines. Therefore, $n_2 = 27$. For higher degree curves, the author is uncertain about whether the standard complex structure on $\mathbb{P}^2$ is regular in the sense of Theorem 1.5 when considering a generic choice of an elliptic curve D. It is noteworthy that the relative Gromov-Witten invariants N_d can be computed using Andreas Gathmann's GROWI software.

As the last calculation indicates, for the curves contributing to N_A, the tangent points with D have finite order in the elliptic curve D. Therefore, in the holomorphic setting, it is possible to refine the invariants N_A and n_A using the order of the contact points in D, as is done in [10] and [3]. Since $\dim_{\mathbb{C}} D = 1$, every $J \in$ AK(X, D) is integrable in a neighborhood of D in X. Therefore, such a refinement of N_A and n_A also seems possible from the analytic perspective.

2. Logarithmic Fredholm theory

In this section, we review the Fredholm setup for the deformation-obstruction theory of relative/log maps in the virtually main stratum $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ introduced in [13]. The main output of this discussion will be the notion of logarithmic normal bundle (2.17) and the super-rigidity Lemma 2.4. We conclude this section with a relatively short proof of Theorem 1.5.

2.1. **Fredholm setup.** Let us start with a brief discussion of the classical Fredholm setup. Suppose Σ is a smooth closed Riemann surface with the complex structure j. Let $\mathrm{Map}_A(\Sigma, X)$ denote the space of all smooth maps $u\colon \Sigma \longrightarrow X$ that represent the homology class A, and

$$\mathcal{E}_A(\Sigma, X) \longrightarrow \mathrm{Map}_A(\Sigma, X)$$

be the infinite dimensional bundle whose fiber over u is $\Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes_{\mathbb{C}} u^*TX)$. The Cauchy-Riemann (or **CR**) equation

$$(2.1) \qquad \bar{\partial}u \equiv \frac{1}{2}\big(\mathrm{d}\,u + J\,\mathrm{d}\,u \circ \mathrm{j}\big)$$

can be seen as a section $\bar{\partial}\colon \mathrm{Map}_A(\Sigma, X) \longrightarrow \mathcal{E}_A(\Sigma, X)$. More precisely, we consider a Sobolev completion of these spaces for the Implicit Function Theorem to apply, but, by elliptic regularity, every solution of $\bar{\partial}u = 0$ will be smooth; see [34, Appendix B]. The linearization of the $\bar{\partial}$-section at any J-holomorphic map u is an $\mathbb{R}$-linear map

$$(2.2) \qquad \mathrm{D}_u\bar{\partial}\colon \Gamma(\Sigma, u^*TX) \longrightarrow \Gamma(\Sigma, \Omega^{0,1}_\Sigma \otimes_{\mathbb{C}} u^*TX)$$

that is the sum of a $\mathbb{C}$-linear $\bar{\partial}$-operator and a compact operator. Therefore, it is a Fredholm operator and Riemann-Roch applies; i.e., it has finite dimensional kernel and co-kernel, and

$$(2.3) \qquad \dim_{\mathbb{R}} \mathrm{Def}(u) - \dim_{\mathbb{R}} \mathrm{Ob}(u) = 2\big(\deg(u^*TX) + \dim_{\mathbb{C}} X(1-g)\big),$$

where

$$\mathrm{Def}(u) = \ker(\mathrm{D}_u\bar{\partial}) \quad \text{and} \quad \mathrm{Ob}(u) = \mathrm{coker}(\mathrm{D}_u\bar{\partial}).$$

The first space corresponds to infinitesimal deformations of u (over the fixed smooth marked domain) and the second one is the obstruction space for integrating elements of $\mathrm{Def}(u)$ to actual deformations. If $\mathrm{Ob}(u) = 0$, by Implicit Function Theorem

[34, Thm. A.3.3], in a small neighborhood U of u in $\text{Map}_A(\Sigma, X)$, the set of J-holomorphic maps $V = U \cap \bar{\partial}^{-1}(0)$ is a smooth manifold of real dimension (2.3), all the elements of $\text{Def}(u)$ are smooth, and $T_u V \cong \text{Def}(u)$; see [34, Thm. 3.1.5]. The manifold V carries a natural orientation.

For $(\omega, J) \in \text{AK}(X, D)$, in [13, Sec. 4], corresponding to every genus g k-marked relative map in the virtually main stratum,

$$(2.4) \qquad f = [u, C = (\Sigma, z_1, \ldots, z_k)] \in \mathcal{M}_{g,\mathfrak{s}}(X, D, A),$$

we derive a **logarithmic linearization** of the CR operator/section at u, denoted by

$$(2.5) \qquad \text{D}_u^{\log} \bar{\partial} : \Gamma(\Sigma, u^* TX(-\log D)) \longrightarrow \Gamma(\Sigma, \Omega_\Sigma^{0,1} \otimes_{\mathbb{C}} u^* TX(-\log D)),$$

such that $TX(-\log D)$ is the logarithmic tangent bundle introduced in [15] and constructed in detail in [17]. After Remark 2.1, we briefly digress from the main discussion and recall the construction of $TX(-\log D)$ and some of its properties. Then, we show how $\text{D}_u^{\log} \bar{\partial}$ can be used to study the deformation-obstruction spaces of $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ at f.

Remark 2.1. In the symplectic category, moduli spaces of relative maps were first studied by Ionel-Parker and Li-Ruan in [29] and [33]. In [18], we give a detailed account of their approaches and streamline some aspects of their construction. The work of Ionel-Parker does not include a dedicated Fredholm setup for relative maps. Deformation spaces are defined as a subspace of the classical deformation space and the obstruction spaces are not precisely identified. The approach of Li-Ruan is based on working with manifolds with cylindrical ends as in Symplectic Field Theory. In this approach, $\text{D}_u^{\log} \bar{\partial}$ is defined over a subspace of

$$\Gamma\left(\Sigma^*, \left(u|_{\Sigma^*}\right)^* T(X - D)\right), \qquad \Sigma^* = \Sigma - u^{-1}(D),$$

that is completed with weighted norms on the cylindrical parts of Σ^* and $X - D$. These weighted norms control the behavior of u at infinity. As the following concrete definitions and structures indicate, our logarithmic Fredholm setup has the advantage of working with closed surfaces and is more user-friendly for applications such as explicitly identifying the deformation/obstruction spaces and localization calculations.

Assume (X, ω) is a symplectic manifold and $D \subset X$ is a smooth symplectic hypersurafce. Our construction of the complex vector bundle $TX(-\log D)$ (as well as an almost complex structure J compatible with D) depends on

- a Hermitian structure $(\rho, \nabla, \mathfrak{i})$ on $\mathcal{N}_X D$, and
- a symplectic identification

$$(2.6) \qquad \Psi : \mathcal{N}' \xrightarrow{\cong} U \subset X$$

of a neighborhood U of D in X with a neighborhood $\mathcal{N}' \subset \mathcal{N}_X D$ of the zero section in the normal bundle

$$\pi : \mathcal{N}_X D \equiv \frac{TX|_D}{TD} \longrightarrow D.$$

Here, $\mathfrak{i}$ is a complex structure compatible with the symplectic structure on the rank 2 real normal bundle

$$\mathcal{N}_X D \cong TD^\omega = \left\{ v \in T_x X : x \in D, \omega(v, w) = 0 \quad \forall w \in T_x D \right\},$$

ρ is a Hermitian metric on $(\mathcal{N}_X D, \mathfrak{i})$, ∇ is a $(\rho, \mathfrak{i})$-Hermitian connection, and the standard symplectic structure on $\mathcal{N}'$ is determined by $\omega|_D$ and (ρ, ∇) as in [18, (3.1)]. The connection ∇ also defines a $\mathbb{C}$-linear injective homomorphism

$$h_\nabla : \pi^* TD \longrightarrow T\mathcal{N}_X D$$

that lifts a tangent vector $w \in T_x D$ to a "horizontal" tangent vector $h_\nabla(w) \in T_v \mathcal{N}_X D$, for all $v \in \mathcal{N}_X D|_x$. The homomorphism h_∇ also gives rise to a similarly denoted isomorphism

$$(2.7) \qquad h_\nabla : \pi^*(TD \oplus \mathcal{N}_X D) \xrightarrow{\cong} T\mathcal{N}_X D, \qquad (v; w \oplus v') \to (v; h_\nabla(w) + v');$$

see [15, (2.6)]. Let $\mathcal{R}$ denote the tuple $(\Psi, (\rho, \nabla, \mathfrak{i}))$ used above which is called a **regularization** in [14]. Via (2.7), an almost complex structure J_D on TD and the complex structure $\mathfrak{i}$ on $\mathcal{N}_X D$ define an almost complex structure J' on $T\mathcal{N}_X D$. Then, the identification Ψ in (2.6) can be used to extend $\Psi_*(J'|_{\mathcal{N}'})$ to an ω-tame almost complex structure J over the entire X. In [12,14], the space of such D-compatible almost Kähler structures (ω, J) on X is denoted by $\mathrm{AK}(X, D)$. The relative moduli spaces are defined for a larger class of D-compatible almost complex structures introduced in [29, Sec. 3]; also see [19, (2.11)].

For any space Y and every complex line bundle $L \longrightarrow Y$, let $Y \times \mathbb{C}$ denote the trivial line bundle on Y and $L \oplus Y \times \mathbb{C}$ denote the direct sum of the two line bundles. The logarithmic tangent bundle arising from a regularization $\mathcal{R}$ is defined to be

$$T_\mathcal{R} X(-\log D) = \left(\left(\Psi_* \pi^* TD \oplus U \times \mathbb{C} \right) \sqcup T(X-D) \right) \Big/ \sim \longrightarrow U \cup (X-D) = X,$$

$$(\Psi_* \pi^* TD) \oplus U \times \mathbb{C} \ni \left(\Psi(v); w \right) \oplus \left(\Psi(v); c \right) \sim \mathrm{d}_v \Psi \left(h_\nabla(w) + cv \right) \in T_{\Psi(v)}(X-D).$$

In particular,

$$(2.8) \qquad \Psi^* T_\mathcal{R} X(-\log D)|_U = \pi^* TD|_{\mathcal{N}'} \oplus \mathcal{N}' \times \mathbb{C}.$$

Furthermore, the map $\iota_\mathcal{R} : T_\mathcal{R} X(-\log D) \longrightarrow TX$ defined by

$$(2.9) \qquad \iota_\mathcal{R}(\zeta) = \begin{cases} \mathrm{d}_v \Psi \left(h_\nabla(w) + cv \right), & \text{if } \zeta = \left[(\Psi(v); w) \oplus (\Psi(v); c) \right], v \in \mathcal{N}'; \\ \zeta, & \text{if } \zeta \in T(X-D); \end{cases}$$

is a complex linear homomorphism whose restriction to $X - D$ is a bundle isomorphism.

In [14] and [16], we introduced topological notions of normal crossings symplectic divisor (and variety). We proved that, at the cost of deforming ω, every normal crossings symplectic divisor $D \subset X$ admits a compatible system $\mathcal{R}$ of local linearizations generalizing the auxiliary data $(\Psi, (\rho, \nabla, \mathfrak{i}))$ used above in the smooth case; see [14, Thm. 2.13] and [16, Thm. 3.4]. We call such a compatible system $\mathcal{R}$ of local linearizations a **regularization**. In [17], we used a regularization $\mathcal{R}$ to construct $T_\mathcal{R} X(-\log D)$ and prove the following and a few other results for an arbitrary normal crossings symplectic divisor.

Theorem 2.2 ([17, Thm. 1.2]). *Suppose (X, ω) is a symplectic manifold and $D \subset X$ is a normal crossings symplectic divisor.*

(1) *An ω-regularization $\mathcal{R}$ for $D \subset X$ determines a vector bundle $T_\mathcal{R} X(-\log D)$ over X with a smooth vector bundle homomorphism*

$$(2.10) \qquad \iota_\mathcal{R} : T_\mathcal{R} X(-\log D) \longrightarrow TX.$$

(2) *An $\mathcal{R}$-compatible almost complex structure J on X determines a complex structure $\mathfrak{i}_{\mathcal{R},J}$ on the vector bundle $T_{\mathcal{R}}X(-\log D)$ so that the bundle homomorphism (2.10) is $\mathbb{C}$-linear.*

(3) *The deformation equivalence class $TX(-\log D)$ of $T_{\mathcal{R}}X(-\log D)$ depends only on the deformation equivalence class of $((X,\omega),D)$.*

(4) *If $D' \subset X$ is a smooth submanifold so that $D \cup D' \subset X$ is also an NC symplectic divisor and $D \cap D'$ contains no open subspace of D, then*

$$(2.11) \qquad TX(-\log(D \cup D')) \oplus \mathcal{O}_X(D') \cong TX(-\log D) \oplus (X \times \mathbb{C}).$$

(5) *We have*

$$(2.12) \quad c\big(TX(-\log D)\big) = \frac{c(TX)}{1+\mathrm{PD}_X([D^{(1)}]_X)+\mathrm{PD}_X([D^{(2)}]_X)+\ldots} \in H^*(X;\mathbb{Q}),$$

where $D^{(k)}$ is the k-fold intersection locus of D. The above equality holds in $H^*(X;\mathbb{Z})$ if $D \subset X$ is a simple normal crossings divisor.

In what follows, in the light of Theorem 2.2 (3), we drop $\mathcal{R}$ from the notation and simply denote $T_{\mathcal{R}}X(-\log D)$ by $TX(-\log D)$ and $\iota_{\mathcal{R}}$ in (2.9) by ι.

Going back to the main discussion, the logarithmic linearization map (2.5) constructed in [12] is a natural lift of (2.2) that makes the following diagram commutes:
(2.13)

$$\begin{array}{ccc}
\Gamma(\Sigma, u^*TX(-\log D)) & \xrightarrow{\mathrm{D}_u^{\log}\bar{\partial}} & \Gamma(\Sigma, \Omega_\Sigma^{0,1} \otimes_{\mathbb{C}} u^*TX(-\log D)) \\
\downarrow{\iota} & & \downarrow{\iota} \\
\Gamma(\Sigma, u^*TX) & \xrightarrow{\mathrm{D}_u\bar{\partial}} & \Gamma(\Sigma, \Omega_\Sigma^{0,1} \otimes_{\mathbb{C}} u^*TX) \,.
\end{array}$$

Similarly to the classical case, if $\mathrm{coker}(\mathrm{D}_u^{\log}\bar{\partial}) = 0$, by Riemann-Roch, the set of relative J-holomorphic maps of any fixed contact type $\mathfrak{s}$ (over the fixed marked domain C) close to f in (2.4)) form an oriented smooth manifold of real dimension

$$2\big(\deg(u^*TX(-\log D))+\dim_{\mathbb{C}}X(1-g)\big).$$

Considering the deformations of the marked domain C of the relative map f, we get the dimension formula (1.1) and the deformation-obstruction long exact sequence

$$(2.14) \qquad \begin{aligned} 0 \longrightarrow \mathrm{aut}(C) &\xrightarrow{\delta} \mathrm{Def}_{\log}(u) \longrightarrow \mathrm{Def}_{\log}(f) \longrightarrow \mathrm{Def}(C) \\ &\xrightarrow{\delta} \mathrm{Ob}_{\log}(u) \longrightarrow \mathrm{Ob}_{\log}(f) \longrightarrow 0, \end{aligned}$$

such that

$$\mathrm{Def}_{\log}(u) = \ker(\mathrm{D}_u^{\log}\bar{\partial}), \qquad \mathrm{Ob}_{\log}(u) = \mathrm{coker}(\mathrm{D}_u^{\log}\bar{\partial}),$$

and $\mathrm{aut}(C)$ is the Lie algebra of the automorphism group of C. If $\mathrm{Ob}_{\log}(f) = 0$, then a small neighborhood of f in $\mathcal{M}_{g,\mathfrak{s}}(X,D,A)$ is a smooth orbifold of the expected dimension (1.1).

The long-exact sequence (2.14) is the long-exact sequence associated to a short-exact sequence of fine sheaves

$$0 \longrightarrow \mathcal{O}(T\Sigma(-\log z)) \longrightarrow \mathcal{O}(u^*TX(-\log D)) \longrightarrow \mathcal{O}(\widetilde{\mathcal{N}}_f) \longrightarrow 0$$

over Σ, defined in the following way.

In [13, (4.18)], we show that the standard complex linear derivative map $\mathrm{d}\, u\colon T\Sigma \longrightarrow TX$ gives rise to a **logarithmic derivative** map

$$\mathrm{d}^{\log} u\colon T\Sigma(-\log \mathfrak{z}) \longrightarrow u^*TX(-\log D), \quad \text{with } \mathfrak{z} = u^{-1}(D) \subset \Sigma,$$

such that the following diagram commutes:

(2.15)
$$
\begin{array}{ccc}
T\Sigma(-\log \mathfrak{z}) & \xrightarrow{\ \mathrm{d}^{\log} u\ } & u^*TX(-\log D) \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\
T\Sigma & \xrightarrow{\ \mathrm{d}\, u\ } & u^*TX \ .
\end{array}
$$

Here, the vertical maps ι are the natural $\mathbb{C}$-linear homomorphisms in (2.9) from the log tangent bundles $T\Sigma(-\log \mathfrak{z})$ and $TX(-\log D)$ into the classical tangent bundles $T\Sigma$ and TX, respectively; also see [15, Thm. 1.2]. By composing $\mathrm{d}^{\log} u$ with the homomorphism $T\Sigma(-\log z) \longrightarrow T\Sigma(-\log \mathfrak{z})$, we can also define

$$\mathrm{d}^{\log} u\colon T\Sigma(-\log z) \longrightarrow u^*TX(-\log D),$$

where

$$z \equiv z_1 + \cdots + z_k$$

is the full divisor corresponding to all marked points. However, in what follows, we will have $z = \mathfrak{z}$; therefore, the two maps are the same.

Briefly, the log derivative map $\mathrm{d}^{\log} u$ is defined in the following way. Away from the contact points $\mathfrak{z} = u^{-1}(D) \subset \{z_1, \ldots, z_k\}$, by the identification

$$TX(-\log D)|_{X-D} \stackrel{\iota}{\cong} TX|_{X-D},$$

we have $\mathrm{d}^{\log} u \cong \mathrm{d}\, u$. For each marked point $z_a \in \mathfrak{z}$, there are

- a local coordinate x on a sufficiently small neighborhood $\Delta \ni z_a$ of $z_a \in \Sigma$ (i.e., z_a is $(x=0)$),
- and, a local chart U_a around $u(z_a) \in D \subset X$,

such that

(1) U_a is identified with a neighborhood of $u(z_a)$ in the complex normal bundle $\pi\colon (\mathcal{N}_X D, \mathfrak{i}) \longrightarrow D$ as in (2.6);
(2) $T\mathcal{N}_X D$ admits a decomposition $T\mathcal{N}_X D \cong \pi^*TD \oplus \pi^*\mathcal{N}_X D$ as in (2.7);
(3) with respect to the identifications in (1) and (2), $J|_{TU_a}$ coincides with $\pi^*\big(J_D \oplus \mathfrak{i}\big)|_{U_a}$;
(4) the J-holomorphic map u decomposes into horizontal and vertical components as

$$u(x) = (\overline{u}_a(x); \zeta_a(x)) \in \mathcal{N}_X D$$

meaning that $\overline{u}_a\colon \Delta \longrightarrow D$ is a J_D-holomorphic map into D and ζ_a is a section of $\overline{u}_a^*\mathcal{N}_X D$;
(5) and, the section ζ_a is holomorphic with respect to a $\mathbb{C}$-linear $\bar{\partial}$-operator $\overline{u}_a^*\mathrm{D}^{\mathcal{N}}\bar{\partial}$ on $\overline{u}_a^*\mathcal{N}_X D$ induced by J along D.

Then,

- the holomorphic section ζ_a decomposes as $\zeta_a(x) = x^{s_a}\eta_a(x)$ with $\eta_a(0) \neq 0$,
- and, via the decomposition (2.8), $\mathrm{d}^{\log} u|_\Delta$ has an equation of the form

$$\mathrm{d}^{\log} u = \mathrm{d}\,\overline{u}_a \oplus \big(s_a \frac{\mathrm{d}\, x}{x} + \text{holomorphic terms}\big),$$

mapping the local generating section $x\partial x$ of $T\Sigma(-\log\mathfrak{z})|_\Delta$ to

$$\partial\overline{u}_a(x\partial x) \oplus \big(s_a + O(x)\big) \in T_{u(z_a)}D \oplus \mathbb{C} \cong TX(-\log D)|_{u(z_a)}.$$

Note that at $x = 0$, we have $(s_a + O(x))|_{x=0} = s_a$ and

$$(2.16) \qquad \mathrm{d}^{\log}_{z_a} u(x\partial x) = 0 \oplus s_a \in T_{u(z_a)}D \oplus \mathbb{C}.$$

Definition 2.3. We say $f \in \mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ is a **log immersion** if u is an immersion away from $\mathfrak{z}$ and $\mathfrak{z} = z$ (i.e., all of the marked points are contact points with D).

By (2.16) and similarly to the classical case [42, p. 284-285], if $f = [u, \Sigma, \vec{z} = (z_1, \ldots, z_k)]$ is a log immersion, then $\mathrm{d}^{\log} u$ is an embedding of vector bundles, the quotient

$$(2.17) \qquad \mathcal{N}_f \equiv u^*TX(-\log D)/\big(\mathrm{d}^{\log}_u T\Sigma(-\log z)\big)$$

is a complex vector bundle, and $\mathrm{D}^{\log}_u\bar{\partial}$ descends to an $\mathbb{R}$-linear Fredholm operator $\mathrm{D}^{\log}_{\mathcal{N}_f}\bar{\partial}$ on smooth sections of $\mathcal{N}_f$ such that

$$(2.18) \qquad \mathrm{Def}_{\log}(f) = \ker(\mathrm{D}^{\log}_{\mathcal{N}_f}\bar{\partial}) \quad \text{and} \quad \mathrm{Ob}_{\log}(f) = \mathrm{coker}(\mathrm{D}^{\log}_{\mathcal{N}_f}\bar{\partial}).$$

Here, the key point is that, by (2.16), $\mathrm{d}^{\log} u$ is automatically injective at every contact point with D, whereas $\mathrm{d}_{z_a} u$ could be zero if $s_a > 1$. We call $\mathcal{N}_f$ the **logarithmic normal bundle** of f. If only the relative marked points are concerned, one can define the logarithmic normal bundle of u to be

$$u^*TX(-\log D)/\big(\mathrm{d}^{\log}_u T\Sigma(-\log\mathfrak{z})\big).$$

In our desired application, $\mathfrak{z} = z$ and thus the two bundles are the same.

As usual, the Fredholm operator $\mathrm{D}^{\log}_{\mathcal{N}_f}\bar{\partial}$ decomposes as

$$(2.19) \qquad \mathrm{D}^{\log}_{\mathcal{N}_f}\bar{\partial} = \bar{\partial}_{\mathcal{N}_f} + R$$

such that $\bar{\partial}_{\mathcal{N}_f}$ is a $\mathbb{C}$-linear $\bar{\partial}$-operator defining a holomorphic structure on the log normal bundle $\mathcal{N}_f$ and R is an anti $\mathbb{C}$-linear zero-order operator depending on the Nijenhueis tensor of J. If $\mathrm{d}^{\log} u$ is not an embedding or $\mathfrak{z} \neq z$, we still obtain a short exact sequence of sheaves of $\mathcal{O}_\Sigma$-modules

$$0 \longrightarrow \mathcal{O}(T\Sigma(-\log z)) \xrightarrow{\mathrm{d}^{\log} u} \mathcal{O}(u^*TX(-\log D)) \longrightarrow \mathcal{O}(\widetilde{\mathcal{N}}_f) \longrightarrow 0$$

such that

$$\widetilde{\mathcal{N}}_f = \mathcal{O}(\mathcal{N}_f) \oplus \mathcal{N}_f^{\mathrm{tor}}$$

is the direct sum of the sheaf of the holomorphic sections of an $(\dim_{\mathbb{C}} X - 1)$-dimensional holomorphic vector bundle $\mathcal{N}_f$ and a skyscraper sheaf $\mathcal{N}_f^{\mathrm{tor}}$. Furthermore, $\mathrm{D}^{\log}_u\bar{\partial}$ descends to a Fredholm operator $\mathrm{D}^{\log}_{\mathcal{N}_f}\bar{\partial}$ on sections of $\mathcal{N}_f$ such that

$$(2.20) \qquad \mathrm{Def}_{\log}(f) = \ker(\mathrm{D}^{\log}_{\mathcal{N}_f}\bar{\partial}) \oplus H^0(\mathcal{N}_f^{\mathrm{tor}}) \quad \text{and} \quad \mathrm{Ob}_{\log}(f) = \mathrm{coker}(\mathrm{D}^{\log}_{\mathcal{N}_f}\bar{\partial});$$

see [38, Sec. 1.4] or [42] for a general account of the discussion above.

Lemma 2.4. *Suppose (X, D) is a symplectic log Calabi-Yau fourfold, $(\omega, J) \in \mathrm{AK}(X, D)$, $f = [u, \mathbb{P}^1, z_1] \in \mathcal{M}_{0,(s_A)}(X, D, A)$, and let*

$$n = \dim_{\mathbb{C}} H^0(\mathcal{N}_f^{\mathrm{tor}}) \geq 0.$$

Then, $\mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(-(1+n))$,

(2.21) $$\mathrm{Def}_{\log}(f) = H^0(\mathcal{N}_f^{\mathrm{tor}}) \quad and \quad \dim_{\mathbb{R}} \mathrm{Ob}_{\log}(f) = 2n.$$

In particular,

$$\mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(-1) \quad \Leftrightarrow \quad \mathrm{Ob}_{\log}(f) = 0 \quad \Leftrightarrow \quad f \text{ is a log immersion,}$$

and any of these implies that u *is somewhere injective and* $\mathrm{Def}_{\log}(f) = 0$.

Proof. First, by (1.1), we have

$$\dim \mathrm{Def}_{\log}(f) - \dim \mathrm{Ob}_{\log}(f) = 2\Big(0 + (1-0)(2-3) + 1\Big) = 0.$$

By Riemann-Roch,

$$\dim_{\mathbb{R}} \ker(\mathrm{D}_{\mathcal{N}_f}^{\log} \bar{\partial}) - \dim_{\mathbb{R}} \mathrm{coker}(\mathrm{D}_{\mathcal{N}_f}^{\log} \bar{\partial}) = 2\big(\deg(\mathcal{N}_f) + 1\big).$$

From (2.20), we conclude that $\mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(-(1+n))$. Since $-(1+n) < 0$, by [26, Thm. 1'] , for any choice of a compact operator R in (2.19), we have $\ker \mathrm{D}_{\mathcal{N}_f}^{\log} \bar{\partial} = 0$, which gives us (2.21). The conditions $\mathcal{N}_f \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, $\mathrm{Ob}_{\log}(f) = 0$, and f being a log immersion are all equivalent to $n = 0$. Every holomorphic map $h \colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2 or higher has at least two branch points. Therefore, since f has only one contact marked point z_1, if f is a log immersion then u must be somewhere injective and an immersion away from z_1; i.e., $n = 0$. $\square$

Remark 2.5. In the context of Lemma 2.4, if u is somewhere injective with cusp points

$$w_1, \ldots, w_k \in u^{-1}(X - D)$$

of orders $b_1, \ldots, b_k$ (see [38, Sec 1.5]), then $\mathcal{N}_f^{\mathrm{tor}}$ is the direct sum sky-scraper sheaf $\oplus_{i=1}^k \mathbb{C}_{w_i}^{b_i}$; in particular, $n = \sum_{i=1}^k b_i$. The marked point $z_1 = \infty$ can also be a cusp point of u, however, since the second term on the righthand side of (2.16) is non-zero at z_1, from the logarithmic perspective, $d_{z_1}^{\log} u$ is non-zero and z_1 behaves like a smooth point.

Remark 2.6. The discussion above only concerns maps in the virtually main stratum $\mathcal{M}_{g,\mathfrak{s}}(X, D, A)$ of the compactified moduli space $\overline{\mathcal{M}}_{g,\mathfrak{s}}(X, D, A)$. Together with some ad-hoc dimension counting, this is sufficient for proving Theorem 1.5. As noted in [22], in order describe the deformation-obstruction spaces along maps in other strata of $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A)$, one must systematically replace TX by $TX(-\log D)$. We illustrate how this should be done in the analysis of the virtual normal bundle to the fixed loci components in the localization calculations of Section 3.

2.2. **Proof of Theorem 1.5.** In [13, Prp. 1.7 and Crl. 1.9], we prove that if a pair (X, D) is positive in the sense of [18, Dfn. 4.7] or equally [13, Dfn. 1.6], and $E > 0$ is an arbitrary large number, then there exists a Baire subset

$$\mathrm{AK}^{\mathrm{reg}}(X, D) \in \mathrm{AK}(X, D)$$

of the second category such that for every (ω, J) in this set and $A \in H_2(X, \mathbb{Z})$ with $\omega(A) \le E$,

(1) the moduli space $\mathcal{M}_{0,(s_A)}^{\star}(X, D, A)$ is cut transversely and is a smooth manifold of real dimension

(2.22) $$2\big(\langle c_1(TX(-\log D)), A \rangle + (\dim_{\mathbb{C}} X - 3) + 1\big);$$

(2) the image of $\overline{\mathcal{M}}^{\mathrm{log}}_{0,(s_A)}(X,D,A) - \mathcal{M}^{\star}_{0,(s_A)}(X,D,A)$ under the evaluation map

$$(2.23) \qquad\qquad \mathrm{ev}\colon \mathcal{M}^{\star}_{0,(s_A)}(X,D,A) \longrightarrow D$$

lies in the image of smooth maps from finitely many smooth even-dimensional manifolds of at least 2 real dimension less than (2.22),

(3) and consequently, the map (2.23) defines a pseudo-cycle of real dimension (2.22) in D.

The same holds for the relative compactification $\overline{\mathcal{M}}^{\mathrm{rel}}_{0,(s_A)}(X,D,A)$ instead of

$$\overline{\mathcal{M}}^{\mathrm{log}}_{0,(s_A)}(X,D,A);$$

see [18, 19, 29]. Similarly to the proof of the classical results [34, Thm. 6.6.1] and [41, Prp. 3.21], the proof of (1)–(3) above is by showing that for generic J,

- each stratum of the simple part $\overline{\mathcal{M}}^{\star}_{0,(s_A)}(X,D,A)$ is cut transversely,
- and, under the positivity condition, non-simple or multiple-cover maps (more precisely, their image under ev) happen in real codimension 2 or higher.

The proof of (2) involves replacing a non-simple map f with an underlying simple map f' with "multi-nodes" (i.e., points at which more than one component intersect). This is done by (i) collapsing the ghost bubbles (ii) replacing each multiple-cover bubble component by its image curve (or the underlying simple map), and (iii) collapsing each sub-tree of the bubbles whose components have the same image. Under the weaker semi-positivity assumption, instead, the codimension of the collapsed stratum is a complicated formula that depends on the number of multi-nodes and a few other factors; see [41, (3.42)] and [13, (4.79)]. In particular, the real codimension will be a positive even number if f' has more than one irreducible component. Therefore, if (X,D) is semi-positive, as is the case in Theorem 1.5, the same argument as in the positive case shows that

(1) each stratum of $\overline{\mathcal{M}}^{\star}_{0,(s_A)}(X,D,A)$ is cut transversely,
(2) and, non-simple or multiple-cover maps f for which f' has at least two components happen in real codimension 2 or higher.

Since, by assumption, the expected dimension (2.22) is zero in Theorem 1.5, we conclude that for generic J

- the moduli space $\mathcal{M}^{\star}_{0,(s_A)}(X,D,A)$ is cut transversely and is a discrete set of points;
- every stratum in $\overline{\mathcal{M}}^{\star}_{0,(s_A)}(X,D,A) - \mathcal{M}^{\star}_{0,(s_A)}(X,D,A)$ has negative dimension (because we get 2 real codimension for each node), thus is empty;
- and, the only strata of multi-cover maps in

$$\mathcal{M}^{\mathrm{mc}}_{0,(s_A)}(X,D,A) \equiv \overline{\mathcal{M}}_{0,(s_A)}(X,D,A) - \overline{\mathcal{M}}^{\star}_{0,(s_A)}(X,D,A)$$

with non-negative dimension are those which are multiple covers of a single somewhere injective curve; i.e.,

$$\mathcal{M}^{\mathrm{mc}}_{0,(s_A)}(X,D,A) \cong \coprod_{\substack{B \in H_2(X,\mathbb{Z}) \\ dB=A, d>1}} \overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^* \times \mathcal{M}^{\star}_{0,(s_B)}(X,D,B).$$

Here, $\overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^*$ are the relative moduli spaces in 1.6,
In order to finish the proof of Theorem 1.5, it remains to show that

- $\mathcal{M}^\star_{0,(s_A)}(X, D, A)$ consists of finitely many points with no accumulation point in $\mathcal{M}^{\mathrm{mc}}_{0,(s_A)}(X, D, A)$,
- and, the subset of such regular almost Kähler structures $\mathrm{AK}^{\mathrm{reg}}(X, D) \subset \mathrm{AK}(X, D)$ is connected in each deformation equivalence class of $\omega \in \mathrm{Symp}_{\log}(X, D)$.

The first statement follows from the super-rigidity/automatic transversailty Lemma 2.4 in the following way. Suppose (X, D) is a symplectic log Calabi-Yau fourfold,

$$f = [u, \Sigma, z_1] \in \overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^* \times \mathcal{M}^\star_{0,(s_B)}(X, D, B) \subset \mathcal{M}^{\mathrm{mc}}_{0,(s_A)}(X, D, A)$$

and $\{f_i\}_{i\in\mathbb{N}}$ is a sequence in $\mathcal{M}^\star_{0,(s_A)}(X, D, A)$ that converges to f. Let

$$\overline{f} \equiv [\overline{u}, \mathbb{P}^1, \infty] \in \mathcal{M}^\star_{0,(s_B)}(X, D, B)$$

denote the simple map underlying f. By Lemma 2.4 and since $\overline{f}$ is cut transversely for all B with $\omega(B) < \omega(A)$, we have

$$\overline{u}^*TX(-\log D) \cong T\mathbb{P}^1(-\log\infty) \oplus \mathcal{N}_{\overline{f}} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1).$$

Therefore,

$$u^*TX(-\log D) \cong \mathcal{O}(d) \oplus \mathcal{O}(-d),$$

and the holomorphic sections of $\mathcal{O}(d)$ correspond to vector fields tangent to the image of u. Consequently, as in [44, Prp. B.1], there is i_0 such that for all $i > i_0$, f_i has the same image as f. That contradicts the simpleness of f_i if $d > 1$. This finishes the proof of the first bullet point above.

In order to prove the second bullet point, suppose (ω_1, J_1) and (ω_2, J_2) are two regular almost Kähler structures on (X, D) such that ω_1 and ω_2 are deformation equivalent in $\mathrm{Symp}_{\log}(X, D)$. By considering the space of all paths $(\omega_t, J_t)_{t\in[0,1]}$ connecting (ω_1, J_1) and (ω_2, J_2) and the 1-parameter family moduli space

$$(2.24) \qquad \overline{\mathcal{M}}_{0,(s_A)}\big(X, D, A; \{J_t\}_{t\in[0,1]}\big) \equiv \bigcup_{t\in[0,1]} \overline{\mathcal{M}}_{0,(s_A)}\big(X, D, A; J_t\big) \longrightarrow [0,1],$$

the same proof as above shows that for a generic path $(\omega_t, J_t)_{t\in[0,1]}$

(1) the moduli space $\mathcal{M}^\star_{0,(s_A)}\big(X, D, A; \{J_t\}_{t\in[0,1]}\big)$ is an (oriented) smooth 1-manifold;

(2) every stratum in

$$\overline{\mathcal{M}}^\star_{0,(s_A)}\big(X, D, A; \{J_t\}_{t\in[0,1]}\big) - \mathcal{M}^\star_{0,(s_A)}\big(X, D, A; \{J_t\}_{t\in[0,1]}\big)$$

has negative dimension (because we get 2 real codimension for each node) and therefore is empty;

(3) and, the only strata of multi-cover maps in $\mathcal{M}^{\mathrm{mc}}_{0,(s_A)}\big(X, D, A; \{J_t\}_{t\in[0,1]}\big)$ with non-negative dimension are those which are multiple covers of a single somewhere injective curve.

More precisely, suppose

$$f = [u, \Sigma, z_1] \in \overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^* \times \mathcal{M}^\star_{0,(s_B)}(X, D, B; J_t) \subset \mathcal{M}^{\mathrm{mc}}_{0,(s_A)}(X, D, A; J_t),$$

for some $t \in [0,1]$ and $\{f_i\}_{i \in \mathbb{N}}$ is a sequence in $\mathcal{M}^\star_{0,(s_A)}\big(X,D,A;J_{t_i}\big)$ that converges to f. We may assume that the underlying simple curve $\overline{f} = [\overline{u}, \mathbb{P}^1, \infty] \in \mathcal{M}^\star_{0,(s_B)}(X,D,B;J_t)$ is not a regular point of the projection

$$(2.25) \qquad \mathcal{M}_{0,(s_B)}\big(X,D,B;\{J_t\}_{t \in [0,1]}\big) \longrightarrow [0,1];$$

otherwise, for t_i sufficiently close to t, the same reasoning as above shows that the only other curves f_i in

$$\overline{\mathcal{M}}_{0,(s_A)}(X,D,A;J_{t_i})$$

near f are of the form $\overline{u'} \circ h'$ such that

$$[\overline{u'}, \mathbb{P}^1, \infty] \in \mathcal{M}^\star_{0,(s_B)}(X,D,B;J_{t_i})$$

is a deformation of $[\overline{u}, \mathbb{P}^1, \infty]$ and h' defines an element of $\overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^*$. Furthermore, for a generic path $(w_t, J_t)_{t \in [0,1]}$, every critical point of the projection map (2.25) satisfies

$$\dim_{\mathbb{R}} \mathrm{Ob}_{\log}(\overline{f}) = 1.$$

By the second identity in (2.21), the latter is impossible. Therefore, for generic path between (ω_1, J_1) and (ω_2, J_2), we have a similar decomposition

$$\overline{\mathcal{M}}_{0,(s_A)}\big(X,D,A;\{J_t\}_{t \in [0,1]}\big) \cong$$

$$\coprod_{\substack{B \in H_2(X,\mathbb{Z}) \\ dB=A, d \in \mathbb{Z}_+}} \overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^* \times \mathcal{M}^\star_{0,(s_B)}\big(X,D,B;\{J_t\}_{t \in [0,1]}\big).$$

This finishes the proof of (2) and thus Theorem 1.5.   □

3. Proof of Theorem 1.7

In this section, we first clarify and derive the first statement of Theorem 1.7 from Theorem 1.5. Subsequently, we briefly review the definition of relative maps and explain the localization calculation of (1.10).

3.1. **Gromov-Witten invariants** N_A. As highlighted in Remark 1.4, the general procedure for defining Gromov-Witten invariants in the symplectic category involves the construction of a virtual fundamental class. This class is subsequently used to define GW invariants through an integration or intersection argument. For dealing with relative moduli spaces $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X,D,A)$, such virtual treatments can be found in [11] and [37]. However, there are instances where the process can be streamlined by considering a workaround, such as considering a nice class of almost complex structures for which the moduli spaces have better geometric properties. Theorem 1.5 serves that purpose when it comes to defining genus zero GW invariants of log Calabi-Yau fourfolds. The theorem states that, for generic compatible almost complex structure, the moduli space $\overline{\mathcal{M}}_{0,(s_A)}(X,D,A)$ decomposes into a finite union of irreducible components with varying dimensions. Each component is an orbifold, and the associated obstruction bundle is an (orbifold) vector bundle. Notably, this decomposition remains invariant within a generic one-parameter family of almost complex structures. Consequently, the ensemble of orbifolds and orbifold vector bundles defines a Kuranishi structure of dimension 0 on $\overline{\mathcal{M}}_{0,(s_A)}\big(X,D,A\big)$. With additional analysis, it could be demonstrated that this Kuranishi structure yields the same virtual fundamental class as defined in [11];

see [47, Thm. 1.2.(1)] for an example of how this comparison can be done. However, such a comparison proves unnecessary, because the numbers n_A are defined geometrically and (1.9), taken as the defining equation of N_A, is clearly related to the algebraic construction. In simpler terms, for symplectic log CY fourfolds, the Gromov-Witten invariants can be straightforwardly defined as the sum of contributions from each component. Within each component, the contribution is represented by the Euler characteristic of the obstruction bundle and can be calculated by a localization argument. This approach bears strong resemblance to the strategy employed by Ionel-Parker in proving the Gopakumar-Vafa conjecture for closed symplectic Calabi-Yau sixfolds, where a similar comparison with existing abstract analytical virtual techniques in the literature is not explicitly provided, but the calculations are clearly related to the algebraic construction. The following explains the argument above in more details and establishes the first statement of Theorem 1.7.

By Theorem 1.5, for generic choice of J, the moduli space $\overline{\mathcal{M}}_{0,(s_A)}(X, D, A)$ decomposes into a disjoint union of closed components

$$(3.1) \qquad \overline{\mathcal{M}}_{0,(s_A)}(X, D, A) = \mathcal{M}^{\star}_{0,(s_A)}(X, D, A) \cup \mathcal{M}^{\mathrm{mc}}_{0,(s_A)}(X, D, A)$$

where $\mathcal{M}^{\star}_{0,(s_A)}(X, D, A)$ is a finite set of (oriented) points and $\mathcal{M}^{\mathrm{mc}}_{0,(s_A)}(X, D, A)$ is a disjoint union of positive-dimensional closed oriented orbifolds. Furthermore, by Theorem 1.5 again, for every two regular almost complex structures J_1 and J_2, there is a path of almost complex structures $\{J_t\}_{t \in [0,1]}$ such that the 1-parameter family moduli space $\overline{\mathcal{M}}_{0,(s_A)}(X, D, A; \{J_t\}_{t \in [0,1]})$ in (2.24) similarly decomposes into a disjoint union of closed components

$$\overline{\mathcal{M}}_{0,(s_A)}\big(X, D, A; \{J_t\}_{t \in [0,1]}\big) \cong$$
$$\mathcal{M}^{\star}_{0,(s_A)}(X, D, A; \{J_t\}_{t \in [0,1]}) \cup \mathcal{M}^{\mathrm{mc}}_{0,(s_A)}(X, D, A; \{J_t\}_{t \in [0,1]}).$$

In conclusion, the relative GW invariant

$$n_A \equiv \#\mathcal{M}^{\star}_{0,(s_A)}(X, D, A) \in \mathbb{Z}$$

is well-defined and counts the number of logarithmically immersed degree A rational curves in X with maximal tangency at a single point with D.

Since a generic path $\{J_t\}_{t \in [0,1]}$ preserves the decomposition (3.1), we may abstractly define

$$\mathbb{Q} \ni N_A \equiv \sum_{\substack{B \in H_2(X, \mathbb{Z}) \\ dB = A, d > 0}} \mathrm{mc}(d, s_B) n_B \qquad \forall A \in H_2(X, \mathbb{Z})$$

as in [1, Thm. 1.5]. However, by considering the Euler class of the obstruction bundle over each closed component of $\overline{\mathcal{M}}_{0,(s_A)}(X, D, A)$, we can indeed equip $\overline{\mathcal{M}}^{\mathrm{rel}}_{0,(s_A)}(X, D, A)$ with a zero-dimensional natural virtual fundamental class (or **vir**) such that

$$N_A = \#[\overline{\mathcal{M}}^{\mathrm{rel}}_{0,(s_A)}(X, D, A)]^{\mathrm{vir}} \equiv \int_{[\overline{\mathcal{M}}^{\mathrm{rel}}_{0,(s_A)}(X, D, A)]^{\mathrm{vir}}} 1.$$

This is a simple case of a similar construction/definition of **vir** in [47, Thm. 1.2.(1)].

Remark 3.1. To be consistent with the setup in the items (1)–(3) including (2.23), we may as well define

$$n_A = \deg\Big(\operatorname{ev} \big(\mathcal{M}^\star_{0,(s_A)}(X, D, A) \big) \Big) \in H_0(D, \mathbb{Z}) \cong \mathbb{Z}.$$

To complete the proof of Theorem 1.7, the remaining task involves elucidating the computation of the multiple-cover contributions (1.10), which will be the focus of the remainder of this section. Prior to delving into that, in Section 3.2 , and for the reader's convenience, we revisit the definition of relative compactification and establish notations that will be explicitly used in the localization calculation in Section 3.3.

3.2. **Relative compactification.**

In this section, following the description in [18, Sec. 4], we briefly review the construction of the relative stable map moduli spaces $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,\mathfrak{s}}(X, D, A)$ and clarify the notation used in the rest of the paper (specifically, the localization calculation below). For a comprehensive exploration of proofs and justifications, we direct the reader to [18, Sec. 4] and [12, 29].

Suppose $D \subset (X, \omega)$ is a smooth symplectic divisor, J is an ω-tame almost complex structure on X compatible with D, and $\bar\partial_{\mathcal{N}_X D}$ is the $\bar\partial$-operator arising from J on the normal bundle

$$\pi \colon \mathcal{N}_X D \equiv \frac{TX|_D}{TD} \longrightarrow D. \tag{3.2}$$

For any space Y and every complex line bundle $L \longrightarrow Y$, let $Y \times \mathbb{C}$ denote the trivial line bundle on Y and $L \oplus Y \times \mathbb{C}$ denote the direct sum of the two line bundles. Let

(3.3)
$$\mathbb{P}_X D = \mathbb{P}(\mathcal{N}_X D \oplus D \times \mathbb{C}), \qquad D_0 = \mathbb{P}(0 \oplus D \times \mathbb{C}) \text{ and } D_\infty = \mathbb{P}(\mathcal{N}_X D \oplus 0) \subset \mathbb{P}_X D.$$

Here, direct sum with 0 denotes restricting to the zero-section in the line bundle $\mathcal{N}_X D$ or $D \times 0$. As the projectivization of $\mathbb{C}$ is a point, both D_0 and D_∞ are naturally isomorphic to D.

The splitting (2.7) extends to a splitting of the exact sequence

$$0 \longrightarrow T^{\mathrm{vrt}}(\mathbb{P}_X D) \longrightarrow T(\mathbb{P}_X D) \xrightarrow{\mathrm{d}\,\pi} \pi^* TD \longrightarrow 0,$$

where $\pi \colon \mathbb{P}_X D \longrightarrow D$ is the bundle projection map induced by π in (3.2); this splitting restricts to the canonical splittings over $D_0 \cong D_\infty \cong D$ and is preserved by the multiplication by $\mathbb{C}^*$. Via this splitting, the almost complex structure $J_D = J|_{TD}$ and the complex structure $\mathfrak{i}$ in the fibers of π induce an almost complex structure $J_{X,D}$ on $\mathbb{P}_X D$ which restricts to J_D on D_0 and D_∞ and is preserved by the $\mathbb{C}^*$-action. In fact, $J_{X,D}|_{\mathcal{N}_X D}$ is the almost complex structure $J_{\bar\partial_{\mathcal{N}_X D}}$ associated to $\bar\partial_{\mathcal{N}_X D}$. The projection $\pi \colon \mathbb{P}_X D \longrightarrow D$ is $(J_D, J_{X,D})$-holomorphic and there is a one-to-one correspondence between the space of $J_{X,D}$-holomorphic maps $u \colon \Sigma \longrightarrow \mathbb{P}_X D$ and tuples (u_D, ζ) where $u_D \colon \Sigma \longrightarrow D$ is a J_D-holomorphic map into D and ζ is a meromorphic section of $u_D^* \mathcal{N}_X D$ with respect to the holomorphic structure defined by $u^* \bar\partial_{\mathcal{N}_X D}$ on $u^* \mathcal{N}_X D$.

For each $\ell \in \mathbb{N}$, the ℓ-th expanded degeneration of X is the normal crossings variety

$$X[\ell] = \big(X \sqcup \{1\} \times \mathbb{P}_X D \sqcup \ldots \sqcup \{\ell\} \times \mathbb{P}_X D \big)/\sim, \tag{3.4}$$

where

$$D \sim \{1\} \times D_\infty, \quad \{r\} \times D_0 \sim \{r+1\} \times D_\infty \quad \forall r = 1, \ldots, \ell-1;$$

see Figure 1. There exists a continuous projection map $\pi_\ell \colon X[\ell] \longrightarrow X$ which is identity on X and π on each $\mathbb{P}_X D$. We denote by J_ℓ the almost complex structure on $X[\ell]$ so that

$$J_\ell|_X = J_X \qquad \text{and} \qquad J_\ell|_{\{r\} \times \mathbb{P}_X D} = J_{X,D} \quad \forall r = 1, \dots, \ell.$$

For each $(t_1, \dots, t_\ell) \in \mathbb{C}^*$, define

$$\Theta_{t_1, \dots, t_\ell} \colon X[\ell] \longrightarrow X[\ell]$$

$$\text{by} \quad \Theta_{t_1, \dots, t_\ell}(x) = \begin{cases} (r, [t_r v, w]), & \text{if } x = (r, [v, w]) \in \{r\} \times \mathbb{P}_X D; \\ x, & \text{if } x \in X. \end{cases}$$

This diffeomorphism is biholomorphic with respect to J_ℓ and preserves the fibers of the projection $\mathbb{P}_X D \longrightarrow D$ and the sections D_0 and D_∞.

Definition 3.2. For $\ell > 0$, a level ℓ k-marked **relative** J-holomorphic map of contact type $\mathfrak{s} = (s_1, \dots, s_k) \in \mathbb{N}^k$ is a continuous map $u \colon \Sigma \longrightarrow X[\ell]$ from a connected marked nodal curve $C = \big(\Sigma, (z_1, \dots, z_k)\big)$ such that

$$u^{-1}\big(\{\ell\} \times D_0\big) \subset \{z^1, \dots, z^k\}, \quad \operatorname{ord}_{z_a}(u, \{\ell\} \times D_0) = s_a \quad \forall z_a \in u^{-1}\big(\{\ell\} \times D_0\big),$$

$s_a = 0$ if and only if $z_a \notin u^{-1}\big(\{\ell\} \times D_0\big)$, and the restriction of u to each irreducible component Σ_j of Σ is either

(1) a J-holomorphic map to X such that the set $u|_{\Sigma_j}^{-1}(D)$ consists of the nodes joining Σ_j to irreducible components of Σ mapped to $\{1\} \times \mathbb{P}_X D$, or

(2) a $J_{X,D}$-holomorphic map to $\{r\} \times \mathbb{P}_X D$ for some $r = 1, \dots, \ell$ such that
   (a) the set $u|_{\Sigma_j}^{-1}(\{r\} \times D_\infty)$ consists of the nodes $q_{j,i}$ joining Σ_j to irreducible components of Σ mapped to $\{r-1\} \times \mathbb{P}_X D$ if $r > 1$ and to X if $r = 1$ and

$$\operatorname{ord}_{q_{j,i}}\big(u, D_\infty\big) = \begin{cases} \operatorname{ord}_{q_{i,j}}(u, D_0), & \text{if } r > 1, \\ \operatorname{ord}_{q_{i,j}}(u, D), & \text{if } r = 1, \end{cases}$$

   where $q_{i,j} \in \Sigma_i$ is the point identified with $q_{j,i}$,
   (b) if $r < \ell$, the set $u|_{\Sigma_j}^{-1}(\{r\} \times D_0)$ consists of the nodes joining Σ_j to irreducible components of Σ mapped to $\{r+1\} \times \mathbb{P}_X D$;

see Figure 1.

The **genus** and the **degree** of such a map $u \colon \Sigma \longrightarrow X[\ell]$ are the arithmetic genus of Σ and the homology class

$$A = \big[\pi_\ell \circ u\big] \in H_2(X, \mathbb{Z}).$$

Two tuples (u, C) and (u', C') as above are **equivalent** if there exist a biholomorphic map $\varphi \colon \Sigma \longrightarrow \Sigma'$ and $t_1, \dots, t_\ell \in \mathbb{C}^*$ so that

$$\varphi(z_a) = z'_a \quad \forall a = 1, \dots, k \quad \text{and} \quad u' = \Theta_{t_1, \dots, t_\ell} \circ u \circ \varphi.$$

A tuple as above is **stable** if it has finitely many automorphisms (self-equivalences). If $A \in H_2(X, \mathbb{Z})$, $g, k \in \mathbb{N}$, and $\mathfrak{s} = (s_1, \dots, s_k) \in \mathbb{N}^k$ is a tuple satisfying

$$\sum_{a=1}^k s_a = A \cdot D,$$

then the **relative moduli space**

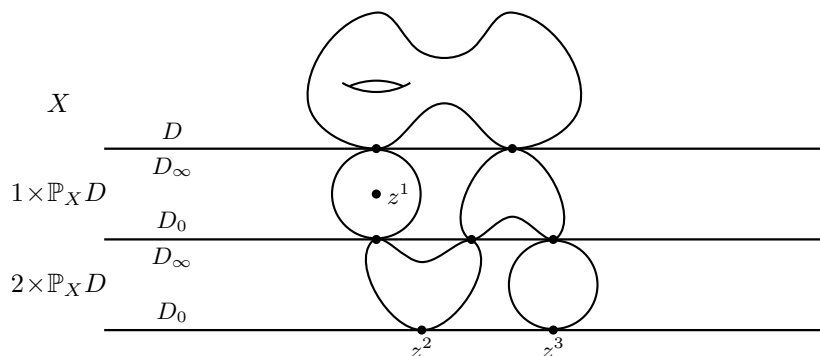$$(3.5) \qquad \overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A)$$

FIGURE 1. A genus 2 relative map with $k=3$ and $\mathfrak{s}=(0,2,2)$ into the expanded degeneration $X[2]$

is the set of equivalence classes of such connected stable k-marked genus g degree A J-holomorphic maps into $X[\ell]$ for any $\ell \in \mathbb{N}$. If X is compact, the latter space has a natural compact Hausdorff topology with respect to which the forgetful map

$$\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\text{rel}}(X, D, A) \longrightarrow \overline{\mathcal{M}}_{g,k}(X, A)$$

is continuous. The logarithmic compactification $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X, D, A)$ defined in [12] replaces the level structure with a partial ordering on the components of the domain and collapses unstable components that are maps from $\mathbb{P}^1$ with less than three special points to fibers of $\mathbb{P}_X D$.

3.3. **Relative localization.** For generic J as in the statement of Theorem 1.5, each component

$$\overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^* \times \mathcal{M}_{0,(s_B)}^\star(X, D, B)$$

in (1.5) is n_B copy of the irreducible complex $(d-1)$-dimensional orbifold

$$\overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^*.$$

The obstruction bundle $\text{Ob}(d, s_B)$ on this component is a complex rank $d-1$ vector bundle and the virtual fundamental class is the top chern class of $\text{Ob}(d, s_B)$, which can be seen as a rational number

$$(3.6) \qquad \text{mc}(d, s_B) \equiv \int_{\overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^*} c_{\text{top}}\big(\text{Ob}(d, s_B)\big)$$

in $H_{\text{top}}(\overline{\mathcal{M}}_{0,(d)}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)^*) \cong \mathbb{Q}$. In the following, we show how a natural action on $\mathbb{P}^1$ lifts to this orbibundle and can be used to calculate $\text{mc}(d, s_B)$. Consider the $\mathbb{C}^*$-action

$$[x_0, x_1] \longrightarrow [x_0, tx_1] \qquad \forall t \in \mathbb{C}^*$$

on $\mathbb{P}^1$ corresponding to weights $(\alpha_0, \alpha_1) = (0, -1)$ in [27, Sec. 27.1]. Let $y_0 = x_1/x_0$ and $y_1 = x_0/x_1$ be the affine coordinates around

$$p_0 \equiv [1, 0] \quad \text{and} \quad p_1 \equiv [0, 1] \in \mathbb{P}^1.$$

The weights of the action on $T_{p_0}\mathbb{P}^1$ (generated by ∂y_0) and $T_{p_1}\mathbb{P}^1$ (generated by ∂y_1) are 1 and -1, respectively. Different lifts of the given action to $\mathcal{O}(1) \longrightarrow \mathbb{P}^1$ are classified by the action-weights $(m+1, m)$ on $\big(\mathcal{O}(1)|_{p_0}, \mathcal{O}(1)|_{p_1}\big)$. Since $T\mathbb{P}^1(-\log p_1)|_{p_0}$ and $T\mathbb{P}^1(-\log p_1)|_{p_1}$ are generated by ∂y_0 and $y_1\partial y_1$, the line bundle

$$T\mathbb{P}^1(-\log p_1) \cong \mathcal{O}(1)$$

has weights $(1, 0)$ at (p_0, p_1).

Let

(3.7) $$h\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad h(y_0) = y_0^d,$$

be the Galois d-covering of $\mathbb{P}^1$ fully ramified at p_0 and p_1. By [27, Exe. 27.2.3], for m as above, the weights of the lifted action on

$$H^1(\mathbb{P}^1, h^*\mathcal{O}(-1)) = H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1} \otimes h^*\mathcal{O}(1))^\vee$$

are

$$-\left(\frac{j}{d} + m\right), \qquad 1 \le j \le d-1,$$

where the $-$ sign comes from the Serre duality.[8] Therefore, the product of the weights of the $\mathbb{C}^*$-action on $H^1(\mathbb{P}^1, h^*\mathcal{O}(-1))$ is

(3.8) $$\frac{(-1)^{d-1}}{d^{d-1}} \prod_{j=1}^{d-1} (j + md).$$

This number will determine the contribution of the obstruction bundle below with $\mathcal{O}(-1)$ being the logarithmic normal bundle of a log immersed curve $\overline{u}$.

The $\mathbb{C}^*$-action on $\mathbb{P}^1$ naturally lifts to an $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_{0,(d)}^{\mathrm{rel}}\big(\mathbb{P}^1, p_1 = \infty, [d\mathbb{P}^1]\big)^*$ by post-composition. The fixed point loci of this action come in families of various dimensions and are characterized by (unordered) partitions

$$\mathcal{P}\colon \big(d = d_1 + \cdots + d_k\big), \quad d_i > 0,$$

of d in the following way.

- If $k = 1$, the fixed point locus is a single point and corresponds to the degree d Galois cover in (3.7) of the fixed base simple curve $\overline{f} = [\overline{u}, \mathbb{P}^1, p_1] \in \mathcal{M}_{0,(s_B)}^\star(X, D, B)$.
- If $k \ge 2$, the fixed locus $\overline{\mathcal{M}}_\mathcal{P}$ corresponding to $\mathcal{P}$ is the compactification of the open dense subset $\mathcal{M}_\mathcal{P} \cong \mathcal{M}_{0,k+1}/\sim$ described below. Figure 2 illustrates the domain and image of an element in $\mathcal{M}_\mathcal{P}$ with $k = 3$.

Following the notation in Section 3.2, the relative curves in $\mathcal{M}_u$ consist of k (unordered) Galois covers

$$h_i\colon (\mathbb{P}^1, \infty) \longrightarrow (\mathbb{P}^1, p_1), \qquad i = 1, \ldots, k,$$

of degrees $d_1, \ldots, d_k$ of $\overline{f}$ connected to a genus 0 $(k+1)$-marked curve that is mapped $(ds_B):1$ to the fiber

$$\mathbb{P}^1 \cong F_q \subset \mathbb{P}_X D = \mathbb{P}(\mathcal{N}_X D \oplus \mathbb{C})$$

over $q = \overline{u}(p_1) \in D$. In a suitable coordinate z (so that the intersection with the

---

[8]The statement of [27, Ex. 27.2.6] is missing the sign factor arising from the Serre duality.
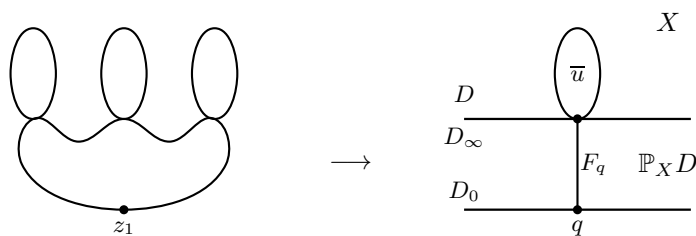
FIGURE 2. A relative map with $k=3$ into the expanded degeneration $X[1] = X \cup \mathbb{P}_X D$

divisor $D_0 \subset \mathbb{P}_X D$ happens at $z = \infty$), the map to F_q is given by the polynomial

$$(3.9) \qquad \varrho(z) = \Big(\prod_{i=1}^{k} (z - z_i)^{d_i} \Big)^{s_B} : \mathbb{P}^1 \to F_q \cong \mathbb{P}^1$$

and is fully ramified at the infinity. Since the nodes of f are unordered, the notation $\mathcal{M}_{0,k+1}/\sim$ means $\mathcal{M}_{0,k+1}$ divided by a permutation group of marked points z_i and z_j with $d_i = d_j$. The identification $\mathcal{M}_{\mathcal{P}} \cong \mathcal{M}_{0,k+1}/\sim$ is given by

$$\varrho(z) \Leftrightarrow [\mathbb{P}^1, z_1, \ldots, z_k, \infty] \in \mathcal{M}_{0,k+1}/\sim .$$

In the discussion above, we may alternatively put an order on the nodal points $z_1, \ldots, z_k$ and divide (3.13) by $k!$.

Note that if $s_B = 1$, then $\overline{\mathcal{M}}_{\mathcal{P}} = \overline{\mathcal{M}}_{0,(d)}^{\mathrm{rel}}\big(\mathbb{P}^1, p_1, [d\mathbb{P}^1]\big)$. However, as we mentioned earlier, if $s_B > 1$, due to the extra factor of s_B in (3.9), the algebraic moduli structure or the Kuranishi structure of the moduli space of d-covers of $\overline{f}$ slightly differs from the standard structure on $\overline{\mathcal{M}}_{0,(d)}^{\mathrm{rel}}\big(\mathbb{P}^1, p_1, [d\mathbb{P}^1]\big)$, and is indicated by a superscript $*$.

Remark 3.3. As highlighted earlier, the log compactification constructed in [12] is smaller than the relative compactification. Theorem 1.5 is applicable to both choices of log and relative compactifications, with the distinction only manifesting in the multiple cover contibutions. More specifically, in the log compactification $\overline{\mathcal{M}}_{0,(d)}^{\mathrm{log}}\big(\mathbb{P}^1, p_1, [d\mathbb{P}^1]\big)^*$, the fixed loci of the $\mathbb{C}^*$-action are still indexed by $\mathcal{P}$, the open dense part $\mathcal{M}_{\mathcal{P}}$ of $\overline{\mathcal{M}}_{\mathcal{P}}^{\mathrm{log}}$ is the same as above, and $\overline{\mathcal{M}}_{\mathcal{P}}^{\mathrm{log}} \cong \overline{\mathcal{M}}_{0,k+1}/\sim$. In the relative compactification, however, $\overline{\mathcal{M}}_{\mathcal{P}}$ is a complicated blowup of $\overline{\mathcal{M}}_{0,k+1}/\sim$ away from $\mathcal{M}_{0,k+1}/\sim$.

In the following, we first explain the localization contribution of the isolated point (3.7) to (3.6) when $k = 1$, and then describe the contributions of other strata corresponding to $k > 1$.

For

$$f = [u = \overline{u} \circ h, C = (\mathbb{P}^1, \infty)] \in \mathcal{M}_{0,(s_A)}(X, D, A),$$

if $\overline{u}$ is a log immersion, we have

$$\overline{u}^* TX(-\log D) = T\mathbb{P}^1(-\log p_1) \oplus \mathcal{N}_{\overline{f}} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$$

and the Deformation-Obstruction long exact sequence

$$0 \longrightarrow \mathrm{aut}(C) \longrightarrow \mathrm{Def}_{\log}(u) \longrightarrow \mathrm{Def}_{\log}(f) \longrightarrow$$
$$\mathrm{Def}(C) \longrightarrow \mathrm{Ob}_{\log}(u) \longrightarrow \mathrm{Ob}_{\log}(f) \longrightarrow 0,$$

in (2.14) reads

$$0 \longrightarrow H^0\big(\mathbb{P}^1, T\mathbb{P}^1(-\log\infty)\big) \longrightarrow H^0\big(\mathbb{P}^1, h^*T\mathbb{P}^1(-\log p_1)\big) \longrightarrow \mathrm{Def}_{\log}(f) \longrightarrow 0,$$
$$0 \longrightarrow H^1\big(\mathbb{P}^1, h^*\mathcal{N}_{\overline{f}}\big) \longrightarrow \mathrm{Ob}_{\log}(f) \longrightarrow 0.$$

Since the line bundle $T\mathbb{P}^1(-\log p_1) \cong \mathcal{O}(1)$ has weights $(1,0)$ at (p_0, p_1), by [27, Exe. 27.2.3], the localization contribution of $\mathrm{Def}_{\log}(f)$ is

$$\frac{d!}{d^{d-1}}.$$

Therefore, by (3.8) and considering the discrete automorphism factor $1/d$ of h in (3.7), the contribution of the d-fold Galois cover to N_A is

$$(3.10) \quad \frac{(-1)^{d-1}}{d^2}\frac{\prod_{j=1}^{d-1}(j+md)}{(d-1)!} = \frac{(-1)^{d-1}}{d^2}\binom{d(m+1)-1}{d-1} = \frac{1}{d^2}\binom{-md-1}{d-1}.$$

In the proof of [24, Prp. 6.1], the authors state that the right value of m to consider for the other contributions to vanish is

$$m = 1 - s_B,$$

which yields the coefficient

$$(3.11) \qquad \mathrm{mc}(d, s_B) = \frac{(-1)^{d-1}}{d^2}\binom{d(2-s_B)-1}{d-1} = \frac{1}{d^2}\binom{d(s_B-1)-1}{d-1}.$$

The justification (for the vanishing of other terms) is left to the reader and the claim is stated to follow from a similar argument as in the proof of [7, Thm. 5.1] and examining the obstruction space in [22].

After the following remarks, we describe a different choice of weight that is in-line with the relative localization of Graber-Vakil in [22] and is applicable to the Fredholm setup introduced above.

Remark 3.4. The moduli space of concern is (more or less) a simple case ($r = 1, \alpha_1 = d, b_1 = (d)$) of [7, Dfn. 3.1]. When $s_B = 1$, the rank $d-1$ obstruction bundle

$$\mathrm{Ob}(d,1) \longrightarrow \overline{\mathcal{M}}_{0,(d)}^{\mathrm{rel}}\big(\mathbb{P}^1, \infty, [d\mathbb{P}^1]\big)$$

is $\mathcal{R}^1\pi_*f^*\mathcal{O}(-1)$; see the proof of [4, Lmm. 5.9]. The author could not find any reference in which the obstruction bundle is explicitely identified as above when $s_B > 1$. The proof of [7, Thm. 5.1] also concerns the case $s_B = 1$ and the chosen weights on the logarithmic normal bundle $\mathcal{O}(-1)$ are $(-1,0)$. Items (i)–(iv) in [7, p. 387] explain the reasons for the vanishing of the other strata in the fixed locus (i.e., $\overline{\mathcal{M}}_{\mathcal{P}}$ with $k > 1$).

Remark 3.5. In [24], the base curve $\overline{u}\colon \mathbb{P}^1 \longrightarrow X$ is (implicitly) assumed to be an immersion so that its (regular) normal bundle is defined and is the line bundle $\mathcal{O}(s_B - 2) \longrightarrow \mathbb{P}^1$. In the light of Lemma 2.4, the map $\overline{u}$ can have a cusp point at the intersection point with D and still be a smooth point of the moduli space $\mathcal{M}_{0,(s_B)}(X, D, A)$. The local contribution should still be the same as (3.11) because the cusp point can be smoothed out without affecting the transversality.

In the setup of Graber-Vakil's paper [22], they require an $\mathbb{C}^*$-action on the entire target X that preserves D (and the induced action on $\mathcal{N}_X D$ has non-zero weights). In the approach of [24] reviewed above, we only have an action on the domain $\mathbb{P}^1$ of $\overline{u}$. Then, we need to identify the obstruction bundle $\mathrm{Ob}(d, s_B)$ and lift the given $\mathbb{C}^*$-action on the base to the fibers of $\mathrm{Ob}(d, s_B)$. This can be done in many ways and different choices correspond to different values of m above.

The weight considered on $\mathcal{N}_X D|_q$ in [24] is zero. Even though we don't have a global $\mathbb{C}^*$-action on X, a neighborhood of $\overline{u}$ can be identified with a neighborhood of $\mathbb{P}^1$ in the normal bundle $\mathcal{O}(s_B - 2)$ and the action can be (infinitesimally) lifted to such a neighborhood. With respect to local coordinates (y_1, c_1) on the total space of $\mathcal{O}(s_B - 2)$ at p_1, the intersection of D with such a local neighborhood is given by the local model equation

$$(c_1 - y_1^{s_B} = 0) \subset \mathbb{C}^2.$$

In order for this equation (in other words, the intersection of D with this neighborhood) to be invariant under the $\mathbb{C}^*$-action, the weight of the action on c_1 must be $-s_B$. That means the weight of the action on $\mathcal{O}(s_B - 2)|_{p_1}$ must be $-s_B$ which corresponds to $m = 1$ and

$$(3.12) \qquad \frac{(-1)^{d-1}}{d^2}\binom{d(m+1)-1}{d-1} = \frac{(-1)^{d-1}}{d^2}\binom{2d-1}{d-1}$$

in (3.10).

In the rest of this section, following the discussion above and by using the resulting weight $(-s_B)$ on the normal bundle $\mathcal{N}_X D$, we explain the localization contributions of the fixed curves in $\mathcal{M}_{\mathcal{P}}$ with $k > 1$. In [22], the authors assume that the $\mathbb{C}^*$-action fixes D to obtain the relative virtual localization formula [22, Thm. 3.6]. The local action described above only preserves D. Nevertheless, if $s_B > 1$, the long-exact sequence in [22, p. 14] splits into easy to understand terms from which we obtain an explicit relative virtual localization formula.

Since the case $s_B = 1$ is well-studied, assume $s_B > 1$. For

$$f = \big(u\colon C = (\Sigma, z_1) \longrightarrow (X[1], D_0)\big) \in \mathcal{M}_{\mathcal{P}}$$

illustrated in Figure 2, let

$$f_1 = \big(u_1\colon C_1 = (\Sigma_1, (z_{1i})_{i=1}^k) \longrightarrow$$
$$(X, D)\big) = \bigcup_{i=1}^k \big(u_{1i} = \overline{u} \circ h_{1i}\colon C_{1i} = (\mathbb{P}^1_{1i}, \infty) \longrightarrow (X, D)\big)$$

denote the union of components corresponding to the k Galois covers and

$$f_2 = \Big(u_2\colon C_2 = \big(\mathbb{P}^1_2, (z_{2i})_{i=1}^k \cup z_1\big) \longrightarrow (\mathbb{P}_X D, D_\infty \cup D_0)\Big)$$

denote the rubber component in $\mathbb{P}_X D$ given by (3.9). As noted in [22], the analysis of the virtual normal bundle to such a component is identical to the case of ordinary stable maps with the bundle TX systematically replaced by $TX(-\log D)$. In fact, the contribution of $\overline{\mathcal{M}}_{\mathcal{P}}$ to $N_{0,dB}$ is of the form

$$(3.13) \qquad \mathrm{Cont}(\mathcal{M}_{\mathcal{P}}) = \frac{\prod_{i=1}^k (d_i s_B)}{|\mathrm{Aut}(\mathcal{P})| \prod_{i=1}^k d_i} \frac{e(\mathrm{Ob}_{\log}(u)^{\mathrm{mov}})e(\mathrm{aut}(C)^{\mathrm{mov}})}{e(\mathrm{Def}_{\log}(u)^{\mathrm{mov}})e(\mathrm{Def}(C)^{\mathrm{mov}})}$$

where

- $\mathrm{Def}_{\log}(u)$ and $\mathrm{Ob}_{\log}(u)$ are calculated via the long-exact sequence

(3.14)
$$0 \longrightarrow \mathrm{Def}_{\log}(u) \longrightarrow H^0(\Sigma_1, u_1^* TX(-\log D)) \oplus \widetilde{H}^0(\mathbb{P}_2^1, u_2^* T\mathbb{P}_X D(-\log D_\infty \cup D_0))$$

$$\xrightarrow{\mathrm{Res}} \bigoplus_{i=1}^{k} T_q D \longrightarrow \mathrm{Ob}_{\log}(u) \longrightarrow H^1(\Sigma_1, u_1^* TX(-\log D))$$

$$\oplus\, H^1(\mathbb{P}_2^1, u_2^* T\mathbb{P}_X D(-\log D_\infty \cup D_0)) \longrightarrow 0$$

as in [22, p. 14];
- $\mathrm{Aut}(\mathcal{P})$ is the symmetry group of $\mathcal{P}$;
- $\prod(d_i s_B)$ is the product of the tangency orders at the nodes because each f arises as a limit of that many distinct curves in the main stratum,
- and, the product $\prod d_i$ in the denominator corresponds to the discrete automorphisms of h_i.

Moreover, in (3.14)

- $q = u_2(z_{2i}) = u_1(z_{1i})$, for all $i = 1, \ldots, k$, is the image of the nodes between f_1 and f_2 in D;
- and, $\widetilde{H}^0(\mathbb{P}_2^1, u_2^* T\mathbb{P}_X D(-\log D_\infty \cup D_0))$ is the quotient of

$$H^0(\mathbb{P}_2^1, u_2^* T\mathbb{P}_X D(-\log D_\infty \cup D_0))$$

by the complex 1-dimensional subspace corresponding to the $\mathbb{C}^*$-action on $\mathbb{P}_X D$. Since

$$u_2^* T\mathbb{P}_X D(-\log D_\infty \cup D_0) = (\mathbb{P}^1 \times T_q D) \oplus (\mathbb{P}^1 \times \mathbb{C}),$$

we have

$$\widetilde{H}^0(\mathbb{P}_2^1, u_2^* T\mathbb{P}_X D(-\log D_\infty \cup D_0)) \cong T_q D$$

$$\text{and} \quad H^1(\mathbb{P}_2^1, u_2^* T\mathbb{P}_X D(-\log D_\infty \cup D_0)) = 0.$$

Since $s_B > 1$, for all $i = 1, \ldots, k$, the residue map

$$\mathrm{Res} \colon H^0(\mathbb{P}_{1i}^1, u_{1i}^* TX(-\log D)) \longrightarrow T_q D$$

is zero. Also

$$\widetilde{H}^0(\mathbb{P}_2^1, u_2^* T\mathbb{P}_X D(-\log D_\infty \cup D_0)) \cong T_q D \xrightarrow{\mathrm{Res}} \bigoplus_{i=1}^{k} T_q D$$

is the diagonal embedding. Therefore,

$$\mathrm{Def}_{\log}(u) \cong H^0(\Sigma_1, u_1^* TX(-\log D))$$

and

$$0 \longrightarrow \frac{\bigoplus_{i=1}^{k} T_q D}{T_q D} \longrightarrow \mathrm{Ob}_{\log}(u) \longrightarrow H^1(\Sigma_1, u_1^* TX(-\log D)) \longrightarrow 0.$$

Note that,

$$H^1(\Sigma_1, u_1^* TX(-\log D)) = \bigoplus_{i=1}^{k} H^1(\mathbb{P}_{1i}^1, h_{1i}^* \mathcal{O}(-1)).$$

Decomposing $\mathrm{Def}(C)$ into the moving and fixed factors, $\mathrm{Def}(C) = \mathrm{Def}(C)^{\mathrm{fix}} \oplus \mathrm{Def}(C)^{\mathrm{mov}}$, the moving part corresponds to the simultaneous smoothing of the nodes and has the equivariant contribution $-s_B - \psi$. Here, $-s_B$ is the weight of the action on $\mathcal{N}_X D|_q$ and ψ is the relative ψ-class in [22, Sec. 2.5]. The fixed

part $\mathrm{Def}(C)^{\mathrm{fix}}$ corresponds to the deformations of $\mathcal{M}_{\mathcal{P}}$. We conclude that the localization contribution $\mathrm{Cont}(\overline{\mathcal{M}}_{\mathcal{P}})$ of $\mathcal{P}$ is given by the formula

$$
\mathrm{Cont}(\overline{\mathcal{M}}_{\mathcal{P}}) = \frac{s_B^k}{|\mathrm{Aut}(\mathcal{P})|}\Big(\prod_{i=1}^{k} d_i \mathrm{Cont}(f_{1i})\Big)(-1)^{k-1}\int_{\overline{\mathcal{M}}_{\mathcal{P}}}\frac{1}{-s_B-\psi}
$$

$$
(3.15) \qquad = \frac{s_B}{|\mathrm{Aut}(\mathcal{P})|}\Big(\prod_{i=1}^{k} d_i \mathrm{Cont}(f_{1i})\Big)\int_{\overline{\mathcal{M}}_{\mathcal{P}}}\psi^{k-2}
$$

$$
= \frac{s_B}{|\mathrm{Aut}(\mathcal{P})|}\prod_{i=1}^{k}\frac{(-1)^{d_i-1}}{d_i}\binom{2d_i-1}{d_i-1}\int_{\overline{\mathcal{M}}_{\mathcal{P}}}\psi^{k-2}
$$

where $\mathrm{Cont}(f_{1i})$ is the contribution of the Galois cover f_{1i} with $m=1$ in (3.12) and $(-1)^{k-1}$ is the contribution of $\frac{\oplus_{i=1}^{k}T_qD}{T_qD}$ to $e\big(\mathrm{Ob}_{\log}(f)^{\mathrm{mov}}\big)$.

For instance, if $d=2$ and $\mathcal{P}=\{1,1\}$, then $\overline{\mathcal{M}}_{\mathcal{P}}$ is a point, $\mathrm{Aut}(\mathcal{P})$ has order 2, and

$$
\mathrm{Cont}(\mathcal{M}_{\mathcal{P}}) = \frac{s_B}{2}.
$$

Together with the contribution of the Galois cover (i.e. $\mathcal{P}=(2)$), which is

$$
\frac{(-1)^{2-1}}{2^2}\binom{2\times 2-1}{2-1} = -3/4,
$$

we get

$$
\mathrm{mc}(2,s_B) = \frac{-3}{4}+\frac{2s_B}{4} = \frac{1}{2^2}\binom{2(s_B-1)-1}{2-1}.
$$

If $d=3$, the partitions of 3 are $\mathcal{P}_1=(3),\mathcal{P}_2=(2,1)$, and $\mathcal{P}_3=(1,1,1)$. Since $\overline{\mathcal{M}}_{\mathcal{P}_2}$ is a point, by (3.12) and (3.15), we have

$$
\mathrm{Cont}(\mathcal{M}_{\mathcal{P}_1}) = \frac{10}{9}, \quad \mathrm{Cont}(\mathcal{M}_{\mathcal{P}_2}) = \frac{-3s_B}{2}.
$$

Also, by [22, Rmk. 3.4], we have

$$
\int_{\overline{\mathcal{M}}_{\mu_2}}\psi = \frac{1}{3!}\int_{\overline{\mathcal{M}}_{0,4}}3s_B\psi_1 = s_B/2;
$$

therefore,

$$
\mathrm{Cont}(\mathcal{M}_{\mathcal{P}_3}) = s_B^2/2.
$$

We conclude that

$$
\mathrm{mc}(3,s_B) = \frac{10}{9}-\frac{3s_B}{2}+\frac{s_B^2}{2} = \frac{1}{3^2}\binom{3(s_B-1)-1}{3-1}.
$$

Remark 3.6. The examples above suggest that the degree k term of (3.11), as a polynomial in s_B, is the sum of all $\mathrm{Cont}(\mathcal{M}_{\mathcal{P}})$ where $\mathcal{P}$ is a partition of d into $k+1$ summands.

4. Higher genus invariants with maximal tangency

In this section, we study a higher genus version of the invariants N_A and n_A, and explain a decomposition of genus one invariants into different contributions.

4.1. **Higher genus integrality conjecture.** Suppose (X, D) is a symplectic log Calabi-Yau fourfold and $g \geq 1$. By (1.1), the real expected dimension of

$$\overline{\mathcal{M}}_{g,(s_A)}(X, D, A)$$

is $2g$. The relative GW invariants

$$(4.1) \qquad N_{g,A} = \int_{\left[\overline{\mathcal{M}}^{\mathrm{rel}}_{g,(s_A)}(X,D,A)\right]^{\mathrm{vir}}} (-1)^g \lambda_g \in \mathbb{Q} \qquad \forall A \in H_2(X, \mathbb{Z}), s_A > 0,$$

are defined by pairing the (algebraic) virtual fundamental class of $\overline{\mathcal{M}}^{\mathrm{rel}}_{g,(s_A)}(X, D, A)$ with the degree $2g$ Hodge class λ_g. The latter is the top chern class of a rank g (orbifold) vector bundle

$$\mathbb{E} \longrightarrow \overline{\mathcal{M}}_{g,(s_A)}(X, D, A)$$

whose fiber over (u, Σ, z_1) with smooth domain is the space of holomorphic 1-forms on Σ; more generally, $\mathbb{E}$ is defined to be the push-forward of the relative dualizing sheaf on the universal curve space $\mathcal{C}_{g,(s_A)}(X, D, A)$. These invariants are defined and studied in [24, Sec. 5.8] and [4]. The fact that the top lambda class is a natural insertion to consider in higher genus stems from the fact that $N_{g,A}$ are related to the higher genus invariants of the symplectic sixfold $X \times \mathbb{P}^1$, and (via localization) the top lambda class with the appropriate sign $(-1)^g$ measures the difference between complex 2 and 3 dimensional obstruction theories; see [4, p.13].
Let

$$(4.2) \qquad F_A(q) = \frac{\sin(h/2)}{h/2} \sum_{\substack{B \in H_2(X,\mathbb{Z}) \\ dB=A, d>0}} (-1)^{s_B - 1} \mu(d) d^{2g-2} \sum_{g \geq 0} N_{g,B} h^{2g},$$

where μ is the Möbius function, $q = \mathrm{e}^{\mathrm{i}h/2}$, and $N_{0,A} \equiv N_A$ are the genus 0 invariants defined in (1.8).

Conjecture 4.1 (c.f. [4, Cnj. 8.3]). *The function F_A in (4.2) is a well-defined rational function of q, invariant under $q \to q^{-1}$; furthermore, it is a Laurent polynomial in q with integer coefficients.*

Note that, letting $h \longrightarrow 0$, only the genus zero terms survive, $\sin(h/2)$ cancels with $h/2$, and thus the value $F_A(1)$, which is the sum of the coefficients of the Laurent polynomial, is the genus 0 BPS invariant

$$\widetilde{n}_A = \sum_{\substack{B \in H_2(X,\mathbb{Z}) \\ dB=A, d>0}} \frac{\mu(d)}{d^2} (-1)^{s_B - 1} N_B \in \mathbb{Z}.$$

The inverse of the formula above is

$$N_A = \sum_{\substack{B \in H_2(X,\mathbb{Z}) \\ dB=A, d>0}} \frac{(-1)^{s_B(d-1)}}{d^2} \widetilde{n}_B.$$

This formula looks different from (1.9) but the integrality of $\widetilde{n}_A$ is equivalent to the integrality of the genus 0 BPS invariants n_A in (1.9). A non-elementary but conceptual proof of this equivalence follows from Donaldson-Thomas theory of quivers; see [4, Sec. 8.5].

In the following, first, we explain how the higher genus invariants $N_{g,A}$ can be analytically defined without using virtual techniques at the cost of using (Ruan-Tian)-type global perturbations. Then, we revisit the unperturbed setup and explain the issue of multi-cover contributions, with a particular focus on genus one invariants.

4.2. **Higher genus invariants of semi-positive pairs.** From the analytic point of view, for $g \geq 1$, in order to construct a (virtual) fundamental class for

$$\overline{\mathcal{M}}_{g,(s_A)}(X, D, A),$$

we can use the logarithmic/relative Ruan-Tian perturbations as in [13,29] in the following way. For $g, k \in \mathbb{N}$ with $2g + k \geq 3$ (which is the case for all $g \geq 1$ and $k = 1$), let $\mathcal{T}_{g,k}$ denote the Teichmüler space of genus g Riemann surfaces with k marked points and by $\mathcal{G}_{g,k}$ the corresponding mapping class group. We have

$$\mathcal{M}_{g,k} = \mathcal{T}_{g,k}/\mathcal{G}_{g,k}.$$

Assume $g = g_1 + g_2$ and $k = k_1 + k_2$ with $2g_i + k_i \geq 3$ for $i = 1, 2$. For any decomposition $S_1 \cup S_2$ of $\{1, 2, \ldots, k\}$ with $|S_i| = k_i$, there exists a canonical immersion

$$(4.3) \qquad \iota = \iota_{S_1,S_2} \colon \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \longrightarrow \partial\overline{\mathcal{M}}_{g,k}$$

which assigns to a pair of marked curves

$$\left(C_i = [\Sigma_i, z_{i,1}, \ldots, z_{i,k_i+1}]\right)_{i=1,2},$$

the marked curve

$$C = [\Sigma, z_1, \ldots, z_k], \quad \Sigma = \Sigma_1 \sqcup \Sigma_2 / z_{1,k_1+1} \sim z_{2,k_2+1}$$
$$\{z_1, \ldots, z_k\} = \{z_{1,1}, \ldots, z_{1,k_1}\} \cup \{z_{2,1}, \ldots, z_{2,k_2}\},$$

so that the remaining marked points are renumbered by $\{1, \ldots, k\}$ according to the decomposition $S_1 \cup S_2$. There is also another natural immersion

$$(4.4) \qquad \delta \colon \overline{\mathcal{M}}_{g-1,k+2} \longrightarrow \partial\overline{\mathcal{M}}_{g,k}$$

which is obtained by gluing together the last two marked points.

Definition 4.2 ([48, Dfn. 2.1]). Let $g, k \in \mathbb{N}$ with $2g + k \geq 3$, and

$$(4.5) \qquad p \colon \overline{\mathfrak{M}}_{g,k} \longrightarrow \overline{\mathcal{M}}_{g,k}$$

be a finite branched cover in the orbifold category. A **universal family over** $\overline{\mathfrak{M}}_{g,k}$ is a tuple

$$(4.6) \qquad \left(\pi \colon \overline{\mathfrak{U}}_{g,k} \longrightarrow \overline{\mathfrak{M}}_{g,k}, \mathfrak{z}_1, \ldots, \mathfrak{z}_k\right),$$

where $\overline{\mathfrak{U}}_{g,k}$ is a complex projective variety and π is a projective morphism with disjoint sections $\mathfrak{z}_1, \ldots, \mathfrak{z}_k$ such that for each $c \in \overline{\mathfrak{M}}_{g,k}$ the tuple

$$C = \left(\Sigma = \pi^{-1}(c), (z_1, \ldots, z_k) = (\mathfrak{z}_1(c), \ldots, \mathfrak{z}_k(c))\right)$$

is a stable k-marked genus g curve whose equivalence class is $[C] = p(c)$.

Definition 4.3. Let $g, k \in \mathbb{N}$ with $2g + k \geq 3$. A cover (4.5) is **regular** if

(1) it admits a universal family,
(2) each topological component of $p^{-1}(\mathcal{M}_{g,k})$ is the quotient of $\mathcal{T}_{g,k}$ by a subgroup of $\mathcal{G}_{g,k}$,

(3) for each boundary divisor (4.3) we have

$$\left(\overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1}\right) \times_{(\iota,p)} \overline{\mathfrak{M}}_{g,k} \approx \overline{\mathfrak{M}}_{g_1,k_1+1} \times \overline{\mathfrak{M}}_{g_2,k_2+1},$$

for some regular covers $\overline{\mathfrak{M}}_{g_i,k_i+1}$ of $\overline{\mathcal{M}}_{g_i,k_i+1}$, and
(4) for the boundary divisor (4.4) we have

$$\overline{\mathcal{M}}_{g-1,k+2} \times_{(\delta,p)} \overline{\mathfrak{M}}_{g,k} \approx \overline{\mathfrak{M}}_{g-1,k+2},$$

for some regular cover $\overline{\mathfrak{M}}_{g-1,k+2}$ of $\overline{\mathcal{M}}_{g-1,k+2}$.

The last two conditions are inductively well-defined. The existence of such regular covers is a consequence of [2, Prp. 2.2, Thm. 2.3, Thm. 3.9]; see also moduli space of curves with *level n structures* in [35, p. 285]. In the genus 0 case, for each $k \geq 3$, the moduli space $\overline{\mathcal{M}}_{0,k}$ itself is smooth and the universal curve

$$\overline{\mathcal{C}}_{0,k} = \overline{\mathcal{M}}_{0,k+1} \longrightarrow \overline{\mathcal{M}}_{0,k}$$

is already a universal family. The regular covers are only branched over the boundaries of the moduli space. Furthermore, the total space of a universal family as in (4.6) over a regular cover only has singularities of the form

$$\{(x,y,t) \in \mathbb{C}^3 \colon xy = t^m\} \longrightarrow \mathbb{C}, \qquad (x,y,t) \longrightarrow t$$

at the nodal points of the fibers of π. In the original approach of [41], for dealing with such singularities they consider embeddings of a universal family into $\mathbb{P}^N$ for sufficiently large N.
Fix a regular covering (4.5) and a universal family (4.6). Denote by

$$\overline{\mathfrak{U}}_{g,k}^\star \subset \overline{\mathfrak{U}}_{g,k}$$

the complement of the nodes of the fibers of the projection map π in (4.6). Denote by

$$T_{g,k} = \ker \mathrm{d}(\pi|_{\overline{\mathfrak{U}}_{g,k}^\star}) \longrightarrow \overline{\mathfrak{U}}_{g,k}^\star$$

the vertical tangent bundle. The latter is a complex line bundle; we denote the complex structure by $\mathrm{j}_{\mathfrak{U}}$. Then

$$\Omega_{g,k}^{0,1} := (T_{g,k}, -\mathrm{j}_{\mathfrak{U}})^* \longrightarrow \overline{\mathfrak{U}}_{g,k}^\star$$

is the complex line bundle of vertical (or relative) $(0,1)$-forms. It is possible to extend this construction to the nodal points by allowing simple poles and dual residues, or by embedding $\overline{\mathfrak{U}}_{g,k}$ into some $\mathbb{P}^M$ as in [41].
Let (X,ω) be a symplectic manifold and J be an ω-tame almost complex structure on X. The classical space of perturbations considered in [41] (following the modification in [48]) is the infinite dimensional linear space

$$(4.7) \quad \mathcal{H}_{g,k}(X,J) = \left\{ \nu \in \Gamma\left(\overline{\mathfrak{U}}_{g,k}^\star \times X, \pi_1^* \Omega_{g,k}^{0,1} \otimes_\mathbb{C} \pi_2^* TX\right) \right.$$
$$\left. \text{s.t. } \mathrm{supp}(\nu) \subset (\overline{\mathfrak{U}}_{g,k}^\star - \bigcup_{a=1}^k \mathrm{Im}(\mathfrak{z}_a)) \times X \right\},$$

where π_1, π_2 are projection maps from $\overline{\mathfrak{U}}_{g,k}^\star \times X$ onto the first and second components, respectively, and $\mathrm{supp}(\nu)$ is the closure of the complement of the vanishing locus of ν in the compact space $\overline{\mathfrak{U}}_{g,k} \times X$. Let $\mathcal{H}_{g,k}(X,\omega)$ denote the space of tuples (J,ν) where J is ω-tame and $\nu \in \mathcal{H}_{g,k}(X,J)$. Note that given ν and a boundary

component as in Definition 4.3 (3) (resp. Definition 4.3 (4)), the restriction of ν to $\overline{\mathfrak{M}}_{g_1,k_1+1}$ gives a perturbation term in $\mathcal{H}_{g_1,k_1}(X,J)$ (resp. $\mathcal{H}_{g-1,k+2}(X,J)$).

Definition 4.4. Suppose $g,k \in \mathbb{N}$ with $2g+k \geq 3$, $\overline{\mathfrak{U}}_{g,k}$ is a universal family as in (4.6), (X,ω) is a symplectic manifold, $A \in H_2(X,\mathbb{Z})$, and $(J,\nu) \in \mathcal{H}_{g,k}(X,\omega)$. A **degree A genus g k-marked (J,ν)-map** is a tuple

$$(4.8) \qquad f = \Big(\phi, u, C = \big(\Sigma, (z_1, \ldots, z_k) \big) \Big),$$

where C is a nodal genus g k-marked complex curve, $\phi \colon \Sigma \longrightarrow \overline{\mathfrak{U}}_{g,k}$ is a holomorphic map onto a fiber of $\overline{\mathfrak{U}}_{g,k}$ preserving the marked points, and $u \colon \Sigma \longrightarrow X$ represents the homology class A and satisfies

$$\bar{\partial} u = (\phi, u)^* \nu.$$

Two k-marked (J,ν)-holomorphic maps $\big(\phi_1, u_1, C_1 \big)$ and $\big(\phi_2, u_2, C_2 \big)$ are **equivalent** if there exists a holomorphic identification h of C_1 and C_2 such that $(\phi_1, u_1) = (\phi_2, u_2) \circ h$. A (J,ν)-holomorphic map is **stable** if it has a finite automorphism group. A **contracted** component of Σ in (4.8) is a smooth component whose image under the map ϕ is just a point. A map (4.8) is stable if and only if the degree of the restriction of u to every contracted component of Σ containing only one or two special (nodal or marked) points is not zero. If (4.8) is stable, every connected cluster of contracted components is a tree of spheres, with a total of at most 2 special[9] points, at least one of which is a nodal point. For generic ν, the only components of Σ contributing non-trivially to the automorphism group of (4.8) are the contracted components.

In order to perturb J-holomorphic curves with tangency condition relative to a (smooth or normal crossing) divisor $D \subset X$, we need to restrict to subset of $\mathcal{H}_{g,k}(X,J)$ consisting of perturbations that are "compatible" with D in suitable sense. In [18, 29], the latter is expressed in terms of a first order condition on ν along D. In [13], we consider the logarithmic perturbation space

$$(4.9) \qquad \left\{ \begin{aligned} &\nu_{\log} \in \Gamma \big(\overline{\mathfrak{U}}_{g,k}^\star \times X, \pi_1^* \Omega_{g,k}^{0,1} \otimes_{\mathbb{C}} \pi_2^* TX(-\log D) \big) \\[2mm] &\text{s.t. } \mathrm{supp}(\nu_{\log}) \subset \big(\overline{\mathfrak{U}}_{g,k}^\star - \bigcup_{a=1}^{k} \mathrm{Im}(\mathfrak{z}_a) \big) \times X \end{aligned} \right\}.$$

Associated to each $\nu_{\log}$ we get a classical perturbation term

$$(4.10) \qquad \nu = \iota(\nu_{\log}) \in \mathcal{H}_{g,k}(X,J),$$

where by abuse of notation ι denotes the $\mathbb{C}$-linear homomorphism

$$\pi_1^*(\Omega_{g,k}^{0,1}) \otimes_{\mathbb{C}} \pi_2^* TX(-\log D) \longrightarrow \pi_1^*(\Omega_{g,k}^{0,1}) \otimes_{\mathbb{C}} \pi_2^* TX$$

induced by (2.9). Define $\mathcal{H}_{g,\mathfrak{s}}(X,D)$ to be the space of tuples $(\omega, J, \nu_{\log})$ where $(\omega, J) \in \mathrm{AK}(X,D)$ and $\nu_{\log}$ belongs to (4.9). For such a tuple $(\omega, J, \nu_{\log})$, it is shown in [13] that the moduli space $\mathcal{M}_{g,\mathfrak{s}}(X,D,A,\nu)$ consisting of equivalence classes of genus g k-marked (J,ν)-holomorphic maps of tangency type $\mathfrak{s}$ is defined and has a natural compactification $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\log}(X,D,A,\nu)$ satisfying properties similarly

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[9]Either a marked point or a nodal point connecting the cluster to an irreducible component of Σ outside the cluster.

to the unperturbed case. The relative moduli space $\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A, \nu)$ can be defined similarly and admits a surjective map

$$\overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{rel}}(X, D, A, \nu) \longrightarrow \overline{\mathcal{M}}_{g,\mathfrak{s}}^{\mathrm{log}}(X, D, A, \nu)$$

as in the unperturbed case.

If (X, D) is a symplectic log CY fourfold, then it is semi-positive in the sense of [18, Dfn. 4.7] or equally [13, Dfn. 1.6]. Therefore, by [13, Prp. 1.7 and Crl. 1.9], we have the following result.

Theorem 4.5. *Suppose (X, D) is a symplectic log CY fourfold, $A \in H_2(X, \mathbb{Z})$, and $g > 0$. Then, for any given choice of universal family in (4.6), there exists a Baire set of second category $\mathcal{H}_{g,(s_A)}^{\mathrm{reg}}(X, D) \subset \mathcal{H}_{g,(s_A)}(X, D)$ such that for each $(\omega, J, \nu_{\log}) \in \mathcal{H}_{g,(s_A)}^{\mathrm{reg}}(X, D)$,*

(1) *the moduli space $\mathcal{M}_{g,(s_A)}^{\star}(X, D, A)$ is cut transversely and is a smooth manifold of real dimension $2g$,*

(2) *the image of $\overline{\mathcal{M}}_{g,(s_A)}(X, D, A) - \mathcal{M}_{g,(s_A)}^{\star}(X, D, A)$ under the forgetful map*

(4.11) $$\mathrm{st} : \overline{\mathcal{M}}_{g,(s_A)}(X, D, A) \longrightarrow \overline{\mathcal{M}}_{g,1}$$

lies in the image of smooth maps from finitely many smooth even-dimensional manifolds of at least 2 real dimension less than $2g$,

(3) *and consequently, the map (4.11) defines a pseudo-cycle of real dimension $2g$ in $\overline{\mathcal{M}}_{g,1}$ whose integral homology class $[\mathrm{st}]$ only depends on the deformation equivalence class of (X, D, ω) and the degree $\deg p$ of the regular covering (4.5) used to define $\nu_{\log}$.*

Corollary 4.6. *The rational homology class*

$$\left[\overline{\mathcal{M}}_{g,(s_A)}(X, D, A) \right]^{\mathrm{vir}} \equiv \frac{1}{\deg p} [\mathrm{st}] \in H_{2g}(\overline{\mathcal{M}}_{g,1}, \mathbb{Q}),$$

is an invariant of the deformation equivalence class of (X, ω, D) and can be used to define (4.1). Furthermore, for every $g > 0$, there exists a constant c_g (independent of the choice of (X, D) and A) such that

$$c_g N_{g,A} \in \mathbb{Z} \qquad \forall A \in H_2(X, \mathbb{Z}), \ \text{with } s_A > 0.$$

Using the perturbed setting above proves to be an effective method for defining the invariants $N_{g,A}$. However, it falls short when it comes to defining integer-valued invariants and comprehending the geometric significance of the integers posited by Conjecture 4.1. In the subsequent section, we eschew these perturbations and explore strategies for addressing the irregularity present in higher genus (unperturbed) moduli spaces. The section concludes with a conjecture concerning the integrality of genus one.

4.3. **Reduced genus one invariants.** For $g > 0$, given a 1-marked smooth curve (Σ, p), the relative space $\overline{\mathcal{M}}_{g,(d)}(\Sigma, p, [d\Sigma])$ of genus g degree d covers fully ramified over p is not necessarily a smooth orbifold of the correct complex dimension. Furthermore, the following observations illustrate the complicated nature of relative multiple-cover maps in higher genus.

First, we have the following weaker analogue of Lemma 2.4 which is again a corollary of [26, Thm 1'].

Lemma 4.7. *Suppose (X, D) is a symplectic log Calabi-Yau fourfold and*

$$f = [u, \Sigma, z_1] \in \mathcal{M}_{g,(s_A)}(X, D, A).$$

If f is a log immersion then $\deg(\mathcal{N}_f) = 2g - 1$,

$$\mathrm{Def}_{\log}(f) = H^0(\mathcal{N}_f) \quad and \quad \mathrm{Ob}_{\log}(f) = H^1(\mathcal{N}_f) = 0,$$

where $\mathcal{N}_f$ is the logarithmic normal bundle defined in (2.17). In other words, $\mathcal{M}_{g,(s_A)}(X, D, A)$ is cut transversally in a neighborhood of every log immersion.

In [40], given an elliptic curve Σ and an integer $g \geq 2$, the author constructs a genus g curve Σ' and a map $h\colon \Sigma' \longrightarrow \Sigma$ of degree $2g - 1$, so that h is ramified above exactly one point of Σ, and so that the local monodromy above that point is of type a $(2g - 1)$-cycle. Together with Lemma 4.7, we get the following observation. Suppose $u\colon (\Sigma, p) \longrightarrow (X, D)$ is a genus 1 log immersion in $\mathcal{M}_{1,(s_A)}(X, D, A)$ and $h\colon (\Sigma', z_1) \longrightarrow (\Sigma, p)$ is a degree $2g - 1$ map ramified at exactly p. Then, by Lemma 4.7, the moduli space $\mathcal{M}_{g,(s_A)}(X, D, A)$ is cut transversely in a neighborhood of the multiple-cover map $u \circ h\colon (\Sigma', z_1) \longrightarrow (X, D)$. Therefore, u can not have a multiple-cover contribution to an integral-valued invariant arising from $\overline{\mathcal{M}}_{g,(s_A)}(X, D, A)$. Nevertheless, for $g = 1$, the possible multiple-covers are well-understood and moduli space $\overline{\mathcal{M}}_{1,(s_A)}(X, D, A)$ and the invariant $N_{1,A}$ can be explicitly described as follows.

For $A \neq 0$, let $\mathcal{M}'_{1,(s_A)}(X, D, A)$ denote the subset of $\overline{\mathcal{M}}_{1,(s_A)}(X, D, A)$ consisting of the relative/log stable maps $[u, C = (\Sigma, z_1)]$ such that Σ is an elliptic curve E with 1 rational component attached directly to it and $u|_E$ is constant (therefore, the restriction of u to the rational component has degree A). In other words,

$$\mathcal{M}'_{1,(s_A)}(X, D, A) \cong \mathcal{M}_{0,(0,s_A)}(X, D, A) \times \mathcal{M}_{1,1}.$$

We denote by $\overline{\mathcal{M}}'_{1,(s_A)}(X, D, A)$ the closure of $\mathcal{M}'_{1,(s_A)}(X, D, A)$ in $\overline{\mathcal{M}}_{1,(s_A)}(X, D, A)$. By [45, Thm 1.2], the moduli space $\overline{\mathcal{M}}_{1,(s_A)}(X, D, A)$ admits a closed subspace $\overline{\mathcal{M}}^{\mathrm{main}}_{1,(s_A)}(X, D, A)$ that contains the virtually main stratum $\mathcal{M}_{1,(s_A)}(X, D, A)$ and is invariant under deformations of J. Furthermore, if (X, D) is a log Calabi-Yau fourfold and J is generic, the moduli space $\overline{\mathcal{M}}_{1,(s_A)}(X, D, A)$ decomposes as

$$(4.12) \qquad \overline{\mathcal{M}}_{1,(s_A)}(X, D, A) = \overline{\mathcal{M}}^{\mathrm{main}}_{1,(s_A)}(X, D, A) \cup \overline{\mathcal{M}}'_{1,(s_A)}(X, D, A).$$

By (4.12), we have

$$N_{1,A} = N^{\mathrm{main}}_{1,A} + N'_{1,A}$$

such that $N'_{1,A}$ is a function of $\{n_B\}_{dB=A}$ determined by the lemma below.

Lemma 4.8. *We have*

$$N'_{1,A} = \sum_{\substack{dB=A \\ d>0}} \frac{(2 - s_B)}{24} \binom{d(s_B - 1) - 1}{d - 1} n_B \in \frac{1}{24}\mathbb{Z}.$$

Proof. For the same choice of weight as in (3.11), the contribution of a logarithmically immersed rational curve $[\overline{u}, (\mathbb{P}^1, \infty)] \in \mathcal{M}^{\star}_{0,(s_B)}(X, D, A)$ to $N'_{1,A=dB}$ has the form

$$(4.13) \qquad \int_{\overline{\mathcal{M}}_{1,1}} \left(\frac{1}{d} \frac{e(\mathrm{Ob}_{\log}(u)^{\mathrm{mov}}) e(\mathrm{aut}(C)^{\mathrm{mov}})}{e(\mathrm{Def}(u)^{\mathrm{mov}}) e(\mathrm{Def}_{\log}(C)^{\mathrm{mov}})} \times (-\lambda_1) \right),$$

where $C = (E \cup_{q \sim 0} \mathbb{P}^1, \infty)$ is the 1-marked nodal domain obtained by attaching $\mathbb{P}^1$ and an elliptic curve E at the points 0 and q, respectively, with the marked point ∞ on the rational component. Over $\mathbb{P}^1$, the map f is the d-fold Galois cover $(\mathbb{P}^1, 0 \cup \infty) \longrightarrow (\mathbb{P}^1, p_0 \cup p_1)$, and over E the map is constant p_0. Compared to the calculations leading to (3.10), the node in the domain contributes a factor of

$$\frac{1}{\frac{1}{d} - \psi_1} = d\frac{1}{1 - d\psi_1}$$

to the moving part $\mathrm{Def}(C)^{\mathrm{mov}}$ and the automorphism factor $\mathrm{aut}(C)$ of $(\mathbb{P}^1, 0 \cup \infty)$ is the weight zero representation

$$H^0(\mathbb{P}^1, T\mathbb{P}^1(-\log 0 \cup \infty)) = \mathbb{C} \cdot (z_0 \partial z_0).$$

These two changes each contribute an extra factor of d to the resulting fraction in (3.10). Since the weights on the tangent space of the (standard) normal bundle to $\overline{u}$ at p_0 are $s_B - 2$ and 1, by [27, p. 550], the obstruction bundle over E contributes a factor of

$$(4.14) \qquad (s_B - 2)c(\mathbb{E}^\vee)(1/(s_B - 2))c(\mathbb{E}^\vee)(1)$$

to $e(\mathrm{Ob}_{\log}(f)^{\mathrm{mov}}/e(\mathrm{Def}_{\log}(f)^{\mathrm{mov}})$ where $\mathbb{E}$ is the Hodge bundle over E and

$$c(Q)(t) = 1 + tc_1(Q) + \cdots + t^r c_r(Q)$$

for any complex vector bundle Q of rank r. By dimensional reason, only the degree zero term $s_B - 2$ of (4.14) contributes to the integral (4.13). Since

$$\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{1}{24},$$

we conclude that (4.13) is equal to

$$\frac{(2 - s_B)}{24} \binom{d(s_B - 1) - 1}{d - 1}.$$

$\square$

Remark 4.9. For $s_B = 1$, we get

$$\frac{(2 - s_B)}{24} \binom{d(s_B - 1) - 1}{d - 1} = \frac{(-1)^{d-1}}{24}$$

which is the number calculated in [7, (13)].

The reduced moduli space $\overline{\mathcal{M}}_{1,(s_A)}^{\mathrm{main}}(X, D, A)$ may still include components of higher than expected dimension. For instance, if $A = dB$, $\overline{\mathcal{M}}_{1,(s_A)}^{\mathrm{main}}(X, D, A)$ includes multiple-covers of the form

$$u\colon (E, q) \xrightarrow{h} (\mathbb{P}^1, \infty) \xrightarrow{\overline{u}} (X, D),$$

where $h \in \mathcal{M}_{1,(d)}(\mathbb{P}^1, \infty, [d])$ is a genus 1 multiple-cover of $\mathbb{P}^1$ fully ramified at ∞. Note that

$$\dim_{\mathbb{C}} \mathcal{M}_{1,(d)}(\mathbb{P}^1, \infty, [d]) = d + 1.$$

However, since the class λ_1 vanishes on the locus of curves with loops (c.f. [24, p. 351]), the (localization) contribution of such multiple-covers to N_1^{main} is zero. Also, unlike in classical GW theory, isogenies of elliptic curve do not appear because of the maximal tangency condition. Furthermore, by fixing the complex structure

of the domain elliptic curve, we can reduce the complex dimension to zero and avoid the integration of λ_1 at the cost of multiplying by the constant

$$\mathrm{PD}_{\overline{\mathcal{M}}_{1,1}}(\lambda_1) \in H_0(\overline{\mathcal{M}}_{1,1}, \mathbb{Q}) \cong \mathbb{Q}.$$

Therefore, we expect the reduced genus-one GW invariants $N_{1,A}^{\mathrm{main}}$ arising from $\overline{\mathcal{M}}_{1,(s_A)}^{\mathrm{main}}(X, D, A)$ to be a fixed multiple of an integer count of genus one maximally-tangent logarithmically immersed curves in (X, D) with a fixed complex structure on the domain.

Conjecture 4.10. *Suppose (X, D) is a log Calabi-Yau surface. Then the reduced genus one invariants $n_{1,A}$ formally derived from $N_{1,A}$ by the formula*

$$(4.15) \qquad 24 N_{1,A} = n_{1,A} + \sum_{\substack{dB=A \\ d>0}} (2 - s_B)\binom{d(s_B - 1) - 1}{d - 1} n_B$$

are integer-valued.[10] *Moreover, for a generic choice of a complex structure $\mathfrak{j}$ on the torus, $n_{1,A}$ represents a geometric count of degree A maximally tangent genus 1 curves in X with the complex structure $\mathfrak{j}$.*

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