SIAM J. MATH. ANAL.
Vol. 55, No. 4, pp. 4089–4139

THREE-WAVE INTERACTION EQUATIONS: CLASSICAL AND NONLOCAL*

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Abstract. A discussion of three-wave interaction systems with rapidly decaying data is provided. Included are the classical and two nonlocal three-wave interaction systems. These three-wave equations are formulated from underlying compatible linear systems and are connected to a third order linear scattering problem. The inverse scattering transform (IST) is carried out in detail for all these three-wave interaction equations. This entails obtaining and analyzing the direct scattering problem, discrete eigenvalues, symmetries, the inverse scattering problem via Riemann–Hilbert methods, minimal scattering data, and time dependence. In addition, soliton solutions illustrating energy sharing mechanisms are also discussed. A crucial step in the analysis is the use of adjoint eigenfunctions which connects the third order scattering problem to key eigenfunctions that are analytic in the upper/lower half planes. The general compatible nonlinear wave system and its classical and nonlocal three-wave reductions are asymptotic limits of physically significant nonlinear equations, including water/gravity waves with surface tension.

Key words. inverse scattering transform, Riemann–Hilbert problems, three-wave systems, solitons

MSC codes. 35Q15, 35Q51, 37K15, 37K40

DOI. 10.1137/22M1488880

1. Introduction.

Three-wave interaction equations are fundamental nonlinear wave systems. In 1967, Benney and Newell [19] showed that the three-wave interaction equations arise whenever an underlying nonlinear dispersive equation has three wave packets of the form

$$\sum_{j=1}^{3} Q_j(x,t)e^{i(k_j\cdot x-\omega(k_j)t)}+c.c.$$

(c.c. stands for complex conjugate), where the three wave amplitudes $Q_j(x,t)$ are associated with wavenumbers k_j and frequencies $\omega_j := \omega(k_j)$ satisfying the following relations:

$$k_1+k_2+k_3=0 \quad \text{and} \quad \omega_1+\omega_2+\omega_3=0.$$

One finds from multiscale perturbation methods that the slowly varying envelope functions $Q_j(x,t)$ satisfy the one space-one time three-wave equations in (1.3) below, where $C_j=\omega'(k_j)$, $j=1,2,3$, are the group velocities of each packet. We also remark

*Received by the editors April 5, 2022; accepted for publication (in revised form) March 6, 2023; published electronically August 22, 2023.
https://doi.org/10.1137/22M1488880
Funding: This work was partially supported by National Science Foundation grants DMS-1712793 and DMS-2005343 and by National Natural Science Foundation of China grant 12101590.

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that there is an extension to a two space-one time system, where in this system $C_j\partial_x$ is replaced by $C_j \cdot \nabla$. Such three-wave equations arise widely in applications including water waves, internal waves, plasma physics, nonlinear optics, amongst many others; cf. [12, 21, 24, 35, 37, 40]. We also note that four-wave interaction equations become important if three-wave conditions are not satisfied; cf. [18, 19].

In 1965, the notion of solitons, i.e., the elastic interaction of solitary waves, was uncovered from numerical simulations of the Korteweg–deVries (KdV) equation [44]. In 1967, the KdV equation subject to rapidly decaying initial data was solved by inverse scattering methods [26]. This led to a linearization of the KdV equation and the important connection between discrete eigenvalues and bound states of the linear time independent Schrödinger equation and soliton solutions of the KdV equation. The solutions of the KdV equation were investigated rigorously, in full detail, by Deift and Trubowitz [25].

In 1972, Zakharov and Shabat [47] using methods introduced by Lax [32] found that the ubiquitous nonlinear Schrödinger (NLS) equation was integrable via the inverse scattering techniques. Afterwards, Ablowitz et al. [1] showed that the KdV, NLS, sine-Gordon, and modified KdV equations were part of a class of such integrable systems. Ablowitz et al. [1] termed this method the inverse scattering transform (IST).

The scattering analysis associated with the above nonlinear equations involves a linear second order system. Soon afterwards, higher order linear scattering equations were shown to lead to solutions of physically interesting nonlinear equations, such as the classical three-wave and the Boussinesq equations [2]. The IST for the classical one dimensional three-wave equations was studied by Kaup and Zakharov & Manakov in 1976 [27, 46]. In 1979, Kaup, Reiman, and Bers reviewed the homogeneous three-wave system in various aspects, including IST and numerical solutions [28]. Also in 1979, Reiman discussed the integrability of the spatial nonhomogeneous-medium three-wave interaction equations and IST methods to solve such systems [39]. The IST for the systems in [28, 39] involve 3×3 scattering problems. Important analysis involving $N \times N$ scattering problems and related nonlinear evolution equations was given in [13, 14]. Subsequently, the field expanded rapidly and many nonlinear equations were found and were amenable to IST methods. This includes PDEs in one space-one time, two space-one time dimensions, discrete equations, singular integro-differential equations, etc. IST methods associated with such equations have been discussed in a number of textbooks; see, e.g., [3, 4, 5, 36]. Importantly, solutions to a number of physically significant equations were found. These nonlinear equations were related to a range of linear scattering problems.

In 2013, a new nonlocal reduction of the Ablowitz–Kaup–Newell–Segur (AKNS) scattering problem was found [6], which gave rise to the following integrable nonlocal NLS equation:

$$iq_t(x,t) = q_{xx}(x,t) - 2\sigma q^2(x,t)q^*(-x,t), \quad \sigma = \mp 1\,, \tag{1.1}$$

where the asterisk is complex conjugate. The equation can alternatively be rewritten as

$$iq_t(x,t) = q_{xx}(x,t) + V[q,x,t]q(x,t), \text{ where } V[q,x,t] = q(x,t)q^*(-x,t)\,. \tag{1.2}$$

Equation (1.1) can be viewed as having a self-induced nonlinear "potential." It has the property of being a PT symmetric nonlinear equation. The linear counterpart (with V being an external potential) was introduced by Bender and Boettcher [17], where the potential $V(x,t)$ has the so-called PT property: $V(x,t) = V^*(-x,t)$.

In other words, one can view (1.1) as a linear Schrödinger equation with a self-induced potential $V[q,x,t] \equiv -2\sigma q(x,t)q^*(-x,t)$ satisfying the PT symmetry condition $V[q,x,t] = V^*[q,-x,t]$. We refer to (1.1) as the PT NLS equation. Subsequently, (1.2) was analyzed by the IST method in 2016 [7], and many new nonlocal equations were identified in 2017 [8]. Among the new equations, the nonlocal three-wave equations in one space-one time and two space-one time dimensions were included. Specifically, there are three versions of integrable three-wave equations in one space-one time, including the classical three-wave system

$$Q_{l,t}(x,t) + C_l Q_{l,x}(x,t) = i\varepsilon_l Q_m^*(x,t)Q_n^*(x,t), \tag{1.3}$$

the complex reverse space-time (RST) three-wave system

$$Q_{l,t}(x,t) + C_l Q_{l,x}(x,t) = i(-1)^{l+1}\varepsilon_l Q_m^*(-x,-t)Q_n^*(-x,-t), \tag{1.4}$$

and the real reverse space-time (RST) three-wave system

$$Q_{l,t}(x,t) + C_l Q_{l,x}(x,t) = (-1)^{l+1}\varepsilon_l Q_m(-x,-t)Q_n(-x,-t), \tag{1.5}$$

where $1 \le l,m,n \le 3$, $l \ne m \ne n$. In 2019, it was shown how nonlocal NLS systems can be derived from the nonlinear Klein–Gordon, KdV, and the water wave equations [9]. The issue discussed in detail here is the formulation and detailed study of the inverse scattering transform associated with the one space-one time classical and nonlocal three-wave equations with suitably decaying data. This includes the classical (1.3) and two nonlocal three-wave systems: the complex and real reverse space-time three-wave equations (1.4) and (1.5).

The outline of the paper is as follows. In section 2, we show that these three-wave equations are connected with underlying the linear system as a third order scalar equation. In section 3, for decaying data, suitable eigenfunctions are defined by their boundary values as $x \to \pm\infty$. As $x \to \pm\infty$, two of the three eigenfunctions are analytic in the upper half or lower half planes (UHP, LHP). Appropriate continuous scattering data are defined in section 4. But in order to carry out the inverse scattering, key information about the third eigenfunction is needed. In section 5, we employ adjoint eigenfunctions which allow us to relate the third eigenfunction to the other two. This method was employed first by Kaup in 1976 [27] and later extended in order to analyze the coupled NLS systems by Prinari et al. in 2006 [38] and investigate the Manakov system by Biondini et al. in 2015 (see [20, 31]).

Asymptotic information about the eigenfunctions that will be needed later is covered in section 6. The time dependence of the data is found in section 7. Important symmetries associated with the data are discussed in section 8, discrete eigenvalues in section 9, and the trace formulae relating scattering data in section 10. The trace formulae are needed in order to allow the inverse problem to be formulated in terms of a minimal number of reflection coefficients and discrete data: eigenvalues and appropriate normalization coefficients.

The inverse scattering leading to the linear integral equations for suitable eigenfunctions is developed via the Riemann–Hilbert (RH) technique in section 11; a Gel'fand Levitan–Marchenko approach was used in [27]. These RH equations are formulated in terms of a set of "minimal" scattering data which consists of the needed reflection coefficients, eigenvalues (poles of "transmission coefficients"), and "reduced" normalization coefficients. In order to reconstruct the potentials, the minimal set of data is required; this is discussed in section 12. We remark that use of the adjoint eigenfunctions admits a direct connection from the scattering data to the initial data in much the same way as for second order systems, though more detailed. A special class of soliton solutions that have nonzero waves Q_1, Q_3 with $Q_2 \sim 0$ as $t \to -\infty$ or $t \to +\infty$ (we refer to these as 1-0-1 solitons at large negative or positive time, respectively) and their interaction properties are discussed in section 13. The soliton solutions illustrate the energy sharing mechanisms in these three-wave systems. The pure soliton case where there are multiple eigenvalues (multiple poles in the transmission coefficients) for the classical three-wave interaction equation was studied in [41, 42]. We note that the IST for the degenerate and nondegenerate 3×3 operators was discussed in 2009 and 2010, respectively [29, 30].

In section 14, we show how the classical and nonlocal reductions and, more generally, the sixth order wave system derived in section 2 (see (2.3)) are asymptotic reductions of a physically based nonlinear PDE. This nonlinear PDE is motivated by water/ocean waves [22]; it exhibits triad resonance phenomena. Indeed, we have recently shown that the sixth order wave system is an asymptotic reduction of the classical water/gravity wave equations with surface tension. This is a more extensive calculation which will appear separately [11]. Moreover, we carry out the IST analysis for three systems: the classical three-wave, nonlocal complex, and nonlocal real reverse space-time three-wave equations. We also remark that the scattering data, symmetries, RH problems, solitons, and energy sharing results for the nonlocal systems are different from the classical three-wave system. As such, there are many important and novel aspects to the direct and inverse scattering associated with the nonlocal three-wave systems discussed here. We note that the inverse scattering transform based on the RH approach for the classical three-wave system was studied nicely in [23]. The methods we use here, employing adjoint functions, was not used; we do not assume disjoint conditions on the initial data.

2. Integrable three-wave interaction equations. We begin with the matrix formulation $v_x = ikDv + Nv$, $v_t = Qv$, where v is a 3×1 matrix and D, N, Q are 3×3 matrices with D, N such that $D = \mathrm{diag}(d_1, d_2, d_3)$, $N_{ii} = 0$. We will assume $d_1 > d_2 > d_3$. Cross differentiation requiring $k_t = 0$ yields

$$Q_x = N_t + ik[D, Q] + [N, Q], \tag{2.1}$$

where $[A, B] := AB - BA$. We expand Q as follows:

$$Q = Q^{(1)}k + Q^{(0)}. \tag{2.2}$$

We substitute (2.2) into (2.1), yielding at k^2: $Q^{(1)}_{lj} = q_l \delta_{lj}$, where q_1, q_2, and q_3 are chosen as constants. At k, we have $Q^{(0)}_{ll} = 0$ and $Q^{(0)}_{lj} = \alpha_{lj} N_{lj}$, $l \neq j$, where $\alpha_{lj} := \frac{1}{i} \frac{q_l - q_j}{d_l - d_j} = \alpha_{jl}$. At k^0, we have

$$N_{lj,t} - \alpha_{lj} N_{lj,x} = \sum_{m=1}^{3} (\alpha_{lm} - \alpha_{mj}) N_{lm} N_{mj}. \tag{2.3}$$

Clearly, system (2.3) represents six coupled equations.

2.1. Classical three-wave system. Under the symmetry reduction $N_{21}(x, t) = \sigma_1 N^*_{12}(x, t)$, $N_{31}(x, t) = \sigma_2 N^*_{13}(x, t)$, $N_{32}(x, t) = \sigma_3 N^*_{23}(x, t)$, where $*$ is the complex conjugate and assuming α_{lj}, σ_1, σ_2, and σ_3 are real, we have $\frac{\sigma_1 \sigma_3}{\sigma_2} = -1$. Without loss of generality, we may assume that $\sigma_j^2 = 1$ $(j = 1, 2, 3)$. Equation (2.3) may be transformed into the classical three-wave interaction equations by a suitable scaling of variables [3]. For example, we find the system

$$Q_{l,t}(x,t)+C_lQ_{l,x}(x,t)=i\varepsilon_lQ_m^*(x,t)Q_n^*(x,t), \tag{2.4}$$

$1\le l,m,n\le 3,\ l\ne m\ne n$, if we take

$$\begin{gathered} N_{12}=-\frac{iQ_3}{\sqrt{\beta_{13}\beta_{23}}},\quad N_{31}=-\frac{iQ_2}{\sqrt{\beta_{12}\beta_{23}}},\quad N_{23}=-\frac{iQ_1}{\sqrt{\beta_{12}\beta_{13}}},\\ N_{13}=-\varepsilon_1\varepsilon_3N_{31}^*(x,t),\quad N_{32}=\varepsilon_2\varepsilon_3N_{23}^*(x,t),\quad N_{21}=\varepsilon_1\varepsilon_2N_{12}^*(x,t), \end{gathered} \tag{2.5}$$

where $d_1=-C_1,\ d_2=-C_2,\ d_3=-C_3,\ \beta_{lj}:=d_l-d_j=-C_l+C_j,\ q_1=-iC_2C_3$, $q_2=-iC_1C_3$, $q_3=-iC_1C_2$, $\alpha_{12}=-C_3$, $\alpha_{13}=-C_2$, $\alpha_{23}=-C_1$,$\sigma_1=\varepsilon_1\varepsilon_2$, $\sigma_2=-\varepsilon_1\varepsilon_3$, $\sigma_3=\varepsilon_2\varepsilon_3,\ \varepsilon_j=\pm1,\ \varepsilon_1\varepsilon_2\varepsilon_3=-1$. The so-called decay instability case with positive definite energy occurs when one chooses one of the ε_j different in sign from the others; the explosive instability case is when $\varepsilon_1=\varepsilon_2=\varepsilon_3=-1$. Here, decay instability is the result of three-wave interactions that exist globally, while the explosive instability refers to the finite-time blow up [45]. Directly from the equations, we can derive the conserved quantities:

$$\varepsilon_n\int_{-\infty}^{\infty}Q_m(x,t)Q_m^*(x,t)dx-\varepsilon_m\int_{-\infty}^{\infty}Q_n(x,t)Q_n^*(x,t)dx=\text{constant} \tag{2.6}$$

for all $1\le m<n\le 3$.

2.2. Complex reverse space-time three-wave system. Under the symmetry reduction $N_{21}(x,t)=\sigma_1N_{12}^*(-x,-t),\ N_{31}(x,t)=\sigma_2N_{13}^*(-x,-t),\ N_{32}(x,t)=\sigma_3N_{23}^*(-x,-t)$ with $\alpha_{lj},\sigma_1,\sigma_2,\sigma_3$ chosen as real numbers, we have $\frac{\sigma_1\sigma_3}{\sigma_2}=1$. Without loss of generality, we may assume that $\sigma_j^2=1\ (j=1,2,3)$. Equation (2.3) can be put into a set of nonlocal three-wave interaction equations by a suitable scaling of variables. For example, we find the system

$$Q_{l,t}(x,t)+C_lQ_{l,x}(x,t)=i(-1)^{l+1}\varepsilon_lQ_m^*(-x,-t)Q_n^*(-x,-t) \tag{2.7}$$

for $1\le l,m,n\le 3,\ l\ne m\ne n$, if we take

$$\begin{gathered} N_{12}=-\frac{iQ_3}{\sqrt{\beta_{13}\beta_{23}}},\quad N_{31}=-\frac{iQ_2}{\sqrt{\beta_{12}\beta_{23}}},\quad N_{23}=-\frac{iQ_1}{\sqrt{\beta_{12}\beta_{13}}},\\ N_{13}=\varepsilon_1\varepsilon_3N_{31}^*(-x,-t),\quad N_{32}=\varepsilon_2\varepsilon_3N_{23}^*(-x,-t),\quad N_{21}=\varepsilon_1\varepsilon_2N_{12}^*(-x,-t), \end{gathered} \tag{2.8}$$

where

$$\begin{gathered} d_1=-C_1,\quad d_2=-C_2,\quad d_3=-C_3,\quad \beta_{lj}:=d_l-d_j=-C_l+C_j,\\ q_1=-iC_2C_3,\ q_2=-iC_1C_3,\ q_3=-iC_1C_2,\ \alpha_{12}=-C_3,\ \alpha_{13}=-C_2,\ \alpha_{23}=-C_1,\\ \sigma_1=\varepsilon_1\varepsilon_2,\quad \sigma_2=\varepsilon_1\varepsilon_3,\quad \sigma_3=\varepsilon_2\varepsilon_3,\quad \varepsilon_j=\pm1,\quad \varepsilon_1\varepsilon_2\varepsilon_3=1. \end{gathered} \tag{2.9}$$

Directly from the equations, one derives the conserved quantities: for $1\le m<n\le 3$,

$$\varepsilon_n\int_{-\infty}^{\infty}Q_m(x,t)Q_m^*(-x,-t)dx+(-1)^{n-m+1}\varepsilon_m\int_{-\infty}^{\infty}Q_n(x,t)Q_n^*(-x,-t)dx=\text{constant}. \tag{2.10}$$

2.3. Real reverse space-time three-wave system. This case occurs under the symmetry reduction $N_{21}(x,t)=\sigma_1 N_{12}(-x,-t)$, $N_{31}(x,t)=\sigma_2 N_{13}(-x,-t)$, and $N_{32}(x,t)=\sigma_3 N_{23}(-x,-t)$, where σ_1, σ_2, σ_3 are real and $\frac{\sigma_1\sigma_3}{\sigma_2}=1$. Without loss of generality, we may assume that $\sigma_j^2=1$ $(j=1,2,3)$. Equation (2.3) can be put into a set of nonlocal three-wave interaction equations by a suitable scaling of variables. We find the system

$$Q_{l,t}(x,t)+C_l Q_{l,x}(x,t)=(-1)^{l+1}\varepsilon_l Q_m(-x,-t)Q_n(-x,-t) \tag{2.11}$$

for $1\le l,m,n\le 3$, $l\ne m\ne n$, if we take

$$\begin{aligned} &N_{12}=-\frac{Q_3}{\sqrt{\beta_{13}\beta_{23}}}, \quad N_{31}=-\frac{Q_2}{\sqrt{\beta_{12}\beta_{23}}}, \quad N_{23}=-\frac{Q_1}{\sqrt{\beta_{12}\beta_{13}}},\\ &N_{13}=\varepsilon_1\varepsilon_3 N_{31}(-x,-t), \quad N_{32}=\varepsilon_2\varepsilon_3 N_{23}(-x,-t), \quad N_{21}=\varepsilon_1\varepsilon_2 N_{12}(-x,-t), \end{aligned} \tag{2.12}$$

where the parameters chosen are the same as (2.9).

Remark 2.1. Equation (2.11) implies that there exist real-valued solutions.

Directly from the equations, the conserved quantities are the same as (2.10), but the envelopes Q_j here are real, $j=1,2,3$. Note that Q_j are real, $j=1,2,3$, and thus (2.12) implies that N_{lj} are also real, where $l\ne j$. Therefore, the real reverse space-time three-wave equations admit additional symmetry reduction:

$$N_{13}=\varepsilon_1\varepsilon_3 N_{31}^*(-x,-t), \quad N_{32}=\varepsilon_2\varepsilon_3 N_{23}^*(-x,-t), \quad N_{21}=\varepsilon_1\varepsilon_2 N_{12}^*(-x,-t). \tag{2.13}$$

It means that the real nonlocal system (2.11) owns the symmetry properties which the complex reverse space-time three-wave system has. In addition, the real case possesses its peculiar symmetries.

3. Direct scattering: Eigenfunctions. We refer to solutions of the scattering problem

$$\begin{pmatrix} v_1\\ v_2\\ v_3 \end{pmatrix}_x = \begin{pmatrix} ikd_1 & N_{12}(x) & N_{13}(x)\\ N_{21}(x) & ikd_2 & N_{23}(x)\\ N_{31}(x) & N_{32}(x) & ikd_3 \end{pmatrix} \begin{pmatrix} v_1\\ v_2\\ v_3 \end{pmatrix} \tag{3.1}$$

as eigenfunctions with respect to the parameter k. As mentioned above, we assume $d_1>d_2>d_3$. The scattering problem (3.1) can be rewritten as

$$v_x=ikDv+Nv=(ikD+N)v, \tag{3.2}$$

where $D=\mathrm{diag}(d_1,d_2,d_3)$ and $N(x)=(N_{ij}(x))_{1\le i,j\le 3}$ with $N_{ii}(x)=0$. When the potentials $N_{12}(x), N_{13}(x), N_{23}(x)\to 0$ rapidly as $x\to\pm\infty$, then the eigenfunctions are asymptotic to the solutions of

$$(v_1\, v_2\, v_3)_x^T=\mathrm{diag}(ikd_1,ikd_2,ikd_3)\,(v_1\, v_2\, v_3)^T$$

when $|x|$ is sufficiently large, where the superscript T denotes a matrix transpose. Therefore, it is natural to introduce the eigenfunctions defined by the following boundary conditions: As $x\to-\infty$, they satisfy

$$\phi_1(x,k)\sim(1\,0\,0)^T e^{ikd_1x},\ \phi_2(x,k)\sim(0\,1\,0)^T e^{ikd_2x},\ \phi_3(x,k)\sim(0\,0\,1)^T e^{ikd_3x}, \tag{3.3}$$

whereas in the limit $x \to +\infty$, they obey

$$\psi_1(x,k) \sim (1\,0\,0)^T e^{ikd_1 x},\ \psi_2(x,k) \sim (0\,1\,0)^T e^{ikd_2 x},\ \psi_3(x,k) \sim (0\,0\,1)^T e^{ikd_3 x}. \tag{3.4}$$

In the following analysis, it is convenient to consider functions with constant boundary conditions. Therefore, we define modified eigenfunctions as follows: $M_j(x,k) = \phi_j(x,k)e^{-ikd_j x}$ and $N_j(x,k) = \psi_j(x,k)e^{-ikd_j x}$, $j = 1,2,3$. Then the Jost functions $M_j(x,k)$ and $N_j(x,k)$ are the solutions of the differential equations

$$\chi_{j,x}(x,k) - ik(D - d_j I)\chi_j(x,k) = (N\chi_j)(x,k), \tag{3.5}$$

where the potential $N(x) = (N_{ij}(x))_{1\le i,j\le 3}$ with $N_{ii}(x) = 0$ and $j = 1,2,3$. Solutions of the differential equations (3.5) can be represented by means of the following integral equations:

$$\chi_j(x,k) = \omega_j + \int_{-\infty}^{+\infty} G_j(x - x', k)(N\chi_j)(x',k)dx', \quad \text{where} \tag{3.6}$$

$$\omega_1 = (1\,0\,0)^T,\ \omega_2 = (0\,1\,0)^T,\ \omega_3 = (0\,0\,1)^T.$$

The Green's function $G_j(x,k)$ satisfies the differential equation

$$I\partial_x G_j(x,k) - ik(D - d_j I)G_j(x,k) = \delta(x)I.$$

Using Fourier transforms, we obtain

$$G_1^{\pm}(x,k) = \mp\theta(\mp x)\ \mathrm{diag}(1, e^{ik(d_2-d_1)x}, e^{ik(d_3-d_1)x}),$$

$$G_3^{\pm}(x,k) = \pm\theta(\pm x)\ \mathrm{diag}(e^{ik(d_1-d_3)x}, e^{ik(d_2-d_3)x}, 1),$$

where $\theta(x)$ is the Heaviside function, i.e., $\theta(x) = 1$ if $x > 0$ and $\theta(x) = 0$ if $x < 0$, and the "$\pm$" function is analytic in the upper (lower) half k-plane, respectively. The eigenfunctions can be represented by means of the following integral equations:

$$M_j(x,k) = \omega_j + \int_{-\infty}^{+\infty} G_j^{\mp}(x - x', k)(NM_j)(x',k)dx', \quad j = 1,3, \tag{3.7}$$

$$N_j(x,k) = \omega_j + \int_{-\infty}^{+\infty} G_j^{\pm}(x - x', k)(NN_j)(x',k)dx', \quad j = 1,3.$$

We remark that the Green's function $G_1^-(G_3^+)$ is associated with the integral equation of $M_1(M_3)$. Similarly, $N_1(N_3)$ corresponds to $G_1^+(G_3^-)$.

Definition 3.1. *We say $f \in L^1(\mathbb{R})$ if $\int_{-\infty}^{+\infty}|f(x)|dx < \infty$. In addition, a matrix $N \in L^1(\mathbb{R})$ if each entry of N belongs to $L^1(\mathbb{R})$.*

Then we have the following result.

Theorem 3.2. *If $N(x) \in L^1(\mathbb{R})$, then for each $x \in \mathbb{R}$, $M_3(x,k)$ and $N_1(x,k)$ are analytic for $\Im k > 0$ and continuous for $\Im k \ge 0$ and $M_1(x,k)$ and $N_3(x,k)$ are analytic for $\Im k < 0$ and continuous for $\Im k \le 0$.*

Proof. We prove the result for $M_3(x,k)$. The proofs are analogous for $N_1(x,k)$, $M_1(x,k)$, and $N_3(x,k)$. We consider the Neumann series $M_3(x,k)=\sum_{n=0}^{\infty} M_{3(n)}(x,k)$, where $M_{3(0)}(x,k)=\omega_3$, $M_{3(n+1)}(x,k)=\int_{-\infty}^{x} G_3^{+}(x-x',k)(NM_{3(n)})(x',k)dx'$, is formally a solution of the integral equation (3.7) for $j=3$. Since all the entries of N belong to $L^1(\mathbb{R})$, using the identities

$$\frac{1}{j!}\int_{-\infty}^{x}|f(\xi)|\left[\int_{-\infty}^{\xi}|f(\xi')|d\xi'\right]^{j}d\xi=\frac{1}{(j+1)!}\left[\int_{-\infty}^{x}|f(\xi)|d\xi\right]^{j+1},$$

and $d_1>d_2>d_3$, we obtain that the Neumann series is uniformly convergent for $\Im k\geq 0$, which implies that $M_3(x,k)$ is analytic for $\Im k>0$ and continuous for $\Im k\geq 0$. ∎

4. Continuous scattering data. The Wronskian of a set $\{u_1,u_2,u_3\}$ of solution of the scattering problem (3.1) is defined as $W(u_1,u_2,u_3)=\det(u_1,u_2,u_3)$, and it satisfies the equation $\partial_x W(u_1,u_2,u_3)=ikdW(u_1,u_2,u_3)$, where $d:=d_1+d_2+d_3$.

With the Wronskian, we find that the two matrices $\Phi(x,k)=(\phi_1,\phi_2,\phi_3)$ and $\Psi(x,k)=(\psi_1,\psi_2,\psi_3)$, each contains a set of three linearly independent solutions of the scattering problem (3.1). Therefore, we can write $\phi_1(x,k)$, $\phi_2(x,k)$, and $\phi_3(x,k)$ as linear combinations of $\psi_1(x,k)$, $\psi_2(x,k)$, and $\psi_3(x,k)$, or vice versa. Hence, the relations

$$\Phi(x,k)=\Psi(x,k)A^T(k), \tag{4.1}$$

$$\Psi(x,k)=\Phi(x,k)B^T(k) \tag{4.2}$$

hold for any k such that all eigenfunctions exist, where $A(k):=(a_{ij})$ is referred to as the 3×3 scattering matrix and $B(k)=(b_{ij})=A^{-1}(k)$; $a_{ij}(k)$ is the scattering data. We call (4.1) the left scattering problem and (4.2) the right scattering problem. Moreover,

$$\sum_{m=1}^{3} a_{lm}b_{mj}=\delta_{lj},\quad \sum_{m=1}^{3} b_{lm}a_{mj}=\delta_{lj} \tag{4.3}$$

and $\det(A)=\det(B)=1$. Then we have the following theorem.

THEOREM 4.1. *If $N(x)\in L^1(\mathbb{R})$, then $a_{33}(k)$, $b_{11}(k)$ are analytic for $\Im k>0$ and continuous for $\Im k\geq 0$, and $a_{11}(k)$, $b_{33}(k)$ are analytic for $\Im k<0$ and continuous for $\Im k\leq 0$.*

Proof. We prove the result for $a_{11}(k)$. The proofs are analogous for $a_{33}(k)$, $b_{11}(k)$, and $b_{33}(k)$. Since $\phi_1(x,k)=a_{11}(k)\psi_1(x,k)+a_{12}(k)\psi_2(x,k)+a_{13}(k)\psi_3(x,k)$, we have a relation amongst their first component, i.e.,

$$\phi_1^{(1)}(x,k)=a_{11}(k)\psi_1^{(1)}(x,k)+a_{12}(k)\psi_2^{(1)}(x,k)+a_{13}(k)\psi_3^{(1)}(x,k).$$

Letting $x\to+\infty$, by (3.4), we obtain

$$a_{11}(k)=\lim_{x\to+\infty}\phi_1^{(1)}(x,k)e^{-ikd_1x}=\lim_{x\to+\infty}M_1^{(1)}(x,k).$$

We deduce from Theorem 3.2 that $M_1(x,k)$ is analytic for $\Im k<0$ and continuous for $\Im k\leq 0$, which yields that $a_{11}(k)$ is analytic for $\Im k<0$ and continuous for $\Im k\leq 0$ ∎

Remark 4.2. In general, the entries $a_{ij}(k),b_{ij}(k)$, $i\neq j$, cannot be extended off the real axis. If $N(x)$ is in Schwartz class, then $a_{ij}(k),b_{ij}(k)$ are in Schwartz class as well for all i,j; cf. [13, 14, 15].

5. Adjoint problem and auxiliary eigenfunctions. In order to formulate and solve the inverse scattering problem, we need two independent sets of analytic eigenfunctions. The main issue at this stage is eliminating the nonanalytic eigenfunctions ϕ_2 and ψ_2. The key idea is to consider a related eigenvalue problem, which we will refer to as an adjoint eigenvalue problem:

$$v_x^{ad} = -(ikD + N^T)v^{ad}, \tag{5.1}$$

where T is a matrix transpose. Then recall a well-known fact (see [16]) that if $u^{ad}(x,k)$ and $w^{ad}(x,k)$ are two arbitrary solutions of (5.1), we have that

$$v(x,k) = e^{ikdx}(u^{ad}(x,k) \times w^{ad}(x,k)) \tag{5.2}$$

satisfies (3.2). Conversely, if $u(x,k)$ and $w(x,k)$ are two arbitrary solutions of (3.2), then

$$v^{ad}(x,k) = e^{-ikdx}(u(x,k) \times w(x,k)) \tag{5.3}$$

solves (5.1), where $\times$ denotes the cross product, and we recall that $d = d_1 + d_2 + d_3$. Details underlying (5.2)–(5.3) are discussed in the appendix.

In order to uniquely define the adjoint eigenfunctions $\phi_n^{ad}(x,k)$ and $\psi_n^{ad}(x,k)$, where $n = 1,2,3$, we impose the following boundary conditions:

$$\phi_j^{ad}(x,k) \sim \gamma_j\omega_j e^{-ikd_jx} \text{ as } x \to -\infty; \;\; \psi_j^{ad}(x,k) \sim \gamma_j\omega_j e^{-ikd_jx} \text{ as } x \to +\infty \tag{5.4}$$

for $j = 1,2,3$, where γ_j will be given for the case of the classical and two nonlocal systems, respectively, in section 8. We define the bounded adjoint eigenfunctions as follows: $M_j^{ad}(x,k) = \phi_j^{ad}(x,k)e^{ikd_jx}$ and $N_j^{ad}(x,k) = \psi_j^{ad}(x,k)e^{ikd_jx}$, $j = 1,2,3$. The analytic properties of the adjoint eigenfunctions are the opposite of the original ones. So we have the following theorem.

THEOREM 5.1. *If $N(x) \in L^1(\mathbb{R})$, then for each $x \in \mathbb{R}$, $M_3^{ad}(x,k)$ and $N_1^{ad}(x,k)$ are analytic for $\Im k < 0$ and continuous for $\Im k \le 0$ and $M_1^{ad}(x,k)$ and $N_3^{ad}(x,k)$ are analytic for $\Im k > 0$ and continuous for $\Im k \ge 0$.*

Similarly, $\phi_1^{ad}(x,k)$, $\phi_2^{ad}(x,k)$, and $\phi_3^{ad}(x,k)$ can be written as linear combinations of $\psi_1^{ad}(x,k)$, $\psi_2^{ad}(x,k)$, and $\psi_3^{ad}(x,k)$, or vice versa. Thus, we have

$$\Phi^{ad}(x,k) = \Psi^{ad}(x,k)\widetilde{B}^T(k), \quad \Psi^{ad}(x,k) = \Phi^{ad}(x,k)\widetilde{A}^T(k), \tag{5.5}$$

where $\Phi^{ad} = (\phi_1^{ad}, \phi_2^{ad}, \phi_3^{ad})$, $\Psi^{ad} = (\psi_1^{ad}, \psi_2^{ad}, \psi_3^{ad})$, $\widetilde{A} = (\widetilde{a}_{ij})$, and $\widetilde{B} = (\widetilde{b}_{ij}) = \widetilde{A}^{-1}$. Applying (5.2), we introduce eigenfunctions $\tau(x,k)$ and $\overline{\tau}(x,k)$, which are defined by

$$\tau(x,k) = e^{ikdx}(\phi_1^{ad}(x,k) \times \psi_3^{ad}(x,k)), \quad \overline{\tau}(x,k) = e^{ikdx}(\phi_3^{ad}(x,k) \times \psi_1^{ad}(x,k)). \tag{5.6}$$

Taking into account boundary conditions, for cyclic indices j,l,m, we obtain

$$\phi_j(x,k) = e^{ikdx}(\phi_l^{ad}(x,k) \times \phi_m^{ad}(x,k)), \quad \psi_j(x,k) = e^{ikdx}(\psi_l^{ad}(x,k) \times \psi_m^{ad}(x,k)), \tag{5.7}$$

$$\phi_j^{ad}(x,k) = e^{-ikdx}(\phi_l(x,k) \times \phi_m(x,k)), \quad \psi_j^{ad}(x,k) = e^{-ikdx}(\psi_l(x,k) \times \psi_m(x,k)). \tag{5.8}$$

Combining (4.1), (4.2), (5.5), and (5.7)–(5.8), we deduce

$$\widetilde{A}^T(k) = A(k), \quad \widetilde{B}^T(k) = B(k), \tag{5.9}$$

$$\text{i.e.,} \quad \phi_j^{ad}(x,k) = \sum_l b_{lj}\psi_l^{ad}, \quad \psi_j^{ad}(x,k) = \sum_l a_{lj}\phi_l^{ad}. \tag{5.10}$$

Substituting (5.5), (5.9) into (5.6), it follows that

$$\tau(x,k) = b_{21}(k)\psi_1(x,k) - b_{11}(k)\psi_2(x,k), \quad \overline{\tau}(x,k) = b_{33}(k)\psi_2(x,k) - b_{23}(k)\psi_3(x,k). \tag{5.11}$$

One also has

$$\tau(x,k) = a_{23}(k)\phi_3(x,k) - a_{33}(k)\phi_2(x,k), \quad \overline{\tau}(x,k) = a_{11}(k)\phi_2(x,k) - a_{21}(k)\phi_1(x,k). \tag{5.12}$$

Moreover, (5.6) implies the corresponding bounded eigenfunctions, which can be defined by

$$\chi(x,k) = e^{-ikd_2x}(b_{21}(k)\psi_1(x,k) - b_{11}(k)\psi_2(x,k)), \tag{5.13}$$

$$\overline{\chi}(x,k) = e^{-ikd_2x}(b_{33}(k)\psi_2(x,k) - b_{23}(k)\psi_3(x,k)). \tag{5.14}$$

Similarly,

$$\chi(x,k) = e^{-ikd_2x}(a_{23}(k)\phi_3(x,k) - a_{33}(k)\phi_2(x,k)), \tag{5.15}$$

$$\overline{\chi}(x,k) = e^{-ikd_2x}(a_{11}(k)\phi_2(x,k) - a_{21}(k)\phi_1(x,k)). \tag{5.16}$$

Then we have the following theorem.

THEOREM 5.2. *If $N(x) \in L^1(\mathbb{R})$, then for each $x \in \mathbb{R}$, $\chi(x,k)$ is analytic for $\Im k > 0$ and continuous for $\Im k \geq 0$ and $\overline{\chi}(x,k)$ is analytic for $\Im k < 0$ and continuous for $\Im k \leq 0$.*

6. Asymptotic behavior of eigenfunctions and scattering data. In order to solve the inverse problem, one has to determine the asymptotic behavior of eigenfunctions and scattering data as $k \to \infty$. From the integral equations in terms of the Green's functions found earlier, we have

$$M_1 \sim \begin{pmatrix} 1 - \frac{\int_{-\infty}^x N_{12}(x')N_{21}(x')dx'}{ik(d_2-d_1)} - \frac{\int_{-\infty}^x N_{13}(x')N_{31}(x')dx'}{ik(d_3-d_1)} & -\frac{N_{21}(x)}{ik(d_2-d_1)} & -\frac{N_{31}(x)}{ik(d_3-d_1)} \end{pmatrix}^T + O(k^{-2}), \tag{6.1}$$

$$N_1 \sim \begin{pmatrix} 1 + \frac{\int_x^{+\infty} N_{12}(x')N_{21}(x')dx'}{ik(d_2-d_1)} + \frac{\int_x^{+\infty} N_{13}(x')N_{31}(x')dx'}{ik(d_3-d_1)} & -\frac{N_{21}(x)}{ik(d_2-d_1)} & -\frac{N_{31}(x)}{ik(d_3-d_1)} \end{pmatrix}^T + O(k^{-2}), \tag{6.2}$$

$$M_3 \sim \begin{pmatrix} -\frac{N_{13}(x)}{ik(d_1-d_3)} & -\frac{N_{23}(x)}{ik(d_2-d_3)} & 1 - \frac{\int_{-\infty}^x N_{13}(x')N_{31}(x')dx'}{ik(d_1-d_3)} - \frac{\int_{-\infty}^x N_{23}(x')N_{32}(x')dx'}{ik(d_2-d_3)} \end{pmatrix}^T + O(k^{-2}), \tag{6.3}$$

$$N_3 \sim \begin{pmatrix} -\frac{N_{13}(x)}{ik(d_1-d_3)} & -\frac{N_{23}(x)}{ik(d_2-d_3)} & 1 + \frac{\int_x^{+\infty} N_{13}(x')N_{31}(x')dx'}{ik(d_1-d_3)} + \frac{\int_x^{+\infty} N_{23}(x')N_{32}(x')dx'}{ik(d_2-d_3)} \end{pmatrix}^T + O(k^{-2}) \tag{6.4}$$

as $k \to \infty$. Note that

$$a_{11}(k) = \frac{W(\phi_1, \psi_2, \psi_3)}{W(\psi_1, \psi_2, \psi_3)}, \ \ a_{33}(k) = \frac{W(\psi_1, \psi_2, \phi_3)}{W(\psi_1, \psi_2, \psi_3)},$$

where we have $a_{11}(k) \sim 1$ for $\Im k < 0$ and as $k \to \infty$, and $a_{33}(k) \sim 1$ for $\Im k > 0$ and as $k \to \infty$.

Similarly, we can deduce $b_{11}(k) \sim 1$ for $\Im k > 0$ and as $k \to \infty$, and $b_{33}(k) \sim 1$ for $\Im k < 0$ and as $k \to \infty$.

Moreover, we obtain $\chi(x,k) \sim -\gamma_1\gamma_3\omega_2$ for $\Im k > 0$ and as $k \to \infty$, and $\overline{\chi}(x,k) \sim \gamma_1\gamma_3\omega_2$ for $\Im k < 0$ and as $k \to \infty$.

7. Time evolution. From (2.2) and the time evolution equation

$$v_t = Qv, \quad Q(x,t) = \begin{pmatrix} -iC_2C_3k & -C_3N_{12}(x,t) & -C_2N_{13}(x,t) \\ -C_3N_{21}(x,t) & -iC_1C_3k & -C_1N_{23}(x,t) \\ -C_2N_{31}(x,t) & -C_1N_{32}(x,t) & -iC_1C_2k \end{pmatrix}, \tag{7.1}$$

one has

$$v_t^{(1)} = -iC_2C_3kv^{(1)} - C_3N_{12}(x,t)v^{(2)} - C_2N_{13}(x,t)v^{(3)},$$

$$v_t^{(2)} = -C_3N_{21}(x,t)v^{(1)} - iC_1C_3kv^{(2)} - C_1N_{23}(x,t)v^{(3)},$$

$$v_t^{(3)} = -C_2N_{31}(x,t)v^{(1)} - C_1N_{32}(x,t)v^{(2)} - iC_1C_2kv^{(3)}.$$

Since $N_{lj}(x,t) \to 0$ as $x \to \pm\infty$, $l \neq j$, we find

$$v_t^{(1)} \sim -iC_2C_3kv^{(1)}, \ \ v_t^{(2)} \sim -iC_1C_3kv^{(2)}, \ \ v_t^{(3)} \sim -iC_1C_2kv^{(3)}$$

as $x \to \pm\infty$.

Note that the eigenfunctions themselves, whose boundary values are space infinities, are not compatible with this time evolution.

Therefore, one introduces time-dependent eigenfunctions

$$\Phi_1(x,t) = e^{iA_\infty t} \cdot \phi_1(x,t), \Phi_2(x,t) = e^{iB_\infty t} \cdot \phi_2(x,t), \Phi_3(x,t) = e^{iC_\infty t} \cdot \phi_3(x,t),$$

$$\Psi_1(x,t) = e^{i\overline{A}_\infty t} \cdot \psi_1(x,t), \Psi_2(x,t) = e^{i\overline{B}_\infty t} \cdot \psi_2(x,t), \Psi_3(x,t) = e^{i\overline{C}_\infty t} \cdot \psi_3(x,t)$$

to be solutions of (7.1). As a result, $\frac{\partial \Phi_1(x,t)}{\partial t} = iA_\infty \Phi_1(x,t) + e^{iA_\infty t} \frac{\partial \phi_1(x,t)}{\partial t}$.

We recall that $\phi_1(x,t) \sim \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T e^{ikd_1x}$ as $x \to -\infty$. From

$$\frac{\partial \Phi_1^{(1)}(x,t)}{\partial t} \sim -iC_2C_3k\Phi_1^{(1)}(x,t) = iA_\infty \Phi_1^{(1)}(x,t) + e^{iA_\infty t} \frac{\partial \phi_1^{(1)}(x,t)}{\partial t},$$

we deduce $A_\infty = -C_2C_3k$. Similarly, $B_\infty = -C_1C_3k$, $C_\infty = -C_1C_2k$, $\overline{A}_\infty = A_\infty$, $\overline{B}_\infty = B_\infty$, $\overline{C}_\infty = C_\infty$. Then

$$\frac{\partial \phi_1}{\partial t} = (Q - iA_\infty I)\phi_1, \quad \frac{\partial \phi_2}{\partial t} = (Q - iB_\infty I)\phi_2, \quad \frac{\partial \phi_3}{\partial t} = (Q - iC_\infty I)\phi_3,$$

$$\frac{\partial \psi_1}{\partial t} = (Q - iA_\infty I)\psi_1, \quad \frac{\partial \psi_2}{\partial t} = (Q - iB_\infty I)\psi_2, \quad \frac{\partial \psi_3}{\partial t} = (Q - iC_\infty I)\psi_3.$$

Noting that

$$\phi_1(x,t) = a_{11}(k;t)\psi_1(x,t) + a_{12}(k;t)\psi_2(x,t) + a_{13}(k;t)\psi_3(x,t),$$

one obtains

$$a_{11,t}\psi_1 + a_{11}(Q - iA_\infty I)\psi_1 + a_{12,t}\psi_2 + a_{12}(Q - iB_\infty I)\psi_2 + a_{13,t}\psi_3 + a_{13}(Q - iC_\infty I)\psi_3 = (Q - iA_\infty I)(a_{11}\psi_1 + a_{12}\psi_2 + a_{13}\psi_3),$$

which implies that

$$\frac{\partial a_{11}(k,t)}{\partial t} = 0, \quad \frac{\partial a_{12}(k,t)}{\partial t} = i(B_\infty - A_\infty)a_{12}(k,t), \quad \frac{\partial a_{13}(k,t)}{\partial t} = i(C_\infty - A_\infty)a_{13}(k,t).$$

Similarly, we deduce

$$\frac{\partial a_{21}(k,t)}{\partial t} = i(A_\infty - B_\infty)a_{21}(k,t), \quad \frac{\partial a_{22}(k,t)}{\partial t} = 0, \quad \frac{\partial a_{23}(k,t)}{\partial t} = i(C_\infty - B_\infty)a_{23}(k,t),$$

$$\frac{\partial a_{31}(k,t)}{\partial t} = i(A_\infty - C_\infty)a_{31}(k,t), \quad \frac{\partial a_{32}(k,t)}{\partial t} = i(B_\infty - C_\infty)a_{32}(k,t), \quad \frac{\partial a_{33}(k,t)}{\partial t} = 0.$$

Using a method similar to that above, we derive

$$\frac{\partial b_{11}(k,t)}{\partial t} = 0, \quad \frac{\partial b_{12}(k,t)}{\partial t} = i(B_\infty - A_\infty)b_{12}(k,t), \quad \frac{\partial b_{13}(k,t)}{\partial t} = i(C_\infty - A_\infty)b_{13}(k,t),$$

$$\frac{\partial b_{21}(k,t)}{\partial t} = i(A_\infty - B_\infty)b_{21}(k,t), \quad \frac{\partial b_{22}(k,t)}{\partial t} = 0, \quad \frac{\partial b_{23}(k,t)}{\partial t} = i(C_\infty - B_\infty)b_{23}(k,t),$$

$$\frac{\partial b_{31}(k,t)}{\partial t} = i(A_\infty - C_\infty)b_{31}(k,t), \quad \frac{\partial b_{32}(k,t)}{\partial t} = i(B_\infty - C_\infty)b_{32}(k,t), \quad \frac{\partial b_{33}(k,t)}{\partial t} = 0.$$

Therefore, $a_{11}(k,t)$, $a_{22}(k,t)$, $a_{33}(k,t)$, $b_{11}(k,t)$, $b_{22}(k,t)$, $b_{33}(k,t)$ are time-independent.

8. Symmetries.

The symmetry in the potential $N(x,t)$ induces symmetry among the eigenfunctions. Note that we add time t since it is needed in subsequent sections. Recall

$$v_x(x,k;t) = (ikD + N(x,t))v(x,k;t), \quad w_x^{ad}(x,k;t) = (-ikD - N^T(x,t))w^{ad}(x,k;t),$$

where $v = (v_1 \ \ v_2 \ \ v_3)^T$ and $w = (w_1 \ \ w_2 \ \ w_3)^T$ We introduce the analogue of reflection coefficients below:

$$\rho_1(k,t) := \frac{a_{12}(k,t)}{a_{11}(k,t)}, \quad \overline{\rho}_1(k,t) := \frac{a_{13}(k,t)}{a_{11}(k,t)}, \quad \rho_2(k,t) := \frac{a_{31}(k,t)}{a_{33}(k,t)}, \quad \overline{\rho}_2(k,t) := \frac{a_{32}(k,t)}{a_{33}(k,t)},$$

$$\rho_3(k,t) := \frac{b_{21}(k,t)}{b_{11}(k,t)}, \quad \overline{\rho}_3(k,t) := \frac{b_{23}(k,t)}{b_{33}(k,t)}, \quad \rho_4(k,t) := \frac{b_{12}(k,t)}{b_{11}(k,t)}, \quad \overline{\rho}_4(k,t) := \frac{b_{13}(k,t)}{b_{11}(k,t)},$$

$$\rho_5(k,t) := \frac{a_{23}(k,t)}{a_{33}(k,t)}, \quad \overline{\rho}_5(k,t) := \frac{a_{21}(k,t)}{a_{11}(k,t)}, \quad \rho_6(k,t) := \frac{b_{31}(k,t)}{b_{33}(k,t)}, \quad \overline{\rho}_6(k,t) := \frac{b_{32}(k,t)}{b_{33}(k,t)}.$$

8.1. Classical three-wave system.

Under the symmetry reduction

$$N_{21}(x,t) = \varepsilon_1\varepsilon_2 N_{12}^*(x,t), \ \ N_{31}(x,t) = -\varepsilon_1\varepsilon_3 N_{13}^*(x,t), \ \ N_{32}(x,t) = \varepsilon_2\varepsilon_3 N_{23}^*(x,t),$$

where $\varepsilon_j^2 = 1$, $j = 1,2,3$, and using the Hermitian conjugate of (3.2), i.e.,

$$v_x^H(x,k;t) = v^H(x,k;t)(-ik^*D + N^H(x,t)),$$

where $N^H(x,t)$ is the Hermitian conjugate of $N(x,t)$, leads to

$$v_{1,x}^*(x,k^*;t) = -ikd_1 v_1^*(x,k^*;t) + \varepsilon_1\varepsilon_2 N_{21}(x,t)v_2^*(x,k^*;t) - \varepsilon_1\varepsilon_3 N_{31}(x,t)v_3^*(x,k^*;t),$$
$$v_{2,x}^*(x,k^*;t) = \varepsilon_1\varepsilon_2 N_{12}(x,t)v_1^*(x,k^*;t) - ikd_2 v_2^*(x,k^*;t) + \varepsilon_2\varepsilon_3 N_{32}(x,t)v_3^*(x,k^*;t),$$
$$v_{3,x}^*(x,k^*;t) = -\varepsilon_1\varepsilon_3 N_{13}(x,t)v_1^*(x,k^*;t) + \varepsilon_2\varepsilon_3 N_{23}(x,t)v_2^*(x,k^*;t) - ikd_3 v_3^*(x,k^*;t).$$

Note that using $\varepsilon_n^2 = 1$, $n = 1,2,3$, from section 5, we obtain that $w_n^{ad}(x,k;t) = -(-1)^n \varepsilon_n v_n^*(x,k^*;t)$ also solve for the classical three-wave system. Taking into account the boundary conditions (3.3), (3.4), and (5.4), we deduce

$$\phi_j^{ad}(x,k;t) = (-1)^{j+1}\varepsilon_j \phi_j^*(x,k^*;t), \quad \psi_j^{ad}(x,k;t) = (-1)^{j+1}\varepsilon_j \psi_j^*(x,k^*;t), \quad j = 1,2,3, \tag{8.1}$$

where $\gamma_1 := \varepsilon_1$, $\gamma_2 := -\varepsilon_2$, and $\gamma_3 := \varepsilon_3$. Thus,

$$b_{1l}(k,t) = \frac{W(\phi_l^{ad}(x,k;t), \psi_2^{ad}(x,k;t), \psi_3^{ad}(x,k;t))}{W(\psi_1^{ad}(x,k;t), \psi_2^{ad}(x,k;t), \psi_3^{ad}(x,k;t))} = -(-1)^l \frac{\varepsilon_l}{\varepsilon_1} a_{l1}^*(k^*,t). \tag{8.2}$$

Similarly, one obtains

$$b_{2l}(k,t) = (-1)^l \frac{\varepsilon_l}{\varepsilon_2} a_{l2}^*(k^*,t), \quad b_{3l}(k,t) = -(-1)^l \frac{\varepsilon_l}{\varepsilon_3} a_{l3}^*(k^*,t). \tag{8.3}$$

It follows that

$$\rho_3(k,t) = -\frac{\varepsilon_1}{\varepsilon_2}\rho_1^*(k^*,t), \quad \overline{\rho}_3(k,t) = -\frac{\varepsilon_3}{\varepsilon_2}\overline{\rho}_2^*(k^*,t). \tag{8.4}$$

Recall

$$a_{31}(k,t)b_{11}(k,t) + a_{32}(k,t)b_{21}(k,t) + a_{33}(k,t)b_{31}(k,t) = 0,$$

which implies $\frac{a_{31}(k,t)}{a_{33}(k,t)} + \frac{\varepsilon_1}{\varepsilon_3}\frac{a_{13}^*(k^*,t)}{a_{11}^*(k^*,t)} - \frac{\varepsilon_1}{\varepsilon_2}\frac{a_{12}^*(k^*,t)}{a_{11}^*(k^*,t)} \cdot \frac{a_{32}(k,t)}{a_{33}(k,t)} = 0$, i.e.,

$$\rho_2(k,t) + \frac{\varepsilon_1}{\varepsilon_3}\overline{\rho}_1^*(k^*,t) - \frac{\varepsilon_1}{\varepsilon_2}\rho_1^*(k^*,t)\overline{\rho}_2(k,t) = 0.$$

8.2. Complex reverse space-time three-wave system. Under the symmetry reduction

$$N_{21}(x,t) = \varepsilon_1\varepsilon_2 N_{12}^*(-x,-t), N_{31}(x,t) = \varepsilon_1\varepsilon_3 N_{13}^*(-x,-t),$$
$$N_{32}(x,t) = \varepsilon_2\varepsilon_3 N_{23}^*(-x,-t),$$

where $\varepsilon_j^2 = 1$, $j = 1,2,3$, and using the Hermitian conjugate of (3.2), i.e.,

$$v_x^H(x,k;t) = v^H(x,k;t)(-ik^*D + N^H(x,t)),$$

we have in component form

$$\begin{aligned}
&\varepsilon_1 v_{1,x}^*(-x,-k^*;-t) = -ikd_1\varepsilon_1 v_1^*(-x,-k^*;-t) - \varepsilon_1^2\varepsilon_2 N_{21}(x,t)v_2^*(-x,-k^*;-t)\\
&- \varepsilon_1^2\varepsilon_3 N_{31}(x,t)v_3^*(-x,-k^*;-t),\\
&\varepsilon_2 v_{2,x}^*(-x,-k^*;-t) = -\varepsilon_1\varepsilon_2^2 N_{12}(x,t)v_1^*(-x,-k^*;-t) - ikd_2\varepsilon_2 v_2^*(-x,-k^*;-t)\\
&- \varepsilon_2^2\varepsilon_3 N_{32}(x,t)v_3^*(-x,-k^*;-t),\\
&\varepsilon_3 v_{3,x}^*(-x,-k^*;-t) = -\varepsilon_1\varepsilon_3^2 N_{13}(x,t)v_1^*(-x,-k^*;-t) - \varepsilon_2\varepsilon_3^2 N_{23}(x,t)v_2^*(-x,-k^*;-t)\\
&- ikd_3\varepsilon_3 v_3^*(-x,-k^*;-t).
\end{aligned}$$

Noting that $\varepsilon_n^2 = 1$, $n = 1,2,3$, from section 5, we obtain that $w_n^{ad}(x,k;t) = \varepsilon_n v_n^*(-x, -k^*;-t)$ satisfy the complex reverse space-time three-wave system. Using the boundary conditions (3.3), (3.4), and (5.4) leads to

$$\psi_j^{ad}(x,k;t) = \varepsilon_j \phi_j^*(-x,-k^*;-t), \quad \phi_j^{ad}(x,k;t) = \varepsilon_j \psi_j^*(-x,-k^*;-t), \tag{8.5}$$

where $\gamma_j := \varepsilon_j$, $j = 1,2,3$. Thus,

$$b_{1l}(k,t) = \frac{W(\phi_l^{ad}(x,k;t), \psi_2^{ad}(x,k;t), \psi_3^{ad}(x,k;t))}{W(\psi_1^{ad}(x,k;t), \psi_2^{ad}(x,k;t), \psi_3^{ad}(x,k;t))} = \frac{\varepsilon_l}{\varepsilon_1} b_{l1}^*(-k^*,-t).$$

Similarly, one can deduce

$$b_{mn}(k,t) = \frac{\varepsilon_n}{\varepsilon_m} b_{nm}^*(-k^*,-t), \quad a_{mn}(k,t) = \frac{\varepsilon_n}{\varepsilon_m} a_{nm}^*(-k^*,-t), \tag{8.6}$$

which induces

$$\begin{aligned} \rho_3(k,t) &= \frac{\varepsilon_1}{\varepsilon_2}\rho_4^*(-k^*,-t), \overline{\rho}_3(k,t) = \frac{\varepsilon_3}{\varepsilon_2}\overline{\rho}_6^*(-k^*,-t), \\ \rho_1(k,t) &= \frac{\varepsilon_2}{\varepsilon_1}\overline{\rho}_5^*(-k^*,-t), \overline{\rho}_2(k,t) = \frac{\varepsilon_2}{\varepsilon_3}\rho_5^*(-k^*,-t). \end{aligned} \tag{8.7}$$

Also, $\rho_2(k,t) + \frac{\varepsilon_1}{\varepsilon_3}\overline{\rho}_4^*(-k^*,-t) + \frac{\varepsilon_1}{\varepsilon_3}\rho_4^*(-k^*,-t)\rho_5^*(-k^*,-t) = 0$.

8.3. Real reverse space-time three-wave system. Under the symmetry reduction

$$\begin{aligned} &N_{21}(x,t) = \varepsilon_1\varepsilon_2 N_{12}(-x,-t), N_{31}(x,t) = \varepsilon_1\varepsilon_3 N_{13}(-x,-t), \\ &N_{32}(x,t) = \varepsilon_2\varepsilon_3 N_{23}(-x,-t), \end{aligned}$$

where $\varepsilon_j^2 = 1$, $j = 1,2,3$, and using

$$v_x^T(x,k;t) = v^T(x,k;t)(ikD + N^T(x,t))$$

implies

$$\begin{aligned} \varepsilon_1 v_{1,x}(-x,k;-t) = &-ikd_1\varepsilon_1 v_1(-x,k;-t) - \varepsilon_1^2\varepsilon_2 N_{21}(x,t)v_2(-x,k;-t) \\ &- \varepsilon_1^2\varepsilon_3 N_{31}(x,t)v_3(-x,k;-t), \end{aligned}$$

$$\begin{aligned} \varepsilon_2 v_{2,x}(-x,k;-t) = &-\varepsilon_1\varepsilon_2^2 N_{12}(x,t)v_1(-x,k;-t) - ikd_2\varepsilon_2 v_2(-x,k;-t) \\ &- \varepsilon_2^2\varepsilon_3 N_{32}(x,t)v_3(-x,k;-t), \end{aligned}$$

$$\begin{aligned} \varepsilon_3 v_{3,x}(-x,k;-t) = &-\varepsilon_1\varepsilon_3^2 N_{13}(x,t)v_1(-x,k;-t) - \varepsilon_2\varepsilon_3^2 N_{23}(x,t)v_2(-x,k;-t) \\ &- ikd_3\varepsilon_3 v_3(-x,k;-t). \end{aligned}$$

Noting that $\varepsilon_n^2 = 1$, $n = 1,2,3$, from section 5, we obtain that $w_n^{ad}(x,k;t) = \varepsilon_n v_n(-x,k;-t)$ are also solutions of the real reverse space-time three-wave system. Taking into account the boundary conditions (3.3), (3.4), and (5.4), one deduces

$$\psi_j^{ad}(x,k;t) = \varepsilon_j\phi_j(-x,k;-t), \quad \phi_j^{ad}(x,k;t) = \varepsilon_j\psi_j(-x,k;-t) \tag{8.8}$$

with $j = 1,2,3$ and $\gamma_j := \varepsilon_j$, and thus

$$a_{mn}(k,t) = \frac{\varepsilon_n}{\varepsilon_m} a_{nm}(k,-t), \quad b_{mn}(k,t) = \frac{\varepsilon_n}{\varepsilon_m} b_{nm}(k,-t), \tag{8.9}$$

which follows from the fact that $\rho_3(k,t) = \frac{\varepsilon_1}{\varepsilon_2}\rho_4(k,-t)$, $\overline{\rho}_3(k,t) = \frac{\varepsilon_3}{\varepsilon_2}\overline{\rho}_6(k.-t)$, $\rho_1(k,t) = \frac{\varepsilon_2}{\varepsilon_1}\overline{\rho}_5(k,-t)$, $\overline{\rho}_2(k,t) = \frac{\varepsilon_2}{\varepsilon_3}\rho_5(k,-t)$. Moreover, $\rho_2(k,t) + \frac{\varepsilon_1}{\varepsilon_3}\overline{\rho}_4(k,-t) + \frac{\varepsilon_1}{\varepsilon_3}\rho_4(k,-t)\rho_5(k,-t) = 0$. In addition, the symmetry reduction

$$N_{21}(x,t)=\varepsilon_1\varepsilon_2 N_{12}^*(-x,-t), N_{31}(x,t)=\varepsilon_1\varepsilon_3 N_{13}^*(-x,-t),$$
$$N_{32}(x,t)=\varepsilon_2\varepsilon_3 N_{23}^*(-x,-t) \tag{8.10}$$

implies that what we have discussed in section 8.2 also works for the real nonlocal case. Hence, (8.5) and (8.8) yield

$$\phi_j(-x,k;-t)=\phi_j^*(-x,-k^*;-t), \psi_j(-x,k;-t)=\psi_j^*(-x,-k^*;-t), \quad j=1,2,3.$$

From (8.6) and (8.9), one has

$$a_{nm}(k,-t)=a_{nm}^*(-k^*,-t), \quad b_{nm}(k,-t)=b_{nm}^*(-k^*,-t). \tag{8.11}$$

In particular, if $k\in i\mathbb{R}$, then $\phi_j(x,k;t)$, $\psi_j(x,k;t)$, $a_{nm}(k,t)$, and $b_{nm}(k,t)$ are real.

Remark 8.1. In this section, the symmetry relations among scattering data involve time; they are consistent with what we obtained in section 7.

9. Zeros of the scattering data and discrete eigenvalues. Zeros of scattering data (i.e., zeros of $a_{11}(k)$, $a_{33}(k)$, $b_{11}(k)$, $b_{33}(k)$) are referred to here as eigenvalues. Given the initial data encoded in the potential $N(x)\in L^1(\mathbb{R})$, the zeros of scattering data form a bounded set in $\mathbb{C}\setminus\mathbb{R}$. Noting that $a_{33}(k)$ and $b_{11}(k)$ are analytic in the upper half plane, we assume they have simple zeros α_j and β_n, respectively, i.e., $a_{33}(\alpha_j)=0$ $(j=1,\ldots,J)$ and $b_{11}(\beta_n)=0$ $(n=1,\ldots,N)$; $b_{33}(k)$ and $a_{11}(k)$ are analytic in the lower half plane, where they have simple zeros (by assumption) $\overline{\alpha}_j$ and $\overline{\beta}_n$, respectively, i.e., $b_{33}(\overline{\alpha}_j)=0$ $(j=1,\ldots,\overline{J})$ and $a_{11}(\overline{\beta}_n)=0$ $(n=1,\ldots,\overline{N})$. In general, the zeros of $a_{11}(k)$, $a_{33}(k)$, $b_{11}(k)$, $b_{33}(k)$ can be multiple, which gives rise to higher-order soliton solutions; higher-order soliton solutions to the classical three-wave system have been discussed in [42]. In this paper, we assume that the zeros are proper; i.e., they are simple and do not lie on the the line $\Im k=0$.

Remark 9.1. As shown in [13, 14, 15], generic potentials are such that zeros of the associated data $a_{11}(k)$, $a_{33}(k)$, $b_{11}(k)$, $b_{33}(k)$ are proper; i.e. each zero is simple and lies off the line $\Im k=0$.

In addition, if we only focus on the zeros in the upper half plane, which come from the zeros of $a_{33}(k)$ and $b_{11}(k)$, then four possibilities may occur:

(1) $J=0$, i.e., $a_{33}(k)$ has no zeros.
(2) $N=0$, i.e., $b_{11}(k)$ has no zeros.
(3) All the zeros of $a_{33}(k)$ and $b_{11}(k)$ are distinct, i.e., $\alpha_j\neq\beta_n$ for all j,n.
(4) $a_{33}(k)$ and $b_{11}(k)$ have common zeros, i.e., there exists at least one $j_0\in\{1,\ldots,J\}$ such that $\alpha_{j_0}=\beta_{n_0}$ for some $n_0\in\{1,\ldots,N\}$.

A similar consequence holds for the zeros in the lower half plane. Thus, it gives rise to many different types of solitons. In section 13, typical examples of 1-0-1 solitons will be discussed. Recall that 1-0-1 solitons at $t=-\infty$ are defined as the nonzero envelopes Q_1, Q_3 with $Q_2\sim 0$ as $t\to-\infty$, which can result in different types of solitons at $t=+\infty$, such as 1-0-1 solitons (the waves Q_1, Q_3 are nonzero and $Q_2\sim 0$ as $t\to+\infty$) and 0-1-0 solitons (Q_2 is nonzero and $Q_1, Q_3\sim 0$ as $t\to+\infty$).

From (5.13)–(5.16), it follows that

$$\begin{aligned}\chi(x,\beta_n;t)&=b_{21}(\beta_n,t)N_1(x,\beta_n;t)e^{i\beta_n(d_1-d_2)x},\\ \overline{\chi}(x,\overline{\alpha}_j;t)&=-b_{23}(\overline{\alpha}_j,t)N_3(x,\overline{\alpha}_j;t)e^{i\overline{\alpha}_j(d_3-d_2)x},\end{aligned} \tag{9.1}$$

$$\begin{aligned}\overline{\chi}(x,\overline{\beta}_n;t)&=-a_{21}(\overline{\beta}_n,t)M_1(x,\overline{\beta}_n;t)e^{i\overline{\beta}_n(d_1-d_2)x},\\ \chi(x,\alpha_j;t)&=a_{23}(\alpha_j,t)M_3(x,\alpha_j;t)e^{i\alpha_j(d_3-d_2)x}.\end{aligned} \tag{9.2}$$

Equivalently,

$$\tau(x,\beta_n;t)=b_{21}(\beta_n,t)\psi_1(x,\beta_n;t), \quad \overline{\tau}(x,\overline{\alpha}_j;t)=-b_{23}(\overline{\alpha}_j,t)\psi_3(x,\overline{\alpha}_j;t), \tag{9.3}$$

$$\overline{\tau}(x,\overline{\beta}_n;t)=-a_{21}(\overline{\beta}_n,t)\phi_1(x,\overline{\beta}_n;t), \quad \tau(x,\alpha_j;t)=a_{23}(\alpha_j,t)\phi_3(x,\alpha_j;t). \tag{9.4}$$

Indeed, for each β_n such that $b_{11}(\beta_n)=0$, there is a nonzero proportionality constant such that the left-and right-hand sides of the first equation in (9.1) are exponentially decaying in opposite directions as $|x|\to\infty$. Similar conclusions hold for α_j, $\overline{\beta}_n$, and $\overline{\alpha}_j$.

We will need the coefficients

$$b_{21}(\beta_n,t),\ b_{23}(\overline{\alpha}_j,t),\ a_{21}(\overline{\beta}_n,t),\ a_{23}(\alpha_j,t) \tag{9.5}$$

as part of the integral equations governing the inverse scattering in section 11 and the soliton solutions in section 13; we call these coefficients reduced normalization constants/coefficients. In order to construct the inverse problem, it is convenient to introduce the 3×3 matrices

$$E_+(x,k,t)=(\psi_1(x,k,t),\tau(x,k,t),\phi_3(x,k,t)),$$
$$E_-(x,k,t)=(\phi_1(x,k,t),\overline{\tau}(x,k,t),\psi_3(x,k,t)).$$

With this notation, $E_\pm(x,k,t)$ collect three eigenfunctions which are analytic in the upper/lower half k-planes, respectively. Combining (3.3), (3.4), (5.11), and (5.12), we obtain $\det(E_+(x,k,t)) = W(\psi_1(x,k,t),\tau(x,k),\phi_3(x,k,t)) = -a_{33}(k,t)b_{11}(k,t)e^{ikdx}$, $\det(E_-(x,k,t))=W(\phi_1(x,k,t),\overline{\tau}(x,k,t),\psi_3(x,k),t)=a_{11}(k,t)b_{33}(k,t)e^{ikdx}$.

This shows that the three solutions $\psi_1(x,k,t)$, $\tau(x,k,t)$, and $\phi_3(x,k,t)$ become linearly dependent at the zeros of $a_{33}(k,t)$ or $b_{11}(k,t)$. Similarly, $\phi_1(x,k,t)$, $\overline{\tau}(x,k,t)$, and $\psi_3(x,k,t)$ are linearly dependent at the zeros of $a_{11}(k,t)$ or $b_{33}(k,t)$.

Next, our analysis will be based on three different symmetry reductions, which correspond to the classical, complex reverse space-time, and real reverse space-time three-wave systems, respectively.

We note certain key formulae: (9.6)–(9.7), (9.8)–(9.11), (9.12)–(9.13).

9.1. Classical three-wave system. We have assumed that $a_{33}(k)$ and $b_{11}(k)$ have simple zeros $\{\alpha_j : \Im\alpha_j > 0\}_{j=1}^J$ and $\{\beta_n : \Im\beta_n > 0\}_{n=1}^N$, respectively. By the symmetry relation mentioned in section 8.1, it implies that $b_{11}(k,t)=a_{11}^*(k^*,t)$ and $b_{33}(k,t)=a_{33}^*(k^*,t)$. Then $b_{33}(k)$ and $a_{11}(k)$ have simple zeros α_j^* and β_n^*, respectively, and therefore $\overline{\alpha}_j=\alpha_j^*$, $\overline{\beta}_n=\beta_n^*$, and $J=\overline{J}$, $N=\overline{N}$. From section 7, one has that $a_{11}(k,t)$, $a_{33}(k,t)$, $b_{11}(k,t)$, and $b_{33}(k,t)$ are time-independent. It yields $b_{11}(\beta_n,t)=0 \Leftrightarrow a_{11}(\beta_n^*,t)=0$, $a_{33}(\alpha_j,t)=0\Leftrightarrow b_{33}(\alpha_j^*,t)=0$. Thus, from (5.8), (8.1), and (9.4), one can write

$$\begin{aligned}\tau^*(x,\alpha_j;t)&=a_{23}^*(\alpha_j,t)\phi_3^*(x,\alpha_j;t)=\varepsilon_3 a_{23}^*(\alpha_j,t)\phi_3^{ad}(x,\alpha_j^*;t)\\&=\varepsilon_3 a_{23}^*(\alpha_j,t)e^{-i\alpha_j^* dx}(\phi_1(x,\alpha_j^*;t)\times\phi_2(x,\alpha_j^*;t)).\end{aligned}$$

Equation (5.12) implies

$$\begin{aligned}\phi_2(x,k;t)&=\frac{a_{23}(k,t)}{a_{33}(k,t)}\phi_3(x,k;t)-\frac{1}{a_{33}(k,t)}\tau(x,k;t)\\&=\frac{a_{21}(k,t)}{a_{11}(k,t)}\phi_1(x,k;t)+\frac{1}{a_{11}(k,t)}\overline{\tau}(x,k;t).\end{aligned}$$

From (8.1) and (9.3), it gives

$$\begin{aligned}\phi_1(x,\alpha_j^*;t)\times\phi_2(x,\alpha_j^*;t) &= \frac{1}{a_{11}(\alpha_j^*,t)}\phi_1(x,\alpha_j^*;t)\times\overline{\tau}(x,\alpha_j^*;t)\\ &= \frac{-b_{23}(\alpha_j^*,t)}{a_{11}(\alpha_j^*,t)}\phi_1(x,\alpha_j^*;t)\times\psi_3(x,\alpha_j^*;t)\\ &= \frac{-\varepsilon_1\varepsilon_3 b_{23}(\alpha_j^*,t)}{a_{11}(\alpha_j^*,t)}\left(\phi_1^{ad}(x,\alpha_j;t)\times\psi_3^{ad}(x,\alpha_j;t)\right)^*.\end{aligned}$$

Combining (5.6), we then have $\tau^*(x,\alpha_j;t)=\frac{-\varepsilon_1 b_{23}(\alpha_j^*,t)a_{23}^*(\alpha_j,t)\tau^*(x,\alpha_j;t)}{a_{11}(\alpha_j^*,t)}$. So we find the additional symmetries

$$a_{23}^*(\alpha_j,t)=-\frac{a_{11}(\alpha_j^*,t)}{\varepsilon_1 b_{23}(\alpha_j^*,t)}. \tag{9.6}$$

Similarly, we deduce

$$b_{21}^*(\beta_n,t)=-\frac{b_{33}(\beta_n^*,t)}{\varepsilon_3 a_{21}(\beta_n^*,t)}. \tag{9.7}$$

9.2. Complex reverse space-time three-wave system. We assume that $a_{33}(k)$ and $b_{33}(k)$ have the simple zeros $\{\alpha_j,-\alpha_j^*:\Im\alpha_j>0 \text{ and } \Re\alpha_j\neq 0\}_{j=1}^{J_1}\cup\{\widetilde{\alpha}_l:\Im\widetilde{\alpha}_l>0 \text{ and } \Re\widetilde{\alpha}_l=0\}_{l=1}^{J_2}$ and $\{\overline{\alpha}_j,-\overline{\alpha}_j^*:\Im\overline{\alpha}_j<0 \text{ and } \Re\overline{\alpha}_j\neq 0\}_{j=1}^{\overline{J}_1}\cup\{\overline{\widetilde{\alpha}}_l:\Im\overline{\widetilde{\alpha}}_l<0 \text{ and } \Re\overline{\widetilde{\alpha}}_l=0\}_{l=1}^{\overline{J}_2}$, respectively, where $2J_1+J_2=J$ and $2\overline{J}_1+\overline{J}_2=\overline{J}$. Indeed, the symmetry relations discussed in section 8.2 give $a_{33}(k,t)=a_{33}^*(-k^*,-t)$ and $b_{33}(k,t)=b_{33}^*(-k^*,-t)$. It shows that $a_{33}(k)$ and $b_{33}(k)$ also have simple zeros $-\alpha_j^*$ and $-\overline{\alpha}_j^*$, respectively, i.e., $\{\alpha_j,-\alpha_j^*\}$ and $\{\overline{\alpha}_j,-\overline{\alpha}_j^*\}$ are pairs of zeros.

One can have a similar assumption for $b_{11}(k)$ and $a_{11}(k)$; that is, both have the simple zeros $\{\beta_n,-\beta_n^*:\Im\beta_n>0 \text{ and } \Re\beta_n\neq 0\}_{n=1}^{N_1}\cup\{\widetilde{\beta}_m:\Im\widetilde{\beta}_m>0 \text{ and } \Re\widetilde{\beta}_m=0\}_{m=1}^{N_2}$ and $\{\overline{\beta}_n,-\overline{\beta}_n^*:\Im\overline{\beta}_n<0 \text{ and } \Re\overline{\beta}_n\neq 0\}_{n=1}^{\overline{N}_1}\cup\{\overline{\widetilde{\beta}}_m:\Im\overline{\widetilde{\beta}}_m<0 \text{ and } \Re\overline{\widetilde{\beta}}_m=0\}_{m=1}^{\overline{N}_2}$, respectively, where $2N_1+N_2=N$ and $2\overline{N}_1+\overline{N}_2=\overline{N}$. Also, $\{\beta_n,-\beta_n^*\}$ and $\{\overline{\beta}_n,-\overline{\beta}_n^*\}$ are pairs of zeros for $b_{11}(k)$ and $a_{11}(k)$, respectively. From section 7, we have that $a_{11}(k,t)$, $a_{33}(k,t)$, $b_{11}(k,t)$, and $b_{33}(k,t)$ are time-independent. Thus, $b_{11}(\beta_n,t)=0\Leftrightarrow b_{11}(-\beta_n^*,-t)=0$, $a_{33}(\alpha_j,t)=0\Leftrightarrow a_{33}(-\alpha_j^*,-t)=0$, $a_{11}(\overline{\beta}_n,t)=0\Leftrightarrow a_{11}(-\overline{\beta}_n^*,-t)=0$, $b_{33}(\overline{\alpha}_j,t)=0\Leftrightarrow b_{33}(-\overline{\alpha}_j^*,-t)=0$. By (5.8), (8.5), and (9.4), we can write $\tau^*(x,\alpha_j;t)=a_{23}^*(\alpha_j,t)\phi_3^*(x,\alpha_j;t)=\varepsilon_3 a_{23}^*(\alpha_j,t)\psi_3^{ad}(-x,-\alpha_j^*;-t)=\varepsilon_3 a_{23}^*(\alpha_j,t)e^{-i\alpha_j^* dx}(\psi_1(-x,-\alpha_j^*;-t)\times\psi_2(-x,-\alpha_j^*;-t))$. Equation (5.11) shows that $\psi_2(x,k;t)=\frac{b_{21}(k,t)}{b_{11}(k,t)}\psi_1(x,k;t)-\frac{\tau(x,k;t)}{b_{11}(k,t)}=\frac{b_{23}(k,t)}{b_{33}(k,t)}\psi_3(x,k;t)+\frac{\overline{\tau}(x,k;t)}{b_{33}(k,t)}$. From (8.5) and (9.4), one derives

$$\begin{aligned}&\psi_1(-x,-\alpha_j^*;-t)\times\psi_2(-x,-\alpha_j^*;-t)\\ &= -\frac{1}{b_{11}(-\alpha_j^*,-t)}\psi_1(-x,-\alpha_j^*;-t)\times\tau(-x,-\alpha_j^*;-t)\\ &= -\frac{\varepsilon_1\varepsilon_3 a_{23}(-\alpha_j^*,-t)}{b_{11}(-\alpha_j^*,-t)}(\phi_1^{ad}(x,\alpha_j;t)\times\psi_3^{ad}(x,\alpha_j;t))^*.\end{aligned}$$

Combining (5.6), one gets $\tau^*(x,\alpha_j;t)=-\frac{\varepsilon_1 a_{23}^*(\alpha_j,t)a_{23}(-\alpha_j^*,-t)}{b_{11}(-\alpha_j^*,-t)}\tau^*(x,\alpha_j;t)$, which yields

$$-\frac{\varepsilon_1 a_{23}^*(\alpha_j,t)a_{23}(-\alpha_j^*-t)}{b_{11}(-\alpha_j^*,-t)}=1. \tag{9.8}$$

Similarly, we deduce the following relations:

$$-\frac{\varepsilon_3 b_{21}^*(\beta_n,t)b_{21}(-\beta_n^*,-t)}{a_{33}(-\beta_n^*,-t)}=1,\quad -\frac{\varepsilon_1 b_{23}^*(\overline{\alpha}_j,t)b_{23}(-\overline{\alpha}_j^*,-t)}{a_{11}(-\overline{\alpha}_j^*,-t)}=1, \tag{9.9}$$

$$-\frac{\varepsilon_3 a_{21}^*(\overline{\beta}_n,t)a_{21}(-\overline{\beta}_n^*,-t)}{b_{33}(-\overline{\beta}_n^*,-t)}=1. \tag{9.10}$$

In particular,

$$\begin{aligned}&-\frac{\varepsilon_1 a_{23}^*(\widetilde{\alpha}_l,t)a_{23}(\widetilde{\alpha}_l,-t)}{b_{11}(\widetilde{\alpha}_l,-t)}=1,\quad -\frac{\varepsilon_3 b_{21}^*(\widetilde{\beta}_m,t)b_{21}(\widetilde{\beta}_m,-t)}{a_{33}(\widetilde{\beta}_m,-t)}=1,\\&-\frac{\varepsilon_1 b_{23}^*(\overline{\widetilde{\alpha}}_l,t)b_{23}(\overline{\widetilde{\alpha}}_l,-t)}{a_{11}(\overline{\widetilde{\alpha}}_l,-t)}=1,\quad -\frac{\varepsilon_3 a_{21}^*(\overline{\widetilde{\beta}}_m,t)a_{21}(\overline{\widetilde{\beta}}_m,-t)}{b_{33}(\overline{\widetilde{\beta}}_m,-t)}=1.\end{aligned} \tag{9.11}$$

9.3. Real reverse space-time three-wave system. Similarly, as above we assume that $a_{33}(k)$ and $b_{33}(k)$ have the simple zeros $\{\alpha_j : \Im\alpha_j > 0\}_{j=1}^{J}$ and $\{\overline{\alpha}_j : \Im\overline{\alpha}_j < 0\}_{j=1}^{\overline{J}}$, respectively. Also, $b_{11}(k)$ and $a_{11}(k)$ have the simple zeros $\{\beta_n : \Im\beta_n > 0\}_{n=1}^{N}$ and $\{\overline{\beta}_n : \Im\overline{\beta}_n < 0\}_{n=1}^{\overline{N}}$, respectively. Thus, (5.8) and (9.4) give $\tau(x,\alpha_j;t) = a_{23}(\alpha_j,t)\phi_3(x,\alpha_j;t) = \varepsilon_3 a_{23}(\alpha_j,t)\psi_3^{ad}(-x,\alpha_j;-t) = \varepsilon_3 a_{23}(\alpha_j,t)e^{i\alpha_j dx}(\psi_1(-x,\alpha_j;-t)\times \psi_2(-x,\alpha_j;-t))$. Also, (5.11) and (9.4) read as $\psi_1(-x,\alpha_j;-t)\times\psi_2(-x,\alpha_j;-t) = -\frac{1}{b_{11}(\alpha_j,-t)}\psi_1(-x,\alpha_j;-t)\times\tau(-x,\alpha_j;-t) = -\frac{a_{23}(\alpha_j,-t)}{b_{11}(\alpha_j,-t)}\psi_1(-x,\alpha_j;-t)\times\phi_3(-x,\alpha_j;-t)$. From (5.6) and (8.8), we obtain

$$\begin{aligned}\tau(x,\alpha_j;t) &= -\frac{\varepsilon_3 a_{23}(\alpha_j,t)a_{23}(\alpha_j,-t)}{b_{11}(\alpha_j,-t)}(e^{i\alpha_j dx}(\psi_1(-x,\alpha_j;-t)\times\phi_3(-x,\alpha_j;-t)))\\&= -\frac{\varepsilon_1 a_{23}(\alpha_j,t)a_{23}(\alpha_j,-t)}{b_{11}(\alpha_j,-t)}(e^{i\alpha_j dx}(\phi_1^{ad}(x,\alpha_j;t)\times\psi_3^{ad}(x,\alpha_j;t)))\\&= -\frac{\varepsilon_1 a_{23}(\alpha_j,t)a_{23}(\alpha_j,-t)}{b_{11}(\alpha_j,-t)}\tau(x,\alpha_j;t).\end{aligned}$$

Hence,

$$-\frac{\varepsilon_1 a_{23}(\alpha_j,t)a_{23}(\alpha_j,-t)}{b_{11}(\alpha_j,-t)}=1. \tag{9.12}$$

Similarly, one has the formulae

$$\begin{aligned}&-\frac{\varepsilon_3 b_{21}(\beta_n,t)b_{21}(\beta_n,-t)}{a_{33}(\beta_n,-t)}=1,\ -\frac{\varepsilon_1 b_{23}(\overline{\alpha}_j,t)b_{23}(\overline{\alpha}_j,-t)}{a_{11}(\overline{\alpha}_j,-t)}=1,\\&-\frac{\varepsilon_3 a_{21}(\overline{\beta}_n,t)a_{21}(\overline{\beta}_n,-t)}{b_{33}(\overline{\beta}_n,-t)}=1.\end{aligned} \tag{9.13}$$

Remark 9.2. Note that as in the nonlocal complex three-wave case, we have

$$\begin{aligned}&\{\alpha_j : \Im\alpha_j > 0\}_{j=1}^{J}\\&= \{\alpha_m, -\alpha_m^* : \Im\alpha_m > 0 \text{ and } \Re\alpha_m \neq 0\}_{m=1}^{J_1} \cup \{\widetilde{\alpha}_l : \Im\widetilde{\alpha}_l > 0 \text{ and } \Re\widetilde{\alpha}_l = 0\}_{l=1}^{J_2},\end{aligned}$$

$$\begin{aligned}&\{\overline{\alpha}_j : \Im\overline{\alpha}_j < 0\}_{j=1}^{\overline{J}}\\&= \{\overline{\alpha}_m, -\overline{\alpha}_m^* : \Im\overline{\alpha}_m < 0 \text{ and } \Re\overline{\alpha}_m \neq 0\}_{m=1}^{\overline{J}_1} \cup \{\overline{\widetilde{\alpha}}_l : \Im\overline{\widetilde{\alpha}}_l < 0 \text{ and } \Re\overline{\widetilde{\alpha}}_l = 0\}_{l=1}^{\overline{J}_2},\end{aligned}$$

$$\{\beta_n : \Im\beta_n > 0\}_{n=1}^{N}$$
$$= \{\beta_p, -\beta_p^* : \Im\beta_p > 0 \text{ and } \Re\beta_p \neq 0\}_{p=1}^{N_1} \cup \{\widetilde{\beta}_m : \Im\widetilde{\beta}_m > 0 \text{ and } \Re\widetilde{\beta}_m = 0\}_{m=1}^{N_2},$$

$$\{\overline{\beta}_n : \Im\overline{\beta}_n < 0\}_{n=1}^{\overline{N}}$$
$$= \{\overline{\beta}_p, -\overline{\beta}_p^* : \Im\overline{\beta}_p < 0 \text{ and } \Re\overline{\beta}_p \neq 0\}_{p=1}^{\overline{N}_1} \cup \{\overline{\widetilde{\beta}}_m : \Im\overline{\widetilde{\beta}}_m < 0 \text{ and } \Re\overline{\widetilde{\beta}}_m = 0\}_{m=1}^{\overline{N}_2};$$

the above symmetry properties hold for eigenvalues both on and off the imaginary axis.

10. Trace formulae. In this section, we show how some of the scattering data needed for reconstruction of the potentials can be constructed from reflection coefficients and eigenvalues; these equations are usually termed trace formulae. Since (b_{ij}) is the inverse of (a_{ij}) and $\det(a_{ij}) = 1$, we obtain $B(k) = A^{-1}(k) = \frac{\mathrm{adj}(A(k))}{\det(A)}$, which implies $b_{22}(k) = a_{11}(k)a_{33}(k) - a_{13}(k)a_{31}(k)$, where $\mathrm{adj}(A)$ is the adjugate matrix of A. By (4.3), one has $b_{21}(k)a_{11}(k) + b_{22}(k)a_{21}(k) + b_{23}(k)a_{31}(k) = 0$, and it yields $b_{21}(k)a_{11}(k) + [a_{11}(k)a_{33}(k) - a_{13}(k)a_{31}(k)]a_{21}(k) + b_{23}(k)a_{31}(k) = 0$.

Using the definitions of the reflection coefficients, we find

$$a_{21}(k)(1 - \rho_2(k)\overline{\rho}_1(k)) = -\frac{b_{33}(k)}{a_{11}(k)}\rho_2(k)\overline{\rho}_3(k) - \frac{b_{11}(k)}{a_{33}(k)}\rho_3(k).$$

Similarly, we have

$$a_{23}(k)(1 - \rho_2(k)\overline{\rho}_1(k)) = -\frac{b_{11}(k)}{a_{33}(k)}\rho_3(k)\overline{\rho}_1(k) - \frac{b_{33}(k)}{a_{11}(k)}\overline{\rho}_3(k),$$
$$a_{22}(k) = \frac{b_{11}(k)}{a_{33}(k)} + \overline{\rho}_2(k)a_{23}(k) = \frac{b_{33}(k)}{a_{11}(k)} + \rho_1(k)a_{21}(k).$$

Thus, we have shown that $a_{21}(k), a_{23}(k), a_{22}(k)$ are found in terms of reflection coefficients and $a_{11}(k), a_{33}(k), b_{11}(k), b_{33}(k)$. The latter four functions have been shown to be expressed in terms of reflection coefficients and eigenvalues. By

$$b_{21}(k)a_{11}(k) + b_{22}(k)a_{21}(k) + b_{23}a_{31}(k) = 0,$$
$$a_{31}(k)b_{13}(k) + a_{32}(k)b_{23}(k) + a_{33}b_{33} = 1,$$
$$b_{21}(k) = a_{31}(k)a_{23}(k) - a_{21}(k)a_{33}(k),$$

we deduce

$$a_{31}(k)b_{23}(k) + b_{21}(k)a_{11}(k)$$
$$= b_{22}(k)\left(b_{21}(k)b_{33}(k) + \frac{a_{31}(k)b_{11}(k)b_{23}(k)}{a_{33}(k)} + \frac{a_{32}(k)b_{23}(k)b_{21}(k)}{a_{33}(k)}\right),$$

which follows that $\frac{\frac{\rho_2(k)\overline{\rho}_3(k)}{a_{11}(k)b_{11}(k)} + \frac{\rho_3(k)}{a_{33}(k)b_{33}(k)}}{1-\rho_2(k)\overline{\rho}_1(k)} = \rho_3(k) + \rho_2(k)\overline{\rho}_3(k) + \overline{\rho}_2(k)\overline{\rho}_3(k)\rho_3(k)$. Thus, $a_{21}(k) = -b_{11}(k)b_{33}(k)[\rho_3(k) + \rho_2(k)\overline{\rho}_3(k) + \overline{\rho}_2(k)\overline{\rho}_3(k)\rho_3(k)]$.

Similarly, we derive $a_{23}(k) = -b_{11}(k)b_{33}(k)[\overline{\rho}_3(k) + \overline{\rho}_1(k)\rho_3(k) + \rho_1(k)\overline{\rho}_3(k)\rho_3(k)]$, $a_{22}(k) = b_{11}(k)b_{33}(k)[1 - (\rho_2(k) + \overline{\rho}_2(k)\rho_3(k))(\overline{\rho}_1(k) + \rho_1(k)\overline{\rho}_3(k))]$. One also obtains

$$a_{11}(k)b_{11}(k) = [1 - \rho_2(k)\overline{\rho}_1(k) - \overline{\rho}_2(k)\overline{\rho}_1(k)\rho_3(k) + \rho_3(k)\rho_1(k)]^{-1}, \tag{10.1}$$

$$a_{33}(k)b_{33}(k) = [1 - \rho_2(k)\overline{\rho}_1(k) - \rho_1(k)\rho_2(k)\overline{\rho}_3(k) + \overline{\rho}_3(k)\overline{\rho}_2(k)]^{-1}. \tag{10.2}$$

Remark 10.1. Equations (10.1) and (10.2) provide us a mechanism to construct trace formulae for $a_{11}(k)$, $b_{11}(k)$, $a_{33}(k)$, and $b_{33}(k)$.

10.1. General trace formulae. In section 9, we have assumed that $a_{33}(k)$ and $b_{33}(k)$ have simple zeros $\{\alpha_j : \Im\alpha_j > 0\}_{j=1}^{J}$ and $\{\overline{\alpha}_j : \Im\overline{\alpha}_j < 0\}_{j=1}^{\overline{J}}$, respectively. Moreover, $b_{11}(k)$ and $a_{11}(k)$ have the simple zeros $\{\beta_n : \Im\beta_n > 0\}_{n=1}^{N}$ and $\{\overline{\beta}_n : \Im\overline{\beta}_n < 0\}_{n=1}^{\overline{N}}$, respectively. Let $J = \overline{J}$ and $N = \overline{N}$; then we define

$$f_3(k) = a_{33}(k)\cdot\prod_{j=1}^{J}\frac{k-\overline{\alpha}_j}{k-\alpha_j}, \quad \overline{f}_3(k) = b_{33}(k)\cdot\prod_{j=1}^{J}\frac{k-\alpha_j}{k-\overline{\alpha}_j}. \tag{10.3}$$

Thus, $f_3(k)$ ($\overline{f}_3(k)$) is analytic in the upper (lower) half k-plane. Moreover, $f_3(k), \overline{f}_3(k) \to 1$ as $k\to\infty$ and have no zeros in their respective half planes. Hence, we have

$$\log f_3(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log f_3(\xi)}{\xi-k}d\xi, \quad \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log \overline{f}_3(\xi)}{\xi-k}d\xi = 0, \quad \Im k > 0;$$

$$\log \overline{f}_3(k) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log \overline{f}_3(\xi)}{\xi-k}d\xi, \quad \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log f_3(\xi)}{\xi-k}d\xi = 0 \quad \Im k < 0.$$

Adding/subtracting the above equations in each half plane, respectively, it yields

$$\log f_3(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log f_3(\xi)\overline{f}_3(\xi)}{\xi-k}d\xi, \ \Im k > 0;$$
$$\log \overline{f}_3(k) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log f_3(\xi)\overline{f}_3(\xi)}{\xi-k}d\xi, \ \Im k < 0.$$

By (10.2) and (10.3), we obtain

$$a_{33}(k) = \prod_{m=1}^{J}\frac{k-\alpha_m}{k-\overline{\alpha}_m}\cdot\exp\left(\Theta_1(k)\right), \quad \Im k > 0, \tag{10.4}$$

$$b_{33}(k) = \prod_{m=1}^{J}\frac{k-\overline{\alpha}_m}{k-\alpha_m}\cdot\exp\left(-\Theta_1(k)\right), \quad \Im k < 0, \tag{10.5}$$

where

$$\Theta_1(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\left[1-\rho_2(\xi)\overline{\rho}_1(\xi)-\rho_1(\xi)\rho_2(\xi)\overline{\rho}_3(\xi)+\overline{\rho}_2(\xi)\overline{\rho}_3(\xi)\right]}{k-\xi}d\xi.$$

Similarly, we have

$$b_{11}(k) = \prod_{l=1}^{N}\frac{k-\beta_l}{k-\overline{\beta}_l}\cdot\exp\left(\Theta_2(k)\right), \quad \Im k > 0, \tag{10.6}$$

$$a_{11}(k) = \prod_{l=1}^{N}\frac{k-\overline{\beta}_l}{k-\beta_l}\cdot\exp\left(-\Theta_2(k)\right), \quad \Im k < 0, \tag{10.7}$$

where

$$\Theta_2(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\left[1-\rho_2(\xi)\overline{\rho}_1(\xi)-\overline{\rho}_1(\xi)\overline{\rho}_2(\xi)\rho_3(\xi)+\rho_1(\xi)\rho_3(\xi)\right]}{k-\xi}d\xi.$$

In order to solve the inverse problem, we need $a'_{33}(\alpha_j)$, $b'_{11}(\beta_n)$, $b'_{33}(\overline{\alpha}_j)$, and $a'_{11}(\overline{\beta}_n)$. These four derivatives are shown below:

$$a'_{33}(\alpha_j) = \frac{a_{33}(k)}{k-\alpha_j}\Big|_{k=\alpha_j}, \quad b'_{33}(\overline{\alpha}_j) = \frac{b_{33}(k)}{k-\overline{\alpha}_j}\Big|_{k=\overline{\alpha}_j}, \tag{10.8}$$

$$b'_{11}(\beta_n) = \frac{b_{11}(k)}{k-\beta_n}\Big|_{k=\beta_n}, \quad a'_{11}(\overline{\beta}_n) = \frac{a_{11}(k)}{k-\overline{\beta}_n}\Big|_{k=\overline{\beta}_n}. \tag{10.9}$$

In general, $a'_{33}(\alpha_j)$ and $b'_{33}(\overline{\alpha}_j)$ depend on the simple zeros $\{\alpha_j : \Im\alpha_j > 0\}_{j=1}^{J}$ and $\{\overline{\alpha}_j : \Im\overline{\alpha}_j < 0\}_{j=1}^{J}$. Similarly, $b'_{11}(\beta_n)$ and $a'_{11}(\overline{\beta}_n)$ rely on $\{\beta_n : \Im\beta_n > 0\}_{n=1}^{N}$ and $\{\overline{\beta}_n : \Im\overline{\beta}_n < 0\}_{n=1}^{N}$. In general, all derivatives also depend on the reflection coefficients $\rho_1(k)$, $\rho_2(k)$, $\rho_3(k)$, $\overline{\rho}_1(k)$, $\overline{\rho}_2(k)$, $\overline{\rho}_3(k)$. Since these derivatives can be determined by eigenvalues and reflection coefficients, therefore they are not contained in the minimal data.

In particular, if $\rho_1(k)=0$, $\rho_2(k)=0$, $\overline{\rho}_1(k)=0$, and $\overline{\rho}_2(k)=0$ on $\mathbb{R}$, i.e., $a_{12}(k)=0$, $a_{13}(k)=0$, $a_{31}(k)=0$, and $a_{32}(k)=0$ on the real axis (via the definitions of ρ_1, ρ_1, $\overline{\rho}_1$, $\overline{\rho}_2$), then it corresponds to the case of pure solitons, and these derivatives only depend on the zeros mentioned above.

10.2. Classical three-wave system. Under the symmetry reduction $N_{21}(x) = \varepsilon_1\varepsilon_2 N^*_{12}(x)$, $N_{31}(x) = -\varepsilon_1\varepsilon_3 N^*_{13}(x)$, and $N_{32}(x) = \varepsilon_2\varepsilon_3 N^*_{23}(x)$, where $\varepsilon_j^2 = 1$, $j = 1,2,3$, we have assumed that $a_{33}(k)$ and $b_{11}(k)$ have simple zeros $\{\alpha_j : \Im\alpha_j > 0\}_{j=1}^{J}$ and $\{\beta_n : \Im\beta_n > 0\}_{n=1}^{N}$, respectively. By the symmetry relations $b_{11}(k) = a^*_{11}(k^*)$ and $b_{33}(k) = a^*_{33}(k^*)$, we deduce that $b_{33}(k)$ and $a_{11}(k)$ have simple zeros α_j^* and β_n^*, respectively, i.e., $\overline{\alpha}_j = \alpha_j^*$, $\overline{\beta}_n = \beta_n^*$ and $J = \overline{J}$, $N = \overline{N}$.

By (8.4) and (10.4)–(10.7), one deduces that

$$a_{33}(k) = \prod_{m=1}^{J} \frac{k-\alpha_m}{k-\alpha_m^*} \cdot \exp\left(\Theta_3(k)\right), \quad b_{33}(k) = \prod_{m=1}^{J} \frac{k-\alpha_m^*}{k-\alpha_m} \cdot \exp\left(-\Theta_3(k)\right), \tag{10.10}$$

$$b_{11}(k) = \prod_{l=1}^{N} \frac{k-\beta_l}{k-\beta_l^*} \cdot \exp\left(\Theta_4(k)\right), \quad a_{11}(k) = \prod_{l=1}^{N} \frac{k-\beta_l^*}{k-\beta_l} \cdot \exp\left(-\Theta_4(k)\right), \tag{10.11}$$

where

$$\Theta_3(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log\left[1-\rho_2(\xi)\overline{\rho}_1(\xi) + \frac{\varepsilon_3}{\varepsilon_2}\rho_1(\xi)\rho_2(\xi)\overline{\rho}_2^*(\xi) - \frac{\varepsilon_3}{\varepsilon_2}\overline{\rho}_2^*(\xi)\overline{\rho}_2(\xi)\right]}{k-\xi} d\xi,$$

$$\Theta_4(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\log\left[1-\rho_2(\xi)\overline{\rho}_1(\xi) + \frac{\varepsilon_1}{\varepsilon_2}\rho_1^*(\xi)\overline{\rho}_1(\xi)\overline{\rho}_2(\xi) - \frac{\varepsilon_1}{\varepsilon_2}\rho_1^*(\xi)\rho_1(\xi)\right]}{k-\xi} d\xi.$$

When we reconstruct potentials, the functions $a'_{33}(\alpha_j)$, $b'_{11}(\beta_n)$, $b'_{33}(\alpha_j^*)$, and $a'_{11}(\beta_n^*)$ are needed. In general, these derivatives are found as follows:

$$a'_{33}(\alpha_j) = \frac{a_{33}(k)}{k-\alpha_j}\Big|_{k=\alpha_j}, \quad b'_{33}(\alpha_j^*) = \frac{b_{33}(k)}{k-\alpha_j^*}\Big|_{k=\alpha_j^*}, \tag{10.12}$$

$$b'_{11}(\beta_n) = \frac{b_{11}(k)}{k-\beta_n}\Big|_{k=\beta_n}, \quad a'_{11}(\beta_n^*) = \frac{a_{11}(k)}{k-\beta_n^*}\Big|_{k=\beta_n^*}. \tag{10.13}$$

If $\rho_1(k) = 0$, $\rho_2(k) = 0$, $\overline{\rho}_1(k) = 0$, and $\overline{\rho}_2(k) = 0$ (equivalently, $a_{12}(k) = 0$, $a_{13}(k) = 0$, $a_{31}(k) = 0$, and $a_{32}(k) = 0$) on $\mathbb{R}$, then it corresponds to pure solitons, and $a'_{33}(\alpha_j)$ and $b'_{33}(\alpha_j^*)$ only depend on the simple zeros $\{\alpha_j : \Im\alpha_j > 0\}_{j=1}^{J}$. Also, $b'_{11}(\beta_n)$ and $a'_{11}(\beta_n^*)$ only depend on $\{\beta_n : \Im\beta_n > 0\}_{n=1}^{N}$. However, in the general case, $a'_{33}(\alpha_j)$ and $b'_{33}(\alpha_j^*)$ rely on zeros $\{\alpha_j : \Im\alpha_j > 0\}_{j=1}^{J}$ and the reflection coefficients $\rho_1(k)$, $\rho_2(k)$, $\overline{\rho}_1(k)$, $\overline{\rho}_2(k)$. Similarly, $b'_{11}(\beta_n)$ and $a'_{11}(\beta_n^*)$ rely on $\{\beta_n : \Im\beta_n > 0\}_{n=1}^{N}$ and $\rho_1(k)$, $\rho_2(k)$, $\overline{\rho}_1(k)$, $\overline{\rho}_2(k)$. We note that $a'_{33}(\alpha_j)$, $b'_{11}(\beta_n)$, $b'_{33}(\alpha_j^*)$, and $a'_{11}(\beta_n^*)$ are not part of the minimal data.

10.3. Complex reverse space-time three-wave system. Under the symmetry reduction $N_{21}(x) = \varepsilon_1\varepsilon_2 N_{12}^*(-x)$, $N_{31}(x) = \varepsilon_1\varepsilon_3 N_{13}^*(-x)$, and $N_{32}(x) = \varepsilon_2\varepsilon_3 N_{23}^*(-x)$, where $\varepsilon_j^2 = 1$, $j = 1, 2, 3$, we assume that $a_{33}(k)$ and $b_{33}(k)$ have simple zeros $\{\alpha_j : \Im\alpha_j > 0 \text{ and } \Re\alpha_j \neq 0\}_{j=1}^{J_1} \cup \{\widetilde{\alpha}_l : \Im\widetilde{\alpha}_l > 0 \text{ and } \Re\widetilde{\alpha}_l = 0\}_{l=1}^{J_2}$ and $\{\overline{\alpha}_j : \Im\overline{\alpha}_j < 0 \text{ and } \Re\overline{\alpha}_j \neq 0\}_{j=1}^{J_1} \cup \{\overline{\widetilde{\alpha}}_l : \Im\overline{\widetilde{\alpha}}_l < 0 \text{ and } \Re\overline{\widetilde{\alpha}}_l = 0\}_{l=1}^{J_2}$, respectively, where $2J_1 + J_2 = J$. Similarly, we assume $b_{11}(k)$ and $a_{11}(k)$ have the simple zeros $\{\beta_n : \Im\beta_n > 0 \text{ and } \Re\beta_n \neq 0\}_{n=1}^{N_1} \cup \{\widetilde{\beta}_m : \Im\widetilde{\beta}_m > 0 \text{ and } \Re\widetilde{\beta}_m = 0\}_{m=1}^{N_2}$ and $\{\overline{\beta}_n : \Im\overline{\beta}_n < 0 \text{ and } \Re\overline{\beta}_n \neq 0\}_{n=1}^{N_1} \cup \{\overline{\widetilde{\beta}}_m : \Im\overline{\widetilde{\beta}}_m < 0 \text{ and } \Re\overline{\widetilde{\beta}}_m = 0\}_{m=1}^{N_2}$, respectively, where $2N_1 + N_2 = N$. Utilizing an analogous method for the general trace formulae, trace formulae for the complex RST three-wave system are stated below:

$$a_{33}(k) = \prod_{m=1}^{J_1} \frac{(k-\alpha_m)(k+\alpha_m^*)}{(k-\overline{\alpha}_m)(k+\overline{\alpha}_m^*)} \prod_{i=1}^{J_2} \frac{k-\tilde{\alpha}_i}{k-\overline{\tilde{\alpha}}_i} \exp\left(\Theta_1(k)\right), \tag{10.14}$$

$$b_{33}(k) = \prod_{m=1}^{J_1} \frac{(k-\overline{\alpha}_m)(k+\overline{\alpha}_m^*)}{(k-\alpha_m)(k+\alpha_m^*)} \prod_{i=1}^{J_2} \frac{k-\overline{\tilde{\alpha}}_i}{k-\tilde{\alpha}_i} \exp\left(-\Theta_1(k)\right), \tag{10.15}$$

$$b_{11}(k) = \prod_{p=1}^{N_1} \frac{(k-\beta_p)(k+\beta_p^*)}{(k-\overline{\beta}_p)(k+\overline{\beta}_p^*)} \prod_{q=1}^{N_2} \frac{k-\tilde{\beta}_q}{k-\overline{\tilde{\beta}}_q} \exp\left(\Theta_2(k)\right), \tag{10.16}$$

$$a_{11}(k) = \prod_{p=1}^{N_1} \frac{(k-\overline{\beta}_p)(k+\overline{\beta}_p^*)}{(k-\beta_p)(k+\beta_p^*)} \prod_{q=1}^{N_2} \frac{k-\overline{\tilde{\beta}}_q}{k-\tilde{\beta}_q} \exp\left(-\Theta_2(k)\right). \tag{10.17}$$

In order to reconstruct soliton solutions, we need $a'_{33}(\alpha_j)$, $a'_{33}(-\alpha_j^*)$, $a'_{33}(\tilde{\alpha}_l)$, $b'_{33}(\overline{\alpha}_j)$, $b'_{33}(-\overline{\alpha}_j^*)$, $b'_{33}(\overline{\tilde{\alpha}}_l)$, $b'_{11}(\beta_n)$, $b'_{11}(-\beta_n^*)$, $b'_{11}(\tilde{\beta}_m)$, $a'_{11}(\overline{\beta}_n)$, $a'_{11}(-\overline{\beta}_n^*)$, $a'_{11}(\overline{\tilde{\beta}}_m)$. These derivatives are found to be

$$a'_{33}(\alpha_j) = \frac{a_{33}(k)}{k-\alpha_j}\Big|_{k=\alpha_j}, \quad a'_{33}(-\alpha_j^*) = \frac{a_{33}(k)}{k+\alpha_j^*}\Big|_{k=-\alpha_j^*}, \quad a'_{33}(\tilde{\alpha}_l) = \frac{a_{33}(k)}{k-\tilde{\alpha}_l}\Big|_{k=\tilde{\alpha}_l}, \tag{10.18}$$

$$b'_{33}(\overline{\alpha}_j) = \frac{b_{33}(k)}{k-\overline{\alpha}_j}\Big|_{k=\overline{\alpha}_j}, \quad b'_{33}(-\overline{\alpha}_j^*) = \frac{b_{33}(k)}{k+\overline{\alpha}_j^*}\Big|_{k=-\overline{\alpha}_j^*}, \quad b'_{33}(\overline{\tilde{\alpha}}_l) = \frac{b_{33}(k)}{k-\overline{\tilde{\alpha}}_l}\Big|_{k=\overline{\tilde{\alpha}}_l}, \tag{10.19}$$

$$b'_{11}(\beta_n) = \frac{b_{11}(k)}{k-\beta_n}\Big|_{k=\beta_n}, \quad b'_{11}(-\beta_n^*) = \frac{b_{11}(k)}{k+\beta_n^*}\Big|_{k=-\beta_n^*}, \quad b'_{11}(\tilde{\beta}_m) = \frac{b_{11}(k)}{k-\tilde{\beta}_m}\Big|_{k=\tilde{\beta}_m}, \tag{10.20}$$

$$a'_{11}(\overline{\beta}_n) = \frac{a_{11}(k)}{k-\overline{\beta}_n}\Big|_{k=\overline{\beta}_n}, \quad a'_{11}(-\overline{\beta}_n^*) = \frac{a_{11}(k)}{k+\overline{\beta}_n^*}\Big|_{k=-\overline{\beta}_n^*}, \quad a'_{11}(\overline{\tilde{\beta}}_m) = \frac{a_{11}(k)}{k-\overline{\tilde{\beta}}_m}\Big|_{k=\overline{\tilde{\beta}}_m}. \tag{10.21}$$

Generally speaking, $a'_{33}(\alpha_j)$, $a'_{33}(-\alpha_j^*)$, $a'_{33}(\tilde{\alpha}_l)$, $b'_{33}(\overline{\alpha}_j)$, $b'_{33}(-\overline{\alpha}_j^*)$, $b'_{33}(\overline{\tilde{\alpha}}_l)$ depend on simple zeros $\{\alpha_j, -\alpha_j^* : \Im\alpha_j > 0 \text{ and } \Re\alpha_j \neq 0\}_{j=1}^{J_1} \cup \{\widetilde{\alpha}_l : \Im\widetilde{\alpha}_l > 0 \text{ and } \Re\widetilde{\alpha}_l = 0\}_{l=1}^{J_2}$,

$\{\overline{\alpha}_j, -\overline{\alpha}_j^* : \Im\overline{\alpha}_j < 0 \text{ and } \Re\overline{\alpha}_j \neq 0\}_{j=1}^{J_1} \cup \{\overline{\widetilde{\alpha}}_l : \Im\overline{\widetilde{\alpha}}_l < 0 \text{ and } \Re\overline{\widetilde{\alpha}}_l = 0\}_{l=1}^{J_2}$ as well as the reflection coefficients $\rho_1(k)$, $\rho_2(k)$, $\rho_3(k)$, $\overline{\rho}_1(k)$, $\overline{\rho}_2(k)$, $\overline{\rho}_3(k)$.

Similarly, $b'_{11}(\beta_n)$, $b'_{11}(-\beta_n^*)$, $b'_{11}(\tilde{\beta}_m)$, $a'_{11}(\overline{\beta}_n)$, $a'_{11}(-\overline{\beta}_n^*)$, $a'_{11}(\overline{\tilde{\beta}}_m)$ rely on simple zeros $\{\beta_n, -\beta_n^* : \Im\beta_n > 0 \text{ and } \Re\beta_n \neq 0\}_{n=1}^{N_1} \cup \{\widetilde{\beta}_m : \Im\widetilde{\beta}_m > 0 \text{ and } \Re\widetilde{\beta}_m = 0\}_{m=1}^{N_2}$ and $\{\overline{\beta}_n, -\overline{\beta}_n^* : \Im\overline{\beta}_n < 0 \text{ and } \Re\overline{\beta}_n \neq 0\}_{n=1}^{N_1} \cup \{\overline{\widetilde{\beta}}_m : \Im\overline{\widetilde{\beta}}_m < 0 \text{ and } \Re\overline{\widetilde{\beta}}_m = 0\}_{m=1}^{N_2}$. Moreover, the above six derivatives also depend on the reflection coefficients $\rho_1(k)$, $\rho_2(k)$, $\rho_3(k)$, $\overline{\rho}_1(k)$, $\overline{\rho}_2(k)$, $\overline{\rho}_3(k)$.

In addition, if $\rho_1(k) = 0$, $\rho_2(k) = 0$, $\overline{\rho}_1(k) = 0$, and $\overline{\rho}_2(k) = 0$ on the real axis, i.e., $a_{12}(k) = 0$, $a_{13}(k) = 0$, and $a_{32}(k) = 0$ on $\mathbb{R}$ (via the definitions of ρ_1, ρ_1, $\overline{\rho}_1$, $\overline{\rho}_2$ and (8.6)), then it yields the case of pure solitons, and the derivatives only depend on the zeros.

10.4. Real reverse space-time three-wave system. Under the symmetry reductions $N_{21}(x) = \varepsilon_1\varepsilon_2 N_{12}(-x)$, $N_{31}(x) = \varepsilon_1\varepsilon_3 N_{13}(-x)$, and $N_{32}(x) = \varepsilon_2\varepsilon_3 N_{23}(-x)$ as well as $N_{21}(x) = \varepsilon_1\varepsilon_2 N_{12}^*(-x)$, $N_{31}(x) = \varepsilon_1\varepsilon_3 N_{13}^*(-x)$, and $N_{32}(x) = \varepsilon_2\varepsilon_3 N_{23}^*(-x)$, where $\varepsilon_j^2 = 1$, $j = 1, 2, 3$, we assume that $a_{33}(k)$ and $b_{33}(k)$ have simple zeros $\{\alpha_j : \Im\alpha_j > 0 \text{ and } \Re\alpha_j \neq 0\}_{j=1}^{J_1} \cup \{\widetilde{\alpha}_l : \Im\widetilde{\alpha}_l > 0 \text{ and } \Re\widetilde{\alpha}_l = 0\}_{l=1}^{J_2}$ and $\{\overline{\alpha}_j : \Im\overline{\alpha}_j < 0 \text{ and } \Re\overline{\alpha}_j \neq 0\}_{j=1}^{J_1} \cup \{\overline{\widetilde{\alpha}}_l : \Im\overline{\widetilde{\alpha}}_l < 0 \text{ and } \Re\overline{\widetilde{\alpha}}_l = 0\}_{l=1}^{J_2}$, respectively. Similarly, $b_{11}(k)$ and $a_{11}(k)$ have the simple zeros $\{\beta_n : \Im\beta_n > 0 \text{ and } \Re\beta_n \neq 0\}_{n=1}^{N_1} \cup \{\widetilde{\beta}_m : \Im\widetilde{\beta}_m > 0 \text{ and } \Re\widetilde{\beta}_m = 0\}_{m=1}^{N_2}$ and $\{\overline{\beta}_n : \Im\overline{\beta}_n < 0 \text{ and } \Re\overline{\beta}_n \neq 0\}_{n=1}^{N_1} \cup \{\overline{\widetilde{\beta}}_m : \Im\overline{\widetilde{\beta}}_m < 0 \text{ and } \Re\overline{\widetilde{\beta}}_m = 0\}_{m=1}^{N_2}$, respectively.

Note that the real and complex reverse space-time three-wave systems have the same symmetry relations among eigenvalues; thus, the statement regarding trace formulae for the real nonlocal case is the same as the complex one, which is discussed in section 10.3.

11. Riemann–Hilbert problem-left scattering problem. From (4.1), one has $\frac{M_1(x,k;t)}{a_{11}(k,t)} = N_1(x,k;t) + \rho_1(k,t)N_2(x,k;t)e^{ik(d_2-d_1)x} + \overline{\rho}_1(k,t)N_3(x,k;t)e^{ik(d_3-d_1)x}$, $\frac{M_3(x,k;t)}{a_{33}(k,t)} = \rho_2(k,t)N_1(x,k;t)e^{ik(d_1-d_3)x} + \overline{\rho}_2(k,t)N_2(x,k;t)e^{ik(d_2-d_3)x} + N_3(x,k;t)$. Combining (5.13) and (5.14), we have

$$\begin{aligned}\frac{M_1(x,k;t)}{a_{11}(k,t)} &= N_1(x,k;t) + (\rho_1(k,t)\overline{\rho}_3(k,t) + \overline{\rho}_1(k,t))\, N_3(x,k;t)e^{ik(d_3-d_1)x} \\ &\quad + \rho_1(k,t)\frac{\overline{\chi}(x,k;t)}{b_{33}(k,t)}e^{ik(d_2-d_1)x},\end{aligned} \tag{11.1}$$

$$\frac{\chi(x,k;t)}{b_{11}(k,t)} = \rho_3(k,t)N_1(x,k;t)e^{ik(d_1-d_2)x} - \frac{\overline{\chi}(x,k;t)}{b_{33}(k,t)} - \overline{\rho}_3(k,t)N_3(x,k;t)e^{ik(d_3-d_2)x}, \tag{11.2}$$

$$\begin{aligned}\frac{M_3(x,k;t)}{a_{33}(k,t)} &= (\rho_2(k,t) + \overline{\rho}_2(k,t)\rho_3(k,t))\, N_1(x,k;t)e^{ik(d_1-d_3)x} \\ &\quad - \overline{\rho}_2(k,t)\frac{\chi(x,k;t)}{b_{11}(k,t)}e^{ik(d_2-d_3)x} + N_3(x,k;t).\end{aligned} \tag{11.3}$$

Note that from Theorems 3.2, 4.1, and 5.2, $\frac{M_3(x,k)}{a_{33}(k)}, N_1(x,k)$, and $\frac{\chi(x,k)}{b_{11}(k)}$, as functions of k, are meromorphic in the upper half plane, where $\frac{M_3(x,k)}{a_{33}(k)}$ and $\frac{\chi(x,k)}{b_{11}(k)}$ have simple poles (by assumption) α_j and β_n respectively, i.e., $a_{33}(\alpha_j) = 0$ and $b_{11}(\beta_n) = 0$; similarly, $\frac{M_1(x,k)}{a_{11}(k)}, N_3(x,k)$, and $\frac{\overline{\chi}(x,k)}{b_{33}(k)}$, as functions of k, are meromorphic in the lower half plane, where $\frac{\overline{\chi}(x,k)}{b_{33}(k)}$ and $\frac{M_1(x,k)}{a_{11}(k)}$ have simple poles (by assumption) $\overline{\alpha}_j$ and $\overline{\beta}_n$ respectively, i.e., $b_{33}(\overline{\alpha}_j) = 0$ and $a_{11}(\overline{\beta}_n) = 0$.

We introduce $m^+(x,k;t) := (N_1(x,k;t), -\frac{\chi(x,k;t)}{b_{11}(k)}, \frac{M_3(x,k;t)}{a_{33}(k)})$ and $m^-(x,k;t) := (\frac{M_1(x,k;t)}{a_{11}(k)}, \frac{\overline{\chi}(x,k;t)}{b_{33}(k)}, N_3(x,k;t))$; then (11.1)–(11.3) can be written as a jump condition:

$$m^+(x,k;t) = m^-(x,k;t)V_0(x,k;t), \tag{11.4}$$

where

$$V_0(x,k;t) = (V_0^{(1)}(x,k;t)\ V_0^{(2)}(x,k;t)\ V_0^{(3)}(x,k;t))\,, \tag{11.5}$$

and

$$\begin{aligned} V_0^{(1)} &= (1 \quad -\rho_1 e^{ik(d_2-d_1)x} \quad -(\rho_1\overline{\rho}_3 + \overline{\rho}_1)e^{ik(d_3-d_1)x})^T, \\ V_0^{(2)} &= (-\rho_3 e^{ik(d_1-d_2)x} \quad 1+\rho_1\rho_3 \quad [\rho_3(\rho_1\overline{\rho}_3+\overline{\rho}_1)+\overline{\rho}_3]\, e^{ik(d_3-d_2)x})^T, \\ V_0^{(3)} &= (\rho_2 e^{ik(d_1-d_3)x} \quad (\overline{\rho}_2 - \rho_1\rho_2)e^{ik(d_2-d_3)x} \quad 1+\overline{\rho}_2\overline{\rho}_3 - \overline{\rho}_1\rho_2 - \rho_1\rho_2\overline{\rho}_3)^T. \end{aligned}$$

Consequently, we can formulate the following generalized matrix Riemann–Hilbert problem in terms of scattering data and discrete eigenvalues, which determines $m(k) := m(x,k;t)$. Essentially, we look for $m(k)$ with the following properties:

1. *Analyticity*. $m(k)$ is analytic for $k \in \mathbb{C}\backslash(\mathbb{R} \cup \{\alpha_j\}_{j=1}^{J} \cup \{\beta_n\}_{n=1}^{N} \cup \{\overline{\alpha}_j\}_{j=1}^{\overline{J}} \cup \{\overline{\beta}_n\}_{n=1}^{\overline{N}})$.
2. *Jump condition*. $m(k)$ takes continuous boundary values $m_\pm(k) := \lim_{\delta\to 0} m(k \pm i\delta)$ for $k \in \mathbb{R}$, and the boundary values are related by (11.4). At the poles,
$$\begin{aligned} \mathrm{Res}_{k=\beta_n} m(k) &= \left(N_1(\beta_n), -\frac{b_{21}(\beta_n)N_1(\beta_n)e^{i\beta_n(d_1-d_2)x}}{b_{11}'(\beta_n)}, \frac{M_3(\beta_n)}{a_{33}(\beta_n)}\right), \\ \mathrm{Res}_{k=\alpha_j} m(k) &= \left(N_1(\alpha_j), -\frac{\chi(x,\alpha_j)}{b_{11}(\alpha_j)}, \frac{\frac{1}{a_{23}(\alpha_j)}\chi(x,\alpha_j)e^{i\alpha_j(d_2-d_3)x}}{a_{33}'(\alpha_j)}\right), \\ \mathrm{Res}_{k=\overline{\beta}_n} m(k) &= \left(-\frac{\frac{1}{a_{21}(\overline{\beta}_n)}\overline{\chi}(x,\overline{\beta}_n)e^{i\overline{\beta}_n(d_2-d_1)x}}{a_{11}'(\overline{\beta}_n)}, \frac{\overline{\chi}(\overline{\beta}_n)}{b_{33}(\overline{\beta}_n)}, N_3(\overline{\beta}_n)\right), \\ \mathrm{Res}_{k=\overline{\alpha}_j} m(k) &= \left(\frac{M_1(\overline{\alpha}_j)}{a_{11}(\overline{\alpha}_j)}, -\frac{b_{23}(\overline{\alpha}_j)N_3(x,\overline{\alpha}_j)e^{i\overline{\alpha}_j(d_3-d_2)x}}{b_{33}'(\overline{\alpha}_j)}, N_3(\overline{\alpha}_j)\right). \end{aligned}$$
3. *Normalization condition*. $m(k) \to \mathbb{I}$ as $k \to \infty$.

Remark 11.1. If the potential $N(x,t=0)$ is generic and is in Schwartz space, then there is a unique $V_0(x,k;t)$ such that the jump condition (11.4) holds. Moreover, the potential $N(x,t)$ is uniquely determined by scattering data and discrete eigenvalues and it exists in the generic sense until possibly blow up; see [13, 14, 15].

Note that the above RH problem can be transformed to a system of linear integral/algebraic equations, once we impose the symmetry reduction for specific three-wave systems. Then the solution of each RH problem suffices to uniquely solve each three-wave system corresponding to the associated scattering data and discrete eigenvalues.

It follows that

$$\begin{aligned} M_3(x,\alpha_j) = &\left[a_{31}(\alpha_j) + \frac{a_{32}(\alpha_j)b_{21}(\alpha_j)}{b_{11}(\alpha_j)}\right] N_1(x,\alpha_j)e^{i\alpha_j(d_1-d_3)x} \\ &- \frac{a_{32}(\alpha_j)}{b_{11}(\alpha_j)}\chi(x,\alpha_j)e^{i\alpha_j(d_2-d_3)x}, \end{aligned}$$

$$\chi(x,\beta_n) = b_{21}(\beta_n)N_1(x,\beta_n)e^{i\beta_n(d_1-d_2)x}, \quad \overline{\chi}(x,\overline{\alpha}_j) = -b_{23}(\overline{\alpha}_j)N_3(x,\overline{\alpha}_j)e^{i\overline{\alpha}_j(d_3-d_2)x},$$

$$M_1(x,\overline{\beta}_n) = \left[\frac{a_{12}(\overline{\beta}_n)b_{23}(\overline{\beta}_n)}{b_{33}(\overline{\beta}_n)} + a_{13}(\overline{\beta}_n)\right] N_3(x,\overline{\beta}_n)e^{i\overline{\beta}_n(d_3-d_1)x} + \frac{a_{12}(\overline{\beta}_n)}{b_{33}(\overline{\beta}_n)}\overline{\chi}(x,\overline{\beta}_n)e^{i\overline{\beta}_n(d_2-d_1)x}.$$

By subtracting the poles, assumed simple in the upper/lower half planes, respectively, at $a_{33}(\alpha_j) = 0$ $(j = 1,2,\dots,J)$, $b_{11}(\beta_n) = 0$ $(n = 1,2,\dots,N)$ and $a_{11}(\overline{\beta}_n) = 0$ $(n = 1,2,\dots,\overline{N})$, $b_{33}(\overline{\alpha}_j) = 0$ $(j = 1,2,\dots,\overline{J})$, combining (9.1), (9.2), and (11.1)–(11.3) and the time independence of $a_{11}(k)$, $a_{33}(k)$, $b_{11}(k)$, $b_{33}(k)$, one obtains

(11.6)
$$\begin{aligned}
&\frac{M_1(x,k;t)}{a_{11}(k)} - \omega_1 - \sum_{n=1}^{\overline{N}} \frac{M_1(x,\overline{\beta}_n;t)}{(k-\overline{\beta}_n)a'_{11}(\overline{\beta}_n)} \\
&= N_1(x,k;t) - \omega_1 + \sum_{n=1}^{\overline{N}} \frac{\frac{1}{a_{21}(\overline{\beta}_n,t)}\overline{\chi}(x,\overline{\beta}_n;t)e^{i\overline{\beta}_n(d_2-d_1)x}}{(k-\overline{\beta}_n)a'_{11}(\overline{\beta}_n)} \\
&+ \left(\rho_1(k,t)\overline{\rho}_3(k,t) + \overline{\rho}_1(k,t)\right) N_3(x,k;t)e^{ik(d_3-d_1)x} + \rho_1(k,t)\frac{\overline{\chi}(x,k;t)}{b_{33}(k)}e^{ik(d_2-d_1)x},
\end{aligned}$$

(11.7)
$$\begin{aligned}
&\frac{M_3(x,k;t)}{a_{33}(k)} - \omega_3 - \sum_{j=1}^{J} \frac{M_3(x,\alpha_j;t)}{(k-\alpha_j)a'_{33}(\alpha_j)} \\
&= N_3(x,k;t) - \omega_3 - \sum_{j=1}^{J} \frac{\frac{1}{a_{23}(\alpha_j,t)}\chi(x,\alpha_j;t)e^{i\alpha_j(d_2-d_3)x}}{(k-\alpha_j)a'_{33}(\alpha_j)} \\
&+ \left(\rho_2(k,t) + \overline{\rho}_2(k,t)\rho_3(k,t)\right) N_1(x,k;t)e^{ik(d_1-d_3)x} - \overline{\rho}_2(k,t)\frac{\chi(x,k;t)}{b_{11}(k)}e^{ik(d_2-d_3)x},
\end{aligned}$$

(11.8)
$$\begin{aligned}
&\frac{\chi(x,k;t)}{b_{11}(k)} + \gamma_1\gamma_3\omega_2 - \sum_{n=1}^{N} \frac{\chi(x,\beta_n;t)}{(k-\beta_n)b'_{11}(\beta_n)} + \sum_{j=1}^{\overline{J}} \frac{\overline{\chi}(x,\overline{\alpha}_j;t)}{(k-\overline{\alpha}_j)b'_{33}(\overline{\alpha}_j)} \\
&= -\frac{\overline{\chi}(x,k;t)}{b_{33}(k)} + \gamma_1\gamma_3\omega_2 - \sum_{n=1}^{N} \frac{\chi(x,\beta_n;t)}{(k-\beta_n)b'_{11}(\beta_n)} + \sum_{j=1}^{\overline{J}} \frac{\overline{\chi}(x,\overline{\alpha}_j;t)}{(k-\overline{\alpha}_j)b'_{33}(\overline{\alpha}_j)} \\
&+ \rho_3(k,t)N_1(x,k;t)e^{ik(d_1-d_2)x} - \overline{\rho}_3(k,t)N_3(x,k;t)e^{ik(d_3-d_2)x}.
\end{aligned}$$

Combining the asymptotic behavior of eigenfunctions and scattering data (section 6), (11.6)–(11.8) can be interpreted as a RH problem for suitable combinations of the three functions $M_3(x,k,t)/a_{33}(k), N_1(x,k,t), \chi(x,k,t)/b_{11}(k)$ meromorphic in the upper half k-plane and $M_1(x,k,t)/a_{11}(k), N_3(x,k,t), \overline{\chi}(x,k,t)/b_{33}(k)$ meromorphic in the lower half k-plane.

We now introduce the projection operators

$$P_\pm(f)(k) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(\xi)}{\xi-(k\pm i0)}d\xi,$$

which are well-defined for any function $f(\xi)$ that is integrable on the real axis. If $f_\pm(\xi)$ is analytic in the upper/lower k-plane and $f_\pm(\xi)$ is decaying at large ξ, then

$P_\pm(f_\pm)(k) = \pm f_\pm(k)$, $P_\mp(f_\pm)(k) = 0$. Applying P_+ to (11.6), in a similar way, one can treat (11.7) and (11.8) by a projector P_-; using (9.1) to (11.8), we obtain

$$\begin{aligned} N_1(x,k;t) &= \omega_1 - \sum_{n=1}^{\overline{N}} \frac{\frac{1}{a_{21}(\overline{\beta}_n,t)}\overline{\chi}(x,\overline{\beta}_n;t)e^{i\overline{\beta}_n(d_2-d_1)x}}{(k-\overline{\beta}_n)a'_{11}(\overline{\beta}_n)} \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)}{\xi-(k+i0)}\frac{\overline{\chi}(x,\xi;t)}{b_{33}(\xi,t)}e^{i\xi(d_2-d_1)x}d\xi \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)\overline{\rho}_3(\xi,t)+\overline{\rho}_1(\xi,t)}{\xi-(k+i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_1)x}d\xi, \end{aligned} \tag{11.9}$$

$$\begin{aligned} N_3(x,k;t) &= \omega_3 + \sum_{j=1}^{J} \frac{\frac{1}{a_{23}(\alpha_j,t)}\chi(x,\alpha_j;t)e^{i\alpha_j(d_2-d_3)x}}{(k-\alpha_j)a'_{33}(\alpha_j)} \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_2(\xi,t)}{\xi-(k-i0)}\frac{\chi(x,\xi;t)}{b_{11}(\xi)}e^{i\xi(d_2-d_3)x}d\xi \\ &\quad + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_2(\xi,t)+\overline{\rho}_2(\xi,t)\rho_3(\xi,t)}{\xi-(k-i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_3)x}d\xi, \end{aligned} \tag{11.10}$$

$$\begin{aligned} \frac{\overline{\chi}(x,k;t)}{b_{33}(k)} &= \gamma_1\gamma_3\omega_2 - \sum_{n=1}^{N} \frac{b_{21}(\beta_n,t)N_1(x,\beta_n;t)e^{i\beta_n(d_1-d_2)x}}{(k-\beta_n)b'_{11}(\beta_n)} \\ &\quad - \sum_{j=1}^{\overline{J}} \frac{b_{23}(\overline{\alpha}_j,t)N_3(x,\overline{\alpha}_j;t)e^{i\overline{\alpha}_j(d_3-d_2)x}}{(k-\overline{\alpha}_j)b'_{33}(\overline{\alpha}_j)} \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_3(\xi,t)}{\xi-(k-i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi \\ &\quad + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_3(\xi,t)}{\xi-(k-i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi. \end{aligned} \tag{11.11}$$

Moreover, one deduces

$$\begin{aligned} \frac{\chi(x,k;t)}{b_{11}(k)} &= -\gamma_1\gamma_3\omega_2 + \sum_{n=1}^{N} \frac{b_{21}(\beta_n,t)N_1(x,\beta_n;t)e^{i\beta_n(d_1-d_2)x}}{(k-\beta_n)b'_{11}(\beta_n)} \\ &\quad + \sum_{j=1}^{\overline{J}} \frac{b_{23}(\overline{\alpha}_j,t)N_3(x,\overline{\alpha}_j;t)e^{i\overline{\alpha}_j(d_3-d_2)x}}{(k-\overline{\alpha}_j)b'_{33}(\overline{\alpha}_j)} \\ &\quad + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_3(\xi,t)}{\xi-(k+i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_3(\xi,t)}{\xi-(k+i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi. \end{aligned} \tag{11.12}$$

11.1. Closing the system. To close the system, we substitute $k=\beta_n$ in (11.9), $k=\overline{\alpha}_j$ in (11.10), $k=\overline{\beta}_n$ in (11.11), and $k=\alpha_j$ in (11.12) to obtain

$$\begin{aligned} N_1(x,\beta_n;t) &= \omega_1 - \sum_{l=1}^{\overline{N}} \frac{\frac{b_{33}(\overline{\beta}_l,t)}{a_{21}(\overline{\beta}_l,t)}\cdot\frac{\overline{\chi}(x,\overline{\beta}_l;t)}{b_{33}(\overline{\beta}_l)}e^{i\overline{\beta}_l(d_2-d_1)x}}{(\beta_n-\overline{\beta}_l)a'_{11}(\overline{\beta}_l)} \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)}{\xi-(\beta_n+i0)}\frac{\overline{\chi}(x,\xi;t)}{b_{33}(\xi,t)}e^{i\xi(d_2-d_1)x}d\xi \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)\overline{\rho}_3(\xi,t)+\overline{\rho}_1(\xi,t)}{\xi-(\beta_n+i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_1)x}d\xi, \end{aligned} \tag{11.13}$$

$$N_3(x,\overline{\alpha}_j;t) = \omega_3 + \sum_{m=1}^{J} \frac{\frac{b_{11}(\alpha_m,t)}{a_{23}(\alpha_m,t)} \cdot \frac{\chi(x,\alpha_m;t)}{b_{11}(\alpha_m)} e^{i\alpha_m(d_2-d_3)x}}{(\overline{\alpha}_j - \alpha_m)a'_{33}(\alpha_m)}$$

$$- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_2(\xi,t)}{\xi-(\overline{\alpha}_j - i0)} \frac{\chi(x,\xi;t)}{b_{11}(\xi)} e^{i\xi(d_2-d_3)x} d\xi$$

$$+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_2(\xi,t)+\overline{\rho}_2(\xi,t)\rho_3(\xi,t)}{\xi-(\overline{\alpha}_j - i0)} N_1(x,\xi;t) e^{i\xi(d_1-d_3)x} d\xi, \tag{11.14}$$

$$\frac{\overline{\chi}(x,\overline{\beta}_n;t)}{b_{33}(\overline{\beta}_n)} = \gamma_1\gamma_3\omega_2 - \sum_{p=1}^{N} \frac{b_{21}(\beta_p,t)N_1(x,\beta_p;t)e^{i\beta_p(d_1-d_2)x}}{(\overline{\beta}_n - \beta_p)b'_{11}(\beta_p)}$$

$$- \sum_{q=1}^{\overline{J}} \frac{b_{23}(\overline{\alpha}_q,t)N_3(x,\overline{\alpha}_q;t)e^{i\overline{\alpha}_q(d_3-d_2)x}}{(\overline{\beta}_n - \overline{\alpha}_q)b'_{33}(\overline{\alpha}_q)}$$

$$- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_3(\xi,t)}{\xi-(\overline{\beta}_n - i0)} N_1(x,\xi;t) e^{i\xi(d_1-d_2)x} d\xi$$

$$+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_3(\xi,t)}{\xi-(\overline{\beta}_n - i0)} N_3(x,\xi;t) e^{i\xi(d_3-d_2)x} d\xi, \tag{11.15}$$

$$\frac{\chi(x,\alpha_j;t)}{b_{11}(\alpha_j)} = -\gamma_1\gamma_3\omega_2 + \sum_{p=1}^{N} \frac{b_{21}(\beta_p,t)N_1(x,\beta_p;t)e^{i\beta_p(d_1-d_2)x}}{(\alpha_j - \beta_p)b'_{11}(\beta_p)}$$

$$+ \sum_{q=1}^{\overline{J}} \frac{b_{23}(\overline{\alpha}_q,t)N_3(x,\overline{\alpha}_q;t)e^{i\overline{\alpha}_q(d_3-d_2)x}}{(\alpha_j - \overline{\alpha}_q)b'_{33}(\overline{\alpha}_q)}$$

$$+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_3(\xi,t)}{\xi-(\alpha_j + i0)} N_1(x,\xi;t) e^{i\xi(d_1-d_2)x} d\xi$$

$$- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_3(\xi,t)}{\xi-(\alpha_j + i0)} N_3(x,\xi;t) e^{i\xi(d_3-d_2)x} d\xi. \tag{11.16}$$

Equations (11.13)–(11.16) are integro-algebraic equations which are used to solve the inverse problem and reconstruct the potentials of the classical and two nonlocal three-wave interaction equations. In fact, the system is of $N+\overline{N}+J+\overline{J}$ vector equations in the same number of unknowns. In order to determine the linear system for the classical or complex reverse space-time or real reverse space-time three-wave system, we need to impose symmetry relations on the scattering data. We will do this below for each case.

11.2. Classical three-wave system. By setting $\overline{\alpha}_j = \alpha_j^*$, $\overline{\beta}_n = \beta_n^*$, $J = \overline{J}$, $N = \overline{N}$ in (11.9)–(11.12) and using (8.4), we obtain the representations of eigenfunctions below:

(11.17)

$$N_1(x,k;t) = \omega_1 - \sum_{n=1}^{N} \frac{\frac{1}{a_{21}(\beta_n^*,t)}\overline{\chi}(x,\beta_n^*;t)e^{i\beta_n^*(d_2-d_1)x}}{(k-\beta_n^*)a'_{11}(\beta_n^*)}$$

$$- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)}{\xi-(k+i0)} \frac{\overline{\chi}(x,\xi;t)}{b_{33}(\xi)} e^{i\xi(d_2-d_1)x} d\xi$$

$$- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{-\frac{\varepsilon_3}{\varepsilon_2}\rho_1(\xi,t)\overline{\rho}_2^*(\xi,t)+\overline{\rho}_1(\xi,t)}{\xi-(k+i0)} N_3(x,\xi;t) e^{i\xi(d_3-d_1)x} d\xi,$$

$$\begin{aligned}N_3(x,k;t) &= \omega_3 + \sum_{j=1}^{J} \frac{\frac{1}{a_{23}(\alpha_j,t)}\chi(x,\alpha_j;t)e^{i\alpha_j(d_2-d_3)x}}{(k-\alpha_j)a'_{33}(\alpha_j)} \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_2(\xi,t)}{\xi-(k-i0)}\frac{\chi(x,\xi;t)}{b_{11}(\xi)}e^{i\xi(d_2-d_3)x}d\xi \\ &\quad + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_2(\xi,t)-\frac{\varepsilon_1}{\varepsilon_2}\overline{\rho}_2(\xi,t)\rho_1^*(\xi,t)}{\xi-(k-i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_3)x}d\xi,\end{aligned} \tag{11.18}$$

$$\begin{aligned}\frac{\overline{\chi}(x,k;t)}{b_{33}(k)} &= \varepsilon_1\varepsilon_3\omega_2 - \sum_{n=1}^{N} \frac{b_{21}(\beta_n,t)N_1(x,\beta_n;t)e^{i\beta_n(d_1-d_2)x}}{(k-\beta_n)b'_{11}(\beta_n)} \\ &\quad - \sum_{j=1}^{J} \frac{b_{23}(\alpha_j^*,t)N_3(x,\alpha_j^*;t)e^{i\alpha_j^*(d_3-d_2)x}}{(k-\alpha_j^*)b'_{33}(\alpha_j^*)} \\ &\quad + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\varepsilon_1}{\varepsilon_2}\rho_1^*(\xi,t)}{\xi-(k-i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\varepsilon_3}{\varepsilon_2}\overline{\rho}_2^*(\xi,t)}{\xi-(k-i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi,\end{aligned} \tag{11.19}$$

$$\begin{aligned}\frac{\chi(x,k;t)}{b_{11}(k)} &= -\varepsilon_1\varepsilon_3\omega_2 + \sum_{n=1}^{N} \frac{b_{21}(\beta_n,t)N_1(x,\beta_n;t)e^{i\beta_n(d_1-d_2)x}}{(k-\beta_n)b'_{11}(\beta_n)} \\ &\quad + \sum_{j=1}^{J} \frac{b_{23}(\alpha_j^*,t)N_3(x,\alpha_j^*;t)e^{i\alpha_j^*(d_3-d_2)x}}{(k-\alpha_j^*)b'_{33}(\alpha_j^*)} \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\varepsilon_1}{\varepsilon_2}\rho_1^*(\xi,t)}{\xi-(k+i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi \\ &\quad + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\varepsilon_3}{\varepsilon_2}\overline{\rho}_2^*(\xi,t)}{\xi-(k+i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi.\end{aligned} \tag{11.20}$$

The closed system is obtained by setting $\overline{\alpha}_j = \alpha_j^*$, $\overline{\beta}_n = \beta_n^*$ and $J = \overline{J}$, $N = \overline{N}$ in (11.13)–(11.16). Combining (8.2), (8.4), (9.6), and (9.7), we find

$$\begin{aligned}N_1(x,\beta_n;t) &= \omega_1 + \sum_{l=1}^{N} \frac{\varepsilon_3 b_{21}^*(\beta_l,t)\cdot\frac{\overline{\chi}(x,(\beta_l^*;t)}{b_{33}((\beta_l^*)}e^{i\beta_l^*(d_2-d_1)x}}{(\beta_n-\beta_l^*)a'_{11}(\beta_l^*)} \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)}{\xi-(\beta_n+i0)}\frac{\overline{\chi}(x,\xi;t)}{b_{33}(\xi,t)}e^{i\xi(d_2-d_1)x}d\xi \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{-\frac{\varepsilon_3}{\varepsilon_2}\rho_1(\xi,t)\overline{\rho}_2^*(\xi,t)+\overline{\rho}_1(\xi,t)}{\xi-(\beta_n+i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_1)x}d\xi,\end{aligned} \tag{11.21}$$

$$\begin{aligned}N_3(x,\alpha_j^*;t) &= \omega_3 - \sum_{m=1}^{J} \frac{\varepsilon_1 b_{23}^*(\alpha_m^*,t)\cdot\frac{\chi(x,\alpha_m;t)}{b_{11}(\alpha_m)}e^{i\alpha_m(d_2-d_3)x}}{(\alpha_j^*-\alpha_m)a'_{33}(\alpha_m)} \\ &\quad - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_2(\xi,t)}{\xi-(\alpha_j^*-i0)}\frac{\chi(x,\xi;t)}{b_{11}(\xi)}e^{i\xi(d_2-d_3)x}d\xi \\ &\quad + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_2(\xi,t)-\frac{\varepsilon_1}{\varepsilon_2}\overline{\rho}_2(\xi,t)\rho_1^*(\xi,t)}{\xi-(\alpha_j^*-i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_3)x}d\xi,\end{aligned} \tag{11.22}$$

$$\frac{\overline{\chi}(x,\beta_n^*;t)}{b_{33}(\beta_n^*)} = \varepsilon_1\varepsilon_3\omega_2 - \sum_{p=1}^{N} \frac{b_{21}(\beta_p,t)N_1(x,\beta_p;t)e^{i\beta_p(d_1-d_2)x}}{(\beta_n^*-\beta_p)b'_{11}(\beta_p)} - \sum_{q=1}^{J} \frac{b_{23}(\alpha_q^*,t)N_3(x,\alpha_q^*;t)e^{i\alpha_q^*(d_3-d_2)x}}{(\beta_n^*-\alpha_q^*)b'_{33}(\alpha_q^*)} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\varepsilon_1}{\varepsilon_2}\rho_1^*(\xi,t)}{\xi-(\beta_n^*-i0)} N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\varepsilon_3}{\varepsilon_2}\overline{\rho}_2^*(\xi,t)}{\xi-(\beta_n^*-i0)} N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi, \tag{11.23}$$

$$\frac{\chi(x,\alpha_j;t)}{b_{11}(\alpha_j)} = -\varepsilon_1\varepsilon_3\omega_2 + \sum_{p=1}^{N} \frac{b_{21}(\beta_p,t)N_1(x,\beta_p;t)e^{i\beta_p(d_1-d_2)x}}{(\alpha_j-\beta_p)b'_{11}(\beta_p)} + \sum_{q=1}^{J} \frac{b_{23}(\alpha_q^*,t)N_3(x,\alpha_q^*;t)e^{i\alpha_q^*(d_3-d_2)x}}{(\alpha_j-\alpha_q^*)b'_{33}(\alpha_q^*)} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\varepsilon_1}{\varepsilon_2}\rho_1^*(\xi,t)}{\xi-(\alpha_j+i0)} N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\frac{\varepsilon_3}{\varepsilon_2}\overline{\rho}_2^*(\xi,t)}{\xi-(\alpha_j+i0)} N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi. \tag{11.24}$$

11.3. Complex reverse space-time three-wave system. Under the symmetry reduction $N_{21}(x)=\varepsilon_1\varepsilon_2 N_{12}^*(-x)$, $N_{31}(x)=\varepsilon_1\varepsilon_3 N_{13}^*(-x)$, and $N_{32}(x)=\varepsilon_2\varepsilon_3 N_{23}^*(-x)$, where $\varepsilon_j^2=1$, $j=1,2,3$, and from the distribution of zeros of $a_{11}(k)$, $b_{11}(k)$, $a_{33}(k)$, $b_{33}(k)$ (see section 9.2) and (11.9)–(11.12), we find

(11.25)

$$N_1(x,k;t) = \omega_1 - \sum_{n=1}^{\overline{N}_1} \frac{\frac{1}{a_{21}(\overline{\beta}_n,t)}\overline{\chi}(x,\overline{\beta}_n;t)e^{i\overline{\beta}_n(d_2-d_1)x}}{(k-\overline{\beta}_n)a'_{11}(\overline{\beta}_n)} - \sum_{n=1}^{\overline{N}_1} \frac{\frac{1}{a_{21}(-\overline{\beta}_n^*,t)}\overline{\chi}(x,-\overline{\beta}_n^*;t)e^{-i\overline{\beta}_n^*(d_2-d_1)x}}{(k+\overline{\beta}_n^*)a'_{11}(-\overline{\beta}_n^*)} - \sum_{m=1}^{\overline{N}_2} \frac{\frac{1}{a_{21}(\overline{\overline{\beta}}_m,t)}\overline{\chi}(x,\overline{\overline{\beta}}_m;t)e^{i\overline{\overline{\beta}}_m(d_2-d_1)x}}{(k-\overline{\overline{\beta}}_m)a'_{11}(\overline{\overline{\beta}}_m)} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)\overline{\rho}_3(\xi,t)+\overline{\rho}_1(\xi,t)}{\xi-(k+i0)} N_3(x,\xi;t)e^{i\xi(d_3-d_1)x}d\xi - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)}{\xi-(k+i0)}\frac{\overline{\chi}(x,\xi;t)}{b_{33}(\xi)}e^{i\xi(d_2-d_1)x}d\xi,$$

(11.26)

$$N_3(x,k;t) = \omega_3 + \sum_{j=1}^{J_1} \frac{\frac{1}{a_{23}(\alpha_j,t)}\chi(x,\alpha_j;t)e^{i\alpha_j(d_2-d_3)x}}{(k-\alpha_j)a'_{33}(\alpha_j)} + \sum_{j=1}^{J_1} \frac{\frac{1}{a_{23}(-\alpha_j^*,t)}\chi(x,-\alpha_j^*;t)e^{-i\alpha_j^*(d_2-d_3)x}}{(k+\alpha_j^*)a'_{33}(-\alpha_j^*)} + \sum_{l=1}^{J_2} \frac{\frac{1}{a_{23}(\widetilde{\alpha}_l,t)}\chi(x,\widetilde{\alpha}_l;t)e^{i\widetilde{\alpha}_l(d_2-d_3)x}}{(k-\widetilde{\alpha}_l)a'_{33}(\widetilde{\alpha}_l)} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_2(\xi,t)+\overline{\rho}_2(\xi,t)\rho_3(\xi,t)}{\xi-(k-i0)} N_1(x,\xi;t)e^{i\xi(d_1-d_3)x}d\xi - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_2(\xi,t)}{\xi-(k-i0)}\frac{\chi(x,\xi;t)}{b_{11}(\xi)}e^{i\xi(d_2-d_3)x}d\xi,$$

(11.27)
$$\frac{\overline{\chi}(x,k;t)}{b_{33}(k)} = \varepsilon_1\varepsilon_3\omega_2 - \sum_{n=1}^{N_1} \frac{b_{21}(\beta_n,t)N_1(x,\beta_n;t)e^{i\beta_n(d_1-d_2)x}}{(k-\beta_n)b'_{11}(\beta_n)} - \sum_{n=1}^{N_1} \frac{b_{21}(-\beta_n^*,t)N_1(x,-\beta_n^*;t)e^{-i\beta_n^*(d_1-d_2)x}}{(k+\beta_n^*)b'_{11}(-\beta_n^*)} - \sum_{m=1}^{N_2} \frac{b_{21}(\widetilde{\beta}_m,t)N_1(x,\widetilde{\beta}_m;t)e^{i\widetilde{\beta}_m(d_1-d_2)x}}{(k-\widetilde{\beta}_m)b'_{11}(\widetilde{\beta}_m)} - \sum_{j=1}^{\overline{J}_1} \frac{b_{23}(\overline{\alpha}_j,t)N_3(x,\overline{\alpha}_j;t)e^{i\overline{\alpha}_j(d_3-d_2)x}}{(k-\overline{\alpha}_j)b'_{33}(\overline{\alpha}_j)} - \sum_{j=1}^{\overline{J}_1} \frac{b_{23}(-\overline{\alpha}_j^*,t)N_3(x,-\overline{\alpha}_j^*;t)e^{-i\overline{\alpha}_j^*(d_3-d_2)x}}{(k+\overline{\alpha}_j^*)b'_{33}(-\overline{\alpha}_j^*)} - \sum_{l=1}^{\overline{J}_2} \frac{b_{23}(\overline{\widetilde{\alpha}}_l,t)N_3(x,\overline{\widetilde{\alpha}}_l;t)e^{i\overline{\widetilde{\alpha}}_l(d_3-d_2)x}}{(k-\overline{\widetilde{\alpha}}_l)b'_{33}(\overline{\widetilde{\alpha}}_l)} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_3(\xi,t)}{\xi-(k-i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_3(\xi,t)}{\xi-(k-i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi,$$

(11.28)
$$\frac{\chi(x,k;t)}{b_{11}(k)} = -\varepsilon_1\varepsilon_3\omega_2 + \sum_{n=1}^{N_1} \frac{b_{21}(\beta_n,t)N_1(x,\beta_n;t)e^{i\beta_n(d_1-d_2)x}}{(k-\beta_n)b'_{11}(\beta_n)} + \sum_{n=1}^{N_1} \frac{b_{21}(-\beta_n^*,t)N_1(x,-\beta_n^*;t)e^{-i\beta_n^*(d_1-d_2)x}}{(k+\beta_n^*)b'_{11}(-\beta_n^*)} + \sum_{m=1}^{N_2} \frac{b_{21}(\widetilde{\beta}_m,t)N_1(x,\widetilde{\beta}_m;t)e^{i\widetilde{\beta}_m(d_1-d_2)x}}{(k-\widetilde{\beta}_m)b'_{11}(\widetilde{\beta}_m)} + \sum_{j=1}^{\overline{J}_1} \frac{b_{23}(\overline{\alpha}_j,t)N_3(x,\overline{\alpha}_j;t)e^{i\overline{\alpha}_j(d_3-d_2)x}}{(k-\overline{\alpha}_j)b'_{33}(\overline{\alpha}_j)} + \sum_{j=1}^{\overline{J}_1} \frac{b_{23}(-\overline{\alpha}_j^*,t)N_3(x,-\overline{\alpha}_j^*;t)e^{-i\overline{\alpha}_j^*(d_3-d_2)x}}{(k+\overline{\alpha}_j^*)b'_{33}(-\overline{\alpha}_j^*)} + \sum_{l=1}^{\overline{J}_2} \frac{b_{23}(\overline{\widetilde{\alpha}}_l,t)N_3(x,\overline{\widetilde{\alpha}}_l;t)e^{i\overline{\widetilde{\alpha}}_l(d_3-d_2)x}}{(k-\overline{\widetilde{\alpha}}_l)b'_{33}(\overline{\widetilde{\alpha}}_l)} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_3(\xi,t)}{\xi-(k+i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}_3(\xi,t)}{\xi-(k+i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi.$$

Recall $2J_1+J_2=J$, $2\overline{J}_1+\overline{J}_2=\overline{J}$, $2N_1+N_2=N$, and $2\overline{N}_1+\overline{N}_2=\overline{N}$. In fact, if J is an odd integer, then J_2 is also odd; that is, at least one eigenvalue $\tilde{\alpha}_l$ lies on the imaginary axis (the actual number is J_2), and the number of eigenvalue pairs $\{\alpha_j,-\alpha_j^*\}$ is J_1. On the other hand, if J is an even integer, then J_2 must be even. In particular, it is possible that there is no eigenvalue on $i\mathbb{R}$ ($J_2=0$, $J_1=J/2$) or no eigenvalue pair ($J_1=0$, $J_2=J$). One has similar conclusions for $\overline{J}$, N, and $\overline{N}$.

For simplification, we only consider the case of $J_1=\overline{J}_1=N_1=\overline{N}_1=0$ here. By setting $J_1=\overline{J}_1=N_1=\overline{N}_1=0$ and substituting $k=\tilde{\beta}_n$ in (11.25), $k=\overline{\tilde{\alpha}}_l$ in (11.26), $k=\overline{\tilde{\beta}}_m$ in (11.27), $k=\tilde{\alpha}_j$ in (11.28) and applying (9.11), one derives the closed system for the complex RST three-wave interaction equations, i.e.,

$$(11.29)\quad \begin{aligned} N_1(x,\tilde{\beta}_n;t) = \omega_1 &+ \sum_{r=1}^{\overline{N}} \frac{\varepsilon_3 a_{21}^*(\overline{\tilde{\beta}}_r,-t)\cdot \frac{\overline{\chi}(x,\overline{\tilde{\beta}}_r;t)}{b_{33}(\overline{\tilde{\beta}}_r,t)} e^{i\overline{\tilde{\beta}}_r(d_2-d_1)x}}{(\tilde{\beta}_n-\overline{\tilde{\beta}}_m)a'_{11}(\overline{\tilde{\beta}}_m)} \\ &- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)}{\xi-(\tilde{\beta}_n+i0)}\frac{\overline{\chi}(x,\xi;t)}{b_{33}(\xi)}e^{i\xi(d_2-d_1)x}d\xi \\ &- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_1(\xi,t)\overline{\rho}_3(\xi,t)+\overline{\rho}_1(\xi,t)}{\xi-(\tilde{\beta}_n+i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_1)x}d\xi, \end{aligned}$$

$$N_3(x,\overline{\widetilde{\alpha}}_l;t)=\omega_3-\sum_{s=1}^{J}\frac{\varepsilon_1 a_{23}^*(\widetilde{\alpha}_s,-t)\cdot\frac{\chi(x,\widetilde{\alpha}_s;t)}{b_{11}(\widetilde{\alpha}_s,t)}e^{i\widetilde{\alpha}_s(d_2-d_3)x}}{(\overline{\widetilde{\alpha}}_l-\widetilde{\alpha}_s)a_{33}'(\widetilde{\alpha}_s)}$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\overline{\rho}_2(\xi,t)}{\xi-(\overline{\widetilde{\alpha}}_l-i0)}\frac{\chi(x,\xi;t)}{b_{11}(\xi)}e^{i\xi(d_2-d_3)x}d\xi \tag{11.30}$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\rho_2(\xi,t)+\overline{\rho}_2(\xi,t)\rho_3(\xi,t)}{\xi-(\overline{\widetilde{\alpha}}_l-i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_3)x}d\xi,$$

$$\frac{\overline{\chi}(x,\overline{\widetilde{\beta}}_m;t)}{b_{33}(\overline{\widetilde{\beta}}_m)}=\varepsilon_1\varepsilon_3\omega_2-\sum_{p=1}^{N}\frac{b_{21}(\widetilde{\beta}_p,t)N_1(x,\widetilde{\beta}_p;t)e^{i\widetilde{\beta}_p(d_1-d_2)x}}{(\overline{\widetilde{\beta}}_m-\widetilde{\beta}_p)b_{11}'(\widetilde{\beta}_p)}$$

$$-\sum_{q=1}^{\overline{J}}\frac{b_{23}(\overline{\widetilde{\alpha}}_q,t)N_3(x,\overline{\widetilde{\alpha}}_q;t)e^{i\overline{\widetilde{\alpha}}_q(d_3-d_2)x}}{(\overline{\widetilde{\beta}}_m-\overline{\widetilde{\alpha}}_q)b_{33}'(\overline{\widetilde{\alpha}}_q)}$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\rho_3(\xi,t)}{\xi-(\overline{\widetilde{\beta}}_m-i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\overline{\rho}_3(\xi,t)}{\xi-(\overline{\widetilde{\beta}}_m-i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi, \tag{11.31}$$

$$\frac{\chi(x,\tilde{\alpha}_j;t)}{b_{11}(\tilde{\alpha}_j)}=-\varepsilon_1\varepsilon_3\omega_2+\sum_{p=1}^{N}\frac{b_{21}(\widetilde{\beta}_p,t)N_1(x,\widetilde{\beta}_p;t)e^{i\widetilde{\beta}_p(d_1-d_2)x}}{(\tilde{\alpha}_j-\widetilde{\beta}_p)b_{11}'(\widetilde{\beta}_p)}$$

$$+\sum_{q=1}^{\overline{J}}\frac{b_{23}(\overline{\widetilde{\alpha}}_q,t)N_3(x,\overline{\widetilde{\alpha}}_q;t)e^{i\overline{\widetilde{\alpha}}_q(d_3-d_2)x}}{(\tilde{\alpha}_j-\overline{\widetilde{\alpha}}_q)b_{33}'(\overline{\widetilde{\alpha}}_q)}$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\rho_3(\xi,t)}{\xi-(\tilde{\alpha}_j+i0)}N_1(x,\xi;t)e^{i\xi(d_1-d_2)x}d\xi$$

$$-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\overline{\rho}_3(\xi,t)}{\xi-(\tilde{\alpha}_j+i0)}N_3(x,\xi;t)e^{i\xi(d_3-d_2)x}d\xi. \tag{11.32}$$

11.4. Real reverse space-time three-wave system. Under the symmetry reductions $N_{21}(x)=\varepsilon_1\varepsilon_2 N_{12}(-x)$, $N_{31}(x)=\varepsilon_1\varepsilon_3 N_{13}(-x)$, and $N_{32}(x)=\varepsilon_2\varepsilon_3 N_{23}(-x)$ as well as $N_{21}(x)=\varepsilon_1\varepsilon_2 N_{12}^*(-x)$, $N_{31}(x)=\varepsilon_1\varepsilon_3 N_{13}^*(-x)$, and $N_{32}(x)=\varepsilon_2\varepsilon_3 N_{23}^*(-x)$, where $\varepsilon_j^2=1$, $j=1,2,3$, and noting again that the symmetry relations among eigenvalues for the real reverse space-time three-wave system are the same as the complex nonlocal case (see section 8.3), the general representations of eigenfunctions are the same as (11.25)–(11.28), and the special case is obtained by choosing $J_1=\overline{J}_1=N_1=\overline{N}_1=0$.

From (8.11) and (11.29)–(11.32), one obtains the closed system for $J_1=\overline{J}_1=N_1=\overline{N}_1=0$, that is, the system (11.29)–(11.32) by setting $a_{21}(\overline{\widetilde{\beta}}_r,-t)$ and $a_{23}(\widetilde{\alpha}_s,-t)$ to be real, i.e., $a_{21}^*(\overline{\widetilde{\beta}}_r,-t)=a_{21}(\overline{\widetilde{\beta}}_r,-t)$ and $a_{23}^*(\widetilde{\alpha}_s,-t)=a_{23}(\widetilde{\alpha}_s,-t)$.

12. Minimal data. In this section, we define the minimal data needed for the inverse scattering reconstruction of the eigenfunctions and potentials. In the general case, we need the following:

(i) *Continuous spectra*: reflection coefficients $\rho_1(k),\overline{\rho}_1(k),\rho_2(k),\overline{\rho}_2(k),\rho_3(k),\overline{\rho}_3(k)$.

(ii) *Discrete spectra*: eigenvalues α_j, $j=1,2,\ldots,J$; $\bar{\alpha}_j$, $j=1,2,\ldots,\bar{J}$; β_n, $n=1,2,\ldots,N$; $\bar{\beta}_n$, $n=1,2,\ldots,\bar{N}$.

(iii) *Reduced normalization coefficients* (see section 9: (9.1)–(9.2) and (9.5)):

$$b_{21}(\beta_n),\ b_{23}(\overline{\alpha}_j),\ a_{21}(\overline{\beta}_n),\ a_{23}(\alpha_j).$$

We remark that the following values are obtained from the trace formulae in terms of eigenvalues and the above reflection coefficients: $a'_{11}(\overline{\beta}_n)$, $a'_{33}(\alpha_j)$, $b'_{11}(\beta_n)$, $b'_{33}(\overline{\alpha}_j)$. These latter values are not part of the minimal data since they are expressed in terms of known data.

Due to symmetry reductions for the potentials, it induces the symmetries among scattering data, reflection coefficients, and eigenvalues. Therefore, the number of minimal data can be reduced. Next, we discuss the minimal data for the case of classical, complex reverse space-time, and real reverse space-time three-wave interaction equations, respectively.

12.1. The classical three-wave system. To solve the inverse problem, we need the following quantities:

(i) *Continuous spectra*: reflection coefficients $\rho_1(k)$, $\rho_2(k)$, $\overline{\rho}_1(k)$, $\overline{\rho}_2(k)$ (via (8.4), the number of reflection coefficients is reduced by two).

(ii) *Discrete spectra*: eigenvalues α_j, $j=1,2,\ldots,J$, and β_n, $n=1,2,\ldots,N$ (via $\overline{\alpha}_j=\alpha_j^*$, $\overline{\beta}_n=\beta_n^*$, the number of eigenvalues is reduced by a factor of two).

(iii) *Reduced normalization coefficients*: $b_{21}(\beta_n)$, $b_{23}(\alpha_j^*)$ (via (9.6) and (9.7), the number of reduced normalization coefficients is also reduced by a factor of two).

In summary, we need reflection coefficients $\rho_1(k)$, $\rho_2(k)$, $\overline{\rho}_1(k)$, $\overline{\rho}_2(k)$, eigenvalues α_j, β_n, and reduced normalization coefficients $b_{21}(\beta_n)$, $b_{23}(\alpha_j^*)$ to recover potentials. In particular, the reflectionless potentials only depend on eigenvalues and reduced normalization coefficients.

12.2. Complex reverse space-time three-wave system. In order to reconstruct the potentials, the following data are needed:

(i) *Continuous spectra*: reflection coefficients $\rho_1(k),\overline{\rho}_1(k),\rho_2(k),\overline{\rho}_2(k),\rho_3(k),\overline{\rho}_3(k)$.

(ii) *Discrete spectra:* eigenvalues

$$\begin{aligned}&\{\alpha_j:\Im\alpha_j>0\}_{j=1}^{J}\\&=\{\alpha_m,-\alpha_m^*:\Im\alpha_m>0 \text{ and } \Re\alpha_m\neq 0\}_{m=1}^{J_1}\cup\{\widetilde{\alpha}_l:\Im\widetilde{\alpha}_l>0 \text{ and } \Re\widetilde{\alpha}_l=0\}_{l=1}^{J_2},\end{aligned}$$

$$\begin{aligned}&\{\overline{\alpha}_j:\Im\overline{\alpha}_j<0\}_{j=1}^{\overline{J}}\\&=\{\overline{\alpha}_m,-\overline{\alpha}_m^*:\Im\overline{\alpha}_m<0 \text{ and } \Re\overline{\alpha}_m\neq 0\}_{m=1}^{\overline{J}_1}\cup\{\overline{\widetilde{\alpha}}_l:\Im\overline{\widetilde{\alpha}}_l<0 \text{ and } \Re\overline{\widetilde{\alpha}}_l=0\}_{l=1}^{\overline{J}_2},\end{aligned}$$

$$\begin{aligned}&\{\beta_n:\Im\beta_n>0\}_{n=1}^{N}\\&=\{\beta_p,-\beta_p^*:\Im\beta_p>0 \text{ and } \Re\beta_p\neq 0\}_{p=1}^{N_1}\cup\{\widetilde{\beta}_m:\Im\widetilde{\beta}_m>0 \text{ and } \Re\widetilde{\beta}_m=0\}_{m=1}^{N_2},\end{aligned}$$

$$\begin{aligned}&\{\overline{\beta}_n:\Im\overline{\beta}_n<0\}_{n=1}^{\overline{N}}\\&=\{\overline{\beta}_p,-\overline{\beta}_p^*:\Im\overline{\beta}_p<0 \text{ and } \Re\overline{\beta}_p\neq 0\}_{p=1}^{\overline{N}_1}\cup\{\overline{\widetilde{\beta}}_m:\Im\overline{\widetilde{\beta}}_m<0 \text{ and } \Re\overline{\widetilde{\beta}}_m=0\}_{m=1}^{\overline{N}_2}.\end{aligned}$$

(iii) *Reduced normalization coefficients*: $b_{21}(\beta_p)$, $b_{23}(\overline{\alpha}_m)$, $a_{21}(\overline{\beta}_p)$, $a_{23}(\alpha_m)$ and phases (defined below)

$$\begin{aligned}&\theta_1^{(m)},\ m=1,2,\ldots,N_2;\quad \theta_2^{(l)},\ l=1,2,\ldots,\overline{J}_2;\\&\theta_3^{(m)},\ m=1,2,\ldots,\overline{N}_2;\quad \theta_4^{(l)},\ l=1,2,\ldots,J_2.\end{aligned}$$

In general, we need $b_{21}(\beta_p)$, $b_{21}(-\beta_p^*)$, $b_{21}(\widetilde{\beta}_m)$, $b_{23}(\overline{\alpha}_m)$, $b_{23}(-\overline{\alpha}_m^*)$, $b_{23}(\overline{\widetilde{\alpha}}_l)$, $a_{21}(\overline{\beta}_p)$, $a_{21}(-\overline{\beta}_p^*)$, $a_{21}(\overline{\widetilde{\beta}}_m)$, $a_{23}(\alpha_m)$, $a_{23}(-\alpha_m^*)$, $a_{23}(\widetilde{\alpha}_l)$.

By (9.8)–(9.11), it follows that $b_{21}(-\beta_p^*)$, $b_{23}(-\overline{\alpha}_m^*)$, $a_{21}(-\overline{\beta}_p^*)$, $a_{23}(-\alpha_m^*)$ are related to $b_{21}(\beta_p)$, $b_{23}(\overline{\alpha}_m)$, $a_{21}(\overline{\beta}_p)$, $a_{23}(\alpha_m)$ and some values obtained from the trace formulae in terms of known data.

Thus, $b_{21}(-\beta_p^*)$, $b_{23}(-\overline{\alpha}_m^*)$, $a_{21}(-\overline{\beta}_p^*)$, $a_{23}(-\alpha_m^*)$ are not part of the minimal data. In fact, (9.11) implies that for purely imaginary eigenvalues, one has

$$-\frac{\varepsilon_1|a_{23}(\widetilde{\alpha}_l)|^2}{b_{11}(\widetilde{\alpha}_l)}=1,-\frac{\varepsilon_3|b_{21}(\widetilde{\beta}_m)|^2}{a_{33}(\widetilde{\beta}_m)}=1,-\frac{\varepsilon_1|b_{23}(\overline{\widetilde{\alpha}}_l)|^2}{a_{11}(\overline{\widetilde{\alpha}}_l)}=1,-\frac{\varepsilon_3|a_{21}(\overline{\widetilde{\beta}}_m)|^2}{b_{33}(\overline{\widetilde{\beta}}_m)}=1.$$

Thus, $|b_{21}(\widetilde{\beta}_m)|$, $|b_{23}(\overline{\widetilde{\alpha}}_l)|$, $|a_{21}(\overline{\widetilde{\beta}}_m)|$, $|a_{23}(\widetilde{\alpha}_l)|$ are able to be represented in terms of $a_{33}(\widetilde{\beta}_m)$, $a_{11}(\overline{\widetilde{\alpha}}_l)$, $b_{33}(\overline{\widetilde{\beta}}_m)$, $b_{11}(\widetilde{\alpha}_l)$, respectively. These four values can be found from the trace formulae, which consist of known data: eigenvalues and reflection coefficients. If the reduced normalization coefficients $b_{21}(\widetilde{\beta}_m)$, $b_{23}(\overline{\widetilde{\alpha}}_l)$, $a_{21}(\overline{\widetilde{\beta}}_m)$, and $a_{23}(\widetilde{\alpha}_l)$ are omitted, then the above phases $\theta_s^{(m)},\theta_j^{(l)}$, $s=1,3$, $j=2,4$, are introduced.

12.3. Real reverse space-time three-wave system. To recover the potentials, we need the data stated below:

(i) *Continuous spectra*: reflection coefficients $\rho_1(k),\overline{\rho}_1(k),\rho_2(k),\overline{\rho}_2(k),\rho_3(k),\overline{\rho}_3(k)$.

(ii) *Discrete spectra*: eigenvalues

$$\begin{aligned}&\{\alpha_j:\Im\alpha_j>0\}_{j=1}^{J}\\&=\{\alpha_m,-\alpha_m^*:\Im\alpha_m>0\text{ and }\Re\alpha_m\neq 0\}_{m=1}^{J_1}\cup\{\widetilde{\alpha}_l:\Im\widetilde{\alpha}_l>0\text{ and }\Re\widetilde{\alpha}_l=0\}_{l=1}^{J_2},\end{aligned}$$

$$\begin{aligned}&\{\overline{\alpha}_j:\Im\overline{\alpha}_j<0\}_{j=1}^{\overline{J}}\\&=\{\overline{\alpha}_m,-\overline{\alpha}_m^*:\Im\overline{\alpha}_m<0\text{ and }\Re\overline{\alpha}_m\neq 0\}_{m=1}^{\overline{J}_1}\cup\{\overline{\widetilde{\alpha}}_l:\Im\overline{\widetilde{\alpha}}_l<0\text{ and }\Re\overline{\widetilde{\alpha}}_l=0\}_{l=1}^{\overline{J}_2},\end{aligned}$$

$$\begin{aligned}&\{\beta_n:\Im\beta_n>0\}_{n=1}^{N}\\&=\{\beta_p,-\beta_p^*:\Im\beta_p>0\text{ and }\Re\beta_p\neq 0\}_{p=1}^{N_1}\cup\{\widetilde{\beta}_m:\Im\widetilde{\beta}_m>0\text{ and }\Re\widetilde{\beta}_m=0\}_{m=1}^{N_2},\end{aligned}$$

$$\begin{aligned}&\{\overline{\beta}_n:\Im\overline{\beta}_n<0\}_{n=1}^{\overline{N}}\\&=\{\overline{\beta}_p,-\overline{\beta}_p^*:\Im\overline{\beta}_p<0\text{ and }\Re\overline{\beta}_p\neq 0\}_{p=1}^{\overline{N}_1}\cup\{\overline{\widetilde{\beta}}_m:\Im\overline{\widetilde{\beta}}_m<0\text{ and }\Re\overline{\widetilde{\beta}}_m=0\}_{m=1}^{\overline{N}_2}.\end{aligned}$$

(iii) *Units*: $\delta_1^{(n)}$, $n=1,2,\dots,\overline{N}$; $\delta_2^{(j)}$, $j=1,2,\dots,J$; $\delta_3^{(n)}$, $n=1,2,\dots,N$; $\delta_4^{(j)}, j=1,2,\dots,\overline{J}$, where $\delta_1^{(n)}=\pm1$, $\delta_2^{(j)}=\pm1$, $\delta_3^{(n)}=\pm1$, and $\delta_4^{(j)}=\pm1$.

Actually, the above units are introduced due to (9.12) and (9.13). In fact, (9.12) and (9.13) imply $-\frac{\varepsilon_1 a_{23}^2(\alpha_j)}{b_{11}(\alpha_j)}=1$, $-\frac{\varepsilon_3 b_{21}^2(\beta_n)}{a_{33}(\beta_n)}=1$, $-\frac{\varepsilon_1 b_{23}^2(\overline{\alpha}_j)}{a_{11}(\overline{\alpha}_j)}=1$, $-\frac{\varepsilon_3 a_{21}^2(\overline{\beta}_n)}{b_{33}(\overline{\beta}_n)}=1$, which shows that the reduced normalization coefficients only rely on $b_{11}(\alpha_j)$, $a_{33}(\beta_n)$, $a_{11}(\overline{\alpha}_j)$, $b_{33}(\overline{\beta}_n)$ and units. Moreover, $b_{11}(\alpha_j)$, $a_{33}(\beta_n)$, $a_{11}(\overline{\alpha}_j)$, and $b_{33}(\overline{\beta}_n)$ are derived from the trace formulae in terms of known data.

In conclusion, the minimal data for reconstructing potentials contain the reflection coefficients, the eigenvalues, and the above units. In addition, the pure soliton solutions are only relevant to the eigenvalues and the units.

13. Reflectionless potentials and soliton solutions. In this section, we discuss reflectionless potentials and hence when time is included in the associated soliton solutions. The soliton solutions for the classical and nonlocal systems provide concrete examples of how energy is shared between the three components. Reflectionless potentials and pure soliton solutions correspond to zero reflection coefficients, i.e., $\rho_1(\xi)=0$, $\overline{\rho}_1(\xi)=0$, $\rho_2(\xi)=0$, $\overline{\rho}_2(\xi)=0$, $\rho_3(\xi)=0$, and $\overline{\rho}_3(\xi)=0$ on $\mathbb{R}$. Recall the notation $d_1=-C_1$, $d_2=-C_2$, and $d_3=-C_3$ for the classical and two nonlocal cases (see section 2).

13.1. General case. Letting $J=N=\overline{J}=\overline{N}=1$, by (11.9) and (11.10), one obtains

$$N_1(x,k;t)=\omega_1-\frac{\frac{b_{33}(\overline{\beta}_1)}{a_{21}(\overline{\beta}_1,t)}\frac{\overline{\chi}(x,\overline{\beta}_1;t)}{b_{33}(\overline{\beta}_1)}e^{i\overline{\beta}_1(d_2-d_1)x}}{(k-\overline{\beta}_1)a_{11}'(\overline{\beta}_1)}, \tag{13.1}$$

$$N_3(x,k;t)=\omega_3+\frac{\frac{b_{11}(\alpha_1)}{a_{23}(\alpha_1,t)}\frac{\chi(x,\alpha_1;t)}{b_{11}(\alpha_1)}e^{i\alpha_1(d_2-d_3)x}}{(k-\alpha_1)a_{33}'(\alpha_1)}. \tag{13.2}$$

By (11.13)–(11.16), one has

$$N_1(x,\beta_1;t)=\omega_1-\frac{\frac{b_{33}(\overline{\beta}_1)}{a_{21}(\overline{\beta}_1,t)}\cdot\frac{\overline{\chi}(x,\overline{\beta}_1;t)}{b_{33}(\overline{\beta}_1)}e^{i\overline{\beta}_1(d_2-d_1)x}}{(\beta_1-\overline{\beta}_1)a_{11}'(\overline{\beta}_1)},$$

$$N_3(x,\overline{\alpha}_1;t)=\omega_3+\frac{\frac{b_{11}(\alpha_1)}{a_{23}(\alpha_1,t)}\cdot\frac{\chi(x,\alpha_1;t)}{b_{11}(\alpha_1)}e^{i\alpha_1(d_2-d_3)x}}{(\overline{\alpha}_1-\alpha_1)a_{33}'(\alpha_1)},$$

$$\begin{aligned}\frac{\overline{\chi}(x,\overline{\beta}_1;t)}{b_{33}(\overline{\beta}_1)}&=\gamma_1\gamma_3\omega_2-\frac{b_{21}(\beta_1,t)N_1(x,\beta_1;t)e^{i\beta_1(d_1-d_2)x}}{(\overline{\beta}_1-\beta_1)b_{11}'(\beta_1)}\\&\quad-\frac{b_{23}(\overline{\alpha}_1,t)N_3(x,\overline{\alpha}_1;t)e^{i\overline{\alpha}_1(d_3-d_2)x}}{(\overline{\beta}_1-\overline{\alpha}_1)b_{33}'(\overline{\alpha}_1)},\end{aligned}$$

$$\begin{aligned}\frac{\chi(x,\alpha_1;t)}{b_{11}(\alpha_1)}&=-\gamma_1\gamma_3\omega_2+\frac{b_{21}(\beta_1,t)N_1(x,\beta_1;t)e^{i\beta_1(d_1-d_2)x}}{(\alpha_1-\beta_1)b_{11}'(\beta_1)}\\&\quad+\frac{b_{23}(\overline{\alpha}_1,t)N_3(x,\overline{\alpha}_1;t)e^{i\overline{\alpha}_1(d_3-d_2)x}}{(\alpha_1-\overline{\alpha}_1)b_{33}'(\overline{\alpha}_1)}.\end{aligned}$$

Combining the trace formulae, the above algebraic system reads as

$$\frac{\overline{\chi}^{(2)}(x,\overline{\beta}_1;t)}{b_{33}(\overline{\beta}_1)}=\frac{\gamma_1\gamma_3}{\tilde{D}(x,t)}\cdot\left[1-\frac{\alpha_1-\beta_1}{\overline{\alpha}_1-\overline{\beta}_1}\cdot\frac{b_{23}(\overline{\alpha}_1,t)}{a_{23}(\alpha_1,t)}e^{i(\alpha_1-\overline{\alpha}_1)(d_2-d_3)x}\right],$$

$$\frac{\overline{\chi}^{(3)}(x,\overline{\beta}_1;t)}{b_{33}(\overline{\beta}_1)}=\frac{1}{\tilde{D}(x,t)}\cdot\frac{\overline{\alpha}_1-\alpha_1}{\overline{\alpha}_1-\overline{\beta}_1}b_{23}(\overline{\alpha}_1,t)e^{i\overline{\alpha}_1(d_3-d_2)x},$$

$$\frac{\chi^{(2)}(x,\alpha_1;t)}{b_{11}(\alpha_1)}=-\frac{\gamma_1\gamma_3}{\tilde{D}(x,t)}\cdot\left[1+\frac{\overline{\beta}_1-\overline{\alpha}_1}{\alpha_1-\beta_1}\cdot\frac{b_{21}(\beta_1,t)}{a_{21}(\overline{\beta}_1,t)}e^{i(\beta_1-\overline{\beta}_1)(d_1-d_2)x}\right], \quad \text{where}$$

$$\begin{aligned}\tilde{D}(x,t) := &\left[1 - \frac{\overline{\beta}_1 - \overline{\alpha}_1}{\overline{\beta}_1 - \alpha_1} \cdot \frac{b_{21}(\beta_1,t)}{a_{21}(\overline{\beta}_1,t)} e^{i(\beta_1-\overline{\beta}_1)(d_1-d_2)x}\right] \\ &\cdot \left[1 - \frac{\alpha_1 - \beta_1}{\alpha_1 - \overline{\beta}_1} \cdot \frac{b_{23}(\overline{\alpha}_1,t)}{a_{23}(\alpha_1,t)} e^{i(\alpha_1-\overline{\alpha}_1)(d_2-d_3)x}\right] \\ &- \frac{(\alpha_1 - \overline{\alpha}_1)(\beta_1 - \overline{\beta}_1)}{(\alpha_1 - \overline{\beta}_1)^2} \cdot \frac{b_{23}(\overline{\alpha}_1,t)}{a_{23}(\alpha_1,t)} \frac{b_{21}(\beta_1,t)}{a_{21}(\overline{\beta}_1,t)} e^{i(\beta_1-\overline{\beta}_1)(d_1-d_2)x + i(\alpha_1-\overline{\alpha}_1)(d_2-d_3)x}.\end{aligned}$$

By substituting $\frac{\overline{\chi}^{(2)}(x,\overline{\beta}_1;t)}{b_{33}(\overline{\beta}_1)}$, $\frac{\overline{\chi}^{(3)}(x,\overline{\beta}_1;t)}{b_{33}(\overline{\beta}_1)}$ to the first equation of (13.1) and $\frac{\chi^{(2)}(x,\alpha_1;t)}{b_{11}(\alpha_1)}$ to the second equation of (13.1), we deduce

$$\begin{aligned}N_1^{(2)}(x,k;t) = &-\frac{(\overline{\beta}_1 - \beta_1)(\overline{\beta}_1 - \overline{\alpha}_1)}{\overline{\beta}_1 - \alpha_1} \frac{\frac{e^{i\overline{\beta}_1(d_2-d_1)x}}{a_{21}(\overline{\beta}_1,t)}}{k - \overline{\beta}_1} \frac{\gamma_1\gamma_3}{\tilde{D}(x,t)} \\ &\cdot \left[1 - \frac{\alpha_1 - \beta_1}{\overline{\alpha}_1 - \overline{\beta}_1} \cdot \frac{b_{23}(\overline{\alpha}_1,t)}{a_{23}(\alpha_1,t)} e^{i(\alpha_1-\overline{\alpha}_1)(d_2-d_3)x}\right],\end{aligned}$$

$$N_1^{(3)}(x,k;t) = -\frac{(\overline{\beta}_1 - \beta_1)(\overline{\beta}_1 - \overline{\alpha}_1)}{\overline{\beta}_1 - \alpha_1} \frac{\frac{e^{i\overline{\beta}_1(d_2-d_1)x}}{a_{21}(\overline{\beta}_1,t)}}{k - \overline{\beta}_1} \frac{1}{\tilde{D}(x,t)} \frac{\overline{\alpha}_1 - \alpha_1}{\overline{\alpha}_1 - \overline{\beta}_1} b_{23}(\overline{\alpha}_1,t) e^{i\overline{\alpha}_1(d_3-d_2)x},$$

$$\begin{aligned}N_3^{(2)}(x,k;t) = &-\frac{(\alpha_1 - \overline{\alpha}_1)(\alpha_1 - \beta_1)}{\alpha_1 - \overline{\beta}_1} \frac{\frac{e^{i\alpha_1(d_2-d_1)x}}{a_{23}(\alpha_1,t)}}{k - \alpha_1} \frac{\gamma_1\gamma_3}{\tilde{D}(x,t)} \\ &\cdot \left[1 + \frac{\overline{\beta}_1 - \overline{\alpha}_1}{\alpha_1 - \beta_1} \frac{b_{21}(\beta_1,t)}{a_{21}(\overline{\beta}_1,t)} e^{i(\beta_1-\overline{\beta}_1)(d_1-d_2)x}\right].\end{aligned}$$

Note that (6.2) and (6.4) imply $N_{21}(x,t) \sim ik(d_1 - d_2)N_1^{(2)}(x,k;t)$, $N_{31}(x,t) \sim ik(d_1 - d_3)N_1^{(3)}(x,k;t)$, and $N_{23}(x,t) \sim ik(d_3 - d_2)N_3^{(2)}(x,k;t)$ as $k \to \infty$. Thus,

$$\begin{aligned}N_{21}(x,t) = &-\frac{i(d_1 - d_2)(\overline{\beta}_1 - \beta_1)(\overline{\beta}_1 - \overline{\alpha}_1)}{\overline{\beta}_1 - \alpha_1} \frac{e^{i\overline{\beta}_1(d_2-d_1)x}}{a_{21}(\overline{\beta}_1,t)} \\ &\cdot \frac{\gamma_1\gamma_3}{\tilde{D}(x,t)} \left[1 - \frac{\alpha_1 - \beta_1}{\overline{\alpha}_1 - \overline{\beta}_1} \frac{b_{23}(\overline{\alpha}_1,t)}{a_{23}(\alpha_1,t)} e^{i(\alpha_1-\overline{\alpha}_1)(d_2-d_3)x}\right],\end{aligned}$$

$$\begin{aligned}N_{31}(x,t) = &-\frac{i(d_1 - d_3)(\overline{\beta}_1 - \beta_1)(\overline{\beta}_1 - \overline{\alpha}_1)}{\overline{\beta}_1 - \alpha_1} \frac{e^{i\overline{\beta}_1(d_2-d_1)x}}{a_{21}(\overline{\beta}_1,t)} \frac{1}{\tilde{D}(x,t)} \frac{\overline{\alpha}_1 - \alpha_1}{\overline{\alpha}_1 - \overline{\beta}_1} \\ &\cdot b_{23}(\overline{\alpha}_1,t) e^{i\overline{\alpha}_1(d_3-d_2)x},\end{aligned}$$

$$\begin{aligned}N_{23}(x,t) = &-\frac{i(d_3 - d_2)(\alpha_1 - \overline{\alpha}_1)(\alpha_1 - \beta_1)}{\alpha_1 - \overline{\beta}_1} \frac{e^{i\alpha_1(d_2-d_1)x}}{a_{23}(\alpha_1,t)} \frac{\gamma_1\gamma_3}{\tilde{D}(x,t)} \\ &\cdot \left[1 + \frac{\overline{\beta}_1 - \overline{\alpha}_1}{\alpha_1 - \beta_1} \cdot \frac{b_{21}(\beta_1,t)}{a_{21}(\overline{\beta}_1,t)} e^{i(\beta_1-\overline{\beta}_1)(d_1-d_2)x}\right].\end{aligned}$$

Once the symmetry reductions for the potentials are imposed, the specific reflectionless potentials $\{Q_1, Q_2, Q_3\}$ will be obtained; the details are discussed below.

13.2. Classical three-wave system. Letting $J = N = 1$, by (8.2), (9.6)–(9.7), and (11.17)–(11.18), we have

$$N_1(x,k;t) = \omega_1 + \frac{\varepsilon_3 b_{21}^*(\beta_1,t)\frac{\overline{\chi}(x,\beta_1^*;t)}{b_{33}(\beta_1^*,t)}e^{i\beta_1^*(d_2-d_1)x}}{(k-\beta_1^*)a_{11}'(\beta_1^*)}, \tag{13.3}$$

$$N_3(x,k;t) = \omega_3 - \frac{\varepsilon_1 b_{23}^*(\alpha_1^*,t)\frac{\chi(x,\alpha_1;t)}{b_{11}(\alpha_1,t)}e^{i\alpha_1(d_2-d_3)x}}{(k-\alpha_1)a_{33}'(\alpha_1)}. \tag{13.4}$$

Now, letting $\alpha_1 = \xi_1 + i\eta_1$ and $\beta_1 = \overline{\xi}_1 + i\overline{\eta}_1$, where $\eta_1 > 0$ and $\overline{\eta}_1 > 0$, adding time dependence (see section 7), and using (10.12) and (11.21)–(11.24), we obtain

$$N_1(x,\beta_1;t) = \omega_1 - \varepsilon_3\overline{c}_1^*\frac{\overline{\chi}(x,\beta_1^*;t)}{b_{33}(\beta_1^*,t)}e^{(\overline{\eta}_1+i\overline{\xi}_1)(C_1-C_2)(x-C_3t)},$$

$$N_3(x,\alpha_1^*;t) = \omega_3 + \varepsilon_1\overline{c}_2^*\frac{\chi(x,\alpha_1;t)}{b_{11}(\alpha_1,t)}e^{(\eta_1-i\xi_1)(C_2-C_3)(x-C_1t)},$$

$$\begin{aligned}\frac{\overline{\chi}(x,\beta_1^*;t)}{b_{33}(\beta_1^*,t)} &= \varepsilon_1\varepsilon_3\omega_2 + \overline{c}_1 N_1(x,\beta_1;t)e^{(\overline{\eta}_1-i\overline{\xi}_1)(C_1-C_2)(x-C_3t)}\\ &\quad + \frac{2i\eta_1\overline{c}_2 N_3(x,\alpha_1^*;t)e^{(\eta_1+i\xi_1)(C_2-C_3)(x-C_1t)}}{(\overline{\xi}_1-\xi_1)+i(\eta_1-\overline{\eta}_1)},\end{aligned}$$

$$\begin{aligned}\frac{\chi(x,\alpha_1;t)}{b_{11}(\alpha_1,t)} &= -\varepsilon_1\varepsilon_3\omega_2 + \frac{2i\overline{\eta}_1\overline{c}_1 N_1(x,\beta_1;t)e^{(\overline{\eta}_1-i\overline{\xi}_1)(C_1-C_2)(x-C_3t)}}{(\xi_1-\overline{\xi}_1)+i(\eta_1-\overline{\eta}_1)}\\ &\quad - \overline{c}_2 N_3(x,\alpha_1^*;t)e^{(\eta_1+i\xi_1)(C_2-C_3)(x-C_1t)},\end{aligned}$$

where $\overline{c}_1 := b_{21}(\beta_1,0)$, $\overline{c}_2 := b_{23}(\alpha_1^*,0)$, $\frac{b_{33}(\beta_1^*,t)}{a_{21}(\beta_1^*,t)} = -\varepsilon_3 b_{21}^*(\beta_1,t)$ $= -\varepsilon_3\overline{c}_1^* e^{C_3(C_2-C_1)(\overline{\eta}_1+i\overline{\xi}_1)t}$, $\frac{b_{11}(\alpha_1,t)}{a_{23}(\alpha_1,t)} = -\varepsilon_1 b_{23}^*(\alpha_1^*,t) = -\varepsilon_1\overline{c}_2^* e^{C_1(C_3-C_2)(\eta_1-i\xi_1)t}$. The above equations are an algebraic system, from which one can get $N_1(x,\beta_1;t)$, $N_3(x,\alpha_1^*;t)$, $\frac{\overline{\chi}(x,\beta_1^*;t)}{b_{33}(\beta_1^*,t)}$, and $\frac{\chi(x,\alpha_1;t)}{b_{11}(\alpha_1,t)}$; in particular, we find

$$\begin{aligned}\frac{\overline{\chi}^{(2)}(x,\beta_1^*;t)}{b_{33}(\beta_1^*,t)} &= \frac{1}{D(x,t)}\varepsilon_1\varepsilon_3\left(1+\varepsilon_1|\overline{c}_2|^2e^{2\eta_1(C_2-C_3)(x-C_1t)}\right)\\ &\quad - \frac{1}{D(x,t)}\frac{2i\varepsilon_3\eta_1|\overline{c}_2|^2e^{2\eta_1(C_2-C_3)(x-C_1t)}}{(\overline{\xi}_1-\xi_1)+i(\eta_1-\overline{\eta}_1)},\end{aligned}$$

$$\frac{\overline{\chi}^{(3)}(x,\beta_1^*;t)}{b_{33}(\beta_1^*,t)} = \frac{1}{D(x,t)}\cdot\frac{2i\eta_1\overline{c}_2e^{(\eta_1+i\xi_1)(C_2-C_3)(x-C_1t)}}{(\overline{\xi}_1-\xi_1)+i(\eta_1-\overline{\eta}_1)},$$

$$\begin{aligned}\frac{\chi^{(2)}(x,\alpha_1;t)}{b_{11}(\alpha_1,t)} &= -\frac{1}{D(x,t)}\varepsilon_1\varepsilon_3\left(1+\varepsilon_3|\overline{c}_1|^2e^{2\overline{\eta}_1(C_1-C_2)(x-C_3t)}\right)\\ &\quad - \frac{1}{D(x,t)}\frac{2i\varepsilon_1\overline{\eta}_1|\overline{c}_1|^2e^{2\overline{\eta}_1(C_1-C_2)(x-C_3t)}}{(\xi_1-\overline{\xi}_1)+i(\eta_1-\overline{\eta}_1)},\end{aligned}$$

where $D(x,t) := \left(1+\varepsilon_1|\overline{c}_2|^2e^{2\eta_1(C_2-C_3)(x-C_1t)}\right)\cdot\left(1+\varepsilon_3|\overline{c}_1|^2e^{2\overline{\eta}_1(C_1-C_2)(x-C_3t)}\right) + \frac{4\varepsilon_1\varepsilon_3\eta_1\overline{\eta}_1|\overline{c}_1|^2|\overline{c}_2|^2e^{2\overline{\eta}_1(C_1-C_2)(x-C_3t)+2\eta_1(C_2-C_3)(x-C_1t)}}{(\xi_1-\overline{\xi}_1)^2+(\eta_1-\overline{\eta}_1)^2}$. In order to reconstruct the potentials, we apply the asymptotics discussed in section 6. For example, (6.2) and (6.4) yield $N_{21}(x,t) \sim ik(C_2 - C_1)N_1^{(2)}(x,k;t)$, $N_{31}(x,t) \sim ik(C_3 - C_1)N_1^{(3)}(x,k;t)$, and $N_{23}(x,t) \sim ik(C_2-C_3)N_3^{(2)}(x,k;t)$ as $k \to \infty$. Indeed, we are able to find $N_1^{(2)}(x,k;t)$, $N_1^{(3)}(x,k;t)$, and $N_3^{(2)}(x,k;t)$ by substituting $\frac{\overline{\chi}^{(2)}(x,\beta_1^*;t)}{b_{33}(\beta_1^*,t)}$, $\frac{\overline{\chi}^{(3)}(x,\beta_1^*;t)}{b_{33}(\beta_1^*,t)}$ to the first equation of (13.3) and $\frac{\chi^{(2)}(x,\alpha_1;t)}{b_{11}(\alpha_1,t)}$ to the second equation of (13.3). It turns out that

$$N_{21}(x,t) = \frac{1}{D(x,t)} \cdot 2\overline{\eta}_1\varepsilon_3(C_2 - C_1)\overline{c}_1^* e^{(\overline{\eta}_1+i\overline{\xi}_1)(C_1-C_2)(x-C_3t)} \cdot \left[\varepsilon_1\varepsilon_3\left(1+\varepsilon_1|\overline{c}_2|^2 e^{2\eta_1(C_2-C_3)(x-C_1t)}\right) - \frac{2i\varepsilon_3\eta_1|\overline{c}_2|^2 e^{2\eta_1(C_2-C_3)(x-C_1t)}}{(\overline{\xi}_1-\xi_1)+i(\eta_1-\overline{\eta}_1)}\right],$$

$$N_{31}(x,t) = \frac{1}{D(x,t)} 2\overline{\eta}_1\varepsilon_3(C_3 - C_1)\overline{c}_1^* e^{(\overline{\eta}_1+i\overline{\xi}_1)(C_1-C_2)(x-C_3t)} \frac{2i\eta_1\overline{c}_2 e^{(\eta_1+i\xi_1)(C_2-C_3)(x-C_1t)}}{(\overline{\xi}_1-\xi_1)+i(\eta_1-\overline{\eta}_1)},$$

$$N_{23}(x,t) = \frac{1}{D(x,t)} \cdot 2\eta_1\varepsilon_1(C_3 - C_2)\overline{c}_2^* e^{(\eta_1-i\xi_1)(C_2-C_3)(x-C_1t)} \cdot \left[\varepsilon_1\varepsilon_3\left(1+\varepsilon_3|\overline{c}_1|^2 e^{2\overline{\eta}_1(C_1-C_2)(x-C_3t)}\right) + \frac{2i\varepsilon_1\overline{\eta}_1|\overline{c}_1|^2 e^{2\overline{\eta}_1(C_1-C_2)(x-C_3t)}}{(\xi_1-\overline{\xi}_1)+i(\eta_1-\overline{\eta}_1)}\right].$$

From (2.5), it yields that

$$Q_1(x,t) = \frac{\sqrt{(C_2-C_1)(C_3-C_1)}}{D(x,t)} \cdot 2i\eta_1(C_3 - C_2)\overline{c}_2^* e^{(\eta_1-i\xi_1)(C_2-C_3)(x-C_1t)} \cdot \left[\varepsilon_3\left(1+\varepsilon_3|\overline{c}_1|^2 e^{2\overline{\eta}_1(C_1-C_2)(x-C_3t)}\right) + \frac{2i\overline{\eta}_1|\overline{c}_1|^2 e^{2\overline{\eta}_1(C_1-C_2)(x-C_3t)}}{(\xi_1-\overline{\xi}_1)+i(\eta_1-\overline{\eta}_1)}\right], \tag{13.5}$$

$$Q_2(x,t) = \frac{\sqrt{(C_2-C_1)(C_3-C_2)}}{D(x,t)} \cdot 2i\overline{\eta}_1\varepsilon_3(C_3 - C_1)\overline{c}_1^* e^{(\overline{\eta}_1+i\overline{\xi}_1)(C_1-C_2)(x-C_3t)} \cdot \frac{2i\eta_1\overline{c}_2 e^{(\eta_1+i\xi_1)(C_2-C_3)(x-C_1t)}}{(\overline{\xi}_1-\xi_1)+i(\eta_1-\overline{\eta}_1)}, \tag{13.6}$$

$$Q_3(x,t) = \frac{\sqrt{(C_3-C_1)(C_3-C_2)}}{D(x,t)} \cdot 2i\overline{\eta}_1\varepsilon_1\varepsilon_2(C_2 - C_1)\overline{c}_1 e^{(\overline{\eta}_1-i\overline{\xi}_1)(C_1-C_2)(x-C_3t)} \cdot \left[\varepsilon_1\left(1+\varepsilon_1|\overline{c}_2|^2 e^{2\eta_1(C_2-C_3)(x-C_1t)}\right) + \frac{2i\eta_1|\overline{c}_2|^2 e^{2\eta_1(C_2-C_3)(x-C_1t)}}{(\overline{\xi}_1-\xi_1)-i(\eta_1-\overline{\eta}_1)}\right] \tag{13.7}$$

solve the classical three-wave system (2.4). In addition, the solutions are nonsingular whenever $\varepsilon_1 = 1$ and $\varepsilon_3 = 1$.

The minimal data we use for constructing 1-0-1 soliton solutions of the classical three-wave interaction equation include the following four values: the reduced normalization coefficients $\overline{c}_1 := b_{21}(\beta_1, 0)$, $\overline{c}_2 := b_{23}(\alpha_1^*, 0)$ and eigenvalues $\alpha_1 = \xi_1 + i\eta_1$, $\beta_1 = \overline{\xi}_1 + i\overline{\eta}_1$, where $\alpha_1 \neq \beta_1$.

We have the following theorem.

THEOREM 13.1. *Given the minimal data $\overline{c}_1$, $\overline{c}_2$, α_1, and β_1 with $\alpha_1 \neq \beta_1$, then the classical three-wave system admits a unique global soliton solution given by* (13.5)–(13.7) *for the case of $\varepsilon_1 = \varepsilon_3 = 1$, while in the remaining cases, there exists a blow-up time t_0 such that the solutions exist for $t < t_0$.*

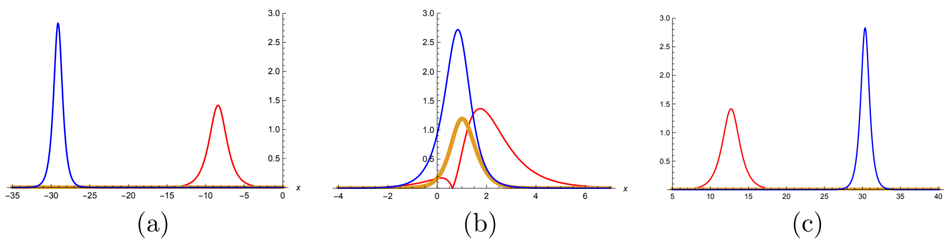


FIG. 1. (a) *The amplitudes (magnitudes) of* $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_1=-10$. (b) *The amplitudes of* $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_2=0$. (c) *The amplitudes of* $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_3=10$. *Here,* $\varepsilon_1=\varepsilon_3=1$, $\varepsilon_2=-1$, $C_1=1$, $C_2=2$, $C_3=3$, $\xi_1=\bar{\xi}_1=0$, $\eta_1=1$, $\bar{\eta}_1=2$, $\bar{c}_1=2$, *and* $\bar{c}_2=5$.

Remark 13.2. For large negative time t, Q_2 is exponentially small; Q_1 and Q_3 are nonzero along characteristics $x-C_1t$ and $x-C_3t$. At a later (interaction) time, Q_2 grows and becomes $O(1)$. Later, Q_2 decays to become exponentially small again, while Q_1 and Q_3 interact and continue traveling with their respective velocities. At this point, Q_3 is to the right of Q_1. Moreover, Q_1 and Q_3 maintain their initial and final amplitudes, but their centers/phases are shifted.

The following figures depict the traveling of $Q_1(x,t)$, $Q_2(x,t)$, and $Q_3(x,t)$. Figure 1(a) illustrates that Q_3 is to the left of Q_1, and Q_2 is small at large negative time t_1; Figure 1(b) describes the three waves interacting at $t_2=0$; Figure 1(c) shows that Q_3 is to the right of Q_1, and Q_2 decays to be small again at large positive time t_3. Note that the initial and final amplitudes of Q_1 and Q_3 do not change; only the phases are shifted.

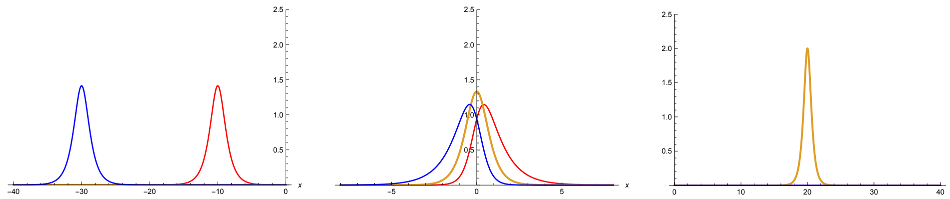
It should be remarked that this 1-0-1 soliton solution is only one type of solution to the classical three-wave system; more general solutions can be obtained according to the discussion below Remark 9.1. A full classification of solutions is outside the scope of this paper.

There is an interesting case for the two eigenvalues satisfying $\alpha_1=\beta_1$, which yields splitting/joining solitons; see also [23]. Using the method described above, one can obtain splitting/joining solitons, for example,

$$Q_1(x,t)=-2i\eta_1\epsilon_1(C_3-C_2)\sqrt{(C_2-C_1)(C_3-C_1)}e^{-i\xi_1(C_3-C_2)(x-C_1t-x_1)}e^{i\gamma_1}/\Delta_1,$$
$$Q_2(x,t)=2i\eta_1\epsilon_2(C_3-C_1)\sqrt{(C_2-C_1)(C_3-C_2)}e^{i\xi_1(C_3-C_1)(x-C_2t-x_2)}e^{i\gamma_2}/\Delta_2,$$
$$Q_3(x,t)=-2i\eta_1\epsilon_3(C_2-C_1)\sqrt{(C_3-C_1)(C_3-C_2)}e^{-i\xi_1(C_2-C_1)(x-C_3t-x_3)}e^{i\gamma_3}/\Delta_3,$$

with $(C_3-C_2)x_1+(C_2-C_1)x_3+(C_1-C_3)x_2=0$ and $\gamma_1+\gamma_2+\gamma_3=0$, where

$$\begin{aligned}\Delta_1 &:= e^{-\eta_1(C_3-C_2)(x-C_1t-x_1)}-\epsilon_2\epsilon_3e^{\eta_1(C_3-C_2)(x-C_1t-x_1)}\\&-\epsilon_1\epsilon_2e^{-\eta_1[(C_3-C_1)(x-C_2t-x_2)+[(C_2-C_1)(x-C_3t-x_3)]]},\\\Delta_2 &:= e^{\eta_1(C_3-C_1)(x-C_2t-x_2)}+\epsilon_1\epsilon_3e^{-\eta_1(C_3-C_1)(x-C_2t-x_2)}\\&-\epsilon_2\epsilon_3e^{\eta_1[(C_2-C_1)(x-C_3t-x_3)-[(C_3-C_2)(x-C_1t-x_1)]]},\\\Delta_3 &:= e^{-\eta_1(C_2-C_1)(x-C_3t-x_3)}-\epsilon_1\epsilon_2e^{\eta_1(C_2-C_1)(x-C_3t-x_3)}\\&+\epsilon_1\epsilon_3e^{\eta_1[(C_3-C_1)(x-C_2t-x_2)+[(C_3-C_2)(x-C_1t-x_1)]]}.\end{aligned}$$



(a) (b) (c)

FIG. 2. (a) *The amplitudes (magnitudes) of* $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_1=-10$. (b) *The amplitudes of* $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_2=0$. (c) *The amplitudes of* $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_3=10$. *Here,* $\varepsilon_1=\varepsilon_3=1$, $\varepsilon_2=-1$, $C_1=1$, $C_2=2$, $C_3=3$, $\xi_1=0$, $\eta_1=1$, $x_1=x_2=0$, *and* $\gamma_1=\gamma_2=0$.

The following figures depict the traveling of $Q_1(x,t)$, $Q_2(x,t)$, and $Q_3(x,t)$ at $t=-10$, $t=0$, and $t=10$, respectively. Figure 2(a) shows that there are 1-0-1 solitons at $t=-10$, and the interaction of all solitons at $t=0$ are displayed in Figure 2(b). Eventually, as t increases further, there is only a 0-1-0 soliton at $t=10$, which is described in Figure 2 (c). In summary, these figures illustrate that the 1-0-1 solitons at large negative time (t $=-10$) merge to a 0-1-0 soliton at large positive time ($t=10$).

13.3. Complex reverse space-time three-wave system. We let $J=\overline{J}=N=\overline{N}=1$, $\widetilde{\alpha}_1=iv_1$, $\widetilde{\beta}_1=iv_2$, $\overline{\widetilde{\alpha}}_1=-i\overline{v}_1$, and $\overline{\widetilde{\beta}}_1=-i\overline{v}_2$, where $v_1>0$, $v_2>0$, $\overline{v}_1>0$, and $\overline{v}_2>0$. For more than one eigenvalue, the detailed analysis was discussed in section 11.3. Then (9.11) and (11.25)–(11.26) give

$$N_1(x,k;t)=\omega_1+\frac{\varepsilon_3 a_{21}^*(-i\overline{v}_2,-t)\frac{\overline{\chi}(x,-i\overline{v}_2;t)}{b_{33}(-i\overline{v}_2,t)}e^{\overline{v}_2(C_1-C_2)x}}{(k+i\overline{v}_2)a_{11}'(-i\overline{v}_2)}, \tag{13.8}$$

$$N_3(x,k;t)=\omega_3-\frac{\varepsilon_1 a_{23}^*(-i\overline{v}_2,-t)\frac{\chi(x,iv_1;t)}{b_{11}(iv_1,t)}e^{v_1(C_2-C_3)x}}{(k-iv_1)a_{33}'(iv_1)}. \tag{13.9}$$

By adding time dependence and combining (10.18), (10.19), (10.20), (10.21) and (11.29)–(11.32), we have

$$N_1(x,iv_2;t)=\omega_1-\varepsilon_3\overline{c}_3^*\frac{\overline{\chi}(x,-i\overline{v}_2;t)}{b_{33}(-i\overline{v}_2,t)}e^{\overline{v}_2(C_1-C_2)(x-C_3t)},$$

$$N_3(x,-i\overline{v}_1;t)=\omega_3+\varepsilon_1\overline{c}_4^*\frac{\chi(x,iv_1;t)}{b_{11}(iv_1,t)}e^{v_1(C_2-C_3)(x-C_1t)},$$

$$\begin{aligned}\frac{\overline{\chi}(x,-i\overline{v}_2;t)}{b_{33}(-i\overline{v}_2,t)}&=\varepsilon_1\varepsilon_3\omega_2+\overline{c}_1N_1(x,iv_2;t)e^{v_2(C_1-C_2)(x-C_3t)}\\&\quad+\frac{\overline{v}_1+v_1}{\overline{v}_1-\overline{v}_2}\overline{c}_2N_3(x,-i\overline{v}_1;t)e^{\overline{v}_1(C_2-C_3)(x-C_1t)},\\\frac{\chi(x,iv_1;t)}{b_{11}(iv_1,t)}&=-\varepsilon_1\varepsilon_3\omega_2-\frac{v_2+\overline{v}_2}{v_2-v_1}\overline{c}_1N_1(x,iv_2;t)e^{v_2(C_1-C_2)(x-C_3t)}\\&\quad-\overline{c}_2N_3(x,-i\overline{v}_1;t)e^{\overline{v}_1(C_2-C_3)(x-C_1t)},\end{aligned}$$

where $\overline{c}_1:=b_{21}(iv_2,0)$, $\overline{c}_2:=b_{23}(-i\overline{v}_1,0)$, $\overline{c}_3:=a_{21}(-i\overline{v}_2,0)$, $\overline{c}_4:=a_{23}(iv_1,0)$, $\frac{b_{33}(-i\overline{v}_2,t)}{a_{21}(-i\overline{v}_2,t)}=-\varepsilon_3a_{21}^*(-i\overline{v}_2,-t)=-\varepsilon_3\overline{c}_3^*e^{-C_3(C_1-C_2)\overline{v}_2t}$, $\frac{b_{11}(iv_1,t)}{a_{23}(iv_1,t)}=-\varepsilon_1a_{23}^*(-i\overline{v}_2,-t)=-\varepsilon_1\overline{c}_4^*e^{-C_1(C_2-C_3)v_1t}$. The above algebraic system yields

$$\frac{\overline{\chi}^{(2)}(x,-i\overline{v}_2;t)}{b_{33}(-i\overline{v}_2,t)} = \frac{1}{D_1(x,t)}\left\{\varepsilon_1\varepsilon_3 + \frac{\overline{v}_2+v_1}{\overline{v}_2-\overline{v}_1}\varepsilon_3\overline{c}_2\overline{c}_4^* e^{(C_2-C_3)(v_1+\overline{v}_1)(x-C_1t)}\right\},$$

$$\frac{\overline{\chi}^{(3)}(x,-i\overline{v}_2;t)}{b_{33}(-i\overline{v}_2,t)} = \frac{1}{D_1(x,t)}\cdot\frac{\overline{v}_1+v_1}{\overline{v}_1-\overline{v}_2}\overline{c}_2 e^{\overline{v}_1(C_2-C_3)(x-C_1t)},$$

$$\frac{\chi^{(2)}(x,iv_1;t)}{b_{11}(iv_1,t)} = \frac{1}{D_1(x,t)}\left\{-\varepsilon_1\varepsilon_3 + \frac{\overline{v}_2+v_1}{v_2-v_1}\varepsilon_1\overline{c}_1\overline{c}_3^* e^{(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}\right\},$$

where $D_1(x,t) := \left[1+\varepsilon_3\overline{c}_1\overline{c}_3^* e^{(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}\right]\cdot\left[1+\varepsilon_1\overline{c}_2\overline{c}_4^* e^{(C_2-C_3)(v_1+\overline{v}_1)(x-C_1t)}\right]+ \frac{\overline{v}_1+v_1}{\overline{v}_2-\overline{v}_1}\cdot\frac{\overline{v}_2+v_2}{v_2-v_1}\cdot\varepsilon_1\varepsilon_3\overline{c}_1\overline{c}_2\overline{c}_3^*\overline{c}_4^*\cdot e^{(C_2-C_3)(v_1+\overline{v}_1)(x-C_1t)+(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)]}$.

By substituting $\frac{\overline{\chi}^{(2)}(x,-i\overline{v}_2;t)}{b_{33}(-i\overline{v}_2,t)}$, $\frac{\overline{\chi}^{(3)}(x,-i\overline{v}_2;t)}{b_{33}(-i\overline{v}_2,t)}$ to (13.8) and $\frac{\chi^{(2)}(x,iv_1;t)}{b_{11}(iv_1,t)}$ to (13.9), we obtain

$$\begin{aligned} N_1^{(2)}(x,k;t) = &-\frac{1}{D_1(x,t)}\frac{i(v_2+\overline{v}_2)\overline{c}_3^* e^{\overline{v}_2(C_1-C_2)(x-C_3t)}}{k+i\overline{v}_2} \\ &\left\{\varepsilon_1 + \frac{\overline{v}_2+v_1}{\overline{v}_2-\overline{v}_1}\overline{c}_2\overline{c}_4^* e^{(C_2-C_3)(v_1+\overline{v}_1)(x-C_1t)}\right\},\end{aligned}$$

$$N_1^{(3)}(x,k;t) = -\frac{1}{D_1(x,t)}\frac{\overline{v}_1+v_1}{\overline{v}_1-\overline{v}_2}\frac{i(v_2+\overline{v}_2)\varepsilon_3\overline{c}_2\overline{c}_3^* e^{\overline{v}_1(C_2-C_3)(x-C_1t)+\overline{v}_2(C_1-C_2)(x-C_3t)}}{k+i\overline{v}_2},$$

$$\begin{aligned} N_3^{(2)}(x,k;t) = &-\frac{1}{D_1(x,t)}\frac{i(v_1+\overline{v}_1)\overline{c}_4^* e^{v_1(C_2-C_3)(x-C_1t)}}{k-iv_1} \\ &\left\{-\varepsilon_3 + \frac{\overline{v}_2+v_1}{v_2-v_1}\overline{c}_1\overline{c}_3^* e^{(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}\right\}.\end{aligned}$$

As we mentioned in the classical case, $N_{21}(x,t)\sim ik(C_2-C_1)N_1^{(2)}(x,k;t)$, $N_{31}(x,t)\sim ik(C_3-C_1)N_1^{(3)}(x,k;t)$, and $N_{23}(x,t)\sim ik(C_2-C_3)N_3^{(2)}(x,k;t)$ as $k\to\infty$. Thus,

$$\begin{aligned} &N_{21}(x,t) \\ &= \frac{C_2-C_1}{D_1(x,t)}\cdot(v_2+\overline{v}_2)\overline{c}_3^* e^{\overline{v}_2(C_1-C_2)(x-C_3t)}\cdot\left\{\varepsilon_1+\frac{\overline{v}_2+v_1}{\overline{v}_2-\overline{v}_1}\overline{c}_2\overline{c}_4^* e^{(C_2-C_3)(v_1+\overline{v}_1)(x-C_1t)}\right\},\end{aligned}$$

$$N_{31}(x,t) = \frac{C_3-C_1}{D_1(x,t)}\cdot\frac{\overline{v}_1+v_1}{\overline{v}_1-\overline{v}_2}\cdot(v_2+\overline{v}_2)\varepsilon_3\overline{c}_2\overline{c}_3^* e^{\overline{v}_1(C_2-C_3)(x-C_1t)+\overline{v}_2(C_1-C_2)(x-C_3t)},$$

$$\begin{aligned} &N_{23}(x,t) \\ &= \frac{C_2-C_3}{D_1(x,t)}\cdot(v_1+\overline{v}_1)\overline{c}_4^* e^{v_1(C_2-C_3)(x-C_1t)}\cdot\left\{-\varepsilon_3+\frac{\overline{v}_2+v_1}{v_2-v_1}\overline{c}_1\overline{c}_3^* e^{(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}\right\}.\end{aligned}$$

By (2.8), one has

$$\begin{aligned} Q_1(x,t) &= i\sqrt{(C_2-C_1)(C_3-C_1)}N_{23}(x,t) = i\sqrt{(C_2-C_1)(C_3-C_1)}\cdot\frac{C_2-C_3}{D_1(x,t)} \\ &\cdot(v_1+\overline{v}_1)\overline{c}_4^* e^{v_1(C_2-C_3)(x-C_1t)}\cdot\left\{-\varepsilon_3+\frac{\overline{v}_2+v_1}{v_2-v_1}\overline{c}_1\overline{c}_3^* e^{(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}\right\},\end{aligned}$$

$$Q_2(x,t) = i\sqrt{(C_2-C_1)(C_3-C_2)}N_{31}(x,t) = i\sqrt{(C_2-C_1)(C_3-C_2)} \cdot \frac{C_3-C_1}{D_1(x,t)} \cdot \frac{\overline{v}_1+v_1}{\overline{v}_1-\overline{v}_2} \cdot (v_2+\overline{v}_2)\varepsilon_3\overline{c}_2\overline{c}_3^* e^{\overline{v}_1(C_2-C_3)(x-C_1t)+\overline{v}_2(C_1-C_2)(x-C_3t)},$$

$$Q_3(x,t) = i\sqrt{(C_3-C_1)(C_3-C_2)}N_{12}(x,t) = i\sqrt{(C_3-C_1)(C_3-C_2)} \cdot \varepsilon_1\varepsilon_2 \cdot N_{21}^*(-x,-t) = i\sqrt{(C_3-C_1)(C_3-C_2)} \cdot \varepsilon_1\varepsilon_2 \cdot \frac{C_2-C_1}{D_1^*(-x,-t)} \cdot (v_2+\overline{v}_2)\overline{c}_3 e^{\overline{v}_2(C_2-C_1)(x-C_3t)} \cdot \left\{\varepsilon_1 + \frac{\overline{v}_2+v_1}{\overline{v}_2-\overline{v}_1}\overline{c}_2^*\overline{c}_4 e^{(C_3-C_2)(v_1+\overline{v}_1)(x-C_1t)}\right\}.$$

In addition, (9.11) shows that $|\overline{c}_1|^2 = -\varepsilon_3\frac{v_2-v_1}{\overline{v}_1+v_2}$, $|\overline{c}_2|^2 = \varepsilon_1\frac{\overline{v}_2-\overline{v}_1}{\overline{v}_1+v_2}$, $|\overline{c}_3|^2 = -\varepsilon_3\frac{\overline{v}_2-\overline{v}_1}{v_1+\overline{v}_2}$, $|\overline{c}_4|^2 = \varepsilon_1\frac{v_2-v_1}{v_1+\overline{v}_2}$. Hence, if $\varepsilon_1 = 1$, then $v_2 > v_1$, $\overline{v}_2 > \overline{v}_1$, and $\varepsilon_3 = -1$. Similarly, if $\varepsilon_1 = -1$, then $v_2 < v_1$, $\overline{v}_2 < \overline{v}_1$, and $\varepsilon_3 = 1$. It implies that ε_1 and ε_3 are of opposite signs, i.e., $\varepsilon_1\varepsilon_3 = -1$. On one hand, when $\varepsilon_1 = 1$ and $\varepsilon_3 = -1$, we obtain $|\overline{c}_1| = \sqrt{\frac{v_2-v_1}{\overline{v}_1+v_2}}$, $|\overline{c}_2| = \sqrt{\frac{\overline{v}_2-\overline{v}_1}{\overline{v}_1+v_2}}$, $|\overline{c}_3| = \sqrt{\frac{\overline{v}_2-\overline{v}_1}{v_1+\overline{v}_2}}$, $|\overline{c}_4| = \sqrt{\frac{v_2-v_1}{v_1+\overline{v}_2}}$. Thus, one can write $\overline{c}_1 = \sqrt{\frac{v_2-v_1}{\overline{v}_1+v_2}}e^{i\theta_1}$, $\overline{c}_2 = \sqrt{\frac{\overline{v}_2-\overline{v}_1}{\overline{v}_1+v_2}}e^{i\theta_2}$, $\overline{c}_3 = \sqrt{\frac{\overline{v}_2-\overline{v}_1}{v_1+\overline{v}_2}}e^{i\theta_3}$, $\overline{c}_4 = \sqrt{\frac{v_2-v_1}{v_1+\overline{v}_2}}e^{i\theta_4}$, where all θ_j are real, $j = 1,2,3,4$. Note that $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$ (see section 2.2), so one has $\varepsilon_2 = -1$. Therefore,

(13.10)
$$Q_1(x,t) = \frac{i}{D_1(x,t)} \cdot (C_2-C_3)\sqrt{(C_2-C_1)(C_3-C_1)}(v_1+\overline{v}_1)\sqrt{\frac{v_2-v_1}{v_1+\overline{v}_2}} \cdot e^{-i\theta_4+v_1(C_2-C_3)(x-C_1t)} \cdot \left\{1 + \sqrt{\frac{(\overline{v}_2+v_1)(\overline{v}_2-\overline{v}_1)}{(v_2-v_1)(\overline{v}_1+v_2)}}e^{i(\theta_1-\theta_3)+(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}\right\},$$

(13.11)
$$Q_2(x,t) = \frac{i}{D_1(x,t)} \cdot \frac{(\overline{v}_1+v_1)(v_2+\overline{v}_2)}{\sqrt{(\overline{v}_1+v_2)(v_1+\overline{v}_2)}} \cdot (C_3-C_1)\sqrt{(C_2-C_1)(C_3-C_2)}e^{i(\theta_2-\theta_3)+\overline{v}_1(C_2-C_3)(x-C_1t)+\overline{v}_2(C_1-C_2)(x-C_3t)},$$

(13.12)
$$Q_3(x,t) = -\frac{i}{D_1^*(-x,-t)} \cdot (C_2-C_1)\sqrt{(C_3-C_1)(C_3-C_2)}(v_2+\overline{v}_2)\sqrt{\frac{\overline{v}_2-\overline{v}_1}{v_1+\overline{v}_2}} \cdot e^{i\theta_3+\overline{v}_2(C_2-C_1)(x-C_3t)} \cdot \left\{1 + \sqrt{\frac{(\overline{v}_2+v_1)(v_2-v_1)}{(\overline{v}_2-\overline{v}_1)(\overline{v}_1+v_2)}}e^{i(\theta_4-\theta_2)+(C_3-C_2)(v_1+\overline{v}_1)(x-C_1t)}\right\}$$

are solutions to the complex reverse space-time system (2.7), where

$$D_1(x,t) = \left[1 - \sqrt{\frac{(v_2-v_1)(\overline{v}_2-\overline{v}_1)}{(\overline{v}_1+v_2)(v_1+\overline{v}_2)}}e^{i(\theta_1-\theta_3)+(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}\right] \cdot \left[1 + \sqrt{\frac{(v_2-v_1)(\overline{v}_2-\overline{v}_1)}{(\overline{v}_1+v_2)(v_1+\overline{v}_2)}}e^{i(\theta_2-\theta_4)+(C_2-C_3)(v_1+\overline{v}_1)(x-C_1t)}\right] - \frac{\overline{v}_1+v_1}{\overline{v}_1+v_2} \cdot \frac{\overline{v}_2+v_2}{v_1+\overline{v}_2} \cdot e^{i(\theta_1+\theta_2-\theta_3-\theta_4)+(C_2-C_3)(v_1+\overline{v}_1)(x-C_1t)+(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}$$

and $v_2 > v_1$, $\overline{v}_2 > \overline{v}_1$.

Remark 13.3. In general, there exist singular points for Q_j, $j=1,2,3$; that is, in general, $D_1(x,t)$ is not zero-free. However, we can choose $\theta_1-\theta_3=\pi+2k_1\pi$ and $\theta_2-\theta_4=2k_2\pi$ such that $D_1(x,t)>0$, where $k_1,k_2\in\mathbb{Z}$. Thus, for these special parameters, Q_j are regular for all x,t.

On the other hand, when $\varepsilon_1=-1$ and $\varepsilon_3=1$, the solution is given by

(13.13)
$$Q_1(x,t)=-\frac{i}{D_1(x,t)}\cdot(C_2-C_3)\sqrt{(C_2-C_1)(C_3-C_1)}(v_1+\overline{v}_1)\sqrt{\frac{v_1-v_2}{v_1+\overline{v}_2}}$$
$$\cdot\, e^{-i\theta_4+v_1(C_2-C_3)(x-C_1t)}\cdot\left\{1+\sqrt{\frac{(\overline{v}_2+v_1)(\overline{v}_1-\overline{v}_2)}{(v_1-v_2)(\overline{v}_1+v_2)}}e^{i(\theta_1-\theta_3)+(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}\right\},$$

(13.14)
$$Q_2(x,t)=\frac{i}{D_1(x,t)}\cdot\frac{(\overline{v}_1+v_1)(v_2+\overline{v}_2)}{\sqrt{(\overline{v}_1+v_2)(v_1+\overline{v}_2)}}$$
$$\cdot\,(C_3-C_1)\sqrt{(C_2-C_1)(C_3-C_2)}e^{i(\theta_2-\theta_3)+\overline{v}_1(C_2-C_3)(x-C_1t)+\overline{v}_2(C_1-C_2)(x-C_3t)},$$

(13.15)
$$Q_3(x,t)=-\frac{i}{D_1^*(-x,-t)}\cdot(C_2-C_1)\sqrt{(C_3-C_1)(C_3-C_2)}(v_2+\overline{v}_2)\sqrt{\frac{\overline{v}_1-\overline{v}_2}{v_1+\overline{v}_2}}$$
$$\cdot\, e^{i\theta_3+\overline{v}_2(C_2-C_1)(x-C_3t)}\cdot\left\{1+\sqrt{\frac{(\overline{v}_2+v_1)(v_1-v_2)}{(\overline{v}_1-\overline{v}_2)(\overline{v}_1+v_2)}}e^{i(\theta_4-\theta_2)+(C_3-C_2)(v_1+\overline{v}_1)(x-C_1t)}\right\},$$

where

$$\begin{aligned}
&D_1(x,t)\\
&=\left[1+\sqrt{\frac{(v_1-v_2)(\overline{v}_1-\overline{v}_2)}{(\overline{v}_1+v_2)(v_1+\overline{v}_2)}}e^{i(\theta_1-\theta_3)+(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)}\right]\\
&\cdot\left[1-\sqrt{\frac{(v_1-v_2)(\overline{v}_1-\overline{v}_2)}{(\overline{v}_1+v_2)(v_1+\overline{v}_2)}}e^{i(\theta_2-\theta_4)+(C_2-C_3)(v_1+\overline{v}_1)(x-C_1t)}\right]\\
&-\frac{\overline{v}_1+v_1}{\overline{v}_1+v_2}\cdot\frac{\overline{v}_2+v_2}{v_1+\overline{v}_2}\cdot e^{i(\theta_1+\theta_2-\theta_3-\theta_4)+(C_2-C_3)(v_1+\overline{v}_1)(x-C_1t)+(C_1-C_2)(v_2+\overline{v}_2)(x-C_3t)},
\end{aligned}$$

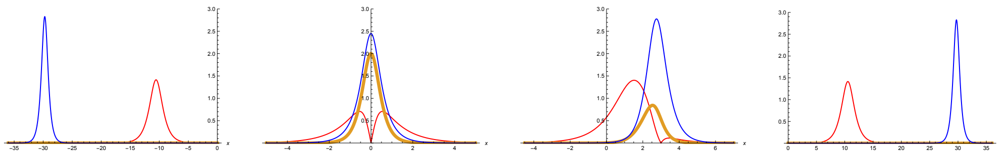
$\overline{c}_1=\sqrt{\frac{v_1-v_2}{\overline{v}_1+v_2}}e^{i\theta_1}$, $\overline{c}_2=\sqrt{\frac{\overline{v}_1-\overline{v}_2}{\overline{v}_1+v_2}}e^{i\theta_2}$, $\overline{c}_3=\sqrt{\frac{\overline{v}_1-\overline{v}_2}{v_1+\overline{v}_2}}e^{i\theta_3}$, $\overline{c}_4=\sqrt{\frac{v_1-v_2}{v_1+\overline{v}_2}}e^{i\theta_4}$, $v_1>v_2$, and $\overline{v}_1>\overline{v}_2$.

Remark 13.4. Similarly, Q_j do not blow up if $\theta_1-\theta_3=2k_1\pi$ and $\theta_2-\theta_4=\pi+2k_2\pi$, where $j=1,2,3$ and $k_1,k_2\in\mathbb{Z}$.

The data needed for recovering 1-0-1 soliton solutions via the IST include the eigenvalues v_1, v_2, $\overline{v}_1$, $\overline{v}_2$ and phases θ_1, θ_2, θ_3, θ_4. In fact, (9.11) implies that the reduced normalization coefficients $\overline{c}_j$ depend on eigenvalues, where $j=1,2,3,4$. Specifically, the moduli of $\overline{c}_j$ are uniquely determined by eigenvalues; however, the phases θ_j are introduced by polar exponential.

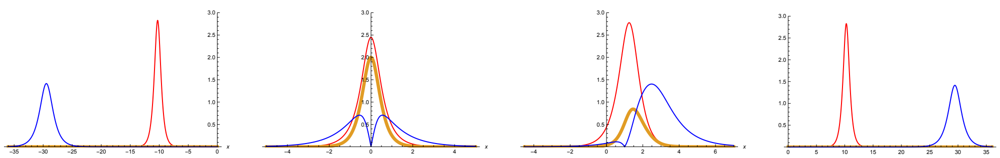
We state these results as a theorem.

THEOREM 13.5. *Given the minimal data v_1, v_2, $\overline{v}_1$, $\overline{v}_2$ and θ_1, θ_2, θ_3, θ_4, then the complex reverse space-time three-wave system admits a unique soliton solution*



(a) (b) (c) (d)

FIG. 3. (a) *The amplitudes of $Q_1(x,t)$ (red), $Q_2(x,t)$ (yellow), and $Q_3(x,t)$ (blue) at $t_1=-10$.* (b) *The amplitudes of $Q_1(x,t)$ (red), $Q_2(x,t)$ (yellow), and $Q_3(x,t)$ (blue) at $t_2=0$.* (c) *The amplitudes of $Q_1(x,t)$ (red), $Q_2(x,t)$ (yellow), and $Q_3(x,t)$ (blue) at $t_3=1$.* (d) *The amplitudes of $Q_1(x,t)$ (red), $Q_2(x,t)$ (yellow), and $Q_3(x,t)$ (blue) at $t_4=10$. Here, $\varepsilon_1=1$, $\varepsilon_2=\varepsilon_3=-1$, $C_1=1$, $C_2=2$, $C_3=3$, $\theta_1=\pi$, $\theta_2=\theta_3=\theta_4=0$, $v_1=\overline{v}_1=1$, $v_2=\overline{v}_2=2$.*



(a) (b) (c) (d)

FIG. 4. (a) *The amplitudes of $Q_1(x,t)$ (red), $Q_2(x,t)$ (yellow), and $Q_3(x,t)$ (blue) at $t_1=-10$.* (b) *The amplitudes of $Q_1(x,t)$ (red), $Q_2(x,t)$ (yellow), and $Q_3(x,t)$ (blue) at $t_2=0$.* (c) *The magnitudes of $Q_1(x,t)$ (red), $Q_2(x,t)$ (yellow), and $Q_3(x,t)$ (blue) at $t_3=1$.* (d) *The magnitudes of $Q_1(x,t)$ (red), $Q_2(x,t)$ (yellow), and $Q_3(x,t)$ (blue) at $t_4=10$. Here, $\varepsilon_1=\varepsilon_2=-1$, $\varepsilon_3=1$, $C_1=1$, $C_2=2$, $C_3=3$, $\theta_1=\theta_2=\theta_3=0$, $\theta_4=\pi$, $v_1=\overline{v}_1=2$, $v_2=\overline{v}_2=1$.*

$\{Q_1,Q_2,Q_3\}$ *given by* (13.10)–(13.12) *and* (13.13)–13.15*) until possibly blow up. In particular, if $\theta_1-\theta_3=2k_1\pi$ and $\theta_2-\theta_4=\pi+2k_2\pi$, then $\{Q_1,Q_2,Q_3\}$ is a global solution.*

The following figures illustrate the traveling of Q_1, Q_2, and Q_3. Specifically, Figure 3 is for the case of $\varepsilon_1=1$ and $\varepsilon_3=-1$. From Figure 3, we see Q_3 is to the left of Q_1, the amplitude (magnitude) of Q_1 is smaller than Q_3, and Q_2 is small at $t_1=-10$. At a later time, e.g., at $t_2=0$ and $t_3=1$, Q_2 grows and interacts with Q_1 and Q_3. In particular, at $t_2=0$, the amplitudes of these three waves are symmetric about $x=0$. Figure 3(d) indicates that Q_2 decays to be small again, and Q_3 is to the right of Q_1 at $t_4=10$. In this process, the initial and final amplitudes of Q_1 and Q_3 are kept, while the positions are shifted. A different case is given by Figure 4 below, where $\varepsilon_1=-1$ and $\varepsilon_3=1$. Compared with Figure 3, now the amplitude of Q_1 is bigger than Q_3.

13.4. Real reverse space-time three-wave system. We let $J=\overline{J}=N=\overline{N}=1$, $\widetilde{\alpha}_1=i\eta_1$, $\widetilde{\beta}_1=i\overline{\eta}_1$, $\overline{\widetilde{\alpha}}_1=-i\eta_2$ and $\overline{\widetilde{\beta}}_1=-i\overline{\eta}_2$, where $\eta_1>0$, $\eta_2>0$, $\overline{\eta}_1>0$, and $\overline{\eta}_2>0$. By a similar analysis for the complex RST three-wave system, one obtains

$$N_{21}(x,t)=\frac{(C_2-C_1)(\overline{\eta}_2+\overline{\eta}_1)\overline{c}_3e^{\overline{\eta}_2(C_1-C_2)(x-C_3t)}}{D_2(x,t)}\left\{\varepsilon_1+\frac{\overline{\eta}_2+\eta_1}{\overline{\eta}_2-\eta_2}\cdot\overline{c}_2\overline{c}_4e^{(C_2-C_3)(x-C_1t)(\eta_1+\eta_2)}\right\},$$

$$N_{31}(x,t)=-\frac{C_3-C_1}{D_2(x,t)}\cdot(\overline{\eta}_2+\overline{\eta}_1)\varepsilon_3\overline{c}_3e^{\overline{\eta}_2(C_1-C_2)(x-C_3t)}\cdot\frac{\eta_1+\eta_2}{\overline{\eta}_2-\eta_2}\cdot\overline{c}_2\cdot e^{\eta_2(C_2-C_3)(x-C_1t)},$$

$$N_{23}(x,t) = -\frac{(C_2-C_3)(\eta_1+\eta_2)\overline{c}_4 e^{\eta_1(C_2-C_3)(x-C_1t)}}{D_2(x,t)}$$
$$\left\{\varepsilon_3 + \frac{\overline{\eta}_2+\eta_1}{\eta_1-\overline{\eta}_1}\cdot \overline{c}_1\overline{c}_3 e^{(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}\right\},$$

where $\overline{c}_1 := b_{21}(\tilde{\beta}_1, 0)$, $\overline{c}_2 := b_{23}(\overline{\tilde{\alpha}}_1, 0)$, $\overline{c}_3 := a_{21}(\overline{\tilde{\beta}}_1, 0)$, $\overline{c}_4 := a_{23}(\tilde{\alpha}_1, 0)$, $\frac{b_{33}(\overline{\tilde{\beta}}_1;t)}{a_{21}(\overline{\tilde{\beta}}_1,t)} = -\varepsilon_3 a_{21}(\overline{\tilde{\beta}}_1,-t) = -\varepsilon_3\overline{c}_3 e^{\overline{\eta}_2 C_3(C_2-C_1)t}$, $\frac{b_{11}(\tilde{\alpha}_1,t)}{a_{23}(\tilde{\alpha}_1,t)} = -\varepsilon_1 a_{23}(\tilde{\alpha}_1,-t) = -\varepsilon_1\overline{c}_4 e^{\eta_1 C_1(C_3-C_2)t}$, $D_2(x,t) := \left[1+\varepsilon_3\overline{c}_1\overline{c}_3 e^{(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}\right]\cdot\left[1+\varepsilon_1\overline{c}_2\overline{c}_4 e^{(C_2-C_3)(x-C_1t)(\eta_1+\eta_2)}\right] + \frac{\eta_1+\eta_2}{\eta_2-\overline{\eta}_2}\cdot\frac{\overline{\eta}_2+\overline{\eta}_1}{\eta_1-\overline{\eta}_1}\cdot\varepsilon_1\varepsilon_3\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4\cdot e^{(C_2-C_3)(x-C_1t)(\eta_1+\eta_2)+(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}$.

From (2.12), one has

$$Q_1(x,t) = \frac{(C_2-C_3)\sqrt{(C_2-C_1)(C_3-C_1)}}{D_2(x,t)}(\eta_1-\eta_2)\overline{c}_4 e^{\eta_1(C_2-C_3)(x-C_1t)}$$
$$\cdot\left\{\varepsilon_3+\frac{\overline{\eta}_2+\eta_1}{\eta_1-\overline{\eta}_1}\cdot\overline{c}_1\overline{c}_3 e^{(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}\right\},$$
$$Q_2(x,t) = \frac{(C_3-C_1)\sqrt{(C_2-C_1)(C_3-C_2)}}{D_2(x,t)}(\overline{\eta}_2+\overline{\eta}_1)\varepsilon_3\overline{c}_2\overline{c}_3 e^{\overline{\eta}_2(C_1-C_2)(x-C_3t)}$$
$$\frac{\eta_1+\eta_2}{\overline{\eta}_2-\eta_2}e^{\eta_2(C_2-C_3)(x-C_1t)},$$
$$Q_3(x,t) = -\frac{(C_2-C_1)\sqrt{(C_3-C_1)(C_3-C_2)}}{D_2(-x,-t)}\varepsilon_1\varepsilon_2\cdot(\overline{\eta}_2+\overline{\eta}_1)\overline{c}_3$$
$$\cdot e^{\overline{\eta}_2(C_2-C_1)(x-C_3t)}\cdot\left\{\varepsilon_1+\frac{\overline{\eta}_2+\eta_1}{\overline{\eta}_2-\eta_2}\cdot\overline{c}_2\overline{c}_4 e^{(C_3-C_2)(x-C_1t)(\eta_1+\eta_2)}\right\}.$$

Besides, (9.12) and (9.13) imply $\overline{c}_1^2 = -\varepsilon_3\frac{\overline{\eta}_1-\eta_1}{\overline{\eta}_1+\eta_2}$, $\overline{c}_2^2 = \varepsilon_1\frac{\overline{\eta}_2-\eta_2}{\eta_2+\overline{\eta}_1}$, $\overline{c}_3^2 = -\varepsilon_3\frac{\overline{\eta}_2-\eta_2}{\overline{\eta}_2+\eta_1}$, $\overline{c}_4^2 = \varepsilon_1\frac{\overline{\eta}_1-\eta_1}{\eta_1+\overline{\eta}_2}$. Recall from section 8.3 that $\overline{c}_j$ are real, $j=1,2,3,4$. Thus, $\overline{c}_j^2>0$ and $\varepsilon_1\varepsilon_3=-1$. When $\varepsilon_1=1$ and $\varepsilon_3=-1$, it implies $\overline{\eta}_1>\eta_1$ and $\overline{\eta}_2>\eta_2$. As a result, $\overline{c}_1 = \delta_1\sqrt{\frac{\overline{\eta}_1-\eta_1}{\overline{\eta}_1+\eta_2}}$, $\overline{c}_2 = \delta_2\sqrt{\frac{\overline{\eta}_2-\eta_2}{\eta_2+\overline{\eta}_1}}$, $\overline{c}_3 = \delta_3\sqrt{\frac{\overline{\eta}_2-\eta_2}{\overline{\eta}_2+\eta_1}}$, $\overline{c}_4 = \delta_4\sqrt{\frac{\overline{\eta}_1-\eta_1}{\eta_1+\overline{\eta}_2}}$, where $\delta_j=\pm1$, $j=1,2,3,4$. Noting that $\varepsilon_1\varepsilon_2\varepsilon_3=1$ (see section 2.3), one has $\varepsilon_2=-1$. Therefore,

(13.16)
$$Q_1(x,t) = -\frac{(C_2-C_3)\sqrt{(C_2-C_1)(C_3-C_1)}}{D_2(x,t)}\cdot(\eta_1+\eta_2)\cdot\delta_4\sqrt{\frac{\overline{\eta}_1-\eta_1}{\eta_1+\overline{\eta}_2}}e^{\eta_1(C_2-C_3)(x-C_1t)}$$
$$\cdot\left\{1+\delta_1\delta_3\sqrt{\frac{(\overline{\eta}_2+\eta_1)(\overline{\eta}_2-\eta_2)}{(\overline{\eta}_1+\eta_2)(\overline{\eta}_1-\eta_1)}}e^{(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}\right\},$$

(13.17)
$$Q_2(x,t) = -\frac{(C_3-C_1)\sqrt{(C_2-C_1)(C_3-C_2)}}{D_2(x,t)}\frac{(\eta_1+\eta_2)(\overline{\eta}_1+\overline{\eta}_2)}{\sqrt{(\eta_1+\overline{\eta}_2)(\overline{\eta}_1+\eta_2)}}$$
$$\cdot\delta_2\delta_3 e^{\overline{\eta}_2(C_1-C_2)(x-C_3t)+\eta_2(C_2-C_3)(x-C_1t)},$$

(13.18)
$$Q_3(x,t) = \frac{(C_2-C_1)\sqrt{(C_3-C_1)(C_3-C_2)}}{D_2(-x,-t)}\cdot(\overline{\eta}_2+\overline{\eta}_1)\delta_3\sqrt{\frac{\overline{\eta}_2-\eta_2}{\overline{\eta}_2+\eta_1}}e^{\overline{\eta}_2(C_2-C_1)(x-C_3t)}$$
$$\cdot\left\{1+\delta_2\delta_4\sqrt{\frac{(\eta_1+\overline{\eta}_2)(\overline{\eta}_1-\eta_1)}{(\overline{\eta}_1+\eta_2)(\overline{\eta}_2-\eta_2)}}e^{(C_3-C_2)(x-C_1t)(\eta_1+\eta_2)}\right\}$$

solve the real reverse space-time three-wave system (2.11), where $D_2(x,t)$ is shown as follows:

$$D_2(x,t) = \left[1 - \delta_1\delta_3\sqrt{\frac{(\overline{\eta}_1-\eta_1)(\overline{\eta}_2-\eta_2)}{(\overline{\eta}_1+\eta_2)(\overline{\eta}_2+\eta_1)}}e^{(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}\right] \cdot \left[1 + \delta_2\delta_4\sqrt{\frac{(\overline{\eta}_2-\eta_2)(\overline{\eta}_1-\eta_1)}{(\eta_2+\overline{\eta}_1)(\eta_1+\overline{\eta}_2)}}e^{(C_2-C_3)(x-C_1t)(\eta_1+\eta_2)}\right] - \delta_1\delta_2\delta_3\delta_4\frac{(\eta_1+\eta_2)(\overline{\eta}_1+\overline{\eta}_2)}{(\overline{\eta}_1+\eta_2)(\eta_1+\overline{\eta}_2)} \cdot e^{(C_2-C_3)(x-C_1t)(\eta_1+\eta_2)+(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}.$$

Remark 13.6. If $\delta_1\delta_3 = -1$ and $\delta_2\delta_4 = 1$, then these three envelopes are regular.

Similarly, when $\varepsilon_1 = -1$ and $\varepsilon_3 = 1$, we deduce that

(13.19)
$$Q_1(x,t) = \frac{(C_2-C_3)\sqrt{(C_2-C_1)(C_3-C_1)}}{D_2(x,t)} \cdot (\eta_1+\eta_2) \cdot \delta_4\sqrt{\frac{\eta_1-\overline{\eta}_1}{\eta_1+\overline{\eta}_2}}e^{\eta_1(C_2-C_3)(x-C_1t)} \cdot \left\{1 + \delta_1\delta_3\sqrt{\frac{(\overline{\eta}_2+\eta_1)(\eta_2-\overline{\eta}_2)}{(\overline{\eta}_1+\eta_2)(\eta_1-\overline{\eta}_1)}}e^{(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}\right\},$$

(13.20)
$$Q_2(x,t) = -\frac{(C_3-C_1)\sqrt{(C_2-C_1)(C_3-C_2)}}{D_2(x,t)}\frac{(\eta_1+\eta_2)(\overline{\eta}_1+\overline{\eta}_2)}{\sqrt{(\eta_1+\overline{\eta}_2)(\overline{\eta}_1+\eta_2)}} \delta_2\delta_3 e^{\overline{\eta}_2(C_1-C_2)(x-C_3t)+\eta_2(C_2-C_3)(x-C_1t)},$$

(13.21)
$$Q_3(x,t) = \frac{(C_2-C_1)\sqrt{(C_3-C_1)(C_3-C_2)}}{D_2(-x,-t)} \cdot (\overline{\eta}_2+\overline{\eta}_1)\delta_3\sqrt{\frac{\eta_2-\overline{\eta}_2}{\overline{\eta}_2+\eta_1}}e^{\overline{\eta}_2(C_2-C_1)(x-C_3t)} \cdot \left\{1 + \delta_2\delta_4\sqrt{\frac{(\eta_1+\overline{\eta}_2)(\eta_1-\overline{\eta}_1)}{(\overline{\eta}_1+\eta_2)(\eta_2-\overline{\eta}_2)}}e^{(C_3-C_2)(x-C_1t)(\eta_1+\eta_2)}\right\}$$

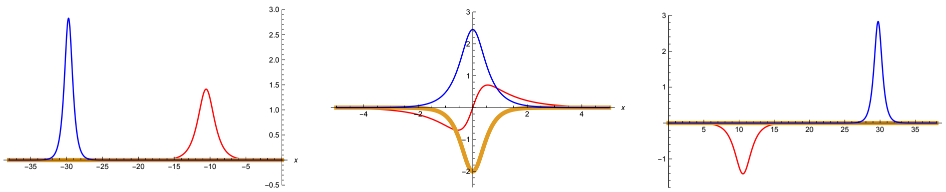
also solve the real nonlocal three-wave interaction system (2.11), where $\overline{\eta}_1 < \eta_1$, $\overline{\eta}_2 < \eta_2$, and

$$D_2(x,t) = \left[1 + \delta_1\delta_3\sqrt{\frac{(\eta_1-\overline{\eta}_1)(\eta_2-\overline{\eta}_2)}{(\overline{\eta}_1+\eta_2)(\overline{\eta}_2+\eta_1)}}e^{(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}\right] \cdot \left[1 - \delta_2\delta_4\sqrt{\frac{(\eta_2-\overline{\eta}_2)(\eta_1-\overline{\eta}_1)}{(\eta_2+\overline{\eta}_1)(\eta_1+\overline{\eta}_2)}}e^{(C_2-C_3)(x-C_1t)(\eta_1+\eta_2)}\right] - \delta_1\delta_2\delta_3\delta_4\frac{(\eta_1+\eta_2)(\overline{\eta}_1+\overline{\eta}_2)}{(\overline{\eta}_1+\eta_2)(\eta_1+\overline{\eta}_2)} \cdot e^{(C_2-C_3)(x-C_1t)(\eta_1+\eta_2)+(C_1-C_2)(x-C_3t)(\overline{\eta}_1+\overline{\eta}_2)}.$$

Remark 13.7. If $\delta_1\delta_3 = 1$ and $\delta_2\delta_4 = -1$, then the three waves do not blow up.

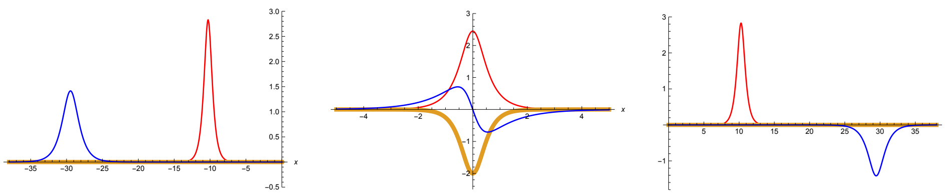
Remark 13.8. The system (2.11) implies that the corresponding solution set $\{Q_1, Q_2, Q_3\}$ must be real-valued. Obviously, our solutions are consistent with the reality of (2.11).

The inverse problem shows that the minimal data for reconstructing 1-0-1 soliton solutions contain the following eight quantities: four eigenvalues η_1, η_2, $\overline{\eta}_1$, $\overline{\eta}_2$ and



(a) (b) (c)

FIG. 5. (a) $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_1=-10$. (b) $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_2=0$. (c) $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_3=10$. *Here,* $\varepsilon_1=1$, $\varepsilon_2=\varepsilon_3=-1$, $C_1=1$, $C_2=2$, $C_3=3$, $\delta_1=-1$, $\delta_2=\delta_3=\delta_4=1$, $\overline{\eta}_1=\overline{\eta}_2=2$, $\eta_1=\eta_2=1$.



(a) (b) (c)

FIG. 6. (a) $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_1=-10$. (b) $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_2=0$. (c) $Q_1(x,t)$ *(red),* $Q_2(x,t)$ *(yellow), and* $Q_3(x,t)$ *(blue) at* $t_3=10$. *Here,* $\varepsilon_1=\varepsilon_2=-1$, $\varepsilon_3=1$, $C_1=1$, $C_2=2$, $C_3=3$, $\delta_1=\delta_2=\delta_3=1$, $\delta_4=-1$, $\overline{\eta}_1=\overline{\eta}_2=1$, $\eta_1=\eta_2=2$.

four units δ_1, δ_2, δ_3, δ_4. The reason is similar to that in the complex reverse space-time three-wave system. Indeed, (9.12)–(9.13) imply that the reduced normalization coefficients $\overline{c}_j$ only depend on four eigenvalues and δ_j, where $\delta_j^2=1$, $j=1,2,3,4$.

We express the above as a theorem.

THEOREM 13.9. *Given the minimal data* η_1, η_2, $\overline{\eta}_1$, $\overline{\eta}_2$ *and* δ_1, δ_2, δ_3, δ_4, *then the real reverse space-time three-wave system admits a unique soliton solution* $\{Q_1,Q_2,Q_3\}$ *given by* (13.16)–(13.18) *and* (13.19)–(13.21) *until possibly blow up. In particular, if* $\delta_1\delta_3=1$ *and* $\delta_2\delta_4=-1$, *then* $\{Q_1,Q_2,Q_3\}$ *is a global solution.*

We plot the interactions of Q_1, Q_2, and Q_3 below for the real reverse space-time three-wave system. Figure 5 is a typical example, where $\varepsilon_1=1$ and $\varepsilon_3=-1$. Figures 5(a)–(c) describe the traveling of three envelopes at $t_1=-10$, $t_2=0$, and $t_3=10$, respectively. At $t_1=-10$, Q_3 is to the left of Q_1 (both are elevation waves), the amplitude (magnitude) of Q_3 is greater than Q_1, and Q_2 is small. Later, at $t_2=0$, these three waves are interacting. Specifically, Q_2 is found to be negative (recall that the values Q_j are real); Q_1 is symmetric about $(0,0)$ (positive for $x>0$) and Q_3 is still positive, where Q_2 and Q_3 are even functions. At $t_3=10$, Q_2 decays back to be small, Q_3 always remains positive, and at this point, Q_1 is a depression wave and is to the left of Q_3. However, their magnitudes are unchanged. In Figure 6, we find similar phenomena for $\varepsilon_1=-1$ and $\varepsilon_3=1$. In conclusion, the one with bigger amplitude is an elevation wave from initial time to final time, but a polarity shift occurs for the other one.

14. Connection with physical models. Equations that are "close" to physical equations such as those presented in this paper are often related to physically significant systems. Thus, it is natural to ask if the nonlocal reductions of the sixth order wave system (2.3) can be related to equations arising from a physical model. In [9], it was shown that the coupled second order "q, r" system derived in [1] arises from an asymptotic reduction of the nonlinear Klein–Gordon, KdV, and water wave equations. In turn, this implies that the classical NLS equation and the nonlocal PT symmetric and the nonlocal reverse space time NLS equations are all asymptotic reductions of these physical systems since the "q, r" system contains the classical and nonlocal reductions [6, 8]. The reduction to the q, r system is, in general, complex. We find the same here. Complex equations are common in the field of integrable systems, e.g., Painlevé equations, self-dual Yang–Mills, self-dual reductions of Einstein's equations, such as those contained in Bianchi IX cosmological models (see, e.g., [4, 34]) and water waves (see, e.g., [33]). The three-wave equations discussed in this paper are different from NLS-type systems in that they are quadratically nonlinear and are derived from evolution equations with linear dispersion relations that admit three-wave/triad resonance. As discussed in the introduction, triad resonance occurs widely in physical applications, e.g., in water waves, plasma physics, and nonlinear optics. Motivated by water/ocean wave phenomena, a simple model was introduced in [22] that illustrates the underlying three-wave resonant mechanisms. We consider the following nonlinear partial differential equation which apart from a sign is the one discussed in [22]:

$$\partial_t^2 u - \partial_x^2 u + \partial_x^4 u + u + \epsilon\sigma u^2 = 0, \quad 0 < \epsilon \ll 1, \tag{14.1}$$

where σ is constant. This equation is motivated by the study of water waves with surface tension (see [10, 22]). We remark that water waves without surface tension do not exhibit triad resonance. Below we show that the sixth order wave system (2.3) is an asymptotic reduction of (14.1). As mentioned in the introduction, in a separate paper we show that the sixth order wave system (2.3) can be derived from the classical water/gravity wave equations with surface tension [11].

The linear part of this equation ($\sigma = 0$) has waves of the form $e^{ikx-\omega(k)t}$, where $\omega(k)$ the dispersion relation is given by $\omega(k)^2 = 1 + k^2 + k^4$.

We need to establish that this dispersion relation has triad resonance, which we will take to be of the form

$$k_3 = k_1 + k_2, \quad \omega_3 = \omega_1 + \omega_2, \quad \text{where } \omega_j = \omega(k_j),$$

with associated group velocities $C_j = \omega'(k_j)$, $j = 1, 2, 3$.

As an example, suppose we take $k_2 = 1$, $\omega_2 = \sqrt{3}$; then $k_3 = 1 + k_1$, $\omega_3 = \omega(1 + k_1)$ solving for k_1 from the triad equation $\omega_3 = \omega_1 + \omega_2$ yields $\sqrt{1 + (1 + k_1)^2 + (1 + k_1)^4} = \sqrt{1 + k_1^2 + k_1^4} + \sqrt{3}$. Numerically, we find a solution $k_1 \approx 0.497$: hence $k_3 \approx 1.497$, and so $k_1 < k_2 < k_3$; we also find $\omega_1 \approx 1.143 < \omega_2 < \omega_3 \approx 2.875$ and $C_1 \approx 0.649 < C_2 = \sqrt{3} < C_3 \approx 2.854$.

Similar arguments and numerical computations show that by varying k_2 there is a broad range of such solutions for k_1, k_2. Therefore, we have shown that (14.1) has triad resonance.

Next we show how to obtain equations governing three-wave resonance. We introduce multiple scales $X = \epsilon x, T = \epsilon t$ so that $\partial_t \to \partial_t + \epsilon\partial_T, \partial_x \to \partial_x + \epsilon\partial_X$. Hence, (14.1) takes the form $((\partial_t + \epsilon\partial_T)^2 - (\partial_x + \epsilon\partial_X)^2 + (\partial_x + \epsilon\partial_X)^4 + 1)u + \epsilon\sigma u^2 = 0$. Expanding $u = u_0 + \epsilon u_1 + \cdots$ yields the following for the first two orders in ϵ:

$$Lu_0 = (\partial_t^2 - \partial_x^2 + \partial_x^4 + 1)u_0 = 0, \quad Lu_1 = -2\left(\partial_t\partial_T - \partial_x\partial_X + 2\partial_x^3\partial_X\right)u_0 - \sigma u_0^2. \tag{14.2}$$

We take as our solution of the leading order equation

$$u_0 \sim \sum_{j=1}^{3} (A_j(X,T)e^{i\theta_j} + B_j(X,T)e^{-i\theta_j}),$$

where $\theta_j = k_j x - \omega_j t$.

Since in general $B_j(X,T) \neq A_j(X,T)$, the solution can be complex. When we proceed to the next order and remove secular terms, we find

$$\begin{aligned} &i\omega_1(\partial_T A_1 + C_1\partial_X A_1) + \sigma B_2 A_3 = 0, \qquad i\omega_1(\partial_T B_1 + C_1\partial_X B_1) - \sigma A_2 B_3 = 0,\\ &i\omega_2(\partial_T A_2 + C_2\partial_X A_2) + \sigma B_1 A_3 = 0, \qquad i\omega_2(\partial_T B_2 + C_2\partial_X B_2) - \sigma A_1 B_3 = 0,\\ &i\omega_3(\partial_T A_3 + C_3\partial_X A_3) + \sigma A_1 A_2 = 0, \qquad i\omega_3(\partial_T B_3 + C_3\partial_X B_3) - \sigma B_1 B_2 = 0. \end{aligned} \tag{14.3}$$

The usual three-wave equations are obtained when we assume $B_j(X,T) = A_j^*(X,T)$, $j = 1,2,3$, where $*$ stands for complex conjugate. But here we allow $B_j(X,T)$, $j = 1,2,3$, to be free; so the solution is, in general, complex.

We rescale the equations by taking $\tilde{A}_j = \gamma_j A_j$, $\tilde{B}_j = \gamma_j B_j$, $j = 1,2,3$; introducing another parameter χ_j, $j = 1,2,3$, leads to $\gamma_j^2 = -\frac{i(\gamma_1\gamma_2\gamma_3)\omega_j\chi_j}{\sigma} = -\frac{\sigma^2\omega_j\chi_j}{\omega_1\omega_2\omega_3\chi_1\chi_2\chi_3}$, $j = 1,2,3$, since $\gamma_1\gamma_2\gamma_3 = \frac{\sigma^3}{i\omega_1\omega_2\omega_3\chi_1\chi_2\chi_3}$.

Dropping the tilde: $\sim$ yields the following A, B system of equations

$$\begin{aligned} &\partial_T A_1 + C_1\partial_X A_1 - \chi_1 B_2 A_3 = 0, \qquad \partial_T B_1 + C_1\partial_X B_1 + \chi_1 A_2 B_3 = 0,\\ &\partial_T A_2 + C_2\partial_X A_2 - \chi_2 B_1 A_3 = 0, \qquad \partial_T B_2 + C_2\partial_X B_2 + \chi_2 A_1 B_3 = 0,\\ &\partial_T A_3 + C_3\partial_X A_3 - \chi_3 A_1 A_2 = 0, \qquad \partial_T B_3 + C_3\partial_X B_3 + \chi_3 B_1 B_2 = 0. \end{aligned} \tag{14.4}$$

Next, consider the sixth order wave system (2.3) and identify the following components: $A_1 = N_{32}$, $A_2 = N_{13}$, $A_3 = N_{12}$, $B_1 = N_{23}$, $B_2 = N_{31}$, $B_3 = N_{21}$. Comparing the A, B equations (14.4) with the wave system (2.3), we identify $C_1 = -\alpha_{23}$, $C_2 = -\alpha_{13}$, $C_3 = -\alpha_{21}$ and $\chi_1 = C_3 - C_2$, $\chi_2 = C_1 - C_3$, $\chi_3 = C_1 - C_2$. We also remark that for the compatibility of the above system, we require $\chi_3 = \chi_1 + \chi_2$.

Therefore, the A, B system is equivalent to the sixth order wave system (2.3). Moreover, since this sixth order wave system has reductions to the classical three-wave equations and the nonlocal complex and real three-wave systems, we have established that these nonlocal reductions are asymptotic limits of the original nonlinear PDE (14.1).

15. Conclusion.

Three-wave interaction equations are extremely important nonlinear wave systems; they arise in many physical problems. In this paper, the classical three-wave and two nonlocal three-wave systems: complex reverse space-time and real reverse space-time equations, are investigated in detail. It is shown how to derive these systems from 3×3 linear compatible systems. The inverse scattering transform with rapidly decaying data is employed to analyze these systems. The direct and inverse problems are carefully analyzed. Considering the behavior at plus and minus infinity of the six possible scattering eigenfunctions in the direct problem, four are shown to be analytic in the upper/lower half planes. The adjoint eigenvalue problem is used to find the remaining two analytic eigenfunctions. Bound states and symmetry relations are then found. The inverse problem is developed via Riemann–Hilbert (RH) methods, and the scattering data is connected to the initial values for data decaying sufficiently fast; to our knowledge, even for the classical three-wave system, this has not been done before. Formulae to reconstruct the potentials, trace formulae, and minimal data are obtained for the general case and for the reductions to the classical, complex and real reverse space-time three-wave interaction systems. Explicit reflectionless potentials/soliton solutions are also found; figures describing various typical interactions and energy sharing are provided as illustrations. In the direct and inverse scattering analysis, as compared with the classical three-wave interaction system, there are numerous differences and new features associated with the nonlocal three-wave interaction equations. Finally, we showed that the underlying sixth order compatible system and its reductions to the classical and nonlocal three-wave equations are asymptotic reductions of a nonlinear PDE that is motivated by physical applications and exhibits triad/three-wave resonance.

16. Appendix.

Suppose

$$v(x,k) = -\widetilde{D}\cdot(u^{ad}(x,k)\times w^{ad}(x,k))\cdot e^{ikdx}$$

is a solution of (3.2), where $\widetilde{D}=\mathrm{diag}(\widetilde{d}_1,\widetilde{d}_2,\widetilde{d}_3)$, $u(x,k)=(u_1(x,k),u_2(x,k),u_3(x,k))^T$, and $w(x,k)=(w_1(x,k),w_2(x,k),w_3(x,k))^T$. Our aim is to determine $\widetilde{D}$. By direct computation, we have

$$\begin{aligned}
v_x &= -\widetilde{D}[(u^{ad}\times w^{ad})_x\cdot e^{ikdx} + ikd(u^{ad}\times w^{ad})\cdot e^{ikdx}]\\
&= e^{ikdx}\cdot\widetilde{D}\{[(ikD+N^T)u^{ad}]\times w^{ad} + u^{ad}\times[(ikD+N^T)w^{ad}] - ikd(u^{ad}\times w^{ad})\}\\
&= e^{ikdx}\cdot(\widetilde{d}_1[N_{12}(w_3^{ad}u_1^{ad}-u_3^{ad}w_1^{ad}) + N_{13}(w_1^{ad}u_2^{ad}-u_1^{ad}w_2^{ad})\\
&\quad - ikd_1(w_3^{ad}u_2^{ad}-u_3^{ad}w_2^{ad})],\\
&\quad \widetilde{d}_2[N_{23}(w_1^{ad}u_2^{ad}-u_1^{ad}w_2^{ad}) + N_{21}(w_2^{ad}u_3^{ad}-u_2^{ad}w_3^{ad}) - ikd_2(w_1^{ad}u_3^{ad}-u_1^{ad}w_3^{ad})],\\
&\quad \widetilde{d}_3[N_{31}(w_2^{ad}u_3^{ad}-u_2^{ad}w_3^{ad}) + N_{32}(w_3^{ad}u_1^{ad}-u_3^{ad}w_1^{ad}) - ikd_3(w_2^{ad}u_1^{ad}-u_2^{ad}w_1^{ad})])^T,
\end{aligned}$$

$$\begin{aligned}
&(ikD+N)v = -e^{ikdx}\cdot(ikD+N)\widetilde{D}(u^{ad}\times w^{ad})\\
&= e^{ikdx}\cdot(\widetilde{d}_2N_{12}(w_3^{ad}u_1^{ad}-u_3^{ad}w_1^{ad}) + \widetilde{d}_3N_{13}(w_1^{ad}u_2^{ad}-u_1^{ad}w_2^{ad})\\
&\quad - ikd_1\widetilde{d}_1(w_3^{ad}u_2^{ad}-u_3^{ad}w_2^{ad}),\\
&\widetilde{d}_3N_{23}(w_1^{ad}u_2^{ad}-u_1^{ad}w_2^{ad}) + \widetilde{d}_1N_{21}(w_2^{ad}u_3^{ad}-u_2^{ad}w_3^{ad}) - ikd_2\widetilde{d}_2(w_1^{ad}u_3^{ad}-u_1^{ad}w_3^{ad}),\\
&\widetilde{d}_1N_{31}(w_2^{ad}u_3^{ad}-u_2^{ad}w_3^{ad}) + \widetilde{d}_2N_{32}(w_3^{ad}u_1^{ad}-u_3^{ad}w_1^{ad}) - ikd_3\widetilde{d}_3(w_2^{ad}u_1^{ad}-u_2^{ad}w_1^{ad}))^T.
\end{aligned}$$

$v_x=(ikD+N)v$ yields $\widetilde{d}_1=\widetilde{d}_2=\widetilde{d}_3$. Without loss of generality, we choose may $\widetilde{d}_1=\widetilde{d}_2=\widetilde{d}_3=-1$, and thus $v(x,k)=e^{ikdx}(u^{ad}(x,k)\times w^{ad}(x,k))$ solves for (3.2).

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