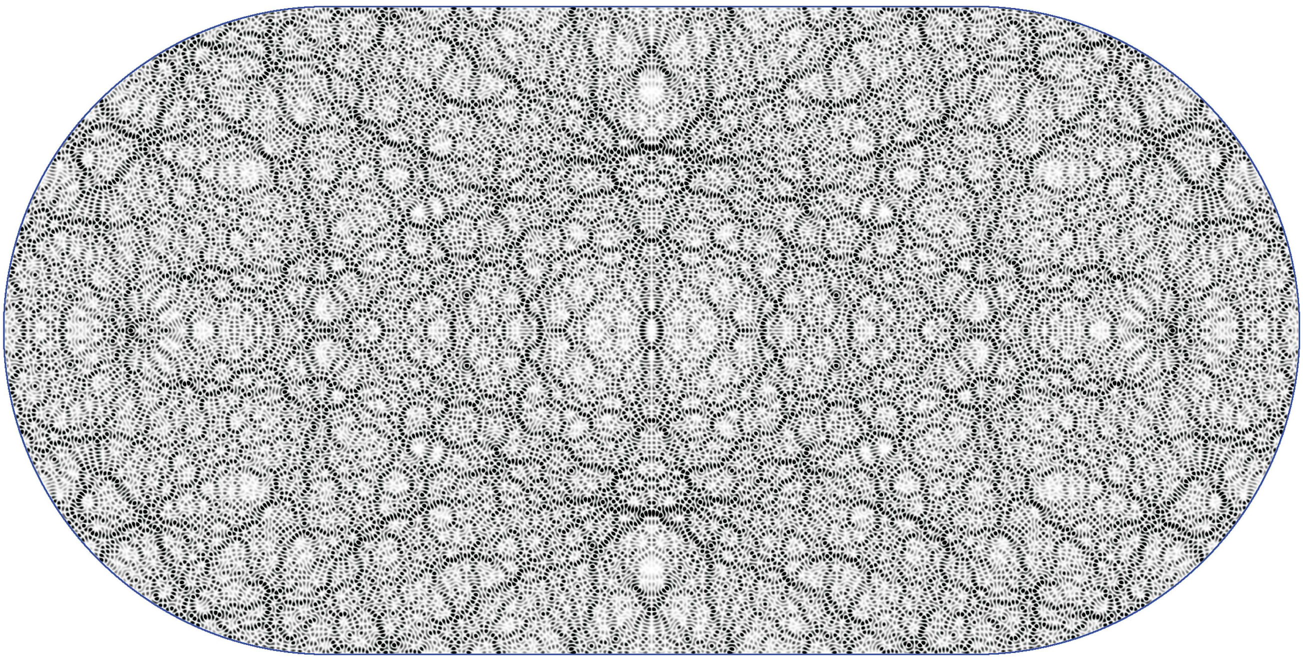
Quantum Ergodicity in Theorems and Pictures



Semyon Dyatlov

A popular culture notion of chaos was summed up by Edward Lorenz: it occurs "when the present determines the future, but the approximate present does not approximately determine the future" (or more dramatically "a butterfly flapping its wings in Brazil could set off a tornado in Texas"). In quantum mechanics there is no clear definition of *quantum chaos* but its manifestations include properties of eigenvalues and eigenfunctions. Here eigenfunctions are interpreted as pure quantum states, yielding the simplest, time-harmonic, solutions to the Schrödinger equation.

It is natural then to look for distinguishing properties between quantum systems with underlying completely integrable (that is, organized and nonchaotic) and chaotic classical dynamics. At high energies or small wavelengths, such classical effects would manifest themselves most clearly. We should stress though that the validity of such asymptotics almost always becomes accurate right away.

One classical notion, present in many chaotic systems, is that of *ergodicity*. A classical system is ergodic if almost all classical trajectories equidistribute—see Definition 1. This article focuses on the corresponding topic in quantum chaos: *macroscopic behavior of high energy eigenfunctions* for systems with ergodic or more strongly chaotic classical dynamics.

We cannot do justice here to the extensive literature on quantum ergodicity but we refer to the reviews by Sarnak [Sar11] and Zelditch [Zel19], as well as the author's ICM proceedings [Dya21], for more references, and for yet another perspective to the article of Rudnick [Rud08].

To see animated versions of the figures illustrating both classical and quantum phenomena, the reader is encouraged to visit `https://math.mit.edu/~dyatlov/chaos-movies.html`.

1. Eigenfunctions on Planar Domains

Eigenfunctions and eigenvalues of the Laplacian on bounded planar domains, with either Dirichlet or

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In memory of Steve Zelditch.

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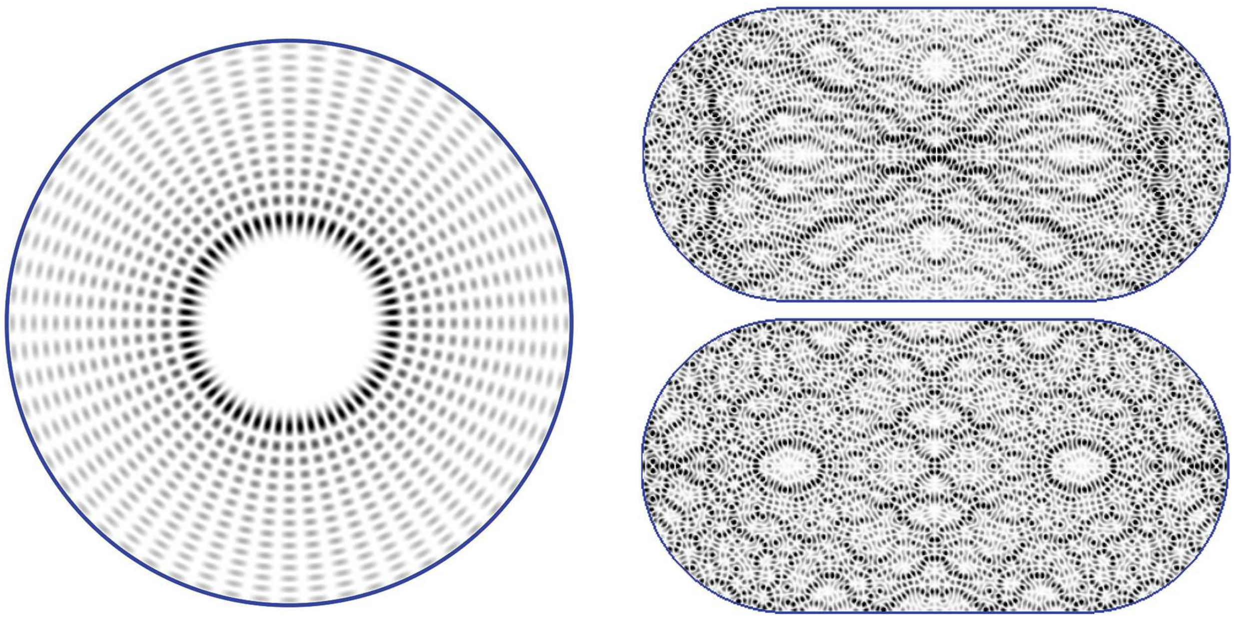

Figure 1. Numerically computed high energy Dirichlet eigenfunctions for two domains: a disk and a stadium. Here darker shading corresponds to larger values of $|u_j(x)|$. The eigenfunctions for the stadium here and in Figure 4 are computed using the method developed by Barnett, see [BH14].

Neumann boundary conditions, are familiar across mathematics and science. Their investigation goes back to the experiments by Chladni over two hundred years ago and includes such popular questions as "Can one hear the shape of a drum?" formulated by Kac over sixty years ago. In the Dirichlet case, these eigenfunctions are solutions to the eigenvalue problem

$$-\Delta u_j(x) = \lambda_j^2 u_j(x), \quad x \in \Omega, \quad u_j|_{\partial\Omega} = 0. \tag{1}$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded open set with smooth enough boundary $\partial\Omega$, $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ is Laplace's operator, and we choose, as we may, u_j's to form an orthonormal basis of the space of square integrable functions, $L^2(\Omega)$. Moreover, $\lambda_j \uparrow \infty$.

The eigenfunction u_j can be thought of as a pure state of a quantum particle confined to the domain Ω, with energy λ_j^2. Since $\|u_j\|_{L^2(\Omega)} = 1$, the expression $|u_j(x)|^2\,dx$ defines a probability measure on Ω. Following a standard interpretation of quantum mechanics, this measure gives the probability distribution of the position of the particle. We will be particularly interested in the quantities

$$\int_\Omega b(x)|u_j(x)|^2\,dx, \quad b \in C(\Omega), \tag{2}$$

which give the expected value of $b(x)$ where x is the position of the particle. (If $b = \mathbf{1}_S$ is the indicator function of a set $S \subset \Omega$, then (2) is the probability of finding the particle in S. However for taking the $j \to \infty$ limit it is better to restrict to continuous b.)

Figure 1 gives an example of Dirichlet eigenfunctions in two domains: a disk and a stadium. We observe that:

- The eigenfunction for the disk has a lot of geometric structure. Moreover, it is small near the center of the disk.
- By contrast, the eigenfunctions for the stadium spread out evenly on the entire domain. The two eigenfunctions are different when looking closely at the pictures but they appear similar from far away.

We also see that both pictures show a lot of oscillation. In fact, u_j oscillates on the scale

$$h_j = 1/\lambda_j, \tag{3}$$

so λ_j can be interpreted as the frequency of oscillation (which is why we denoted the eigenvalue by λ_j^2 and not λ_j). To illustrate this, consider the case when $\Omega = (0,\pi)^2$ is a square, with eigenfunctions $u_{k\ell} = \sin(kx_1)\sin(\ell x_2)$ where $k, \ell \in \mathbb{N}$. Then $u_{k\ell}$ oscillates at frequency $\lambda_{k\ell} = \sqrt{k^2+\ell^2}$.

What makes eigenfunctions look so different for the disk and for the stadium? The answer lies in the behavior of the corresponding classical dynamical system. For domains with boundary, this system is the *billiard ball flow*, modeling a classical particle in Ω which moves in a straight line until collision with the boundary and then follows the standard law of reflection.

Figure 2 shows a single longtime billiard ball trajectory in the disk and two such trajectories in the stadium. In the disk, the trajectory follows a regular pattern (perhaps reminding one of a ball of twine) and leaves out a region near the center. In the stadium, the trajectories appear chaotic, in particular covering the whole domain. In fact, they *equidistribute*: the amount of time the trajectory

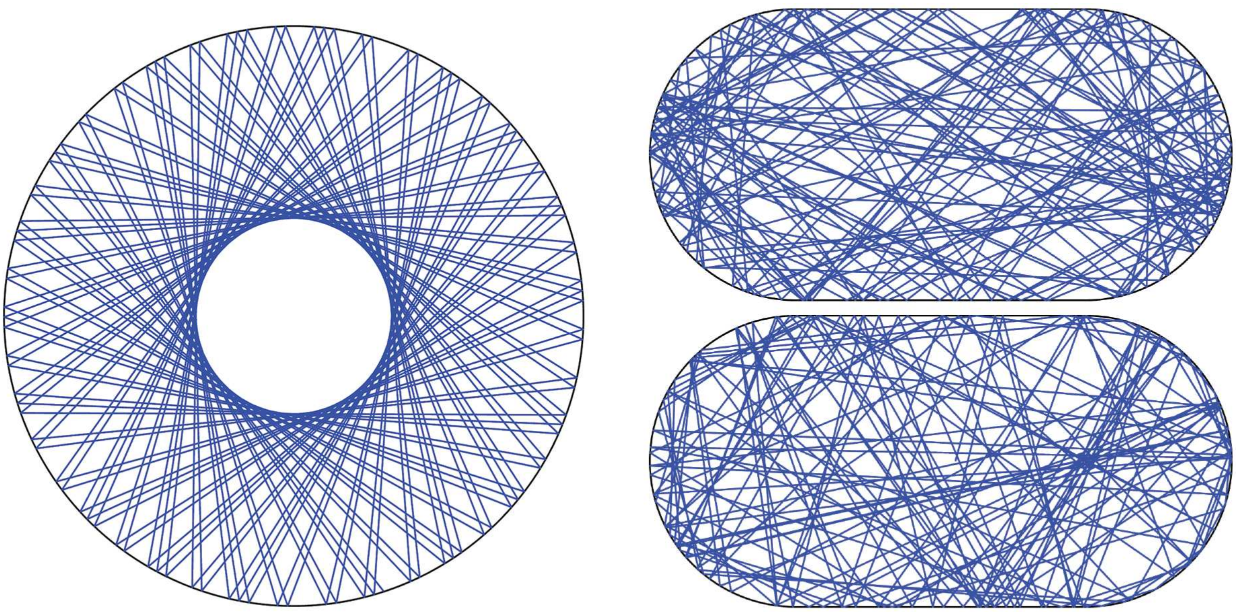

Figure 2. Typical billiard ball trajectories in a disk and a stadium after many bounces.

spends in a set S tends to the ratio of the area of S to the area of the domain, asymptotically as the length of the trajectory tends to infinity.

From now on we focus on the chaotic case. The goal of this section is to formulate precisely a result known as *quantum ergodicity*, which informally states that

If most billiard trajectories equidistribute,
then most eigenfunctions equidistribute.

We first explain what it means for most billiard trajectories to equidistribute, which is naturally given by the concept of *ergodicity*. Denote the billiard ball flow by (see Figure 3)

$$\varphi^t : \Omega \times \mathbb{S}^1 \to \Omega \times \mathbb{S}^1, \quad t \in \mathbb{R}. \tag{4}$$

Here $\Omega \times \mathbb{S}^1$ consists of all possible positions and (unit) velocity vectors and $\varphi^t(x_0, \xi_0)$ gives the position and the velocity after time t of the billiard ball particle starting at position x_0 and velocity ξ_0. The billiard ball flow might be undefined for some (x_0, ξ_0) and t because of various problems that can happen at the boundary, but under reasonable assumptions these form a measure 0 set and thus will not matter for the definition below—see [ZZ96]. We use the natural φ^t-invariant volume measure on $\Omega \times \mathbb{S}^1$

$$\mu_L = c\, dx dS(\xi)$$

with the constant $c > 0$ chosen so that μ_L is a probability measure.

Definition 1. *We say that the billiard ball flow φ^t is* ergodic *(with respect to μ_L) if for μ_L-almost every (x_0, ξ_0), the trajectory $\varphi^t(x_0, \xi_0)$ equidistributes, namely for any $a \in C(\Omega \times \mathbb{S}^1)$ we have as $T \to \infty$*

$$\frac{1}{T} \int_0^T a(\varphi^t(x_0, \xi_0))\, dt \to \int_{\Omega \times \mathbb{S}^1} a(x, \xi)\, d\mu_L.$$

Note that we require equidistribution in both position (x) and velocity (ξ) variables.

Coming back to Figure 2, we remark that the billiard ball flow is not ergodic for the disk (in fact, it has a conserved quantity: the angle at which the trajectory intersects the boundary circle stays the same with each bounce), but it is ergodic for the stadium, as proved by Bunimovich in 1974.

Next, we give a definition of equidistribution for eigenfunctions, taking the limits of expressions (2):

Definition 2. *Assume that u_{j_k}, $j_k \to \infty$, is a sequence of eigenfunctions from* (1). *We say that u_{j_k}* equidistributes in position *if for each $b \in C(\Omega)$*

$$\int_\Omega b(x) |u_{j_k}(x)|^2\, dx \to \frac{1}{\mathrm{vol}(\Omega)} \int_\Omega b(x)\, dx.$$

The above definition talks about the *macroscopic behavior* of u_{j_k} since we first fix the classical observable b and then take the limit $k \to \infty$. A quantum mechanical interpretation of equidistribution of eigenfunctions is as follows: in the high energy limit, the probability of observing the pure state quantum particle in a "nice" set $S \subset \Omega$ becomes proportional to the area of S.

We are now ready to state a version of quantum ergodicity. In the present setting it is due to Zelditch–Zworski [ZZ96], with an earlier contribution by Gérard–Leichtnam which covered the example of the stadium. In the setting of manifolds without boundary, the result goes back to the seminal works of Shnirelman, Zelditch, and Colin de Verdière in the 1970s–1980s.

Theorem 1. *Assume that the billiard ball flow φ^t is ergodic. Then there exists a density 1 increasing sequence $j_k \to \infty$ such*

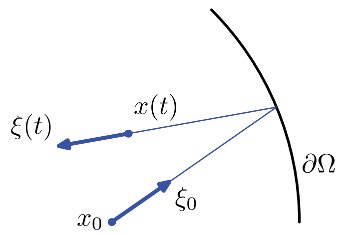

Figure 3. The billiard ball flow trajectory $(x(t), \xi(t)) = \varphi^t(x_0, \xi_0)$. Here t is the distance traveled by the billiard ball. The study of such billiard ball flows is an old and subtle subject—see for instance Avila–De Simoi–Kaloshin [ADSK16] for recent progress.

that the corresponding sequence of eigenfunctions u_{j_k} equidistributes in position. Here "density 1" means that

$$\frac{\#\{k \mid j_k \le N\}}{N} \to 1 \quad \text{as } N \to \infty.$$

2. Semiclassical Measures

We now discuss semiclassical quantization and classical/quantum correspondence, which underlie the proof of Theorem 1 and other results given below. This leads us to *semiclassical measures*, which are a way to capture the concentration of high-energy eigenfunctions simultaneously in position and frequency, and to a more refined version of quantum ergodicity.

A quantization maps smooth functions $a(x, \xi)$ on $\mathbb{R}^{2n}$, interpreted as classical observables, to operators $a(x, D_x)$ on $C^\infty(\mathbb{R}^n)$, interpreted as the corresponding quantum observables. Here the coordinate functions x_ℓ should be mapped to the multiplication operators $u \mapsto x_\ell u$, while ξ_ℓ should be mapped to the differentiation operators $D_{x_\ell} = -i\partial_{x_\ell}$. One can define a quantization procedure using the Fourier transform:

$$a(x, D_x)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x, \xi)\hat{u}(\xi)\, d\xi.$$

By the Fourier inversion formula, if a is a function of x only, then $a(x, D_x)u = au$ is the corresponding multiplication operator; in particular, $1(x, D_x)$ is the identity. More generally, if a is a polynomial in ξ, then $a(x, D_x)$ is a differential operator. Since differential operators do not in general commute with each other, the map $a \mapsto a(x, D_x)$ cannot be an algebra homomorphism; however, $a(x, D_x)b(x, D_x) - (ab)(x, D_x)$ consists of lower-order terms. This is related to the product rule (6) below.

In the theory of PDE, operators of the form $a(x, D_x)$ are called *pseudodifferential operators*. In mathematics they were originally motivated by singular integral operators, boundary value problems, and several complex variables. Eventually, that mathematical theory merged with the the theories of quantization from quantum mechanics—see [Zwo12] for general properties of quantization and for pointers to the vast literature on the subject.

As remarked in (3) above, the eigenfunction u_j oscillates on scale $h_j = \lambda_j^{-1}$. Thus we expect $D_{x_\ell} u_j$ to be roughly of size λ_j. It then makes sense to multiply D by h_j, which gives the *semiclassical quantization procedure*

$$\mathrm{Op}_h(a) := a(x, hD_x). \tag{5}$$

Semiclassical quantization has several algebraic properties, such as the product rule

$$\mathrm{Op}_h(a)\,\mathrm{Op}_h(b) = \mathrm{Op}_h(ab) + \mathcal{O}(h) \tag{6}$$

and the commutator rule

$$[\mathrm{Op}_h(a), \mathrm{Op}_h(b)] = -ih\,\mathrm{Op}_h(\{a, b\}) + \mathcal{O}(h^2). \tag{7}$$

Here $\{a, b\} = \sum_{\ell=1}^n (\partial_{\xi_\ell} a)(\partial_{x_\ell} b) - (\partial_{x_\ell} a)(\partial_{\xi_\ell} b)$ is the Poisson bracket of a and b, and the remainders are understood in the sense of operator norm on appropriate spaces. Another key property, connecting classical and quantum dynamics, is *Egorov's theorem*:

$$U(-t)\,\mathrm{Op}_h(a)U(t) = \mathrm{Op}_h(a \circ \varphi^t) + \mathcal{O}(h) \tag{8}$$

where a is a smooth compactly supported function on $\Omega \times \mathbb{R}^2$, $U(t) = e^{ith\Delta/2}$ is the Schrödinger group associated to the Dirichlet Laplacian on the domain Ω, and φ^t is the billiard ball flow (4) extended appropriately to $(x, \xi) \in \Omega \times \mathbb{R}^2$. Note that $U(t)$ describes evolution of quantum wave functions by the Schrödinger equation and φ^t describes evolution of classical particles in Ω. (Some care is needed at the boundary of Ω but we omit the details here.)

We now introduce semiclassical measures corresponding to eigenfunctions:

Definition 3. *Assume that u_{j_k} is a sequence of eigenfunctions. We say that u_{j_k}* converges semiclassically *to a Borel measure μ on $\Omega \times \mathbb{R}^2$ if for each (sufficiently regular) function $a(x, \xi)$ on $\Omega \times \mathbb{R}^2$ we have (putting $h_{j_k} := \lambda_{j_k}^{-1}$)*

$$\langle \mathrm{Op}_{h_{j_k}}(a)u_{j_k}, u_{j_k}\rangle_{L^2(\Omega)} \to \int_{\Omega\times\mathbb{R}^2} a\, d\mu. \tag{9}$$

We say that a measure μ on $\Omega \times \mathbb{R}^2$ is a semiclassical measure *if there exists a sequence of eigenfunctions converging to it.*

The left-hand side of (9) has a natural quantum mechanical interpretation: it is the expected value of the observable $a(x, \xi)$ where x is the position and ξ is the momentum of the quantum particle. Thus the limiting measure μ describes the probability distribution of the particle in position and momentum in the high energy limit along the sequence of quantum pure states u_{j_k}. From a mathematical point of view, μ captures the distribution of mass of u_{j_k} in position (x) and frequency (ξ).

Each semiclassical measure μ has the following properties:

(a) μ is a probability measure;
(b) the support of μ is contained in $\Omega \times \mathbb{S}^1$;
(c) μ is invariant under the billiard ball flow φ^t.

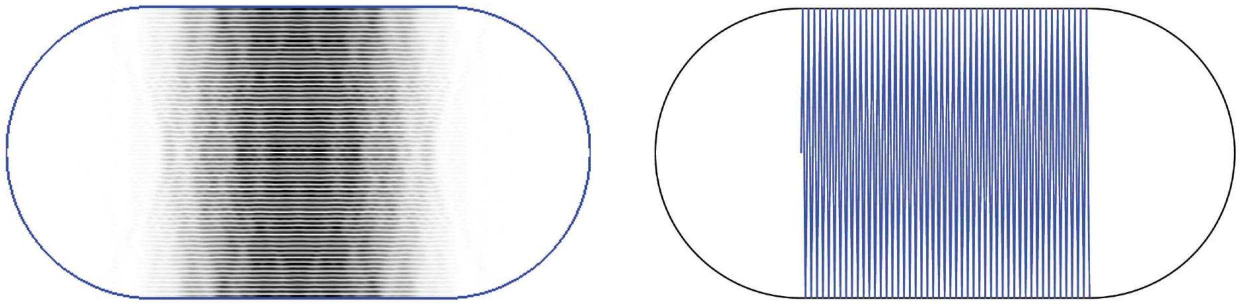

Figure 4. An example of an anomalous, nonequidistributing, eigenfunction for the stadium (left); such eigenfunctions were numerically observed by Heller in *Physical Review Letters* in 1984. Its existence is related to the presence of "mildly chaotic" billiard ball trajectories which take a long time to exhibit chaotic behavior, like the one pictured on the right. A generic stadium has a sequence of nonequidistributing eigenfunctions. However, it is an open problem to show that such eigenfunctions localize precisely on the mildly chaotic trajectories.

Here (a) corresponds to the normalization $\|u_{j_k}\|_{L^2(\Omega)} = 1$ and the fact that $\mathrm{Op}_h(1)$ is the identity. The property (b) corresponds to the correct choice of the semiclassical scaling parameter $h_{j_k} = \lambda_{j_k}^{-1}$, so that after rescaling u_{j_k} oscillates at unit length frequency. Finally, the property (c) follows from Egorov's theorem: indeed, pairing both sides of (8) with u_{j_k} and passing to the limit we see that $\int (a \circ \varphi^t)\, d\mu = \int a\, d\mu$ for all a.

There are many measures satisfying properties (a)–(c) above. Of particular importance is the *Liouville measure* $\mu_L = c\, dx dS(\xi)$ featured in Definition 1, which is in some sense the most "spread-out" invariant measure. The opposite, most "concentrated" case, is the delta measure on a periodic trajectory of φ^t. One of the central questions in quantum chaos is:

What measures can arise as semiclassical limits of high energy eigenfunctions?

This question is discussed in more detail in §4. It is not restricted to the chaotic case: even for tori it is a nontrivial question which attracted the attention of many including Jean Bourgain; see Lester–Rudnick [LR17] for a recent contribution.

We can now state a stronger version of quantum ergodicity, giving equidistribution in both position and frequency. Following Definition 2, we say that a sequence of eigenfunctions *semiclassically equidistributes* if it converges to the Liouville measure in the sense of (9).

Theorem 2. *Assume that the billiard ball flow φ^t is ergodic. Then there exists a density 1 sequence $j_k \to \infty$ such that the corresponding sequence of eigenfunctions u_{j_k} semiclassically equidistributes.*

Note that semiclassical equidistribution implies equidistribution in position of Definition 2, taking observables of the form $a(x,\xi) = b(x)$ in (9); thus Theorem 2 implies Theorem 1. On the other hand, for general (not necessarily ergodic) domains one might have equidistribution in position without semiclassical equidistribution, see Marklof–Rudnick [MR12].

We also mention briefly the case of *mixed systems*, having a positive measure subset of $\Omega \times \mathbb{S}^1$ on which the billiard ball flow is ergodic. For a special class of these systems, namely generic mushroom billiards, Galkowski [Gal14] and Gomes [Gom18] showed *Percival's conjecture*, giving a positive density sequence of eigenfunctions equidistributing in the ergodic region; for earlier numerics in this setting, see Barnett–Betcke [BB07].

3. QUE and Strongly Chaotic Systems

A natural question to ask, known as the quantum unique ergodicity (QUE) conjecture, is whether Theorem 2 holds *without* passing to a density 1 subsequence:

Is Liouville measure the only semiclassical measure?

For general ergodic settings this can fail. In fact, this is the case for a generic stadium domain as shown by Hassell [Has10]; see Figure 4.

A natural setting in which QUE is more feasible (and was explicitly conjectured by Rudnick–Sarnak in 1994) is that of strongly chaotic systems, which is a subclass of ergodic systems for which small perturbations of any trajectory lead to exponentially fast divergence from the original trajectory. More precisely, for such a system the tangent space to $\Omega \times \mathbb{S}^1$ splits into the flow, unstable, and stable subspaces, and the differential of the flow is exponentially expanding on the unstable spaces and contracting on the stable spaces as time goes to infinity. This implies that the flow has a positive Lyapunov exponent and is related to the "butterfly effect" mentioned in the opening paragraph of this article.

To give an example of a strongly chaotic system, we move away from planar domains to the setting of manifolds without boundary. Let (M, g) be a compact Riemannian manifold. The analog of Dirichlet eigenfunctions (1) is given by eigenfunctions of the Laplace–Beltrami

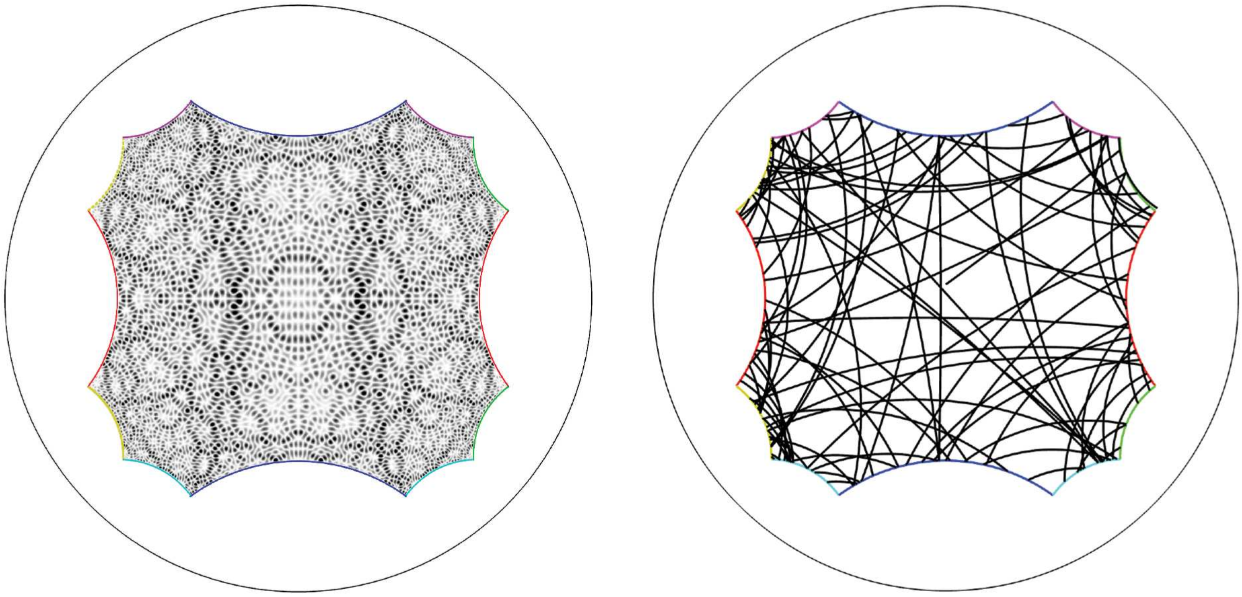

Figure 5. An eigenfunction (left) and a geodesic (right) on a genus 2 hyperbolic surface obtained by gluing together the same color sides of the pictured dodecagon, embedded in the Poincaré disk model of the hyperbolic plane. The eigenfunction is computed using the method developed by Strohmaier–Uski [SU13].

operator Δ_g induced by the metric g:

$$-\Delta_g u_j(x) = \lambda_j^2 u_j(x), \quad u_j \in C^\infty(M). \tag{10}$$

The semiclassical quantization introduced in (5) can be defined on manifolds, if we take a to be a function on the cotangent bundle T^*M. The appearance of the cotangent bundle is already evident for differential operators: if X is a vector field on M, then the first order differential operator $-ihX$ is equal to $\mathrm{Op}_h(\chi) + \mathcal{O}(h)$ where $\chi(x,\xi) = \langle \xi, X(x)\rangle$ is the linear function on the fibers of T^*M defined by X. Note also that the Poisson bracket featured in the commutator rule (7) is well-defined on functions on T^*M since the latter has a natural symplectic form.

The corresponding classical dynamical system is the geodesic flow

$$\varphi^t : S^*M \to S^*M$$

where S^*M is the unit cotangent bundle of M, consisting of pairs (x,ξ) where $x \in M$ and $\xi \in T_x^*M$ satisfies $|\xi|_g = 1$. Here ξ is the cotangent vector dual to the velocity vector of the geodesic via the metric g.

It is a result of Anosov in the 1960s that if the metric g has *negative curvature*, then the geodesic flow φ^t is strongly chaotic. An important family of examples of negatively curved manifolds, appearing in many areas of mathematics, is given by *hyperbolic surfaces* which are surfaces of Gauss curvature -1; see Figure 5.

In the setting of Riemannian manifolds, semiclassical measures are supported on S^*M and invariant under the flow φ^t, and an analog of quantum ergodicity (Theorem 2) holds.

Coming back to QUE, in a special setting of arithmetic hyperbolic surfaces QUE for joint eigenfunctions of the Laplacian and all Hecke operators (which are additional symmetries commuting with the Laplacian) was proved by Lindenstrauss [Lin06]. However, in general this conjecture is completely open and in fact there are toy models where it fails; the most celebrated one is described in §5 below.

4. More on Semiclassical Measures

With QUE seeming out of reach, we return to the question asked in §2, now in the setting of manifolds without boundary: what measures can arise as semiclassical limits of high energy eigenfunctions? We discuss two results giving restrictions on such measures.

We start with the more recent result, due to the author, Jin, and Nonnenmacher [DJN22], and relying on earlier work of Bourgain and the author on the fractal uncertainty principle:

Theorem 3. *Let μ be a semiclassical measure on a negatively curved surface. Then $\operatorname{supp}\mu = S^*M$, that is $\mu(U) > 0$ for any nonempty open set $U \subset S^*M$.*

Theorem 3 together with the unique continuation principle implies a lower bound on the mass of eigenfunctions: for any nonempty open set $V \subset M$ we have

$$\| \mathbb{1}_V u_j \|_{L^2} \geq c_V > 0$$

where the constant c_V is independent of the eigenvalue λ_j. This can be thought of as having no whitespace in Figure 5: for any given macroscopic ball, the probability of finding the quantum particle in that ball is separated away from 0.

$t = 0$ $t = 1$ $t = 2$ $t = 3$ $t = 4$ $t = 5$ $t = 6$

Figure 6. Evolution of an image by the map A^t, $t = 0, \dots, 6$, where $A : \mathbb{T}^2 \to \mathbb{T}^2$ corresponds to the matrix $A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$.

It is an open question whether Theorem 3 holds in dimensions ≥ 3.

Theorem 3 also implies that the delta measure on a closed geodesic cannot be a semiclassical measure. However, the latter fact (conjectured by Colin de Verdière in the 1980s) was already known as a corollary of *entropy bounds* of Anantharaman and Nonnenmacher. These bounds are true for general strongly chaotic systems, but for simplicity we state the result of [AN07] in a special case:

Theorem 4. *Let μ be a semiclassical measure on a hyperbolic surface. Then the Kolmogorov–Sinai entropy of μ satisfies*

$$\mathrm{h}_{\mathrm{KS}}(\mu) \geq \tfrac{1}{2}. \tag{11}$$

We do not give a definition of the entropy $\mathrm{h}_{\mathrm{KS}}(\mu)$ here but remark that it measures the complexity of the flow φ^t with respect to the measure μ. In particular, the entropy of a delta measure on a closed geodesic is equal to 0, while the entropy of the Liouville measure is equal to 1, so in some sense (11) excludes half of φ^t-invariant measures as candidates for semiclassical measures.

5. Quantum Cat Maps

We finally discuss semiclassical measures in the toy model setting of quantum cat maps, where a striking counterexample to QUE is known.

For quantum cat maps, the phase space T^*M is replaced by the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and the geodesic flow φ^t, by a linear map. This has the advantage that the underlying dynamics, while still strongly chaotic, is simpler to understand; moreover, it is easier to compute eigenvalues and eigenfunctions numerically. On the other hand, it is harder to explain the analog of the eigenvalue problems (1), (10).

We first discuss linear maps on the torus, which in this setting are analogs of the time-one map of the geodesic flow. Let $A \in \mathrm{SL}(2, \mathbb{Z})$ be a 2×2 matrix with integer entries and determinant 1. The linear map on $\mathbb{R}^2$ induced by A descends to a diffeomorphism of the torus $\mathbb{T}^2$, which we we still denote by A. The matrix A is called *hyperbolic*, and the corresponding map on $\mathbb{T}^2$ is called a *cat map* (a term coined by Arnold), if $|\operatorname{tr} A| > 2$. In this case A has two real eigenvalues ω, ω^{-1} with $|\omega| > 1$; the corresponding eigenspaces give the unstable and stable directions for the cat map and can be used to show that it is ergodic. See Figure 6.

We next discuss discrete microlocal analysis and semiclassical quantization. The space $L^2(M)$ of square-integrable functions on a manifold is replaced by the finite dimensional space $\mathbb{C}^N$. Here the semiclassical parameter is

$$h = \frac{1}{2\pi N}.$$

The discrete version of the Fourier transform, $\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N$, is given by

$$(\mathcal{F}_N f)_j = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} e^{-\frac{2\pi i j \ell}{N}} f_\ell.$$

We note that this is the Fourier transform used in signal processing and FFT algorithms.

One can define an analog of the quantization procedure (5), mapping a smooth function $a(x, \xi)$ on the torus $\mathbb{T}^2$ to a sequence of operators

$$\mathrm{Op}_N(a) : \mathbb{C}^N \to \mathbb{C}^N.$$

We do not give a proper definition here but note that similarly to semiclassical quantization on $\mathbb{R}^n$

- if a is a function of x only, then $\mathrm{Op}_N(a)$ is a multiplication operator: $(\mathrm{Op}_N(a) f)_j = a(j/N) f_j$;
- if a is a function of ξ only, then $\mathrm{Op}_N(a)$ is a Fourier multiplier: $(\mathcal{F}_N \mathrm{Op}_N(a) f)_j = a(j/N)(\mathcal{F}_N f)_j$.

One also has analogues of the product rule (6) and the commutator rule (7). For the latter, the Poisson bracket is defined as before and corresponds to the symplectic form $d\xi \wedge dx$. Implicit in the construction below is the fact that the map A preserves the symplectic form on $\mathbb{T}^2$, just as the geodesic flow φ^t preserves the symplectic form on T^*M.

We now introduce quantizations of a linear map on $\mathbb{T}^2$ induced by a matrix $A \in \mathrm{SL}(2, \mathbb{Z})$, which in this setting are analogous to the time-one map of the Schrödinger group. For technical reasons we restrict to the case of even N. Quantizations of A are sequences of unitary operators $U_N : \mathbb{C}^N \to \mathbb{C}^N$ which satisfy the following exact version of Egorov's theorem (8): for all $a \in C^\infty(\mathbb{T}^2)$

$$U_N^{-1} \mathrm{Op}_N(a) U_N = \mathrm{Op}_N(a \circ A). \tag{12}$$

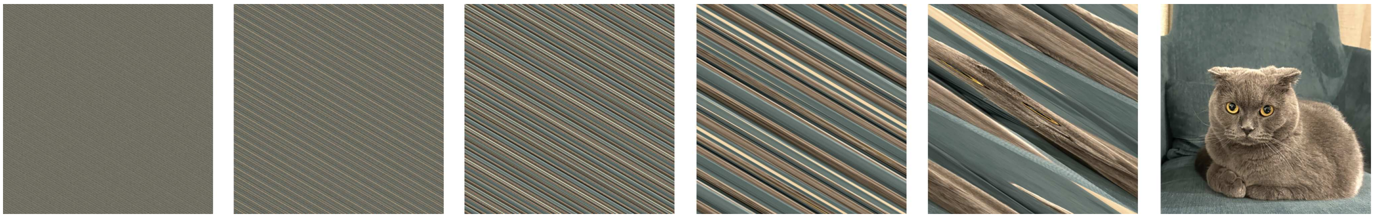

Figure 7. Continuation of Figure 6, showing the times $t = 7, \dots, 12$. Here we took the special resolution $N = 1560$ points per side of the square; the picture illustrates the fact that $A^{12} \bmod N = I$. A short period of the discretized classical cat map implies that the associated quantum cat map also has a short period, which is used in the example (14) below.

One way to compute these explicitly is as follows. Consider the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then a quantization of A_1 is given by the multiplication operator

$$(U_N f)_j = e^{\frac{\pi i j^2}{N}} f_j \tag{13}$$

and a quantization of A_2 is given by the discrete Fourier transform $\mathcal{F}_N$. The matrices A_1, A_2 generate the group $\mathrm{SL}(2, \mathbb{Z})$ so this gives a way to quantize every linear map on the torus. One explanation for the formula (13) is as follows: in the continuous setting the map $f(x) \mapsto e^{\frac{i\varphi(x)}{h}} f(x)$ is a phase shift, quantizing the transformation $(x, \xi) \mapsto (x, \xi + \varphi'(x))$ (which is most evident for the case when $\varphi(x) = x\eta$ is a linear function); putting $\varphi(x) := x^2/2$, $h := (2\pi N)^{-1}$, and $x := j/N$ we get the operator (13) and the associated transformation $(x, \xi) \mapsto (x, \xi + x)$ is linear with the matrix A_1.

From now on, let $U_N : \mathbb{C}^N \to \mathbb{C}^N$ be a quantization of the linear map on $\mathbb{T}^2$ corresponding to a hyperbolic matrix $A \in \mathrm{SL}(2, \mathbb{Z})$. We call this sequence of operators a *quantum cat map*. Let $N_k \to \infty$ and $u_{N_k} \in \mathbb{C}^{N_k}$ be a sequence of normalized eigenfunctions of the map U_{N_k}:

$$U_{N_k} u_{N_k} = z_k u_{N_k}, \quad |z_k| = 1, \quad \|u_{N_k}\| = 1.$$

Similarly to (9), we say that u_{N_k} converges semiclassically to a measure μ on $\mathbb{T}^2$ if for all $a \in C^\infty(\mathbb{T}^2)$ we have

$$\langle \mathrm{Op}_{N_k}(a) u_{N_k}, u_{N_k} \rangle \to \int_{\mathbb{T}^2} a \, d\mu.$$

The resulting limiting measures are called *semiclassical measures* for the quantum cat map U_N. It follows from the normalization and Egorov's theorem (12) that each semiclassical measure μ is a probability measure invariant under the map A.

In the setting of quantum cat maps, there are versions of quantum ergodicity (Theorem 2), due to Bouzouina–De Bièvre in 1996, the full support property (Theorem 3), due to Schwartz in 2021, and entropy bounds (Theorem 4), due to Faure–Nonnenmacher in 2004 and Brooks in 2010.

However, there is a remarkable counterexample to QUE due to Faure–Nonnenmacher–De Bièvre [FNDB03]. More precisely, if γ is any given closed orbit of the map A, δ_γ is the A-invariant probability measure on γ, and $dxd\xi$ is the volume measure on $\mathbb{T}^2$, then there exists a sequence of eigenfunctions u_{N_k} converging semiclassically to the measure

$$\mu = \tfrac{1}{2}\delta_\gamma + \tfrac{1}{2}\, dxd\xi. \tag{14}$$

Note that the entropy of μ is half the entropy of $dxd\xi$ and (14) shows that the entropy bound for quantum cat maps is sharp. See Figure 8 for a numerical illustration.

The construction of (14) relies on the fact (observed by Bonechi–De Bièvre in 2000) that there exists a sequence $N_k \to \infty$ such that the restriction of the classical cat map A to the discrete set of points $N_k^{-1}\mathbb{Z}^2/\mathbb{Z}^2 \subset \mathbb{T}^2$ is periodic with a short period $\sim \log N_k$, and correspondingly the quantum cat map U_{N_k} also has a short period—see Figure 7.

There is also an analogue of arithmetic QUE, due to Kurlberg–Rudnick in 2000: there exists a basis of eigenfunctions of U_N which converges semiclassically to the measure $dxd\xi$. This does not contradict the counterexample (14) since the operator U_{N_k} used there has eigenvalues of high multiplicity, and the eigenfunctions used in (14) do not belong to the arithmetic basis.

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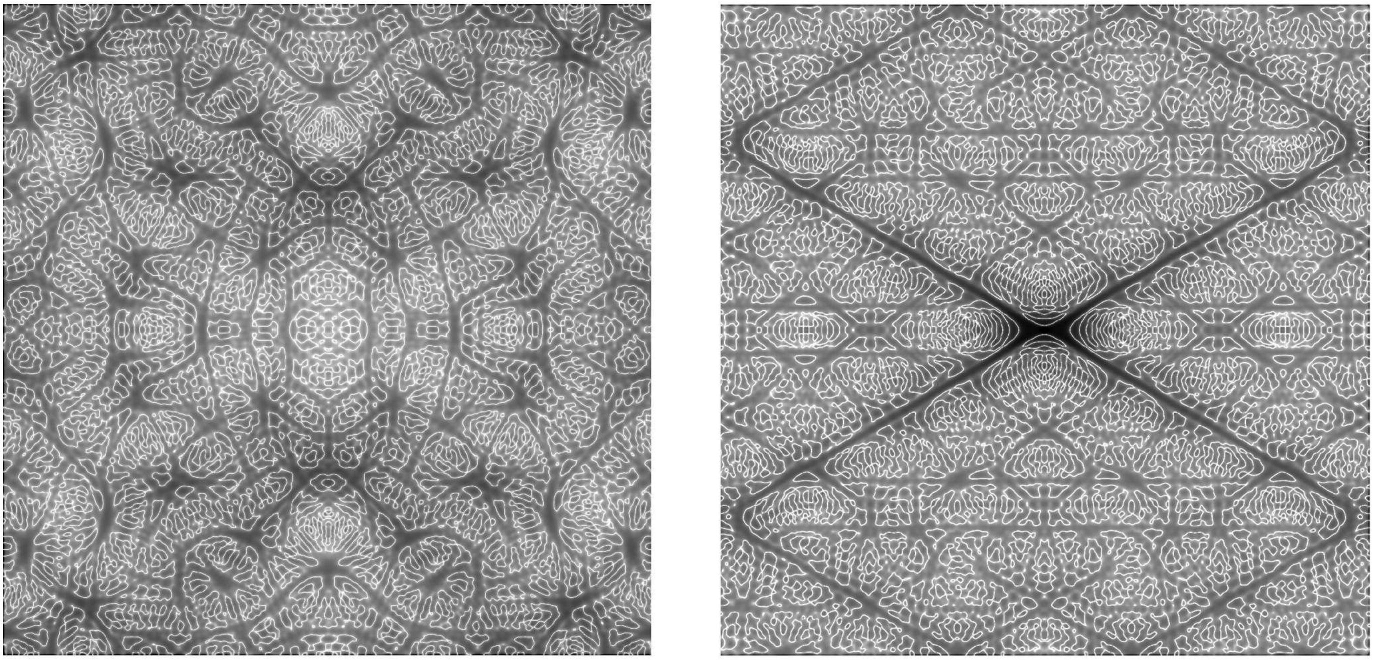

Figure 8. Modified Wigner functions of two eigenfunctions of the quantum cat map with the same matrix A as in Figure 6 and $N = 780$ (which is half of the special N in Figure 7). These pictures show the concentration of the eigenfunction simultaneously in position and frequency, see e.g. [DJ23]; darker shading corresponds to larger Wigner transform. On the left is a typical eigenfunction, showing equidistribution. On the right is an anomalous eigenfunction corresponding to the semiclassical measure (14) with γ being the fixed point $(\frac{1}{2}, \frac{1}{2})$. Note that the scars seen on the stable/unstable manifolds of the fixed point carry about $(\log N)^{-1}$ portion of the mass of the eigenfunction and thus do not contribute to the limiting semiclassical measure.

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