



PAPER

Bounds on eigenfunctions of quantum cat maps

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Elena Kim¹  and Robert Koirala²¹ Massachusetts Institute of Technology, Department of Mathematics, Cambridge, MA 02142, United States of America² University of California San Diego, Department of Mathematics, La Jolla, CA 92093, United States of AmericaE-mail: elenakim@mit.edu and rkoirala@ucsd.edu**Keywords:** semiclassical analysis, quantum chaos, cat map**Abstract**

We study ℓ^∞ norms of ℓ^2 -normalized eigenfunctions of quantum cat maps. For maps with short quantum periods (constructed by Bonechi and de Bièvre in F Bonechi and S De Bièvre (2000, *Communications in Mathematical Physics*, **211**, 659–686)) we show that there exists a sequence of eigenfunctions u with $\|u\|_\infty \gtrsim (\log N)^{-1/2}$. For general eigenfunctions we show the upper bound $\|u\|_\infty \lesssim (\log N)^{-1/2}$. Here the semiclassical parameter is $h = (2\pi N)^{-1}$. Our upper bound is analogous to the one proved by Bérard in P Bérard (1977, *Mathematische Zeitschrift*, **155**, 249–276) for compact Riemannian manifolds without conjugate points.

1. Introduction

In this paper, we build on an existing body of work that examines the extreme values of eigenfunctions of classically chaotic quantum systems. Specifically, we examine the quantum cat map, one of the best-studied models. First introduced by Berry and Hannay in [1], cat maps are the quantization of hyperbolic linear maps in $SL(2, \mathbb{Z})$ on the 2-dimensional torus.

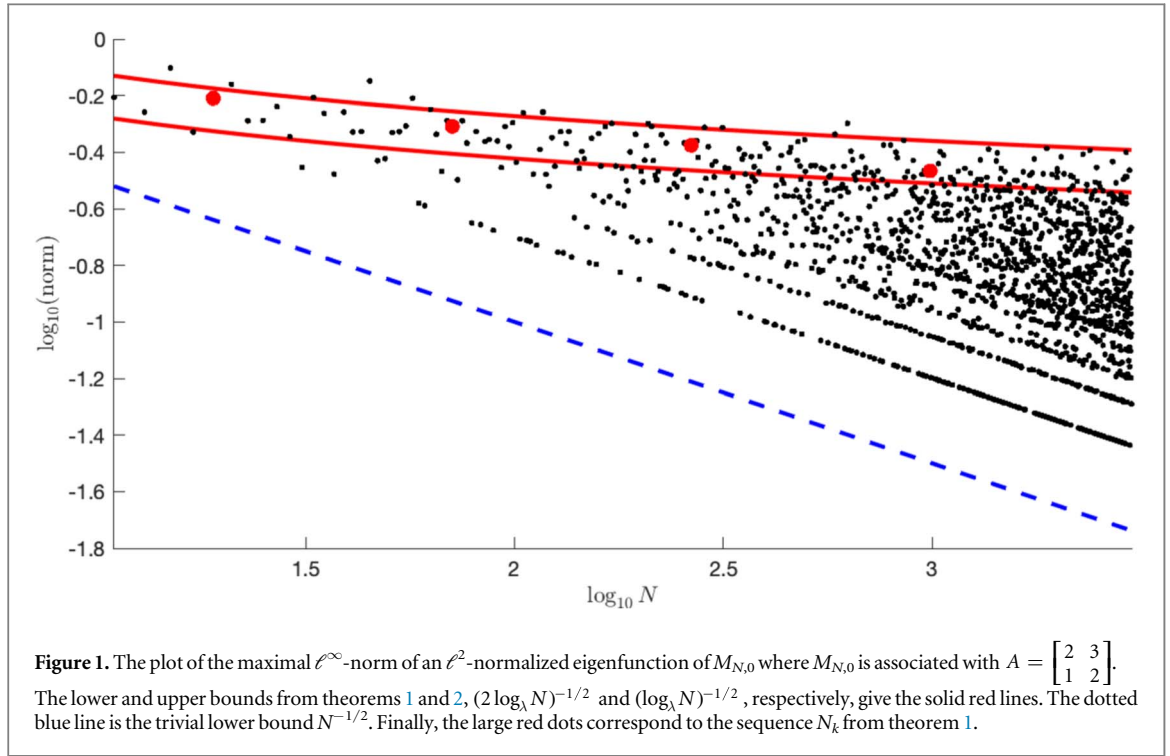
The quantum cat map is a toy model in quantum chaos. Another standard class of quantum chaos models is given by Laplacian eigenfunctions on negatively curved compact manifolds (M, g) , satisfying $-\Delta_g u = \lambda^2 u$ and normalized so that $\|u\|_{L^2} = 1$. Estimating the suprema of these Laplacian eigenfunctions has been an area of great interest. For example, Rudnick and Sarnak in [2] showed that on hyperbolic 3-manifolds, there exists a sequence of eigenstates u_k such that $\|u_k\|_{L^\infty} \gg \lambda^{1/4}$. In regard to upper bounds, Levitan, Avakumović and Hörmander in [3–5], respectively, proved the well-known result that on a compact Riemannian manifold, $\|u\|_{L^\infty} \lesssim \lambda^{\frac{n-1}{2}}$ where $n = \dim M$. However, it is conjectured that much stronger results hold. Specifically, in [6], Iwaniec and Sarnak conjectured that for surfaces of constant negative curvature, $\|u\|_{L^\infty} \lesssim_\varepsilon \lambda^\varepsilon$ for all $\varepsilon > 0$; in the special case of Hecke eigenfunctions on arithmetic surfaces they obtained the bound $\|u\|_{L^\infty} \lesssim_\varepsilon \lambda^{\frac{5}{12} + \varepsilon}$. The best known bound outside of the arithmetic cases is the result of Bérard [7]: when (M, g) has no conjugate points, $\|u\|_{L^\infty} = O(\lambda^{\frac{n-1}{2}} / \sqrt{\log \lambda})$.

In this paper, we are concerned with metaplectic transformations, the quantizations of hyperbolic maps

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z}), \quad |a + d| > 2, \quad ab, cd \in 2\mathbb{Z}. \quad (1)$$

We decompose $L^2(\mathbb{R}^n)$ into a direct integral of finite-dimensional spaces $\mathcal{H}_N(\theta)$, where $\theta \in \mathbb{T}^2$ and $N \in \mathbb{N}$. As shown in section 2, the condition that ab, cd be even is needed to make sure that the metaplectic map associated to A descends to a map from $\mathcal{H}_N(0)$ to itself. We denote the resulting quantum cat map by $M_{N,0}$. An explicit basis for each $\mathcal{H}_N(\theta)$ is given by lemma 3. For $u \in \mathcal{H}_N(\theta)$, we use $\|u\|_{\ell^p}$ to denote the standard ℓ^p norms applied to coefficients of this basis. We prove theorem 1 and theorem 2, bounds on the ℓ^∞ norm of eigenfunctions on $\mathcal{H}_N(0)$. See figure 1 for a numerical illustration. Note that due to the normalization in the spaces ℓ^2 and ℓ^∞ , the standard bound on Laplacian eigenfunctions $\|u\|_{L^2} \lesssim \|u\|_{L^\infty} \lesssim \lambda^{\frac{n-1}{2}} \|u\|_{L^2}$ becomes the bound $\frac{1}{\sqrt{N}} \|u\|_{\ell^2} \leq \|u\|_{\ell^\infty} \leq \|u\|_{\ell^2}$.

In [8], Bonechi and De Bièvre prove that for each A , there exists a sequence of $M_{N,0}$ with ‘short’ periods. Faure, Nonnenmacher, and Bièvre use this result in [9] to show that there exists a sequence of eigenfunctions



that are localized, as quantified by semiclassical measures. We also utilize [8] to show there exists a sequence of $M_{N,0}$ with localized eigenfunctions, as demonstrated by the following lower bound.

Theorem 1. Suppose A is a matrix satisfying (1) with positive eigenvalues, even trace, and coprime off-diagonal terms. Then we can find a sequence of odd integers $N_k \rightarrow \infty$ such that for all $\varepsilon > 0$, there exists k_0 such that for all $k \geq k_0$, there exists an eigenfunction u of $M_{N,0}$ with $\|u\|_{\ell^2} = 1$ and

$$\|u\|_{\ell^\infty} \geq \frac{1 - \varepsilon}{\sqrt{2 \log_\lambda N_k}}. \quad (2)$$

Under more general assumptions, for all odd N , we have the following upper bound.

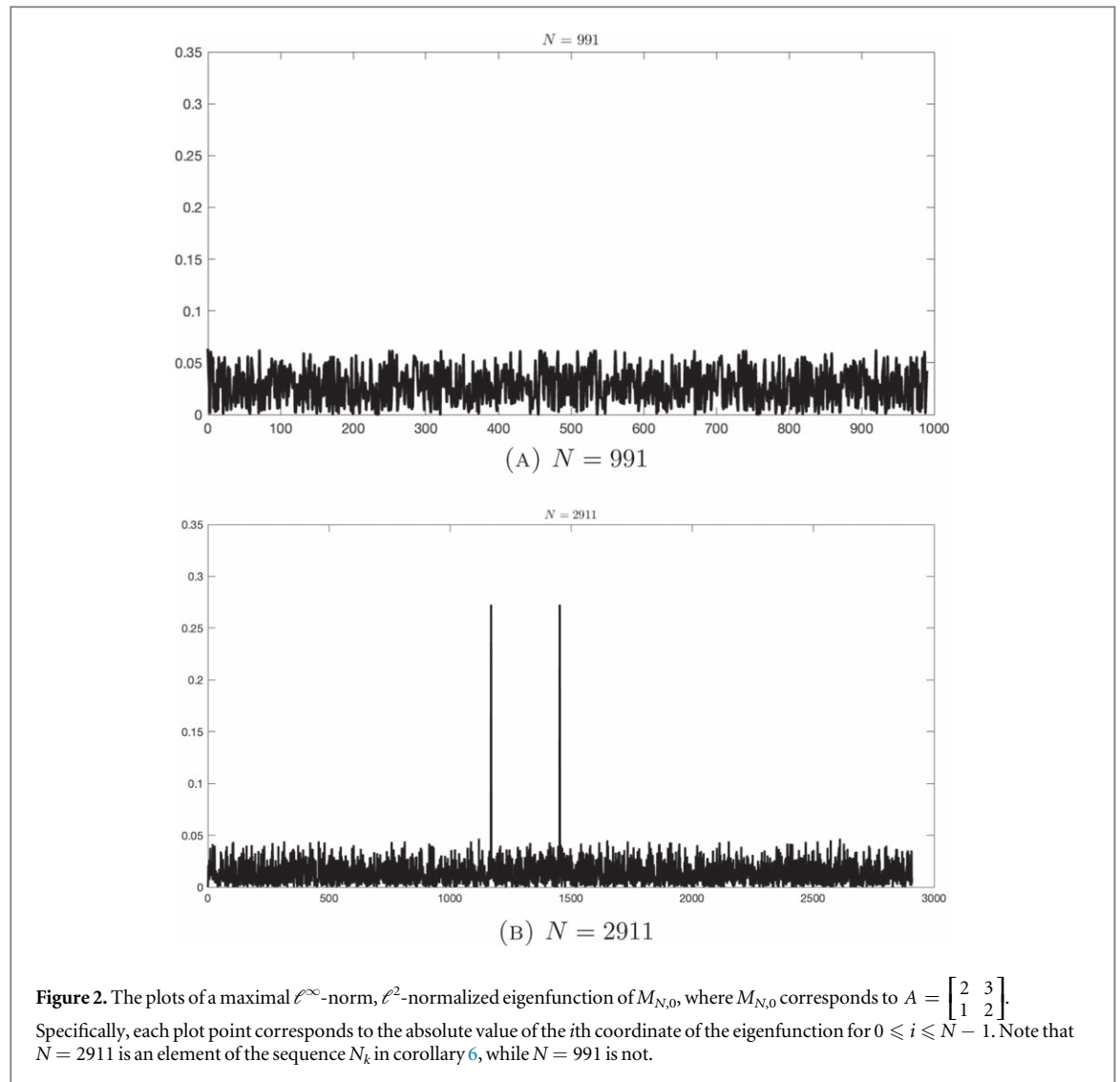
Theorem 2. Suppose A is a matrix satisfying (1). Then for $0 < \varepsilon < 1$, there exists N_0 such that for all odd $N \geq N_0$, if u is an eigenfunction of $M_{N,0}$ with $\|u\|_{\ell^2} = 1$ then

$$\|u\|_{\ell^\infty} \leq \frac{1}{\sqrt{(1 - \varepsilon) \log_\lambda N}}. \quad (3)$$

Analogous statements can be proven for a sequence of even N and any θ , using similar proofs to those of theorems 1 and 2. However, we exclude these arguments as they are overly technical and do not introduce any novel ideas.

The ℓ^∞ bounds on eigenfunctions of quantum cat maps have been extensively studied in arithmetic quantum chaos, see [10–14]. These works have focused on *Hecke eigenfunctions*, which are joint eigenfunctions of the quantum cat map $M_{N,0}$ and the Hecke operators, constructed in this setting by Kurlberg–Rudnick [15]. There always exists an orthonormal basis of $\mathcal{H}_N(0)$ consisting of Hecke eigenfunctions; however, due to the possibility of large multiplicities of the eigenspaces of $M_{N,0}$ (see [15], footnote3) an upper bound on the ℓ^∞ norm of Hecke eigenfunctions does not imply the same bound for general eigenfunctions. We list below the known bounds on ℓ^2 -normalized Hecke eigenfunctions u :

- Kurlberg–Rudnick [10] proved the upper bound $\|u\|_{\ell^\infty} \lesssim_\varepsilon N^{-\frac{1}{8} + \varepsilon}$.
- Building on [10] (which handled roughly half of the prime values of N), Kurlberg [11] showed that for all but finitely many *prime* values of N one has the upper bound $\|u\|_{\ell^\infty} \leq 2N^{-\frac{1}{2}}$, and deduced the upper bound $\|u\|_{\ell^\infty} \lesssim_\varepsilon N^{-\frac{1}{2} + \varepsilon}$ for *square-free* values of N .



- Olofsson [12, 13] showed an upper bound $\|u\|_{\ell^\infty} \lesssim N^{-\frac{1}{4}}$ for *most* values of N (in the sense of density as $N \rightarrow \infty$). On the other hand, these papers also construct eigenfunctions satisfying a lower bound $\|u\|_{\ell^\infty} \gtrsim N^{-\frac{1}{4}}$ for most values of N which are *not square-free*.

2. Preliminaries

We begin with a review of the necessary definitions for this paper. First, recall the semiclassical Weyl quantization. For $a \in \mathcal{S}(\mathbb{R}^2)$ and a semiclassical parameter $h \in (0, 1]$,

$$\text{Op}_h(a)f(x) := \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{i(x-x')\xi} a\left(\frac{x+x'}{2}, \xi\right) f(x') dx' d\xi, \quad f \in \mathcal{S}(\mathbb{R}).$$

Define the symbol class

$$S(1) = \left\{ a \in C^\infty(\mathbb{R}^2): \sup_{(x,\xi) \in \mathbb{R}^2} |\partial_{(x,\xi)}^\alpha a| < \infty \text{ for all } \alpha \in \mathbb{N}^2 \right\},$$

which naturally induces the seminorms $\|a\|_{C^m} := \max_{|\alpha| \leq m} \sup_{\mathbb{R}^2} |\partial_{(x,\xi)}^\alpha a|$ for $m \in \mathbb{N}_0$. From [16, Theorem 4.16], we know for $a \in S(1)$, $\text{Op}_h(a)$ acts on both $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$.

Now, let $\omega = (\gamma, \eta)$, $z = (x, \xi) \in \mathbb{R}^2$. Define the *standard symplectic form* σ on $\mathbb{R}^2$ by $\sigma(z, \omega) := \xi\gamma - x\eta$ and define the *quantum translation* by $U_\omega := \text{Op}_h(a_\omega)$, where $a_\omega(z) := \exp\left(\frac{i}{h}\sigma(\omega, z)\right)$. Noting that $a_\omega(z) \in S(1)$, we see U_ω is well-defined and acts on $\mathcal{S}(\mathbb{R})$. In [16, Theorem 4.7], it is shown that

$$U_\omega f(x) = e^{\frac{i}{h}\eta x - \frac{i}{2h}\eta^2} f(x - \eta).$$

Thus, U_ω is a unitary operator on $L^2(\mathbb{R})$ that satisfies the following exact Egorov's theorem,

$$U_\omega^{-1} \text{Op}_h(a) U_\omega = \text{Op}_h(\tilde{a}) \quad \text{for all } a \in S(1), \quad \tilde{a}(z) := a(z + \omega). \quad (4)$$

From the fact that $U_\omega U_{\omega'} = e^{\frac{i}{2h}\sigma(\omega, \omega')} U_{\omega + \omega'}$, we deduce the following commutator formula,

$$U_\omega U_{\omega'} = e^{\frac{i}{h}\sigma(\omega, \omega')} U_{\omega'} U_\omega. \quad (5)$$

Now let $\text{Sp}(2, \mathbb{R})$ be the group of real symplectic 2×2 matrices. In other words, $A \in \text{Sp}(2, \mathbb{R})$ if and only if $\sigma(Az, A\omega) = \sigma(z, \omega)$. Note that in this specific 2-dimensional case, $\text{Sp}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})$. For each $A \in \text{SL}(2, \mathbb{R})$, denote by $\mathcal{M}_A$ the set of all unitary transformations $M: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ satisfying the following exact Egorov's theorem,

$$M^{-1} \text{Op}_h(a) M = \text{Op}_h(a \circ A) \quad \text{for all } a \in S(1). \quad (6)$$

From [16, Theorem 11.9], we have both existence of these transformations and uniqueness up to a unit factor.

Then, $\mathcal{M} := \bigcup_{A \in \text{SL}(2, \mathbb{R})} \mathcal{M}_A$ is a subgroup of unitary transformations of $L^2(\mathbb{R})$ called the *metaplectic group* and the map $M \mapsto A$ is a group homomorphism $\mathcal{M} \rightarrow \text{SL}(2, \mathbb{R})$. As a corollary of equation (6), we obtain the following intertwining of the metaplectic and quantum transformations: $M^{-1} U_\omega M = U_{A^{-1}\omega}$ for all $M \in \mathcal{M}_A$, $\omega \in \mathbb{R}^2$.

We turn our attention to quantizations of functions on the torus $\mathbb{T}^2 := \mathbb{R}^2 / \mathbb{Z}^2$. Each $a \in C^\infty(\mathbb{T}^2)$ can be identified with a $\mathbb{Z}^2$ -periodic function on $\mathbb{R}^2$. Note that any $a \in C^\infty(\mathbb{T}^2)$ is also an element of $S(1)$, therefore its Weyl quantization $\text{Op}_h(a)$ is an operator on $L^2(\mathbb{R})$.

By equation (5), we have the following commutation relations,

$$\text{Op}_h(a) U_\omega = U_\omega \text{Op}_h(a) \quad \text{for all } a \in C^\infty(\mathbb{T}^2), \quad \omega \in \mathbb{Z}^2. \quad (7)$$

These commutation relations motivate a decomposition of $L^2(\mathbb{R})$ into a direct integral of finite dimensional spaces $\mathcal{H}_N(\theta)$, where $\theta \in \mathbb{T}^2$, such that $\text{Op}_h(a)$ descends onto these spaces. From [17, Proposition 2.1], to ensure the these spaces are nontrivial, for the rest of the paper, we assume

$$h = (2\pi N)^{-1} \quad \text{where } N \in \mathbb{N}.$$

We call $\mathcal{H}_N(\theta)$ the space of *quantum states*. Specifically, for each $\theta \in \mathbb{T}^2$, set

$$\mathcal{H}_N(\theta) := \{f \in \mathcal{S}'(\mathbb{R}) : U_\omega f = e^{2\pi i \sigma(\theta, \omega) + N\pi i Q(\omega)} f \text{ for all } \omega \in \mathbb{Z}^2\},$$

where the quadratic form Q on $\mathbb{R}^2$ is defined by $Q(\omega) = \eta\eta$ for $\omega = (\eta, \eta) \in \mathbb{R}^2$. The following lemma gives an explicit basis for $\mathcal{H}_N(\theta)$.

Lemma 3 [18], lemma 2.5. *The space $\mathcal{H}_N(\theta)$ is N -dimensional with a basis $\{e_j^\theta\}$ defined for $j \in \{0, \dots, N-1\}$ and $\theta = (\theta_x, \theta_\xi) \in \mathbb{R}^2$. In particular,*

$$e_j^\theta(x) := \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}} e^{-2\pi i \theta_\xi k} \delta\left(x - \frac{Nk + j - \theta_x}{N}\right).$$

We fix an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ on each quantum state $\mathcal{H}_N(\theta)$ by requiring $\{e_j^\theta\}$ to be an orthonormal basis. It can be shown using translation identities for e_j^θ (see [18, (2.35)]) that although each $\{e_j^\theta\}$ depends on the choice of the representative $\theta_x \in \mathbb{R}$, the inner product depends only on $\theta \in \mathbb{T}^2$. We often denote the norm defined by this inner product by $\|\cdot\|_{\ell^2}$. Using the bases $\{e_j^\theta\}$, we can consider the spaces $\mathcal{H}_N(\theta)$ as fibers of a smooth N dimensional vector bundle over $\mathbb{T}^2$, which we call $\mathcal{H}_N$.

For $u = \sum_{j=0}^{N-1} \alpha_j e_j^\theta$, we set

$$\|u\|_{\ell^p} := \|(\alpha_0, \dots, \alpha_{N-1})\|_{\ell^p}.$$

Fix $N \in \mathbb{N}$ and $a \in C^\infty(\mathbb{T}^2)$ to define the quantization

$$\text{Op}_{N,\theta}(a) := \text{Op}_h(a)|_{\mathcal{H}_N(\theta)} : \mathcal{H}_N(\theta) \rightarrow \mathcal{H}_N(\theta), \quad \theta \in \mathbb{T}^2,$$

which depends smoothly on θ . This restriction holds by definition of $\mathcal{H}_N(\theta)$ and the commutation relations given in equation (7).

We set

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{Z}) \quad (8)$$

and choose a metaplectic operator $M \in \mathcal{M}_A$. Recall that for $\omega = (\eta, \eta)$, $z = (x, \xi) \in \mathbb{Z}^2$, $\sigma(z, \omega) = \xi\eta - x\eta$ and $Q(\omega) = \eta\eta$. By [18, Lemma 2.9], there exists a unique $\varphi_A \in (\mathbb{Z}/2)^2$ such that for all $\omega \in \mathbb{Z}^2$,

$Q(A^{-1}\omega) - Q(\omega) = \sigma(\varphi_A, \omega) \bmod 2\mathbb{Z}$. Using the definition of $\mathcal{H}_N(\theta)$ and that fact that $M^{-1}U_\omega M = U_{A^{-1}\omega}$, we can verify that $M(\mathcal{H}_N(\theta)) \subset \mathcal{H}(A\theta + \frac{N\varphi_A}{2})$ for all $\theta \in \mathbb{T}^2$.

Denote $M_{N,\theta} := M|_{\mathcal{H}_N(\theta)}: \mathcal{H}_N(\theta) \rightarrow \mathcal{H}_N(A\theta + \frac{N\varphi_A}{2})$, which depends smoothly on $\theta \in \mathbb{T}^2$. We require the domain and range of $M_{N,\theta}$ to be the same, in other words, we must have

$$(I - A)\theta = \frac{N\varphi_A}{2} \bmod \mathbb{Z}^2. \quad (9)$$

Thus, when $\theta = 0$, condition (9) is satisfied when N is even or $\varphi_A = 0$.

We henceforth assume that $\varphi_A = 0$, which gives $\sigma(\varphi_A, \omega) = 0 \bmod \mathbb{Z}/2$. Noting that for all $\omega = (y, \eta)$,

$$Q(A^{-1}\omega) - Q(\omega) = (dy - b\eta)(-cy + a\eta) - y\eta = -dcy^2 + 2bcy\eta - ba\eta^2,$$

we must have $dcy^2 + ba\eta^2 = 0 \bmod 2\mathbb{Z}$. We conclude that dc and ab must be even.

Assuming condition (9), we have the following exact Egorov's theorem for all $a \in C^\infty(\mathbb{T}^2)$,

$$M_{N,\theta}^{-1} \text{Op}_{N,\theta}(a) M_{N,\theta} = \text{Op}_{N,\theta}(a \circ A).$$

Essential to our proof of theorem 2 is the following explicit formula for M .

Lemma 4. When A is given by (8) and $b \neq 0$, an element of $\mathcal{M}_A$ is given by

$$Mu(x) = \frac{\sqrt{N}}{\sqrt{|b|}} \int_{\mathbb{R}} e^{2\pi Ni\Phi(x,y)} u(y) dy, \quad (10)$$

where

$$\Phi(x, y) = \frac{d}{2b}x^2 - \frac{xy}{b} + \frac{a}{2b}y^2. \quad (11)$$

We provide a short outline of this formula's derivation; for further details see theorem 11.10 and its following remark in [16]. Set $\Lambda = \{(x, y, \xi, -\eta): (x, \xi) = A(y, \eta)\}$ and note that Λ is a Lagrangian submanifold of $\mathbb{R}^4$. Additionally, as $b \neq 0$, $\Lambda \ni (x, y, \xi, \eta) \mapsto (x, \eta)$ is surjective. Therefore, there exists a generating function $\Phi(x, y)$ such that $\Lambda = \{(x, y, \partial_x \Phi, \partial_y \Phi)\}$. As $\eta = -\partial_y \Phi$ and $\xi = \partial_x \Phi$, we know $\Phi(x, y)$ is given by (11). We then have (10), where the coefficient ensures M is unitary, up to a unit factor.

3. Proof of theorem 1

First, following the presentation in [8], in section 3.1, we show that for a sequence N_k , $M_{N_k,0}$ has a large degenerate eigenspace. In section 3.2, we then reduce the proof of theorem 1 to a linear algebra argument.

3.1. Eigenspace of $M_{N_k,0}$

Consider A given by (8) such that b and c are coprime and $\text{Tr} A$ is even and greater than 2. Let λ be the largest eigenvalue of A . Then for each $t \in \mathbb{N}$,

$$A^t = p_t A - p_{t-1} I, \quad p_{t+1} = \text{Tr}(A)p_t - p_{t-1}, \quad \text{where } p_t = \frac{\lambda^t - \lambda^{-t}}{\lambda - \lambda^{-1}} \in \mathbb{Z}. \quad (12)$$

Set $T_N = \min\{t: A^t = I \bmod N\}$ and, for $k \in \mathbb{N}$, define

$$N'_k := \max\{N: A^k = I \bmod N\}.$$

Essential to our proof of theorem 1 is the following statement from [8]. For the reader's convenience, we replicate their proof here.

Theorem 5 [8], Prop. 11. For each $k \in \mathbb{N}$, we have $N'_{2k} = 2p_k$, $N'_{2k+1} = p_k + p_{k+1}$, and $T_{N'_k} = k$.

Proof. Using equation (12), we see that N'_k is the greatest integer such that

$$\begin{bmatrix} p_k a - p_{k-1} - 1 & p_k b \\ p_k c & p_k d - p_{k-1} - 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \bmod N'_k.$$

Recall that we assumed b and c are coprime. Therefore, $p_k = 0 \bmod N'_k$ and $p_{k-1} = -1 \bmod N'_k$, which gives $N'_k = \gcd(p_k, p_{k-1} + 1)$. We claim that for $s = 0, \dots, k-1$,

$$N'_k = \gcd(p_{k-s} + p_s, p_{k-(s+1)} + p_{s+1}). \quad (13)$$

We proceed by induction. Note that $p_0 = 0$ and $p_1 = 1$, therefore, (13) clearly holds for $s = 0$. Now suppose we know (13) for some $s \geq 0$. Using (12) and the identity $\gcd(a, ca - b) = \gcd(a, b)$, we have

$$\begin{aligned} N'_k &= \gcd(\text{Tr}A_{p_{k-s-1} - p_{k-s-2} + p_s, p_{k-s-1} + p_{s+1}}) \\ &= \gcd(\text{Tr}A(p_{k-s-1} + p_{s+1}) - \text{Tr}A_{p_{k-s-2} + p_s, p_{k-s-1} + p_{s+1}}) \\ &= \gcd(p_{s+2} + p_{k-(s+2)}, p_{k-(s+1)} + p_{s+1}), \end{aligned}$$

which completes the induction. We set $k = 2\ell$, $s = \ell$ in (13) to conclude

$$N'_{2\ell} = \gcd(2p_\ell, p_{\ell-1} + p_{\ell+1}) = \gcd(2p_\ell, \text{Tr}A_\ell) = 2p_\ell,$$

where the last equality follows from our assumption that $\text{Tr}A$ is even. Similarly, setting $k = 2\ell + 1$ and $s = \ell$ in (13) gives $N'_{2\ell+1} = p_\ell + p_{\ell+1}$.

Now note that for each k , we have

$$A^k = 1 \bmod N'_k, \quad (14)$$

$$A^{T_{N'_k}} = 1 \bmod N'_k, \quad \text{and} \quad A^{T_{N'_k}} = 1 \bmod N'_{T_{N'_k}}. \quad (15)$$

From the definition of $T_{N'_k}$ and (14), we see that $T_{N'_k} \leq k$. From the definition of N'_k and (15), we see that $N'_k \leq N'_{T_{N'_k}}$. As $\{N'_k\}$ is increasing, we conclude that $T_{N'_k} \geq k$. Therefore, $T_{N'_k} = k$. $\square$

Now, let $n(N)$ denote the period of $M_{N,0}$; specifically

$$n(N) := \min \{t: M_{N,0}^t = e^{i\varphi} \text{ for some } \varphi \in \mathbb{R}\}.$$

Suppose A_N is the matrix with integer entries that satisfies $A^{T_N} = 1 + NA_N$. From [1, (36)–(46)], we know that $n(N) = T_N$ if N is odd or if N is even and $(A_N)_{12}$ and $(A_N)_{21}$ are even. Otherwise, $n(N) = 2T_N$.

Using this formula for $n(N)$ and theorem 5, following [8], we show an upper bound for $n(N)$ that depends only on λ and N .

Corollary 6. *Let λ be the largest eigenvalue of A . Then, there exists a sequence of odd N_k such that $2 \log_\lambda N_k + 1 \geq n(N_k)$.*

Proof. Using our assumption that $\text{Tr}A$ is even, we note that p_{2k} is even and p_{2k+1} is odd.

Thus, $n(N'_{2k+1}) = T_{N'_{2k+1}} = 2k + 1$.

We now formulate a bound for $n(N'_{2k+1})$ in terms of N'_{2k+1} . As $\text{Tr}A > 2$, we know $\lambda > 1$. Additionally using theorem 5, $N'_{2k+1} = \frac{\lambda^{k+1} + \lambda^k - \lambda^{-k-1} - \lambda^{-k}}{\lambda - \lambda^{-1}} \geq \lambda^k$. Therefore, $\log_\lambda N'_{2k+1} \geq k$, which gives $2 \log_\lambda (N'_{2k+1}) + 1 \geq 2k + 1 = n(N'_{2k+1})$. Labeling, N'_{2k+1} as N_k , we are done. $\square$

Note that the same proof ideas can be adapted for a sequence of even N , using the fact that N'_{2k} is even. However, as the odd case is the simpler of the two, our main theorem is proven for a sequence of odd N .

For ease of notation, again using $N_k := N'_{2k+1}$, set

$$t_k := n(N_k).$$

Therefore, if u is an eigenfunction of $M_{N_k,0}$ with eigenvalue λ , $M_{N_k,0}^{t_k} u = e^{i\varphi_k} u = \lambda^{t_k} u$. Thus, each eigenvalue of $M_{N_k,0}$ satisfies $\lambda^{t_k} = e^{i\varphi_k}$, giving an eigenvalue whose multiplicity is at least $\frac{N_k}{t_k}$. In other words, $M_{N_k,0}$ has a degenerate eigenspace with dimension at least $\frac{N_k}{t_k}$.

3.2. Lower bound

Now we are ready to state the lower bound in linear algebraic terms

Proposition 7. *Suppose V is subspace of $\mathbb{C}^N$ with dimension at least $\frac{N}{t}$. Then,*

$$\frac{1}{\sqrt{t}} \leq \max_{u \in V, \|u\|_{\ell^2} = 1} \|u\|_{\ell^\infty}. \quad (16)$$

Proof. Define $\Pi: \mathbb{C}^N \rightarrow V$ to be the orthogonal projection onto V and let e_j denote the j th coordinate vector. Note that

$$\sum_{j=1}^N \|\Pi e_j\|_{\ell^2}^2 = \text{Tr}(\Pi^* \Pi) = \text{Tr} \Pi = \dim V \geq \frac{N}{t}.$$

Therefore, there exists a j such that $\|\Pi e_j\|_{\ell^2} \geq \frac{1}{\sqrt{t}}$. We then have

$$\sup_{\substack{u \in V \\ \|u\|_{\ell^2}=1}} \|u\|_{\ell^\infty} = \max_j \sup_{\substack{u \in \mathbb{C}^N \\ \|u\|_{\ell^2}=1}} |\langle \Pi u, e_j \rangle| = \max_j \sup_{\substack{u \in \mathbb{C}^N \\ \|u\|_{\ell^2}=1}} |\langle u, \Pi e_j \rangle| = \max_j \|\Pi e_j\|_{\ell^2} \geq \frac{1}{\sqrt{t}},$$

which completes the proof. $\square$

We claim that proposition 7 implies theorem 1. In particular, fixing V to be the degenerate eigenspace of $M_{N_k,0}$ with dimension at least $\frac{N_k}{t_k}$ and using the fact that $\mathbb{C}^{N_k} \simeq \mathcal{H}_{N_k}(0)$, proposition 7 implies

$$\frac{1}{\sqrt{2 \log_\lambda N_k + 1}} \leq \frac{1}{\sqrt{t_k}} \leq \max_{u \in V, \|u\|_{\ell^2}=1} \|u\|_{\ell^\infty}. \quad (17)$$

For an explicit demonstration of this lower bound, see figure 2.

3.3. Eigenfunctions in proposition 7

We further examine the eigenspace V , which achieved the lower bound in proposition 7. Recall we defined Π to be the orthogonal projection onto V and examined eigenfunctions of the form $\Pi(\sum_{j=0}^{N-1} \alpha_j e_j^0)$. To find a more precise formula for these eigenfunctions, we deduce a formula for Π .

Suppose V corresponds to eigenvalue λ . We know that the spectrum of $M_{N_k,0}$ is contained in $\{z: z^{t_k} = e^{i\varphi_k}\}$. Therefore, for some $s \in \mathbb{Z}$, λ is of the form $e^{i\frac{\varphi_k+2\pi s}{t_k}}$. Thus, on the spectrum,

$$\frac{1}{t_k} \left(1 + \frac{z}{\lambda} + \dots + \frac{z^{t_k-1}}{\lambda^{t_k-1}} \right) = \begin{cases} 1, & \text{if } z = \lambda \\ 0, & \text{else} \end{cases}.$$

We then see

$$\Pi = \frac{1}{t_k} \left(I + e^{-i\frac{\varphi_k+2\pi s}{t_k}} M + \dots + (e^{-i\frac{\varphi_k+2\pi s}{t_k}} M)^{t_k-1} \right).$$

Therefore in our proof of the lower bound, we are actually studying the eigenfunctions of the form

$$\frac{1}{t_k} \sum_{t=0}^{t_k-1} e^{-i\frac{\varphi_k+2\pi s}{t_k} t} M^t e_j^0. \quad (18)$$

Similar eigenfunctions were studied in [9]. Specifically, the paper examines eigenfunctions of the form

$$\frac{1}{t_k} \sum_{t=0}^{t_k-1} e^{-i\frac{\varphi_k+2\pi s}{t_k} t} M^t g(x), \quad (19)$$

where $g(x)$ is a Gaussian function. The limit of these eigenfunctions was shown to be half ergodic and half localized, in the sense of semiclassical measures. Although (18) and (19) appear very similar, they actually exhibit distinct behavior, as demonstrated by figures 3 and 4. In these figures, $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $k = 14$, and $N_k = 5822$.

4. Proof of theorem 2

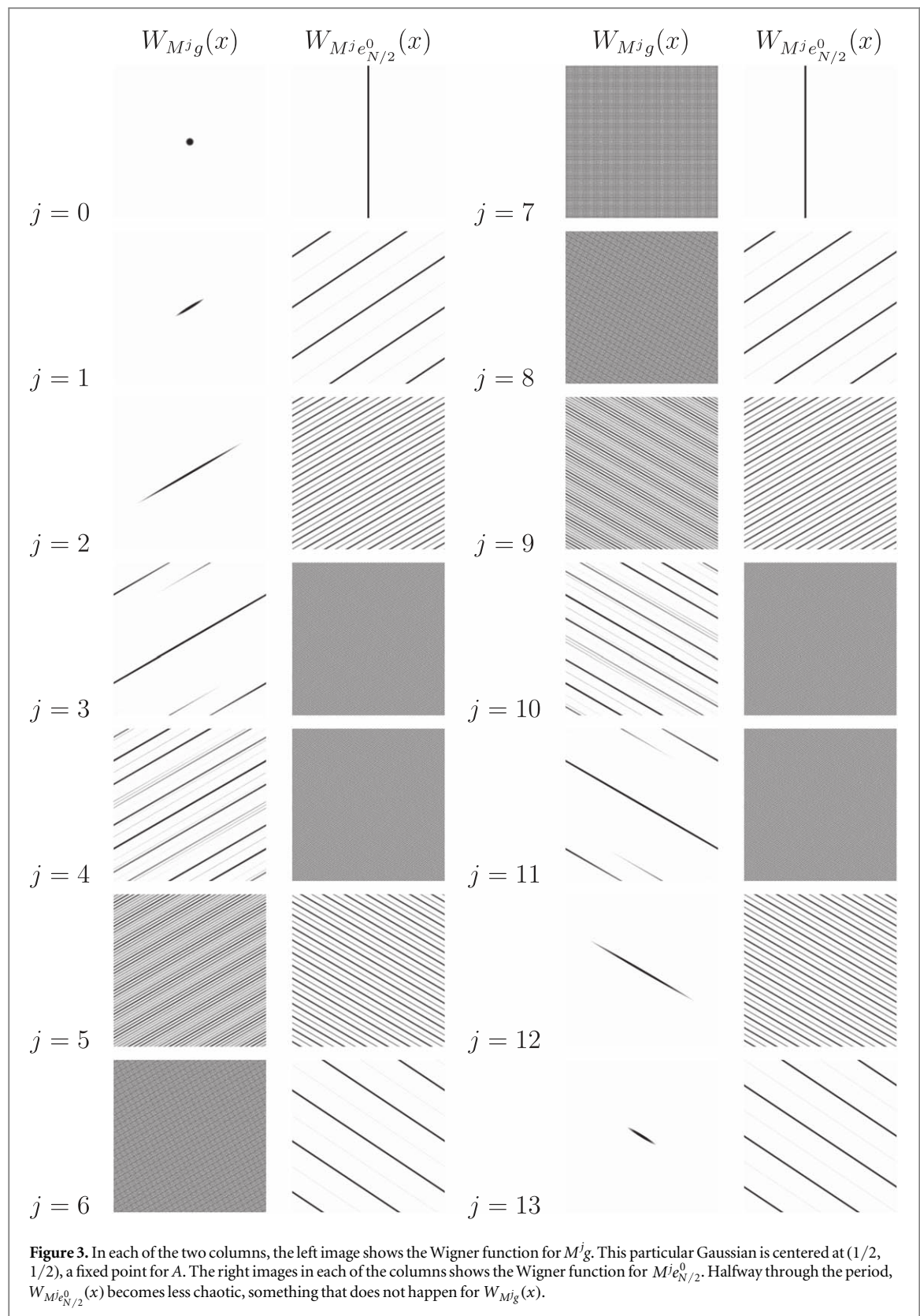
We examine $A \in \text{SL}(2, \mathbb{Z})$ of the form (1). Note that the conditions on A are less restrictive than those in section 3. Let N be odd. We begin with the following dispersive estimate.

Proposition 8. For N odd and $M_{N,0}: \mathcal{H}_N(0) \rightarrow \mathcal{H}_N(0)$, we have

$$|(M_{N,0} e_j^0, e_k^0)_{\mathcal{H}}| \leq \frac{\sqrt{|b|}}{\sqrt{N}}.$$

Proof. We begin by calculating $M_{N,0} e_j^0$. By lemma 4, for $\Phi(x, y) = \frac{d}{2b} x^2 - \frac{xy}{b} + \frac{a}{2b} y^2$,

$$M_{N,0} e_j^0(x) = \frac{1}{\sqrt{|b|}} \int_{\mathbb{R}} e^{2\pi i N \Phi(x,y)} \sum_{k \in \mathbb{Z}} \delta\left(y - \frac{Nk+j}{N}\right) dy = \frac{1}{\sqrt{|b|}} \sum_{k \in \mathbb{Z}} e^{2\pi i N \Phi\left(x, \frac{Nk+j}{N}\right)}.$$



Thus, we focus on calculating $\Phi\left(x, \frac{Nk+j}{N}\right) \bmod N\mathbb{Z}$. In the following, we set $k = mb + r$ for $0 \leq r \leq |b| - 1$. We have

$$\Phi\left(x, \frac{Nk+j}{N}\right) = \frac{d}{2b}x^2 - xm - \frac{xr}{b} - \frac{xj}{Nb} + \frac{ar^2}{2b} + \frac{arj}{bN} + \frac{aj}{2bN^2} \bmod N\mathbb{Z}.$$

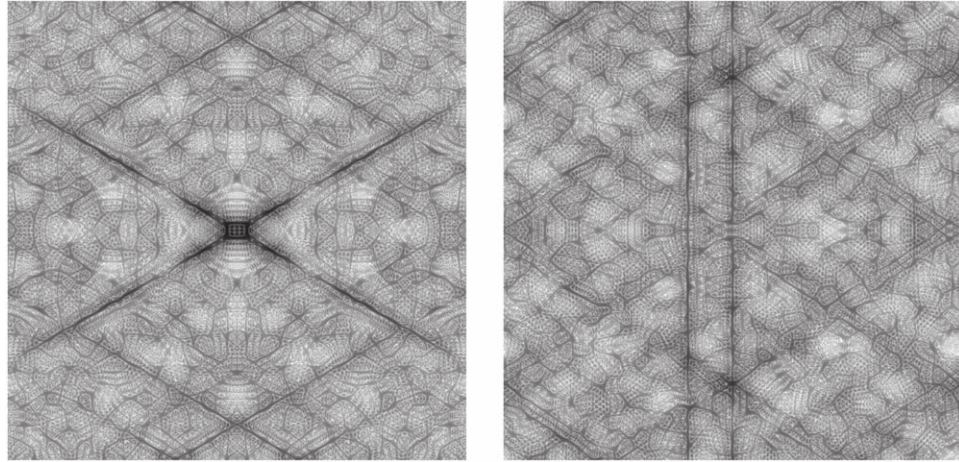


Figure 4. The left image shows the Wigner function for the eigenfunction (19), where the Gaussian is centered at $(1/2, 1/2)$, a fixed point. The right image shows the Wigner function for the eigenfunction (18). Notice how the left image illustrates localization at $(1/2, 1/2)$, while the right image does not. We conjecture that the limit of eigenfunctions (18) equidistributes in the sense of semiclassical measures.

Thus, using the Poisson summation formula and the fact that $\delta(x - x_0)f(x) = \delta(x - x_0)f(x_0)$, we know

$$\begin{aligned} M_{N,0}e_j^0(x) &= \frac{1}{\sqrt{|b|}} \sum_{r=0}^{|b|-1} e^{\frac{2\pi i}{b} \left(\frac{aNr^2}{2} + arj + \frac{aj}{2N} \right)} \sum_{m \in \mathbb{Z}} e^{\frac{2\pi i N}{b} \left(\frac{d}{2} x^2 - xr - \frac{xj}{N} \right)} e^{-2\pi i N x m} \\ &= \frac{1}{N\sqrt{|b|}} \sum_{l \in \mathbb{Z}} \sum_{r=0}^{|b|-1} e^{\frac{2\pi i}{b} \left(\frac{dl^2}{2N} - lr - \frac{l j}{N} + \frac{ar^2 N}{2} + arj + \frac{aj}{2N} \right)} \delta\left(x - \frac{l}{N}\right). \end{aligned}$$

Now setting

$$c_l = \sum_{r=0}^{|b|-1} e^{\frac{2\pi i}{b} \left(\frac{dl^2}{2N} - lr - \frac{l j}{N} + \frac{ar^2 N}{2} + arj + \frac{aj}{2N} \right)},$$

we want to show that c_l has period N , in other words $c_l = c_{l+N}$. We know that

$$c_{l+N} = e^{\frac{2\pi i}{b} \left(\frac{dl^2}{2N} + dl + \frac{dN}{2} - \frac{l j}{N} - j + \frac{aj}{2N} \right)} \sum_{r=0}^{|b|-1} e^{\frac{2\pi i}{b} \left(-lr - Nr + \frac{ar^2 N}{2} + arj \right)} = \alpha_{l+N} \sum_{r=0}^{|b|-1} \beta_{l+N,r}.$$

Recall in section 2, we showed in order for $M_{N,0}$ to descend $\mathcal{H}_N(0)$ to itself, ab must even. Therefore,

$$\beta_{l,r+b} = e^{\frac{2\pi i}{b} \left(-lr - lb + \frac{ar^2 N}{2} + abrN + \frac{ab^2 N}{2} + arj + abj \right)} = e^{\frac{2\pi i}{b} \left(-lr + \frac{ar^2 N}{2} + arj \right)} = \beta_{l,r}.$$

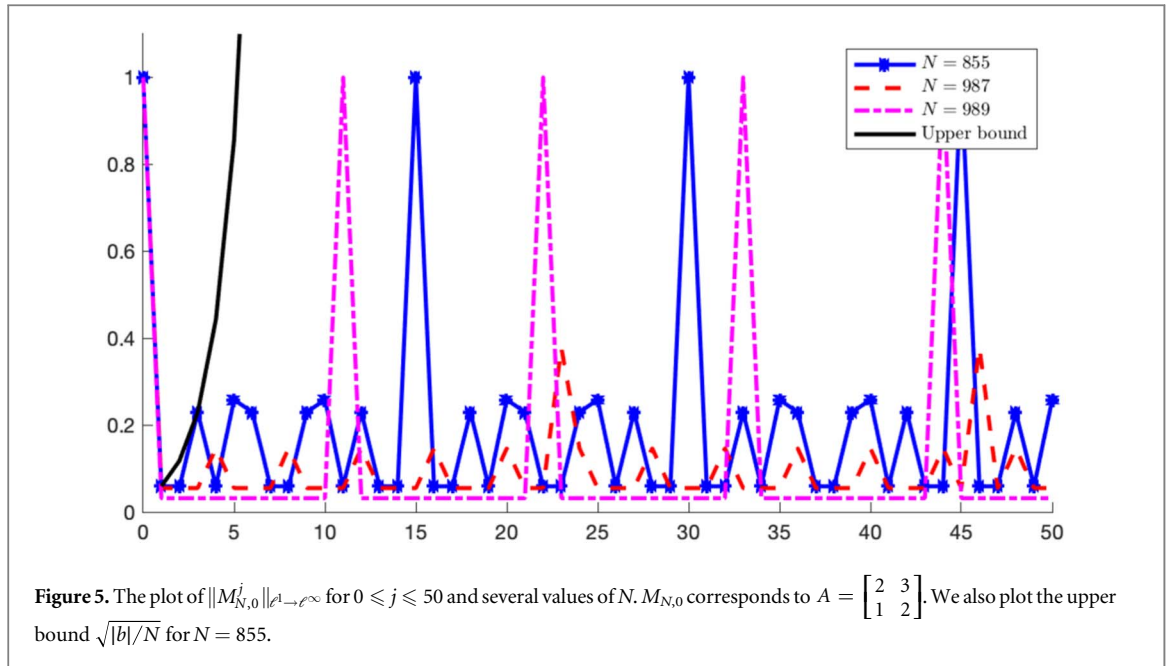
Therefore, $c_{l+N} = \alpha_{l+N} \sum_{r=d}^{|b|-1+d} \beta_{l+N,r}$. As $ad - bc = 1$, we know $ad = 1 \pmod{b}$. Then,

$$c_{l+N} = \alpha_{l+N} \sum_{r=d}^{|b|-1+d} e^{\frac{2\pi i}{b} \left(-lr - Nr + \frac{ar^2 N}{2} + arj \right)} = \alpha_{l+N} \sum_{r=0}^{|b|-1} e^{\frac{2\pi i}{b} \left(-lr - ld - Nd + \frac{ar^2 N}{2} + \frac{dN}{2} + arj + j \right)} = c_l.$$

Using the fact that $c_k = c_{k+IN}$ for all $l \in \mathbb{Z}$ and that $\{e_j^0\}$ is an orthonormal basis, we have

$$\begin{aligned} &\langle M_{N,0}e_j^0(x), e_k^0(x) \rangle_{\mathcal{H}} \\ &= \frac{1}{\sqrt{N|b|}} \left\langle \frac{1}{\sqrt{N}} \sum_{l \in \mathbb{Z}} c_{k+IN} \delta\left(x - \frac{Nl+k}{N}\right), \frac{1}{\sqrt{N}} \sum_{l \in \mathbb{Z}} \delta\left(x - \frac{Nl+k}{N}\right) \right\rangle_{\mathcal{H}} \\ &= \frac{1}{\sqrt{N|b|}} \sum_{r=0}^{|b|-1} e^{\frac{2\pi i}{b} \left(\frac{ar^2 N}{2} + arj + \frac{aj}{2N} + \frac{dk^2}{2N} - kr - \frac{kj}{N} \right)}. \end{aligned}$$

Using the triangle inequality, we conclude $|\langle M_{N,0}e_j^0(x), e_k^0(x) \rangle_{\mathcal{H}}| \leq \frac{\sqrt{|b|}}{\sqrt{N}}$. □



For $j > 0$, we use the notation $A^j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}$. As a direct consequence of proposition 8, for $j > 0$,

$$\|M_{N,0}^j\|_{\ell^1 \rightarrow \ell^\infty} \leq \frac{\sqrt{|b_j|}}{\sqrt{N}}.$$

Figure 5 compares this upper bound to actual values of $\|M_{N,0}^j\|_{\ell^1 \rightarrow \ell^\infty}$. Now as $\text{Tr} A > 2$, A has eigenvalues λ, λ^{-1} with $\lambda > 1$. Then we have $|b_j| \sim \lambda^j$, giving

$$\|M_{N,0}^j\|_{\ell^1 \rightarrow \ell^\infty} \leq \frac{C\lambda^{j/2}}{\sqrt{N}}.$$

Suppose u is an eigenfunction of $M_{N,0}$ with eigenvalue μ . As $M_{N,0}$ is unitary, $|\mu| = 1$. We have $u = \mu^{-n} M_{N,0}^n u$, which gives $u = \frac{1}{T} (\sum_{n=0}^{T-1} \mu^{-n} M_{N,0}^n) u$.

Setting $B = \frac{1}{T} \sum_{n=0}^{T-1} \mu^{-n} M_{N,0}^n$, we see $u = Bu$. Again, as $M_{N,0}$ is unitary,

$$B^* B = \frac{1}{T^2} \sum_{m,n=0}^{T-1} \mu^{m-n} M_{N,0}^{n-m}.$$

Then

$$\begin{aligned} \|B^* B\|_{\ell^1 \rightarrow \ell^\infty} &\leq \frac{1}{T} + \frac{1}{T^2} \sum_{\substack{m \neq n \\ 0 \leq m, n \leq T-1}} \|M_{N,0}^{n-m}\|_{\ell^1 \rightarrow \ell^\infty} \\ &\leq \frac{1}{T} + \frac{C}{T^2} \sum_{\substack{m \neq n \\ 0 \leq m, n \leq T-1}} \frac{\lambda^{\frac{n-m}{2}}}{\sqrt{N}} \\ &\leq \frac{1}{T} + \frac{C}{\sqrt{N}} \lambda^{\frac{T}{2}}. \end{aligned}$$

We set $T = (1 - \varepsilon/2) \log_\lambda N$ to get

$$\|B^* B\|_{\ell^1 \rightarrow \ell^\infty} \leq \frac{1}{(1 - \varepsilon/2) \log_\lambda N} + \frac{C}{\sqrt{\lambda N^\varepsilon}}.$$

Finally, we know that $\|B\|_{\ell^2 \rightarrow \ell^\infty}^2 = \|B^* B\|_{\ell^1 \rightarrow \ell^\infty}$. Therefore, for $0 < \varepsilon < 1$, there exists an N_0 such that for odd $N \geq N_0$,

$$\|u\|_{\ell^\infty} \leq \|B\|_{\ell^2 \rightarrow \ell^\infty} \|u\|_{\ell^2} \leq \frac{1}{\sqrt{(1 - \varepsilon) \log_\lambda N}}.$$

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Data availability statement

No new data were created or analysed in this study. Data will be available from 22 February 2023.

ORCID iDs

Elena Kim  <https://orcid.org/0000-0003-4231-0659>

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