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Data-Driven Superstabilization of Linear Systems under Quantization

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***Abstract*— This paper focuses on the stabilization and regulation of linear systems affected by quantization in state-transition data and in actuated input. The observed data are composed of tuples of current state, input, and the next state's interval ranges based on sensor quantization. Using an established characterization of input-logarithmically-quantized stabilization based on robustness to sector-bounded uncertainty, we formulate a nonconservative robust linear program that enforces superstabilization of all possible consistent systems under assumed priors. We solve this problem by posing a pair of exponentially-scaling linear programs, and demonstrate the success of our method on example systems.**

I. Introduction

This paper performs Data Driven Control (DDC) of discrete-time linear systems under data quantization in the state-transition records and logarithmic quantization in the input. Input quantization can be encountered in data-rate constraints for network models when sending instructions to digital actuators, and its presence adds a nonlinearity to system dynamics [1], [2], [3]. The logarithmic input quantizer offers the coarsest possible quantization density [2] among all possible quantization schemes. These logarithmic quantizers admit a nonconservative characterization as a Luré-type sector-bounded input [4], [5], [6]. Data quantization could occur in the storage of sensor data into bits on a computer, and admits the mixed-precision setting of sensor fusion with different per-sensor precisions.

DDC is a design method to synthesize control laws directly from acquired system observations and model/noise priors, without first performing system-identification/robust-synthesis pipeline [7], [8]. This paper utilizes a Set-Membership approach to DDC: furnishing a controller along with a certificate that the set of all quantized data-consistent plants are contained within the set of all commonly-stabilized plants. Certificate methods for set-membership DDC approaches include Farkas certificates for polytope-in-polytope containment [9], a Matrix S-Lemma for Quadratic Matrix Inequalities (QMIs) to prove quadratic and robust stabilization [10], [11], [12], and Sum-of-Squares certificates of polynomial nonnegativity [13], [14], [15]. Other methods for DDC include Iterative Feedback Tuning [16], Virtual Reference Feedback Tuning [17], [18], Behavioral characterizations (Willem's Fundamental Lemma) with applications to Model-Predictive Control [19], [20], [21], moment proofs for switching control [22], learning with Lipschitz bounds [23], and kernel regression [24].

The most relevant prior work to the quantized DDC approach in this paper is the research in [25], which casts quantized control as an an H_∞ small-gain task [4] and enforces common stabilization over a a QMI of consistent plants using a Matrix S-Lemma [10]. In contrast, our work includes individually-quantized data as well as quantized control by developing a polytopic description of the plant consistency set. We then restrict to superstabilization [26], [27] to formulate DDC Linear Programs (LPs) over the polytopic consistency set. The QMI approach in [25] would over-approximate the polytopic consistency constraint with a single ellipsoidal region in the case of data quantization.

The contributions of this work are:

- A formulation for superstabilizing DDC under input and data quantization
- A sign-based LP for data-driven quantized superstabilization that grows exponentially in n and m
- A more tractable Affinely-Adjustable Robust Counterpart (AARC) that is exponential in m alone.

This paper has the following structure: Section II introduces notation and superstabilization. Section III provides an overview of the data and logarithmic-input quantization schemes considered in this work. Section IV formulates superstabilizing DDC under quantization as a pair of equivalent LPs. Section V demonstrates these algorithms on example quantized systems. Section VI concludes the paper.

II. Preliminaries

A. Notation

$a..b$	Natural numbers between a and b
$\mathbb{R}^n$	n-dimensional real Euclidean space
$\mathbb{R}^n_{\geq 0}$ ($\mathbb{R}^n_{>0}$)	n-dimensional nonnegative (positive) orthant
$\mathbb{R}^{n\times m}$	$n \times m$-dimensional real matrix space
$\mathbf{1}_n,\ \mathbf{0}_n$	Vector of all ones or zeros
I_n	Identity matrix
$\otimes$	Kronecker product
$\text{vec}(X)$	Column-wise vectorization of a matrix
X^T	Matrix transpose
$\lVert x\rVert_\infty$	L_∞-norm (vector): $\max_i \lvert x\rvert_i$
$\lVert X\rVert_\infty$	Induced L_∞ norm (matrix): $\max_i \sum_j \lvert X_{ij}\rvert$
$x./y$	Element-wise division between x and y
$A \leq B$	Element-wise $\leq$ between $A, B \in \mathbb{R}^{n\times m}$

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B. Superstabilization

A discrete-time system $x_+ = Ax$ is *Extended Superstable* if there exists nonnegative weights $v > 0$ such that $\|x./v\|_\infty$ is a Lyapunov function [28]. This condition may be expressed using an operator norm through the definition $Y = \text{diag}(v)$ and the constraint $\|YAY^{-1}\|_\infty < 1$. Standard *superstability* is the restriction of extended superstability when $v = \mathbf{1}_n$.

A discrete-time linear system with input of

$$x^+ = Ax + Bu \tag{1}$$

is extended-superstabilized by the full-state-feedback controller $u = Kx$ if there exists [28] $v \in \mathbb{R}^n_{>0},\ S \in \mathbb{R}^{m\times n}$ with

$$\begin{aligned} &\forall i \in 1..n,\ \ \alpha \in \{-1, 1\}^n : \\ &\quad \textstyle\sum_{j=1}^n \alpha_i \left(A_{ij} v_j + \sum_{k=1} B_{ik} S_{kj}\right) < v_i. \end{aligned} \tag{2}$$

The controller K forming the input $u = Kx$ is then recovered by $K = S\text{diag}(1./v)$. Problem (2) is a set of $n2^n$ strict linear inequality constraints. A more efficient method of imposing extended-superstability is by introducing a new matrix $M \in \mathbb{R}^{n\times n}$ [29],

$$\textstyle\sum_{j=1}^n M_{ij} < v_i \qquad \forall i \in 1..n \tag{3a}$$

$$|A_{ij} v_j + \textstyle\sum_{k=1} B_{ik} S_{kj}| \leq M_{ij} \qquad \forall i, j \in 1..n. \tag{3b}$$

Problems (2) and (3) are equivalent, in which an admissible selection for M is $M_{ij} = |A_{ij} v_j + \sum_{k=1} B_{ik} S_{kj}|$. The conditions in (2) and (3) are necessary and sufficient for full-state feedback extended superstabilization.

If the system in (1) is superstabilized ($v = \mathbf{1}_n$) and $\|A+BK\|_\infty \leq \lambda$ with $\lambda < 1$, then any closed-loop trajectory x_t starting at x_0 with $\forall t : u_t = 0$ will satisfy $\|x_t\|_\infty \leq \lambda^t \|x_0\|_\infty$ [30], [26]. The quantity λ can be interpreted as a decay rate, and the controller K can be designed using an LP to minimize λ and ensure the fastest possible convergence. A similar minimal peak-to-peak design task for extended superstabilization requires the solution of parametric LP with a single free parameter [28].

III. Quantization

This section will introduce the two sources of quantization considered in this paper.

A. Quantization of Data

Our data $\mathcal{D}$ with N_s samples is composed of the current state $\hat{x}$, input $\hat{u}$, and bounds on the subsequent state $[p, q]$, forming the N_s tuples $\mathcal{D} = \cup_{s=1}^{N_s}(\hat{x}_s, \hat{u}_s, p_s, q_s)$. We define the polytope $\mathcal{P}(A, B)$ as the set of all plants that are consistent with the data in $\mathcal{D}$:

$$\mathcal{P} = \{(A, B) \mid \forall s \in 1..N_s : A\hat{x}_s + B\hat{u}_s \in [p_s, q_s]\}. \tag{4}$$

The bounds p_s, q_s at each sample-index s may arise from interval quantization. In the case where a quantization process performs rounding to the first decimal place, the true state transition $x^+ = 0.368$ would be restricted to the interval to the interval described by $p = 0.3$ and $q = 0.4$.

This data-quantization framework in $\mathcal{D}$ allows for the integration of L_∞-bounded process-noise. In the case where there exists a process-noise w_s such that $A\hat{x}_s + B\hat{u}_s + w_s \in [p_s, q_s]$ with $\|w_s\|_\infty \leq \epsilon$, interval arithmetic can be used to express the data constraint as $A\hat{x}_s + B\hat{u}_s \in [p_s - \epsilon, q_s + \epsilon]$.

B. Quantization of Input

A scalar logarithmic quantizer with density $\rho \in (0, 1)$ and step $\delta = (1-\rho)/(1+\rho)$ is defined by $g_\rho : \mathbb{R} \to \mathbb{R}$ [4, Equation 7]:

$$g_\rho(z) = \begin{cases} \rho^i & \exists i \in \mathbb{N} \mid \frac{1}{1+\delta}\rho^i \leq z \leq \frac{1}{1-\delta}\rho^i \\ 0 & z = 0 \\ -g_\rho(-z) & z < 0. \end{cases} \tag{5}$$

We will obey the convention of [4] in referring to ρ as the *quantization density*, in which a larger ρ refers to a coarser quantizer with wider intervals. A ρ-logarithmically-quantized linear system has dynamics

$$x_{t+1} = Ax_t + Bg_\rho(u_t), \tag{6}$$

where the quantization in g_ρ should be understood to occur elementwise in u_t.

The following proposition establishes a sector-bound characterization of logarithmic quantization (for $m = 1$):

Proposition 3.1 (Eq. (21)-(22) in [4]): For any $z \geq 0$ and logarithmic quantization density $\rho > 0$ with

$$\delta = (1-\rho)/(1+\rho), \tag{7}$$

the quantization error at z satisfies a multiplicative bound

$$z - g_\rho(z) \in [-\delta z, \delta z]. \tag{8}$$

Figure 1 plots the graph of a logarithmic quantizer with $\rho = 0.4,\ \delta = 0.4286$ along with the error bound in (8) over the interval $u \in [-8, 8]$.

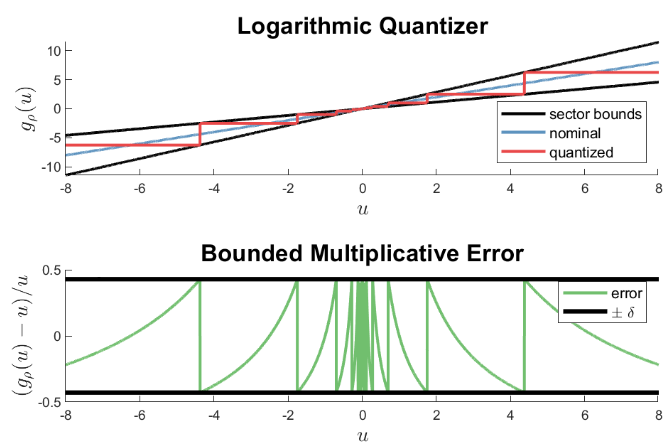

Fig. 1: Logarithmic quantizer ($\rho = 0.4$) and error bound

The trajectories of a logarithmically quantized systems with $m = 1$ are therefore contained in the class of scalar-Δ sector-bounded models:

$$x_{t+1} = [A + (1+\Delta)BK]\, x_t \qquad \forall \Delta \in [-\delta, \delta]. \tag{9}$$

Theorem 2.1 of [4] proves that the state-feedback controller $u_t = Kx_t$ with $K \in \mathbb{R}^{1\times n}$ **quadratically** stabilizes (1) iff $u = Kx$ can **quadratically** stabilize (6).

For systems in which each input channel u_j has a separate quantization density (ρ_j, δ_j), quadratic state-feedback stabilization of the quantized system will occur if [4, Theorem 3.2]:

$$\begin{aligned}\forall \Delta \in \textstyle\prod_{j=1}^m [-\delta_j, \delta_j]&\\ x_{t+1} = [A + B(I_m + \text{diag}(\Delta))K]\, x_t.&\end{aligned} \tag{10}$$

The work in [4] and [25] treat common stabilization of (9) as an H_∞ optimization using the small-gain theorem for a sector-bounded uncertainty. The multi-input small-gain formulation in (10) is posed and solved using a conservative multi-block S-Lemma.

C. Combined Superstability and Input-Quantization

We can apply the extended superstabilization method Section II-B towards the control of input-quantized systems, as represented by the sector-bounded model class in (10).

Theorem 3.2: A logarithmically quantized system in (6) is extended superstabilized by a controller $u = Kx$ if there exists a $v \in \mathbb{R}^n_{>0},\ S \in \mathbb{R}^{m\times n}, M \in \mathbb{R}^{n\times n}$ with $Y = \text{diag}(v)$

$$\begin{aligned}&\forall i \in 1..n\\ &\qquad \textstyle\sum_{j=1}^n M_{ij} < v_i && \text{(11a)}\\ &\forall \Delta \in \textstyle\prod_{j=1}^m [-\delta_j, \delta_j]\\ &- M \leq AY + B(I_m + \text{diag}(\Delta))S \leq M. && \text{(11b)}\end{aligned}$$

The recovered controller is $K = SY^{-1}$.

Proof: In the case where $\Delta = 0$, then the quantized program (11) is equivalent to the unquantized program (3). We can apply Proposition 3.1 to generate a sector-bound description of quantization, together with separate input channel quantization based on Equation (10) regarding the multiplicative perturbations Δ. The linear inequality constraints (11) are convex for each box-constrained Δ, such that a common M is a worst-case certificate over all possible closed-loop matrices $AY + B(I_m + \text{diag}(\Delta))S \leq M$. Such a certificate ensures extended superstability of all systems in (9). ∎

Corollary 1: We can enumerate the convex constraint (11) over the vertices of the hypercube formed by δ, producing the equivalent statement of

$$\begin{aligned}&\forall \gamma \in \textstyle\prod_{j=1}^m \{-\delta_j, \delta_j\}\\ &- M \leq AY + B(I_m + \text{diag}(\gamma))S \leq M.\end{aligned} \tag{12}$$

Corollary 2: An equivalent formulation to (11) with respect to sign-enumeration in (2) and substitution $\beta = 1 + \gamma$ is the following LP in $n2^{n+m}$ constraints:

$$\begin{aligned}&\forall i \in 1..n,\ \alpha \in \{-1, 1\}^n,\ \beta \in \textstyle\prod_{j=1}^m \{1 - \delta_j, 1 + \delta_j\}:\\ &\quad \textstyle\sum_{j=1}^n \alpha_j \left(A_{ij} v_j + \sum_{k=1}^m \beta_k B_{ik} S_{kj}\right) < v_i.\end{aligned} \tag{13}$$

Proposition 3.3: A controller K that is feasible for quantization $\delta \in \mathbb{R}^m_{\geq 0}$ in (13) will also be feasible for every $\delta' \in \mathbb{R}^m_{\geq 0}$ with $\delta' \leq \delta$.

IV. Quantized DDC

This section will outline a DDC approach towards quantized superstability.

Given data in $\mathcal{D}$, let $\mathcal{P}$ in (4) be the polytopic consistency of plants (A, B) in agreement with $\mathcal{D}$.

Our task is to solve the following problem:

Problem 4.1: Find a state-feedback controller $u = Kx$ such that the quantized system (9) is (extended) superstable for all $(A, B) \in \mathcal{P}$.

A. Consistency Polytope Representation

Let us define $\mathbf{X}, \mathbf{U}, \mathbf{p}, \mathbf{q}$ as the following concatenations of data in $\mathcal{D}$:

$$\begin{aligned}\mathbf{X} &= \begin{bmatrix}\hat{x}_1; & \hat{x}_2; & \dots & \hat{x}_{N_s}\end{bmatrix} && \text{(14a)}\\ \mathbf{U} &= \begin{bmatrix}\hat{u}_1; & \hat{u}_2; & \dots & \hat{u}_{N_s}\end{bmatrix} && \text{(14b)}\\ \mathbf{p} &= \begin{bmatrix}p_1; & p_2; & \dots & p_{N_s}\end{bmatrix} && \text{(14c)}\\ \mathbf{q} &= \begin{bmatrix}q_1; & q_2; & \dots & q_{N_s}\end{bmatrix}. && \text{(14d)}\end{aligned}$$

The data-consistency polytope in (4) may be represented using the data matrices in (14) as

$$\begin{aligned}G_{\mathcal{D}} &= \begin{bmatrix}-\mathbf{X}^T \otimes I_n & -\mathbf{U}^T \otimes I_n\\ \mathbf{X}^T \otimes I_n & \mathbf{U}^T \otimes I_n\end{bmatrix}\\ h_{\mathcal{D}} &= \begin{bmatrix}-\mathbf{p}; & \mathbf{q}\end{bmatrix}\\ \mathcal{P} &= \{(A, B) \mid G_{\mathcal{D}}[\text{vec}(A); \text{vec}(B)] \leq h_{\mathcal{D}}\},\end{aligned} \tag{15}$$

using the Kronecker identity $\text{vec}(PXQ) = (Q^T \otimes P)\text{vec}(X)$ for matrices (P, X, Q) of compatible dimensions. We will denote $L \leq 2nN_s$ as the number of faces in (15) ($h_{\mathcal{D}} \in \mathbb{R}^{1\times L}$). The number of faces L can be reduced from $2nN_s$ by pruning redundant constraints from $\mathcal{P}$ [31] through iterative LPs.

B. Sign-Based Approach

The sign-based program in (13) in the DDC case can be considered as a finite-dimensional robust LP:

$$\begin{aligned}&\forall i \in 1..n,\ \alpha \in \{-1, 1\}^n,\ \beta \in \textstyle\prod_{j=1}^m \{1 - \delta_j, 1 + \delta_j\}:\\ &\textstyle\sum_{j=1}^n \alpha_j \left(A_{ij} v_j + \sum_{k=1} \beta_k B_{ik} S_{kj}\right) < v_i,\ \forall (A, B) \in \mathcal{P}.\end{aligned} \tag{16}$$

Program (16) features a total of $n2^{n+m}$ strict robust inequalities. We will add a stability tolerance $\eta > 0$ in order to modify the comparator and right-hand side of (16) into a nonstrict inequality $\leq v_i - \eta$. Each nonstrict robust inequality in α, β may be formulated as a polytope:

$$\begin{aligned}G_{\alpha\beta} &= \begin{bmatrix}(\text{diag}(v)\alpha)^T \otimes I_n & (\text{diag}(\beta)S\alpha)^T \otimes I_n\end{bmatrix}\\ h_{\alpha\beta} &= v - \eta\mathbf{1}\\ \mathcal{P}_{\alpha\beta} &= \{(A, B) \mid G_{\alpha\beta}[\text{vec}(A); \text{vec}(B)] \leq h_{\alpha\beta}\}.\end{aligned} \tag{17}$$

We will enforce containment of $\mathcal{P}$ in each $\mathcal{P}_{\alpha\beta}$ using the Extended Farkas Lemma:

Lemma 4.2 (Extended Farkas Lemma [32], [33]): Let $P_1 = \{x \mid G_1 x \leq h_1\}$ and $P_2 = \{x \mid G_2 x \leq h_2\}$ be a pair of polytopes with $G_1 \in \mathbb{R}^{m\times n}$ and $G_2 \in \mathbb{R}^{p\times n}$. Then

$P_1 \subseteq P_2$ if and only if there exists a matrix $Z \in \mathbb{R}^{p\times m}_{\geq 0}$ such that,

$$ZG_1 = G_2, \qquad\qquad Zh_1 \leq h_2. \tag{18}$$

Remark 1: The Extended Farkas Lemma is a particular instance of a robust counterpart [34] when certifying validity of a system of linear inequalities over polytopic uncertainty.

A sign-based program to solve Problem 4.1 is:

$$\begin{aligned}
&\underset{v,S,Z}{\text{find}} \quad \forall \alpha \in \{-1,1\}^n,\ \beta \in \textstyle\prod_{j=1}^m \{1-\delta_j, 1+\delta_j\}: && \text{(19a)}\\
&\qquad Z_{\alpha\beta} G_{\mathcal{D}} = G_{\alpha\beta}, \quad Z_{\alpha\beta} h_{\mathcal{D}} \leq h_{\alpha\beta} && \text{(19b)}\\
&\qquad Z_{\alpha\beta} \in \mathbb{R}^{n\times L}_{\geq 0} && \text{(19c)}\\
&\qquad v - \eta \mathbf{1}_n \in \mathbb{R}^n_{\geq 0},\ S \in \mathbb{R}^{n\times m}. && \text{(19d)}
\end{aligned}$$

C. Lifted Approach

We can solve Problem 4.1 by posing (11) as an infinite-dimensional LP in terms of a function $M : \mathcal{P} \to \mathbb{R}^{n\times n}$.

Theorem 4.3: A state-feedback controller $u = Kx$ will solve Problem 4.1 if the following infinite-dimensional LP has a feasible solution with $v \in \mathbb{R}^n_{>0},\ S \in \mathbb{R}^{m\times n}, M : \mathcal{P} \to \mathbb{R}^{n\times n}$ with $Y = \text{diag}(v)$

$$\begin{aligned}
&\forall i \in 1..n\\
&\qquad \textstyle\sum_{j=1}^n M_{ij}(A,B) < v_i && \text{(20a)}\\
&\forall \beta \in \textstyle\prod_{j=1}^m \{1-\delta_j, 1+\delta_j\}\\
&-M(A,B) \leq AY + B(\text{diag}(\beta))S \leq M(A,B). && \text{(20b)}
\end{aligned}$$

Proof: Each plant $(A,B) \in \mathcal{P}$ has a certificate of extended superstabilizability $(v, M(A,B))$ by Theorem 3.2. If (20) is feasible, then all plants in $\mathcal{P}$ simultaneously extended superstabilized by a common $K = SY^{-1}$. ■

Remark 2: The function $M(A,B)$ may be treated as an adjustable decision variable given the a-priori unknown $(A,B) \in \mathcal{P}$ [35].

The infinite-dimensional LP in (20) must be truncated into a finite-dimensional convex program in order to admit computationally tractable formulations. One method to perform this truncation is to restrict $M(A,B)$ to an affine function by defining $M^0, M^A_{ij}, M^B_{ik} \in \mathbb{R}^{n\times n}$ to form

$$M(A,B) = M^0 + \textstyle\sum_{ij} M^A_{ij} A_{ij} + \sum_{ik} M^B_{ik} B_{ik}, \tag{21}$$

We can define the quantities $\mathbf{m} = (m^0, m^a, m^b)$

$$\begin{aligned}
m^0 &= \text{vec}(M^0) && \text{(22a)}\\
m^a &= \begin{bmatrix}\text{vec}(M^A_{11}), & \text{vec}(M^A_{21}), & \ldots, & \text{vec}(M^A_{nn})\end{bmatrix} && \text{(22b)}\\
m^b &= \begin{bmatrix}\text{vec}(M^B_{11}), & \text{vec}(M^B_{21}), & \ldots, & \text{vec}(M^B_{nm})\end{bmatrix}. && \text{(22c)}
\end{aligned}$$

in order to obtain a vectorized expression for (21) with

$$\text{vec}(M(A,B)) = m^0 + m^a \text{vec}(A) + m^b \text{vec}(B). \tag{23}$$

The row-sums of M can be expressed as

$$\begin{aligned}
\text{vec}(M(A,B)\mathbf{1}_n) = (\mathbf{1}_n^T \otimes I_n)(m^0 + m^a \text{vec}(A))&\\
+ (\mathbf{1}_n^T \otimes I_n)(m^b \text{vec}(B)).&
\end{aligned} \tag{24}$$

The constraint in (20a) with stability factor $\eta > 0$ can be reformulated as membership in the following polytope $\mathcal{P}_M$:

$$\begin{aligned}
G_M &= (\mathbf{1}_n^T \otimes I_n)\begin{bmatrix} m^a, & m^b\end{bmatrix}\\
h_M &= [v - \eta - (\mathbf{1}_n^T \otimes I_n) m^0]\\
\mathcal{P}_M &= \{(A,B) \mid G_M[\text{vec}(A); \text{vec}(B)] \leq h_M\}.
\end{aligned} \tag{25}$$

The polytopic constraint region in (20b) for each $\beta \in \prod_{j=1}^m \{1-\delta_j, 1+\delta_j\}$ is

$$\begin{aligned}
G^s_\beta &= \begin{bmatrix} -m^a - \text{diag}(v)^T \otimes I_n & -m^b - (\text{diag}(\beta)S)^T \otimes I_n\\ -m^a + \text{diag}(v)^T \otimes I_n & -m^b + (\text{diag}(\beta)S)^T \otimes I_n\end{bmatrix}\\
h_\beta &= \begin{bmatrix} m^0; & m^0\end{bmatrix}\\
\mathcal{P}_\beta &= \{(A,B) \mid G_\beta[\text{vec}(A); \text{vec}(B)] \leq h_\beta\}.
\end{aligned} \tag{26}$$

The affine restriction of M in (21) results in an AARC program for (20):

$$\begin{aligned}
&\underset{v,S,Z,\mathbf{m}}{\text{find}} \quad \forall \beta \in \textstyle\prod_{j=1}^m \{1-\delta_j, 1+\delta_j\}: && \text{(27a)}\\
&\qquad Z_\beta G_{\mathcal{D}} = G_\beta, \quad Z_\beta h_{\mathcal{D}} \leq h_\beta && \text{(27b)}\\
&\qquad Z_\beta \in \mathbb{R}^{2n^2\times L}_{\geq 0} && \text{(27c)}\\
&\qquad Z_M G_{\mathcal{D}} = G_M, \quad Z_M h_{\mathcal{D}} \leq h_M && \text{(27d)}\\
&\qquad Z_M \in \mathbb{R}^{n\times L}_{\geq 0} && \text{(27e)}\\
&\qquad m^0 \in \mathbb{R}^{n^2\times 1},\ m^A \in \mathbb{R}^{n^2\times n^2} && \text{(27f)}\\
&\qquad m^B \in \mathbb{R}^{n^2\times nm} && \text{(27g)}\\
&\qquad v - \eta\mathbf{1}_n \in \mathbb{R}^n_{\geq 0},\ S \in \mathbb{R}^{n\times m}. && \text{(27h)}
\end{aligned}$$

D. Computational Complexity

We will quantify the computational complexity (19) and (27) based on the number of robust inequalities (for (16) and (20)), scalar variables $(v, S, Z, \mathbf{m})$, slack variables/constraints introduced in reformulations of scalar inequality constraints (e.g., $v - \eta\mathbf{1}_n \in \mathbb{R}^n_{\geq 0} \mapsto q \in \mathbb{R}^n_{\geq 0},\ v - \eta\mathbf{1}_n = q$), scalar inequality constraints $(\in \mathbb{R}_{\geq 0})$, and scalar equality constraints. These counts (up to the highest order terms to save space) are listed in Table I.

TABLE I: Comparison between LPs (19) and (27)

	sign-based (19)	AARC (27)
robust ineq.	$n2^{n+m}$	$n + n^2 2^{m+1}$
scalar vars.	$n(m+1+2^{n+m}L)$	$n(m+1) + n^2(1+nm+n^2)$ $+n^2 L 2^{m+1}$
slack vars.	$n + 2^{n+m+1}n^2$	$n(1+2^{m+1})$
eq. cons.	$n^2(n+m)2^{n+m} + n$	$(2^{m+1}n^2+n)(n+m)+n$
ineq. cons.	$2^{n+m}(n+nL)+n$	$n(L+2) + n^2(L+1)2^{m+1}$

Note how n appears exponentially in the sign-based scheme (19), while n enters only polynomially for quantities in the AARC (27).

Given that the running-time of an Interior Point Method for N-variable LPs up γ-optimality is approximately $O(N^{\omega+0.5}|\log(1/\gamma)|)$ (with matrix multiplication constant ω) [36], the AARC is more computationally efficient than the sign-based scheme as n increases.

V. Numerical Examples

MATLAB (2021a) code to execute all examples is publicly available [1]. The convex optimization problems (19) and (27) are modeled in YALMIP [37] (including the robust programming module [38] with option ‘`lplp.duality`’) and solved in Mosek 9.2 [39].

A. 3-state 2-input

The first example will involve superstabilization of the following system 3-state 2-input discrete-time linear system:

$$A = \begin{bmatrix} -0.1300 & -0.3974 & 0.2030 \\ -0.3974 & -0.5000 & 0.2990 \\ 0.2030 & 0.2990 & -0.5262 \end{bmatrix}, \tag{28a}$$

$$B = \begin{bmatrix} 0.2179 & 1.2300 \\ 0.3592 & 0 \\ -1.1553 & 0 \end{bmatrix}. \tag{28b}$$

System (28) is open-loop unstable with eigenvalues of $[-1.0185, -0.2613, 0.1236]$.

We collect $T = 100$ input-state-transition observations of system (28) to form $\mathcal{D}$. The transition observations are quantized according to the following partition with 9 bins:

$$(-\infty, -4] \cup [-4, -3] \cup [-3, -2] \ldots [3, 4] \cup [4, \infty). \tag{29}$$

Superstabilization ($v = \mathbf{1}_3$) is performed by solving the sign-based scheme in (16). An objective is added to minimize $\lambda \in \mathbb{R}$ such that $\forall i : \sum_j M_{ij} \leq \lambda$, in which $\lambda < 1$ indicates a successful worst-case superstabilization under input and data quantization.

Figure 2 plots worst-case optimal values of λ as a function of the quantization density ρ, in which ρ is the same for all inputs. The $T = 60$ data preserves the first 60 elements of the 100 observations in $\mathcal{D}$ (with a similar process for $T = 80$). Gain values for the ground truth (model-based case when (28) is known) are presented as a comparison. We note that $\rho \to 1$ results in $\delta \to 0$ by (7), for which the (limiting) quantization law at $\rho = 1$ is $g_{\rho=1}(u) = u$.

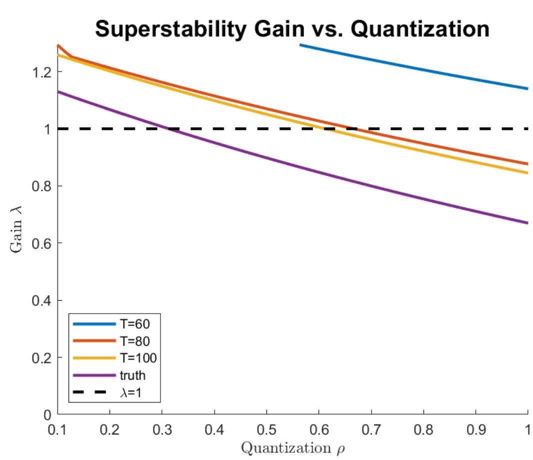

Fig. 2: Peak-to-peak gain (λ) vs. quantization density (ρ)

[1] https://github.com/Jarmill/quantized_ddc

Table II lists the minimal feasible ρ (up to four decimal places) such that the sign-based formulation in (16) returns a feasible superstabilizing (SS) or extended superstabilizing (ESS) controller. The symbol $\varnothing$ indicates primal infeasiblility of the LP for all $\rho \leq 1$.

TABLE II: Minimal ρ with sign-based of (28)

T	60	80	100	Truth
SS	$\varnothing$	0.6727	0.6182	0.3182
ESS	0.9397	0.3494	0.2081	0.1422

Table III lists the minimal ρ for AARC-based quantized superstabilization. There is no difference between the ground-truth values in Tables II and III, because the underlying finite-dimensional LPs with nonrobust inequality constraints are equivalent.

TABLE III: Minimal ρ with AARC stabilization of (28)

T	60	80	100	Truth
SS	$\varnothing$	$\varnothing$	0.9500	0.3182
ESS	$\varnothing$	$\varnothing$	0.7723	0.1422

B. 5-state 3-input

The second example performs extended superstabilization over the following system with 5 states and 3 inputs:

$$A = (1/5)[\min(i/j, j/i)]_{ij} + (1/2)I_5, \tag{30a}$$

$$B = [I_3; \mathbf{0}_{2\times 3}]. \tag{30b}$$

System (30) is open-loop unstable with purely real eigenvalues of $[1.0633, 0.6507, 0.5502, 0.5046, 0.4812]$. The nominal system in (30) can be extended-superstabilized until $\rho = 0.2245$.

The $T = 350$ state-input collected transitions of (30) are quantized according to the following partition with 26 bins:

$$(-\infty, -6] \cup [-6, -5.5] \cup [-5.5, -5] \ldots [5.5, 6] \cup [6, \infty). \tag{31}$$

The polytope in (4) has $2nT = 3500$ faces in $n(n+m) = 40$ dimensions, of which 185 of these faces are nonredundant.

We successfully solve the data-driven common-extended-superstabilizing AARC program in (27) at $\rho = 0.8$ to acquire a feasible controller with parameters

$$K = -\begin{bmatrix} 0.6434 & 0.0943 & 0.0785 & 0.0609 & 0.0330 \\ 0.0965 & 0.6513 & 0.1409 & 0.0899 & 0.0842 \\ 0.0650 & 0.1392 & 0.6528 & 0.1463 & 0.1183 \end{bmatrix}$$
$$v = \begin{bmatrix} 0.0137, & 0.0069 & 0.0058 & 0.0289 & 0.0289 \end{bmatrix}. \tag{32}$$

VI. Conclusion

This paper presented a method to perform superstabilizing control of linear systems under state-transition data quantization and actuated-input quantization. The generated sign-based finite-dimensional LP and lifted infinite-dimensional LPs are nonconservative with respect to the common superstabilization task. This infinite-dimensional LP has a number of constraints that is polynomial in the number of states n

and exponential in the number of inputs m. An AARC was employed to truncate the infinite-dimensional LP, in order to gain tractability at the expense of conservatism.

The logarithmic-quantization approach laid out in this paper involves an infinite number of quantization levels. Future work includes adapting the adaptive finite-level quantizing method of [5] for DDC-superstabilization. Other investigations aim to decrease the computational impact of the presented scheme by formulating nonconservative LP formulations that scale in a polynomial manner with m rather than in an exponential manner, by reducing the conservatism of the AARC truncation by allowing M to be polynomial (using sum-of-squares certificates of nonnegativity), and by formulating control laws in the setting where $(\hat{x}, \hat{u})$ are also data-quantized (resulting in an Error-in-Variables model [40] addressable by polynomial optimization [15]).

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