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Normal Crossings Singularities for Symplectic Topology: Structures

Mohammad FARAJZADEH-TEHRANI[1)]
The University of Iowa, MacLean Hall, Iowa City, IA 52241
E-mail: *mohammad-tehrani@uiowa.edu*

Mark MCLEAN **Aleksey ZINGER**
Department of Mathematics, Stony Brook University, Stony Brook, NY 11794
E-mail: *markmclean@math.stonybrook.edu* *azinger@math.stonybrook.edu*

Abstract Our previous papers introduce topological notions of normal crossings symplectic divisor and variety, show that they are equivalent, in a suitable sense, to the corresponding geometric notions, and establish a topological smoothability criterion for normal crossings symplectic varieties. The present paper constructs a blowup, a complex line bundle, and a logarithmic tangent bundle naturally associated with a normal crossings symplectic divisor and determines the Chern class of the last bundle. These structures have applications in constructions and analysis of various moduli spaces. As a corollary of the Chern class formula for the logarithmic tangent bundle, we refine Aluffi's formula for the Chern class of the tangent bundle of the blowup at a complete intersection to account for the torsion and extend it to the blowup at the deepest stratum of an arbitrary normal crossings divisor.

Keywords Normal crossings divisor, Chern class, Logarithmic tangent bundle

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1 Introduction

Divisors, i.e., subvarieties of codimension 1 over the ground field, and related structures, are among the central objects of study in algebraic geometry. They appear in the study of curves (as dual objects), singularities (particularly in the Minimal Model Program), and semistable degenerations of smooth varieties (as the singular locus). The complex line bundle $\mathcal{O}_X(V)$ corresponding to a Cartier divisor $V \subset X$ and the log tangent bundle $TX(-\log V)$ (or dually the sheaf of log 1-forms $\Omega^1_X(\log V)$) corresponding to a normal crossings (or NC) divisor are among such useful and well-studied structures. They play important roles in the relative/log Gromov–Witten theories of Li [22, 23], Gross–Siebert [21], and Abramovich–Chen [1, 7]. Divisors also play an important role in symplectic topology, including as representatives of the Poincare duals of symplectic forms [10], in symplectic sum constructions [20, 24], in relative Gromov–Witten theory and symplectic sum formulas [11, 19, 30, 31, 35], in affine symplectic geometry [27, 28], in homological mirror symmetry [3, 33], and in relative Fukaya category [8, 9].

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1) Corresponding author

A *smooth* symplectic divisor is simply a symplectic submanifold of real codimension 2. Applications involving a smooth symplectic divisor $V \subset X$ typically rely on the Symplectic Neighborhood Theorem [25, Theorem 3.4.10]. It provides an identification Ψ (which we call a symplectic regularization) of a neighborhood of V in X with a neighborhood of V in its normal bundle $\mathcal{N}_X V$. Such an identification can then be used to construct an auxiliary V-compatible "geometric" data/structure, such as a V-compatible almost complex structure J or a complex line bundle $\mathcal{O}_X(V)$ over X with first Chern class $[V]_X$. One then shows that an invariant defined using J or the deformation equivalence class of an object constructed using Ψ depends only on (X, V, ω). This approach is relatively straightforward to carry out in the case of smooth divisors.

Singular symplectic divisors/varieties and structures associated with them are generally hard to define and work with because there is no direct analogue of the Symplectic Neighborhood or Darboux Theorem in this setting. Following an alternative approach, [14, 15] introduce topological notions of NC symplectic divisor and variety and geometric notions of regularization for NC symplectic divisors and varieties. The latter is basically a "nice" neighborhood identification of the divisor/singular locus, analogous to that provided by the Symplectic Neighborhood Theorem in the smooth case. Every NC symplectic divisor/variety is deformation equivalent to one admitting a regularization. For this reason, in our approach, we work with the entire deformation equivalence classes of NC symplectic divisors/varieties, as opposed to a fixed NC symplectic divisor/variety; see the end of Section 4.2.

As stated in [13], a regularization of an NC symplectic variety $V \subset X$ can be used to construct an associated complex line bundle $\mathcal{O}_X(V)$ and a log tangent bundle $TX(-\log V)$. The present paper carries out these constructions in detail. While the constructions of $\mathcal{O}_X(V)$ and $TX(-\log V)$ involve the auxiliary data of regularizations and other choices, their deformation equivalence classes depend on the deformation equivalence class of the symplectic structure only.

We denote the smooth locus of an NC symplectic divisor $V \subset X$ by V^*. For $r \in \mathbb{Z}^{\geq 0}$, let $V^{(r)} \subset V$ be r-fold locus of V, i.e., the locus that locally is the intersection of at least r branches of V; see (4.1). For example, $V^{(0)} = X$, $V^{(1)} = V$, $V^{(2)}$ is the singular locus of V, and $V^* = V^{(1)} - V^{(2)}$. The subspace $V^{(r)} - V^{(r+1)}$ is a smooth submanifold of X of codimension $2r$. We denote its inclusion into X by $\iota_{V^{(r)}-V^{(r+1)}}$. If X is compact and of (real) dimension $2n$, $\iota_{V^{(r)}-V^{(r+1)}}$ is a pseudocycle in X of dimension $2(n-r)$; see [37]. Thus, V^* and $V^{(r)} - V^{(r+1)}$ determine homology classes

$$[V]_X \in H_{2n-2}(X;\mathbb{Z}) \quad \text{and} \quad [V^{(r)}]_X \in H_{2(n-r)}(X;\mathbb{Z}),$$

respectively.

Proposition 1.1 *Let (X, ω) be a symplectic manifold and $V \subset X$ be an NC symplectic divisor.*

(1) *An ω-regularization $\mathcal{R}$ for $V \subset X$ determines a complex line bundle $(\mathcal{O}_{\mathcal{R};X}(V), \mathfrak{i}_{\mathcal{R}})$ over X with a smooth section $s_{\mathcal{R}}$ so that $s_{\mathcal{R}}^{-1}(0) = V$ and*

$$\mathrm{D}s_{\mathcal{R}} : \mathcal{N}_X(V^*) \longrightarrow \mathcal{O}_{\mathcal{R};X}(V)|_{V^*}$$

is an orientation-preserving isomorphism along the smooth locus V^* of V.

(2) *The deformation equivalence class* $(\mathcal{O}_X(V),\mathfrak{i})$ *of* $(\mathcal{O}_X(V),\mathfrak{i}_{\mathcal{R}})$ *depends only on the deformation equivalence class of* (X,V,ω).

(3) *If* $V'\subset X$ *is another NC symplectic divisor so that* $V\cup V'\subset X$ *is also an NC symplectic divisor and* $V\cap V'$ *contains no open subspace of* V, *then*

$$(\mathcal{O}_X(V\cup V'),\mathfrak{i})\approx(\mathcal{O}_X(V),\mathfrak{i})\otimes(\mathcal{O}_X(V'),\mathfrak{i}). \tag{1.1}$$

The complex line bundle $\mathcal{O}_X(V)$ appears in the smoothability criterion for SC symplectic varieties in [16] and for general NC symplectic varieties in [17]. For a simple (normal) crossings (or SC) symplectic divisor $V=\bigcup_{i\in S}V_i$ as in Definition 3.1, (1.1) gives

$$\mathcal{O}_X(V)\cong\bigotimes_{i\in S}\mathcal{O}_X(V_i)\longrightarrow X.$$

If X is compact and of (real) dimension $2n$, the stated properties of $s_{\mathcal{R}}$ imply that

$$c_1(\mathcal{O}_X(V))=\mathrm{PD}_X([V]_X)\in H^2(X;\mathbb{Z}). \tag{1.2}$$

If X is not compact, this identity holds with $[V]_X$ denoting the element of the Borel-Moore homology of X determined by V^*; see [4].

Theorem 1.2 *Let* (X,ω) *be a symplectic manifold and* $V\subset X$ *be an NC symplectic divisor.*

(1) *An* ω*-regularization* $\mathcal{R}$ *for* $V\subset X$ *determines a vector bundle* $T_{\mathcal{R}}X(-\log V)$ *over* X *with a smooth vector bundle homomorphism*

$$\iota_{\mathcal{R}}:T_{\mathcal{R}}X(-\log V)\longrightarrow TX \tag{1.3}$$

so that for every $r\in\mathbb{Z}^{\ge 0}$

$$T(V^{(r)}-V^{(r+1)})\subset T_{\mathcal{R}}X(-\log V)|_{V^{(r)}-V^{(r+1)}},\quad \iota_{\mathcal{R}}|_{T(V^{(r)}-V^{(r+1)})}=d\iota_{V^{(r)}-V^{(r+1)}},$$
$$\text{and}\quad \iota_{\mathcal{R}}(T_{\mathcal{R}}X(-\log V)|_{V^{(r)}-V^{(r+1)}})=T(V^{(r)}-V^{(r+1)}).$$

(2) *An* $\mathcal{R}$*-compatible almost complex structure* J *on* X *determines a complex structure* $\mathfrak{i}_{\mathcal{R},J}$ *on the vector bundle* $T_{\mathcal{R}}X(-\log V)$ *so that the bundle homomorphism* (1.3) *is* $\mathbb{C}$*-linear.*

(3) *The deformation equivalence class* $(TX(-\log V),\mathfrak{i})$ *of* $(T_{\mathcal{R}}X(-\log V),\mathfrak{i}_{\mathcal{R},J})$ *depends only on the deformation equivalence class of* (X,V,ω).

(4) *If* $V'\subset X$ *is a smooth submanifold so that* $V\cup V'\subset X$ *is also an NC symplectic divisor and* $V\cap V'$ *contains no open subspace of* V, *then*

$$(TX(-\log(V\cup V')),\mathfrak{i})\oplus(\mathcal{O}_X(V'),\mathfrak{i})\approx(TX(-\log V),\mathfrak{i})\oplus(X\times\mathbb{C}). \tag{1.4}$$

(5) *We have*

$$c(TX(-\log V),\mathfrak{i})=\frac{c(TX,\omega)}{1+\mathrm{PD}_X([V^{(1)}]_X)+\mathrm{PD}_X([V^{(2)}]_X)+\cdots}\in H^*(X;\mathbb{Q}). \tag{1.5}$$

The above equality holds in $H^*(X;\mathbb{Z})$ *if* $V\subset X$ *is an SC divisor.*

As shown in [12], the complex vector bundle $(TX(-\log V),\mathrm{i})$ plays the same role in the deformation-obstruction theory of pseudoholomorphic curves relative to an NC symplectic divisor as the complex vector bundle (TX,J) in the standard deformation-obstruction theory of pseudoholomorphic curves; see [26, Chapter 3], for example. This provides a symplectic topology perspective on the constructions of log stable maps in [1, 21, 30]. In [18], the vector bundle $TX(-\log V)$ is used to define Seiberg–Witten invariants of a closed oriented 4-manifold X relative to a smooth oriented Riemann surface V. This perspective on the standard constructions of relative Seiberg–Witten invariants reveals additional structures.

For an SC symplectic divisor $V=\bigcup_{i\in S}V_i$ as in Definition 3.1, (1.4) gives

$$(TX(-\log V),\mathrm{i})\oplus\bigoplus_{i\in S}(\mathcal{O}_X(V_i),\mathrm{i})\approx(TX,J)\oplus(X\times\mathbb{C}^S,\mathrm{i}). \tag{1.6}$$

This immediately implies that

$$\begin{aligned} c(TX(-\log V)) &= \frac{c(TX,\omega)}{\prod_{i\in S}(1+\mathrm{PD}_X([V_i]_X))} \\ &= \frac{c(TX,\omega)}{1+\mathrm{PD}_X([V^{(1)}]_X)+\mathrm{PD}_X([V^{(2)}]_X)+\cdots}\in H^*(X;\mathbb{Z}). \end{aligned} \tag{1.7}$$

The direct sum vector bundle on the left-hand side of (1.6) does not even exist as a vector bundle in the general NC case; see the example in the second half of Section 7. In Section 6.4, we instead establish the de Rham cohomology analogue of (1.5) by expressing the Chern classes on the two sides of (1.5) in terms of the curvatures of connections in the vector bundles TX and $TX(-\log V)$. We construct a $2k$-form τ_k on X, supported in a neighborhood of $V^{(k)}$ and representing $\mathrm{PD}_X([V^{(k)}]_X)$ in $H^{2k}_{\mathrm{deR}}(X)$, and $\mathbb{C}$-linear connections ∇ in TX and ∇' in $TX(-\log V)$ so that the curvature of ∇ is the correct combination of the curvature of ∇' and $\tau_1,\ldots,\tau_r$ to yield (1.5); see (6.38), Lemma 6.10, and Proposition 6.11. Our proofs of (1.4) and (1.5) are carried out in the almost complex category.

When $V\subset X$ is either an NC complex divisor in a complex manifold or an NC almost complex divisor in an almost complex manifold compatible with a regularization for V, in the sense defined in Section 4.2, we can pass to the blowup $\widetilde{X}$ of X along the deepest stratum $V^{(r)}$ of V and to the proper transform $\overline{V}$ of V. We can then compare the log tangent bundles for (X,V) and $(\widetilde{X},\overline{V})$ and their Chern classes via (1.4), (1.5), and Lemma 1.3 below. This immediately yields Corollary 1.4 below, except for the refinement in the vanishing torsion case. This corollary refines [2, Lemma 1.3] in the SC case and extends it to the general NC case.

Lemma 1.3 *Let (X,J) be an almost complex manifold, $V\subset X$ be an NC almost complex divisor with a regularization $\mathcal{R}$, and $r\in\mathbb{Z}^+$ be such that $V^{(r+1)}=\emptyset$. If $\pi:(\widetilde{X},\widetilde{J})\to(X,J)$ is the blowup of (X,J) along $V^{(r)}$ determined by $\mathcal{R}$, $\mathbb{E}$ is the exceptional divisor, and $\overline{V}\subset\widetilde{X}$ is the proper transform of X, then $\widetilde{V}\equiv\overline{V}\cup\mathbb{E}$ is an NC almost complex divisor in $(\widetilde{X},\widetilde{J})$. In this case, there are a regularization $\widetilde{\mathcal{R}}$ of $\widetilde{V}$ in $(\widetilde{X},\widetilde{J})$ and an isomorphism*

$$d^{\log}_{\mathcal{R}\widetilde{\mathcal{R}}}\pi: T_{\widetilde{\mathcal{R}}}\widetilde{X}(-\log\widetilde{V})\xrightarrow{\cong}\pi^*(T_{\mathcal{R}}X(-\log V)) \tag{1.8}$$

so that the diagram

$$\begin{array}{ccc} T_{\widetilde{\mathcal{R}}}\widetilde{X}(-\log\widetilde{V}) & \xrightarrow{\iota_{\widetilde{\mathcal{R}}}} & T\widetilde{X} \\ \downarrow{\scriptstyle d^{\log}_{\widetilde{\mathcal{R}}\widetilde{\mathcal{R}}}\pi} & & \downarrow{\scriptstyle d\pi} \\ \pi^*(T_{\mathcal{R}}X(-\log V)) & \xrightarrow{\pi^*\iota_{\mathcal{R}}} & \pi^*TX \end{array}$$

commutes. The first and third claims above also hold in the category of complex manifolds with NC divisors.

Corollary 1.4 *With the assumptions as in Lemma* 1.3*,*

$$\begin{aligned} &\frac{c(T\widetilde{X})}{(1+\mathrm{PD}_{\widetilde{X}}([\overline{V}^{(1)}]_{\widetilde{X}})+\mathrm{PD}_{\widetilde{X}}([\overline{V}^{(2)}]_{\widetilde{X}})+\cdots)(1+\mathrm{PD}_{\widetilde{X}}([\mathbb{E}]_{\widetilde{X}}))} \\ &\quad=\pi^*\left(\frac{c(TX)}{1+\mathrm{PD}_X([V^{(1)}]_X)+\mathrm{PD}_X([V^{(2)}]_X)+\cdots}\right)\in H^*(\widetilde{X};\mathbb{Q})\,. \end{aligned} \tag{1.9}$$

The above equality holds in $H^*(\widetilde{X};\mathbb{Z})$ if $V\subset X$ is an SC divisor or the torsion in $H_*(\mathbb{E};\mathbb{Z})$ lies in the kernel of the homomorphism $\widetilde{\iota}_*$ induced by the inclusion $\mathbb{E}\to\widetilde{X}$.

The statement of Lemma 1.3 in the complex category is well-known; its proof is recalled at the end of Section 2.1. We establish this lemma in the almost complex category with regularizations in Sections 5.2–5.4. The regularizations are used to construct the bundles $TX(-\log V)$ and $T\widetilde{X}(-\log(\overline{V}\cup\mathbb{E}))$ and the blowup $\widetilde{X}$.

We can also pass to *a* blowup $\widetilde{X}$ of X along the deepest stratum $V^{(r)}$ of V and the proper transform $\overline{V}\subset\widetilde{X}$ of V if $V\subset X$ is an NC symplectic divisor with $V^{(r)}$ admitting a tubular symplectic neighborhood that contains the disk subbundle of $\mathcal{N}_XV^{(r)}$ of a fixed radius; see Section 5.5. This is automatically the case if $V^{(r)}$ is the compact. If so, each deformation equivalence class $[\omega]$ of symplectic forms on (X,V) determines a deformation equivalence class $[\widetilde{\omega}]$ of symplectic forms on the blowup $(\widetilde{X},\widetilde{V})$ of (X,V) along $V^{(r)}$, a homotopy class of blowdown maps

$$\pi:(\widetilde{X},\widetilde{V})\longrightarrow(X,V),$$

and a homotopy class of isomorphisms

$$d^{\log}\pi:T\widetilde{X}(-\log\widetilde{V})\xrightarrow{\cong}\pi^*(TX(-\log V)) \tag{1.10}$$

between the log tangent bundles associated with $[\omega]$ and $[\widetilde{\omega}]$. If $V^{(r)}$ is not compact, the existence of a tubular symplectic neighborhood that contains the disk subbundle of $\mathcal{N}_XV^{(r)}$ of a fixed radius is unclear, even after deforming the symplectic form. We suspect that the answers to the following closely related questions are negative in general; the affirmative answer to Question 1.6 would imply the affirmative answer to Question 1.5.

Question 1.5 Let (V,ω) be a symplectic manifold, $\mathcal{N}\to V$ be a direct sum of Hermitian line bundles $(L_i,\rho_i,\nabla^{(i)})$ determining fiberwise symplectic forms Ω_i on L_i, and $\mathcal{N}'\subset\mathcal{N}$ be a neighborhood of V. Is there a deformation of ω through symplectic forms and of $(\rho_i,\nabla^{(i)})$ through Hermitian structures so that the induced 2-form $\widetilde{\omega}$ as in (3.3) on the total space of $\mathcal{N}$ is symplectic on the unit ball subbundle of $\mathcal{N}$, with respect to the deformed metric, and this subbundle is contained in $\mathcal{N}'$?

Question 1.6 Let (V,ω) be a symplectic manifold, J be an ω-compatible almost complex structure, and $C : V \to \mathbb{R}^+$ be a smooth function. Are there a symplectic form ω' on V deformation equivalent to ω and an ω'-compatible almost complex structure J' so that

$$\omega'(v, J'v) \geq C(x)\omega(v, Jv) \quad \forall x \in V,\ v \in T_xV.$$

We review the complex geometric constructions of the complex line bundle $(\mathcal{O}_X(V), \mathfrak{i})$ and the complex vector bundle $(TX(-\log V), \mathfrak{i})$ associated with an NC divisor V in a complex manifold V and some of their properties in Section 2.1. As a warmup to the general case, we construct these bundles for a smooth symplectic divisor in Section 2.2. For the reader's convenience, Sections 3.1 and 3.2 recall the notions of SC symplectic divisor and regularization, respectively, introduced in [14]. Section 3.3 contains the constructions of the vector bundles $\mathcal{O}_X(V)$ and $TX(-\log V)$ for an SC symplectic divisor $V \subset X$ and establishes Proposition 1.1 and the first three statements of Theorem 1.2 in this setting. Section 3.4 establishes Theorem 1.2 (4) for an SC symplectic divisor V. The constructions and proofs in the SC case illustrate the arguments in the general NC case, which are more notionally involved. Sections 4.1–4.3 and 6.1–6.3 are the analogues of Sections 3.1–3.3 in the local and global perspectives, respectively, on the NC symplectic divisors introduced in [15]. Section 4.3 also shows why the proof of Theorem 1.2 (4) for SC symplectic divisors in Section 3.4 immediately extends to the general NC case. Sections 5 and 6.4 establish Lemma 1.3 and Theorem 1.2 (5), respectively. In Section 7, we establish the remaining statement of Corollary 1.4 and show that (1.5) and (1.9) do not need to hold with $\mathbb{Z}$-coefficients for arbitrary NC divisors.

2 Standard Settings

2.1 NC Complex Divisors

Let X be a complex manifold of (complex) dimension n with structure sheaf $\mathcal{O}_X$ (the sheaf of local holomorphic functions). An NC divisor in X is a subvariety $V \subset X$ locally defined by an equation of the form

$$z_1 \cdots z_k = 0 \tag{2.1}$$

in a holomorphic coordinate chart $(z_1, \ldots, z_n)$ on X. The sheaf of local meromorphic functions with simple poles along the smooth locus of V is freely generated in such a coordinate chart by the meromorphic function $1/z_1 \cdots z_k$ as a module over $\mathcal{O}_X$. Since this sheaf is locally free of rank 1, it is the sheaf of local holomorphic sections of a holomorphic line bundle $\mathcal{O}_X(V)$. The constant function 1 on X determines a holomorphic section s of this sheaf satisfying the properties of $s_{\mathcal{R}}$ in Proposition 1.1 (1). It is immediate that (1.1) holds as well. The dual of $\mathcal{O}_X(V)$ is the holomorphic line bundle $\mathcal{O}_X(-V)$; the sheaf of its local holomorphic sections is freely generated in a coordinate chart as above by the holomorphic function $z_1 \cdots z_k$.

In a local chart as in (2.1), the sheaf $\mathcal{T}X \equiv \mathcal{O}(TX)$ of local holomorphic sections of the tangent bundle TX is generated by the coordinate vector fields $\partial_{z_1}, \ldots, \partial_{z_n}$. The logarithmic tangent sheaf $\mathcal{T}X(-\log V)$ is the subsheaf of $\mathcal{T}X$ generated by the vector fields

$$\partial^{\log}_{z_1} \equiv z_1\partial_{z_1}, \ldots, \partial^{\log}_{z_k} \equiv z_k\partial_{z_k}, \partial_{z_{k+1}}, \ldots, \partial_{z_n}.$$

The dual of this subsheaf is the sheaf of logarithmic 1-forms $\Omega^1_X(\log V)$ is the sheaf generated

by

$$\frac{dz_1}{z_1},\ldots,\frac{dz_k}{z_k},dz_{k+1},\ldots,dz_n.$$

Since $\mathcal{T}X(-\log V)$ is locally free, it is the sheaf of local holomorphic sections of a holomorphic vector bundle $TX(-\log V)$. The inclusion of $\mathcal{T}X(-\log V)$ into $\mathcal{T}X$ gives rise to a holomorphic homomorphism

$$\iota : TX(-\log V) \longrightarrow TX$$

that realizes every section of $TX(-\log V)$ as a section of TX with values in TV along V.

The normalization $\iota : \widetilde{V} \to V \subset X$ of V is an immersion. The inclusion of the sheaf Ω^1_X of 1-forms on X into $\Omega^1_X(\log V)$ and the Poincare residue map induce an exact sequence

$$0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log V) \longrightarrow \iota_*\mathcal{O}_{\widetilde{V}} \longrightarrow 0$$

of sheaves on X, where $\iota_*\mathcal{O}_{\widetilde{V}}$ is the direct image (or push-forward) sheaf of the structure sheaf $\mathcal{O}_{\widetilde{V}}$ of $\widetilde{V}$. Therefore,

$$c(\Omega^1_X(\log V)) = c(\Omega^1_X)\, c(\iota_*\mathcal{O}_{\widetilde{V}}). \tag{2.2}$$

If $V = \bigcup_{i\in S} V_i$ is an SC divisor, then $\iota_*\mathcal{O}_{\widetilde{V}} = \bigoplus_{i\in S}\mathcal{O}_{V_i}$ as sheaves on X. Furthermore, there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-V_i) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{V_i} \longrightarrow 0$$

of sheaves on X for each $i \in S$. Thus,

$$c(\iota_*\mathcal{O}_{\widetilde{V}})\prod_{i\in S}(1-\mathrm{PD}_X([V_i]_X)) = 1 \in H^2(X;\mathbb{Z})$$

in this case. We thus obtain (1.7) in the complex setting.

For an arbitrary NC divisor $V \subset X$, the derived direct image sheaf $\iota_!\mathcal{O}_{\widetilde{V}}$ of $\mathcal{O}_{\widetilde{V}}$ coincides with the direct image sheaf $\iota_*\mathcal{O}_{\widetilde{V}}$ because the higher derived functors for an immersion vanish. Along with the Grothendieck–Riemann–Roch theorem, this gives

$$\mathrm{ch}(\iota_*\mathcal{O}_{\widetilde{V}})\,\mathrm{td}(X) = \mathrm{ch}(\iota_!\mathcal{O}_{\widetilde{V}})\,\mathrm{td}(X) = \iota_*(\mathrm{ch}(\mathcal{O}_{\widetilde{V}})\,\mathrm{td}(\widetilde{V})) = \iota_*(\mathrm{td}(\widetilde{V})),$$

where ch is the Chern character and td is the Todd class; see [32, Theorem 1.3]. Thus,

$$\mathrm{ch}(\iota_*\mathcal{O}_{\widetilde{V}}) = \frac{\iota_*(\mathrm{td}(\widetilde{V}))}{\mathrm{td}(X)} \in H^*(X,\mathbb{Q}). \tag{2.3}$$

This formula holds only with $\mathbb{Q}$-coefficients because the Chern character is a map from the K-theory of X to the rational Chow group of X. The proof of (2.3) in [32, Corollary 5.22] uses blowups to reduce the problem to SC divisors; we are not aware of a direct way for obtaining (2.3). For the purposes of computing $c(TX(-\log V))$ via (2.2), it is still necessary to translate $\mathrm{ch}(\iota_*\mathcal{O}_{\widetilde{V}})$ into $c(\iota_*\mathcal{O}_{\widetilde{V}})$. Nevertheless, it is feasible to directly study the change in $c(TX(-\log V))$ under blowups, as is done in the proofs of Lemma 1.3 in Section 5.4, and to relate $c(TX(-\log V))$ to $c(TX)$ in the spirit of (2.2), as is done in the proof of (1.5) in Section 6.4.

Let $r \in \mathbb{Z}^+$, $\pi : \widetilde{X} \to X$, and $\mathbb{E}, \overline{V} \subset \widetilde{X}$ be as in Lemma 1.3 and $\widetilde{U} \equiv \pi^{-1}(U)$ be the preimage of a coordinate chart U as in the sentence (2.1). For each $i = 1,\ldots,k$,

$$\widetilde{U}_i \equiv \{(z_i,(u_j)_{j\in[k]-i},(z_j)_{j\in[n]-[k]}) : (u_1z_1,\ldots,u_{i-1}z_i,z_i,u_{i+1}z_i,\ldots,u_kz_i,(z_j)_{j\in[n]-[k]}) \in U\},$$

where $[k] \equiv \{1, \ldots, k\}$, is a coordinate chart on $\widetilde{U} \subset \widetilde{X}$; these charts cover $\widetilde{U}$. Since

$$\overline{V} \cap \widetilde{U}_i = (u_1 \ldots u_{i-1} u_{i+1} \ldots u_k = 0), \quad \mathbb{E} \cap \widetilde{U}_i = (z_i = 0),$$
$$\text{and} \quad \frac{dz_j}{z_j} = \frac{du_j}{u_j} + \frac{dz_i}{z_i} \quad \forall j \in [k] - i,$$

we obtain that

$$\pi^* \Omega^1_X(\log V) = \Omega^1_{\widetilde{X}}(\log(\overline{V} \cup \mathbb{E})).$$

This establishes Lemma 1.3 in the complex setting.

2.2 Smooth Symplectic Divisors

It is fairly straightforward to adapt the constructions of $\mathcal{O}_X(V)$ and $TX(-\log V)$ in Section 2.1 via the Symplectic Neighborhood Theorem. Before doing so below, we carefully formulate the relevant notions.

Let V be a smooth manifold. For a vector bundle $\pi : \mathcal{N} \to V$, we denote by $\zeta_{\mathcal{N}}$ the radial vector field on the total space of $\mathcal{N}$; it is given by

$$\zeta_{\mathcal{N}}(v) = (v, v) \in \pi^* \mathcal{N} = T\mathcal{N}^{\mathrm{ver}} \longhookrightarrow T\mathcal{N}\,.$$

Let Ω be a fiberwise 2-form on $\mathcal{N} \to V$. A connection ∇ on $\mathcal{N}$ induces a projection $T\mathcal{N} \to \pi^*\mathcal{N}$ and thus determines an extension Ω_∇ of Ω to a 2-form on the total space of $\mathcal{N}$. If ω is a closed 2-form on V, the 2-form

$$\widetilde{\omega} \equiv \pi^*\omega + \frac{1}{2} d\iota_{\zeta_{\mathcal{N}}} \Omega_\nabla \equiv \pi^*\omega + \frac{1}{2} d(\Omega_\nabla(\zeta_{\mathcal{N}}, \cdot)) \tag{2.4}$$

on the total space of $\mathcal{N}$ is also closed and restricts to Ω on $\pi^*\mathcal{N} = T\mathcal{N}^{\mathrm{ver}}$. If ω is a symplectic form on V and Ω is a fiberwise symplectic form on $\mathcal{N}$, then $\widetilde{\omega}$ is a symplectic form on a neighborhood of V in $\mathcal{N}$.

We call $\pi : (L, \rho, \nabla) \to V$ a Hermitian line bundle if $L \to V$ is a smooth complex line bundle, ρ is a Hermitian metric on L, and ∇ is a ρ-compatible connection on L. We use the same notation ρ to denote the square of the norm function on L and the Hermitian form on L which is $\mathbb{C}$-antilinear in the second input. Thus,

$$\rho(v) \equiv \rho(v, v), \quad \rho(\mathrm{i}v, w) = \mathrm{i}\rho(v, w) = -\rho(v, \mathrm{i}w) \quad \forall (v, w) \in L \times_V L.$$

Let $\rho^{\mathbb{R}}$ denote the real part of the form ρ.

A Riemannian metric on an oriented real vector bundle $L \to V$ of rank 2 determines a complex structure on the fibers of V. A Hermitian structure on an oriented real vector bundle $L \to V$ of rank 2 is a pair (ρ, ∇) such that (L, ρ, ∇) is a Hermitian line bundle with the complex structure i_ρ determined by the Riemannian metric $\rho^{\mathbb{R}}$. If Ω is a fiberwise symplectic form on an oriented vector bundle $L \to V$ of rank 2, an Ω-compatible Hermitian structure on L is a Hermitian structure (ρ, ∇) on L such that $\Omega(\cdot, \mathrm{i}_\rho \cdot) = \rho^{\mathbb{R}}(\cdot, \cdot)$.

Definition 2.1 *Let X be a manifold and $V \subset X$ be a submanifold with normal bundle $\mathcal{N}_X V \to V$. A (smooth) regularization for V in X is a diffeomorphism $\Psi : \mathcal{N}' \to X$ from a neighborhood of V in $\mathcal{N}_X V$ onto a neighborhood of V in X such that $\Psi(x) = x$ and the isomorphism*

$$\mathcal{N}_X V|_x = T^{\mathrm{ver}}_x \mathcal{N}_X V \longhookrightarrow T_x \mathcal{N}_X V \xrightarrow{d_x \Psi} T_x X \longrightarrow \frac{T_x X}{T_x V} \equiv \mathcal{N}_X V|_x$$

is the identity for every $x \in V$.

Let V be a closed symplectic submanifold of a symplectic submanifold in (X,ω). The normal bundle $\mathcal{N}_X V$ of V in X then inherits a fiberwise symplectic form $\omega|_{\mathcal{N}_X V}$ from ω via the isomorphism

$$\pi_{\mathcal{N}_X V} : TV^\omega \equiv \{v \in T_x X : x \in V,\, \omega(v,w) = 0\ \forall w \in T_x V\} \xrightarrow{\approx} \frac{TX|_V}{TV} \equiv \mathcal{N}_X V.$$

The symplectic form $\omega|_V$ on V, the fiberwise 2-form $\Omega \equiv \omega|_{\mathcal{N}_X V}$ on $\mathcal{N}_X V$, and a connection ∇ on $\mathcal{N}_X V$ thus determine a 2-form $\widetilde{\omega}_\nabla$ on the total space of $\mathcal{N}_X V$ via (2.4). By the Symplectic Neighborhood Theorem, there exists a regularization $\Psi : \mathcal{N}' \to X$ for V in X so that $\Psi^*\omega = \widetilde{\omega}_\nabla|_{\mathcal{N}'}$.

Suppose in addition that V is of codimension 2, i.e., V is a smooth symplectic divisor in (X,ω). If (ρ,∇) is an $\omega|_{\mathcal{N}_X V}$-compatible Hermitian structure on $\mathcal{N}_X V$, the triple $\mathcal{R} = ((\rho,\nabla),\Psi)$ is an ω-regularization for V in X in the sense of Definition 3.5 and determines a fiberwise complex structure $\mathfrak{i}_\rho$ on $\mathcal{N}_X V$. Let

$$\begin{aligned} &\mathcal{O}_{\mathcal{R};X}(V) = (\{\Psi^{-1}\}^*\pi^*\mathcal{N}_X V \sqcup (X-V)\times\mathbb{C})/\sim \longrightarrow \Psi(\mathcal{N}')\cup(X-V) = X,\\ &\{\Psi^{-1}\}^*\pi^*\mathcal{N}_X V \ni (\Psi(v),v,cv) \sim (\Psi(v),c) \in (X-V)\times\mathbb{C}, \end{aligned} \tag{2.5}$$

where $\pi : \mathcal{N}' \to V$ is the bundle projection map. This defines a smooth complex line bundle over X. The smooth section $s_{\mathcal{R}}$ of this bundle given by

$$s_{\mathcal{R}}(x) = \begin{cases} [x,v,v], & \text{if } x = \Psi(v),\ v \in \mathcal{N}';\\ [x,1], & \text{if } x \in X - V; \end{cases}$$

satisfies the properties stated in Proposition 1.1 (1).

For each $v \in \mathcal{N}_X V$, the connection ∇ determines an injective homomorphism

$$\mathrm{h}_{\nabla;v} : T_{\pi(v)}V \longrightarrow T_v(\mathcal{N}_X V) \tag{2.6}$$

with the image complementary to the image of $\mathcal{N}_X V$. Let

$$\begin{aligned} &T_{\mathcal{R}}X(-\log V) = ((((\{\Psi^{-1}\}^*\pi^*TV)\oplus\Psi(\mathcal{N}')\times\mathbb{C})\sqcup T(X-V))/\sim \longrightarrow \Psi(\mathcal{N}')\cup(X-V) = X,\\ &(\{\Psi^{-1}\}^*\pi^*TV)\oplus\Psi(\mathcal{N}')\times\mathbb{C} \ni (\Psi(v),v,w)\oplus(\Psi(v),c) \sim d_v\Psi(\mathrm{h}_{\nabla;v}(w)+cv) \in T(X-V). \end{aligned}$$

This defines a smooth vector bundle over X. The smooth bundle homomorphism (1.3) defined by

$$\iota_{\mathcal{R}}(\dot{x}) = \begin{cases} d_v\Psi(\mathrm{h}_{\nabla;v}(w)+cv), & \text{if } \dot{x} = [(\Psi(v),v,w)\oplus(\Psi(v),c)],\ v \in \mathcal{N}';\\ \dot{x}, & \text{if } \dot{x} \in T(X-V); \end{cases}$$

satisfies the properties stated in Theorem 1.2 (1).

An almost complex structure J on V and the fiberwise complex structure $\mathfrak{i}_\rho$ on $\mathcal{N}_X V$ determine an almost complex structure $J_{\mathcal{R}}$ on the total space of $\mathcal{N}_X V$ via the connection ∇. We call an almost complex structure J on X $\mathcal{R}$-compatible if J preserves $TV \subset TX|_V$ and Ψ intertwines J and $J_{\mathcal{R}} \equiv (J|_{TV})_{\mathcal{R}}$, i.e.,

$$J(TV) \subset TV \quad \text{and} \quad J \circ d\Psi = d\Psi \circ J_{\mathcal{R}}|_{\mathcal{N}'}\,.$$

Such an almost complex structure J induces a fiberwise complex structure $\mathfrak{i}_{\mathcal{R},J}$ on $T_{\mathcal{R}}X(-\log V)$ satisfying Theorem 1.2 (2). It can be constructed by pasting together $J_{\mathcal{R}} \circ \{d\Psi\}^{-1}$ and an almost complex structure on $X - V$.

We denote by $\mathrm{Symp}^+(X,V)$ the space of symplectic forms on X that restrict to symplectic forms on V, by $\mathrm{Aux}(X,V)$ the space of pairs $(\omega,\mathcal{R})$ consisting of $\omega\in\mathrm{Symp}^+(X,V)$ and an ω-compatible regularization $\mathcal{R}$ for V in X, and by $\mathrm{AK}(X,V)$ the space of triples $(\omega,\mathcal{R},J)$ consisting of $(\omega,\mathcal{R})\in\mathrm{Aux}(X,V)$ and an almost complex structure J on X compatible with ω and $\mathcal{R}$. Since the Symplectic Neighborhood Theorem can be applied with families of symplectic forms parametrized by compact manifolds, the projection

$$\mathrm{Aux}(X,V)\longrightarrow\mathrm{Symp}^+(X,V),\quad(\omega,\mathcal{R})\longrightarrow\omega, \tag{2.7}$$

is a weak homotopy equivalence. It is straightforward, by adapting the proof of [25, Prop. 4.1], for example, to show that the projection

$$\mathrm{AK}(X,V)\longrightarrow\mathrm{Aux}(X,V),\quad(\omega,\mathcal{R},J)\longrightarrow(\omega,\mathcal{R}), \tag{2.8}$$

is also a weak homotopy equivalence.

The above constructions of the complex line bundle $(\mathcal{O}_X(V),\mathfrak{i})$ and the vector bundle $TX(-\log V)$ can be applied with compact families in $\mathrm{Aux}(X,V)$. The construction of the complex vector bundle $(TX(-\log V),\mathfrak{i}_{\mathcal{R},J})$ can be applied with compact families in $\mathrm{AK}(X,V)$. Along with the previous paragraph, this confirms the statements of Proposition 1.1 (2) and Theorem 1.2 (3) for smooth symplectic divisors V.

The constructions of the complex line bundle $(\mathcal{O}_X(V),\mathfrak{i})$ and the vector bundle $TX(-\log V)$ do not involve the symplectic form ω directly. The first construction can be carried out for any closed codimension 2 submanifold V of a smooth manifold X endowed with a complex structure on the normal bundle $\mathcal{N}_XV$ and a smooth regularization Ψ. The constructions of the vector bundle $TX(-\log V)$ and of the complex structure $\mathfrak{i}_{\mathcal{R},J}$ on it require in addition a connection on $\mathcal{N}_XV$ in the first case and also an $\mathcal{R}$-compatible almost complex structure J on X in the second case.

Corollary 2.3 below is used later in this paper. We deduce it from the following observation.

Lemma 2.2 *Suppose V is a smooth manifold, $\pi:\mathcal{N}\to V$ is a vector bundle, and ∇ is a connection in $\mathcal{N}$. Let $T\mathcal{N}^{\mathrm{hor}}\subset T\mathcal{N}$ be the horizontal tangent subbundle determined by ∇. If $\widetilde{\nabla}\equiv\pi^*\nabla$ is the connection in $\pi^*\mathcal{N}\to\mathcal{N}$ determined by ∇, then*

$$\widetilde{\nabla}\zeta_{\mathcal{N}}|_{T\mathcal{N}^{\mathrm{hor}}}=0:T\mathcal{N}^{\mathrm{hor}}\longrightarrow\pi^*\mathcal{N}. \tag{2.9}$$

If in addition $\mathcal{N}$ is a complex vector bundle (and ∇ is a complex linear connection), then

$$\widetilde{\nabla}\zeta_{\mathcal{N}}|_{T\mathcal{N}^{\mathrm{ver}}}\circ\mathfrak{i}=\mathfrak{i}\widetilde{\nabla}\zeta_{\mathcal{N}}|_{T\mathcal{N}^{\mathrm{ver}}}:T\mathcal{N}^{\mathrm{ver}}\longrightarrow\pi^*\mathcal{N}. \tag{2.10}$$

Corollary 2.3 *Suppose (V,J) is an almost complex manifold, $\pi:\mathcal{N}\to V$ is a complex line bundle, ∇ is a connection in $\mathcal{N}$, and $\widetilde{\nabla}\equiv\pi^*\nabla$. If*

$$\Phi:(\mathcal{N}-V)\times\mathbb{C}\xrightarrow{\approx}\pi^*\mathcal{N}|_{\mathcal{N}-V},\quad\Phi(v,c)=(v,cv),$$

then $\Phi^*\widetilde{\nabla}-d$ is a $(1,0)$-form on $\mathcal{N}-V$ with respect to the almost complex structure J_∇ on $\mathcal{N}-V$ determined by J and ∇.

Proof The 1-form $\Phi^*\widetilde{\nabla}-d$ is given by

$$\{\Phi^*\widetilde{\nabla}-d\}1=\Phi^{-1}\circ\widetilde{\nabla}(\Phi\circ1)=\Phi^{-1}\circ\widetilde{\nabla}\zeta_{\mathcal{N}}. \tag{2.11}$$

The almost complex structure J_∇ restricts to $\{d\pi\}^*J$ on $T\mathcal{N}^{\rm hor}$ and to $\pi^*\mathfrak{i}$ on $T\mathcal{N}^{\rm ver}$. The claim thus follows from Lemma 2.2. □

Proof of Lemma 2.2 Suppose U is an open subset of V and $\xi_1,\ldots,\xi_n \in \Gamma(U;\mathcal{N})$ is a frame for $\mathcal{N}$ over U. Let $\theta_l^k \in \Gamma(U;T^*U)$ be such that

$$\nabla\xi_l = \sum_{k=1}^{k=n} \xi_k\theta_l^k \equiv \sum_{k=1}^{k=n} \theta_l^k \otimes \xi_k \in \Gamma(U;T^*U\otimes\mathcal{N}) \quad \forall l=1,\ldots,n. \tag{2.12}$$

The frame $\xi_1,\ldots,\xi_n$ determines an identification $\mathcal{N}|_U = U\times\mathbb{R}^n$ so that

$$\begin{aligned} \zeta_{\mathcal{N}}(v) &= \sum_{l=1}^{l=n} c_l\xi_l(x), \\ T_v\mathcal{N}^{\rm hor} &= \left\{\left(\dot{x}, -\sum_{l=1}^{l=n} c_l\theta_l^1(\dot{x}),\ldots -\sum_{l=1}^{l=n} c_l\theta_l^n(\dot{x})\right) : \dot{x}\in T_xV\right\}, \end{aligned} \qquad \forall v\equiv(x,c_1,\ldots,c_n)\in\mathcal{N}|_U; \tag{2.13}$$

see the proof of [38, Lemma 1.1].

For each $l=1,\ldots,n$, let $\widetilde{\xi}_l = \pi^*\xi_l \in \Gamma(\mathcal{N}|_U;\pi^*\mathcal{N})$. By the definition of $\widetilde{\nabla}$ and (2.12),

$$\widetilde{\nabla}\widetilde{\xi}_l = \sum_{k=1}^{k=n} (\theta_l^k\circ d\pi)\otimes\widetilde{\xi}_k \quad \forall l=1,\ldots,n.$$

Thus,

$$\widetilde{\nabla}\zeta_{\mathcal{N}}|_{(x,c_1,\ldots,c_n)} = \sum_{l=1}^{l=n}\sum_{k=1}^{k=n} c_l(\theta_l^k\circ d\pi)\otimes\widetilde{\xi}_k + \sum_{l=1}^{l=n}(dc_l)\otimes\widetilde{\xi}_l. \tag{2.14}$$

Along with the second statement in (2.13), this gives (2.9).

The first summand on the right-hand side of (2.14) vanishes on $T\mathcal{N}^{\rm ver}$. If $\mathcal{N}$ is a complex vector bundle, the above applies with $\mathbb{R}$ replaced by $\mathbb{C}$. The second summand on the right-hand side of (2.14) is $\mathbb{C}$-linear on $T\mathcal{N}^{\rm ver}$ in this case. This gives (2.10). □

Remark 2.4 By the proof of Lemma 2.2, the 1-form in (2.11) is given by

$$(\Phi^*\widetilde{\nabla} - d)|_{(x,z)} = \theta_1^1|_x + \frac{dz}{z}.$$

Thus, the curvature $F^{\Phi^*\widetilde{\nabla}}$ of the connection $\Phi^*\widetilde{\nabla}$ on $(\mathcal{N}-V)\times\mathbb{C}$ is given by

$$F^{\Phi^*\widetilde{\nabla}} = d(\pi^*\theta_1^1) = \pi^*F^\nabla.$$

3 SC Symplectic Divisors

For $N\in\mathbb{Z}^{\ge 0}$, let

$$[N] = \{1,\ldots,N\}.$$

If $\mathcal{N}\to V$ is a vector bundle, $\mathcal{N}'\subset\mathcal{N}$, and $V'\subset V$, we define $\mathcal{N}'|_{V'} = \mathcal{N}|_{V'}\cap\mathcal{N}'$.

3.1 Definitions

Let X be a (smooth) manifold. For a collection $\{V_i\}_{i\in S}$ of submanifolds of X and $I\subset S$, let

$$V_I \equiv \bigcap_{i\in I} V_i \subset X, \quad V_{I,\partial} = \bigcup_{I'\supsetneq I} V_{I'}, \quad V_I^\circ = V_I - V_{I,\partial}.$$

Such a collection is called transverse if any subcollection $\{V_i\}_{i\in I}$ of these submanifolds intersects transversely, i.e., the homomorphism

$$T_xX \oplus \bigoplus_{i\in I} T_xV_i \longrightarrow \bigoplus_{i\in I} T_xX, \quad (v,(v_i)_{i\in I}) \longrightarrow (v+v_i)_{i\in I}\,, \tag{3.1}$$

is surjective for all $x \in V_I$. By the Inverse Function Theorem, each subspace $V_I \subset X$ is then a submanifold of X of codimension

$$\mathrm{codim}_X V_I = \sum_{i\in I} \mathrm{codim}_X V_i$$

and the homomorphisms

$$\begin{aligned} &\mathcal{N}_X V_I \longrightarrow \bigoplus_{i\in I} \mathcal{N}_X V_i|_{V_I} \quad \forall I \subset S, \quad \mathcal{N}_{V_{I-i}} V_I \longrightarrow \mathcal{N}_X V_i|_{V_I} \quad \forall i \in I \subset S, \\ &\bigoplus_{i\in I-I'} \mathcal{N}_{V_{I-i}} V_I \longrightarrow \mathcal{N}_{V_{I'}} V_I \quad \forall I' \subset I \subset S \end{aligned} \tag{3.2}$$

induced by inclusions of the tangent bundles are isomorphisms.

As detailed in [14, Section 2.1], a transverse collection $\{V_i\}_{i\in S}$ of oriented submanifolds of an oriented manifold X of even codimensions induces an orientation on each submanifold $V_I \subset X$ with $|I| \geq 2$; we call it the intersection orientation of V_I. If V_I is zero-dimensional, it is a discrete collection of points in X and the homomorphism (3.1) is an isomorphism at each point $x \in V_I$; the intersection orientation of V_I at $x \in V_I$ then corresponds to a plus or minus sign, depending on whether this isomorphism is orientation-preserving or orientation-reversing. We call the original orientations of $X = V_\emptyset$ and $V_i = V_{\{i\}}$ the intersection orientations of these submanifolds V_I of X with $|I| < 2$.

Suppose (X,ω) is a symplectic manifold and $\{V_i\}_{i\in S}$ is a transverse collection of submanifolds of X such that each V_I is a symplectic submanifold of (X,ω). Each V_I then carries an orientation induced by $\omega|_{V_I}$, which we call the ω-orientation. If V_I is zero-dimensional, it is automatically a symplectic submanifold of (X,ω); the ω-orientation of V_I at each point $x \in V_I$ corresponds to the plus sign by definition. By the previous paragraph, the ω-orientations of X and V_i with $i \in I$ also induce intersection orientations on all V_I.

Definition 3.1 *Let (X,ω) be a symplectic manifold. A simple crossings (or SC) symplectic divisor in (X,ω) is a finite transverse union $V = \bigcup_{i\in S} V_i$ of closed submanifolds of X of codimension 2 such that V_I is a symplectic submanifold of (X,ω) for every $I \subset S$ and the intersection and ω-orientations of V_I agree.*

The singular locus $V_\partial \subset V$ of an SC symplectic divisor $V \subset X$ is the union

$$V_\partial \equiv \bigcup_{I\subset S, |I|\geq 2} V_I.$$

An SC symplectic divisor V with $|S| = 1$ is a smooth symplectic divisor in the usual sense. If (X,ω) is a 4-dimensional symplectic manifold, a finite transverse union $V = \bigcup_{i\in S} V_i$ of closed symplectic submanifolds of X of codimension 2 is an SC symplectic divisor if all points of the pairwise intersections $V_{i_1} \cap V_{i_2}$ with $i_1 \neq i_2$ are positive. By [14, Example 2.7], it is not sufficient to consider the deepest (non-empty) intersections in higher dimensions.

Definition 3.2 *Let X be a manifold and $V=\bigcup_{i\in S}V_i$ be a finite transverse union of closed submanifolds of X of codimension 2. A* symplectic structure on V in X *is a symplectic form ω on X such that V_I is a symplectic submanifold of (X,ω) for all $I\subset S$.*

For X and $\{V_i\}_{i\in S}$ as in Definition 3.2, we denote by $\mathrm{Symp}(X,\{V_i\}_{i\in S})$ the space of all symplectic structures on $\{V_i\}_{i\in S}$ in X and by

$$\mathrm{Symp}^+(X,\{V_i\}_{i\in S})\subset\mathrm{Symp}(X,\{V_i\}_{i\in S})$$

the subspace of the symplectic forms ω such that $\{V_i\}_{i\in S}$ is an SC symplectic divisor in (X,ω).

3.2 Regularizations

Let V be a smooth manifold with a 2-form ω and $(L_i,\rho_i,\nabla^{(i)})_{i\in I}$ be a finite collection of Hermitian line bundles over V. If each $(\rho_i,\nabla^{(i)})$ is compatible with a fiberwise symplectic form Ω_i on L_i and

$$(\mathcal{N},\Omega,\nabla)\equiv\bigoplus_{i\in I}(L_i,\Omega_i,\nabla^{(i)}),$$

then the 2-form (2.4) is given by

$$\widetilde{\omega}=\omega_{(\rho_i,\nabla^{(i)})_{i\in I}}\equiv\pi^*\omega+\frac{1}{2}\bigoplus_{i\in I}\pi_{I;i}^*d((\Omega_i)_{\nabla^{(i)}}(\zeta_{L_i},\cdot)),\tag{3.3}$$

where $\pi_{I;i}:\mathcal{N}\to L_i$ is the component projection map.

If in addition $\Psi:V'\to V$ is a smooth map and $(L_i',\rho_i',\nabla'^{(i)})_{i\in I}$ is a finite collection of Hermitian line bundles over V', we call a (fiberwise) vector bundle isomorphism

$$\widetilde{\Psi}:\bigoplus_{i\in I}L_i'\longrightarrow\bigoplus_{i\in I}L_i$$

covering Ψ a product Hermitian isomorphism if

$$\widetilde{\Psi}:(L_i',\rho_i',\nabla'^{(i)})\longrightarrow\Psi^*(L_i,\rho_i,\nabla^{(i)})$$

is an isomorphism of Hermitian line bundles over V' for every $i\in I$.

If V is a symplectic submanifold of a symplectic manifold (X,ω), we denote the restriction of $\omega|_{\mathcal{N}_XV}$ to a subbundle $L\subset\mathcal{N}_XV$ by $\omega|_L$.

Definition 3.3 *Let X be a manifold, $V\subset X$ be a submanifold, and*

$$\mathcal{N}_XV=\bigoplus_{i\in I}L_i$$

be a fixed splitting into oriented rank 2 subbundles. If ω is a symplectic form on X such that V is a symplectic submanifold and $\omega|_{L_i}$ is nondegenerate for every $i\in I$, then an ω-regularization for V in X *is a tuple $((\rho_i,\nabla^{(i)})_{i\in I},\Psi)$, where $(\rho_i,\nabla^{(i)})$ is an $\omega|_{L_i}$-compatible Hermitian structure on L_i for each $i\in I$ and Ψ is a regularization for V in X in the sense of Definition 2.1, such that*

$$\Psi^*\omega=\omega_{(\rho_i,\nabla^{(i)})_{i\in I}}|_{\mathrm{Dom}(\Psi)}.$$

Suppose $\{V_i\}_{i\in S}$ is a transverse collection of codimension 2 submanifolds of X. For each $I\subset S$, the last isomorphism in (3.2) with $I'=\emptyset$ provides a natural decomposition

$$\pi_I:\mathcal{N}_XV_I=\bigoplus_{i\in I}\mathcal{N}_{V_{I-i}}V_I\longrightarrow V_I\tag{3.4}$$

of the normal bundle of V_I in X into oriented rank 2 subbundles. We take this decomposition as given for the purposes of applying Definition 3.3. If in addition $I' \subset I$, let

$$\pi_{I;I'} : \mathcal{N}_{I;I'} \equiv \bigoplus_{i \in I - I'} \mathcal{N}_{V_{I-i}} V_I = \mathcal{N}_{V_{I'}} V_I \longrightarrow V_I \tag{3.5}$$

be the bundle projection. There are canonical identifications

$$\mathcal{N}_{I;I-I'} = \mathcal{N}_X V_{I'}|_{V_I}, \quad \mathcal{N}_X V_I = \pi_{I;I'}^* \mathcal{N}_{I;I-I'} = \pi_{I;I'}^* \mathcal{N}_X V_{I'} \quad \forall I' \subset I \subset [N]. \tag{3.6}$$

The first equality in the second statement above is used in particular in (3.10).

Definition 3.4 *Let X be a manifold and $\{V_i\}_{i \in S}$ be a transverse collection of submanifolds of X. A system of regularizations for $\{V_i\}_{i \in S}$ in X is a tuple $(\Psi_I)_{I \subset S}$, where Ψ_I is a regularization for V_I in X in the sense of Definition* 2.1*, such that*

$$\Psi_I(\mathcal{N}_{I;I'} \cap \mathrm{Dom}(\Psi_I)) = V_{I'} \cap \mathrm{Im}(\Psi_I) \quad \text{and} \quad \mathrm{Im}(\Psi_I) \cap \mathrm{Im}(\Psi_J) = \mathrm{Im}(\Psi_{I \cup J}) \tag{3.7}$$

for all $I' \subset I \subset S$ and $J \subset S$.

Given a system of regularizations as in Definition 3.4 and $I' \subset I \subset S$, let

$$\mathcal{N}'_{I;I'} = \mathcal{N}_{I;I'} \cap \mathrm{Dom}(\Psi_I), \quad \Psi_{I;I'} \equiv \Psi_I|_{\mathcal{N}'_{I;I'}} : \mathcal{N}'_{I;I'} \longrightarrow V_{I'}. \tag{3.8}$$

The map $\Psi_{I;I'}$ is a regularization for V_I in $V_{I'}$. As explained in [14, Section 2.2], Ψ_I determines an isomorphism

$$\mathfrak{D}\Psi_{I;I'} : \pi_{I;I'}^* \mathcal{N}_{I;I-I'}|_{\mathcal{N}'_{I;I'}} \longrightarrow \mathcal{N}_X V_{I'}|_{V_{I'} \cap \mathrm{Im}(\Psi_I)} \tag{3.9}$$

of vector bundles covering $\Psi_{I;I'}$ and respecting the natural decompositions of $\mathcal{N}_{I;I-I'} = \mathcal{N}_X V_{I'}|_{V_I}$ and $\mathcal{N}_X V_{I'}$. By the last assumption in Definition 2.1,

$$\mathfrak{D}\Psi_{I;I'}|_{\pi_{I;I'}^* \mathcal{N}_{I;I-I'}|_{V_I}} = \mathrm{id} : \mathcal{N}_{I;I-I'} \longrightarrow \mathcal{N}_X V_{I'}|_{V_I}$$

under the canonical identification of $\mathcal{N}_{I;I-I'}$ with $\mathcal{N}_X V_{I'}|_{V_I}$.

Definition 3.5 *Let X be a manifold and $\{V_i\}_{i \in S}$ be a transverse collection of submanifolds of X.*

(1) *A regularization for $\{V_i\}_{i \in S}$ in X is a system of regularizations $(\Psi_I)_{I \subset S}$ for $\{V_i\}_{i \in S}$ in X such that*

$$\begin{aligned} &\mathrm{Dom}(\Psi_I) \subset \pi_{I;I'}^* \mathcal{N}_{I;I-I'}|_{\mathcal{N}'_{I;I'}}, \quad \mathfrak{D}\Psi_{I;I'}(\mathrm{Dom}(\Psi_I)) = \mathrm{Dom}(\Psi_{I'})|_{V_{I'} \cap \mathrm{Im}(\Psi_I)}, \\ &\Psi_I = \Psi_{I'} \circ \mathfrak{D}\Psi_{I;I'}|_{\mathrm{Dom}(\Psi_I)} \end{aligned} \tag{3.10}$$

for all $I' \subset I \subset S$.

(2) *Suppose in addition that $V = \bigcup_{i \in S} V_i$ is an SC symplectic divisor in (X, ω). An ω-regularization for V in X is a tuple*

$$(\mathcal{R}_I)_{I \subset S} \equiv ((\rho_{I;i}, \nabla^{(I;i)})_{i \in I}, \Psi_I)_{I \subset S}$$

such that $\mathcal{R}_I$ is an ω-regularization for V_I in X for each $I \subset S$, $(\Psi_I)_{I \subset S}$ is a regularization for $\{V_i\}_{i \in S}$ in X, and the induced vector bundle isomorphisms (3.9) *are product Hermitian isomorphisms for all $I' \subset I \subset S$.*

If $(\Psi_I)_{I\subset S}$ is a regularization for $\{V_i\}_{i\in S}$ in X, then

$$\begin{aligned}&\mathcal{N}'_{I;I''},\pi^*_{I;I''}\mathcal{N}_{I;I-I''}|_{\mathcal{N}'_{I;I''}}\subset\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\mathcal{N}'_{I;I'}},\\&\Psi_{I;I''}=\Psi_{I';I''}\circ\mathfrak{D}\Psi_{I;I'}|_{\mathcal{N}'_{I;I''}},\quad \mathfrak{D}\Psi_{I;I''}=\mathfrak{D}\Psi_{I';I''}\circ\mathfrak{D}\Psi_{I;I'}|_{\pi^*_{I;I''}\mathcal{N}_{I;I-I''}|_{\mathcal{N}'_{I;I''}}}\end{aligned}\tag{3.11}$$

for all $I''\subset I'\subset I\subset S$.

An almost complex structure J on X preserving $TV_I\subset TX|_{V_I}$ and an ω-regularization $\mathcal{R}_I$ for V_I in X as in Definition 3.3 determine an almost complex structure $J_{\mathcal{R};I}$ on the total space of $\mathcal{N}_XV_I$ via the connection $\nabla^{(I)}\equiv\bigoplus_{i\in I}\nabla^{(I;i)}$. We call an almost complex structure J on X compatible with an ω-regularization $(\mathcal{R}_I)_{I\subset S}$ as in Definition 3.5 (2) if

$$J(TV_I)\subset TV_I\quad\text{and}\quad J\circ d\Psi_I=d\Psi_I\circ J_{\mathcal{R};I}|_{\mathrm{Dom}(\Psi_I)}\quad\forall\, I\subset S.$$

The notion of regularization of Definition 3.5 (2) readily extends to families of symplectic forms; see [14, Definition 2.12 (2)]. We define the spaces $\mathrm{Aux}(X,V)$ of pairs $(\omega,\mathcal{R})$ and $\mathrm{AK}(X,V)$ of triples $(\omega,\mathcal{R},J)$ as in Section 2.2. By [14, Theorem 2.17], the map (2.7) is a weak homotopy equivalence in the present setting as well. On the other hand, it is still straightforward to show that the map (2.8) is also a weak homotopy equivalence in the present setting.

3.3 Constructions

We now construct the bundles $\mathcal{O}_X(V)$ and $TX(-\log V)$ for an SC symplectic divisor V in a symplectic manifold (X,ω). We fix an ω-regularization $\mathcal{R}$ for V in X as in Definition 3.5 (2). For the purposes of constructing a complex structure on $TX(-\log V)$, we also fix an almost complex structure J on X compatible with ω and $\mathcal{R}$.

For $I'\subset I\subset S$, let π_I, $\pi_{I;I'}$, $\mathcal{N}'_{I;I'}$, $\Psi_{I;I'}$, and $\mathfrak{D}\Psi_{I;I'}$ be as in (3.4), (3.5), (3.8), and (3.9). In what follows, we write an element $v_I\equiv(v_i)_{i\in I}$ of $\mathcal{N}_XV_I$ as

$$v_I=(v_{I;I'},v_{I;I-I'})\quad\text{with}\quad v_{I;I'}\equiv(v_i)_{i\in I-I'}\in\mathcal{N}_{I;I'}\ \text{ and }\ v_{I;I-I'}\equiv(v_i)_{i\in I'}\in\mathcal{N}_{I;I-I'}.$$

We denote by $\nabla^{(I)}$ and $\nabla^{(I;I')}$ the connections on $\mathcal{N}_XV_I$ and $\mathcal{N}_{I;I'}$ induced by the connections $\nabla^{(I;i)}$ on the direct summands of these vector bundles. Let

$$\mathrm{h}_{\nabla^{(I)};v_I}:T_{\pi_I(v_I)}V_I\longrightarrow T_{v_I}(\mathcal{N}_XV_I)\quad\text{and}\quad\mathrm{h}_{\nabla^{(I;I')};v_{I;I'}}:T_{\pi_{I;I'}(v_{I;I'})}V_I\longrightarrow T_{v_{I;I'}}\mathcal{N}_{I;I'}$$

be the corresponding injective homomorphisms as in (2.6). Define

$$\begin{aligned}&\Pi_I:\mathcal{N}_XV_I\longrightarrow\bigotimes_{i\in I}\mathcal{N}_{V_{I-i}}V_I,\quad\Pi_I((v_i)_{i\in I})=\bigotimes_{i\in I}v_i,\\&U_I^\circ=\Psi_I(\mathrm{Dom}(\Psi_I)|_{V_I^\circ})=\mathrm{Im}\,\Psi_I-\bigcup_{J\not\subset I}V_J;\end{aligned}\tag{3.12}$$

the last equality follows from (3.7) and (3.10). For every $I\subset S$, let

$$\begin{aligned}&\mathcal{O}_{\mathcal{R};I}(V)=\{\Psi_I^{-1}|_{U_I^\circ}\}^*\pi_I^*\bigg(\bigotimes_{i\in I}\mathcal{N}_{V_{I-i}}V_I\bigg)\longrightarrow U_I^\circ,\\&T_{\mathcal{R}}U_I^\circ(-\log V)=(\{\Psi_I^{-1}|_{U_I^\circ}\}^*\pi_I^*TV_I)\oplus(U_I^\circ\times\mathbb{C}^I)\longrightarrow U_I^\circ.\end{aligned}\tag{3.13}$$

The complex structures $\mathfrak{i}_{\rho_{I;i}}$ on $\mathcal{N}_{V_{I-i}}V_I=\mathcal{N}_XV_i|_{V_I}$ encoded in $\mathcal{R}$ determine a complex structure on the complex line bundle $\mathcal{O}_{\mathcal{R};I}(V)$. The almost complex structure $J|_{TV_I}$ on V_I and

the standard complex structure on $\mathbb{C}^I$ determine a complex structure on the vector bundle $T_{\mathcal{R}}U_I^\circ(-\log V)$.

Let $I' \subset I \subset S$. If $x \in U_I^\circ \cap U_{I'}^\circ$, then

$$\begin{aligned} &x = \Psi_I(v_I) = \Psi_{I'}(\mathfrak{D}\Psi_{I;I'}(v_{I;I'}, v_{I;I-I'})) \quad \text{with} \\ &v_I = (v_{I;I'}, v_{I;I-I'}) \equiv ((v_i)_{i\in I-I'}, (v_i)_{i\in I'}) \in \mathcal{N}_{I;I'} \oplus \mathcal{N}_{I;I-I'} \quad \text{s.t. } v_i \neq 0 \ \forall i \in I - I'. \end{aligned}$$

Since $\mathfrak{D}\Psi_{I;I'}$ is a product Hermitian isomorphism, it follows that the map

$$\begin{aligned} &\theta_{I'I} : \mathcal{O}_{\mathcal{R};I}(V)|_{U_I^\circ \cap U_{I'}^\circ} \longrightarrow \mathcal{O}_{\mathcal{R};I'}(V)|_{U_I^\circ \cap U_{I'}^\circ}, \\ &\theta_{I'I}(x, v_I, \Pi_I(v_{I;I'}, w_{I;I-I'})) = (x, \mathfrak{D}\Psi_{I;I'}(v_I), \Pi_{I'}(\mathfrak{D}\Psi_{I;I'}(v_{I;I'}, w_{I;I-I'}))), \end{aligned} \tag{3.14}$$

is a well-defined isomorphism of complex line bundles. The map

$$\begin{aligned} &\vartheta_{I'I} : T_{\mathcal{R}}U_I^\circ(-\log V)|_{U_I^\circ \cap U_{I'}^\circ} \longrightarrow T_{\mathcal{R}}U_{I'}^\circ(-\log V)|_{U_I^\circ \cap U_{I'}^\circ}, \\ &\vartheta_{I'I}((x, v_I, w) \oplus (x, (c_i)_{i\in I})) = (x, \mathfrak{D}\Psi_{I;I'}(v_I), d_{v_{I;I'}}\Psi_{I;I'}\Big(\mathrm{h}_{\nabla^{(I;I')};v_{I;I'}}(w) + \sum_{i\in I-I'} c_i v_i)\Big) \\ &\qquad\qquad \oplus (x, (c_i)_{i\in I'}), \end{aligned} \tag{3.15}$$

is similarly a well-defined isomorphism of vector bundles. Since J is an $\mathcal{R}$-compatible almost complex structure on X, this isomorphism is $\mathbb{C}$-linear. By (3.11),

$$\begin{aligned} \theta_{I''I}|_{U_I^\circ \cap U_{I'}^\circ \cap U_{I''}^\circ} &= \theta_{I''I'}|_{U_I^\circ \cap U_{I'}^\circ \cap U_{I''}^\circ} \circ \theta_{I'I}|_{U_I^\circ \cap U_{I'}^\circ \cap U_{I''}^\circ}, \\ \vartheta_{I''I}|_{U_I^\circ \cap U_{I'}^\circ \cap U_{I''}^\circ} &= \vartheta_{I''I'}|_{U_I^\circ \cap U_{I'}^\circ \cap U_{I''}^\circ} \circ \vartheta_{I'I}|_{U_I^\circ \cap U_{I'}^\circ \cap U_{I''}^\circ} \end{aligned} \tag{3.16}$$

for all $I'' \subset I' \subset I$.

Let $I, K \subset S$. By (3.7) and (3.10), $U_I^\circ \cap U_K^\circ \subset U_{I\cup K}^\circ$. If $I \not\subset K$, the maps

$$\begin{aligned} \theta_{IK} &= \theta_{I(I\cup K)}|_{U_I^\circ \cap U_K^\circ} \circ \theta_{K(I\cup K)}^{-1}|_{U_I^\circ \cap U_K^\circ} : \mathcal{O}_{\mathcal{R};K}(V)|_{U_I^\circ \cap U_K^\circ} \longrightarrow \mathcal{O}_{\mathcal{R};I}(V)|_{U_I^\circ \cap U_K^\circ}, \\ \vartheta_{IK} &= \vartheta_{I(I\cup K)}|_{U_I^\circ \cap U_K^\circ} \circ \vartheta_{K(I\cup K)}^{-1}|_{U_I^\circ \cap U_K^\circ} : T_{\mathcal{R}}U_K^\circ(-\log V)|_{U_I^\circ \cap U_K^\circ} \longrightarrow T_{\mathcal{R}}U_I^\circ(-\log V)|_{U_I^\circ \cap U_K^\circ} \end{aligned}$$

are thus well-defined isomorphisms of complex vector bundles. By (3.16), the collections $\{\theta_{IK}\}_{I,K\subset S}$ and $\{\vartheta_{IK}\}_{I,K\subset S}$ satisfy the cocycle condition. The first collection thus determines a complex line bundle

$$\begin{aligned} &\pi : \mathcal{O}_{\mathcal{R};X}(V) = \Big(\bigsqcup_{I\subset S} \mathcal{O}_{\mathcal{R};I}(V)\Big)\Big/\sim \longrightarrow X, \quad \pi([x, v_I, \Pi_I(w_I)]) = x, \\ &\mathcal{O}_{\mathcal{R};I}(V)|_{U_I^\circ \cap U_K^\circ} \ni \theta_{IK}(v) \sim v \in \mathcal{O}_{\mathcal{R};K}(V)|_{U_I^\circ \cap U_K^\circ} \quad \forall I, K \subset S. \end{aligned} \tag{3.17}$$

The second collection similarly determines a complex vector bundle

$$\begin{aligned} &\pi : T_{\mathcal{R}}X(-\log V) = \Big(\bigsqcup_{I\subset S} T_{\mathcal{R}}U_I^\circ(-\log V)\Big)\Big/\sim \longrightarrow X, \ \pi([(x, v_I, w) \oplus (x, (c_i)_{i\in I})]) = x, \\ &T_{\mathcal{R}}U_I^\circ(-\log V)|_{U_I^\circ \cap U_K^\circ} \ni \vartheta_{IK}(v) \sim v \in T_{\mathcal{R}}U_K^\circ(-\log V)|_{U_I^\circ \cap U_K^\circ} \quad \forall I, K \subset S. \end{aligned} \tag{3.18}$$

The smooth section $s_{\mathcal{R}}$ of the complex line bundle (3.17) given by

$$s_{\mathcal{R}}(x) = [x, v_I, \Pi_I(v_I)] \quad \forall x = \Psi_I(v_I) \in U_I^\circ, I \subset S, \tag{3.19}$$

satisfies the properties stated in Proposition 1.1 (1). The smooth bundle homomorphism (1.3) defined by

$$\iota_{\mathcal{R}}([(x,v_I,w)\oplus(x,(c_i)_{i\in I})]) = d_{v_I}\Psi_I\bigg(\mathrm{h}_{\nabla^{(I)};v_I}(w)+\sum_{i\in I}c_iv_i\bigg)$$
$$\forall x=\Psi_I(v_I)\in U_I^\circ, I\subset S, \tag{3.20}$$

satisfies the properties stated in Theorem 1.2 (1)–(2). By the same reasoning as at the end of Section 2.2, the bundles (3.17) and (3.18) also satisfy the properties in Proposition 1.1 (2) and Theorem 1.2 (3). Along with Proposition 1.1 (2), Lemma 3.6 below implies the claim of Proposition 1.1 (3) for SC symplectic divisors.

Lemma 3.6 *Suppose V and V' are SC symplectic divisors in a symplectic manifold (X,ω) so that $V\cup V'\subset X$ is also an SC symplectic divisor and $V\cap V'$ contains no open subspace of V. An ω-regularization*

$$\widetilde{\mathcal{R}}\equiv((\rho_{I;i},\nabla^{(I;i)})_{i\in I},\widetilde{\Psi}_I)_{I\subset S\sqcup S'}$$

for $V\cup V'$ in X as in Definition 3.5 (2) determines ω-regularizations $\mathcal{R}$ for V in X and $\mathcal{R}'$ for V' in X and an isomorphism

$$\psi_{\mathcal{R}\mathcal{R}'}:(\mathcal{O}_{\widetilde{\mathcal{R}};X}(V\cup V'),\mathfrak{i}_{\widetilde{\mathcal{R}}})\longrightarrow(\mathcal{O}_{\mathcal{R};X}(V),\mathfrak{i}_{\mathcal{R}})\otimes(\mathcal{O}_{\mathcal{R}';X}(V'),\mathfrak{i}_{\mathcal{R}'}) \tag{3.21}$$

natural with respect to the restrictions to the open subsets of X.

Proof Let $V=\bigcup_{i\in S}V_i$ and $V'=\bigcup_{i\in S'}V_i'$. If $\widetilde{\mathcal{R}}=(\mathcal{R}_I)_{I\subset S\sqcup S'}$, then $\mathcal{R}\equiv(\mathcal{R}_I)_{I\subset S}$ is an ω-regularization for V in X and $\mathcal{R}'\equiv(\mathcal{R}_I)_{I\subset S'}$ is an ω-regularization for V' in X. By the assumptions,

$$\widetilde{V}_{I\sqcup K}\equiv(V\cup V')_{I\sqcup K}=V_I\cap V_K',\quad \mathcal{N}_X\widetilde{V}_{I\sqcup K}=\mathcal{N}_XV_I|_{\widetilde{V}_{I\sqcup K}}\oplus\mathcal{N}_XV_K'|_{\widetilde{V}_{I\sqcup K}}\quad\forall I\subset S,\ K\subset S'.$$

For $I'\subset I\subset S'$ (resp. $I'\subset I\subset S\sqcup S'$), we denote by $U_I'^\circ\subset X$ and $\theta'_{I'I}$ (resp. $\widetilde{U}_I^\circ\subset X$ and $\widetilde{\theta}_{I'I}$) the analogues of the open subsets U_I° and the transition maps $\theta_{I'I}$ in (3.14) for the trivialization $\mathcal{R}'$ (resp. $\widetilde{\mathcal{R}}$). Thus,

$$U_I^\circ=\bigcup_{K\subset S'}\widetilde{U}_{I\sqcup K}^\circ\ \ \forall I\subset S\quad\text{and}\quad U_K'^\circ=\bigcup_{I\subset S}\widetilde{U}_{I\sqcup K}^\circ\ \ \forall K\subset S'.$$

For $I\subset S$ and $K\subset S'$, define

$$\psi_{\mathcal{R}\mathcal{R}';IK}:\mathcal{O}_{\widetilde{\mathcal{R}};I\sqcup K}(V\cup V')\longrightarrow\mathcal{O}_{\mathcal{R};I}(V)|_{\widetilde{U}_{I\sqcup K}^\circ}\otimes_{\mathbb{C}}\mathcal{O}_{\mathcal{R}';K}(V')|_{\widetilde{U}_{I\sqcup K}^\circ},$$
$$\psi_{\mathcal{R}\mathcal{R}';IK}(x,v_{I\sqcup K},\Pi_I(w_{I\sqcup K}))=(x,\mathfrak{D}\widetilde{\Psi}_{I\sqcup K;I}(v_{I\sqcup K}),\Pi_I(\mathfrak{D}\widetilde{\Psi}_{I\sqcup K;I}(w_{I\sqcup K})))$$
$$\otimes(x,\mathfrak{D}\widetilde{\Psi}_{I\sqcup K;K}(v_{I\sqcup K}),\Pi_K(\mathfrak{D}\widetilde{\Psi}_{I\sqcup K;K}(w_{I\sqcup K}))).$$

By (3.11) for the regularization $\widetilde{\mathcal{R}}$,

$$\{\vartheta_{I'I}\otimes\vartheta'_{K'K}\}\circ\psi_{\mathcal{R}\mathcal{R}';IK}=\psi_{\mathcal{R}\mathcal{R}';I'K'}\circ\widetilde{\vartheta}_{(I'\sqcup K')(I\sqcup K)}:$$
$$\mathcal{O}_{\widetilde{\mathcal{R}};I\sqcup K}(V\cup V')|_{\widetilde{U}_{I\sqcup K}^\circ\cap\widetilde{U}_{I'\sqcup K'}^\circ}\longrightarrow\mathcal{O}_{\mathcal{R};I'}(V)|_{\widetilde{U}_{I\sqcup K}^\circ\cap\widetilde{U}_{I'\sqcup K'}^\circ}\otimes_{\mathbb{C}}\mathcal{O}_{\mathcal{R}';K'}(V')|_{\widetilde{U}_{I\sqcup K}^\circ\cap\widetilde{U}_{I'\sqcup K'}^\circ}$$

for all $I'\subset I\subset S$ and $K'\subset K\subset S'$. Along with (3.17), this implies that the collection $\{\psi_{\mathcal{R}\mathcal{R}';IK}\}$ induces a well-defined bundle homomorphism (3.21). □

3.4 Proof of Theorem 1.2(4)

We now establish (1.4) under the assumption that V is an SC symplectic divisor. The reasoning in the general case is identical; see the end of Section 4.3. As noted after Theorem 1.2, (1.4) implies (1.5) if V is an SC symplectic divisor.

Suppose X is a smooth manifold, $V \subset X$ is a submanifold, and $\psi : E \to F$ is a homomorphism of complex vector bundles over X that vanishes on a complex subbundle $L' \subset E|_V$. The restriction of ψ to an extension of L' to a complex subbundle L of E over a neighborhood U of $V \subset X$ then determines a section $\psi_{L'}$ of the complex vector bundle $L^* \otimes_{\mathbb{C}} F$ over U that vanishes on V and well-defined derivative bundle homomorphisms

$$\mathrm{D}_x\psi_{L'} : \mathcal{N}_X V|_x \longrightarrow L'^*_x \otimes_{\mathbb{C}} (\operatorname{cok}\psi_x) \quad \text{and} \quad \mathrm{D}_x\psi_{L'} : \mathcal{N}_X V|_x \otimes_{\mathbb{R}} L'_x \longrightarrow \operatorname{cok}\psi_x \tag{3.22}$$

for each $x \in V$; these homomorphisms do not depend on the extension of L'. Lemma 3.7 below, proved at the end of this section, is the key topological input we use to establish (1.4).

Lemma 3.7 *Suppose X is a smooth manifold, $V \subset X$ is a closed submanifold of codimension 2 with a complex structure on $\mathcal{N}_X Y$, $\psi : E \to F$ is a homomorphism of complex vector bundles over X, and $L \to X$ is a complex vector bundle so that*

$$L|_V = \ker(\psi : E|_V \longrightarrow F|_V) \subset E|_V.$$

If ψ is an isomorphism over $X - V$, the first homomorphism in (3.22) with $L' = L|_V$ is $\mathbb{C}$-linear for every $x \in V$, and the second homomorphism (3.22) is surjective, then there exists an isomorphism

$$\widetilde{\psi} : E \oplus \mathcal{O}_X(V) \otimes_{\mathbb{C}} L \longrightarrow F \oplus L \tag{3.23}$$

of complex vector bundles over X.

Lemma 3.8 *Let V and $V' \equiv \bigcup_{i\in S'} V'_i$, $\widetilde{\mathcal{R}}$, and $\mathcal{R}$ be as in Lemma 3.6. There exists a unique vector bundle homomorphism*

$$\iota_{\mathcal{R}\widetilde{\mathcal{R}}} : T_{\widetilde{\mathcal{R}}} X(-\log(V \cup V')) \longrightarrow T_{\mathcal{R}} X(-\log V) \tag{3.24}$$

so that $\iota_{\widetilde{\mathcal{R}}} = \iota_{\mathcal{R}} \circ \iota_{\mathcal{R}\widetilde{\mathcal{R}}}$. For every $K \subset S'$,

(1) *the kernel of $\iota_{\mathcal{R}\widetilde{\mathcal{R}}}$ over V'°_K is a trivial subbundle $L'_K \approx V'^{\circ}_K \times \mathbb{C}^K$;*

(2) *the cokernel of $\iota_{\mathcal{R}\widetilde{\mathcal{R}}}$ over V'°_K is canonically isomorphic to $\mathcal{N}_X V'^{\circ}_K$;*

(3) *the composition of the above isomorphism with the first homomorphism in (3.22) with $\psi = \iota_{\mathcal{R}\widetilde{\mathcal{R}}}$ and $L' = L'_K$,*

$$\mathcal{N}_X V'^{\circ}_K|_x \xrightarrow{\mathrm{D}_x(\iota_{\mathcal{R}\widetilde{\mathcal{R}}})_{L'_K}} (L'_K)^*_x \otimes_{\mathbb{C}} (\operatorname{cok}\iota_{\mathcal{R}\widetilde{\mathcal{R}}})_x \xrightarrow{\approx} (L'_K)^*_x \otimes_{\mathbb{C}} \mathcal{N}_X V'^{\circ}_K|_x,$$

is $\mathbb{C}$-linear with respect to the complex structures on its domain and target determined by the regularization $\widetilde{\mathcal{R}}$ for every $x \in V'^{\circ}_K$;

(4) *the second homomorphism in (3.22) with $\psi = \iota_{\mathcal{R}\widetilde{\mathcal{R}}}$ and $L' = L'_K$,*

$$\mathrm{D}_x(\iota_{\mathcal{R}\widetilde{\mathcal{R}}})_{L'_K} : \mathcal{N}_X V'^{\circ}_K|_x \otimes_{\mathbb{R}} L'_K|_x \longrightarrow (\operatorname{cok}\iota_{\mathcal{R}\widetilde{\mathcal{R}}})_x$$

is surjective for every $x \in V'^{\circ}_K$.

Suppose J is an almost complex structure on X compatible with $\widetilde{\mathcal{R}}$ (and thus with $\mathcal{R}$). Since $\iota_{\widetilde{\mathcal{R}}} = \iota_{\mathcal{R}} \circ \iota_{\mathcal{R}\widetilde{\mathcal{R}}}$ and the homomorphisms $\iota_{\widetilde{\mathcal{R}}}$ and $\iota_{\mathcal{R}}$ are $\mathbb{C}$-linear isomorphisms over $X - V \cup V'$, the homomorphism (3.24) is $\mathbb{C}$-linear with respect to the complex structures $\mathfrak{i}_{\widetilde{\mathcal{R}},J}$ on the domain and $\mathfrak{i}_{\mathcal{R},J}$ on the target. The isomorphism in Lemma 3.8 (2) is $\mathbb{C}$-linear with respect to the complex structures on its target and domain determined by $\widetilde{\mathcal{R}}$. If $V' \subset X$ is a smooth symplectic divisor, the bundle homomorphism $\psi = \iota_{\mathcal{R}\widetilde{\mathcal{R}}}$ thus satisfies the conditions of Lemma 3.7 with V replaced by V' and $L = X \times \mathbb{C}$. The conclusion of this lemma is the claim of (1.4).

Proof of Lemma 3.8 We continue with the notation in the proof of Lemma 3.6 and denote by $\widetilde{\vartheta}_{I'I}$ the analogues of the transition maps $\vartheta_{I'I}$ in (3.15) for the regularization $\widetilde{\mathcal{R}}$. For $I \subset S$ and $K \subset S'$, define

$$\begin{aligned}
&\iota_{\mathcal{R}\widetilde{\mathcal{R}};IK} : T_{\widetilde{\mathcal{R}}}\widetilde{U}^{\circ}_{I\sqcup K}(-\log(V\cup V')) \longrightarrow T_{\mathcal{R}}U^{\circ}_{I}(-\log V)|_{\widetilde{U}^{\circ}_{I\sqcup K}},\\
&\iota_{\mathcal{R}\widetilde{\mathcal{R}};IK}((x,v_{I\sqcup K},w)\oplus(x,(c_i)_{i\in I\sqcup K}))\\
&\quad = \left(x, \mathfrak{D}\widetilde{\Psi}_{I\sqcup K;I}(v_{I\sqcup K}), d_{v_{I\sqcup K;I}}\widetilde{\Psi}_{I\sqcup K;I}\left(\mathrm{h}_{\nabla^{(I\sqcup K)};v_{I\sqcup K}}(w) + \sum_{i\in K} c_i v_i\right)\right)\oplus(x,(c_i)_{i\in I}).
\end{aligned}$$

By (3.11) for the regularization $\widetilde{\mathcal{R}}$,

$$\begin{aligned}
\vartheta_{I'I}\circ\iota_{\mathcal{R}\widetilde{\mathcal{R}};IK} = \iota_{\mathcal{R}\widetilde{\mathcal{R}};I'K'}\circ\widetilde{\vartheta}_{(I'\sqcup K')(I\sqcup K)} &: T_{\widetilde{\mathcal{R}}}\widetilde{U}^{\circ}_{I\sqcup K}(-\log(V\cup V'))|_{\widetilde{U}^{\circ}_{I\sqcup K}\cap\widetilde{U}^{\circ}_{I'\sqcup K'}}\\
&\longrightarrow T_{\mathcal{R}}U^{\circ}_{I'}(-\log V)|_{\widetilde{U}^{\circ}_{I\sqcup K}\cap\widetilde{U}^{\circ}_{I'\sqcup K'}}
\end{aligned}$$

for all $I' \subset I \subset S$ and $K' \subset K \subset S'$. Along with (3.18), this implies that the collection $\{\iota_{\mathcal{R}\widetilde{\mathcal{R}};IK}\}$ induces a well-defined bundle homomorphism (3.24).

It is immediate from the definitions and (3.11) for the regularization $\widetilde{\mathcal{R}}$ that $\iota_{\widetilde{\mathcal{R}}} = \iota_{\mathcal{R}}\circ\iota_{\mathcal{R}\widetilde{\mathcal{R}}}$. If $I \subset S$, $K \subset S'$, and $x \in V'_K \cap \widetilde{U}^{\circ}_{I\sqcup K}$ with $x = \widetilde{\Psi}_{I\sqcup K}(v_{I\sqcup K;K}) = \Psi_I(v_{I\sqcup K;K})$, then

$$\ker(\iota_{\mathcal{R}\widetilde{\mathcal{R}};IK})_x = \{(x, v_{I\sqcup K;K}, 0)\}\oplus(\{x\}\times\{0\}^I\times\mathbb{C}^K).$$

The homomorphism

$$\begin{aligned}
&T_{\mathcal{R}}U^{\circ}_{I}(-\log V)|_{V'_K\cap\widetilde{U}^{\circ}_{I\sqcup K}} \longrightarrow \mathcal{N}_X V'^{\circ}_K|_{V'_K\cap\widetilde{U}^{\circ}_{I\sqcup K}},\\
&(x, v_{I\sqcup K;K}, w)\oplus(x,(c_i)_{i\in I\sqcup K}) \longrightarrow d_{v_{I\sqcup K;K}}\Psi_I(h_{\nabla^{(I)};v_{I\sqcup K;K}}(w)) + T_x V'^{\circ}_K,
\end{aligned}$$

induces an isomorphism $\mathrm{cok}(\iota_{\mathcal{R}\widetilde{\mathcal{R}}})_x \to \mathcal{N}_X V'^{\circ}_K|_x$. The composition of this isomorphism with the second homomorphism in (3.22) with $\psi = \iota_{\mathcal{R}\widetilde{\mathcal{R}}}$ and $L' = L'_K$ is given by

$$\mathrm{D}_x(\iota_{\mathcal{R}\widetilde{\mathcal{R}}})_{L'_K}((w_i)_{i\in K}\otimes(x,(c_i)_{i\in K})) = \sum_{i\in K} c_i w_i.$$

In particular, this homomorphism is surjective, as claimed. □

Proof of Lemma 3.7 Let $\Psi : \mathcal{N}' \to X$ be a regularization for V in X as in Definition 2.1 and $U = \Psi(\mathcal{N}')$. Let $\mathcal{O}_X(V) \to X$ be the complex line bundle (2.5) and $E' \subset E|_V$ be a complex subbundle complementary to $L|_V$. Extend E' to a subbundle of $E|_U$, which we still denote by E'. By the assumptions of the lemma, ψ is injective on E' and $\psi(E') \subset F|_U$ is a complex subbundle. Let $Q \subset F|_U$ be a complex subbundle complementary to $\psi(E')$. After shrinking $\mathcal{N}'$ if necessary, we can extend the identification of $L|_V$ with a subbundle of $E|_V$ to an identification

of $L|_U$ with a subbundle of $E|_U$ so that $\psi(L|_U) \subset Q$. Since ψ vanishes on $L|_V$, the derivative of the associated section of $L^*|_U \otimes_{\mathbb{C}} Q$ induces a homomorphism

$$\mathrm{D}\psi_L : \mathcal{N}_X V \longrightarrow L^*|_V \otimes_{\mathbb{C}} Q|_V \subset (L^* \otimes_{\mathbb{C}} F)|_V$$

of real vector bundles over V. By the assumptions on (3.22) with $L' = L|_V$, this homomorphism is $\mathbb{C}$-linear and induces an isomorphism

$$\mathrm{D}\psi_L : \mathcal{N}_X V \otimes_{\mathbb{C}} L|_V \longrightarrow Q|_V \subset F|_V$$

of an isomorphism of complex vector bundles over V. Using parallel transport, we identify $L|_U$ and Q with the vector bundles $\{\Psi^{-1}\}^*\pi^*L$ and $\{\Psi^{-1}\}^*\pi^*Q$, respectively.

Choose a smooth function $\beta : X \to [0,1]$ such that $\beta = 1$ on a neighborhood $U' \subset U$ of V and $\operatorname{supp}\beta \subset U$. We define (3.23) over $X - \operatorname{supp}\beta \subset X - V$ and on $E|_U = E' \oplus (L|_U)$ by

$$\begin{aligned}
&\widetilde{\psi}(e,w) = (\psi(e), w) \quad \forall (e,w) \in (E \oplus \mathcal{O}_X(V) \otimes_{\mathbb{C}} L)|_{X-\operatorname{supp}\beta} = (E \oplus L)|_{X-\operatorname{supp}\beta},\\
&\widetilde{\psi}(e' + e'', 0) = (\psi(e') + \psi(e''), -\beta(x)e'') \quad \forall e' \in E'_x,\ e'' \in L_x,\ x \in U.
\end{aligned}$$

We extend this definition to $(\mathcal{O}_X(V) \otimes_{\mathbb{C}} L)|_U$ by

$$\begin{aligned}
\widetilde{\psi}(0,(x,v,w)) &= (x, v, (\beta(x) D\psi_L(v \otimes w), (1-\beta(x))w)) \in \{\Psi^{-1}\}^*\pi^*(Q \oplus L)\\
&\qquad \forall (x,v,w) \in (\mathcal{O}_X(V) \otimes_{\mathbb{C}} L)|_{U-V} = \{\Psi^{-1}\}^*\pi^*L|_{U-V},\\
\widetilde{\psi}(0,(x,v,w)) &= (x, v, (\beta(x) D\psi_L(w), 0)) \in \{\Psi^{-1}\}^*\pi^*(Q \oplus L)\\
&\qquad \forall (x,v,w) \in (\mathcal{O}_X(V) \otimes_{\mathbb{C}} L)|_{U'} = \{\Psi^{-1}\}^*\pi^*(\mathcal{N}_X V \otimes L)|_{U'}.
\end{aligned}$$

The bundle homomorphism $\widetilde{\psi}$ is well-defined and smooth. It remains to verify that the homomorphism

$$\widetilde{\psi} : (L \oplus \mathcal{O}_X(V) \otimes_{\mathbb{C}} L)|_U \longrightarrow Q \oplus (L|_U) \tag{3.25}$$

is injective over $\operatorname{supp}\beta \subset U$ if $\operatorname{supp}\beta$ is sufficiently small.

Over U', the homomorphism (3.25) is given by

$$\begin{aligned}
&\widetilde{\psi} : \{\Psi^{-1}\}^*\pi^*(L \oplus \mathcal{N}_X V \otimes L)|_{U'} \longrightarrow \{\Psi^{-1}\}^*\pi^*(Q \oplus L),\\
&\widetilde{\psi}(x,v,(e'', v' \otimes w)) = (\psi(x,v,e''),0) + (x,v,(D\psi_L(v' \otimes w), -e'')),
\end{aligned}$$

and is thus injective. Over $\operatorname{supp}\beta - V \subset U - V$, (3.25) is given by

$$\begin{aligned}
&\widetilde{\psi} : \{\Psi^{-1}\}^*\pi^*(L \oplus L)|_{\operatorname{supp}\beta - V} \longrightarrow \{\Psi^{-1}\}^*\pi^*(Q \oplus L),\\
&\widetilde{\psi}(x,v,(e'',w)) = (\psi(x,v,e''),0) + (x,v,(\beta(x)D\psi_L(v \otimes w), -\beta(x)e'' + (1-\beta(x))w)).
\end{aligned} \tag{3.26}$$

Choose norms on $L|_U$ and Q. Let $\mathcal{C} : V \to \mathbb{R}^+$ and $\varepsilon : (U,V) \to (\mathbb{R},0)$ be smooth functions such that

$$\begin{aligned}
&\mathcal{C}(\pi(v))|\mathrm{D}\psi_L(v \otimes w)| \geq |v||w|,\\
&|\psi(\Psi(v),v,w) - (\Psi(v),v,\mathrm{D}\psi_L(v \otimes w))| \leq \varepsilon(v)||v||w|,
\end{aligned} \qquad \forall (v,w) \in \mathcal{N}' \oplus L|_V.$$

If $(x,v,(e'',w))$ with $x = \Psi(v) \in \operatorname{supp}\beta - V$ lies in the kernel of (3.26), then

$$\begin{aligned}
&\beta(x)e'' = (1-\beta(x))w, \quad D\psi_L(v \otimes (\beta(x)w + e'')) = D\psi_L(v \otimes e'') - \psi(x,v,e''),\\
&|v||\beta(x)w + e''| \leq \mathcal{C}(\pi(v))|\mathrm{D}\psi_L(v \otimes (\beta(x)w + e''))| \leq \mathcal{C}(\pi(v))\varepsilon(v)||v||e''|.
\end{aligned}$$

If $\mathcal{C}(\pi(v))\varepsilon(v) < 1$, this implies that $e'', w = 0$. Thus, the homomorphism (3.25) is injective everywhere over U if the support of β is sufficiently small. We conclude that $\widetilde{\psi}$ is an isomorphism everywhere over X. □

4 NC Symplectic Divisors: Local Perspective

Arbitrary normal crossings (or NC) divisors are spaces that are locally SC divisors. This local perspective, reviewed below, makes it straightforward to define NC divisors and regularizations for them. It is also readily applicable to local statements, such as Theorem 1.2 (4) and Lemma 1.3.

For a set S, denote by $\mathcal{P}(S)$ the collection of subsets of S. If in addition $i \in S$, let

$$\mathcal{P}_i(S) = \{I \in \mathcal{P}(S) : i \in S\}.$$

4.1 Definitions

We begin by extending the definitions of Section 3.1 to the general NC setting.

Definition 4.1 *Let (X, ω) be a symplectic manifold. A subspace $V \subset X$ is an NC symplectic divisor in (X, ω) if for every $x \in X$ there exist an open neighborhood U of x in X and a finite transverse collection $\{V_i\}_{i\in S}$ of closed submanifolds of U of codimension 2 such that*

$$V \cap U = \bigcup_{i\in S} V_i$$

is an SC symplectic divisor in $(U, \omega|_U)$.

Every NC divisor $V \subset X$ is a closed subspace; its singular locus $V_\partial \subset V$ consists of the points $x \in V$ such that there exists a chart $(U, \{V_i\}_{i\in S})$ as in Definition 4.1 and $I \subset S$ with $|I| = 2$ and $x \in V_I$. Figure 1 shows an NC divisor V, a chart around a singular point of V, and a chart around a smooth point of V.

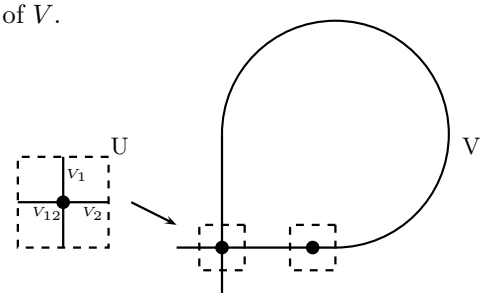


Figure 1 A non-SC normal crossings divisor

For each chart $(U, \{V_i\}_{i\in S})$ as in Definition 4.1 and each $x \in U$, let

$$S_x = \{i \in S : x \in V_i\}.$$

The cardinality $|x| \equiv |S_x|$ is independent of the choice of a chart around x. For each $r \in \mathbb{Z}^{\geq 0}$, let

$$V^{(r)} \equiv \{x \in X : |x| \geq r\}. \tag{4.1}$$

If $(U', \{V'_i\}_{i\in S'})$ is another chart for V in X and $x \in U \cap U'$, there exist a neighborhood U_x of x in $U \cap U'$ and a bijection

$$h_x : S_x \longrightarrow S'_x \quad \text{s.t.} \quad V_i \cap U_x = V'_{h_x(i)} \cap U_x \quad \forall i \in S_x. \tag{4.2}$$

We also denote by h_x the induced bijection $\mathcal{P}(S_x) \to \mathcal{P}(S'_x)$. By (4.2),

$$\mathcal{N}_{V_{I-i}} V_I|_{V_I \cap U_x} = \mathcal{N}_{V'_{h_x(I)-h_x(i)}} V'_{h_x(I)}|_{V'_{h_x(I)} \cap U_x} \quad \forall i \in I \subset S_x. \tag{4.3}$$

We denote by $\mathrm{Symp}^+(X,V)$ the space of all symplectic structures ω on X such that V is an NC symplectic divisor in (X,ω).

4.2 Regularizations

Suppose $V \subset X$ is an NC divisor, $(U, \{V_i\}_{i\in S})$ and $(U', \{V'_i\}_{i\in S'})$ are charts for V in (X,ω), and

$$(\mathcal{R}_I)_{I\subset S} \equiv ((\rho_{I;i}, \nabla^{(I;i)})_{i\in I}, \Psi_I)_{I\subset S} \quad \text{and} \quad (\mathcal{R}'_I)_{I\subset S'} \equiv ((\rho'_{I;i}, \nabla'^{(I;i)})_{i\in I}, \Psi'_I)_{I\subset S'}$$

are an $\omega|_U$-regularization for $\bigcup_{i\in S} V_i$ in U and an $\omega|_{U'}$-regularization for $\bigcup_{i\in S'} V'_i$ in U', respectively. We define

$$(\mathcal{R}_I)_{I\subset S} \cong_X (\mathcal{R}'_I)_{I\subset S'}$$

if for every $x \in U \cap U'$ there exist U_x and h_x as in (4.2) such that

$$\begin{aligned} &(\rho_{I;i}, \nabla^{(I;i)})|_{V_I\cap U_x} = (\rho'_{h(I);h(i)}, \nabla'^{(h(I);h(i))})|_{V_{h(I)}\cap U_x} \quad \forall i \in I \subset S_x \quad \text{and} \\ &\Psi_I = \Psi'_{I'} \ \text{ on } \mathrm{Dom}(\Psi_I)|_{V_I\cap U_x} \cap \mathrm{Dom}(\Psi'_{h(I)})|_{V_{h(I)}\cap U_x} \quad \forall I \subset S_x. \end{aligned} \tag{4.4}$$

Definition 4.2 *Let (X,ω) be a symplectic manifold, $V \subset X$ be an NC symplectic divisor, and $(U_y, \{V_{y;i}\}_{i\in S_y})_{y\in\mathcal{A}}$ be a collection of charts for V in X as in Definition* 4.1. *An ω-regularization for V in X (with respect to the atlas $\mathcal{A}$) is a collection*

$$\mathcal{R} \equiv (\mathcal{R}_{y;I})_{y\in\mathcal{A}, I\subset S_y} \equiv ((\rho_{y;I;i}, \nabla^{(y;I;i)})_{i\in I}, \Psi_{y;I})_{y\in\mathcal{A}, I\subset S_y}$$

such that $(\mathcal{R}_{y;I})_{I\subset S_y}$ is an $\omega|_{U_y}$-regularization for V_y in U_y as in Definition 3.5 (2) *for each $y \in \mathcal{A}$ and*

$$(\mathcal{R}_{y;I})_{I\subset S_y} \cong_X (\mathcal{R}_{y';I})_{I\subset S_{y'}} \quad \forall y, y' \in \mathcal{A}. \tag{4.5}$$

We call an almost complex structure J on X compatible with a regularization $\mathcal{R}$ as in Definition 4.2 if $J|_{U_y}$ is compatible with the regularization $(\mathcal{R}_{y;I})_{y\in\mathcal{A}, I\subset S_y}$ for V_y in U_y for each $y \in \mathcal{A}$ as defined at the end of Section 3.2. In particular, every open stratum $V^{(r)} - V^{(r+1)}$ is an almost complex submanifold of X with respect to an $\mathcal{R}$-compatible almost complex structure on X.

There are natural notions of equivalence classes of regularizations on the level of germs and families of such equivalence classes; see [14, Section 4.1]. We denote by $\mathrm{Aux}(X,V)$ the space of pairs $(\omega,\mathcal{R})$ consisting of $\omega \in \mathrm{Symp}^+(X,V)$ and the equivalence class of an ω-compatible regularization $\mathcal{R}$ for V in X. Let $\mathrm{AK}(X,V)$ be the space of triples $(\omega,\mathcal{R},J)$ consisting of $(\omega,\mathcal{R}) \in \mathrm{Aux}(X,V)$ and an almost complex structure J on X compatible with ω and $\mathcal{R}$. By [15, Theorem 4.5], the map (2.7) is again a weak homotopy equivalence in the present setting. It remains straightforward to show that the map (2.8) is also a weak homotopy equivalence in the present setting.

4.3 Constructions

In this section, we extend the constructions of Section 3.3 to arbitrary NC divisors in the local perspective. We then note that the proof of Theorem 1.2 (4) in Section 3.4 readily extends to the general NC case.

Suppose V is an NC symplectic divisor in a symplectic manifold (X,ω), $(U_y,\{V_{y;i}\}_{i\in S_y})_{y\in\mathcal{A}}$ is a collection of charts for V in X as in Definition 4.1, and $\mathcal{R}\equiv(\mathcal{R}_y)_{y\in\mathcal{A}}$ is an ω-regularization with respect to the atlas $\mathcal{A}$ in the sense of Definition 4.2. For $y\in\mathcal{A}$ and $I'\subset I\subset S_y$, let $V_y\equiv V\cap U_y$, $U^\circ_{y;I}\subset U_y$ be as in (3.12), $\theta_{y';I'I}=\theta_{I'I}$ be as in (3.14), and $\vartheta_{y';I'I}=\vartheta_{I'I}$ be as in (3.15). Let

$$\pi_y:\mathcal{O}_{\mathcal{R}_y;U_y}(V_y)\longrightarrow U_y \quad\text{and}\quad \pi_y:T_{\mathcal{R}_y}U_y(-\log V_y)\longrightarrow U_y$$

be the complex line bundle (3.17) and the logarithmic tangent bundle (3.18) determined by the $\omega|_{U_y}$-regularization $\mathcal{R}_y$ for the NC divisor V_y for $(U_y,\omega|_{U_y})$. Let $s_{\mathcal{R};y}$ and

$$\iota_{\mathcal{R};y}:T_{\mathcal{R}_y}U_y(-\log V_y)\longrightarrow TU_y$$

be the associated section of $\mathcal{O}_{\mathcal{R}_y;U_y}(V_y)$ and the vector bundle homomorphism (1.3), respectively.

Suppose $y,y'\in\mathcal{A}$. For $x\in U_y\cap U_{y'}$, let $U_{yy';x}\equiv U_x\subset U_y\cap U_{y'}$ and

$$h_{y'y;x}\equiv h_x:S_{y;x}\equiv S_x\longrightarrow S_{y';x}\equiv S_x$$

be as in (4.2). By (4.3),

$$\mathcal{N}_{V_{y;S_{y;x}-i}}V_{y;S_{y;x}}|_{V_{y;S_{y;x}}\cap U_{yy';x}}=\mathcal{N}_{V_{y';S_{y';x}}-h_{y'y;x}(i)}V_{y';S_{y';x}}|_{V_{y';S_{y';x}}\cap U_{yy';x}}\quad\forall i\in S_{y;x}. \tag{4.6}$$

We can choose $U_{yy';x}$ sufficiently small so that

$$U_{yy';x}\subset U^\circ_{y;S_{y;x}}\cap U^\circ_{y';S_{y';x}} \tag{4.7}$$

and $U_x\equiv U_{yy';x}$ satisfies (4.4). By (3.13), (4.7), and (4.6), there are canonical identifications

$$\begin{aligned}
\theta_{y'y;x}&:\mathcal{O}_{\mathcal{R}_y;S_{y;x}}(V_y)|_{U_{yy';x}}\xrightarrow{=}\mathcal{O}_{\mathcal{R}_y|_{U_{yy';x}};S_{y;x}}(V_y\cap U_{yy';x})\xrightarrow{\approx}\mathcal{O}_{\mathcal{R}_{y'};S_{y';x}}(V_{y'})|_{U_{yy';x}},\\
\vartheta_{y'y;x}&:T_{\mathcal{R}_y}U^\circ_{S_{y;x}}(-\log V_y)|_{U_{yy';x}}\xrightarrow{=}T_{\mathcal{R}_y|_{U_{yy';x}}}U_{yy';x}(-\log(V_y\cap U_{yy';x}))\\
&\qquad\xrightarrow{\approx}T_{\mathcal{R}_{y'}}U^\circ_{S_{y';x}}(-\log V_{y'})|_{U_{yy';x}}.
\end{aligned}\tag{4.8}$$

Suppose $x'\in U_{yy';x}$. By (4.7) and the uniqueness of $h_{y'y;x'}$,

$$S_{y;x'}\subset S_{y;x},\quad S_{y';x'}\subset S_{y';x},\quad\text{and}\quad h_{y'y;x'}=h_{y'y;x}|_{S_{y;x'}}.$$

Combining these statements with (3.14), (3.15), and (4.4), we obtain

$$\begin{aligned}
&\theta_{y';S_{y';x'}S_{y';x}}\circ\theta_{y'y;x}=\theta_{y'y;x'}\circ\theta_{y;S_{y;x'}S_{y;x}}:\\
&\quad\mathcal{O}_{\mathcal{R}_y;S_{y;x}}(V_y)|_{U_{yy';x}\cap U_{yy';x'}}\longrightarrow\mathcal{O}_{\mathcal{R}_{y'};S_{y';x'}}(V_{y'})|_{U_{yy';x}\cap U_{yy';x'}},\\
&\vartheta_{y';S_{y';x'}S_{y';x}}\circ\vartheta_{y'y;x}=\vartheta_{y'y;x'}\circ\vartheta_{y;S_{y;x'}S_{y;x}}:\\
&\quad T_{\mathcal{R}_y}U^\circ_{S_{y;x}}(-\log V_y)|_{U_{yy';x}\cap U_{yy';x'}}\longrightarrow T_{\mathcal{R}_{y'}}U^\circ_{S_{y';x'}}(-\log V_{y'})|_{U_{yy';x}\cap U_{yy';x'}}.
\end{aligned}$$

Along with (3.17) and (3.18), this implies that the collections $\{\theta_{y'y;x}\}_{x\in U_y\cap U_{y'}}$ and $\{\vartheta_{y'y;x}\}_{x\in U_y\cap U_{y'}}$ determine bundle isomorphisms

$$\begin{aligned}
\theta_{y'y}&:\mathcal{O}_{\mathcal{R}_y;U_y}(V_y)|_{U_y\cap U_{y'}}\longrightarrow\mathcal{O}_{\mathcal{R}_{y'};U_{y'}}(V_{y'})|_{U_y\cap U_{y'}},\\
\vartheta_{y'y}&:T_{\mathcal{R}_y}U_y(-\log V_y)|_{U_y\cap U_{y'}}\longrightarrow T_{\mathcal{R}_{y'}}U_{y'}(-\log V_{y'})|_{U_y\cap U_{y'}},
\end{aligned}\tag{4.9}$$

respectively.

Suppose $y'' \in \mathcal{A}$ is another element and $x \in U_y \cap U_{y'} \cap U_{y''}$. Let

$$U_{yy'y'';x} = U_{yy';x} \cap U_{yy';x} \cap U_{y'y'';x}.$$

By the uniqueness of $h_{y''y;x}$,

$$h_{y''y;x} = h_{y''y';x} \circ h_{y'y;x} : S_{y;x} \longrightarrow S_{y'';x}.$$

This implies that the identifications (4.8) satisfy

$$\theta_{y''y;x} = \theta_{y''y';x} \circ \theta_{y'y;x} : \mathcal{O}_{\mathcal{R}_y;S_{y;x}}(V_y)|_{U_{yy'y'';x}} \longrightarrow \mathcal{O}_{\mathcal{R}_{y''};S_{y'';x}}(V_{y''})|_{U_{yy'y'';x}},$$
$$\vartheta_{y''y;x} = \vartheta_{y''y';x} \circ \vartheta_{y'y;x} : T_{\mathcal{R}_y}U^\circ_{S_{y;x}}(-\log V_y)|_{U_{yy'y'';x}} \longrightarrow T_{\mathcal{R}_y}U^\circ_{S_{y'';x}}(-\log V_{y''})|_{U_{yy'y'';x}}.$$

Thus, the collections $\{\theta_{y'y}\}_{y,y'\in\mathcal{A}}$ and $\{\vartheta_{y'y}\}_{y,y'\in\mathcal{A}}$ satisfy the cocycle condition. The first collection thus determines a complex line bundle

$$\begin{aligned} &\pi : \mathcal{O}_{\mathcal{R};X}(V) = \Big(\bigsqcup_{y\in\mathcal{A}} \mathcal{O}_{\mathcal{R}_y;U_y}(V_y)\Big)\Big/\sim \longrightarrow X, \quad \pi|_{\mathcal{O}_{\mathcal{R}_y;U_y}(V_y)} = \pi_y, \\ &\mathcal{O}_{\mathcal{R}_y;U_y}(V_y)|_{U_y\cap U_{y'}} \ni v \sim \theta_{y'y}(v) \in \mathcal{O}_{\mathcal{R}_{y'};U_{y'}}(V_{y'})|_{U_y\cap U_{y'}} \quad \forall y, y' \in \mathcal{A}. \end{aligned} \tag{4.10}$$

The second collection similarly determines a vector bundle

$$\begin{aligned} &\pi : T_{\mathcal{R}}X(-\log V) = \Big(\bigsqcup_{y\in\mathcal{A}} T_{\mathcal{R}_y}U_y(-\log V_y)\Big)\Big/\sim \longrightarrow X, \quad \pi|_{T_{\mathcal{R}_y}U_y(-\log V_y)} = \pi_y, \\ &T_{\mathcal{R}_y}U_y(-\log V_y)|_{U_y\cap U_{y'}} \ni v \sim \vartheta_{y'y}(v) \in T_{\mathcal{R}_{y'}}U_{y'}(-\log V_{y'})|_{U_y\cap U_{y'}} \quad \forall y, y' \in \mathcal{A}. \end{aligned} \tag{4.11}$$

By (3.19), (3.20), and (4.4),

$$\theta_{y'y} \circ s_{\mathcal{R};y} = s_{\mathcal{R};y'} : U_y \cap U_{y'} \longrightarrow \mathcal{O}_{\mathcal{R}_{y'};U_{y'}}(V_{y'})|_{U_y\cap U_{y'}},$$
$$\iota_{\mathcal{R};y} = \iota_{\mathcal{R};y'} \circ \vartheta_{y'y} : T_{\mathcal{R}_y}U_y(-\log V_y)|_{U_y\cap U_{y'}} \longrightarrow TX|_{U_y\cap U_{y'}}.$$

The collections $\{s_{\mathcal{R};y}\}_{y\in\mathcal{A}}$ and $\{\iota_{\mathcal{R};y}\}_{y\in\mathcal{A}}$ thus determine a section $s_{\mathcal{R}}$ of the line bundle (4.10) and a bundle homomorphism $\iota_{\mathcal{R}}$ as in (1.3). Since the sections $s_{\mathcal{R};y}$ and the homomorphisms $\iota_{\mathcal{R};y}$ satisfy the properties of $s_{\mathcal{R}}$ and $\iota_{\mathcal{R}}$ stated in Proposition 1.1 (1) and Theorem 1.2 (1), so do the just constructed sections $s_{\mathcal{R}}$ and $\iota_{\mathcal{R}}$.

Suppose J is an $\mathcal{R}$-compatible almost complex structure on X. For each $y \in \mathcal{A}$, $J_y \equiv J|_{U_y}$ is then an $\mathcal{R}_y$-compatible almost complex structure on U_y and determines a complex structure on the vector bundle

$$T_{\mathcal{R}_y}U^\circ_I(-\log V_y) \longrightarrow U^\circ_I$$

for every $I \subset S_y$. Since $J_y = J_{y'}$ on $U_{yy';x}$, (4.4) implies that the bundle identification $\vartheta_{y'y;x}$ in (4.8) is $\mathbb{C}$-linear. Thus, J determines a complex structure $\mathfrak{i}_{\mathcal{R};J}$ on the vector bundle (4.11) which restricts to the complex structure $\mathfrak{i}_{\mathcal{R}_y;J_y}$ on the vector bundle

$$T_{\mathcal{R}_y}U_y(-\log V_y) \longrightarrow U_y$$

for every $y \in \mathcal{A}$. Since the bundle homomorphism $\iota_{\mathcal{R}_y}$ is $\mathbb{C}$-linear with respect to the complex structure $\mathfrak{i}_{\mathcal{R}_y;J_y}$ on its domain and the complex structure J_y on its target, we obtain Theorem 1.2 (2).

By the same reasoning as at the end of Section 2.2, the bundles (4.10) and (4.11) also satisfy the properties in Proposition 1.1 (2) and Theorem 1.2 (3).

Suppose V' is another NC symplectic divisor in (X,ω) so that $V\cup V'\subset X$ is also an NC symplectic divisor and $V\cap V'$ contains no open subspace of V. An atlas for $V\cup V'$ in X is a collection of the form $(U_y,\{V_{y;i}\}_{i\in S_y\sqcup S'_y})_{y\in\mathcal{A}}$ so that $(U_y,\{V_{y;i}\}_{i\in S_y})_{y\in\mathcal{A}}$ and $(U_y,\{V_{y;i}\}_{i\in S'_y})_{y\in\mathcal{A}}$ are atlases for V and V', respectively. An ω-regularization $\widetilde{\mathcal{R}}\equiv(\widetilde{\mathcal{R}}_y)_{y\in\mathcal{A}}$ for such an atlas for $V\cup V'$ restricts to ω-regularizations

$$\mathcal{R}\equiv(\mathcal{R}_y)_{y\in\mathcal{A}}\quad\text{and}\quad\mathcal{R}'\equiv(\mathcal{R}'_y)_{y\in\mathcal{A}}$$

for the associated atlases for V and V' so that the $\omega|_{U_y}$-regularization $\widetilde{\mathcal{R}}_y$ for $(V\cup V')\cap U_y$ in U_y restricts to the $\omega|_{U_y}$-regularizations $\mathcal{R}_y$ for $V\cap U_y$ in U_y and $\mathcal{R}'_y$ for $V'\cap U_y$. We denote by $\theta'_{y'y}$ and $\vartheta'_{y'y}$ (resp., $\widetilde{\theta}_{y'y}$ and $\widetilde{\vartheta}_{y'y}$) the analogues of the transition maps $\theta_{y'y}$ and $\vartheta_{y'y}$ in (4.9) for the regularization $\mathcal{R}'$ (resp., $\widetilde{\mathcal{R}}$). For each $y\in\mathcal{A}$, let

$$\begin{aligned}\psi_{\mathcal{R}_y\mathcal{R}'_y}&:(\mathcal{O}_{\widetilde{\mathcal{R}}_y;U_y}(V_y\cup V'_y),\mathfrak{i}_{\widetilde{\mathcal{R}}_y})\longrightarrow(\mathcal{O}_{\mathcal{R}_y;U_y}(V_y),\mathfrak{i}_{\mathcal{R}_y})\otimes(\mathcal{O}_{\mathcal{R}'_y;U_y}(V'_y),\mathfrak{i}_{\mathcal{R}'_y}),\\ \iota_{\mathcal{R}_y\widetilde{\mathcal{R}}_y}&:T_{\widetilde{\mathcal{R}}_y}U_y(-\log(V_y\cup V'_y))\longrightarrow T_{\mathcal{R}_y}X(-\log V_y)\end{aligned}$$

be the bundle isomorphism (3.21) and the bundle homomorphism (3.24) determined by the regularization $\widetilde{\mathcal{R}}_y$.

Since the maps (3.21) and (3.24) are natural with respect to the restrictions to open subsets,

$$\begin{aligned}&\{\theta_{y'y}\otimes\theta'_{y'y}\}\circ\psi_{\mathcal{R}_y\mathcal{R}'_y}=\psi_{\mathcal{R}_{y'}\mathcal{R}'_{y'}}\circ\widetilde{\theta}_{y'y}:\\ &\quad\mathcal{O}_{\widetilde{\mathcal{R}}_y;U_y}(V_y\cup V'_y)|_{U_y\cap U_{y'}}\longrightarrow\mathcal{O}_{\mathcal{R}_{y'};U_{y'}}(V_{y'})|_{U_y\cap U_{y'}}\otimes\mathcal{O}_{\mathcal{R}'_{y'};U_{y'}}(V'_{y'})|_{U_y\cap U_{y'}},\\ &\vartheta_{y'y}\circ\iota_{\mathcal{R}_y\widetilde{\mathcal{R}}_y}=\iota_{\mathcal{R}_{y'}\widetilde{\mathcal{R}}_{y'}}\circ\widetilde{\vartheta}_{y'y}:T_{\widetilde{\mathcal{R}}_y}U_y(-\log(V_y\cup V'_y))|_{U_y\cap U_{y'}}\longrightarrow T_{\mathcal{R}_{y'}}U_{y'}(-\log V_{y'})|_{U_y\cap U_{y'}}.\end{aligned}$$

The collection $\{\psi_{\mathcal{R}_y\mathcal{R}'_y}\}_{y\in\mathcal{A}}$ thus determines an isomorphism

$$\psi_{\mathcal{R}\mathcal{R}'}:(\mathcal{O}_{\widetilde{\mathcal{R}};X}(V\cup V'),\mathfrak{i}_{\widetilde{\mathcal{R}}})\longrightarrow(\mathcal{O}_{\mathcal{R};X}(V),\mathfrak{i}_{\mathcal{R}})\otimes(\mathcal{O}_{\mathcal{R}';X}(V'),\mathfrak{i}_{\mathcal{R}'})$$

of complex line bundles over X. In light of Proposition 1.1 (2), this establishes Proposition 1.1 (3).

The collection $\{\iota_{\mathcal{R}_y\widetilde{\mathcal{R}}_y}\}_{y\in\mathcal{A}}$ similarly determines a homomorphism

$$\iota_{\mathcal{R}\widetilde{\mathcal{R}}}:T_{\widetilde{\mathcal{R}}}X(-\log(V\cup V'))\longrightarrow T_{\mathcal{R}}X(-\log V)$$

of vector bundles over X. By the properties of the homomorphisms $\iota_{\mathcal{R}\widetilde{\mathcal{R}}}\equiv\iota_{\mathcal{R}_y\widetilde{\mathcal{R}}_y}$ stated in Lemma 3.8, this homomorphism satisfies the same properties with $K\subset S'$, V'°_K, and $V'^{\circ}_K\times\mathbb{C}^K$ replaced by $r\in\mathbb{Z}^{\ge0}$, $V'^{(r)}-V'^{(r+1)}$, and some rank r complex vector bundle over $V'^{(r)}-V'^{(r+1)}$, respectively. By the same reasoning as in the paragraph after Lemma 3.8, $\iota_{\mathcal{R}\widetilde{\mathcal{R}}}$ is $\mathbb{C}$-linear with respect to the complex structures $\mathfrak{i}_{\widetilde{\mathcal{R}},J}$ on its domain and $\mathfrak{i}_{\mathcal{R},J}$ on its target determined by an $\widetilde{\mathcal{R}}$-compatible almost complex structure J on X. Furthermore, the $\mathbb{C}$-linear homomorphism $\psi=\iota_{\mathcal{R}\widetilde{\mathcal{R}}}$ satisfies the conditions of Lemma 3.7 if V' is smooth (in which case $V'^{(1)}-V'^{(2)}=V'$). In light of Theorem 1.2 (3), this establishes Theorem 1.2 (4).

5 Almost Complex and Symplectic Blowups

Let X be a smooth manifold and $V\subset X$ be an NC smooth divisor, i.e., a subspace admitting a collection $(U_y,\{V_{y;i}\}_{i\in S_y})_{y\in\mathcal{A}}$ of charts as in Definition 4.1 with each $V_{y;i}\subset V\cap U_y$ being a smooth submanifold of U_y of real codimension 2. We fix such a collection. Let $r\in\mathbb{Z}^+$ be such

that $V^{(r+1)}=\emptyset$. We also fix a Hermitian regularization

$$\mathcal{R}\equiv(\mathcal{R}_y)_{y\in\mathcal{A}}\equiv((\rho_{y;I;i},\nabla^{(y;I;i)})_{i\in I},\Psi_{y;I})_{y\in\mathcal{A},I\subset S_y} \tag{5.1}$$

for V in X, i.e., a tuple satisfying the conditions of Definitions 3.5 (2) and 4.2 that do not involve the symplectic form ω.

By shrinking the open sets U_y, we may assume that $|S_y|\le r$ for all $y\in\mathcal{A}$, $\mathrm{Im}(\Psi_{y;S_y})=U_y$ if $y\in\mathcal{A}$ with $|S_y|=r$,

$$\mathrm{Im}(\Psi_{y;S_y})\cap\mathrm{Im}(\Psi_{y';S_{y'}})\subset\Psi_{y;S_y}(\mathrm{Dom}(\Psi_{y;S_y})|_{V_{y;S_y}\cap V_{y';S_{y'}}})\cup\Psi_{y';S_{y'}}(\mathrm{Dom}(\Psi_{y';S_{y'}})|_{V_{y;S_y}\cap V_{y';S_{y'}}}),$$

and there exists an open neighborhood U_r' of $V^{(r)}\subset X$ so that $U_r'\cap U_{y'}=\emptyset$ for all $y'\in\mathcal{A}$ with $|S_{y'}|<r$. Let

$$\mathcal{A}_r=\{y\in\mathcal{A}:|S_y|=r\},\quad \widetilde{\mathcal{A}}=(\mathcal{A}-\mathcal{A}_r)\sqcup\{(y,i):y\in\mathcal{A}_r,\ i\in S_y\}.$$

5.1 Smooth Complex Blowup

Since $V^{(r+1)}=\emptyset$, $V^{(r)}\subset X$ is a smooth submanifold. Let

$$\pi_r:\mathcal{N}_XV^{(r)}\longrightarrow V^{(r)} \tag{5.2}$$

be its normal bundle. By (4.5), the Hermitian metrics $\rho_{y;S_y;i}$ and the connections $\nabla^{(y;S_y;i)}$ in the complex line bundles $\mathcal{N}_{V_{y;S_y-i}}V_{y;S_y}$ with $y\in\mathcal{A}_r$ and $i\in S_y$ determine a complex structure, a Hermitian metric ρ_r, and a compatible connection $\nabla^{(r)}$ on $\mathcal{N}_XV^{(r)}$. Furthermore, the map

$$\Psi_r:\mathcal{N}_r'\equiv\bigcup_{y\in\mathcal{A}_r}\mathrm{Dom}(\Psi_{y;S_y})\longrightarrow X,\quad \Psi_r(v)=\Psi_{y;S_y}(v)\quad\forall v\in\mathrm{Dom}(\Psi_{y;S_y}),\ y\in\mathcal{A}_r, \tag{5.3}$$

is a well-defined regularization for $V^{(r)}$ in X in the sense of Definition 2.1. We note that the complex vector bundle $\mathcal{N}_XV^{(r)}$ does not necessarily split as a sum of complex line bundles.

We denote by $\mathbb{E}\equiv\mathbb{P}(\mathcal{N}_XV^{(r)})$ the complex projectivization of $\mathcal{N}_XV^{(r)}$ and by

$$\widetilde{\pi}_0:\gamma\equiv\{(\ell,v)\in\mathbb{E}\times\mathcal{N}_XV^{(r)}:v\in\ell\}\longrightarrow\mathbb{E} \tag{5.4}$$

the complex tautological line bundle. Let

$$\pi:\mathbb{E}\longrightarrow V^{(r)},\quad \pi(\ell)=\pi_r(v)\quad\text{if }(\ell,v)\in\gamma, \tag{5.5}$$

be the bundle projection. The connection $\nabla^{(r)}$ and the Hermitian metric ρ_r on $\mathcal{N}_XV^{(r)}$ determine a splitting

$$\begin{aligned}T(\mathcal{N}_XV^{(r)})|_{\mathcal{N}_XV^{(r)}-V^{(r)}}\approx{}&\pi_r^*TV^{(r)}|_{\mathcal{N}_XV^{(r)}-V^{(r)}}\oplus\{(v,w)\in\pi_r^*\mathcal{N}_XV^{(r)}|_{\mathcal{N}_XV^{(r)}-V^{(r)}}:w\in\mathbb{C}v\}\\&\oplus\{(v,w)\in\pi_r^*\mathcal{N}_XV^{(r)}|_{\mathcal{N}_XV^{(r)}-V^{(r)}}:\rho_r(v,w)=0\}\end{aligned} \tag{5.6}$$

of the vector bundle $T(\mathcal{N}_XV^{(r)})|_{\mathcal{N}_XV^{(r)}-V^{(r)}}$ so that the middle summand above is identified with the tangent bundle to the orbits of the $\mathbb{C}^*$-action on $\mathcal{N}_XV^{(r)}$ and the last summand is its complement in the vertical tangent bundle of π_r restricted to $\mathcal{N}_XV^{(r)}-V^{(r)}$. By [38, Lemma 1.1], the above splitting is $\mathbb{C}^*$-invariant. It thus induces a splitting

$$T\mathbb{E}\approx\pi^*TV^{(r)}\oplus(\ker d\pi) \tag{5.7}$$

so that the last summand above corresponds to the vertical tangent subbundle of π and a complex structure on the last summand.

Since $\gamma \subset \pi^*\mathcal{N}_X V^{(r)}$, the Hermitian metric ρ_r and the compatible connection $\nabla^{(r)}$ on $\mathcal{N}_X V^{(r)}$ determine a Hermitian structure $(\widetilde{\rho}_0, \widetilde{\nabla}^{(0)})$ on the complex line bundle γ and a splitting

$$T\gamma \approx \widetilde{\pi}_0^* T\mathbb{E} \oplus \widetilde{\pi}_0^*\gamma \tag{5.8}$$

so that the last summand above corresponds to the vertical tangent bundle of $\widetilde{\pi}_0$. The composition of this splitting with the splitting (5.7) restricts to the splitting (5.6) under the identification

$$(\gamma - \mathbb{E}, \pi \circ \widetilde{\pi}_0|_{\gamma-\mathbb{E}}) = (\mathcal{N}_X V^{(r)} - V^{(r)}, \pi_r|_{\mathcal{N}_X V^{(r)} - V^{(r)}}), \quad (\ell, v) \longrightarrow v. \tag{5.9}$$

We define the smooth complex blowup $\pi : \widetilde{X} \to X$ of X along $V^{(r)}$ with respect to Ψ_r by

$$\widetilde{\mathcal{N}}_0' = \{(\ell, v) \in \gamma : v \in \mathcal{N}_r'\}, \quad \widetilde{X} \equiv ((X - V^{(r)}) \sqcup \widetilde{\mathcal{N}}_0')/\sim, \quad \widetilde{\mathcal{N}}_0' - \mathbb{E} \ni (\ell, v) \sim \Psi_r(v) \in X - V^{(r)},$$

$$\pi([\widetilde{x}]) = \begin{cases} \widetilde{x}, & \text{if } \widetilde{x} \in X - V^{(r)}; \\ \Psi_r(v), & \text{if } \widetilde{x} \equiv (\ell, v) \in \widetilde{\mathcal{N}}_0'. \end{cases}$$

The exceptional divisor $\mathbb{E}$ is a codimension 2 submanifold of $\widetilde{X}$ with a smooth regularization

$$\widetilde{\Psi}_0 : \widetilde{\mathcal{N}}_0' \longrightarrow \widetilde{X}, \quad \widetilde{\Psi}_0(v) = [v].$$

5.2 Almost Complex Blowup

Suppose now that J is an almost complex structure on X, $V \subset X$ is an NC almost complex divisor, each $V_{y;i} \subset V \cap U_y$ is an almost complex submanifold of $(U_y, J|_{U_y})$ of real codimension 2, and the almost complex structure J on X is $\mathcal{R}$-compatible in the sense defined at the end of Section 4.2. The smooth submanifold $V^{(r)} \subset X$ is then almost complex. The induced complex structure on its normal bundle agrees with the fiberwise complex structure determined by the complex line bundles $\mathcal{N}_{V_{y;S_y - i}} V_{y;S_y}$ with $y \in \mathcal{A}_r$ and $i \in S_y$.

Along with the fiberwise complex structure and the connection $\nabla^{(r)}$ on $\mathcal{N}_X V^{(r)}$, $J|_{V^{(r)}}$ determines a complex structure $J_{\mathcal{R};r}$ on the total space of $\mathcal{N}_X V^{(r)}$ such that

$$J \circ d\Psi_r = d\Psi_r \circ J_{\mathcal{R};r}|_{\mathcal{N}_r'}. \tag{5.10}$$

Along with the splitting (5.7), $J|_{V^{(r)}}$ determines an almost complex structure $\widetilde{J}_{\mathbb{E}}$ on $\mathbb{E}$ so that the bundle projection (5.5) is $(J, \widetilde{J}_{\mathbb{E}})$-holomorphic. Along with the splitting (5.8), $\widetilde{J}_{\mathbb{E}}$ in turn determines an almost complex structure $\widetilde{J}_{\mathcal{R};0}$ on the total space of γ. By the sentence containing (5.9), the restrictions of the almost complex structures $\widetilde{J}_{\mathcal{R};0}$ to $\gamma - \mathbb{E}$ and $J_{\mathcal{R};r}$ to $\mathcal{N}_X V^{(r)} - V^{(r)}$ agree under the identification (5.9). Furthermore, the projection

$$\pi_2 : \gamma \longrightarrow \mathcal{N}_X V^{(r)}$$

to the second component in (5.4) is $(J_{\mathcal{R};r}, \widetilde{J}_{\mathcal{R};0})$-holomorphic.

We define an almost complex structure $\widetilde{J}$ on the blowup $\widetilde{X}$ of X constructed in Section 5.1 by

$$\widetilde{J}_{[\widetilde{x}]} = \begin{cases} J_{\widetilde{x}}, & \text{if } \widetilde{x} \in X - V^{(r)}; \\ \widetilde{J}_{\mathcal{R};0}|_{\widetilde{x}}, & \text{if } \widetilde{x} \in \widetilde{\mathcal{N}}_0'. \end{cases}$$

By the conclusion of the previous paragraph and (5.10), the definitions of $\widetilde{J}$ agree on $\widetilde{\mathcal{N}}_0' - \mathbb{E}$. The exceptional divisor $\mathbb{E}$ is an almost complex submanifold of $(\widetilde{X}, \widetilde{J})$. The almost complex structure $\widetilde{J}$ is compatible with the Hermitian regularization $(\widetilde{\rho}_0, \widetilde{\nabla}^{(0)}, \widetilde{\Psi}_0)$ for $\mathbb{E}$ in $\widetilde{X}$.

Let $\overline{V} \subset \widetilde{X}$ be the proper transform of V, i.e., the closure of $V - V^{(r)}$, and

$$\widetilde{V} = \mathbb{E} \cup \overline{V}.$$

We show below that $\widetilde{V}$ is an NC almost complex divisor in $(\widetilde{X}, \widetilde{J})$ with a collection

$$(\widetilde{U}_y, \{\widetilde{V}_{y;i}\}_{i\in\widetilde{S}_y})_{y\in\widetilde{\mathcal{A}}}$$

of charts and a regularization

$$\widetilde{\mathcal{R}} \equiv (\widetilde{\mathcal{R}}_{y;I})_{y\in\widetilde{\mathcal{A}},I\subset\widetilde{S}_y} \equiv ((\widetilde{\rho}_{y;I;i}, \widetilde{\nabla}^{(y;I;i)})_{i\in I}, \widetilde{\Psi}_{y;I})_{y\in\widetilde{\mathcal{A}},I\subset\widetilde{S}_y} \tag{5.11}$$

obtained from the atlas $(U_y, \{V_{y;i}\}_{i\in S_y})_{y\in\mathcal{A}}$ and the regularization $\mathcal{R}$ for V in X. If $y \in \mathcal{A}-\mathcal{A}_r$, then

$$U_y \subset X - V^{(r)} = \widetilde{X} - \mathbb{E}.$$

In this case, we simply take

$$\widetilde{S}_y = S_y, \quad (\widetilde{U}_y, \{\widetilde{V}_{y;i}\}_{i\in\widetilde{S}_y}) = (U_y, \{V_{y;i}\}_{i\in S_y}), \quad (\widetilde{\mathcal{R}}_{y;I})_{I\subset\widetilde{S}_y} = (\mathcal{R}_{y;I})_{I\subset S_y}.$$

Suppose $y \in \mathcal{A}_r$. Let $\widetilde{U}_y \equiv \pi^{-1}(U_y)$ be the blowup of U_y along $V^{(r)} \cap U_y$. Since $\mathrm{Im}(\Psi_{y;S_y}) = U_y$, we can identify U_y with $\mathrm{Dom}(\Psi_{y;S_y})$ via Ψ_r and $\widetilde{U}_y$ with $\pi^{-1}(U_y) \subset \widetilde{\mathcal{N}}'_0$ via $\widetilde{\Psi}_0$. For each $i \in S_y$, let $\overline{V}_{y;i} \subset \widetilde{U}_y$ be the proper transform of $V_{y;i}$ and define

$$\widetilde{S}_{(y,i)} = \{0\}\sqcup(S_y-\{i\}),\ \mathcal{N}_{y;i} = \bigoplus_{j\in S_y-i} \mathcal{N}_{V_{y;S_y-i}} V_{y;S_y},\ \widetilde{U}_{(y,i)} = \widetilde{U}_y - \gamma|_{\mathbb{P}\mathcal{N}_{y;i}},\ \widetilde{V}_{(y,i);0} = \mathbb{E} - \mathbb{P}\mathcal{N}_{y;i}.$$

For $j \in S_y - i$, let $\widetilde{V}_{(y,i);j} = \overline{V}_{y;j} \cap \widetilde{U}_{(y,i)}$. We note that

$$V_{y;i} = \mathcal{N}_{y;i} \cap \mathrm{Dom}(\Psi_{y;S_y}), \quad \overline{V}_{y;i} = \gamma|_{\mathbb{P}\mathcal{N}_{y;i}} \cap \widetilde{U}_y, \quad \text{and} \quad \overline{V} \cap \widetilde{U}_{(y,i)} = \bigcup_{j\in S_y-i} \widetilde{V}_{(y,i);j} \tag{5.12}$$

under the above identifications. Since $\{\widetilde{V}_{(y,i);j}\}_{j\in\widetilde{S}_{(y,i)}}$ is a transverse collection of codimension 2 almost complex submanifolds of $\widetilde{U}_{(y,i)}$, $\widetilde{V}\cap\widetilde{U}_{(y,i)}$ is an SC almost complex divisor in $\widetilde{U}_{(y,i)}$. Thus, $\widetilde{V}$ is an NC almost complex divisor in $(\widetilde{X}, \widetilde{J})$ and $(\widetilde{U}_y, \{\widetilde{V}_{y;i}\}_{i\in\widetilde{S}_y})_{y\in\widetilde{\mathcal{A}}}$ is an atlas of local charts for $\widetilde{V}$.

5.3 Regularizations

For $y \in \mathcal{A}_r$ and $I' \subset I \subset S_y$, let

$$\mathcal{N}'_{y;I;I'} \subset \mathcal{N}_{y;I;I'} \subset \mathcal{N}_{U_y} V_{y;I} = \bigoplus_{i\in I} \mathcal{N}_{V_{y;I-i}} V_{y;I}$$

be the analogues of the subspaces $\mathcal{N}'_{I;I'} \subset \mathcal{N}_{I;I'} \subset \mathcal{N}_X V_I$ for the regularization $(\Psi_{y;I})_{I\subset S_y}$ of $\{V_{y;i}\}_{i\in S_y}$ in U_y as defined in Section 3.2. Suppose $i \in S_y$ and $I \subset \widetilde{S}_{(y,i)}$. If $0 \in I$ (and so $\widetilde{V}_{(y,i);I} \subset \mathbb{E}$), then

$$\begin{aligned}\mathcal{N}_{\widetilde{U}_{(y,i)}} \widetilde{V}_{(y,i);I} &= \gamma|_{\widetilde{V}_{(y,i);I}} \oplus (\gamma^*|_{\widetilde{V}_{(y,i);I}} \otimes_{\mathbb{C}} \pi^* \mathcal{N}_{U_y} V_{y;I-0}|_{\widetilde{V}_{(y,i);I}}) \\ &\subset \{\pi|_{\widetilde{V}_{(y,i);I}}\}^* \mathcal{N}_{y;S_y;I-0} \oplus (\gamma^*|_{\widetilde{V}_{(y,i);I}} \otimes_{\mathbb{C}} \{\pi|_{\widetilde{V}_{(y,i);I}}\}^* \mathcal{N}_{y;S_y;S_y-I}).\end{aligned}$$

In this case, we define

$$\widetilde{\Psi}_{(y,i);I} : \{(\mathbb{C}v', v, u\otimes w) \in \mathcal{N}_{\widetilde{U}_{(y,i)}} \widetilde{V}_{(y,i);I} : v + u(v)w \in \mathcal{N}'_{y;S_y}\} \longrightarrow \widetilde{U}_{(y,i)} \subset \{\pi|_{\mathbb{E}}\}^* \mathcal{N}_{U_y} V_{y;S_y},$$

$$\widetilde{\Psi}_{(y,i);I}(\mathbb{C}v',v,u\otimes w)=(\mathbb{C}(v'+u(v')w),v+u(v)w).$$

Suppose $0\notin I$. Thus, $\widetilde{V}_{(y,i);0\sqcup I}\equiv\widetilde{V}_{(y,i);I}\cap\mathbb{E}$ is a smooth submanifold of $\widetilde{V}_{(y,i);I}$ with normal bundle $\gamma|_{\widetilde{V}_{(y,i);0\sqcup I}}$. The smooth regularization

$$\widetilde{\Psi}_{(y,i);0\sqcup I;I}:\widetilde{\mathcal{N}}'_{(y,i);0\sqcup I;I}\equiv\{(\ell,v)\in\gamma:v\in\mathcal{N}'_{y;S_y;I}\}\longrightarrow\widetilde{V}_{(y,i);I},\quad\widetilde{\Psi}_{(y,i);0\sqcup I;I}(\ell,v)=(\ell,v),$$

of $\widetilde{V}_{(y,i);0\sqcup I}$ in $\widetilde{V}_{(y,i);I}$ is surjective. Let

$$\mathcal{O}_{\widetilde{\mathcal{R}};(y,i);I}(\mathbb{E})\equiv\{\widetilde{\Psi}^{-1}_{(y,i);0\sqcup I;I}\}^*\{\widetilde{\pi}_0|_{\widetilde{\mathcal{N}}'_{(y,i);0\sqcup I;I}}\}^*\gamma\longrightarrow\widetilde{V}_{(y,i);I}$$

be the analogue of the complex line bundle (2.5) determined by $\widetilde{\Psi}_{(y,i);0\sqcup I;I}$ and the fiberwise complex structure of γ. In this case,

$$\mathcal{N}_{\widetilde{U}_{(y,i)}}\widetilde{V}_{(y,i);I}=\mathcal{O}_{\widetilde{\mathcal{R}};(y,i);I}(\mathbb{E})^*\otimes_{\mathbb{C}}\pi^*\mathcal{N}_{U_y}V_{y;I}|_{\widetilde{V}_{(y,i);I}}=\mathcal{O}_{\widetilde{\mathcal{R}};(y,i);I}(\mathbb{E})^*\otimes_{\mathbb{C}}\pi^*\mathcal{N}_{y;S_y;S_y-I}|_{\widetilde{V}_{(y,i);I}}.$$

We define

$$\begin{aligned}&\widetilde{\Psi}_{(y,i);I}:\{(x,(\mathbb{C}v',v),u)\otimes w\in\mathcal{N}_{\widetilde{U}_{(y,i)}}\widetilde{V}_{(y,i);I}:v+u(v)w\in\mathcal{N}'_{y;S_y}\}\longrightarrow\widetilde{U}_{(y,i)},\\&\widetilde{\Psi}_{(y,i);I}((x,(\mathbb{C}v',v),u)\otimes w)=(\mathbb{C}(v'+u(v')w),v+u(v)w).\end{aligned}$$

Since $(\Psi_{y;I})_{I\subset S_y}$ is a regularization for $\{V_{y;i}\}_{i\in S_y}$ in U_y, $(\widetilde{\Psi}_{(y,i);I})_{I\subset\widetilde{S}_{(y,i)}}$ is a regularization for $\{\widetilde{V}_{(y,i);j}\}_{j\in\widetilde{S}_{(y,i)}}$ in $\widetilde{U}_{(y,i)}$. Since the collection $(\Psi_{y;I})_{y\in\mathcal{A},I\subset S_y}$ satisfies the last condition in (4.4), so does the collection $(\widetilde{\Psi}_{y;I})_{y\in\widetilde{\mathcal{A}},I\subset\widetilde{S}_y}$.

Let $y\in\mathcal{A}_r$ as before. The Hermitian structures $(\widetilde{\rho}_0,\widetilde{\nabla}^{(0)})$ on γ and $(\rho_{y;I;j},\nabla^{(y;I;j)})$ on $\mathcal{N}_{V_y;I-j}V_{y;I}$ with $j\in I\subset S_y$ determine Hermitian structures $(\widehat{\rho}_{(y,i);I;j},\widehat{\nabla}^{((y,i);I;j)})$ on the complex line bundles

$$\mathcal{N}_{\widetilde{V}_{(y,i);I-j}}\widetilde{V}_{(y,i);I}=\begin{cases}\gamma|_{\widetilde{V}_{(y,i);I}},&\text{if } j=0\in I\subset\widetilde{S}_{(y,i)};\\ \gamma^*|_{\widetilde{V}_{(y,i);I}}\otimes_{\mathbb{C}}\pi^*\mathcal{N}_{V_{y;S_y-j}}V_{y;S_y}|_{\widetilde{V}_{(y,i);I}},&\text{if } 0,j\in I\subset\widetilde{S}_{(y,i)},\ j\neq 0;\\ \mathcal{O}_{\widetilde{\mathcal{R}};(y,i);I}(\mathbb{E})^*\otimes_{\mathbb{C}}\pi^*\mathcal{N}_{V_y;I-j}V_{y;I}|_{\widetilde{V}_{(y,i);I}},&\text{if } j\in I\subset\widetilde{S}_{(y,i)},\ 0\notin I.\end{cases}$$

The almost complex structure $\widetilde{J}|_{\widetilde{V}_{(y,i);I}}$ and the connection $\widehat{\nabla}^{((y,i);I;j)}$ determine an almost complex structure $\widetilde{J}_{\mathcal{R}_y;I}$ on the total space of the normal bundle $\mathcal{N}_{\widetilde{U}_{(y,i)}}\widetilde{V}_{(y,i);I}$ of $\widetilde{V}_{(y,i);I}$ in $\widetilde{U}_{(y,i)}$. If $0\notin I\subset\widetilde{S}_{(y,i)}$, Corollary 2.3 implies that the isomorphism

$$\mathcal{N}_{\widetilde{U}_{(y,i)}}\widetilde{V}_{(y,i);I}|_{\widetilde{V}_{(y,i);I}-\widetilde{V}_{(y,i);0\sqcup I}}\longrightarrow\mathcal{N}_{U_y}V_{y;I}|_{V_{y;I}-V_{y;S_y}},\quad u\otimes(v,w)\longrightarrow u(v)w,\tag{5.13}$$

intertwines $\widetilde{J}_{\mathcal{R}_y;I}$ with the almost complex structure $J_{\mathcal{R}_y;I}$ determined by the almost complex structure $J|_{V_{y;I}}$ and the connections $\nabla^{(y;I;j)}$ with $j\in I$. Since the regularization $\Psi_{y;I}$ intertwines $J_{\mathcal{R}_y;I}$ and J, it follows that the regularization $\widetilde{\Psi}_{(y,i);I}$ intertwines $\widetilde{J}_{\mathcal{R}_y;I}$ and $\widetilde{J}$ if $0\notin I\subset\widetilde{S}_{(y,i)}$. The same is the case for $I=\{0\}$ by the definition of $\widetilde{J}|_{\widetilde{\mathcal{N}}'_0}$. Since the differentials $\mathfrak{D}\widetilde{\Psi}_{(y,i);I;I-0}$ with $0\in I\subset\widetilde{S}_{(y,i)}$ are product Hermitian isomorphisms, it follows that they intertwine the almost complex structures $\widetilde{J}_{\mathcal{R}_y;I}$ and $\widetilde{J}_{\mathcal{R}_y;I-0}$. Thus, the regularization $\widetilde{\Psi}_{(y,i);I}$ intertwines $\widetilde{J}_{\mathcal{R}_y;I}$ and $\widetilde{J}$ for all $I\subset\widetilde{S}_{(y,i)}$, while $\mathfrak{D}\widetilde{\Psi}_{(y,i);I;I'}$ intertwines $\widetilde{J}_{\mathcal{R}_y;I}$ and $\widetilde{J}_{\mathcal{R}_y;I'}$ for all $I'\subset I\subset\widetilde{S}_{(y,i)}$.

Below we define smooth functions $f_{(y,i);I} : \widetilde{V}_{(y,i);I} \to \mathbb{R}^+$ so that the metrics

$$\widetilde{\rho}_{(y,i);I;j} \equiv \begin{cases} f_{(y,i);I}\widehat{\rho}_{(y,i);I}, & \text{if } j \in I,\ j \neq 0; \\ (1/f_{(y,i);I})\widehat{\rho}_{(y,i);I}, & \text{if } j = 0 \in I \end{cases} \tag{5.14}$$

on the complex line bundles $\mathcal{N}_{\widetilde{V}_{(y,i);I-j}}\widetilde{V}_{(y,i);I}$ are preserved by the isomorphisms $\mathfrak{D}\widetilde{\Psi}_{(y,i);I;I'}$ with $j \in I' \subset I \subset \widetilde{S}_{(y,i)}$, after shrinking the domain of $\widetilde{\Psi}_{(y,i);I}$. The connection

$$\widetilde{\nabla}^{((y,i);I;j)} \equiv \begin{cases} \widehat{\nabla}^{((y,i);I;j)} + f^{-1}_{(y,i);I}\partial_{\widetilde{J}} f_{(y,i);I}, & \text{if } j \in I,\ j \neq 0; \\ \widehat{\nabla}^{((y,i);I;j)} - f^{-1}_{(y,i);I}\partial_{\widetilde{J}} f_{(y,i);I}, & \text{if } j = 0 \in I \end{cases} \tag{5.15}$$

on $\mathcal{N}_{\widetilde{V}_{(y,i);I-j}}\widetilde{V}_{(y,i);I}$ is compatible with $\widetilde{\rho}_{(y,i);I;j}$. Along with $\widetilde{J}|_{\widetilde{V}_{(y,i);I}}$, it determines the same almost complex structure $\widetilde{J}_{\mathcal{R}_y;I}$ on $\mathcal{N}_{\widetilde{V}_{(y,i);I-j}}\widetilde{V}_{(y,i);I}$ as $\widehat{\nabla}^{((y,i);I;j)}$ (because the two connections differ by a $(1,0)$-form). Since the isomorphisms $\mathfrak{D}\widetilde{\Psi}_{(y,i);I;I'}$ preserve the metrics $\widetilde{\rho}_{(y,i);I;j}$ and the almost complex structures $\widetilde{J}_{\mathcal{R}_y;I}$, it follows that they preserve the connections $\widetilde{\nabla}^{((y,i);I;j)}$ as well.

We choose the functions $f_{(y,i);I}$ so that

$$f_{(y,i);I}(x) = f_{(y,i');I}(x) \quad \forall x \in \widetilde{V}_{(y,i);I} \cap \widetilde{V}_{(y,i');I},\ I \subset \widetilde{S}_{(y,i)}, \widetilde{S}_{(y,i')},\ i,i' \in S_y,\ y \in \mathcal{A}_r, \tag{5.16}$$

$$f_{(y,i);I}(x) = f_{(y',h_{y'y;x}(i));h_{y'y;x}(I)}(x) \quad \forall x \in \widetilde{V}_{(y,i);I} \cap \widetilde{V}_{(y',h_{y'y;x}(i));h_{y'y;x}(I)},$$
$$i \in S_y,\ y,y' \in \mathcal{A}_r, \tag{5.17}$$

with $h_{y'y;x}$ as above (4.6). For $I \not\ni 0$, we choose $f_{(y,i);I}$ so that the isomorphism (5.13) identifies the restriction of $\widetilde{\rho}_{(y,i);I;j}$ to $\mathcal{N}_{\widetilde{V}_{(y,i);I-j}}\widetilde{V}_{(y,i);I}|_{\widetilde{V}_{(y,i);I}-U'_r}$ with the restriction of the metric $\rho_{y;I;j}$ to $\mathcal{N}_{V_{y;I-j}}V_{y;I}|_{V_{y;I}-U'_r}$. Along with the assumption on U'_r, this implies that the resulting collection $\widetilde{\mathcal{R}}$ in (5.11) satisfies the first condition in (4.4).

Lemma 5.1 *Let $r \in \mathbb{Z}^+$. There exist a smooth function $h : \mathbb{P}^{r-1} \to \mathbb{R}^+$ and $\delta \in \mathbb{R}^+$ such that*

(1) *h is invariant under the permutations of the homogeneous coordinates on $\mathbb{P}^{r-1}$ and under the standard $(S^1)^r$-action on $\mathbb{P}^{r-1}$;*

(2) *for all $s \in [r]$, $[Z_1,\ldots,Z_s] \in \mathbb{P}^{s-1}$, and $[Z_1,\ldots,Z_r] \in \mathbb{P}^{r-1}$ with $\sum_{i=s+1}^r |Z_i|^2 \le \delta \sum_{i=1}^s |Z_i|^2$,*

$$\frac{h(Z_1,\ldots,Z_r)}{h(Z_1,\ldots,Z_s,0,\ldots,0)} = \frac{|Z_1|^2+\cdots+|Z_r|^2}{|Z_1|^2+\cdots+|Z_s|^2}. \tag{5.18}$$

This lemma is established in Section 5.6. Let $h : \mathbb{P}^{r-1} \to \mathbb{R}^+$ and $\delta \in \mathbb{R}^+$ be as in Lemma 5.1. Define

$$W_{y;I} = \left\{ [(v_i)_{i\in S_y}] \in \mathbb{P}\mathcal{N}_X V^{(r)}|_{V_{y;S_y}} : \sum_{i\in S_y - I} \rho_{y;S_y;i}(v_i) < \delta \sum_{i\in I} \rho_{y;S_y;i}(v_i) \right\} \quad \forall I \subset S_y,\ y \in \mathcal{A}_r.$$

We can identify each fiber of (5.2) with $\mathbb{C}^r$ respecting the splittings and the metrics. This induces an identification of each fiber of $\pi : \mathbb{E} \to V^{(r)}$ with $\mathbb{P}^{r-1}$. By the invariance properties of h, the composition of this identification with h is independent of the choice of the former. Thus, we obtain a smooth function $h_{\mathbb{E}} : \mathbb{E} \to \mathbb{R}^+$ so that

$$\frac{h_{\mathbb{E}}([(v_i)_{i\in S_y}])}{h_{\mathbb{E}}([(v_i)_{i\in I}])} = \frac{\rho_r([(v_i)_{i\in S_y}])}{\rho_r([(v_i)_{i\in I}])} \quad \forall [(v_i)_{i\in S_y}] \in W_{y;I},\ I \subset S_y,\ y \in \mathcal{A}_r. \tag{5.19}$$

For $y \in \mathcal{A}_r$, $i \in S_y$, and $I \subset \widetilde{S}_{(y,i)}$ so that $0 \in I$, let $U_{y;I} \subset \widetilde{U}_y$ be an open neighborhood of $\widetilde{V}_{y;I-0}$ so that $U_{y;I} \cap \mathbb{E} = W_{y;I-0}$. We define

$$f_{(y,i);I} = h_{\mathbb{E}}|_{\widetilde{V}_{(y,i);I}}. \tag{5.20}$$

By (5.19), the restrictions of the differentials $\mathfrak{D}\widetilde{\Psi}_{(y,i);I;I'}$ with $0 \in I' \subset I \subset \widetilde{S}_{(y,i)}$ to $\widetilde{\Psi}^{-1}_{(y,i);I}(U_{y;I})$ preserve the metrics (5.14). By (5.20), the functions $f_{(y,i);I}$ satisfy (5.16) and (5.17) whenever $0 \in I$.

Let $\beta : \mathbb{R} \to \mathbb{R}^{\ge 0}$ and $\varepsilon : V^{(r)} \to \mathbb{R}^+$ be smooth functions so that

$$\beta(t) = \begin{cases} 1, & \text{if } t \le 1; \\ 0, & \text{if } t \ge 2; \end{cases} \quad \text{and} \tag{5.21}$$

$$\bigcup_{y \in \mathcal{A}_r} \{(v_i)_{i \in S_y} \in \mathcal{N}'_r|_{V_{y;S_y}} : \rho_{y;S_y;i}(v) < 2\varepsilon(\pi_r((v_i)_{i \in S_y}))\ \forall i \in S_y\} \subset U'_r \subset U_r = \mathcal{N}'_r.$$

Define

$$U''_r = \bigcup_{y \in \mathcal{A}_r} \{(v_i)_{i \in S_y} \in \mathcal{N}'_r|_{V_{y;S_y}} : \rho_{y;S_y;i}(v_i) < \varepsilon(\pi_r((v_i)_{i \in S_y}))\ \forall i \in S_y\} \subset U'_r \subset \mathcal{N}'_r,$$

$$U_{y;I} = \{(v_i)_{i \in S_y} \in \mathcal{N}'_r|_{V_{y;S_y}} : \rho_{y;S_y;i}(v_i) < \varepsilon(\pi_r((v_i)_{i \in S_y}))\ \forall i \in I\} \quad \forall I \subset S_y,\ y \in \mathcal{A}_r;$$

$$\beta_y : \mathcal{N}'_r|_{V_{y;S_y}} \longrightarrow \mathbb{R}^{\ge 0}, \quad \beta_y((v_i)_{i \in S_y}) = \prod_{j \in S_y} \beta(\rho_{y;S_y;j}(v_j)/\varepsilon(\pi_r((v_i)_{i \in S_y}))) \quad \forall y \in \mathcal{A}_r.$$

By the first condition in (4.4), $\beta_y = \beta_{y'}$ on $\mathcal{N}'_r|_{V_{y;S_y}} \cap \mathcal{N}'_r|_{V_{y';S_{y'}}}$ for all $y, y' \in \mathcal{A}_r$. Thus, the function

$$\beta_r : \mathcal{N}'_r \longrightarrow \mathbb{R}^{\ge 0}, \quad \beta_r(v) = \beta_y(v) \quad \forall v \in \mathcal{N}'_r|_{V_{y;S_y}},\ y \in \mathcal{A}_r,$$

is well-defined and smooth. It satisfies

$$\begin{aligned} &\beta_r|_{U''_r} = 1, \quad \beta_r|_{\mathcal{N}'_r - U'_r} = 0, \quad \beta_r((v_i)_{i \in S_y}) = \beta_r((v_i)_{i \in I}) \\ &\forall [(v_i)_{i \in S_y}] \in U_{y;I},\ I \subset S_y,\ y \in \mathcal{A}_r. \end{aligned} \tag{5.22}$$

For $y \in \mathcal{A}_r$, $i \in S_y$, and $I \subset \widetilde{S}_{(y,i)}$ so that $0 \notin I$, define

$$\begin{aligned} f_{(y,i);I}(\widetilde{\Psi}_{(y,i);0\sqcup I;I}(v)) &= \beta_r(\pi_2(v)) h_{\mathbb{E}}(\widetilde{\pi}_0(v)) + (1 - \beta_r(\pi_2(v)))\widetilde{\rho}_0(v) \\ &\forall v \in \mathcal{N}'_{(y,i);0\sqcup I;I} \subset \gamma|_{\widetilde{V}_{(y,i);0\sqcup I}}. \end{aligned} \tag{5.23}$$

By (5.19) and the last property in (5.22), the restrictions of the differentials $\mathfrak{D}\widetilde{\Psi}_{(y,i);I;I'}$ with $I' \subset I \subset \widetilde{S}_{(y,i)}$ such that $0 \notin I$ to $\widetilde{\Psi}^{-1}_{(y,i);I}(U_{y;I} \cap U_{y;0\sqcup I})$ preserve the metrics (5.14). By (5.19) and the first property in (5.22), the restrictions of the differentials $\mathfrak{D}\widetilde{\Psi}_{(y,i);I;I'}$ with $I' \subset I \subset \widetilde{S}_{(y,i)}$ such that $0 \in I$ to $\widetilde{\Psi}^{-1}_{(y,i);I}(U_{y;I} \cap U''_r)$ preserve the metrics (5.14). By (5.23), the functions $f_{(y,i);I}$ satisfy (5.16) and (5.17) whenever $0 \notin I$. By the middle property in (5.22), the isomorphism (5.13) identifies the restriction of $\widetilde{\rho}_{(y,i);I;j}$ to $\mathcal{N}_{\widetilde{V}_{(y,i);I-j}} \widetilde{V}_{(y,i);I}|_{\widetilde{V}_{(y,i);I} - U'_r}$ with the restriction of the metric $\rho_{y;I;j}$ to $\mathcal{N}_{V_{y;I-j}} V_{y;I}|_{V_{y;I} - U'_r}$ whenever $j \ne 0 \notin I$.

By [14, Lemma 5.8], we can shrink the domains $\mathrm{Dom}(\widetilde{\Psi}_{(y,i);I})$ of $\widetilde{\Psi}_{(y,i);I}$ with $i \in S_y$ and $I \subset \widetilde{S}_{(y,i)}$ to open neighborhoods $\mathcal{N}''_{(y,i);I}$ of $\widetilde{V}_{(y,i);I} \subset \mathrm{Dom}(\widetilde{\Psi}_{(y,i);I})$ so that

$$\widetilde{\Psi}_{(y,i);I}(\mathcal{N}''_{(y,i);I}) \subset \begin{cases} U_{y;I} \cap U''_r, & \text{if } 0 \in I; \\ U_{y;I}, & \text{if } 0 \notin I; \end{cases}$$

and the collection $(\widetilde{\Psi}_{(y,i);I}|_{\mathcal{N}''_{(y,i);I}})_{I\subset\widetilde{S}_{(y,i)}}$ is still a regularization for $\{\widetilde{V}_{(y,i);j}\}_{j\in\widetilde{S}_{(y,i)}}$ in $\widetilde{U}_{y,i}$. Replacing $\widetilde{\Psi}_{y;I}$ with $\widetilde{\Psi}_{y;I}|_{\mathcal{N}''_{y;I}}$ in (5.11) whenever $y\in\widetilde{\mathcal{A}}-\mathcal{A}$, we obtain an almost complex regularization for $\widetilde{V}$ in $(\widetilde{X},\widetilde{J})$.

5.4 Proof of Lemma 1.3

The substance of the last claim of Lemma 1.3 is that the canonical identification

$$T_{\widetilde{\mathcal{R}}}\widetilde{X}(-\log\widetilde{V})|_{\widetilde{X}-\mathbb{E}}=T_{\mathcal{R}}X(-\log V)|_{X-V^{(r)}} \tag{5.24}$$

extends smoothly to a bundle homomorphism as in (1.8) and that this bundle homomorphism is an isomorphism. For each $y\in\mathcal{A}_r$, $i\in S_y$, and $I\subset\widetilde{S}_{(y,i)}$ with $0\in I$, we verify this over the open subspace $U^{\circ}_{(y,i);I}\subset\widetilde{\mathcal{N}}'_0$ defined as in (3.12) via the regularization $\widetilde{\mathcal{R}}$ constructed in Section 5.3.

Let $y\in\mathcal{A}_r$, $i\in S_y$, and $I\subset\widetilde{S}_{(y,i)}$ be as above, $[v]\equiv[(v_i)_{i\in S_y-I}]\in\widetilde{V}_{(y,i);I}$ with $v_i\neq 0$ for all $i\in S_y-I$, and $u\in(\mathbb{C}v)^*-\{0\}$. Let $a\in\mathbb{C}^*$ and $v'\in\mathcal{N}_{y;S_y;S_y-I}$ be sufficiently small. Let

$$h_{\widetilde{\nabla}^{((y,i);I;0)};av}:T_{[v]}\widetilde{V}_{(y,i);I}\longrightarrow T_{av}\gamma\quad\text{and}\quad h_{\nabla^{(r)};av}:T_{[v]}\widetilde{V}_{(y,i);I}\longrightarrow T_{av}(\mathcal{N}_XV^{(r)})$$

be the injective homomorphisms determined by the connections $\widetilde{\nabla}^{((y,i);I;0)}$ and $\nabla^{(r)}$ as in (2.6). The isomorphism (5.7) gives

$$T_{[v]}\widetilde{V}_{(y,i);I}=T_{\pi(v)}V^{(r)}\oplus(\mathbb{C}v)^*\otimes_{\mathbb{C}}(\mathbb{C}v)^{\perp},$$

where $(\mathbb{C}v)^{\perp}\subset\mathcal{N}_{y;S_y;I}|_{\pi(v)}$ is the ρ_r-orthogonal complement of $\mathbb{C}v$. By (5.15) and the sentence containing (5.9),

$$\begin{aligned}h_{\widetilde{\nabla}^{((y,i);I;0)};av}(w,u\otimes v^{\perp})&=h_{\nabla^{(r)};av}(w)+u(av)v^{\perp}+\theta(w,u\otimes v^{\perp})(av)\\&=h_{\nabla^{(r)};av}(w)+a(u(v)v^{\perp}+\theta(w,u\otimes v^{\perp})v)\end{aligned}$$

for some 1-form θ on $\mathbb{E}$. With the notation as in (3.15), we thus obtain

$$\begin{aligned}&\vartheta_{(y,i);(I-0)I}((\widetilde{\Psi}_{(y,i);I}(av,u\otimes v'),(av,u\otimes v'),(w,u\otimes v^{\perp})),(c_i)_{i\in I})\\&=\vartheta_{y;(I-0)S_y}((\Psi_{y;S_y}(av+u(v)v'),av+u(v)v',w),(c_i)_{i\in S_y}),\end{aligned}$$

with $c_i\equiv c_i([v];w,u\otimes v^{\perp},c_0)\in\mathbb{C}$ for $i\in S_y-I$ defined by

$$\sum_{i\in S_y-I}c_iv_i=u(v)v^{\perp}+(\theta(w,u\otimes v^{\perp})+c_0)v.$$

The identification (5.24) thus extends smoothly over $[v]$ as the vector space isomorphism

$$\begin{aligned}&T_{[v]}\mathbb{E}\oplus\mathbb{C}^I\longrightarrow T_{\pi([v])}V^{(r)}\oplus\mathbb{C}^{S_y-I}\oplus\mathbb{C}^{I-0},\\&(w,(c_i)_{i\in I})\longrightarrow(d_{[v]}\pi(w),(c_i([v];w,c_0))_{i\in S_y-I},(c_i)_{i\in I-0}).\end{aligned}$$

5.5 Symplectic Setting

Suppose now that ω is a symplectic form on (X,V) and $\mathcal{R}$ is an ω-regularization for V in X in the sense of Definition 4.2. The smooth submanifold $V^{(r)}\subset X$ is then symplectic. Let $\omega_r\equiv\widetilde{\omega}$ be the closed 2-form on $\mathcal{N}_XV^{(r)}$ determined by $\omega|_{V^{(r)}}$, $\Omega\equiv\omega|_{\mathcal{N}_r}$, and $\nabla^{(r)}$ as in (2.4). Let J be an $\mathcal{R}$-compatible almost complex structure on X,

$$\pi:(\widetilde{X}\equiv((X-V^{(r)})\sqcup\widetilde{\mathcal{N}}'_0)/\sim,\widetilde{J})\longrightarrow(X,J)$$

be the corresponding almost complex blowup with the exceptional divisor $\mathbb{E} \equiv \mathbb{P}(\mathcal{N}_X V^{(r)})$ as in Section 5.2, and $\overline{V} \subset \widetilde{X}$ be the proper transform of V. Suppose also that there exists $\epsilon \in (0,1)$ such that

$$\mathcal{N}_r(2\epsilon) \equiv \{v \in \mathcal{N}_X V^{(r)} : \rho_r(v) < 2\epsilon\} \subset \Psi_r^{-1}(U_r') \subset \mathcal{N}_r'. \tag{5.25}$$

This is automatically the case if $V^{(r)}$ is compact.

The subgroup $S^1 \subset \mathbb{C}^*$ acts on $\mathcal{N}_r(2\epsilon) \times \mathbb{C}$ by

$$u : \mathcal{N}_r(2\epsilon) \times \mathbb{C} \longrightarrow \mathcal{N}_r(2\epsilon) \times \mathbb{C}, \quad u \cdot (v,c) = (uv, c/u). \tag{5.26}$$

This S^1-action preserves the submanifolds

$$\widetilde{\mathcal{Z}}(\epsilon) \equiv \{(v,c) \in \mathcal{N}_r(2\epsilon) \times \mathbb{C} : \rho_r(v) - |c|^2 = \epsilon\} \quad \text{and} \quad \widetilde{\mathbb{E}}_\epsilon \equiv \{(v,0) \in \mathcal{N}_r(2\epsilon) \times \mathbb{C} : \rho_r(v) = \epsilon\}.$$

We identify $\widetilde{\mathbb{E}}_\epsilon$ with the $\sqrt{\epsilon}$-sphere bundle of $\mathcal{N}_X V^{(r)}$ in the obvious way. Let

$$\begin{aligned} &\mathcal{Z}(\epsilon) = \widetilde{\mathcal{Z}}(\epsilon)/S^1, \quad \mathbb{E}_\epsilon = \widetilde{\mathbb{E}}_\epsilon/S^1, \quad X_\epsilon = ((X - \Psi_r(\overline{\mathcal{N}_r(\epsilon)})) \sqcup \mathcal{Z}(\epsilon))/\sim, \\ &\mathcal{Z}(\epsilon) - \mathbb{E}_\epsilon \ni [v, \sqrt{\rho_r(v) - \epsilon}] \sim \Psi_r(v) \in X - \Psi_r(\overline{\mathcal{N}_r(\epsilon)}) \quad \forall v \in \mathcal{N}_r(2\epsilon) - \overline{\mathcal{N}_r(\epsilon)}. \end{aligned} \tag{5.27}$$

By the Symplectic Reduction Theorem [6, Theorem 23.1], there is a unique symplectic form $\omega_{r;\epsilon}$ on the smooth manifold $\mathcal{Z}(\epsilon)$ so that

$$q_\epsilon^* \omega_{r;\epsilon} = (\pi_1^* \omega_r + \pi_2^* \omega_{\mathbb{C}})|_{\widetilde{\mathcal{Z}}(\epsilon)}, \tag{5.28}$$

where $q_\epsilon : \widetilde{\mathcal{Z}}(\epsilon) \to \mathcal{Z}(\epsilon)$ is the quotient projection,

$$\pi_1, \pi_2 : \mathcal{N}_r(2\epsilon) \times \mathbb{C} \longrightarrow \mathcal{N}_r(2\epsilon), \mathbb{C}$$

are the component projections, and $\omega_{\mathbb{C}}$ is the standard symplectic form on $\mathbb{C}$. Since $\Psi_r^* \omega = \omega_r$ on $\mathcal{N}_r(2\epsilon)$ and the S^1-action (5.26) preserves the 2-form $\pi_1^* \omega_r + \pi_2^* \omega_{\mathbb{C}}$, (5.28) implies that the identification (5.27) intertwines $\omega_{r;\epsilon}$ and ω. We thus obtain a symplectic form ω_ϵ on X_ϵ such that

$$\omega_\epsilon|_{X - \Psi_r(\overline{\mathcal{N}_r(\epsilon)})} = \omega|_{X - \Psi_r(\overline{\mathcal{N}_r(\epsilon)})} \quad \text{and} \quad \omega_\epsilon|_{\mathcal{Z}(\epsilon)} = \omega_{r;\epsilon}.$$

It restricts to a symplectic form on $\mathbb{E}_\epsilon \subset X_\epsilon$.

The $\mathbb{P}^{r-1}$-fiber bundle $\pi_\epsilon : \mathbb{E}_\epsilon \to V^{(r)}$ is canonically identified with $\pi : \mathbb{E} \to V^{(r)}$. This identification canonically lifts to an identification of the complex line bundle

$$\widetilde{\pi}_0 : \mathcal{N}_{X_\epsilon} \mathbb{E}_\epsilon = \widetilde{\mathbb{E}}_\epsilon \times_{S^1} \mathbb{C} \longrightarrow \widetilde{\mathbb{E}}_\epsilon/S^1 \equiv \mathbb{E}_\epsilon, \quad (v,c) \sim (uv, c/u) \ \ \forall (v,c) \in \widetilde{\mathbb{E}}_\epsilon \times \mathbb{C},\, u \in S^1,$$

with the tautological line bundle $\gamma \subset \pi^* \mathcal{N}_X V^{(r)}$ as in (5.4). The differential

$$dq_\epsilon : \widetilde{\mathbb{E}}_\epsilon \times \mathbb{C} = \mathcal{N}_{\widetilde{\mathcal{Z}}(\epsilon)} \widetilde{\mathbb{E}}_\epsilon \longrightarrow q_\epsilon^* \mathcal{N}_{X_\epsilon} \mathbb{E}_\epsilon$$

is an isomorphism of complex line bundles. It intertwines the fiberwise symplectic form $\omega_{\mathbb{C}}$ with the fiberwise symplectic form

$$\omega_\epsilon|_{\mathcal{N}_{X_\epsilon} \mathbb{E}_\epsilon} = \omega_{r;\epsilon}|_{\mathcal{N}_{X_\epsilon} \mathbb{E}_\epsilon}$$

on $\mathcal{N}_{X_\epsilon} \mathbb{E}_\epsilon$ induced by ω_ϵ as below Definition 2.1. Thus, the complex orientation on $\mathcal{N}_{X_\epsilon} \mathbb{E}_\epsilon$ agrees with the orientation induced by the symplectic form ω_ϵ. It is straightforward to see that the map

$$\widetilde{\Psi}_{\epsilon;0} : \widetilde{\mathcal{N}}_{\epsilon;0}' \equiv \{[v,c] \in \mathcal{N}_{X_\epsilon} \mathbb{E}_\epsilon : |c|^2 < \epsilon\} \longrightarrow \mathcal{Z}(\epsilon) \subset X_\epsilon, \quad \widetilde{\Psi}_{\epsilon;0}([v,c]) = [\sqrt{1 + |c|^2/\epsilon}\, v, c],$$

is a well-defined smooth regularization for $\mathbb{E}_\epsilon$ in X_ϵ.

Remark 5.2 Via the above identification of the complex line bundles $\mathcal{N}_{\widetilde{X}_\epsilon}\mathbb{E}$ and γ, the Hermitian metric ρ_r and connection $\nabla^{(r)}$ on $\mathcal{N}_X V^{(r)}$ determine a connection $\widetilde{\nabla}^{(0)}$ on $\mathcal{N}_{\widetilde{X}_\epsilon}\mathbb{E}$, as in Section 5.1. The standard Hermitian metric on $\mathbb{C}$ determines a Hermitian metric $\widetilde{\rho}_0$ on $\mathcal{N}_{\widetilde{X}_\epsilon}\mathbb{E}$ compatible with $\widetilde{\nabla}^{(0)}$ and the fiberwise symplectic form $\widetilde{\omega}_\epsilon|_{\mathcal{N}_{\widetilde{X}_\epsilon}\mathbb{E}}$. The metric $\widetilde{\rho}_0$ corresponds to the Hermitian metric $\epsilon^{-1}\pi^*\rho_r|_\gamma$ via the above identification of the complex line bundles $\mathcal{N}_{\widetilde{X}_\epsilon}\mathbb{E}$ and γ. For the record, we show in Section 5.6 that $((\widetilde{\rho}_0, \widetilde{\nabla}^{(0)}), \widetilde{\Psi}_{\epsilon;0})$ is an ω_ϵ-regularization for $\mathbb{E}_\epsilon$ in X_ϵ in the sense of Definition 3.3.

The open subspaces $X - \Psi_r(\overline{\mathcal{N}_r(\epsilon)})$ of X and $X_\epsilon - \mathbb{E}_\epsilon$ of X_ϵ are canonically identified. Let $\overline{V}_\epsilon \subset X_\epsilon$ be the closure of $V - \Psi_r(\overline{\mathcal{N}_r(\epsilon)})$ and $\widetilde{V}_\epsilon = \mathbb{E}_\epsilon \cup \overline{V}_\epsilon$. We now show that $\widetilde{V}_\epsilon$ is an NC symplectic divisor in $(X_\epsilon, \omega_\epsilon)$ with a collection $(\widetilde{U}_y, \{\widetilde{V}_{y;i}\}_{i\in\widetilde{S}_y})_{y\in\widetilde{\mathcal{A}}}$ of charts. If $y \in \mathcal{A} - \mathcal{A}_r$, then

$$U_y \subset X - V^{(r)} = X_\epsilon - \mathbb{E}_\epsilon.$$

In this case, we again take

$$\widetilde{S}_y = S_y \quad \text{and} \quad (\widetilde{U}_y, \{\widetilde{V}_{y;i}\}_{i\in\widetilde{S}_y}) = (U_y, \{V_{y;i}\}_{i\in S_y}).$$

As before, we identify $\mathcal{N}'_r \subset \mathcal{N}_X V^{(r)}$ with $\Psi_r(\mathcal{N}'_r) \subset X$ via Ψ_r. Let

$$\widetilde{\mathcal{N}}'_{r;\epsilon} = (\mathcal{N}'_r - \overline{\mathcal{N}_r(\epsilon)}) \cup \mathcal{Z}(\epsilon) \subset X_\epsilon$$

and $\widetilde{\pi}_{r;\epsilon} : \widetilde{\mathcal{N}}'_{r;\epsilon} \to V^{(r)}$ be the projection induced by π_r. Suppose $y \in \mathcal{A}_r$. For each $i \in S_y$, define $\widetilde{S}_{(y,i)}$, $\mathcal{N}_{y;i}$, and $\widetilde{V}_{(y,i);0}$ as at the end of Section 5.2, with $\mathbb{E}$ replaced by $\mathbb{E}_\epsilon$, and set

$$\widetilde{U}_y = \widetilde{\mathcal{N}}'_{r;\epsilon}|_{V_{y;S_y}} \equiv \widetilde{\pi}^{-1}_{r;\epsilon}(V_{y;S_y}), \quad \overline{V}_{y;i} = \gamma|_{\mathbb{P}\mathcal{N}_{y;i}} \cap \widetilde{U}_y.$$

For $j \in S_y - i$, let $\widetilde{V}_{(y,i);j} = \overline{V}_{y;j} \cap \widetilde{U}_{(y,i)}$ as before. We again have (5.12), with $\overline{V}$ replaced by $\overline{V}_\epsilon$ in the last statement. In this case, $\{\widetilde{V}_{(y,i);j}\}_{j\in\widetilde{S}_{(y,i)}}$ is a transverse collection of codimension 2 symplectic submanifolds of $(\widetilde{U}_{(y,i)}, \omega_{r;\epsilon})$ so that their intersection and symplectic orientations agree. Thus, $\widetilde{V}_\epsilon \cap \widetilde{U}_{(y,i)}$ is an SC symplectic divisor in $(\widetilde{U}_{(y,i)}, \omega_{r;\epsilon})$ in the sense of Definition 3.1, $\widetilde{V}_\epsilon$ is an NC symplectic divisor in $(X_\epsilon, \omega_\epsilon)$, and $(\widetilde{U}_y, \{\widetilde{V}_{y;i}\}_{i\in\widetilde{S}_y})_{y\in\widetilde{\mathcal{A}}}$ is an atlas of local charts for $\widetilde{V}_\epsilon$.

Let $f_\epsilon : \mathbb{R} \to \mathbb{R}$ be a smooth function so that

$$f'_\epsilon(t) > 0 \ \ \forall t \in \mathbb{R}, \quad f_\epsilon(t) = \begin{cases} \sqrt{\epsilon + t^2/\epsilon}, & \text{if } t \le \epsilon/2; \\ t, & \text{if } t \ge 5\sqrt{\epsilon}/4. \end{cases}$$

The map

$$\widetilde{\pi}_\epsilon : \widetilde{X} \longrightarrow X_\epsilon, \quad \widetilde{\pi}_\epsilon(\widetilde{x}) = \begin{cases} \widetilde{\Psi}_{\epsilon;0}(\widetilde{x}) \in \mathcal{Z}(\epsilon), & \text{if } \widetilde{x} \in \widetilde{\mathcal{N}}'_0,\ \rho_r(\widetilde{x}) < \epsilon^2/4; \\ \Psi_r(f_\epsilon(\sqrt{\rho_r(\widetilde{x})})\frac{\widetilde{x}}{\sqrt{\rho_r(\widetilde{x})}}), & \text{if } \widetilde{x} \in \widetilde{\mathcal{N}}'_0,\ \rho_r(\widetilde{x}) > 0; \\ \widetilde{x} \in X - \Psi_r(\overline{\mathcal{N}_r(25\epsilon/16)})), & \text{if } \widetilde{x} \in X - \Psi_r(\overline{\mathcal{N}_r(25\epsilon/16)}) \end{cases}$$

is then an orientation-preserving diffeomorphism. It identifies the NC almost complex divisor $\widetilde{V} \subset \widetilde{X}$ with the NC symplectic divisor $\widetilde{V}_\epsilon \subset X_\epsilon$. Thus,

$$\widetilde{\pi}^*_\epsilon \omega_\epsilon \in \mathrm{Symp}^+(\widetilde{X}, \widetilde{V}).$$

Let $\widetilde{\mathcal{R}}$ be a regularization for $\overline{V}$ in $(\widetilde{X},\widetilde{J})$ obtained as in Section 5.3 from the regularization $\mathcal{R}$ for V in (X,ω) and thus in (X,J). Repeated applications of [14, Theorem 3.1], starting from the deepest strata of $\overline{V}\cap\mathbb{E}$, provide a smooth family $(\mu_\tau)_{\tau\in[0,1]}$ of 1-forms on $\widetilde{X}$ so that

- $\widetilde{\omega}_{\epsilon;\tau}\equiv\widetilde{\pi}_\epsilon^*\omega_\epsilon+d\mu_\tau\in\mathrm{Symp}^+(\widetilde{X},\widetilde{V})$ for all $\tau\in[0,1]$;
- $\mu_0=0$ and $\mathrm{supp}\,\mu_\tau\subset\mathcal{N}_0'$ for all $\tau\in[0,1]$;
- a tuple $\widehat{\mathcal{R}}$ obtained from $\widetilde{\mathcal{R}}$ by restricting the domains of the maps $\widetilde{\Psi}_{y;I}$ with $y\in\widetilde{\mathcal{A}}-(\mathcal{A}-\mathcal{A}_r)$ is an $\widetilde{\omega}_{\epsilon;1}$-regularization for $\widetilde{V}$ in $\widetilde{X}$.

The bundle isomorphism (1.8) thus determines a homotopy class of isomorphisms (1.10) between the log tangent bundles associated with the deformation equivalence classes of ω in $\mathrm{Symp}^+(X,V)$ and of $\widetilde{\pi}_\epsilon^*\omega_\epsilon$ in $\mathrm{Symp}^+(\widetilde{X},\widetilde{V})$.

The (deformation equivalence class of the) NC symplectic divisor $\widetilde{V}\subset\widetilde{X}$ constructed above does not depend on the choices of J, f_ϵ, and ϵ. Since the projection (2.7) is a weak homotopy equivalence, it does not depend on the choices of the regularization $\mathcal{R}$ and $\omega\in\mathrm{Symp}^+(X,V)$ in the given equivalence class if $V^{(r)}$ is compact. Thus, if $V^{(r)}$ is compact, an NC symplectic divisor structure $[\omega]$ on $V\subset X$ determines an NC symplectic divisor structure $[\widetilde{\omega}]$ on $\widetilde{V}\subset\widetilde{X}$.

5.6 Proofs of Technical Statements

We conclude our discussion of blowups with proofs of the claims of Lemma 5.1 and Remark 5.2.

Proof of Lemma 5.1 Suppose $r\ge2$ and the claim is true with r replaced by $r-1$. We denote by $\mathbb{S}_r$ the group of permutations of the homogeneous coordinates of $\mathbb{P}^{r-1}$ and by $\tau_r\in\mathbb{S}_r$ the transposition of the last two coordinates. We identify $\mathbb{P}^{r-2}$ with the subspace $(Z_r=0)$ of $\mathbb{P}^{r-1}$. It is preserved by the subgroup $\mathbb{S}_{r-1}\subset\mathbb{S}_r$ and by the $(S^1)^r$-action on $\mathbb{P}^{r-1}$.

Let $h:\mathbb{P}^{r-2}\to\mathbb{R}^+$ and $\delta\in\mathbb{R}^+$ be a smooth function and a positive number satisfying the conditions in the lemma with r replaced by $r-1$. Let $U\subset\mathbb{P}^{r-1}$ be an open neighborhood of $\mathbb{P}^{r-2}$ preserved by the subgroup $\mathbb{S}_{r-1}\subset\mathbb{S}_r$ and by the $(S^1)^r$-action so that

$$U\cap\tau_r(U)\subset\{[Z_1,\ldots,Z_r]\in\mathbb{P}^{r-1}:|Z_{r-1}|^2+|Z_r|^2\le\delta(|Z_1|^2+\cdots+|Z_{r-2}|^2)\}. \tag{5.29}$$

We extend h over U by

$$\widetilde{h}:U\longrightarrow\mathbb{R}^+,\quad \widetilde{h}([Z_1,\ldots,Z_r])=h([Z_1,\ldots,Z_{r-1},0])\frac{|Z_1|^2+\cdots+|Z_r|^2}{|Z_1|^2+\cdots+|Z_{r-1}|^2}.$$

For each permutation $g\in\mathbb{S}_r$ of the homogeneous coordinates of $\mathbb{P}^{r-1}$, define

$$\widetilde{h}_g:g(U)\longrightarrow\mathbb{R}^+,\quad \widetilde{h}_g([Z])=\widetilde{h}([g^{-1}Z]).$$

By the invariance assumptions on U and h, $\widetilde{h}_{g_1}=\widetilde{h}_{g_2}$ if $g_1^{-1}g_2\in\mathbb{S}_{r-1}$.

Suppose $[Z_1,\ldots,Z_r]\in U\cap\tau_r(U)$. By (5.29) and (5.18) with r replaced by $r-1$,

$$\begin{aligned}\widetilde{h}_{\tau_r}([Z_1,\ldots,Z_r])&=\widetilde{h}([Z_1,\ldots,Z_{r-2},Z_r,Z_{r-1}])\\&=\widetilde{h}([Z_1,\ldots,Z_{r-2},0,0])\frac{|Z_1|^2+\cdots+|Z_{r-2}|^2+|Z_r|^2}{|Z_1|^2+\cdots+|Z_{r-2}|^2}\frac{|Z_1|^2+\cdots+|Z_r|^2}{|Z_1|^2+\cdots+|Z_{r-2}|^2+|Z_r|^2}\\&=\widetilde{h}([Z_1,\ldots,Z_{r-2},0,0])\frac{|Z_1|^2+\cdots+|Z_{r-1}|^2}{|Z_1|^2+\cdots+|Z_{r-2}|^2}\frac{|Z_1|^2+\cdots+|Z_r|^2}{|Z_1|^2+\cdots+|Z_{r-1}|^2}=\widetilde{h}([Z_1,\ldots,Z_r]).\end{aligned}$$

Thus, $\widetilde{h}_{\tau_r}=\widetilde{h}$ on $U\cap\tau_r(U)$. Along with the invariance assumptions on U and h, this implies that $\widetilde{h}_{g_1}=\widetilde{h}_{g_2}$ on $g_1(U)\cap g_2(U)$ for all $g_1,g_2\in\mathbb{S}_r$.

We thus obtain a well-defined smooth function

$$H : W \equiv \bigcup_{g\in\mathbb{S}_r} g(U) \longrightarrow \mathbb{R}^+, \quad H([Z]) = \widetilde{h}_g([Z]) \ \ \forall g \in \mathbb{S}_r, \ [Z] \in g(U),$$

which is invariant under the $\mathbb{S}_r$- and $(S^1)^r$-actions on $\mathbb{P}^{r-1}$ and satisfies the last property in the lemma for some $\delta \in \mathbb{R}^+$. Let $\beta : \mathbb{P}^{r-1} \to [1, 0]$ be a smooth function which is invariant under these two actions, restricts to 1 on a neighborhood of $\mathbb{P}^{r-2}$, and is supported in W. The smooth function

$$\beta H + 1 - \beta : \mathbb{P}^{r-1} \longrightarrow \mathbb{R}^+$$

then has the desired properties for some $\delta \in \mathbb{R}^+$. □

Proof of Remark 5.2 Let

$$\widehat{\omega}_\epsilon = \widetilde{\pi}_0^*(\omega_\epsilon|_{\mathbb{E}_\epsilon}) + \frac{1}{2} d\iota_{\zeta_{\mathcal{N}_{X_\epsilon}\mathbb{E}_\epsilon}}(\omega_\epsilon|_{\mathcal{N}_{X_\epsilon}\mathbb{E}_\epsilon})_{\widetilde{\nabla}^{(0)}},$$

$$\phi, \varphi : \mathcal{N}_X V^{(r)} - V^{(r)} \longrightarrow \mathcal{N}_X V^{(r)}, \quad \phi(v) = \frac{v}{\sqrt{\rho_r(v)}}, \quad \varphi(v) = \sqrt{1+\rho_r(v)}\frac{v}{\sqrt{\rho_r(v)}},$$

$$m_c : \mathcal{N}_X V^{(r)} \longrightarrow \mathcal{N}_X V^{(r)}, \quad m_c(v) = cv, \quad \forall c \in \mathbb{C}.$$

The composition of the restriction of $\widetilde{\Psi}_{\epsilon;0}$ to $\widetilde{\mathcal{N}}'_{\epsilon;0} - \mathbb{E}_\epsilon$ with the identification in (5.27) is given by

$$\widetilde{\mathcal{N}}'_{\epsilon;0} - \mathbb{E}_\epsilon \xrightarrow{m_{\sqrt{\epsilon}}\circ\varphi\circ m_{1/\epsilon}} \mathcal{N}'_r - V^{(r)} \xrightarrow{\Psi_r} X - V^{(r)}.$$

It thus remains to show that $\varphi^*(m^*_{\sqrt{\epsilon}}\omega_r) = m^*_\epsilon\widehat{\omega}_\epsilon$ on $m_{1/\epsilon}(\widetilde{\mathcal{N}}'_{\epsilon;0} - \mathbb{E}_\epsilon) \subset \mathcal{N}_X V^{(r)} - V^{(r)}$.

For each $v \in \mathcal{N}_X V^{(r)} - V^{(r)}$, let

$$\pi_v, \pi_v^\perp : \mathcal{N}_X V^{(r)}|_{\widetilde{\pi}_0(v)} = (\mathbb{C}v)^\perp \oplus (\mathbb{C}v) \longrightarrow \mathbb{C}v, (\mathbb{C}v)^\perp$$

be the ρ_r-orthogonal projections. Let

$$\pi_\nabla : T_v(\mathcal{N}_X V^{(r)}) \longrightarrow \mathcal{N}_X V^{(r)}|_{\pi_r(v)}$$

be the projection corresponding to the decomposition (5.6) determined by the connection $\nabla^{(r)}$. As noted below (5.6), this splitting also encodes the decomposition associated with the connection $\widetilde{\nabla}^{(0)}$. By the properties of a connection, the map m_c preserves the decomposition (5.6) for any $c \in \mathbb{C}^*$; see [38, Lemma 1.1]. Since the connection $\nabla^{(r)}$ is compatible with the Hermitian metric ρ_r, its connection 1-form is purely imaginary in any Hermitian trivialization. It follows that the maps ϕ and φ also preserve the decomposition (5.6); see the proof of [38, Lemma 1.1]. Thus,

$$\pi_\nabla \circ m_c = m_c \circ \pi_\nabla \ \ \forall c \in \mathbb{C}, \quad \pi_\nabla \circ \phi = \phi \circ \pi_\nabla, \quad \pi_\nabla \circ \varphi = \varphi \circ \pi_\nabla. \tag{5.30}$$

We also note that

$$m_{c*}\zeta_{\mathcal{N}} = \zeta_{\mathcal{N}} \circ m_c \quad \text{and} \quad m_c^*\Omega = c^2\Omega \quad \forall c \in \mathbb{R} \tag{5.31}$$

for any vector bundle $\mathcal{N}$ and a fiberwise 2-form Ω on $\mathcal{N}$.

Since $\widetilde{\pi}_0 = q_\epsilon \circ m_{\sqrt{\epsilon}} \circ \phi \circ m_{1/\epsilon}$, $\omega_\epsilon|_{\mathbb{C}v} = \epsilon^{-1}\omega_r|_{\mathbb{C}v}$, and $\zeta_{\mathcal{N}_{X_\epsilon}\mathbb{E}_\epsilon} = \zeta_{\mathcal{N}_X V^{(r)}}$ on $\mathcal{N}_{X_\epsilon}\mathbb{E}_\epsilon - \mathbb{E}_\epsilon = \mathcal{N}_X V^{(r)} - V^{(r)}$, the first identity in (5.30) and (5.31) give

$$\begin{aligned} m_\epsilon^*\widehat{\omega}_\epsilon &= \phi^* m^*_{\sqrt{\epsilon}} q_\epsilon^*(\omega_\epsilon|_{\mathbb{E}_\epsilon}) + \frac{1}{2} d\iota_{\zeta_{\mathcal{N}_{X_\epsilon}\mathbb{E}_\epsilon}} m_\epsilon^*(\epsilon^{-1}\omega_r|_{\mathcal{N}_X V^{(r)}} \circ \pi_v)_{\nabla^{(r)}} \\ &= \phi^*(m^*_{\sqrt{\epsilon}}\omega_r) + \frac{1}{2} d\iota_{\zeta_{\mathcal{N}_X V^{(r)}}}((m^*_{\sqrt{\epsilon}}\omega)|_{\mathcal{N}_X V^{(r)}} \circ \pi_v \circ \pi_\nabla). \end{aligned}$$

Thus, it is sufficient to assume that $\epsilon = 1$ and show that

$$\varphi^*\omega_r = \phi^*\omega_r + \frac{1}{2}d\iota_{\zeta_{\mathcal{N}_X V^{(r)}}}(\omega|_{\mathcal{N}_X V^{(r)}} \circ \pi_v \circ \pi_\nabla) \tag{5.32}$$

on $\mathcal{N}_X V^{(r)} - V^{(r)}$.

By the compatibility of $\omega|_{\mathcal{N}_X V^{(r)}}$ with the Hermitian metric ρ_r determining the projection π_v,

$$\iota_{\zeta_{\mathcal{N}_X V^{(r)}}}(\omega|_{\mathcal{N}_X V^{(r)}}) = \iota_{\zeta_{\mathcal{N}_X V^{(r)}}}(\omega|_{\mathcal{N}_X V^{(r)}} \circ \pi_v).$$

We also note that

$$(\iota_{\zeta_{\mathcal{N}_X V^{(r)}}}(\omega|_{\mathcal{N}_X V^{(r)}})) \circ \phi_* = \frac{1}{\rho_r(v)}(\iota_{\zeta_{\mathcal{N}_X V^{(r)}}}(\omega|_{\mathcal{N}_X V^{(r)}})),$$
$$\varphi_*\left(\frac{1+\rho_r(v)}{\rho_r(v)}\zeta_{\mathcal{N}_X V^{(r)}}\right) = \zeta_{\mathcal{N}_X V^{(r)}} \circ \varphi, \quad \varphi^*(\omega|_{\mathcal{N}_X V^{(r)}}) = \omega|_{\mathcal{N}_X V^{(r)}} + \frac{1}{\rho_r(v)}\omega|_{\mathcal{N}_X V^{(r)}} \circ \pi_v^\perp.$$

Along with the last two identities in (5.30) and

$$\omega_r = \pi_r^*(\omega|_{V^{(r)}}) + \frac{1}{2}d\iota_{\zeta_{\mathcal{N}_X V^{(r)}}}(\omega|_{\mathcal{N}_X V^{(r)}} \circ \pi_\nabla),$$

the above statements give

$$\phi^*\omega_r = \pi_r^*(\omega|_{V^{(r)}}) + \frac{1}{2}d\left(\frac{1}{\rho_r(v)}(\iota_{\zeta_{\mathcal{N}_X V^{(r)}}}\omega|_{\mathcal{N}_X V^{(r)}}) \circ \pi_\nabla\right),$$
$$\varphi^*\omega_r = \pi_r^*(\omega|_{V^{(r)}}) + \frac{1}{2}d\left(\frac{1+\rho_r(v)}{\rho_r(v)}(\iota_{\zeta_{\mathcal{N}_X V^{(r)}}}\omega|_{\mathcal{N}_X V^{(r)}}) \circ \pi_\nabla\right).$$

This establishes (5.32). □

6 NC Symplectic Divisors: Global Perspective

An NC divisor can also be viewed as the image of a transverse immersion ι with certain properties. Following [15], we review the global analogues of the notions of Sections 4.1 and 4.2 in Sections 6.1 and 6.2 below. This global perspective leads to a more succinct notion of regularizations for NC divisors and fits better with global statements, such as Theorem 1.2 (5). The local and global perspectives are shown to be equivalent in [15, Lemma 3.5].

6.1 Definition

For a finite set I, denote by $\mathbb{S}_I$ the symmetric group of permutations of the elements of I. For $k \in \mathbb{Z}^{\geq 0}$, denote by $\mathbb{S}_k \equiv \mathbb{S}_{[k]}$ the k-th symmetric group. For $k' \in [k]$, there is a natural subgroup

$$\mathbb{S}_{k'} \times \mathbb{S}_{[k]-[k']} \subset \mathbb{S}_k.$$

We denote its first factor by $\mathbb{S}_{k;k'}$ and the second by $\mathbb{S}^c_{k;k'}$. For each $\sigma \in \mathbb{S}_k$ and $i \in [k]$, let $\sigma_i \in \mathbb{S}_{k-1}$ be the permutation obtained from the bijection

$$[k] - \{i\} \longrightarrow [k] - \{\sigma(i)\}, \quad j \longrightarrow \sigma(j), \tag{6.1}$$

by identifying its domain and target with $[k-1]$ in the order-preserving fashions.

For any map $\iota : \widetilde{V} \to X$ and $k \in \mathbb{Z}^{\geq 0}$, let

$$\widetilde{V}_\iota^{(k)} = \{(x, \widetilde{v}_1, \ldots, \widetilde{v}_k) \in X \times (\widetilde{V}^k - \Delta_{\widetilde{V}}^{(k)}) : \iota(\widetilde{v}_i) = x \ \forall i \in [k]\}, \tag{6.2}$$

where $\Delta_{\widetilde{V}}^{(k)} \subset \widetilde{V}^k$ is the big diagonal (at least two of the coordinates are the same). Define

$$\iota_k : \widetilde{V}_\iota^{(k)} \longrightarrow X, \quad \iota_k(x, \widetilde{v}_1, \ldots, \widetilde{v}_k) = x, \tag{6.3}$$

$$V_\iota^{(k)} = \iota_k(\widetilde{V}_\iota^{(k)}) = \{x \in X : |\iota^{-1}(x)| \geq k\}. \tag{6.4}$$

For example,

$$\widetilde{V}_\iota^{(0)}, V_\iota^{(0)} = X, \quad \widetilde{V}_\iota^{(1)} \approx \widetilde{V}, \quad V_\iota^{(1)} = \iota(\widetilde{V}).$$

For $k', k \in \mathbb{Z}^{\geq 0}$ and $i \in \mathbb{Z}^+$ with $i, k' \leq k$, define

$$\widetilde{\iota}_{k;k'} : \widetilde{V}_\iota^{(k)} \longrightarrow \widetilde{V}_\iota^{(k')}, \quad \widetilde{\iota}_{k;k'}(x, \widetilde{v}_1, \ldots, \widetilde{v}_k) = (x, \widetilde{v}_1, \ldots, \widetilde{v}_{k'}), \tag{6.5}$$

$$\widetilde{\iota}_{k;k-1}^{(i)} : \widetilde{V}_\iota^{(k)} \longrightarrow \widetilde{V}_\iota^{(k-1)}, \quad \widetilde{\iota}_{k;k-1}^{(i)}(x, \widetilde{v}_1, \ldots, \widetilde{v}_k) = (x, \widetilde{v}_1, \ldots, \widetilde{v}_{i-1}, \widetilde{v}_{i+1}, \ldots, \widetilde{v}_k), \tag{6.6}$$

$$\widetilde{\iota}_k^{(i)} : \widetilde{V}_\iota^{(k)} \longrightarrow \widetilde{V}, \quad \widetilde{\iota}_k^{(i)}(x, \widetilde{v}_1, \ldots, \widetilde{v}_k) = \widetilde{v}_i. \tag{6.7}$$

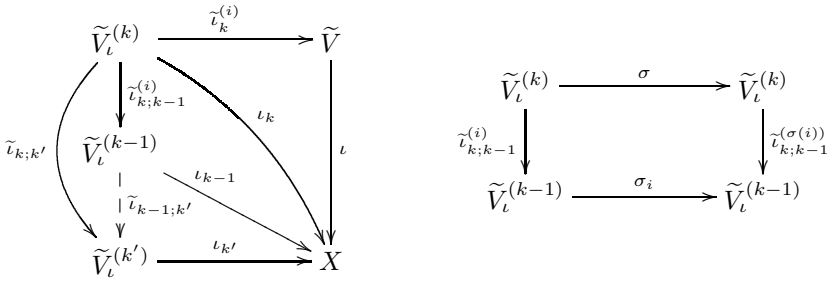
For example,

$$\widetilde{\iota}_{k;k'} = \widetilde{\iota}_{k'+1;k'}^{(k'+1)} \circ \cdots \circ \widetilde{\iota}_{k;k-1}^{(k)} : \widetilde{V}_\iota^{(k)} \longrightarrow \widetilde{V}_\iota^{(k')}, \quad \widetilde{\iota}_{k;1} \approx \widetilde{\iota}_k^{(1)} : \widetilde{V}_\iota^{(k)} \longrightarrow \widetilde{V}_\iota^{(1)} \approx \widetilde{V},$$
$$\widetilde{\iota}_{k;0} = \iota_k : \widetilde{V}_\iota^{(k)} \longrightarrow \widetilde{V}_\iota^{(0)} = X, \quad \widetilde{\iota}_{1;0} \approx \iota : \widetilde{V}_\iota^{(1)} \approx \widetilde{V} \longrightarrow X.$$

We define an $\mathbb{S}_k$-action on $\widetilde{V}_\iota^{(k)}$ by requiring that

$$\widetilde{\iota}_k^{(i)} = \widetilde{\iota}_k^{(\sigma(i))} \circ \sigma : \widetilde{V}_\iota^{(k)} \longrightarrow \widetilde{V} \tag{6.8}$$

for all $\sigma \in \mathbb{S}_k$ and $i \in [k]$. The diagrams



(6.9)

of solid arrows then commute; the entire first diagram commutes if $i > k'$.

Example 6.1 If $V = \bigcup_{i \in S} V_i$ is an SC symplectic divisor, then

$$\iota : \widetilde{V} \equiv \bigsqcup_{i \in S} V_i \longrightarrow X$$

restricts by the inclusion on each V_i, and

$$\widetilde{V}_\iota^{(k)} \approx \bigsqcup_{I \subset S,\, |I| = k} (V_I \times \vec{I}),$$

where $\vec{I} \subset I^k$ is the subcollection of tuples with all entries distinct. The action of $\sigma \in \mathbb{S}_k$ on $\widetilde{V}_\iota^{(k)}$ is by reordering the element of each tuple in $\vec{I}$:

$$(i_1, \ldots, i_k) \longrightarrow (i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(k)}).$$

The maps $\widetilde{\iota}_{k;k'} : \widetilde{V}_\iota^{(k)} \to \widetilde{V}_\iota^{(k')}$ in (6.5) are given by

$$V_I \times \vec{I} \ni (x, (i_1, \ldots, i_k)) \longrightarrow (x, (i_1, \ldots, i_{k'})) \in V_J \times \vec{J}, \quad \text{where } \vec{J} = \vec{I} - \{i_{k'+1}, \ldots, i_k\}.$$

A smooth map $\iota : \widetilde{V} \to X$ is an immersion if the differential $d_x\iota$ of ι at x is injective for all $x \in \widetilde{V}$. This implies that

$$\operatorname{codim}\iota \equiv \dim X - \dim V \geq 0.$$

Such a map has a well-defined normal bundle,

$$\mathcal{N}\iota \equiv \iota^* TX/\mathrm{Im}(d\iota) \longrightarrow \widetilde{V}. \tag{6.10}$$

If ι is a closed immersion, then the subspace $V_\iota^{(k)} \subset X$ and $\widetilde{V}_\iota^{(k)} \subset X \times \widetilde{V}^k$ are closed.

An immersion $\iota : \widetilde{V} \to X$ is transverse if the homomorphism

$$T_x X \oplus \bigoplus_{i=1}^k T_{\widetilde{v}_i}\widetilde{V} \longrightarrow \bigoplus_{i=1}^k T_x X, \quad (w, (w_i)_{i\in[k]}) \longrightarrow (w + d_{\widetilde{v}_i}\iota(w_i))_{i\in[k]},$$

is surjective for all $(x, \widetilde{v}_1, \ldots, \widetilde{v}_k) \in \widetilde{V}_\iota^{(k)}$ and $k \in \mathbb{Z}^+$. By the Inverse Function Theorem, in such a case

- each $\widetilde{V}_\iota^{(k)}$ is a smooth submanifold of $X \times \widetilde{V}^k$,
- the maps $\widetilde{\iota}_{k;k-1}$ in (6.5) and the maps (6.6) are transverse immersions,
- the homeomorphisms σ of $\widetilde{V}_\iota^{(k)}$ determined by the elements of $\mathbb{S}_k$ as in (6.8) are diffeomorphisms.

By the commutativity of the upper and middle triangles in the first diagram in (6.9), the inclusion of $\mathrm{Im}(d\iota_k)$ into $\widetilde{\iota}_k^{(i)*}\mathrm{Im}(d\iota)$ and the homomorphism $d\iota_{k-1}$ induce homomorphisms

$$\mathcal{N}\iota_k \longrightarrow \iota_k^{(i)*}\mathcal{N}\iota, \quad \mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)} \longrightarrow \mathcal{N}\iota_k \quad \forall i \in [k].$$

By the Inverse Function Theorem, the resulting homomorphisms

$$\mathcal{N}\iota_k \longrightarrow \bigoplus_{i\in[k]} \widetilde{\iota}_k^{(i)*}\mathcal{N}\iota \quad \text{and} \quad \mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)} \longrightarrow \widetilde{\iota}_k^{(i)*}\mathcal{N}\iota \quad \forall i \in [k] \tag{6.11}$$

are isomorphisms. If $\widetilde{V}$ is the disjoint union of submanifolds $V_i \subset X$, they correspond to the first two isomorphisms in (3.2). For $\sigma \in \mathbb{S}_k$ and $i \in [k]$, the homomorphisms $d\sigma$ and $d\sigma_i$ of the second diagram in (6.9) induces an isomorphism

$$D_i\sigma : \mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)} \longrightarrow \mathcal{N}\widetilde{\iota}_{k;k-1}^{(\sigma(i))} \tag{6.12}$$

covering σ.

If $\iota : \widetilde{V} \to X$ is any immersion between oriented manifolds of even dimensions, the short exact sequence of vector bundles

$$0 \longrightarrow T\widetilde{V} \xrightarrow{d\iota} \iota^* TX \longrightarrow \mathcal{N}\iota \longrightarrow 0 \tag{6.13}$$

over $\widetilde{V}$ induces an orientation on $\mathcal{N}\iota$. If in addition ι is a transverse immersion, the orientation on $\mathcal{N}\iota$ induced by the orientations of X and $\widetilde{V}$ induces an orientation on $\mathcal{N}\iota_k$ via the first isomorphism in (6.11). The orientations of X and $\mathcal{N}\iota_k$ then induce an orientation on $\widetilde{V}_\iota^{(k)}$ via the short exact sequence (6.13) with $\iota = \iota_k$ for all $k \in \mathbb{Z}^+$, which we call the intersection orientation of $\widetilde{V}_\iota^{(k)}$. For $k = 1$, it agrees with the original orientation of $\widetilde{V}$ under the canonical identification $\widetilde{V}_\iota^{(1)} \approx \widetilde{V}$.

Suppose (X, ω) is a symplectic manifold. If $\iota : \widetilde{V} \to X$ is a transverse immersion such that $\iota_k^*\omega$ is a symplectic form on $\widetilde{V}_\iota^{(k)}$ for all $k \in \mathbb{Z}^+$, then each $\widetilde{V}_\iota^{(k)}$ carries an orientation induced

by $\iota_k^*\omega$, which we call the ω-orientation. By the previous paragraph, the ω-orientations of X and $\widetilde{V}$ also induce intersection orientations on all $\widetilde{V}_\iota^{(k)}$. By definition, the intersection and ω-orientations of $\widetilde{V}_\iota^{(1)}$ are the same.

Proposition 6.2 ([15, Proposition 3.6]) *Suppose (X,ω) is a symplectic manifold and $\iota:\widetilde{V}\to X$ is a transverse immersion of codimension 2. Then, $V=\iota(\widetilde{V})$ is an NC symplectic divisor in (X,ω) in the sense of Definition 4.1 if and only if $\iota_k^*\omega$ is a symplectic form on $\widetilde{V}_\iota^{(k)}$ for all $k\in\mathbb{Z}^+$ and the intersection and ω-orientations of $\widetilde{V}_\iota^{(k)}$ are the same.*

In the global description of an NC divisor $V\subset X$, the singular locus $V_\partial\subset X$ is $V_\iota^{(2)}$.

6.2 Regularizations

Suppose $\iota:\widetilde{V}\to X$ is a transverse immersion and $k,k'\in\mathbb{Z}^{\ge 0}$ with $k'\le k$. With the notation as in (6.2)–(6.7), define

$$\pi_{k;k'}:\mathcal{N}_{k;k'}\iota=\bigoplus_{i\in[k]-[k']}\mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)}\longrightarrow\widetilde{V}_\iota^{(k)}\quad\text{and}\quad\pi_{k;k'}^c:\mathcal{N}_{k;k'}^c\iota=\bigoplus_{i\in[k']}\mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)}\longrightarrow\widetilde{V}_\iota^{(k)}. \tag{6.14}$$

By the commutativity of the first diagram in (6.9), the homomorphisms $d\widetilde{\iota}_{k-1;k'}$ and $d\iota_{k-1}$ induce homomorphisms

$$\mathcal{N}_{k;k'}\iota\longrightarrow\mathcal{N}\widetilde{\iota}_{k;k'}\quad\text{and}\quad\mathcal{N}_{k;k'}^c\iota\longrightarrow\widetilde{\iota}_{k;k'}^*\mathcal{N}\iota_{k'}.$$

By the Inverse Function Theorem, these homomorphisms are isomorphisms. If $\widetilde{V}$ is the disjoint union of submanifolds $V_i\subset X$, they correspond to the last isomorphism in (3.2) and the first identification in (3.6). For each $\sigma\in\mathbb{S}_k$, the isomorphisms (6.11) and (6.12) induce an isomorphism

$$D\sigma=(D_i\sigma)_{i\in[k]}:\mathcal{N}\iota_k\approx\mathcal{N}_{k;0}\iota\equiv\bigoplus_{i\in[k]}\mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)}\longrightarrow\bigoplus_{i\in[k]}\mathcal{N}\widetilde{\iota}_{k;k-1}^{(\sigma(i))}\equiv\mathcal{N}_{k;0}\iota\approx\mathcal{N}\iota_k \tag{6.15}$$

lifting the action of σ on $\widetilde{V}_\iota^{(k)}$. The isomorphism (6.15) permutes the components of the direct sum so that the subbundles

$$\mathcal{N}_{k;k'}\iota,\mathcal{N}_{k;k'}^c\iota\subset\mathcal{N}\iota_k$$

are invariant under the action of the subgroup $\mathbb{S}_{k'}\times\mathbb{S}_{[k]-[k']}$ of $\mathbb{S}_k$, but not under the action of the full group $\mathbb{S}_k$.

Definition 6.3 *A regularization for an immersion $\iota:\widetilde{V}\to X$ is a smooth map $\Psi:\mathcal{N}'\to X$ from a neighborhood of $\widetilde{V}$ in $\mathcal{N}\iota$ such that for every $\widetilde{v}\in\widetilde{V}$, there exists a neighborhood $U_{\widetilde{v}}$ of $\widetilde{v}$ in $\widetilde{V}$ so that the restriction of Ψ to $\mathcal{N}'|_{U_{\widetilde{v}}}$ is a diffeomorphism onto its image, $\Psi(\widetilde{v})=\iota(\widetilde{v})$, and the homomorphism*

$$\mathcal{N}\iota|_{\widetilde{v}}=T_{\widetilde{v}}^{\mathrm{ver}}\mathcal{N}\iota\hookrightarrow T_{\widetilde{v}}\mathcal{N}\iota\xrightarrow{d_{\widetilde{v}}\Psi}T_{\widetilde{v}}X\longrightarrow\frac{T_{\widetilde{v}}X}{\mathrm{Im}(d_{\widetilde{v}}\iota)}\equiv\mathcal{N}\iota|_{\widetilde{v}}$$

is the identity.

Definition 6.4 *A system of regularizations for a transverse immersion $\iota:\widetilde{V}\to X$ is a tuple $(\Psi_k)_{k\in\mathbb{Z}^{\ge 0}}$, where each Ψ_k is a regularization for the immersion ι_k, such that*

$$\Psi_k(\mathcal{N}_{k;k'}\iota\cap\mathrm{Dom}(\Psi_k))=V_\iota^{(k')}\cap\mathrm{Im}(\Psi_k)\quad\forall k\in\mathbb{Z}^{\ge 0},\ k'\in[k], \tag{6.16}$$

$$\Psi_k=\Psi_k\circ D\sigma|_{\mathrm{Dom}(\Psi_k)}\quad\forall k\in\mathbb{Z}^{\ge 0},\ \sigma\in\mathbb{S}_k; \tag{6.17}$$

$$\{x \in X : |\Psi_k^{-1}(x)| \ge k\} \subset \mathrm{Im}(\Psi_{k+1}) \quad \forall k \in \mathbb{Z}^{\ge 0}. \tag{6.18}$$

The stratification condition (6.16) replaces the first condition in (3.7) and implies that there exists a smooth map

$$\begin{aligned} &\Psi_{k;k'} : \mathcal{N}'_{k;k'}\iota \equiv \mathcal{N}_{k;k'}\iota \cap \mathrm{Dom}(\Psi_k) \longrightarrow \widetilde{V}_\iota^{(k')} \quad \text{s.t.} \\ &\Psi_{k;k'}|_{\widetilde{V}_\iota^{(k)}} = \widetilde{\iota}_{k;k'}, \quad \Psi_k|_{\mathcal{N}'_{k;k'}\iota} = \iota_{k'} \circ \Psi_{k;k'}; \end{aligned} \tag{6.19}$$

see Proposition 1.35 and Theorem 1.32 in [36]. Similarly to (3.9), $\Psi_{k;k'}$ lifts to a (fiberwise) vector bundle isomorphism

$$\mathfrak{D}\Psi_{k;k'} : \pi_{k;k'}^*\mathcal{N}_{k;k'}^c\iota|_{\mathcal{N}'_{k;k'}\iota} \longrightarrow \mathcal{N}\iota_{k'}|_{\mathrm{Im}(\Psi_{k;k'})}. \tag{6.20}$$

This bundle isomorphism preserves the second splitting below in (6.14) and is $\mathbb{S}_{k;k'}$-equivariant and $\mathbb{S}_{k;k'}^c$-invariant. The condition (6.21) below replaces (3.10) in the present setting.

Definition 6.5 *A refined regularization for a transverse immersion $\iota : \widetilde{V} \to X$ is a system $(\Psi_k)_{k\in\mathbb{Z}^{\ge 0}}$ of regularizations for ι such that*

$$\begin{aligned} &\mathrm{Dom}(\Psi_k) \subset \pi_{k;k'}^*\mathcal{N}_{k;k'}^c\iota|_{\mathcal{N}'_{k;k'}\iota}, \quad \mathfrak{D}\Psi_{k;k'}(\mathrm{Dom}(\Psi_k)) = \mathrm{Dom}(\Psi_{k'})|_{\mathrm{Im}(\Psi_{k;k'})}, \\ &\Psi_k = \Psi_{k'} \circ \mathfrak{D}\Psi_{k;k'}|_{\mathrm{Dom}(\Psi_k)} \end{aligned} \tag{6.21}$$

whenever $0 \le k' \le k$.

If $(\Psi_k)_{k\in\mathbb{Z}^{\ge 0}}$ is a refined regularization for a transverse immersion $\iota : \widetilde{V} \to X$, then

$$\begin{aligned} &\mathcal{N}'_{k;k''}\iota, \pi_{k;k''}^*\mathcal{N}_{k;k''}^c\iota|_{\mathcal{N}'_{k;k''}\iota} \subset \pi_{k;k'}^*\mathcal{N}_{k;k'}^c\iota|_{\mathcal{N}'_{k;k'}\iota}, \\ &\Psi_{k;k''} = \Psi_{k';k''} \circ \mathfrak{D}\Psi_{k;k'}|_{\mathcal{N}'_{k;k''}\iota}, \quad \mathfrak{D}\Psi_{k;k''} = \mathfrak{D}\Psi_{k';k''} \circ \mathfrak{D}\Psi_{k;k'}|_{\pi_{k;k''}^*\mathcal{N}_{k;k''}^c\iota|_{\mathcal{N}'_{k;k''}\iota}} \end{aligned} \tag{6.22}$$

whenever $0 \le k'' \le k' \le k$.

Suppose (X,ω) is a symplectic manifold and $\iota : \widetilde{V} \to X$ is an immersion so that $\iota^*\omega$ is a symplectic form on V. The normal bundle

$$\mathcal{N}\iota \equiv \frac{\iota^*TX}{\mathrm{Im}(d\iota)} \approx (\mathrm{Im}(d\iota))^\omega \equiv \{w \in T_{\iota(\widetilde{v})}X : \widetilde{v} \in \widetilde{V},\, \omega(w, d_x\iota(w')) = 0\ \forall w' \in T_{\widetilde{v}}V\}$$

of ι then inherits a fiberwise symplectic form $\omega|_{\mathcal{N}\iota}$ from ω. We denote the restriction of $\omega|_{\mathcal{N}\iota}$ to a subbundle $L \subset \mathcal{N}\iota$ by $\omega|_L$.

Definition 6.6 *Suppose (X,ω) is a symplectic manifold, $\iota : \widetilde{V} \to X$ is an immersion so that $\iota^*\omega$ is a symplectic form on V, and*

$$\mathcal{N}\iota = \bigoplus_{i\in I} L_i$$

is a fixed splitting into oriented rank 2 subbundles. If $\omega|_{L_i}$ is nondegenerate for every $i \in I$, then an ω-regularization for ι is a tuple $((\rho_i, \nabla^{(i)})_{i\in I}, \Psi)$, where $(\rho_i, \nabla^{(i)})$ is an $\omega|_{L_i}$-compatible Hermitian structure on L_i for each $i \in I$ and Ψ is a regularization for ι, such that

$$\Psi^*\omega = (\iota^*\omega)_{(\rho_i,\nabla^{(i)})_{i\in I}}|_{\mathrm{Dom}(\Psi)}.$$

Definition 6.7 *Suppose (X,ω) is a symplectic manifold and $\iota : \widetilde{V} \to X$ is a transverse immersion of codimension 2 so that $\iota_k^*\omega$ is a symplectic form on $\widetilde{V}_\iota^{(k)}$ for each $k \in \mathbb{Z}^+$. A refined ω-regularization for ι is a tuple*

$$\mathcal{R} \equiv (\mathcal{R}_k)_{k\in\mathbb{Z}^{\ge 0}} \equiv ((\rho_{k;i}, \nabla^{(k;i)})_{i\in[k]}, \Psi_k)_{k\in\mathbb{Z}^{\ge 0}} \tag{6.23}$$

such that $(\Psi_k)_{k\in\mathbb{Z}^{\geq 0}}$ *is a refined regularization for* ι, $\mathcal{R}_k$ *is an* ω*-regularization for* ι_k *with respect to the splitting* (6.15) *for every* $k \in \mathbb{Z}^{\geq 0}$,

$$(\rho_{k;i}, \nabla^{(k;i)}) = \{D_i\sigma\}^*(\rho_{k;\sigma(i)}, \nabla^{(k;\sigma(i))}) \quad \forall k \in \mathbb{Z}^{\geq 0},\, \sigma \in \mathbb{S}_k,\, i \in [k], \tag{6.24}$$

and the induced vector bundle isomorphisms (6.20) *are product Hermitian isomorphisms for all* $k' \leq k$.

An almost complex structure J on X that preserves $\operatorname{Im} d\iota$ determines an almost complex structure $J_{\iota;k}$ on $\widetilde{V}_\iota^{(k)}$ for every $k \in \mathbb{Z}^{\geq 0}$, with $J_0 = J$. The maps ι, ι_k, $\widetilde{\iota}_{k;k'}$, $\widetilde{\iota}_{k;k-1}^{(i)}$, $\widetilde{\iota}_k^{(i)}$, and σ from $\widetilde{V}_\iota^{(k)}$ respect these almost complex structures. A refined ω-regularization $\mathcal{R}$ for ι determines a fiberwise complex structure $\mathfrak{i}_k$ on $\mathcal{N}\iota_k$ and a splitting

$$T(\mathcal{N}\iota_k) \approx \pi_{k;0}^* T\widetilde{V}_\iota^{(k)} \oplus \pi_{k;0}^* \mathcal{N}\iota_k\,,$$

which are preserved by the action of $\mathbb{S}_k$. Along with $J_{\iota;k}$, they determine an almost complex structure $J_{\mathcal{R};k}$ on the total space of $\mathcal{N}\iota_k$. We call an almost complex structure J on X compatible with an ω-regularization $\mathcal{R}$ as in (6.23) if

$$J(\operatorname{Im} d\iota) \subset \operatorname{Im} d\iota \quad \text{and} \quad J \circ d\Psi_k = d\Psi_k \circ J_{\mathcal{R};k}|_{\mathrm{Dom}(\Psi_k)} \quad \forall k \in \mathbb{Z}^{\geq 0}. \tag{6.25}$$

Under the correspondence between the local and global perspectives provided by Proposition 6.2, this notion is the global version of the $\mathcal{R}$-compatibility defined in Section 4.2.

Remark 6.8 Let $(\Psi_k)_{k\in\mathbb{Z}^{\geq 0}}$ be a refined regularization for a transverse immersion $\iota : \widetilde{V} \to X$ as in Definition 6.5. For $k \in \mathbb{Z}^+$, the limit set $\overline{\operatorname{Im}\Psi_k} - \operatorname{Im}\Psi_k$ of Ψ_k is closed and disjoint from the closed subspace $V_\iota^{(k)}$ of X. There are thus disjoint open neighborhoods W_k of $V_\iota^{(k)}$ and W_k' of the limit set. By [14, Lemma 5.8], we can shrink the domains of Ψ_k so that $\operatorname{Im}\Psi_k \subset W_k$ for every $k \in \mathbb{Z}^+$ and the new collection $(\Psi_k)_{k\in\mathbb{Z}^{\geq 0}}$ is still a refined regularization for ι. Each map Ψ_k is then closed (in addition to being open).

6.3 Constructions

We now describe the vector bundles

$$\mathcal{O}_{\mathcal{R};X}(V) = \mathcal{O}_{\mathcal{R};X}(\iota) \longrightarrow X \quad \text{and} \quad T_{\mathcal{R}}X(-\log V) = T_{\mathcal{R}}X(-\log\iota) \longrightarrow X,$$

the section $s_{\mathcal{R}}$ of $\mathcal{O}_{\mathcal{R};X}(V)$, and the vector bundle homomorphism $\iota_{\mathcal{R}}$ on $T_{\mathcal{R}}X(-\log V)$ constructed in Section 4.3 from the global perspective on NC divisors and regularizations presented in Sections 6.1 and 6.2. We fix a symplectic manifold (X,ω), a normalization $\iota : \widetilde{V} \to X$ of an NC symplectic divisor V as in Proposition 6.2, and a refined ω-regularization $\mathcal{R}$ for ι in X as in (6.23). For the purposes of constructing a complex structure on $T_{\mathcal{R}}X(-\log\iota)$, we also fix an almost complex structure J on X compatible with ω and $\mathcal{R}$.

With the notation as in (6.2) and (6.4), the restrictions

$$\Psi_k : \mathrm{Dom}(\Psi_k)|_{\widetilde{V}_\iota^{(k)} - \mathrm{Im}(\iota_{k+1;k})} \longrightarrow W_k \equiv \Psi_k(\mathrm{Dom}(\Psi_k)|_{\widetilde{V}_\iota^{(k)} - \mathrm{Im}(\iota_{k+1;k})})$$

are not even $\mathbb{S}_k$-covering maps. In other words, there could exist

$$v, v' \in \mathrm{Dom}(\Psi_k)|_{\widetilde{V}_\iota^{(k)} - \mathrm{Im}(\iota_{k+1;k})} \quad \text{s.t.} \quad \Psi_k(v) = \Psi_k(v') \in X \quad \text{and} \quad D\sigma(v) \neq v' \quad \forall \sigma \in \mathbb{S}_k;$$

see the left diagram in Figure 2. This makes it difficult to describe a vector bundle on W_k as a pushdown of a vector bundle on $\mathrm{Dom}(\Psi_k)|_{\widetilde{V}_\iota^{(k)} - \mathrm{Im}(\iota_{k+1;k})}$. For this reason, we shrink the last

space to an $\mathbb{S}_k$-invariant open subspace $\mathcal{N}^\circ_{k;0}\iota$ so that

$$\Psi_k : \mathcal{N}^\circ_{k;0}\iota \longrightarrow U^\circ_k \equiv \Psi_k(\mathcal{N}^\circ_{k;0}\iota) \tag{6.26}$$

is an $\mathbb{S}_k$-covering map and the collection $\{U^\circ_k\}_{k\in\mathbb{Z}^{\geq 0}}$ is an open covering of X. Figure 2 illustrates this shrinking procedure.

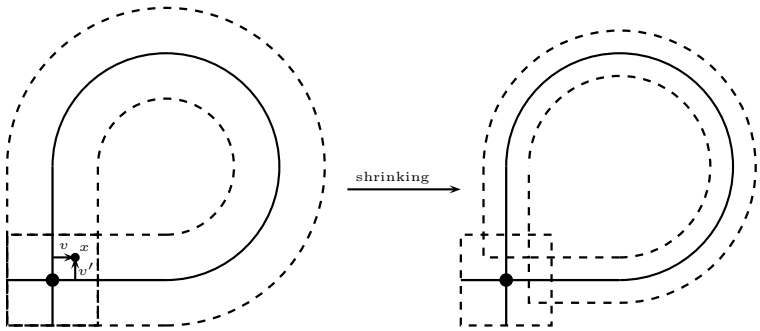


Figure 2 Shrinking regularizations

For $k', k \in \mathbb{Z}^{\geq 0}$ with $k' \leq k$, let $\pi_{k;k'} : \mathcal{N}_{k;k'}\iota \to \widetilde{V}^{(k)}_\iota$ be as in (6.14). For a continuous function $\varepsilon : \widetilde{V}^{(k)}_\iota \to \mathbb{R}^+$, define

$$\mathcal{N}_{k;k'}\iota(\varepsilon) = \{(v_i)_{i\in[k]-[k']} \in \mathcal{N}_{k;k'}\iota : \rho_{k;i}(v_i) < \varepsilon^2(\pi_{k;k'}(v)) \ \forall i \in [k]-[k']\}.$$

By Remark 6.8 and the proof of [14, Lemma 5.8], there exists a smooth function $\varepsilon : X \to \mathbb{R}^+$ such that

$$\mathcal{N}_{k;0}\iota(2^k\varepsilon\circ\iota_k) \subset \mathrm{Dom}(\Psi_k), \quad \varepsilon(\Psi_k(v)) = \varepsilon(\iota_k(\pi_{k;0}(v))) \quad \forall v \in \mathcal{N}_{k;0}\iota(2^k\varepsilon\circ\iota_k), \tag{6.27}$$

and $\Psi_k(\overline{2^{k-1}\varepsilon\circ\iota_k}) \subset X$ is closed for every $k \in \mathbb{Z}^{\geq 0}$. If V is compact, ε can taken to be a constant. Define

$$\begin{aligned} \widetilde{V}^{(k),\circ}_\iota &= \widetilde{V}^{(k)}_\iota - \bigcup_{\ell>k} \Psi_{\ell;k}(\overline{\mathcal{N}_{\ell;k}\iota(2^{\ell-1}\varepsilon\circ\iota_\ell)}), \\ \mathcal{N}^\circ_{k;0}\iota &= \mathcal{N}_{k;0}\iota(2^k\varepsilon\circ\iota_k)|_{\widetilde{V}^{(k),\circ}_\iota}, \quad U^\circ_k = \Psi_k(\mathcal{N}^\circ_{k;0}\iota). \end{aligned} \tag{6.28}$$

By (6.17) and (6.18), the restriction (6.26) is an $\mathbb{S}_k$-covering map.

For $k, k' \in \mathbb{Z}^{\geq 0}$ with $k' \leq k$, let

$$\mathcal{N}^{k'}_{k;0}\iota = \mathcal{N}^\circ_{k;0}\iota \cap \mathfrak{D}\Psi^{-1}_{k;k'}(\mathcal{N}^\circ_{k';0}\iota) \quad \text{and} \quad \mathcal{N}^k_{k';0}\iota = \mathfrak{D}\Psi_{k,k'}(\mathcal{N}^{k'}_{k;0}\iota) = \mathfrak{D}\Psi_{k;k'}(\mathcal{N}^\circ_{k;0}\iota) \cap \mathcal{N}^\circ_{k';0}\iota.$$

Since the map $\mathfrak{D}\Psi_{k;k'}$ is $\mathbb{S}_{k;k'}$-equivariant and $\mathbb{S}^c_{k;k'}$-invariant, the subspace $\mathcal{N}^{k'}_{k;0}\iota \subset \mathcal{N}^k_{k;0}\iota$ is $\mathbb{S}_{k;k'} \times \mathbb{S}^c_{k;k'}$-invariant. The restriction

$$\mathfrak{D}\Psi_{k;k'} : \mathcal{N}^{k'}_{k;0}\iota \longrightarrow \mathcal{N}^k_{k';0}\iota \tag{6.29}$$

is an $\mathbb{S}_{k;k'}$-equivariant $\mathbb{S}^c_{k;k'}$-covering map. By the last equality in (6.22),

$$\begin{aligned} &\mathfrak{D}\Psi_{k;k'}(\mathcal{N}^{k'}_{k;0}\iota \cap \mathcal{N}^{k''}_{k;0}\iota) = \mathcal{N}^k_{k';0}\iota \cap \mathcal{N}^{k''}_{k';0}\iota, \quad \mathfrak{D}\Psi_{k;k''}(\mathcal{N}^{k'}_{k;0}\iota \cap \mathcal{N}^{k''}_{k;0}\iota) = \mathcal{N}^k_{k'';0}\iota \cap \mathcal{N}^{k'}_{k'';0}\iota, \\ &\mathfrak{D}\Psi_{k';k''}(\mathcal{N}^k_{k';0}\iota \cap \mathcal{N}^{k''}_{k';0}\iota) = \mathcal{N}^k_{k'';0}\iota \cap \mathcal{N}^{k'}_{k'';0}\iota, \end{aligned} \tag{6.30}$$

whenever $0 \leq k'' \leq k' \leq k$.

Suppose

$$\{\widetilde{\pi}_k : \widetilde{E}_k \longrightarrow \mathcal{N}^\circ_{k;0}\iota\}_{k\in\mathbb{Z}^{\geq 0}} \quad \text{and} \quad \{\widetilde{F}_{k;k'} : \widetilde{E}_k|_{\mathcal{N}^{k'}_{k;0}\iota} \longrightarrow \widetilde{E}_{k'}|_{\mathcal{N}^{k}_{k';0}\iota}\}_{k,k'\in\mathbb{Z}^{\geq 0},k'\leq k}$$

is a collection of $\mathbb{S}_k$-equivariant (complex) vector bundles and a collection of $\mathbb{S}_{k;k'}$-equivariant $\mathbb{S}^c_{k;k'}$-invariant (smooth) vector bundle maps that lift the covering maps (6.29), restrict to an isomorphism on each fiber, and satisfy

$$\widetilde{F}_{k;k''}|_{\mathcal{N}^{k'}_{k;0}\iota\cap\mathcal{N}^{k''}_{k;0}\iota} = \widetilde{F}_{k';k''}\circ\widetilde{F}_{k;k'}|_{\mathcal{N}^{k'}_{k;0}\iota\cap\mathcal{N}^{k''}_{k;0}\iota} \quad \forall k''\leq k'\leq k. \tag{6.31}$$

By (6.30), the composition on the right-hand side above is well-defined. Since the map (6.29) is an $\mathbb{S}_k$-covering map,

$$E_k \equiv \widetilde{E}_k/\mathbb{S}_k \longrightarrow \mathcal{N}^\circ_{k;0}\iota/\mathbb{S}_k = U^\circ_k$$

is a vector bundle. The maps $\widetilde{F}_{k;k'}$ induce vector bundle isomorphisms

$$F_{k;k'} : E_k|_{U^\circ_k\cap U^\circ_{k'}} \longrightarrow E_{k'}|_{U^\circ_k\cap U^\circ_{k'}}$$

covering the identity on $U^\circ_k\cap U^\circ_{k'}$. By (6.31),

$$F_{k;k''}|_{U^\circ_k\cap U^\circ_{k'}\cap U^\circ_{k''}} = F_{k';k''}\circ F_{k;k'}|_{U^\circ_k\cap U^\circ_{k'}\cap U^\circ_{k''}}.$$

We can thus form a vector bundle

$$\begin{aligned}&\pi : E\equiv\bigg(\bigsqcup_{k\in\mathbb{Z}^{\geq 0}} E_k\bigg)\Big/\sim\longrightarrow X, \quad \pi([v]) = \Psi_k(\widetilde{\pi}_k(v)) \quad \forall k\in\mathbb{Z}^{\geq 0}, v\in E_k,\\ &E_k|_{U^\circ_k\cap U^\circ_{k'}} \ni w\sim F_{k;k'}(w)\in E_{k'}|_{U^\circ_k\cap U^\circ_{k'}} \quad \forall k,k'\in\mathbb{Z}^{\geq 0}, k'\leq k.\end{aligned} \tag{6.32}$$

A collection $\{\widetilde{s}_k\}_{k\in\mathbb{Z}^{\geq 0}}$ of $\mathbb{S}_k$-equivariant sections of the vector bundles $\widetilde{E}_k$ such that

$$\widetilde{F}_{k;k'}\circ\widetilde{s}_k|_{\mathcal{N}^{k'}_{k;0}\iota} = \widetilde{s}_{k'}\circ\Psi_{k;k'}|_{\mathcal{N}^{k'}_{k;0}\iota} \quad \forall k,k'\in\mathbb{Z}^{\geq 0}, k'\leq k, \tag{6.33}$$

determines a section s on the corresponding vector bundle E in (6.32) so that

$$s([v]) = [\widetilde{s}_k(v)] \quad \forall k\in\mathbb{Z}^{\geq 0}, v\in\mathcal{N}^\circ_{k;0}\iota.$$

If $E'\to X$ is a (complex) vector bundle, the $\mathbb{S}_k$-action on $\mathcal{N}^\circ_{k;0}\iota$ lifts to an action on the vector bundle

$$\widetilde{\pi}'_k : \widetilde{E}'_k \equiv \Psi_k^* E' \longrightarrow \mathcal{N}^\circ_{k;0}\iota$$

so that $\widetilde{E}'_k/\mathbb{S}_k = E'|_{U^\circ_k}$. The covering maps (6.29) lift to $\mathbb{S}_{k;k'}$-equivariant $\mathbb{S}^c_{k;k'}$-invariant vector bundle maps

$$\widetilde{F}'_{k;k'} : \widetilde{E}'_k|_{\mathcal{N}^{k'}_{k;0}\iota} \longrightarrow \widetilde{E}'_{k'}|_{\mathcal{N}^{k}_{k';0}\iota}$$

that restrict to an isomorphism on each fiber and satisfy (6.31) with $\widetilde{F}$ replaced by $\widetilde{F}'$ so that the corresponding vector bundle (6.32) is canonically identified with E'. A collection

$$\{\widetilde{\Phi}_k : \widetilde{E}_k \longrightarrow \widetilde{E}'_k\}_{k\in\mathbb{Z}^{\geq 0}}$$

of $\mathbb{S}_k$-equivariant vector bundle homomorphisms covering the identity on $\mathcal{N}_{k;0}\iota$ such that

$$\widetilde{F}_{k;k'}\circ\widetilde{\Phi}_k|_{\mathcal{N}^{k'}_{k;0}\iota} = \widetilde{\Phi}_{k'}\circ\widetilde{F}_{k;k'}|_{\mathcal{N}^{k'}_{k;0}\iota} \quad \forall k,k'\in\mathbb{Z}^{\geq 0},\ k'\leq k, \tag{6.34}$$

determines a vector bundle homomorphism $\Phi : E\to E'$ covering the identity on X so that

$$\Phi([w]) = [\widetilde{\Phi}_k(w)] \quad \forall k\in\mathbb{Z}^{\geq 0},\ w\in\widetilde{E}_k.$$

For every $k \in \mathbb{Z}^{\geq 0}$, let

$$\begin{aligned}\widetilde{\pi}_k : \mathcal{O}_{\mathcal{R};k}(\iota) &= \left(\pi_{k;0}^* \bigotimes_{i\in[k]} \mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)}\right)\Big|_{\mathcal{N}_{k;0}^\circ\iota} \longrightarrow \mathcal{N}_{k;0}^\circ\iota,\\ \widetilde{\pi}_k : T_{\mathcal{R};k}(-\log\iota) &= (\pi_{k;0}^* T\widetilde{V}_\iota^{(k),\circ})|_{\mathcal{N}_{k;0}^\circ\iota} \oplus (\mathcal{N}_{k;0}^\circ\iota \times \mathbb{C}^{[k]}) \longrightarrow \mathcal{N}_{k;0}^\circ\iota.\end{aligned} \tag{6.35}$$

The complex structures $\mathfrak{i}_{k;i}$ on $\mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)}$ encoded in $\mathcal{R}$ determine a complex structure on the complex line bundle $\mathcal{O}_{\mathcal{R};k}(\iota)$. The almost complex structure $J_{\iota;k}$ on $\widetilde{V}_k$, induced by J, and the standard complex structure on $\mathbb{C}^{[k]}$ determine a complex structure on the vector bundle $T_{\mathcal{R};k}(-\log\iota)$. The $\mathbb{S}_k$-action on $\mathcal{N}_{k;0}^\circ\iota$ naturally lifts to both bundles.

Let $k, k' \in \mathbb{Z}^{\geq 0}$ with $k' \leq k$ and

$$\Pi_k : \mathcal{N}\widetilde{\iota}_{k;0} \longrightarrow \bigotimes_{i\in[k]} \mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)}, \quad \Pi_k((v_i)_{i\in[k]}) = \bigotimes_{i\in[k]} v_i.$$

We denote by $\nabla^{(k)}$ and $\nabla^{(k;k')}$ the connections on $\mathcal{N}\iota_k \approx \mathcal{N}_{k;0}\iota$ and $\mathcal{N}\widetilde{\iota}_{k;k'} \approx \mathcal{N}_{k;k'}\iota$ induced by the connections $\nabla^{(k;i)}$ on the direct summands of these vector bundles. We write an element $v \equiv (v_i)_{i\in[k]}$ of $\mathcal{N}_{k;0}\iota$ as

$$v = (v_{k;k'}, v_{k;k'}^c) \quad \text{with} \quad v_{k;k'} \equiv (v_i)_{i\in[k]-[k']} \in \mathcal{N}_{k;k'}\iota \ \text{ and } \ v_{k;k'}^c \equiv (v_i)_{i\in[k']} \in \mathcal{N}_{k;k'}^c\iota$$

below. Let

$$\mathrm{h}_{\nabla^{(k)};v} : T_{\pi_{k;0}(v)}\widetilde{V}_\iota^{(k)} \longrightarrow T_v(\mathcal{N}_{k;0}\iota) \quad \text{and} \quad \mathrm{h}_{\nabla^{(k;k')};v_{k;k'}} : T_{\pi_{k;k'}(v_{k;k'})}\widetilde{V}_\iota^{(k)} \longrightarrow T_{v_{k;k'}}(\mathcal{N}_{k;k'}\iota)$$

be the injective homomorphisms as in (2.6) corresponding to the connections $\nabla^{(k)}$ and $\nabla^{(k;k')}$.

By the first equation in (6.28),

$$v_i \neq 0 \quad \forall i \in [k]-[k'],\ (v_j)_{j\in[k]} \in \mathcal{N}_{k;0}^{k'}\iota \subset \bigoplus_{j\in[k]} \mathcal{N}\widetilde{\iota}_{k;k-1}^{(j)}.$$

Since $\mathfrak{D}\Psi_{k;k'}$ is a product Hermitian isomorphism, the map

$$\begin{aligned}\widetilde{\theta}_{k;k'} &: \mathcal{O}_{\mathcal{R};k}(\iota)|_{\mathcal{N}_{k;0}^{k'}\iota} \longrightarrow \mathcal{O}_{\mathcal{R};k'}(\iota)|_{\mathcal{N}_{k';0}^{k}\iota},\\ \widetilde{\theta}_{k;k'}(v, \Pi_k(v_{k;k'}, w_{k;k'}^c)) &= (\mathfrak{D}\Psi_{k;k'}(v), \Pi_{k'}(\mathfrak{D}\Psi_{k;k'}(v_{k;k'}, w_{k;k'}^c))),\end{aligned} \tag{6.36}$$

is a well-defined homomorphism of complex line bundles that lifts the covering map (6.29) and restricts to an isomorphism on each fiber. The map

$$\begin{aligned}\widetilde{\vartheta}_{k;k'} &: T_{\mathcal{R};k}(-\log\iota)|_{\mathcal{N}_{k;0}^{k'}\iota} \longrightarrow T_{\mathcal{R};k}(-\log\iota)|_{\mathcal{N}_{k';0}^{k}\iota},\\ \widetilde{\vartheta}_{k;k'}((v,w)\oplus(v,(c_i)_{i\in[k]})) &= \left(\mathfrak{D}\Psi_{k;k'}(v), d_{v_{k;k'}}\Psi_{k;k'}\left(\mathrm{h}_{\nabla^{(k;k')};v_{k;k'}}(w) + \sum_{i\in[k]-[k']} c_i v_i\right)\right)\\ &\qquad \oplus (\Psi_{k;k'}(v_{k;k'}), (c_i)_{i\in[k']}),\end{aligned} \tag{6.37}$$

is similarly a well-defined homomorphism of vector bundles that lifts the covering map (6.29) and restricts to an isomorphism on each fiber. Since J is an $\mathcal{R}$-compatible almost complex structure on X, this homomorphism is $\mathbb{C}$-linear.

By the commutativity of the diagrams in Figure 2, the bundle homomorphisms (6.36) and (6.37) are $\mathbb{S}_{k;k'}$-equivariant $\mathbb{S}_{k;k'}^c$-invariant. By (6.22), the collections $\{\widetilde{\theta}_{k;k'}\}_{k'\leq k}$ and

$\{\widetilde{\vartheta}_{k;k'}\}_{k'\le k}$ satisfy (6.31) with $\widetilde{F}$ replaced by $\widetilde{\theta}$ and $\widetilde{\vartheta}$. The first collection thus determines a complex line bundle

$$\pi : \mathcal{O}_{\mathcal{R};X}(\iota) = \Big(\bigsqcup_{k\in\mathbb{Z}^{\ge 0}} \mathcal{O}_{\mathcal{R};k}(\iota)/\mathbb{S}_k \Big) \Big/ \sim \longrightarrow X, \quad \pi([w]) = \Psi_k(\widetilde{\pi}_k(w)) \quad \forall k \in \mathbb{Z}^{\ge 0},\, w \in \mathcal{O}_{\mathcal{R};k}(\iota),$$
$$[w] \sim [\widetilde{\theta}_{k;k'}(w)] \quad \forall k, k' \in \mathbb{Z}^{\ge 0}, k' \le k, w \in \mathcal{O}_{\mathcal{R};k}(\iota)|_{\mathcal{N}^{k'}_{k;0}\iota}.$$

The second collection similarly determines a complex vector bundle

$$\pi : T_{\mathcal{R}}X(-\log\iota) = \Big(\bigsqcup_{k\in\mathbb{Z}^{\ge 0}} T_{\mathcal{R};k}(-\log\iota)/\mathbb{S}_k \Big) \Big/ \sim \longrightarrow X,$$
$$\pi([w]) = \Psi_k(\widetilde{\pi}_k(w)) \quad \forall k \in \mathbb{Z}^{\ge 0},\, w \in T_{\mathcal{R}}X(-\log\iota),$$
$$[w] \sim [\widetilde{\vartheta}_{k;k'}(w)] \quad \forall k, k' \in \mathbb{Z}^{\ge 0}, k' \le k, w \in T_{\mathcal{R}}X(-\log\iota)|_{\mathcal{N}^{k'}_{k;0}\iota}.$$

The smooth sections

$$\widetilde{s}_k : \mathcal{N}^{\circ}_{k;0}\iota \longrightarrow \mathcal{O}_{\mathcal{R};k}(\iota), \quad \widetilde{s}_k(v) = (v, \Pi_k(v)),$$

are $\mathbb{S}_k$-equivariant and satisfy (6.33) with $\widetilde{F}$ replaced by $\widetilde{\theta}$. They thus determine a section $s_{\mathcal{R}}$ of the complex line bundle $\mathcal{O}_{\mathcal{R};X}(\iota)$. The smooth bundle homomorphisms

$$\widetilde{\Phi}_k : T_{\mathcal{R};k}(-\log\iota) \longrightarrow \Psi_k^* TX, \quad \widetilde{\Phi}_k((v,w) \oplus (v,(c_i)_{i\in[k]})) = d_v\Psi_k\bigg(\mathrm{h}_{\nabla^{(k)};v}(w) + \sum_{i\in[k]} c_i v_i \bigg)$$

are $\mathbb{S}_k$-equivariant. By (6.29), these bundle homomorphisms satisfy (6.34) with $\widetilde{F}$ replaced by $\widetilde{\vartheta}$. They thus determine a vector bundle homomorphism

$$\iota_{\mathcal{R}} : T_{\mathcal{R}}X(-\log\iota) \longrightarrow TX.$$

6.4 Proof of Theorem 1.2(5)

Let (X,ω), $V \subset X$, $\iota : \widetilde{V} \to X$, $\mathcal{R}$, and J be as in Section 6.3. We denote the curvature 2-form of a connection ∇ on a complex vector bundle $E \to Y$ over a smooth manifold by

$$F^{\nabla} \in \Gamma(Y; \Lambda^2(T^*Y) \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{C}}(E)).$$

For $i \in \mathbb{Z}^{\ge 0}$, we define

$$c_i(\nabla) \in \Gamma(Y; \Lambda^{2i}(T^*Y) \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{C}}(E)) \quad \text{and} \quad c(\nabla) \in \bigoplus_{i=0}^{\infty} \Gamma(Y; \Lambda^{2i}(T^*Y) \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{C}}(E))$$
$$\text{by} \quad 1 + c_1(\nabla) + c_2(\nabla) + \cdots \equiv c(\nabla) \equiv \det{}_{\mathbb{C}}\bigg(I + \frac{\mathfrak{i}}{2\pi} F^{\nabla} \bigg).$$

By [29, p. 206],

$$[c^{\nabla}(E)] = c(E) \in H^{2*}_{\mathrm{deR}}(Y) \equiv \bigoplus_{i=0}^{\infty} H^{2i}_{\mathrm{deR}}(Y). \tag{6.38}$$

We compare $c(TX)$ and $c(TX(-\log V)) = c(TX(-\log\iota))$ at the level of differential forms, which can be done locally; see Proposition 6.11. This essentially reduces the computation to the SC case. The de Rham cohomology version of (1.5), which is equivalent to (1.5) itself, follows from (6.38), Lemma 6.10, and Proposition 6.11.

A connection ∇ on a complex line bundle $\pi : L \to Y$ determines a horizontal tangent subbundle $TL^{\rm hor} \subset TL$ and an $\mathbb{R}$-valued angular 1-form α_∇ on $L-Y$. The latter is characterized by

$$\ker \alpha_\nabla = (TL^{\rm hor} \oplus \mathbb{R}\zeta_L)|_{L-Y} \quad \text{and} \quad \alpha_\nabla\bigg(\frac{d}{d\theta}\mathrm{e}^{\mathrm{i}\theta}v\Big|_{\theta=0}\bigg) = 1.$$

By the proof of [38, Lemma 1.1],

$$d\alpha_\nabla = \pi^*\eta_\nabla|_{L-Y} \tag{6.39}$$

for some 2-form η_∇ on Y.

Remark 6.9 If ∇ is compatible with a Hermitian metric ρ on L, then the connection 1-form θ_1^1 in the proof of Lemma 2.2 is purely imaginary. In the local chart of this proof, the angular 1-form α_∇ of ∇ is then given by

$$\alpha_\nabla|_{(y,z)} = -\mathfrak{i}\frac{dz}{z} - \mathfrak{i}\pi^*\theta_1^1.$$

Thus, $\eta_\nabla = -\mathfrak{i}F^\nabla$ above.

Let $\beta : \mathbb{R} \to \mathbb{R}^{\ge 0}$ be as in (5.21). For a Hermitian metric ρ and a smooth function $\varepsilon : Y \to \mathbb{R}^+$, define

$$\beta_{\rho;\varepsilon} : L \longrightarrow \mathbb{R}^{\ge 0}, \quad \beta_{\rho;\varepsilon}(v) = \beta(2\varepsilon(\pi(v))^{-2}\rho(v)). \tag{6.40}$$

By (6.39), the 2-form

$$\tau_{\rho;\nabla;\varepsilon} \equiv -\frac{1}{2\pi}d(\beta_{\rho;\varepsilon}\alpha_\nabla) = -\frac{1}{2\pi}(d\beta_{\rho;\varepsilon}\wedge\alpha_\nabla + \beta_{\rho;\varepsilon}\pi^*\eta_\nabla)$$

is well-defined on the entire total space of L. This closed 2-form is compactly supported in the vertical direction and

$$\int_{\pi^{-1}(y)}\tau_{\rho;\nabla;\varepsilon} = -\frac{1}{2\pi}\bigg(\int_0^\infty d\beta_{\rho;\varepsilon}\bigg)\bigg(\int_{S^1}d\theta\bigg) = 1 \quad \forall y \in Y. \tag{6.41}$$

Thus, $\tau_{\rho;\nabla;\varepsilon}$ represents the Thom class of the complex line bundle $\pi : L \to Y$; see [5, p. 64].

For each $k \in \mathbb{Z}^+$, let $\varepsilon_k : \widetilde{V}^{(k)} \to \mathbb{R}^+$ be the composition of the function ε in (6.28) with ι_k. For each $i \in [k]$, let

$$\pi_{k;i} : \mathcal{N}\iota_k \approx \mathcal{N}_{k;0}\iota \longrightarrow \mathcal{N}\widetilde{\iota}_{k;k-1}^{(i)}$$

be the component projection. For $k' \in \mathbb{Z}^{\ge 0}$, we denote by $\widetilde{\tau}_{k;k'} \in \Omega^{2k'}(\mathcal{N}_{k;0})$ the k'-th elementary symmetric polynomial on the set $\{\pi_{k;i}^*\tau_{\rho_{k;i};\nabla^{(k;i)};2\varepsilon_k}\}_{i\in[k]}$, i.e.,

$$\widetilde{\tau}_{k;k'} = \sum_{\substack{i_1,\ldots,i_{k'}\in[k]\\ i_1<\cdots<i_{k'}}}(\pi_{k;i_1}^*\tau_{\rho_{k;i_1};\nabla^{(k;i_1)};2\varepsilon_k})\wedge\cdots\wedge(\pi_{k;i_{k'}}^*\tau_{\rho_{k;i_{k'}};\nabla^{(k;i_{k'})};2\varepsilon_k}).$$

This $2k'$-form on the total space of $\mathcal{N}\iota_k$ is $\mathbb{S}_k$-invariant and closed. The $2k$-form

$$\widetilde{\tau}_{k;k} \equiv \bigwedge_{i\in[k]}\pi_{k;i}^*\tau_{\rho_{k;i};\nabla^{(k;i)};2\varepsilon_k} \equiv (\pi_{k;1}^*\tau_{\rho_{k;1};\nabla^{(k;1)};2\varepsilon_k})\wedge\cdots\wedge(\pi_{k;k}^*\tau_{\rho_{k;k};\nabla^{(k;k)};2\varepsilon_k})$$

is in addition compactly supported in the vertical direction. By (6.41), it represents the Thom class of the complex vector bundle $\pi_{k;0} : \mathcal{N}\iota_k \to \widetilde{V}_\iota^{(k)}$.

For an open subset $U \subset X$ such that

$$\begin{aligned}&\Psi_k^{-1}(U) = \widetilde{U}_0 \sqcup \widetilde{U}_1 \sqcup \cdots \sqcup \widetilde{U}_\ell \quad \text{with}\\ &\widetilde{U}_0 \subset \mathcal{N}\iota_k - \overline{\mathcal{N}\iota_k(\varepsilon_k)} \quad \text{and} \quad \Psi_k : \widetilde{U}_i \longrightarrow U \ \text{ a diffeomorphism } \ \forall i \in [\ell],\end{aligned} \tag{6.42}$$

we define

$$\Psi_{k*}\widetilde{\tau}_{k;k}|_U = \sum_{i=1}^{\ell}\{\{\Psi_k|_{\widetilde{U}_i}\}^{-1}\}^*\widetilde{\tau}_{k;k} \in \Omega^{2k}(U;\mathbb{R}). \tag{6.43}$$

Since $\widetilde{\tau}_{k;k}$ vanishes on $\widetilde{U}_0$, $\Psi_{k*}\widetilde{\tau}_{k;k}|_U$ does not depend on an admissible decomposition of $\Psi_k^{-1}(U)$ as above. Thus, $\Psi_{k*}\widetilde{\tau}_{k;k}|_U$ is well-defined and

$$(\Psi_{k*}\widetilde{\tau}_{k;k}|_U)|_{U\cap U'} = \Psi_{k*}\widetilde{\tau}_{k;k}|_{U\cap U'} = (\Psi_{k*}\widetilde{\tau}_{k;k}|_{U'})|_{U\cap U'}$$

for all open subsets $U, U' \subset X$ with admissible decompositions. By Lemma 6.12 at the end of this section, such open subsets cover X. We thus obtain a closed $2k$-form $\Psi_{k*}\widetilde{\tau}_{k;k}$ on X. If μ is a closed form on X, then

$$\int_X (\Psi_{k*}\widetilde{\tau}_{k;k})\wedge\mu = \int_{\mathcal{N}\iota_k}\widetilde{\tau}_{k;k}\wedge(\Psi_k^*\mu) = \int_{\widetilde{V}^{(k)}}\iota_k^*\mu; \tag{6.44}$$

the first equality above holds for any differential form μ on X. The next straightforward lemma is also proved at the end of this section.

Lemma 6.10 *With the notation as above,*

$$\tau_k \equiv \frac{1}{k!}\Psi_{k*}\widetilde{\tau}_{k;k} = \mathrm{PD}_X([V_\iota^{(k)}]_X), \quad \Psi_k^*\tau_{k'}|_{\mathcal{N}^\circ_{k;0}} = \widetilde{\tau}_{k;k'}|_{\mathcal{N}^\circ_{k;0}} \quad \forall k,k' \in \mathbb{Z}^+. \tag{6.45}$$

Proposition 6.11 *There exist connections ∇ and ∇' in the complex vector bundles (TX,J) and $(T_{\mathcal{R}}X(-\log\iota),\mathfrak{i}_{\mathcal{R},J})$ so that*

$$c(\nabla) = c(\nabla')(1+\tau_1+\tau_2+\cdots). \tag{6.46}$$

Proof We construct ∇ and ∇' using the global perspective of Section 6.3. For $k \in \mathbb{Z}^{\geq 0}$ and $i \in [k]$, let

$$\beta_{k;i} \equiv \beta_{\rho_{k;i};\varepsilon_k} : \mathcal{N}\widetilde{\iota}^{(i)}_{k;k-1} \longrightarrow \mathbb{R}^{\geq 0}$$

be a smooth function as in (6.40) and

$$\Phi_{k;i} : (\mathcal{N}\widetilde{\iota}^{(i)}_{k;k-1} - \widetilde{V}_\iota^{(k)})\times\mathbb{C} \longrightarrow \pi^*\mathcal{N}\widetilde{\iota}^{(i)}_{k;k-1}, \quad \Phi_{k;i}(v,c) = cv,$$

where $\pi : \mathcal{N}\widetilde{\iota}^{(i)}_{k;k-1} \to \widetilde{V}_\iota^{(k)}$ is the bundle projection. We define a connection $\nabla'^{(k;i)}$ in the trivial complex line bundle $\mathcal{N}\widetilde{\iota}^{(i)}_{k;k-1}\times\mathbb{C}$ over $\mathcal{N}\widetilde{\iota}^{(i)}_{k;k-1}$ by

$$\nabla'^{(k;i)} = \beta_{k;i}d + (1-\beta_{k;i})\Phi_{k;i}^*\pi^*\nabla^{(k;i)};$$

the last summand above is well-defined because $\beta_{k;i} = 1$ in a neighborhood of the zero section $\widetilde{V}_\iota^{(k)} \subset \mathcal{N}\widetilde{\iota}^{(i)}_{k;k-1}$. We note that

$$\begin{aligned} F^{\nabla'^{(k;i)}} &= \beta_{k;i}F^d + (1-\beta_{k;i})F^{\Phi_{k;i}^*\pi^*\nabla^{(k;i)}} + d(1-\beta_{k;i})\wedge(\Phi_{k;i}^*\pi^*\nabla^{(k;i)} - d) \\ &= 0 + (1-\beta_{k;i})\pi^*F^{\nabla^{(k;i)}} - \mathfrak{i}\,d\beta_{k;i}\wedge\alpha_{\nabla^{(k;i)}} = \pi^*F^{\nabla^{(k;i)}} + 2\pi\mathfrak{i}\tau_{\rho_{k;i};\nabla^{(k;i)};\varepsilon_k}; \end{aligned} \tag{6.47}$$

the last two equalities follow from Remarks 2.4 and 6.9.

For each $k \in \mathbb{Z}^{\geq 0}$, the connections $\nabla^{(k;i)}$ on the complex line bundles $\mathcal{N}\widetilde{\iota}^{(i)}_{k;k-1}$ determine a splitting

$$T(\mathcal{N}\iota_k) = \pi_{k;0}^*T\widetilde{V}_\iota^{(k)} \oplus \bigoplus_{i=1}^{k}\pi_{k;0}^*\mathcal{N}\widetilde{\iota}^{(i)}_{k;k-1}. \tag{6.48}$$

Let $r \in \mathbb{Z}^+$ be such that $V_\iota^{(r+1)} = \emptyset$. By Definition 6.7, the vector bundle isomorphisms (6.20) are product Hermitian isomorphisms for all $k' \le k$. Furthermore, $\varepsilon_k = \varepsilon_{k'} \circ \Psi_{k;k'}$. Starting with a J-compatible connection on $TV_\iota^{(r)}$ and possibly shrinking the domains of the regularizations Ψ_k, we can thus inductively construct a connection ∇ on TX so that $\{d\Psi_k\}^*\nabla$ agrees with the restriction of the connection

$$\widetilde{\nabla}^{(k)} \equiv \pi_{k;0}^* \nabla^{T\widetilde{V}_\iota^{(k)}} \oplus \bigoplus_{i=1}^{k} (\pi_{k;0}^* \nabla^{(k;i)} + \mathfrak{i}(\beta_{\rho_{k;i};2\varepsilon_k} - \beta_{\rho_{k;i};\varepsilon_k})\alpha_{\nabla^{(k;i)}})$$

on (6.48) to $\mathrm{Dom}(\Psi_k)$ for every $k \in \mathbb{Z}^+$. By Remark 6.9, the curvature of the last summand $\widetilde{\nabla}^{(k;i)}$ above satisfies

$$\begin{aligned} 1 + \frac{\mathfrak{i}}{2\pi} F^{\widetilde{\nabla}^{(k;i)}} &= \left(1 + \frac{\mathfrak{i}}{2\pi}\pi_{k;0}^* F^{\nabla^{(k;i)}} - \tau_{\rho_{k;i};\nabla^{(k;i)};\varepsilon_k}\right)(1 + \pi_{k;i}^* \tau_{\rho_{k;i};\nabla^{(k;i)};2\varepsilon_k}) \\ &\quad - \frac{1}{4\pi} d((1 - \beta_{\rho_{k;i};\varepsilon_k})\alpha_{\nabla^{(k;i)}}) \wedge d(\beta_{\rho_{k;i};2\varepsilon_k}\alpha_{\nabla^{(k;i)}}); \end{aligned}$$

the term on the last line above vanishes because $\beta_{\rho_{k;i};\varepsilon_k} \equiv 1$ on $\mathrm{supp}\,\beta_{\rho_{k;i};2\varepsilon_k}$. Along with (6.47), this gives

$$\begin{aligned} \Psi_k^* c(\nabla)|_{\mathcal{N}^\circ_{k;0}\iota} &= \pi_{k;0}^* c(\nabla^{T\widetilde{V}_\iota^{(k)}})|_{\mathcal{N}^\circ_{k;0}\iota} \prod_{i=1}^{k} (\pi_{k;0}^* c(\nabla'^{(k;i)})(1 + \pi_{k;i}^* \tau_{\rho_{k;i};\nabla^{(k;i)};2\varepsilon_k}))|_{\mathcal{N}^\circ_{k;0}\iota} \\ &= \pi_{k;0}^* c(\nabla^{T\widetilde{V}_\iota^{(k)}})|_{\mathcal{N}^\circ_{k;0}\iota} \left(\prod_{i=1}^{k} \pi_{k;0}^* c(\nabla'^{(k;i)})\right) \Psi_k^*(1 + \tau_1 + \tau_2 + \cdots)|_{\mathcal{N}^\circ_{k;0}\iota}; \end{aligned} \tag{6.49}$$

the last equality follows from the second statement of Lemma 6.10.

By (6.24), the connection

$$\widetilde{\nabla}'^{(k)} \equiv \pi_{k;0}^* \nabla^{T\widetilde{V}_\iota^{(k)}}|_{\mathcal{N}^\circ_{k;0}\iota} \oplus \bigoplus_{i=1}^{k} \pi_{k;0}^* \widetilde{\nabla}'^{(k;i)}|_{\mathcal{N}^\circ_{k;0}\iota}$$

on the complex vector bundle (6.35) is $\mathbb{S}_k$-equivariant. Since $\varepsilon_k = \varepsilon_{k'} \circ \Psi_{k;k'}$ and $\mathfrak{D}\Psi_{k;k'}$ is a product Hermitian isomorphisms, the bundle isomorphisms (6.24) intertwine these connections. Thus, they determine a connection ∇' on the complex vector bundle $T_{\mathcal{R}}X(-\log\iota)$. Since

$$\Psi_k^* c(\nabla')|_{\mathcal{N}^\circ_{k;0}\iota} = c(\widetilde{\nabla}'^{(k)}) = \pi_{k;0}^* c(\nabla^{T\widetilde{V}_\iota^{(k)}})|_{\mathcal{N}^\circ_{k;0}\iota} \prod_{i=1}^{k} \pi_{k;0}^* c(\nabla'^{(k;i)}),$$

the identity (6.46) follows from (6.49). □

Lemma 6.12 *Let $k \in \mathbb{Z}^+$. Every point $x \in X$ admits a neighborhood U as in* (6.42).

Proof Let $\Psi_k^{-1}(x) = \{\widetilde{w}_1, \ldots, \widetilde{w}_\ell\}$. For each $i \in [\ell]$, let $U_i \subset \widetilde{V}_\iota^{(k)}$ be a neighborhood of $\pi_{k;0}(\widetilde{w}_i)$ so that the restriction

$$\Psi_k : \mathrm{Dom}(\Psi_k)|_{U_i} \longrightarrow X$$

is a diffeomorphism onto its image; see Definition 6.3. By Remark 6.8,

$$W \equiv X - \Psi_k(\overline{\mathcal{N}\iota_k(\varepsilon_k)}|_{\widetilde{V}_\iota^{(k)} - U_1 \cup \ldots \cup U_\ell})$$

is an open subspace of X. The open neighborhood

$$U \equiv W \cap \bigcap_{i=1}^{\ell} \Psi_k(\mathrm{Dom}(\Psi_k)|_{U_i})$$

of $x \in X$ then satisfies the condition in (6.42). □

Proof of Lemma 6.10 Let μ be a closed form on X. We show that

$$\int_{V_\iota^{(k)}} \alpha = \int_X \tau_k \wedge \mu. \tag{6.50}$$

Since $\iota_k : \widetilde{V}^{(k)} \to V^{(k)}$ is a $k!$-covering map outside of a codimension 2 subspace,

$$\int_{V_\iota^{(k)}} \alpha = \frac{1}{k!} \int_{\widetilde{V}^{(k)}} \iota_k^* \alpha.$$

Combining this with (6.44), we obtain (6.50) and establish the first statement in (6.45).

If $k' > k$, the left-hand side of the second identity in (6.45) vanishes because

$$\operatorname{supp} \tau_{k'} \subset \Psi_{k'}(\overline{\mathcal{N}_{k;0}(\varepsilon_k/2)}) \quad \text{and} \quad \Psi_{k'}(\overline{\mathcal{N}_{k;0}(\varepsilon_k/2)}) \cap \Psi_k(\mathcal{N}_{k;0}^\circ) = \emptyset.$$

The right-hand side of the second identity in (6.45) vanishes by definition in this case.

Suppose $k' \le k$. Let $S_{k;k'} \subset \mathbb{S}_k$ be a collection of representatives for the right cosets of $\mathbb{S}_{k;k'} \times \mathbb{S}_{k;k'}^c$ in $\mathbb{S}_k$ preserving the order of the first k' elements. Since (6.26) is an $\mathbb{S}_k$-covering map, every point of U_k° has a neighborhood U so that

$$\Psi_k^{-1}(U) = \bigsqcup_{\sigma \in \mathbb{S}_k} \sigma(W) \subset \mathcal{N}_{k;0}^\circ \iota$$

for some open subset $W \subset \mathcal{N}_{k;0}^\circ \iota$. Since $\Psi_k = \Psi_{k'} \circ \mathfrak{D}\Psi_{k;k'}$ and $\mathfrak{D}\Psi_{k;k'}$ is $\mathbb{S}_{k;k'}^c$-invariant, (6.18) implies that

$$\Psi_{k'}^{-1}(U) = \bigcup_{\varpi \in \mathbb{S}_{k;k'}} \bigsqcup_{\sigma \in S_{k;k'}} \Psi_{k;k'}(\varpi\sigma(W)).$$

Since $\mathfrak{D}\Psi_{k;k'}$ is $\mathbb{S}_{k;k'}$-equivariant and $\widetilde{\tau}_{k';k'}$ is $\mathbb{S}_{k'}$-invariant,

$$\Psi_k^* \tau_{k'} = \frac{1}{k!} \sum_{\sigma \in S_{k;k'}} \sum_{\varpi \in \mathbb{S}_{k'}} \{D\sigma\}^* \{\mathfrak{D}\Psi_{k;k'}\}^* \{D\varpi\}^* \widetilde{\tau}_{k';k'} = \sum_{\sigma \in S_{k;k'}} \{D\sigma\}^* \{\mathfrak{D}\Psi_{k;k'}\}^* \widetilde{\tau}_{k';k'}.$$

Since $\epsilon_k = \epsilon_{k'} \circ \Psi_{k;k'}$ and $\mathfrak{D}\Psi_{k;k'}$ is a product Hermitian isomorphism, it follows that

$$\Psi_k^* \tau_{k'} = \sum_{\sigma \in S_{k;k'}} \{D\sigma\}^* \left(\bigwedge_{i \in [k']} \pi_{k;i}^* \tau_{\rho_{k;i}; \nabla^{(k;i)}; 2\varepsilon_k} \right).$$

Since $D\sigma$ is also a product Hermitian isomorphism, the last expression equals $\widetilde{\tau}_{k;k'}$. □

Remark 6.13 If X and $V_\iota^{(k)}$ are not compact, the above proof of the first identity in (6.45) goes through if μ is compactly supported. If X is not compact, but $V_\iota^{(k)}$ is compact, then this identity holds in the compactly de Rham cohomology of X, as well as in the usual de Rham cohomology of X.

7 On the Sharpness of (1.5) and (1.9)

We conclude by establishing the remaining statement of Corollary 1.4 and showing that (1.9) does not need to hold in $H^*(\widetilde{X}; \mathbb{Z})$ for arbitrary NC divisors. The latter implies that (1.5) does not hold in $H^*(X; \mathbb{Z})$ for arbitrary NC divisors either.

Continuing with the notation and setup as in Lemma 1.3, we denote by

$$\widetilde{\iota} : \mathbb{E} \longrightarrow \widetilde{X} \quad \text{and} \quad \widetilde{\pi}_0 : \gamma \longrightarrow \mathbb{E}$$

the inclusion map and the tautological line bundle, respectively. For $\mu \in H_*(V^{(r)};R)$, we denote by $\mathbb{E}|_\mu \in H_*(\mathbb{E};R)$ the fiber product of μ with $\pi|_{\mathbb{E}}$.

Lemma 7.1 *If R is a commutative ring with unity,*

$$\ker(\pi_*: H_*(\widetilde{X};R) \longrightarrow H_*(X;R)) = \widetilde{\iota}_*(\ker(\pi_*: H_*(\mathbb{E};R) \longrightarrow H_*(V^{(r)};R))) \quad \textit{and} \tag{7.1}$$

$$\ker(\pi_*: H_*(\mathbb{E};R) \longrightarrow H_*(V^{(r)};R)) = \bigoplus_{i=1}^{r-1}\{c_1(\gamma)^{r-1-i} \cap (\mathbb{E}|_{\mu_i}) : \mu_i \in H_{*-2i}(V^{(r)};R)\}. \tag{7.2}$$

Proof We omit the coefficient ring R below. Let $U \subset X$ be an open neighborhood of $Y \equiv V^{(r)}$ which deformation retracts onto Y and $\widetilde{U} \subset \pi^{-1}(U)$. Since $\mathbb{E}$ is a deformation retract of $\widetilde{U}$, the Mayer–Vietoris sequences for $\widetilde{X} = (\widetilde{X} - \mathbb{E}) \cup \widetilde{U}$ and $X = (X - Y) \cup U$ induce a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_*(\widetilde{U} - \mathbb{E}) & \longrightarrow & H_*(\widetilde{X} - \mathbb{E}) \oplus H_*(\mathbb{E}) & \longrightarrow & H_*(\widetilde{X}) & \longrightarrow & H_{*-1}(\widetilde{U} - \mathbb{E}) \longrightarrow \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\oplus\pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \mathrm{id}} \\
\cdots & \longrightarrow & H_*(U - Y) & \longrightarrow & H_*(X - Y) \oplus H_*(Y) & \longrightarrow & H_*(X) & \longrightarrow & H_{*-1}(U - Y) \longrightarrow \cdots
\end{array}$$

of exact sequences of R-modules. This gives (7.1).

For every $y \in Y$, the collection $\{1|_{\mathbb{E}_y}, c_1(\gamma)|_{\mathbb{E}_y}, \ldots, c_1(\gamma)^{r-1}|_{\mathbb{E}_y}\}$ is a basis for $H^*(\mathbb{E}_y)$. By [34, Theorem 5.7.9], the homomorphism

$$H_*(\mathbb{E}) \longrightarrow \bigoplus_{i=0}^{r-1} H_{*-2i}(Y), \quad \widetilde{\mu} \longrightarrow (\pi_*(c_1(\gamma)^i \cap \widetilde{\mu})_{i=0,\ldots,r-1}),$$

is thus an isomorphism. This implies (7.2). □

For each $k \in [r-1]$, let

$$\widetilde{\eta}_k = \pi^*(\mathrm{PD}_X([V^{(k)}]_X)) - \mathrm{PD}_{\widetilde{X}}([\overline{V}^{(k)}]_{\widetilde{X}}) \in H^{2k}(\widetilde{X};R).$$

By Lemma 7.1, there exist $\eta^c_{k;i} \in H^{2(k-i)}(V^{(r)};R)$ with $i \in [k]$ so that

$$\widetilde{\eta}_k \cap [\widetilde{X}] = \sum_{i=1}^{k} \widetilde{\iota}_*(c_1(\gamma)^{i-1} \cap (\mathbb{E}|_{\eta^c_{k;i} \cap [V^{(r)}]})) \in H_*(\widetilde{X};R). \tag{7.3}$$

For example,

$$(\pi^*(\mathrm{PD}_X([V^{(1)}]_X))) \cap [\widetilde{X}] = [\overline{V}^{(1)}]_{\widetilde{X}} + r[\mathbb{E}]_{\widetilde{X}} \quad \text{if } r \geq 2; \tag{7.4}$$

the coefficient $\eta^c_{1;1} = r$ above is obtained by intersecting both sides with $\widetilde{\iota}_*(c_1(\gamma)^{r-2} \cap \mathbb{E}_y)$.

We set

$$\mathcal{PD}(V) = 1 + \mathrm{PD}_X([V^{(1)}]_X) + \mathrm{PD}_X([V^{(2)}]_X) + \cdots \in \bigoplus_{i=0}^{\infty} H^{2i}(X;R),$$

$$\eta^c_i = \eta^c_{i;i} + \cdots + \eta^c_{r-1;i} \in \bigoplus_{k=0}^{\infty} H^{2k}(V^{(r)};R) \quad \forall i \in [r-1], \quad \eta^c_r = 1.$$

The identity (1.9) is equivalent to

$$1 - \frac{c(T\widetilde{X})}{\pi^* c(TX)(1 + \mathrm{PD}_{\widetilde{X}}([\mathbb{E}]_{\widetilde{X}}))} = \sum_{i=1}^{r-1} \mathrm{PD}_{\widetilde{X}}(\widetilde{\iota}_*(c_1(\gamma)^{i-1} \cap (\mathbb{E}|_{\eta^c_i \cap \mathcal{PD}(V)^{-1} \cap [V^{(r)}]})))$$

$$+\,\mathrm{PD}_{\widetilde{X}}(\pi^*\mathrm{PD}_X(\iota_*(\mathcal{PD}(V)^{-1}\cap[V^{(r)}]))), \tag{7.5}$$

where $\iota: V^{(r)}\to X$ is the inclusion. Via (7.4), (7.5) in particular gives

$$c_1(T\widetilde{X}) = \pi^*c_1(TX) - (r-1)\mathrm{PD}_{\widetilde{X}}([\mathbb{E}]_{\widetilde{X}}) \quad \text{if } r\ge 2; \tag{7.6}$$

$$c_2(T\widetilde{X}) = \pi^*c_2(TX) - \mathrm{PD}_{\widetilde{X}}(\widetilde{\iota}_*((\pi^*c_1(TX)+2c_1(\gamma))\cap[\mathbb{E}]) + \pi^*(\mathrm{PD}_X[V^{(2)}]_X) - 2\mathbb{E}|_{\mathrm{PD}_X([V^{(1)}]_X)\cap[V^{(2)}]}) \quad \text{if } r=2. \tag{7.7}$$

With the dimension of X and the codimension of the blowup locus $V^{(r)}$ fixed, the left-hand side of (7.5) must be of the form

$$\sum_{i=1}^{r-1}\mathrm{PD}_{\widetilde{X}}(\widetilde{\iota}_*(c_1(\gamma)^{i-1}\cap(\mathbb{E}|_{P_i(c_1(\mathcal{N}_XV^{(r)}),\ldots,c_r(\mathcal{N}_XV^{(r)}))\cap[V^{(r)}]})))$$
$$+\,\mathrm{PD}_{\widetilde{X}}(\pi^*\mathrm{PD}_X(\iota_*(P_r(c_1(\mathcal{N}_XV^{(r)}),\ldots,c_r(\mathcal{N}_XV^{(r)}))\cap[V^{(r)}])))$$

for some universal polynomials $P_1,\ldots,P_r\in\mathbb{Z}[c_1,\ldots,c_r]$. Since (1.9) holds with $\mathbb{Z}$-coefficients in the SC case, the right-hand side of (7.5) reduces to the above form in this case. For example,

$$\mathrm{PD}_X([V^{(k)}]_X)\cap[V^{(r)}] = c_k(\mathcal{N}_XV^{(r)})\cap[V^{(r)}]\in H_*(V^{(r)};\mathbb{Z}) \tag{7.8}$$

in the SC case. This also occurs if the branches of V at $V^{(r)}$ can be distinguished globally, i.e., $\mathcal{N}_XV^{(r)}$ splits into r subbundles which restrict to the subbundles $\mathcal{N}_{V_{y;I-i}}V_{y;I}$ with $i\in I$ for every chart $(U_y,\{V_{y;i}\}_{i\in S_y})$ as in Definition 4.1 and every $I\subset S_y$ with $|S_y|=r$.

Since (1.9) holds with $\mathbb{Q}$-coefficients in general, the differences

$$\eta_i^c\cup\mathcal{PD}(V)^{-1} - P_i(c_1(\mathcal{N}_XV^{(r)}),\ldots,c_r(\mathcal{N}_XV^{(r)}))\in H_*(V^{(r)};\mathbb{Z})$$

are torsion. If the torsion in $H_*(\mathbb{E};\mathbb{Z})$ lies in the kernel $\widetilde{\iota}_*$, then the torsion in $H_*(V^{(r)};\mathbb{Z})$ lies in the kernel ι_*. In such a case,

$$\widetilde{\iota}_*(\mathbb{E}_{\eta_i^c\cap\mathcal{PD}(V)^{-1}\cap[V^{(r)}]}) = \widetilde{\iota}_*(\mathbb{E}_{P_i(c_1(\mathcal{N}_XV^{(r)}),\ldots,c_r(\mathcal{N}_XV^{(r)}))\cap[V^{(r)}]})\in H_*(\widetilde{X};\mathbb{Z}),$$
$$\iota_*(\mathcal{PD}(V)^{-1}\cap[V^{(r)}]) = \iota_*(P_r(c_1(\mathcal{N}_XV^{(r)}),\ldots,c_r(\mathcal{N}_XV^{(r)}))\cap[V^{(r)}])\in H_*(X;\mathbb{Z}).$$

This implies that (1.9) holds in $H^*(\widetilde{X};\mathbb{Z})$ if the torsion in $H_*(V^{(r)};\mathbb{Z})$ lies in the kernel ι_*.

We now give an example in which (1.9) does not hold in $H^*(\widetilde{X};\mathbb{Z})$. Let Σ be a compact connected Riemann surface and $z^*\in\Sigma$. Let $\widetilde{S}$ be a K3 surface and $\psi\in\mathrm{Aut}(\widetilde{S})$ be a fixed-point-free involution so that $S\equiv\widetilde{S}/\psi$ is an Enriques surface. Define

$$\widetilde{\psi}:\widetilde{X}\equiv\widetilde{S}\times\Sigma^2\longrightarrow\widetilde{S},\quad \widetilde{\psi}(y,z_1,z_2)=(\psi(y),z_2,z_1),$$
$$X=\widetilde{X}/\widetilde{\psi},\quad \widetilde{V}=\widetilde{S}\times\Sigma\times\{z^*\}\subset\widetilde{X}.$$

The image $V\subset X$ of $\widetilde{V}$ under the quotient map $q:\widetilde{X}\to X$ is an NC divisor in X. Its 2-fold locus $V^{(2)}\approx S$ is the image of $\widetilde{S}\times\{z^*\}^2$ under q. Since the 3-fold locus of V is empty, $r=2$ in this case. We show below that

$$\iota_*(\mathrm{PD}_X([V^{(1)}]_X)\cap[V^{(2)}]) = 0\in H_2(X;\mathbb{Z}),\quad c_1(\mathcal{N}_XV^{(2)})\neq 0\in H^2(V^{(2)};\mathbb{Z}), \tag{7.9}$$

and the torsion of $H_2(V^{(2)};\mathbb{Z})$ does not vanish under ι_*. This implies that the last term in (7.7) is not determined by the Chern class of $\mathcal{N}_XV^{(2)}$. Thus, (7.8), (7.7), (1.9), and (1.5) do not hold with $\mathbb{Z}$ coefficients in this case.

The image of $\widetilde{S}\times\{z\}^2$ under q is homologous to $V^{(2)}$ for any $z\in\Sigma$. Since it is disjoint from V for $z\neq z^*$,

$$\iota_*(\mathrm{PD}_X([V^{(1)}]_X)\cap[V^{(2)}])=\mathrm{PD}_X([V^{(1)}]_X)\cap[V^{(2)}]_X=0.$$

This confirms the first statement in (7.9).

The normal bundle $\mathcal{N}_XV^{(2)}$ of $V^{(2)}$ in X is the quotient of the trivial bundle $\widetilde{S}\times\mathbb{C}^2$ by the involution

$$\widetilde{S}\times\mathbb{C}^2\longrightarrow\widetilde{S}\times\mathbb{C}^2,\quad (y,c_1,c_2)\longrightarrow(\psi(y),c_2,c_1).$$

Thus, $\mathcal{N}_XV^{(2)}\approx L_+\oplus L_-$, where

$$L_\pm=\{[y,c,\pm c]: y\in\widetilde{S},\, c\in\mathbb{C}\}\subset\mathcal{N}_XV^{(2)}.$$

The complex line bundle L_+ over S is isomorphic to $S\times\mathbb{C}$. The flat complex line bundle L_- corresponds to the non-trivial homomorphism

$$\pi_1(S)=\mathbb{Z}_2\longrightarrow S^1.$$

This confirms the second statement in (7.9).

Let $f_+:\Sigma_+\to S$ be a smooth map from a compact Riemann surface representing the unique torsion element of $H_2(S;\mathbb{Z})$. Let $f_-:\Sigma_-\to S$ be a smooth map from a compact (unorientable) surface representing a class in $H_2(S;\mathbb{Z}_2)$ so that $[f_+]_{\mathbb{Z}_2}\cdot_S[f_-]_{\mathbb{Z}_2}\neq 0\in\mathbb{Z}_2$, where $\cdot_S$ denotes the $\mathbb{Z}_2$-intersection product on S. The involution $\widetilde{\psi}$ pulls back to a smooth involution

$$\widetilde{\psi}_-:\widetilde{Z}_-\equiv\{(x,y)\in\Sigma_-\times\widetilde{S}: f_-(x)=q(y)\}\times\Sigma^2\longrightarrow\widetilde{Z}_-,\quad \widetilde{\psi}_-(x,y,z_1,z_2)=(x,\psi(y),z_2,z_1).$$

The map $F_-:Z_-\equiv\widetilde{Z}_-/\widetilde{\psi}_-\longrightarrow X,\quad F_-([x,y,z_1,z_2])=[y,z_1,z_2]$, is smooth and determines an element of $H_6(X;\mathbb{Z})$ so that

$$\iota_*([f_+]_{\mathbb{Z}_2})\cdot_X[F_-]_{\mathbb{Z}_2}=[f_+]_{\mathbb{Z}_2}\cdot_S[f_-]_{\mathbb{Z}_2}\neq 0\in\mathbb{Z}_2.$$

This confirms the claim just after (7.9).

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