Inter-Modal Coding in Broadcast Packet Erasure Channels with Varying Statistics

Alireza Vahid, *Senior Member, IEEE*, and Shih-Chun Lin, *Senior Member, IEEE*

***Abstract*—We study the capacity region of the canonical two-user broadcast packet erasure channel when the erasure probabilities vary over the course of the communication block. In particular, we assume the network statistics may be in two distinct modes with a a priori known transition time between the two. We further consider the scenario in which the transmitter is informed of the delivery status of the previously transmitted packets through the feedback channel. We first derive a new set of the outer-bounds for this problem where the slope of the boundaries of the outer-bound region is dominated by the mode with the larger of the two erasure probabilities, and the corner points come from the average probability of each link being active. We show that under certain ratios of the lengths of the modes, these outer-bounds are achievable and thus the capacity region is known. We also discuss the behavior of the inner and outer bounds in other regimes and analyze the gap between the two. One key observation is that coding across the modes is superior to treating each mode as an individual problem.**

***Index Terms*—Broadcast erasure channels, varying erasure probabilities, CSI feedback, channels with memory.**

I. Introduction

The continual increase in wireless data traffic is pushing the communications to higher frequency bands, such as mmWave and (sub-)THz [1]. This increase in demand is also deriving the development of new technologies, such as reconfigurable intelligent surface (RIS) [2], movable antenna [3], and low earth orbit (LEO) satellite communications. Communications in higher frequency band is highly sensitive to the environment resulting in channel statistics variations [4], [5], which is further pronounced by the need to change the directionality or the frequency of communications to protect the incumbent users such as radio telescopes or utility links [6]. Movable antenna and RIS provide the opportunity to alter the channel statistics to meet such needs. For instance, an RIS may be used to enhance coverage, protect radio telescopes by neutralizing interference, and/or improve multi-user scaling and security [7]–[11]. In LEO communications, time-varying and intermittent access links between the satellite and terrestrial users also exist [12]. Thus, a theoretical understanding of the capacity region of wireless networks with varying channel statistics, whether natural or induced, is of great relevance.

Alireza Vahid is with the Electrical and Microelectronic Engineering Department at Rochester Institute of Technology, Rochester, NY 14623, USA. Email: alireza.vahid@rit.edu. The work of A. Vahid was in part supported by NSF grants CNS-2343959, CNS-2343964, and AST-2348589.

Shih-Chun Lin is with the Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan. Emails: sclin2@ntu.edu.tw. The work of Shih-Chun Lin is supported in part by the National Science and Technology Council, Taiwan, under Grant NSTC 112-2218-E-002-037; and in part by the National Taiwan University under Grant NTU-CC-112L893202.

To shed light on the fundamental limits of communications in networks with varying channel statistics, we turn our attention to the canonical broadcast packet erasure channel (BPEC), which models each hop of packet-based communications [13] and as such, is useful in understanding multi-session unicasting [14]–[16]. To model and incorporate variations in channel statistics, we assume the BPEC experiences two distinct modes over a block-length of n, namely modes A and B, with corresponding lengths n_{A} and $n_{\mathrm{B}} = n - n_{\mathrm{A}}$. During mode A, erasure probabilities are governed by independently and identically distributed (i.i.d.) across space and time Bernoulli $(1-\delta_{\mathrm{A}})$ processes, whereas during mode B, they are governed by i.i.d. Bernoulli $(1-\delta_{\mathrm{B}})$ processes. We assume the erasure probabilities and the lengths of the modes are known a priori throughout the network. This latter assumption is justified as for instance an RIS is controlled by the transmitter and the changes may be communicated to other nodes. We finally assume the availability of ACK/NACK signaling (*i.e.*, delayed channel state feedback) from each receiver to the other nodes. Our contributions are then two-fold.

We first present a set of outer-bounds for the two-user BPEC with varying channel statistics (the bi-modal BPEC). In broad terms, the defining factor in the outer-bound is the solution to the following extremal entropy optimization:

$$\begin{aligned} &\min \beta \\ &\text{s. t. } \beta H\left(Y_2^n | W_2, S^n, \mathrm{SI}\right) \geq H\left(Y_1^n | W_2, S^n, \mathrm{SI}\right) \end{aligned} \tag{1}$$

over *all* encoding functions having access to causal feedback. In short, Y_i^n is the received signal at user i, W_i is the message for user i, S^n is the channel information, and SI is any available side information. When the erasure probabilities are constant over the communication block, this is essentially the ratio of the probability of a stronger link in a degraded version of the BPEC to that of the weaker link. In the bi-modal BPEC, on the other hand, the ratio is dominated by the mode with the weaker forward links. The corner points come from the average probability of each link being active.

We then present an opportunistic inter-modal coding strategy, *i.e.*, coding across the two modes, that achieves a strictly larger rate region compared to separate treatment of the modes, *i.e.*, intra-modal coding. We show that under certain conditions, the inner and outer bounds match, thus characterizing the capacity region in such cases. The key idea in the inter-modal coding is to shift the multi-cast phase of network coding to the mode with stronger channels. This is intuitively beneficial as multi-cast satisfies multiple users at the same time and if it can be done at a higher rate, then the overall achievable region would be larger. When the relative lengths of the modes are such that it is not possible to strike the perfect balance, the inner and outer bounds deviate.

Related work: Shannon feedback does not increase the capacity of discrete memoryless point-to-point channels [17], and provides bounded gains in the multiple-access channel [18], [19]. On the other hand, feedback can provide significant gains when it comes to broadcast [20], [21] and interference [22]–[25] channels. In the context of BPECs, feedback capacity has been an active area of research. Most results consider the case in which all receivers provide ACK/NACK feedback to the transmitter [14]–[16]; while the study of BPEC with one-sided [26]–[30] and partial/intermittent [31]–[37] feedback has revealed the usefulness of imperfect feedback. Further, BPECs with receiver cache have been studied in [38], [39]. Perhaps, the most relevant result to this paper is the study of BPECs in [40], which we will further discuss in Remark 2.

II. Problem Formulation

We consider the canonical two-user broadcast packet erasure channel (BPEC) with channel state information (CSI) feedback in Figure 1 but with a twist: the erasure probabilities change during communications.

Bi-modal BPEC: Over n channel uses, the erasure probabilities of the forward links change at some a priori known time, $1 \leq n_{\mathrm{A}} \leq n$. More specifically, for $1 \leq t \leq n_{\mathrm{A}}$, the forward erasure probabilities, $S_i[t], i = 1, 2$, are governed by i.i.d. (over time and across users) Bernoulli $(1-\delta_{\mathrm{A}})$ processes, and for $n_{\mathrm{A}} \leq t \leq n$, the forward erasure probabilities are governed by i.i.d. (over time and across users) Bernoulli $(1-\delta_{\mathrm{B}})$ processes, $0 \leq \delta_{\mathrm{A}}, \delta_{\mathrm{B}} \leq 1$.

Messages: The transmitter, Tx, wishes to transmit two independent messages (files), W_1 and W_2, to two receiving terminals Rx_1 and Rx_2, respectively, over n channel uses. Each message, W_i, contains $|W_i| = m_i = nR_i$ data packets (or bits) where R_i is the rate for user i, $i = 1, 2$. For simplicity and when convenient, we also denote message W_1 and W_2 as bit vectors $\vec{a} = (a_1, a_2, \ldots, a_{m_1})$ and $\vec{b} = (b_1, b_2, \ldots, b_{m_2})$, respectively. Here, we note that each packet is a collection of encoded bits, however, for simplicity and without loss of generality, we assume each packet is in the binary field, and we refer to them as bits. Extensions to broadcast packet erasure channels where packets are in large finite fields are straightforward as done in [41] [42].

Input-output relationship: At time instant t, the messages are mapped to channel input $X[t] \in \mathbb{F}_2$, and the corresponding received signals at Rx_1 and Rx_2 are

$$Y_1[t] = S_1[t]X[t] \quad \text{and} \quad Y_2[t] = S_2[t]X[t], \tag{2}$$

respectively, where $\{S_i[t]\}$ denote the forward channels described above and are known to the receivers. When $S_i[t] = 1$, Rx_i receives $X[t]$ noiselessly; and when $S_i[t] = 0$, the receiver understands an erasure has occurred.

CSI and other assumptions: We assume the receivers are aware of the instantaneous channel state information (*i.e.* global CSIR). We assume the transmitter learns the forward channel realizations with unit delay but is aware of the erasure probabilities. We further assume the duration of mode $\mathrm{n_A}$ is known globally. The knowledge of erasure probability and n_A is considered as side-information, denoted by SI. Note that in LEO communication, the satellite trajectory would be known a priori with the help of the ephemeris [43], which is very helpful to predict the channel statistics and n_{A}.

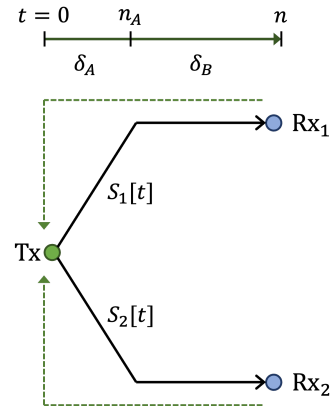

Fig. 1. Two-user bi-modal BPEC with channel state feedback. The erasure probabilities differ between the two modes.

Encoding: The constraint imposed at time index t on the encoding function $f_t(.)$ is

$$X[t] = f_t\left(W_1, W_2, S^{t-1}, \mathrm{SI}\right), \tag{3}$$

where $S^{t-1} = (S_1^{t-1}, S_2^{t-1})$.

Decoding: Each receiver Rx_i, $i = 1, 2$ knows the CSI across the entire transmission block, S^n. Then, the decoding function is $\varphi_{i,n}\left(Y_i^n, S^n, \mathsf{SI}\right)$. An error occurs whenever $\widehat{W}_i \neq W_i$. The average probability of error is given by

$$\lambda_{i,n} = \mathbb{E}[P(\widehat{W}_i \neq W_i)], \tag{4}$$

where the expectation is taken with respect to the random choice of the transmitted messages.

Capacity region: We say that a rate-pair (R_1, R_2) is achievable, if there exist a block encoder at the transmitter and a block decoder at each receiver, such that $\lambda_{i,n}$ goes to zero as the block length n goes to infinity. The capacity region, $\mathcal{C}$, is the closure of the set of all achievable rate-pairs.

III. Main Results & Insights

In this section, we present our main contributions, and provide further insights and intuitions about the findings.

First, we provide a set of outer bounds on the capacity region of the bi-modal BPEC with CSI feedback. Next, we provide a tailored example where these outer-bounds are achievable and thus, the capacity is characterized.

A. Parameters

We define the average erasure probability, $\bar{\delta}$, as

$$\bar{\delta} \triangleq \frac{n_{\mathrm{A}}\delta_{\mathrm{A}} + (n - n_{\mathrm{A}})\delta_{\mathrm{B}}}{n}. \tag{5}$$

We further define

$$\beta_{\mathrm{A}} \triangleq 1+\delta_{\mathrm{A}}, \quad \beta_{\mathrm{B}} \triangleq 1+\delta_{\mathrm{B}}, \quad \bar{\beta} \triangleq 1+\bar{\delta}. \tag{6}$$

We note that β_{A} and β_{B} would have defined the slope of the boundaries of the capacity region of BPECs with CSI feedback where forward channels where governed only by $(1-\delta_{\mathrm{A}})$ or $(1-\delta_{\mathrm{B}})$, respectively. We set:

$$\beta_{\max} \triangleq \max\{\beta_{\mathrm{A}}, \beta_{\mathrm{B}}\}. \tag{7}$$

B. Statement of the Main Results

We start with the new outer-bounds.

Theorem 1. *For the two-user bi-modal BPEC with CSI feedback as described in Section II, we have*

$$\mathcal{C} \subseteq \mathcal{C}^{\mathrm{out}} \equiv \left\{ \begin{array}{l} 0 \leq \beta_{\max} R_1 + R_2 \leq \beta_{\max}\left(1-\bar{\delta}\right), \\ 0 \leq R_1 + \beta_{\max} R_2 \leq \beta_{\max}\left(1-\bar{\delta}\right), \\ 0 \leq R_1 + R_2 \leq \frac{n_{\mathrm{A}}(1-\delta_{\mathrm{A}}^2)+(n-n_{\mathrm{A}})(1-\delta_{\mathrm{B}}^2)}{n}. \end{array} \right. \tag{8}$$

If the forward channels were governed by Bernoulli$(1-\bar{\delta})$ with $\bar{\delta}$ being the average of δ_{A} and δ_{B}, then $\bar{\beta}$ would define the slope of the boundaries of the region. Interestingly, the slope of the boundaries of $\mathcal{C}^{\mathrm{out}}$ is dominated instead by the larger of the two erasure probabilities, *i.e.*, one that results in $\beta_{\max}$, but the corner points come from the average probability of each link being active, *i.e.* $(1-\bar{\delta})$.

Remark 1 (Room for improvement)**.** *One may identify a weakness in the presented outer-bound, which is the fact that the first two bounds are oblivious to the relative lengths of the modes, meaning that even if the length of the mode with weaker links is significantly shorter than the other mode, the region is still defined by the former mode. Nonetheless, we will show that in non-trivial cases, this outer-bound region is indeed the capacity region of the problem, but based on the point highlighted above, it could potentially be improved.*

The following theorem shows there exist non-trivial instances in which the outer-bound region in Theorem 1 is achievable and matches the capacity region.

Theorem 2. *There exist $\delta_{\mathrm{A}}, \delta_{\mathrm{B}}$, and n_{A} such that $\delta_{\mathrm{A}} \neq \delta_{\mathrm{B}}$, $(1-\delta_{\mathrm{A}})(1-\delta_{\mathrm{B}}) > 0$, and $0 < n_{\mathrm{A}} < n$ for which the outer-bounds in* (8) *are achievable.*

To prove Theorem 2, it suffices to show one example meeting the corresponding conditions for which the outer-bound region is achievable. The general idea in achieving the capacity region of the two-user BPEC with feedback is to send packets intended for each user in separate phases and then recycle these packets into "multi-cast" packets that upon delivery would simultaneously benefit both users [14]. For the bi-modal BPEC, the key idea is to push the multi-cast phase as much as possible to the mode with lower erasure probabilities to boost the achievable rates. This way the network coding spans across different modes, which we refer to as inter-modal coding. We show this approach strictly outperforms the strategy the treats the modes separately.

Remark 2 (Comparison to prior work [40])**.** *In [40, Sec IV-C], a finite-state memoryless BPEC is studied where the erasure probabilities of $S_i[t]$ were specified through the conditional distribution given the current state. In other words, there is another channel state that controls the time-varying erasure statistics of the BPEC. Note that our model can also be similarly described by choosing a two-state non-random state sequence where the state changes at fixed time n_{A}. However, in [40], the capacity region is found only when the random state sequence is independently and identically generated as [44], which is fundamentally different from our non-ergodic setting where the state sequence is fixed.*

The proof of Theorem 1 and Theorem 2 are presented in Sections IV and V, respectively.

C. Illustration of the results

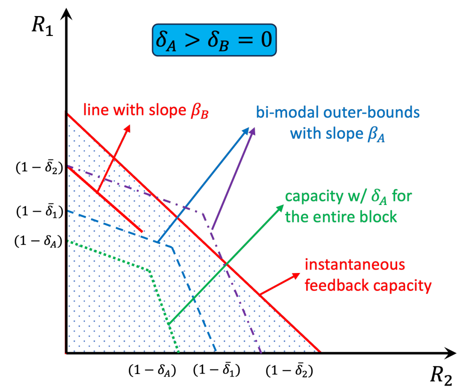

Fig. 2. Illustration of the outer-bound region of Theorem 1 for $\delta_{\mathrm{A}} > \delta_{\mathrm{B}} = 0$ and various values of n_{A}.

In this sub-section, we illustrate the results and provide further insights. For simplicity, we consider a bi-modal BPEC with $\delta_{\mathrm{A}} > \delta_{\mathrm{B}} = 0$, which means during the second mode no erasures occur and that $\beta_{\max} = \beta_{\mathrm{A}} > \beta_{\mathrm{B}} = 1$. Then, if $n_{\mathrm{A}} = 0$, then the capacity region is described by:

$$R_1 + R_2 \leq 1, \tag{9}$$

which matches the instantaneous feedback capacity (as no erasures would occur); if $n_{\mathrm{A}} = n$, then the capacity region is described by:

$$\left\{ \begin{array}{l} 0 \leq \beta_{\mathrm{A}} R_1 + R_2 \leq \beta_{\mathrm{A}}\left(1-\delta_{\mathrm{A}}\right), \\ 0 \leq R_1 + \beta_{\mathrm{A}} R_2 \leq \beta_{\mathrm{A}}\left(1-\delta_{\mathrm{A}}\right), \end{array} \right. \tag{10}$$

which is the capacity region of a BPEC with homogeneous erasure probability of δ_{A} for the entire communication block. Figure 2 includes these two baselines. Further, this figure shows the outer-bounds for two cases where $0 < n_{\mathrm{A}} < n$. The first case, results in an average erasure probability of δ_1, which determines the corner points of the region. Then, the slope of the outer-bounds is dominated by β_{A} as opposed to $\bar{\beta}$. The second case has a smaller average erasure probability, δ_2, compared to the first case (meaning that n_{A} is smaller in this case). For this case, the instantaneous feedback outer-bound becomes active.

IV. Converse Proof of Theorem 1

Here, we derive the outer-bounds presented in Theorem 1. The last inequality in (8) of Theorem 1 is the outer-bound with instantaneous feedback (as opposed to the delayed feedback assumed in this work), and is derived as follows:

$$\begin{aligned} n(R_1+R_2) &\leq H(W_1,W_2) \\ &\overset{\text{Fano}}{\leq} I(W_1,W_2;Y_1^n,Y_2^n|S^n,\text{SI}) + n\xi_n \\ &= H(Y_1^n,Y_2^n|S^n,\text{SI}) - \underbrace{H(Y_1^n,Y_2^n|W_1,W_2,S^n,\text{SI})}_{=\,0} + n\xi_n \\ &\leq n_A(1-\delta_{\text{A}}^2) + (n-n_A)(1-\delta_{\text{B}}^2) + n\xi_n. \end{aligned} \tag{11}$$

Below, we derive the following outer-bound and the other bound is immediately derived by interchanging user IDs:

$$0 \leq R_1 + \beta_{\max} R_2 \leq \beta_{\max}\left(1-\bar{\delta}\right). \tag{12}$$

Suppose rate-tuple (R_1,R_2) is achievable. We have

$$\begin{aligned} n\left(R_1+\beta_{\max}R_2\right) &= H(W_1) + \beta_{\max}H(W_2) \\ &\overset{(a)}{=} H(W_1|W_2,S^n,\text{SI}) + \beta_{\max}H(W_2|S^n,\text{SI}) \\ &\overset{(\text{Fano})}{\leq} I(W_1;Y_1^n|W_2,S^n,\text{SI}) + \beta_{\max}I(W_2;Y_2^n|S^n,\text{SI}) + n\xi_n \\ &= H(Y_1^n|W_2,S^n,\text{SI}) - \underbrace{H(Y_1^n|W_1,W_2,S^n,\text{SI})}_{=\,0} \\ &\quad + \beta_{\max}H(Y_2^n|S^n,\text{SI}) - \beta_{\max}H(Y_2^n|W_2,S^n,\text{SI}) + n\xi_n \\ &\overset{(b)}{\leq} \beta_{\max}H(Y_2^n|S^n,\text{SI}) + n\xi_n \\ &\overset{(c)}{\leq} \beta_{\max}n_{\text{A}}\left(1-\delta_{\text{A}}\right) + \beta_{\max}(n-n_{\text{A}})\left(1-\delta_{\text{B}}\right) + \xi_n \\ &\overset{(d)}{\leq} n\beta_{\max}\left(1-\bar{\delta}\right) + \xi_n, \end{aligned} \tag{13}$$

where $\xi_n \to 0$ as $n \to \infty$; (a) follows from the independence of messages; (b) follows from Lemma 1 below, which captures the interplay between the varying forward erasure probabilities and the delayed feedback; (c) is true since the entropy of a binary random variable is at most 1 (or $\log_2(q)$ for packets in $\mathbb{F}_q$) and the channel to the forward erasure probabilities are in the two modes as described earlier; and (d) follows form (5). Dividing both sides by n and let $n \to \infty$, we get (12).

Lemma 1. *For the two-user bi-modal BPEC with delayed channel state feedback as described in Section II, we have*[1]

$$H\left(Y_1^n|W_2,S^n,\text{SI}\right) - \beta_{\max}H\left(Y_2^n|W_2,S^n,\text{SI}\right) \leq 0, \tag{14}$$

for any encoding function satisfying (3).

Proof. We have

$$\begin{aligned} H\left(Y_2^n|W_2,S^n,\text{SI}\right) &= \sum_{t=1}^{n_{\text{A}}} H\left(Y_2[t]|Y_2^{t-1},W_2,S^n,\text{SI}\right) \\ &\quad + \sum_{t=n_{\text{A}}+1}^{n} H\left(Y_2[t]|Y_2^{t-1},W_2,S^n,\text{SI}\right) \\ &= \sum_{t=1}^{n_{\text{A}}} (1-\delta_{\text{A}}) H\left(X[t]|Y_2^{t-1},W_2,S[t]=1,S^{t-1},S^{t+1:n},\text{SI}\right) \\ &\quad + \sum_{t=n_{\text{A}}+1}^{n} (1-\delta_{\text{B}}) H\left(X[t]|Y_2^{t-1},W_2,S[t]=1,S^{t-1},S^{t+1:n},\text{SI}\right) \\ &\overset{(a)}{=} \sum_{t=1}^{n_{\text{A}}} (1-\delta_{\text{A}}) H\left(X[t]|Y_2^{t-1},W_2,S^n,\text{SI}\right) \\ &\quad + \sum_{t=n_{\text{A}}+1}^{n} (1-\delta_{\text{B}}) H\left(X[t]|Y_2^{t-1},W_2,S^n,\text{SI}\right) \\ &\geq \sum_{t=1}^{n_{\text{A}}} (1-\delta_{\text{A}}) H\left(X[t]|Y_1^{t-1},Y_2^{t-1},W_2,S^n,\text{SI}\right) \\ &\quad + \sum_{t=n_{\text{A}}+1}^{n} (1-\delta_{\text{B}}) H\left(X[t]|Y_1^{t-1},Y_2^{t-1},W_2,S^n,\text{SI}\right) \\ &= \sum_{t=1}^{n_{\text{A}}} \frac{(1-\delta_{\text{A}})}{(1-\delta_{\text{A}}^2)} H\left(Y_1[t],Y_2[t]|Y_1^{t-1},Y_2^{t-1},W_2,S^n,\text{SI}\right) \\ &\quad + \sum_{t=n_{\text{A}}+1}^{n} \frac{(1-\delta_{\text{B}})}{(1-\delta_{\text{B}}^2)} H\left(Y_1[t],Y_2[t]|Y_1^{t-1},Y_2^{t-1},W_2,S^n,\text{SI}\right) \\ &\overset{(b)}{\geq} \min\{1/\beta_{\text{A}},1/\beta_{\text{B}}\} H\left(Y_1^n,Y_2^n|W_2,S^n,\text{SI}\right) \\ &\overset{(c)}{\geq} 1/\beta_{\max} H\left(Y_1^n|W_2,S^n,\text{SI}\right), \end{aligned} \tag{15}$$

where (a) holds since $X[t]$ is independent of $S[t]$ (with the conditioned terms); (b) holds since the omitted term is the product of a discrete entropy term and

$$\max\{1/\beta_A,1/\beta_B\} - \min\{1/\beta_A,1/\beta_B\}, \tag{16}$$

which are both non-negative; and (c) follows from the non-negativity of the discrete entropy function and (7). ☐

[1] Regarding (1), $\beta_{\max}$ is the minimum value satisfying the inequality.

V. Achievability of the outer-bounds in Theorem 1

To prove Theorem 2, it suffices to come up with an example in which $0 < \delta_{\text{A}}, \delta_{\text{B}} < 1$ and $0 < n_{\text{A}} < n$ and for which the outer-bound region of Theorem 1 is achievable. Through this example, we will also showcase the opportunities of inter-modal coding in BPECs with varying statistics and feedback. We then briefly discuss the situation in which the inner and outer bounds diverge, and investigate potential future steps.

A. Inter-modal coding

Setup: Consider a bi-modal BPEC with channel state feedback having the following parameters: $\delta_{\text{A}} = 0.75$, $\delta_{\text{B}} = 0$, $n_A = \lfloor 32/35n \rfloor$. In this particular example, the first mode has a high erasure probability, while in the second mode, both links are always on and no erasure occurs. Further, the lengths of the modes are chosen carefully as it becomes clear shortly.

Benchmarks: The first benchmark is the one that ignores the feedback and achieves a sum-rate of 0.25 in the first mode and 1 in the second mode for an approximate average sum-rate of 0.31. Perhaps the more relevant benchmark is to treat the two modes separately, *i.e.*, intra-modal coding. In other words, we can treat the first mode as a BPEC with delayed channel state feedback having an optimal sum-rate of $7/22 \approx 0.318$ using the well-known three-phase network coding strategy [14], and the second mode as a BEC having an optimal sum-rate of 1. This intra-modal coding strategy results in a weighted average sum-rate of approximately 0.38.

Inter-modal coding: Instead of the separate treatment of the two modes, for *inter-modal* coding, we start with $m = n/5$ packets[2] for each receiver. We then create two phases during the first mode. The first phase is dedicated to transmitting the packets intended for user 1 until *at least* one receiver obtains that packet. The transmitter meanwhile keeps track of the status of each transmitted packet: delivered to the intended user, delivered to the unintended user and not the intended user, needs repeating. The second phase is similar but dedicated to the packets of user 2. We note that these phases are similar to the first two phases of the typical network coding strategy for BPECs with feedback. Each phase will take an average time of[3]

$$\frac{1}{(1-\delta_{\mathrm{A}}^2)} m = \frac{16}{35} n. \tag{17}$$

At the end of the second phase (which coincides with the end of the first mode), there will be on average

$$\frac{\delta_{\mathrm{A}}(1-\delta_{\mathrm{A}})}{(1-\delta_{\mathrm{A}}^2)} m = \frac{3}{35} n \tag{18}$$

packets intended for user i available only at user $\bar{i}$ for $\bar{i} = 2-i$ and $i = 1, 2$. Such packets are tracked at the transmitter in a queue represented by $v_{i|\bar{i}}$.

During phase 3, which coincides with the second mode where no erasure occurs, the transmitter at each time sends the summation (XOR in the binary domain) of the packets at the head of queues $v_{1|2}$ and $v_{2|1}$. The resulting packets are referred to as multi-cast packets as they assist *both* receivers. As the links in the second mode are always on, these multi-cast packets are delivered each time and the packets move through the queues one at a time. The length of the second mode in this example is carefully chosen to match the (average) number of multi-cast packets that would result from the first mode. It is straightforward to verify that the receivers can resolve the interference and recover their intended packets at the end of the multi-cast phase. Since each user will recover $n/5$ bits successfully over a block-length of n, the achievable sum-rate would be 0.4, which is larger than the weighted average when we treated the two modes separately.

For the parameters given here, the outer-bounds in Theorem 1 become:

$$\mathcal{C}^{\text{out}} \equiv \begin{cases} 0 \leq \frac{7}{4} R_1 + R_2 \leq \frac{7}{4}\left(1 - \frac{24}{35}\right), \\ 0 \leq R_1 + \frac{7}{4} R_2 \leq \frac{7}{4}\left(1 - \frac{24}{35}\right). \end{cases} \tag{19}$$

These outer-bounds result in a maximum symmetric sum-rate of 0.4 and as the corner points are easily achievable, the capacity region in this example is known and matches the region described in Theorem 1. This example demonstrates the existence of a non-trivial example in which the outer-bound region of Theorem 1 is achievable, and thus, proves Theorem 2.

[2]Without loss of generality, we assume $m \in \mathbb{Z}^+$. Any impact on analysis would vanish as $n \to \infty$.

[3]Here, we limit our analysis to the average values of different random variables. A careful analysis of the communication strategy will entail concentration inequalities similar to [29].

B. Challenges and further discussion

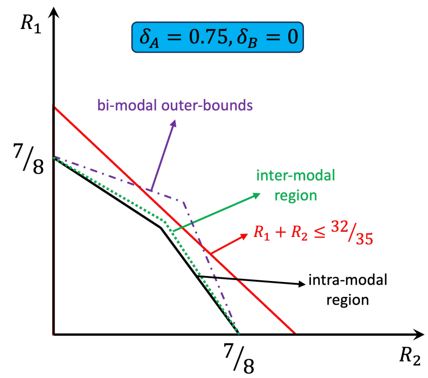

Fig. 3. An example with $\delta_A = 0.75$, $\delta_B = 0$, and $n_A = \lfloor n/6 \rfloor$ for which the inner and outer bounds deviate. Inter-modal coding nonetheless outperforms intra-modal coding.

For this part, we keep $\delta_A = 0.75$ and $\delta_B = 0$, but set $n_A = \lfloor n/6 \rfloor$ resulting in $\bar{\delta} = 1/8$. For these parameters, the instantaneous feedback outer-bound becomes active and maximum sum-rate is bounded by

$$R_1 + R_2 \leq \frac{32}{35} \approx 0.91. \tag{20}$$

Following a similar approach to the previous example, meaning that using the first mode to send new packets intended for each user in two phases and then using part of the second mode for delivering multi-cast packets, leaves most of the second mode untouched, which is then amended by delivering fresh packets at a sum-rate of 1. This strategy results in a sum-rate of approximately 0.89, which falls short of the outer-bound as shown in Figure 3. Nonetheless, this achievable rate is above the weighted average sum-rate of separate treatment of the two modes, which gives us an achievable rate of 0.875.

VI. Conclusion

We considered the bi-modal BPEC and provided a new set of outer-bounds as well as an inter-modal coding strategy. We showed that the outer-bounds could be tight under non-trivial conditions and thus, characterizing the capacity in such cases. We also discussed the challenges in general and observed the potential shortcoming of the outer-bounds in that they are oblivious to the relative length of the modes. Therefore, the first future direction would be to improve the inner and the outer bounds for the bi-modal BPEC. Further, the extension of the work to BPECs with different feedback and side-information assumptions [39] and to multi-modal BPECs (as opposed to bi-modal) would be an interesting direction. Finally, this idea could eventually be extended to erasure interference channels [24], [45].

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