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ON THE TWO-DIMENSIONAL JACOBIAN CONJECTURE: MAGNUS' FORMULA REVISITED, I

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To the memory of Shreeram Shankar Abhyankar

Let K be an algebraically closed field of characteristic 0. For $f, g \in K[x, y]$, when the Jacobian $(\partial f/\partial x)(\partial g/\partial y) - (\partial g/\partial x)(\partial f/\partial y)$ is a constant, Magnus' formula describes the relations between the homogeneous degree pieces f_i and g_i. We show a more general version of Magnus' formula, which could provide a potentially useful tool to prove the Jacobian conjecture.

1. Introduction

The Jacobian conjecture, raised by Keller [8], has been studied by many mathematicians. A survey is given in [4; 5]. In this paper, we exclusively deal with the plane case. Hence, whenever we write the Jacobian conjecture, we mean the two-dimensional Jacobian conjecture.

Let K be an algebraically closed field of characteristic 0, and let $\mathcal{R} = K[x, y]$.

Jacobian conjecture. *Let $f(x, y), g(x, y) \in \mathcal{R}$. Consider the polynomial map $\pi : \mathcal{R} \longrightarrow \mathcal{R}$ given by* $\pi(x) = f(x, y)$ and $\pi(y) = g(x, y)$. *If the Jacobian of the map*

$$\det\begin{pmatrix} \partial f/\partial x & \partial g/\partial x \\ \partial f/\partial y & \partial g/\partial y \end{pmatrix}$$

is a nonzero constant, then the map is bijective.

For simplicity, let

$$[f, g] := \det\begin{pmatrix} \partial f/\partial x & \partial g/\partial x \\ \partial f/\partial y & \partial g/\partial y \end{pmatrix} \in \mathcal{R}$$

for any pair of polynomials $f, g \in \mathcal{R}$. Similarly $[f, g]$ is defined for $f, g \in \mathcal{R}[\![t]\!] = K[\![t]\!][x, y]$.

A useful tool to study this conjecture is the Newton polygon. One source for this is [3], but we redefine it here. Let

$$f = \sum_{i,j\geq 0} f_{ij} x^i y^j$$

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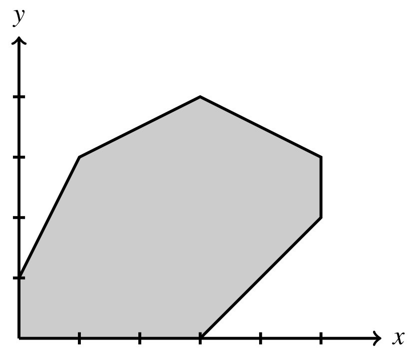

Figure 1. $N(f)$ for $f = y + 7xy^3 + \sqrt{7}x^3y^4 - 4x^5y^3 + 2x^5y^2 - \frac{1}{2}x^3 + xy + 1$.

be a polynomial in $\mathcal{R}$. The *support* of f is defined as

$$\operatorname{supp}(f) = \{(i, j) \mid f_{ij} \neq 0\} \subset \mathbb{Z}^2 \subset \mathbb{R}^2.$$

The *Newton polygon* for f, which we denote $N(f)$, is defined to be the convex hull of $\operatorname{supp}(f)$[1] in $\mathbb{R}^2$. Note that $N(f) \subset \mathbb{R}^2_{\geq 0}$. See Figure 1 for an example of a Newton polygon. The support and Newton polygon of a Laurent polynomial in $K[x^{\pm 1}, y^{\pm 1}]$ are similarly defined.

Throughout this paper, whenever we consider pairs of polynomials f, g with $[f, g] \in K$, we only consider such pairs for which both $N(f)$ and $N(g)$ contain $(1, 0)$, $(0, 1)$, and $(0, 0)$. As long as $\deg(f)$ and $\deg(g)$ are positive, it is always possible to obtain such a pair by adding a generic constant to f and g and applying some linear change of variables x and y, which does not change $[f, g]$.

It is known that the following conjecture implies the Jacobian conjecture; for instance, see [5, Theorem 10.2.23]:

Conjecture A. Let $a, b \in \mathbb{Z}_{>0}$ be relatively prime. Suppose that $F, G \in \mathcal{R}$ satisfy the following:

(1) $[F, G] \in K$;

(2) $\{(1, 0), (0, 1), (0, 0)\} \subset N(F) \cap N(G)$ and $N(F)$ is similar to $N(G)$ with the origin as the center of similarity and with ratio $\deg(F) : \deg(G) = a : b$; and

(3) $\min(a, b) \geq 2$.

Then $[F, G] = 0$.

Let $W = \{(u, v) \in \mathbb{Z}^2 : u > 0 \text{ or } v > 0, \text{ and } \gcd(|u|, |v|) = 1\}$. An element $w = (u, v) \in W$ is called a *direction*. To each such direction, we consider its w-grading on $\mathcal{R}$ by defining the w-degree of the monomial $x^i y^j$ as $n = ui + vj$. Define $\mathcal{R}_n \subset \mathcal{R}$ to be the K-subspace generated by monomials of w-degree n. Then $\mathcal{R} = \oplus_{n \in \mathbb{Z}} \mathcal{R}_n$. A nonzero element P of $\mathcal{R}_n$ is called a w-homogeneous element of $\mathcal{R}$; the integer n is called the w-degree of P and is denoted w-$\deg(P)$. The element of the highest w-degree in the homogeneous decomposition of a nonzero polynomial P is called its w-leading form and is denoted by P_+. The w-degree of P is by definition w-$\deg(P_+)$.

[1] In [3], the Newton polygon was defined as the convex hull of $\operatorname{supp}(f) \cup \{(0, 0)\}$ in $\mathbb{R}^2$, which is different from our definition.

Magnus [9, Theorem 1] produced a formula which inspired much of the work for this paper. His formula was published almost 70 years ago, but has not been used in almost any paper but [10]. Even in [10], only a small piece of information from the formula was utilized. The first main result in our paper is a more general version of Magnus' formula, given in Theorem 1.1. In what follows, the binomial coefficient $\binom{A}{B}$ is defined by

$$\binom{A}{B} := \frac{A(A-1)\cdots(A-B+1)}{B!}$$

for any real number A and any nonnegative integer B.

Theorem 1.1. Suppose $[F, G] \in K$. For any direction $w = (u, v) \in W$, let $d = w\text{-deg}(F_+)$ and $e = w\text{-deg}(G_+)$. Write the w-homogeneous degree decompositions $F = \sum_{i \le d} F_i$ and $G = \sum_{i \le e} G_i$. Then there exists a unique[2] sequence of constants $c_0, c_1, \ldots, c_{d+e-u-v-1} \in K$ such that $c_0 \neq 0$ and

$$G_{e-\mu} = \sum_{\gamma=0}^{\mu} c_\gamma \sum \binom{(e-\gamma)/d}{\sum_{\alpha \le d-1} \nu_{\gamma,\alpha}} \frac{\left(\sum_{\alpha \le d-1} \nu_{\gamma,\alpha}\right)!}{\prod_{\alpha \le d-1} \nu_{\gamma,\alpha}!} F_d^{(e-\gamma)/d - \sum_{\alpha \le d-1} \nu_{\gamma,\alpha}} \prod_{\alpha \le d-1} F_\alpha^{\nu_{\gamma,\alpha}} \tag{1-1}$$

for every integer $\mu \in \{0, 1, \ldots, d+e-u-v-1\}$, where the inner sum is to run over all combinations of nonnegative integers $\nu_{\gamma,\alpha}$ satisfying $\sum_{\alpha \le d-1}(d-\alpha)\nu_{\gamma,\alpha} = \mu - \gamma$. Furthermore, $c_\gamma = 0$ if $r(e-\gamma)/d \notin \mathbb{Z}$, where $r \in \mathbb{Z}_{>0}$ is the largest integer such that $F_d^{1/r} \in K[x, y]$.

In a series of forthcoming papers, we will make progress toward Conjecture A, using Theorem 1.1. The purpose of this paper is to write out a proof of Theorem 1.1[3] and illustrate how useful this theorem is.

Let F and G satisfy assumptions (1) and (2) in Conjecture A. Then $[F, G] \in K$ implies $F_+^{1/a} \in K[x, y]$ for any direction $w \in W$ (for instance, see [1; 2; 11]). Let $\mathcal{T}$ be the set of polynomials $f \in K[x, y]$ such that $N(f)$ contains exactly two distinct lattice points, i.e., $N(f)$ is a line segment containing no lattice points other than its endpoints.

Definition 1.2. We say that the pair (F, G) has *generic boundaries* if it satisfies (1) and (2) in Conjecture A, and the polynomial $F_+^{1/a}$ is not divisible by the square of any polynomial in $\mathcal{T}$ for any direction $w \in W$.[4]

As an application of the generalization of Magnus' theorem, we have the following:

Proposition 1.3. If (F, G) has generic boundaries with $a = 2$, then Conjecture A is true. More precisely,

$$F = P^2 + u_0$$

for some $P \in K[x, y]$ and some $u_0 \in K$. In particular, $[F, G] = 0$.

Remark 1.4. As suggested by the referee, the condition that (F, G) has generic boundaries is very strong, and the above proposition can be proved using Theorem 7.6 and Proposition 5.18 of [12] instead of using the generalization of Magnus' theorem, as we do in this paper. However, the strategy used in this paper motivated the introduction of the remainder vanishing conjecture, appearing in [6]. One can use the

[2]Note that there is some ambiguity in the notation $F_d^{1/r}$, since it is unique up to an r-th root of unity. We fix a choice of $F_d^{1/r}$. Then the fractional power $F_d^{c/r} := \left(F_d^{1/r}\right)^c$ is unambiguous for any integer c.

[3]In [9], a detailed proof was not given. Moreover, the original statement in [9] was written only for $w = (1, 1)$, and did not contain the statement that starts with "Furthermore", which will play a pivotal role in a series of our papers including this one.

[4]Note that the latter condition is equivalent to $F_+^{1/a}$ not having as a divisor any square of a nonmonomial polynomial, since K is algebraically closed.

generalization of Magnus' theorem to show that the remainder vanishing conjecture implies the Jacobian conjecture, which we believe is a fruitful contribution for future work on this topic. In [7], we illustrate how to prove the vanishing conjecture for some special cases.

For a real number $r \in \mathbb{R}$ and a subset $S \subseteq \mathbb{R}^2$, denote $rS := \{rs \ : s \in S\} \subseteq \mathbb{R}^2$.

Corollary 1.5. Suppose that each edge of $(1/a)N(F)$ contains either the origin or no lattice points other than its endpoints. If $a = 2$, then Conjecture A is true.

2. Magnus' formula revisited

The goal of this section is to prove Theorem 1.1 and present Proposition 2.4, a useful application of this theorem. We start by reinterpreting (1-1) in Theorem 1.1 as follows: Recall that $\mathcal{R} = K[x, y]$, where K is an algebraically closed field of characteristic 0. For any $\widetilde{F} \in \mathcal{R}[\![t]\!]$, denote

$$[\widetilde{F}]_{t^i} = \text{ the coefficient of } t^i \text{ in } \widetilde{F},$$

which is a polynomial in x and y.

Recall the generalized multinomial theorem in the formal power series ring $K[[x_1, \dots, x_n]]$: For $A \in \mathbb{Q}$,

$$(1 + x_1 + \cdots + x_n)^A = \sum_{v_1,\dots,v_n \in \mathbb{Z}_{\geq 0}} \frac{A(A-1)\cdots(A - \sum_{i=1}^n v_i)}{\prod_{i=1}^n v_i!} x_1^{v_1} \cdots x_n^{v_n}.$$

Consider a variation of this. For $x_1, \dots, x_n \in \mathcal{R}$, we have the following expansion in the ring $\mathcal{R}[\![t]\!]$:

$$(1 + x_1 t + \cdots + x_n t^n)^A = \sum_{v_1,\dots,v_n \in \mathbb{Z}_{\geq 0}} \frac{A(A-1)\cdots(A - \sum_{i=1}^n v_i)}{\prod_{i=1}^n v_i!} x_1^{v_1} \cdots x_n^{v_n} t^{v_1 + 2v_2 + \cdots + nv_n}.$$

In general, for $x_0, \dots, x_n \in \mathcal{R}$ and for $A = a/b$ where $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{>0}$, we have the following identity in the ring $\mathcal{R}[x_0^{\pm 1/b}][\![t]\!]$ (where we fix a choice of $x_0^{1/b}$):

$$(x_0 + x_1 t + \cdots + x_n t^n)^A = \sum_{v_1,\dots,v_n \in \mathbb{Z}_{\geq 0}} \frac{A\cdots(A - \sum_{i=1}^n v_i)}{\prod_{i=1}^n v_i!} x_0^{A - \sum_{i=1}^n v_i} x_1^{v_1} \cdots x_n^{v_n} t^{v_1 + \cdots + nv_n}. \tag{2-1}$$

Lemma 2.1. Equation (1-1) can be rewritten as the following in $\mathcal{R}[F_d^{\pm 1/d}]$:

$$G_{e-\mu} = \sum_{\gamma=0}^{\mu} c_\gamma \big[(F_d + F_{d-1}t + F_{d-2}t^2 + \cdots)^{(e-\gamma)/d}\big]_{t^{\mu-\gamma}}. \tag{2-2}$$

Proof. Let $A = (e-\gamma)/d$ and $s = \sum_{\alpha \leq d-1} v_{\gamma,\alpha}$. Then

$$\binom{(e-\gamma)/d}{\sum_{\alpha\leq d-1} v_{\gamma,\alpha}} \frac{\big(\sum_{\alpha \leq d-1} v_{\gamma,\alpha}\big)!}{\prod_{\alpha \leq d-1} v_{\gamma,\alpha}!} = \frac{A(A-1)\cdots(A-s+1)}{s!} \frac{s!}{\prod_{\alpha\leq d-1} v_{\gamma,\alpha}!} = \frac{A\cdots(A-s+1)}{\prod_{\alpha \leq d-1} v_{\gamma,\alpha}!}.$$

So the right side of (1-1), without the constraint $\sum_{\alpha\leq d-1}(d-\alpha)v_{\gamma,\alpha} = \mu - \gamma$, is

$$\sum_{\gamma=0}^{\mu} c_\gamma (F_d + F_{d-1}t + F_{d-2}t^2 + \cdots)^{(e-\gamma)/d},$$

thanks to (2-1). Then, note that the constraint $\sum_{\alpha\leq d-1}(d-\alpha)v_{\gamma,\alpha} = \mu - \gamma$ is equivalent to the restriction to degree $t^{\mu-\gamma}$. Thus the right side of (1-1) is equal to the right side of (2-2). □

We will prove the following statement, which is equivalent to Theorem 1.1:

Theorem 2.2. Suppose $[F, G] \in K$. For any direction $w = (u, v) \in W$, let $d = w\text{-deg}(F_+)$ and $e = w\text{-deg}(G_+)$. Assume $d > 0$. Write the w-homogeneous decompositions $F = \sum_{i \le d} F_i$ and $G = \sum_{i \le e} G_i$. Define

$$\widetilde{F} = F_d + F_{d-1}t + \cdots \quad \text{and} \quad \widetilde{G} = G_e + G_{e-1}t + \cdots .$$

Let $r \in \mathbb{Z}_{>0}$ be the largest integer such that $F_d^{1/r} \in K[x, y]$. Then there exists a unique sequence of constants $c_0, c_1, \ldots, c_{d+e-u-v-1} \in K$ such that $c_0 \neq 0$ and

$$G_{e-\mu} = \sum_{\gamma=0}^{\mu} c_\gamma [\widetilde{F}^{(e-\gamma)/d}]_{t^{\mu-\gamma}} \tag{2-3}$$

for every integer $\mu \in \{0, 1, \ldots, d+e-u-v-1\}$. Moreover, $c_\gamma = 0$ if $r(e-\gamma)/d \notin \mathbb{Z}$.

Note that the last condition implies that every nonzero summand appearing on the right side of (2-3) is in $\mathcal{R}[F_d^{-1/r}]$, so it must be a rational function.

In order to prove Theorem 2.2, we need the following lemma (see [11, Propositions 1 and 2], [2, Lemma 22], and [9, p. 258]), and for the reader's convenience, we reproduce the proof here:

Lemma 2.3. Let $w = (u, v) \in W$. Let R be any polynomial ring over K, $f \in R$ be a w-homogeneous polynomial of degree $d_f > 0$, and g be a nonzero w-homogeneous function of degree $d_g \in \mathbb{Z}$ in the fractional field of R such that the Jacobian $[g, f] = 0$. Define $r \in \mathbb{Z}_{>0}$ to be the largest integer such that $h = f^{1/r}$ is a polynomial. Then there exists a unique $c \in K \setminus \{0\}$ so that $g = c \cdot h^s$, where $s = r d_g / d_f$ is an integer.

Proof. If $g \in K$, the statement is trivial where $s = 0$, $c = g$. So for the rest of the proof, we assume that g is not a constant. By Euler's lemma, $uxf_x + vyf_y = d_f f$ and $uxg_x + vyg_y = d_g g$. Then

$$\begin{bmatrix} d_f f \\ d_g g \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} ux \\ vy \end{bmatrix},$$

so

$$\begin{bmatrix} d_f f g_y - d_g g f_y \\ -d_f f g_x + d_g g f_x \end{bmatrix} = \begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix} \begin{bmatrix} d_f f \\ d_g g \end{bmatrix} = \begin{bmatrix} g_y & -f_y \\ -g_x & f_x \end{bmatrix} \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \begin{bmatrix} ux \\ vy \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} ux \\ vy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, $(g^{d_f}/f^{d_g})_x = d_f g^{d_f - 1} g_x f^{-d_g} - d_g f^{-d_g - 1} f_x g^{d_f} = g^{d_f - 1} f^{-d_g - 1}(d_f f g_x - d_g g f_x) = 0$, and similarly $(g^{d_f}/f^{d_g})_y = 0$. So, $g^{d_f}/f^{d_g} = c'$ for some $c' \in K \setminus \{0\}$.

Let $a_1, a_2 \in K \setminus \{0\}$, $p_1, \ldots, p_n$ be distinct irreducible polynomials, $r_1, \ldots, r_n \in \mathbb{Z}_{\ge 0}$, $s_1, \ldots, s_n \in \mathbb{Z}$, such that we have the prime factorizations $f = a_1 p_1^{r_1} \cdots p_n^{r_n}$ and $g = a_2 p_1^{s_1} \cdots p_n^{s_n}$. Then $d_f s_i = d_g r_i$ for $1 \le i \le n$, and $r = \gcd(r_1, \ldots, r_n)$. Let $s' = \gcd(s_1, \ldots, s_n) > 0$. We have $d_f s' = \gcd(d_f s_1, \ldots, d_f s_n) = \gcd(d_g r_1, \ldots, d_g r_n) = |d_g| r$. So $s = \pm s'$ is an integer, and $r_i : s_i = d_f : d_g = r : s$. So the exponent of p_i in the prime factorization of $c = g/h^s$ is $s_i - s(r_i/r) = 0$; thus, c is a constant. The uniqueness of c follows from the previous sentences. □

Proof of Theorem 2.2. We proceed by induction on μ. The base case of $\mu = 0$ is $G_e = c_0 F_d^{e/d}$, which follows from Lemma 2.3. Note that $c_0 \neq 0$, because otherwise $G_e = c_0 F_d^{e/d} = 0$, which contradicts the assumption that $e = w\text{-deg}(G_+)$.

For the inductive step, assume $\mu > 0$. By the inductive assumption, $c_0, \ldots, c_{\mu-1}$ are uniquely determined. The assumption $[F, G] \in K$ implies that each positive w-degree component of $[F, G]$ is 0. Note that if the w-degrees of homogeneous rational functions f and g are i and j, respectively, then the w-degree of $[f, g]$ is $i + j - u - v$. On the other hand, the component in $[F, G]$ of w-degree $d + e - u - v - \mu$ is just $\big[[\widetilde{F}, \widetilde{G}]\big]_{t^\mu}$. Since $\mu < d + e - u - v$, we have

$$0 = \big[[\widetilde{F}, \widetilde{G}]\big]_{t^\mu} = \Bigg[\Big[\sum_{i\ge 0} F_{d-i}t^i, \sum_{j\ge 0} G_{e-j}t^j\Big]\Bigg]_{t^\mu} = \sum_{\substack{i,j\ge 0,\\ i+j=\mu}} [F_{d-i}, G_{e-j}],$$

therefore

$$[G_{e-\mu}, F_d] = -[F_d, G_{e-\mu}] = \sum_{\substack{i>0,\, j\ge 0,\\ i+j=\mu}} [F_{d-i}, G_{e-j}]. \tag{2-4}$$

Define

$$H = G_{e-\mu} - \sum_{\gamma=0}^{\mu-1} c_\gamma [\widetilde{F}^{(e-\gamma)/d}]_{t^{\mu-\gamma}}.$$

Note that (2-3) holds if and only if $H = c_\mu F_d^{(e-\mu)/d}$ $\big(= c_\mu[\widetilde{F}^{(e-\mu)/d}]_{t^0}\big)$.

If $H = 0$, then $H = c_\mu F_d^{(e-\mu)/d}$ holds exactly when $c_\mu = 0$, so the choice of c_μ is unique.

Now assume $H \neq 0$. It is a homogeneous rational function in $\mathcal{R}[F_d^{-1/r}]$ with w-deg $= e - \mu$ by the inductive hypothesis. We claim that $[H, F_d] = 0$. Indeed,

$$\begin{aligned}
[H, F_d] &= [G_{e-\mu}, F_d] - \Bigg[\sum_{\gamma=0}^{\mu-1} c_\gamma [\widetilde{F}^{(e-\gamma)/d}]_{t^{\mu-\gamma}}, F_d\Bigg] \quad \text{(by the definition of } H\text{)}\\
&= \sum_{\substack{i>0,\, j\ge 0\\ i+j=\mu}} [F_{d-i}, G_{e-j}] - \sum_{\gamma=0}^{\mu-1}\big[c_\gamma[\widetilde{F}^{(e-\gamma)/d}]_{t^{\mu-\gamma}}, F_d\big] \quad \text{(by (2-4))}\\
&= \sum_{j=0}^{\mu-1}\Bigg[F_{d-\mu+j}, \sum_{\gamma=0}^{j} c_\gamma[\widetilde{F}^{(e-\gamma)/d}]_{t^{j-\gamma}}\Bigg] - \sum_{\gamma=0}^{\mu-1}\big[c_\gamma[\widetilde{F}^{(e-\gamma)/d}]_{t^{\mu-\gamma}}, F_d\big] \quad \text{(by the inductive hypothesis)}\\
&= \sum_{\gamma=0}^{\mu-1} c_\gamma\Bigg(\sum_{j=\gamma}^{\mu-1}\big[F_{d-\mu+j}, [\widetilde{F}^{(e-\gamma)/d}]_{t^{j-\gamma}}\big] - \big[[\widetilde{F}^{(e-\gamma)/d}]_{t^{\mu-\gamma}}, F_d\big]\Bigg)\\
&= \sum_{\gamma=0}^{\mu-1} c_\gamma\Bigg(\sum_{j=\gamma}^{\mu}\big[F_{d-\mu+j}, [\widetilde{F}^{(e-\gamma)/d}]_{t^{j-\gamma}}\big]\Bigg) = \sum_{\gamma=0}^{\mu-1} c_\gamma\Bigg(\sum_{j=\gamma}^{\mu}\big[[\widetilde{F}]_{t^{\mu-j}}, [\widetilde{F}^{(e-\gamma)/d}]_{t^{j-\gamma}}\big]\Bigg)\\
&= \sum_{\gamma=0}^{\mu-1} c_\gamma\Big(\big[[\widetilde{F}, \widetilde{F}^{(e-\gamma)/d}]\big]_{t^{\mu-\gamma}}\Big) = 0 \quad \text{(because } [\widetilde{F}, \widetilde{F}^{(e-\gamma)/d}] = 0\text{)}.
\end{aligned}$$

This equality together with Lemma 2.3 (with $g = H$, $f = F_d$, and $h = F_d^{1/r}$) implies that there is a unique element $c_\mu \in K \setminus \{0\}$ such that

$$H = c_\mu h^s = c_\mu F_d^{\deg H/\deg F_d} = c_\mu F_d^{(e-\mu)/d},$$

where $s = r(e - \mu)/d \in \mathbb{Z}$. In other words, $c_\mu = 0$ if $r(e - \mu)/d \notin \mathbb{Z}$. □

For any real numbers $r_1 \le r_2$, we use the usual notation for a closed interval $[r_1, r_2] := \{x \in \mathbb{R} : r_1 \le x \le r_2\}$ and introduce the notation $[r_1, r_2]_{\mathbb{Z}} := [r_1, r_2] \cap \mathbb{Z}$. For a line segment $\overline{AB} \subset \mathbb{R}^2$ whose endpoints are both in $\mathbb{Z}^2 \subset \mathbb{R}^2$, we define the length $\operatorname{len}(\overline{AB}) \in \mathbb{Z}_{\ge 0}$ to be one less than the number of lattice points on $\overline{AB}$. For any direction $w = (u, v) \in W$ and for any w-homogeneous Laurent polynomial $h \in K[x^{\pm 1}, y^{\pm 1}]$, we define $\operatorname{len}(h)$ to be the length of $N(h)$; that is, if $h = a_0 x^b y^c + a_1 x^{b+v} y^{c-u} + a_2 x^{b+2v} y^{c-2u} + \cdots + a_l x^{b+lv} y^{c-lu}$ with $a_0 \ne 0$ and $a_l \ne 0$, then $\operatorname{len}(h) = l$.

In the following statement, for each polynomial F, P, and R, we fix $w = (u, v) \in W$ and write the w-homogeneous degree decompositions $F = \sum_i F_i$, $P = \sum_i P_i$, and $R = \sum_i R_i$.

Proposition 2.4. Let $k \in \mathbb{Z}_{>0}$ and assume that (F, G) has generic boundaries with $a = 2$ and $b = 2k + 1$. Denote $d := w\text{-deg}(F_+)$ and $e := w\text{-deg}(G_+)$. Let $P \in \mathcal{R}$ such that $w\text{-deg}(P_+) = m = d/2$ and $F_d = P_m^2$. Let $R = F - P^2$ and $h \in [1, 2m - 1]_{\mathbb{Z}}$. If $R_{d-\ell} = 0$ for all $\ell < h$, then $P_m^{-1} R_{d-h} \in K[x^{\pm 1}, y^{\pm 1}]$.

Proof. We assume P_m is not a monomial (thus, F_d is not a monomial) since the statement is trivial otherwise. Let

$$\begin{aligned} \widetilde{F} &= F_d + F_{d-1} t + F_{d-2} t^2 + \cdots \in \mathcal{R}[[t]], \\ Q &= P_m + P_{m-1} t + \cdots + P_{m-h} t^h \in \mathcal{R}[[t]], \text{ and} \\ T &= \widetilde{F} - Q^2. \end{aligned}$$

For each positive integer z, let $\mathbb{O}(t^z)$ denote an element of the form $\sum_{i \ge z} f_i t^i$ in $\mathcal{R}[[t]]$, where each f_i is a polynomial in $\mathcal{R}$ which we do not have to care about. Since $R_{d-\ell} = 0$ for $\ell < h$, we get

$$\begin{aligned} T &= \Bigl(F_{d-h} - \sum_{j=0}^{h} P_{m-j} P_{m-h+j}\Bigr) t^h + \mathbb{O}(t^{h+1}) \\ &= R_{d-h} t^h + \mathbb{O}(t^{h+1}). \end{aligned}$$

We will apply Theorem 2.2 to the case of $\mu = h(k + 1)$. For that purpose, we need to check that $h(k + 1) \le d + e - u - v - 1$. Observe that

$$\begin{aligned} d + e - u - v - 1 - h(k + 1) &\ge 2m + (2k + 1)m - u - v - 1 - (2m - 1)(k + 1) \\ &= m - u - v + k \\ &\ge m - u - v + 1 \\ &= m + 1 - (w\text{-deg}(xy)) \\ &\ge 0, \end{aligned}$$

where the last inequality holds because $m - (w\text{-deg}(xy)) \ge 0$ if the lattice point $(1, 1)$ is contained in $N(P)$, or $w\text{-deg}(xy) = m + 1$ otherwise. Indeed, if $(1, 1)$ is not contained in $N(P)$, then $N(P)$ is either the triangle with vertices $(0, 0)$, $(c, 0)$, and $(0, 1)$ or the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, c)$ for some $c \in \mathbb{Z}_{>0}$. Without loss of generality, we assume the former. Since P_m is not a monomial, both the points $(c, 0)$ and $(0, 1)$ must lie in the support of P_m. So $w = (1, c)$, $m = c$, and thus $w\text{-deg}(xy) = 1 + c = m + 1$.

Since (F, G) has generic boundaries, Theorem 2.2 gives

$$\begin{aligned} G_{e-h(k+1)} &= \sum_{r=0}^{\lfloor h(k+1)/m \rfloor} c_{rm}[\widetilde{F}^{k+1/2-r/2}]_{t^{h(k+1)-rm}} \\ &= \sum_{r=0}^{\lfloor h(k+1)/m \rfloor} c_{rm}[(Q^2+T)^{k+1/2-r/2}]_{t^{h(k+1)-rm}} \\ &= \sum_{r=0}^{\lfloor h(k+1)/m \rfloor} c_{rm}\Bigg[\sum_{i=0}^{\infty}\binom{k+1/2-r/2}{i} Q^{2k+1-r-2i}T^i\Bigg]_{t^{h(k+1)-rm}}. \end{aligned}$$

Note that $T = \mathbb{O}(t^h)$, which implies $T^i = \mathbb{O}(t^{hi})$. Then it is enough to look at r and i such that $hi \le h(k+1) - rm$, or equivalently $i \le k+1-rm/h$. Then the exponent of Q satisfies

$$2k+1-r-2i \ge 2k+1-r-2(k+1-rm/h) = 2rm/h-r-1 \ge 2rm/d-r-1 = r-r-1 = -1.$$

Here, the first "$\ge$" becomes "$=$" only when $i = k+1-rm/h$, and the second "$\ge$" becomes "$=$" only when $r = 0$. Since the exponent of Q is an integer, it is always nonnegative except when "$r = 0$ and $i = k+1$" (in which case the exponent is -1). Since $G_{e-h(k+1)}$ and $[Q^{2k+1-r-2i}T^i]_{t^{h(k+1)-rm}}$ are polynomials in $K[x, y]$ whenever $2k+1-r-2i \ge 0$, the following must also be a polynomial in $K[x, y]$:

$$\begin{aligned} c_0\binom{k+1/2}{k+1}[Q^{-1}T^{k+1}]_{t^{h(k+1)}} &= c_0\binom{k+1/2}{k+1}\big[(P_m+\mathbb{O}(t))^{-1}\big(R_{d-h}t^h+\mathbb{O}(t^{h+1})\big)^{k+1}\big]_{t^{h(k+1)}} \\ &= c_0\binom{k+1/2}{k+1}P_m^{-1}(R_{d-h})^{k+1}. \end{aligned}$$

Since $c_0 \ne 0$ and $\binom{k+1/2}{k+1} \ne 0$, we get that $P_m^{-1}(R_{d-h})^{k+1}$ is a polynomial. Since (F, G) has generic boundaries, the polynomial P_m is not divisible by the square of any polynomial in $\mathcal{T}$. This implies $P_m^{-1}R_{d-h} \in K[x^{\pm 1}, y^{\pm 1}]$, because K is algebraically closed. □

Corollary 2.5. Assume the same hypotheses as above. If $\mathrm{len}(R_{d-h}) < \mathrm{len}(P_m)$, then $R_{d-h} = 0$.

Proof. If $R_{d-h} \ne 0$ then $\mathrm{len}(R_{d-h}) \ge \mathrm{len}(P_m)$, because $P_m^{-1}R_{d-h} \in K[x^{\pm 1}, y^{\pm 1}]$. □

The next lemma is elementary but makes Proposition 2.4 useful. Let $F \in \mathcal{R}$ be a polynomial with a nonzero constant term such that all vertices of $N(F)$ are in $(2\mathbb{Z})^2$, and let C be any vertex of $N(F)$ other than the point of origin O. Let $\mathcal{N}' = \frac{1}{2}N(F)$, which is defined at the end of Section 1, and $\mathcal{N}'' = \mathcal{N}' + \frac{1}{2}\overrightarrow{OC}$. (For example, see the polygons shown in Figure 4.)

Lemma 2.6. Let $F = \sum_{i,j}\lambda_{ij}x^i y^j$. There exists a polynomial $P = \sum_{(i,j)\in\mathcal{N}'} p_{ij}x^i y^j$, unique up to a sign, such that $\mathrm{supp}(F - P^2) \cap \mathcal{N}'' = \varnothing$.

Proof. First consider the case that $N(F)$ is a rectangle $[0, 2m'] \times [0, 2m]$ for $m', m \in \mathbb{Z}_{>0}$. In particular, $\lambda_{2m',2m} \ne 0$. See Figure 2. Let $C = (2m', 2m)$. The required property gives a system of $(m+1)(m'+1)$ quadratic equations with $(m+1)(m'+1)$ variables p_{ij}. We can solve p_{ij} recursively, in the following "graded lex order": $p_{m',m} > p_{m',m-1} > p_{m'-1,m} > p_{m',m-2} > p_{m'-1,m-1} > p_{m'-2,m} > \cdots$. Namely, first use $\lambda_{2m',2m} - p_{m',m}^2 = 0$ to determine $p_{m',m} \ne 0$ up to a sign; next use $\lambda_{2m',2m-1} - 2p_{m',m}p_{m',m-1} = 0$ to uniquely determine $p_{m',m-1}$, etc.

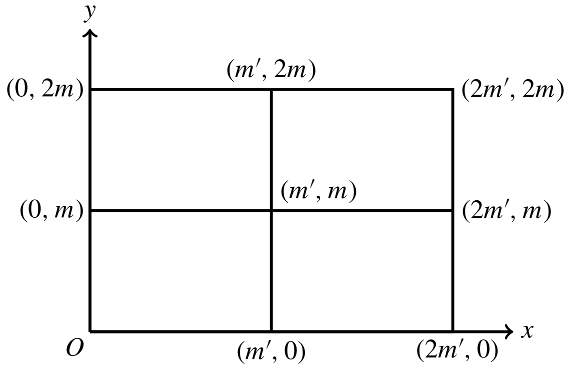

Figure 2. The case where $N(F)$ is a rectangle.

Even if $N(F)$ is arbitrary, we can still solve p_{ij} recursively with respect to an appropriate order in the same way as follows: Let $L_c = \{(x, y) \mid \alpha x + \beta y = c\}$, for some $c > 0$, be a line with irrational slope that passes C and intersects with $N(F)$ only at C. Then $N(F)$ lies in the half-plane $\alpha x + \beta y \le c$. Arrange the points $\{z_i = (x_i, y_i)\}_{1\le i\le n}$ in $\mathcal{N}'$ such that $\alpha x_1 + \beta y_1 > \alpha x_2 + \beta y_2 > \cdots > \alpha x_n + \beta y_n$. Then $z_1 = \frac{1}{2}C$. Denote $\boldsymbol{x}^{z_i} = x^{x_i} y^{y_i}$. Then in our new notation, $P = \sum_{z_i} p_{z_i} \boldsymbol{x}^{z_i}$. We claim that we can solve $p_{z_1}, p_{z_2}, \ldots, p_{z_n}$ recursively. First use $\lambda_{2z_1} - p_{z_1}^2 = 0$ to determine p_{z_1} up to a sign. If $p_{z_1}, \ldots, p_{z_{k-1}}$ are determined, then using

$$\lambda_{z_1+z_k} = \sum_{z_i+z_j=z_1+z_k} p_{z_i} p_{z_j} = 2p_{z_1} p_{z_k} + \sum_{z_i+z_j=z_1+z_k, 1<i,j<k} p_{z_i} p_{z_j},$$

we can uniquely determine p_{z_k}. Since $\mathcal{N}'' = \{z_1 + z_k \mid 1 \le k \le n\}$, we have found a unique P (up to a sign) such that $\operatorname{supp}(F - P^2) \cap \mathcal{N}'' = \varnothing$. □

3. Proof of Proposition 1.3 for the case where $N(F)$ is a rectangle

Let $F = \sum_{i,j} \lambda_{ij} x^i y^j$. In this section, we prove Proposition 1.3, assuming that $N(F)$ is a rectangle $[0, 2m'] \times [0, 2m]$ for $m', m \in \mathbb{Z}_{>0}$. In particular, $\lambda_{2m',2m} \ne 0$. See Figure 2.

Let P be a polynomial given by Lemma 2.6, and let $R = F - P^2$. Write the w-homogeneous degree decomposition $R = \sum_i R_i$ for $w = (0, 1) \in W$, and the w'-homogeneous degree decomposition $R = \sum_i R'_i$ for $w' = (1, 0) \in W$.

Proposition 3.1. Suppose that (F, G) has generic boundaries with $a = 2$. Let $d' = 2m'$, $d = 2m$, and $e' = (2k+1)m'$, $e = (2k+1)m$ for some positive integer k. Then we have $R_{d-h} = 0$ for $h \in [0, d-1]_{\mathbb{Z}}$, and $R'_{d'-h} = 0$ for $h \in [0, d'-1]_{\mathbb{Z}}$.

Proof. We will use induction on h. The base case of $h = 0$ is well known (for instance, see [1; 2; 11]), and it also follows from Theorem 1.1 for $\mu = 0$.

Let $h \in [1, m]_{\mathbb{Z}}$, and assume the inductive hypothesis that $R_{d-\ell} = 0$ for $\ell < h$. Lemma 2.6 implies that $\operatorname{len}(R_{d-h}) < m' = \operatorname{len}(P_+)$. Hence, $R_{d-h} = 0$ by Corollary 2.5.

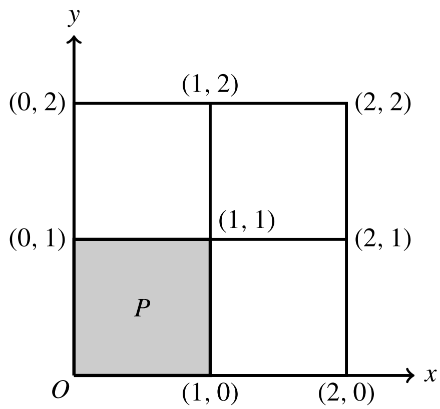

Figure 3. The Newton polygons associated with F and P, with P in bold.

Applying the same argument, we also get $R'_{d'-h} = 0$ for $h \in [0, m']_{\mathbb{Z}}$. Then the support of R is contained in $[0, m'-1] \times [0, m-1]$, which in turn implies that $R_{d-h} = 0$ for $h \in [m+1, d-1]_{\mathbb{Z}}$, and $R'_{d'-h} = 0$ for $h \in [m'+1, d'-1]_{\mathbb{Z}}$. □

In light of this proposition, we can show the following:

Corollary 3.2. Suppose that (F, G) has generic boundaries with $a = 2$. If $N(F)$ is a rectangle, then $F = P^2 + u_0$ for some constant $u_0 \in K$. In particular, $[F, G] = 0$.

Proof. Proposition 3.1 implies that the support of R is either the origin or empty, so R is equal to a constant, say $u_0 \in K$. That is, $F = P^2 + u_0$. This gives $[F, G] = [P^2, G] = 2P[P, G] \in K$. So $P \in K$, which gives $[F, G] = 0$. □

Example 3.3. Assume that $a = 2$, $b = 3$, $N(F)$ is the 2×2 square, $N(G)$ is the 3×3 square, and F takes the form

$$F = \lambda_{2,2}x^2y^2 + \lambda_{2,1}x^2y + \lambda_{1,2}xy^2 + \lambda_{1,1}xy + \lambda_{2,0}x^2 + \lambda_{0,2}y^2 + \lambda_{1,0}x + \lambda_{0,1}y + \lambda_{0,0},$$

where $\lambda_{i,j} \in K$ and $\lambda_{2,2}\lambda_{2,0}\lambda_{0,2} \neq 0$. See Figure 3.

By Lemma 2.6, there exists a polynomial $P = P_{1,1}xy + P_{1,0}x + P_{0,1}y + P_{0,0}$ such that $\operatorname{supp}(F - P^2)$ is contained in $\{(0, 2), (0, 1), (0, 0), (1, 0), (2, 0)\}$. We immediately deduce that

$$\begin{aligned}
\lambda_{2,2} &= P_{1,1}^2, \\
\lambda_{2,1} &= 2P_{1,1}P_{1,0}, \\
\lambda_{1,2} &= 2P_{1,1}P_{0,1}, \\
\lambda_{1,1} &= 2P_{1,0}P_{0,1} + 2P_{1,1}P_{0,0}.
\end{aligned}$$

Suppose that $[F, G] \in K$. Applying (1-1) to $w = (0, 1)$ and $\mu = 0$, we have

$$\lambda_{2,2}x^2y^2 + \lambda_{1,2}xy^2 + \lambda_{0,2}y^2 = P_{1,1}^2x^2y^2 + 2P_{1,1}P_{0,1}xy^2 + \lambda_{0,2}y^2 = (P'_{1,1}xy + P'_{0,1}y)^2$$

for some constants $P'_{1,1}, P'_{0,1} \in K$, which implies $\lambda_{0,2} = P_{0,1}^2$ and $(P_{1,1}xy + P_{0,1}y)^2 = (P'_{1,1}xy + P'_{0,1}y)^2$.

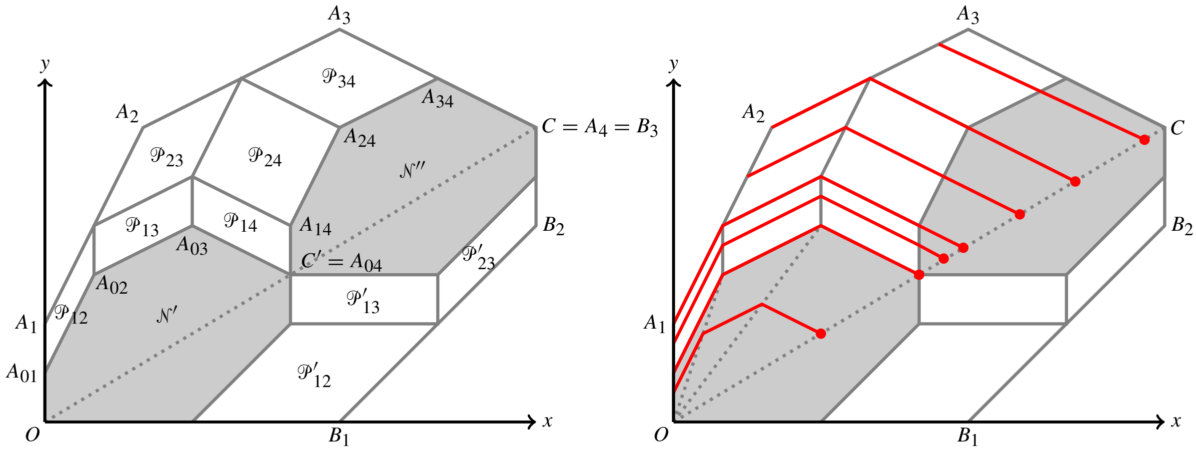

Figure 4. The case where $N(F)$ is arbitrary, $\alpha = 4$, and $\beta = 3$. Regions $\mathcal{N}'$, $\mathcal{N}''$, and $\mathcal{P}_{ij}$ are shown on the left; and various broken lines T_D, where the red dots are the various positions of the point D are shown on the right.

Applying (1-1) to $w = (0, 1)$ and $\mu = 2$, we get that $\lambda_{2,1}x^2y + \lambda_{1,1}xy + \lambda_{0,1}y$ is divisible by $P'_{1,1}xy + P'_{0,1}y$, hence divisible by $P_{1,1}xy + P_{0,1}y$. This means that

$$\begin{aligned}\lambda_{2,1}x^2y + \lambda_{1,1}xy + \lambda_{0,1}y &= 2P_{1,1}P_{1,0}x^2y + (2P_{1,0}P_{0,1} + 2P_{1,1}P_{0,0})xy + \lambda_{0,1}y\\ &= 2(P_{1,1}xy + P_{0,1}y)(P'_{1,0}x + P'_{0,0})\end{aligned}$$

for some constants $P'_{1,0}, P'_{0,0} \in K$, which implies $\lambda_{0,1} = 2P_{0,1}P_{0,0}$.

Similarly, applying Magnus' formula to $w = (1, 0)$, we also have $\lambda_{2,0} = P^2_{1,0}$ and $\lambda_{1,0} = 2P_{1,0}P_{0,0}$. From here, it follows that $F - P^2$ is a constant.

4. Construction of broken lines and the proof of Proposition 1.3

In this section, we prove Proposition 1.3. Besides from using Proposition 2.4, Corollary 2.5, and Lemma 2.6, the rest is a purely combinatorial analysis on subsets of $\mathbb{R}^2$ and $\mathbb{Z}^2$. Suppose that (F, G) has generic boundaries with $a = 2$.

Denote the point of origin by $O = A_0 = B_0$. Now we extend the result from Section 3 to the general case that $N(F)$ is of arbitrary shape. We say that a vertex $C = (c_x, c_y) \in N(F)$ is *northeastern* if $(v_x - c_x, v_y - c_y) \notin \mathbb{Z}^2_{\geq 0}$ for any other vertex $V = (v_x, v_y) \in N(F)$. Observe that a northeastern vertex exists.

Let $N(F) = OA_1 \cdots A_{\alpha-1}CB_{\beta-1} \cdots B_1$, where C is a northeastern vertex of $N(F)$. We set $A_\alpha = B_\beta = C$ and $C' = \frac{1}{2}C$. Since $(1, 0), (0, 1) \in N(F)$, we must have that A_1 lies on the y-axis and B_1 lies on the x-axis. Without loss of generality, assume $\alpha > 1$ (but we allow $\beta = 1$).

Let $\mathcal{N}' = \frac{1}{2}N(F)$ and $\mathcal{N}'' = \mathcal{N}' + \frac{1}{2}\overrightarrow{OC}$ be the polygons shown in Figure 4.

4.1. Construction of parallelograms associated with $N(F)$.

For $0 \leq i \leq j \leq \alpha$, define

$$A_{ij} = \tfrac{1}{2}(A_i + A_j) \in \mathbb{R}^2.$$

In particular, $A_{i,i} = A_i$.

We define the parallelogram $\mathcal{P}_{ij}$, where $1 \le i < j \le \alpha$, by its four vertices $A_{i-1,j-1}$, $A_{i-1,j}$, $A_{i,j-1}$, and $A_{i,j}$. For convenience, we call the line segments $\overline{A_{i-1,j-1}A_{i,j-1}}$, $\overline{A_{i,j-1}A_{ij}}$, $\overline{A_{ij}A_{i-1,j}}$, and $\overline{A_{i-1,j}A_{i-1,j-1}}$ the west, north, east, and south edges of $\mathcal{P}_{ij}$, respectively. Similarly, we also define the parallelogram $\mathcal{P}'_{ij}$, where $1 \le i < j \le \beta$. See Figure 4.

One can verify that:

- $\mathcal{P}_{ij}$ is indeed a parallelogram.
- The lengths of the edges of $\mathcal{P}_{ij}$ are equal to $\frac{1}{2}A_{i-1}A_i$ and $\frac{1}{2}A_{j-1}A_j$.
- $\mathcal{P}_{i,j}$ shares edges with $\mathcal{P}_{i,j\pm1}$ and $\mathcal{P}_{i\pm1,j}$ (if the latter are defined).

We make the following claim:

Lemma 4.1. (a) The union of the parallelograms $\mathcal{P}_{ij}$ is the (closed and nonconvex) polygon

$$\mathcal{P} := A_{01}A_{02}\cdots A_{0\alpha}A_{1\alpha}A_{2\alpha}\cdots A_{\alpha-1,\alpha}A_{\alpha-1}A_{\alpha-2}\cdots A_1$$

(b) These parallelograms do not overlap with each other. More precisely, $\mathcal{P}_{ij} \cap \mathcal{P}_{i,j+1} = \overline{A_{i-1,j}A_{ij}}$, $\mathcal{P}_{ij} \cap \mathcal{P}_{i+1,j} = \overline{A_{i,j-1}A_{ij}}$, $\mathcal{P}_{ij} \cap \mathcal{P}_{i+1,j+1} = A_{ij}$, $\mathcal{P}_{ij} \cap \mathcal{P}_{i+1,j-1} = A_{i,j-1}$, and $\mathcal{P}_{ij} \cap \mathcal{P}_{i',j'} = \varnothing$ if $|i-i'| > 1$ or $|j-j'| > 1$.

Proof. (a) Given any point $r \in \mathcal{P}$, we assert that r is in some $\mathcal{P}_{ij}$.

Let $\boldsymbol{v}_i$ be the vector $\frac{1}{2}\overrightarrow{A_{i-1}A_i}$ for $1 \le i \le \alpha$. Then $A_{ij} = O + \sum_{k=1}^{i} \boldsymbol{v}_k + \sum_{k=1}^{j} \boldsymbol{v}_k$, and the parallelogram $\mathcal{P}_{ij} = \{A_{i-1,j-1} + s\boldsymbol{v}_i + t\boldsymbol{v}_j \mid 0 \le s, t \le 1\}$. The assumption that C is a northeastern vertex of $N(F)$ has the following consequences: $\mathcal{N}' \cap \mathcal{N}'' = \{C'\}$, and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\alpha$ are in clockwise order and (strictly) in the same half plane $y + \lambda x > 0$ for a sufficiently large constant $\lambda \gg 0$.

One can see that $\mathcal{P} = L_1 + L_2$, the Minkowski sum of the two broken lines

$$L_1 := OA_{01}A_{02}\cdots A_{0,\alpha-1} \quad \text{and} \quad L_2 := A_{01}A_{02}\cdots A_{0,\alpha-1}A_{0,\alpha}.$$

It can be visualized as follows: as a point p moves along L_1, the broken line $p + L_2$ sweeps out the region $\mathcal{P}$. As a consequence, any point $r \in \mathcal{P}$ is the sum of a point $p \in L_1$ and a point $q \in L_2$. Without loss of generality, we may assume that p lies in $\overline{A_{0,i-1}A_{0,i}}$ and q lies in $\overline{A_{0,j-1}A_{0,j}}$ for some $i \le j$. Then $p = O + \sum_{k=1}^{i-1} \boldsymbol{v}_k + s\boldsymbol{v}_i$ and $q = O + \sum_{k=1}^{j-1} \boldsymbol{v}_k + t\boldsymbol{v}_j$. We consider three cases:

- If $i < j$, then $r = p+q = A_{i-1,j-1} + s\boldsymbol{v}_i + t\boldsymbol{v}_j$ lies in $\mathcal{P}_{i,j}$.
- If $i = j$ and $s+t \le 1$, then $r = p+q = A_{i-1,i-1} + (s+t)\boldsymbol{v}_i = A_{i-2,i-1} + 1\boldsymbol{v}_{i-1} + (s+t)\boldsymbol{v}_i$ lies in $\mathcal{P}_{i-1,i}$.
- If $i = j$ and $s+t > 1$, then $r = p+q = A_{i-1,i-1} + (s+t)\boldsymbol{v}_i = A_{i-1,i} + (s+t-1)\boldsymbol{v}_i + 0\boldsymbol{v}_{i+1}$ lies in $\mathcal{P}_{i,i+1}$.

So we have proved that in all cases, the point r lies in some parallelogram $\mathcal{P}_{ij}$. This proves (a).

(b) Assume that there exists a point $r \in \mathcal{P}_{ij} \cap \mathcal{P}_{i'j'}$, where $i < j$, $i' < j'$ and $i \le i'$. Then

$$r = O + \sum_{k=1}^{i-1} \boldsymbol{v}_k + \sum_{k=1}^{j-1} \boldsymbol{v}_k + s\boldsymbol{v}_i + t\boldsymbol{v}_j = O + \sum_{k=1}^{i'-1} \boldsymbol{v}_k + \sum_{k=1}^{j'-1} \boldsymbol{v}_k + s'\boldsymbol{v}_{i'} + t'\boldsymbol{v}_{j'}$$

for some $s, t, s', t' \in [0, 1]$.

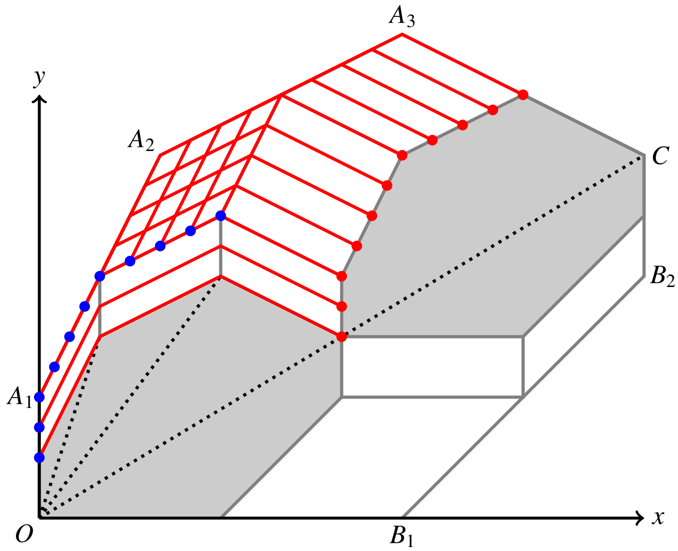

Figure 5. The red broken lines are $p + L_2$ whose left endpoint p is blue and right endpoint is red.

- If $i' = i$ and $j' > j$, then $s\boldsymbol{v}_i + t\boldsymbol{v}_j = \sum_{k=j}^{j'-1} \boldsymbol{v}_k + s'\boldsymbol{v}_i + t'\boldsymbol{v}_{j'}$, hence $(s - s')\boldsymbol{v}_i = (1-t)\boldsymbol{v}_j + \sum_{k=j+1}^{j'-1} \boldsymbol{v}_k + t'\boldsymbol{v}_{j'}$. Note that $\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_{j+1}, \ldots, \boldsymbol{v}_{j'}$ lie in a half-plane and are in the clockwise order. So we must have $j' = j + 1$ and $s - s' = 1 - t = t' = 0$. In this case, $r = A_{i-1,j} + s\boldsymbol{v}_i \in \overline{A_{i-1,j} A_{ij}}$.
- If $j' = j$ and $i' > i$, then by a similar argument we have $i' = i + 1$ and $r \in \overline{A_{i,j-1} A_{ij}}$.
- If $i' > i$ and $j' > j$, then

$$s\boldsymbol{v}_i + t\boldsymbol{v}_j = \sum_{k=i}^{i'-1} \boldsymbol{v}_k + \sum_{k=j}^{j'-1} \boldsymbol{v}_k + s'\boldsymbol{v}_{i+1} + t'\boldsymbol{v}_{j+1},$$

$$(1-s)\boldsymbol{v}_i + (1-t)\boldsymbol{v}_j + \sum_{k=i+1}^{i'-1} \boldsymbol{v}_k + \sum_{k=j+1}^{j'-1} \boldsymbol{v}_k + s'\boldsymbol{v}_{i+1} + t'\boldsymbol{v}_{j+1} = 0.$$

  Note that all $\boldsymbol{v}_k$ lie in a half-plane, so we must have $i' = i+1$, $j' = j+1$, and $1-s = 1-t = s' = t' = 0$. In this case, $r = A_{ij}$.
- If $i' > i$ and $j' < j$, then

$$\sum_{k=j'}^{j-1} \boldsymbol{v}_k + s\boldsymbol{v}_i + t\boldsymbol{v}_j = \sum_{k=i}^{i'-1} \boldsymbol{v}_k + s'\boldsymbol{v}_{i'} + t'\boldsymbol{v}_{j'}$$

$$(1-t')\boldsymbol{v}_{j'} + \sum_{k=j'+1}^{j-1} \boldsymbol{v}_k + t\boldsymbol{v}_j = (1-s)\boldsymbol{v}_i + \sum_{k=i+1}^{i'-1} \boldsymbol{v}_k + s'\boldsymbol{v}_{i'}.$$

  Note that $\boldsymbol{v}_i, \ldots, \boldsymbol{v}_{i'}, \boldsymbol{v}_{j'}, \ldots, \boldsymbol{v}_j$ lie in a half plane and are in the clockwise order, so we must have $i' = i+1$, $j' = j-1$, and $1 - s' = s = 1 - t = t' = 0$. In this case, $r = A_{i,j-1}$.

This proves all cases for (b). □

4.2. Construction of broken lines. Next we construct broken lines T_D and T'_D:

(a) For every point D on the line segment $\overline{C'C}$, we draw a broken line T_D that goes to the left until it reaches the boundary of $N(F)$, as follows:

Step 1: first, it goes in the direction $\overrightarrow{A_\alpha A_{\alpha-1}}$ until it reaches a point $D_{\alpha-1}$ on the west boundary of $\mathcal{P}_{r,\alpha}$ for some r;

Step 2: then, it goes in the direction $\overrightarrow{A_{\alpha-1} A_{\alpha-2}}$ until it reaches a point $D_{\alpha-2}$ on the west boundary of $\mathcal{P}_{r,\alpha-1}$;

Step 3: then, it goes in the direction $\overrightarrow{A_{\alpha-2} A_{\alpha-3}}$ until it reaches a point $D_{\alpha-3}$ on the west boundary of $\mathcal{P}_{r,\alpha-2}$;

$\vdots$

Step $(\alpha - r)$: finally, it goes in the direction $\overrightarrow{A_{r+1} A_r}$ until it reaches a point D_r on the west boundary of $\mathcal{P}_{r,r+1}$.

If $\overline{DD_{\alpha-1}}$ does not contain the north edge of $P_{r,\alpha}$ for any $r \geq 1$, then define $T_D = DD_{\alpha-1}D_{\alpha-2}\cdots D_r$. Note that it now reaches the boundary of $N(F)$.

If $\overline{DD_{\alpha-1}}$ contains the north edge of $P_{r,\alpha}$ for some $r \geq 1$, we define T_D to be the broken line consisting of $\overline{DD_{\alpha-1}}$ and the north edges of $P_{r,\alpha-1}$, $P_{r,\alpha-2}$, ..., $P_{r,r+1}$, and ending at A_r.

(b) Still for every point D on the line segment $\overline{C'C}$, similarly we define a broken line T'_D that goes down and whose linear pieces are parallel to $\overrightarrow{B_i B_{i-1}}$ for $i \leq \beta$. Note that for $D = C'$,

$$T_{C'} = \tfrac{1}{2} A_\alpha A_{\alpha-1} \cdots A_1 = A_{0\alpha} A_{0,\alpha-1} \cdots A_{01} \quad \text{and} \quad T'_{C'} = \tfrac{1}{2} B_\beta B_{\beta-1} \cdots B_1 = B_{0\beta} B_{0,\beta-1} \cdots B_{01}.$$

(c) For every $D \in \overline{OC'} \setminus \{O\}$, define

$$T_D = D_\alpha D_{\alpha-1} \cdots D_1 = \frac{||OD||}{||OC||} A_\alpha A_{\alpha-1} \cdots A_1,$$

rescaling the boundary broken line from A_α to A_1 so that it passes the point $D = D_\alpha$.

(d) For every $D \in \overline{OC'} \setminus \{O\}$, define

$$T'_D = D_\beta D_{\beta-1} \cdots D_1 = \frac{||OD||}{||OC||} B_\beta B_{\beta-1} \cdots B_1,$$

rescaling the boundary broken line from B_β to B_1 so that it passes the point $D = D_\beta$.

Note that every point on $N(F) \setminus \overline{OC}$ lies in a unique T_D or T'_D.

4.3. Proof of Proposition 1.3. Let P be a polynomial given by Lemma 2.6, and let $R = F - P^2$. We have the following generalization of Proposition 3.1:

Proposition 4.2. Suppose that (F, G) has generic boundaries with $a = 2$. In the above setting, R must be a constant; that is, $F = P^2 + u_0$ for some constant $u_0 \in K$.

Proof. We consider the finite sequence of broken lines $T_1, T_2, \ldots, T_s$ such that:

- each T_i is either T_D or T'_D for some $D \in \overline{OC} \setminus \{O\}$;
- $\operatorname{supp}(F) \setminus \{O\} \subset \cup_{i=1}^{s} T_i$;
- each T_i contains a lattice point; and
- the distance from O to $T_i \cap \overline{OC}$ is no less than the distance from O to $T_{i+1} \cap \overline{OC}$.

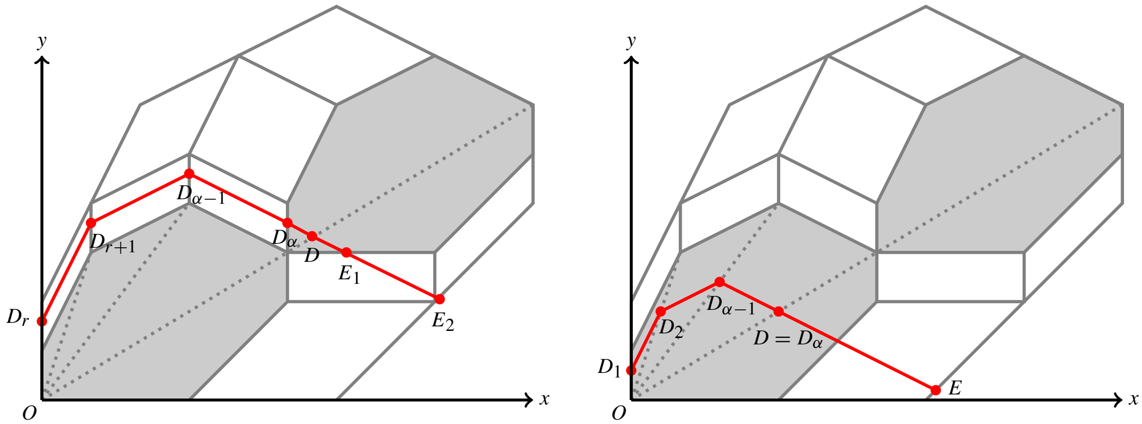

Figure 6. Case (b1) is on the left; and Case (b2) is on the right.

To make the order unique, we assume that if $T_i \cap \overline{OC} = T_{i+1} \cap \overline{OC} = D$, then $T_i = T_D$ and $T_{i+1} = T'_D$. In particular, $T_1 = \overline{A_\alpha A_{\alpha-1}}$ (recall that $\alpha > 1$) and $T_2 = \overline{B_\beta B_{\beta-1}}$ if $\beta > 1$.

We will inductively prove that $\mathrm{supp}(R) \cap T_i = \varnothing$.

(a) Let $w = (u, v)$ be the normal direction of the edge $\overline{A_\alpha A_{\alpha-1}}$. Let $d = w\text{-deg}(F_+)$ and $m = w\text{-deg}(P_+)$, so $d = 2m$. The assumption $[F, G] \in K$ implies that $F_d = P_m^2$. This is well known (see [1; 2; 11]), and also follows from Theorem 1.1 for $\mu = 0$. Thus, $\mathrm{supp}(R) \cap \overline{A_\alpha A_{\alpha-1}} = \mathrm{supp}(R) \cap T_1 = \varnothing$. Similarly, $\mathrm{supp}(R) \cap \overline{B_\beta B_{\beta-1}} = \mathrm{supp}(R) \cap T_2 = \varnothing$ if $\beta > 1$.

(b) Assume the inductive hypothesis that $\mathrm{supp}(R) \cap (T_1 \cup \cdots \cup T_{i-1}) = \varnothing$. We consider the broken line T_i. There are two cases.

(b1) Suppose $T_i = T_D$ or T'_D for $D = (d_1, d_2)$ in the segment $C'C$. Without loss of generality, assume $T_i = T_D = DD_{\alpha-1}D_{\alpha-2} \cdots D_r$.

By Proposition 2.4, we see that $P_m^{-1} R_{ud_1+vd_2} \in K[x^{\pm 1}, y^{\pm 1}]$. Denote by D_α the intersection of the line segment $\overline{DD_{\alpha-1}}$ with the boundary of $\mathcal{N}''$. Let E_1 be the intersection point of the (unbounded) half-line $\overrightarrow{D_\alpha D}$ and the boundary of $\mathcal{N}''$. Let E_2 be the intersection of $\overrightarrow{D_\alpha D}$ with the boundary of $N(F)$. See the picture on the left in Figure 6.

Since $\mathrm{supp}(R) \cap \mathcal{N}'' = \varnothing$, we have $\mathrm{supp}(R) \cap \overline{D_\alpha E_1} = \varnothing$. Since the lattice points on $\overline{E_1 E_2}$ must be in some T_ℓ for some $\ell < i$, we also have $\mathrm{supp}(R) \cap \overline{E_1 E_2} = \varnothing$. Hence, $\mathrm{supp}(R_{ud_1+vd_2}) \in \overline{D_\alpha D_{\alpha-1}} \setminus \{D_\alpha\}$, so $\mathrm{len}(R_{ud_1+vd_2}) < \mathrm{len}(P_m)$. Then $\mathrm{supp}(R) \cap \overline{D_\alpha D_{\alpha-1}} = \varnothing$ due to Corollary 2.5. Applying a similar argument, we see that $\mathrm{supp}(R) \cap \overline{D_{\alpha-1} D_{\alpha-2}} = \cdots = \mathrm{supp}(R) \cap \overline{D_{r+1} D_r} = \varnothing$. Therefore, $\mathrm{supp}(R) \cap T_i = \varnothing$.

(b2) Suppose $T_i = T_D$ or T'_D for $D = (d_1, d_2) \in \overline{OC'} \setminus \{O, C'\}$. Without loss of generality, assume $T_i = T_D = D_\alpha D_{\alpha-1} \cdots D_1$, where $D_\alpha = D$. Let E be the intersection of $\overrightarrow{D_{\alpha-1} D}$ with the boundary of $N(F)$. See the picture on the right in Figure 6. Since the lattice points on $\overline{DE}$ must be in some T_ℓ for some $\ell < i$, we also have $\mathrm{supp}(R) \cap \overline{DE} = \varnothing$. Hence, $\mathrm{supp}(R_{ud_1+vd_2}) \in \overline{D_\alpha D_{\alpha-1}}$, so $\mathrm{len}(R_{ud_1+vd_2}) < \mathrm{len}(P_m)$. From here, we get the same conclusion as in (b1). □

Proof of Proposition 1.3. The proof is analogous to the proof of Corollary 3.2. □

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