THE CONJUGATE UNIFORMIZATION VIA 1-MOTIVES

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Abstract. We use the p-divisible group attached to a 1-motive to generalize the conjugate p-adic uniformization of Iovita–Morrow–Zaharescu to arbitrary p-adic formal semi-abelian schemes or p-divisible groups over the ring of integers in a p-adic field. This mirrors a mixed Hodge theory construction of the inverse uniformization map for complex semi-abelian varieties.

Contents

1. Introduction, statement of results, and context

Let K be a p-adic field; that is, a complete discretely valued extension of $\mathbb{Q}_p$ with perfect residue field κ. Let $\overline{K}$ be an algebraic closure of K and let C be the completion of $\overline{K}$. We write $K^{\mathrm{ur}} \subset \overline{K}$ for the maximal unramified extension and $\overline{\kappa}$ for the common residue field of K^{ur}, $\overline{K}$, and C. For G a p-divisible group or p-adic formal semi-abelian scheme over

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$\mathcal{O}_K$, we may evaluate points on a p-adically complete $\mathcal{O}_K$-algebra R by $G(R) := \lim_k G(R/p^k)$ (for a p-divisible group, the points à la Tate [16, §2.4]). We write $G[p^\infty]$ for the p-divisible group, $T_pG = \lim_n G[p^n]$ for the Tate module, and $V_pG = T_pG \otimes \mathbb{Q}_p$; the latter two are functors on $\mathcal{O}_K$-algebras (since even in the formal case $G[p^n]$ promotes automatically to a scheme over $\mathcal{O}_K$), but if we do not include an argument they are evaluated on $\overline{K}$ and equipped with the action of $\mathrm{Gal}(\overline{K}/K)$.

For G as above, we have the functorial Hodge–Tate exact sequence [16, §4]

$$0 \to \mathrm{Lie}\, G \otimes_{\mathcal{O}_K} C(1) \to T_pG \otimes_{\mathbb{Z}_p} C \to \omega_{G[p^\infty]^\vee} \otimes_{\mathcal{O}_K} C \to 0.$$

It depends only on $G[p^\infty]$, and by [16, Corollary 2] there is a unique Galois equivariant splitting

$$T_pG \to \mathrm{Lie}\, G \otimes_{\mathcal{O}_K} C(1). \tag{1.0.1}$$

As in [10, Theorem A], the kernel of Eq. (1.0.1) is $T_pG(\mathcal{O}_{K^{\mathrm{ur}}}) = T_pG(K^{\mathrm{ur}})$, i.e. the Tate module of the maximal étale p-divisible subgroup of G: this is evidently contained in the kernel by functoriality, but the kernel is crystalline as a sub-representation of T_pG and of Hodge–Tate weight zero by definition, thus unramified by a standard result in p-adic Hodge theory[1].

We will usually assume that $T_pG(K^{\mathrm{ur}}) = 0$, equivalently that G does not contain a non-trivial étale p-divisible subgroup, equivalently that Eq. (1.0.1) is injective. This condition is invariant under extension of p-adic fields and is not too serious, as, e.g., in the p-divisible case one can just quotient by the maximal étale p-divisible subgroup to reduce to this setting. However, it simplifies statements considerably (see Lemma 5.8 for why!).

In this work, we construct an integration homomorphism

$$\overline{I} \colon G(\mathcal{O}_K) \to (\mathrm{Lie}\, G \otimes_{\mathcal{O}_K} C(1))/T_pG,$$

by imitating with p-divisible groups a construction of the inverse of the uniformization map for a complex semi-abelian variety via the extensions of mixed Hodge structures attached to 1-motives. The map $\overline{I}$ is functorial and compatible with extensions $K \subseteq K' \subseteq C$ of p-adic fields. The integration homomorphism $\overline{I}$ is continuous (but see Remarks 1.5 and 6.9 for a subtlety regarding compatibility with infinite extensions). Our main result is

Theorem 1.1. *Let K be a p-adic field and let G be a p-divisible group or p-adic formal semi-abelian scheme over $\mathcal{O}_K$. If $T_pG(K^{\mathrm{ur}}) = 0$, then*

1. $\mathrm{Ker}(\overline{I}) = G(\mathcal{O}_K)^{p-\mathrm{div}}$, *the subgroup of p-divisible elements (see Definition 2.3).*
2. $\overline{I}(G[p^\infty](\mathcal{O}_K))$ *is the set of $y \in (\mathrm{Lie}\, G \otimes_{\mathcal{O}_K} C(1))/T_pG$ such that*
   (a) *y is stabilized by* $\mathrm{Gal}(\overline{K}/K)$,

[1] To wit, if the Hodge–Tate weights are zero then the filtration on the associated filtered isocrystal is trivial, thus it corresponds to an unramified Galois representation under Fontaine's equivalence. See [10, Appendix A] for another argument.

and, for $V_y := T_y \otimes \mathbb{Q}_p$ *where* T_y *is the extension of* $\mathbb{Z}_p$ *by* T_pG *obtained by pulling back along* $\mathbb{Z}_p \to (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$, $1 \mapsto y$,

(b) V_y *is a crystalline representation of* $\operatorname{Gal}(\overline{K}/K)$, *and*

(c) *the maximal unramified quotient of* V_y *is a split extension of* $\mathbb{Q}_p$ *by* $V_pG(\overline{\kappa})$ *as a* $\operatorname{Gal}(\overline{\kappa}/\kappa)$*-representation.*

We make some clarifying remarks on the statement of Theorem 1.1.

Remark 1.2.

(1) Because $T_pG(\mathcal{O}_K) = 0$, any p-divisible $\mathcal{O}_K$-point is uniquely p-divisible, and the Fontaine construction (Example 2.4) induces

$$\operatorname{Hom}(\mathbb{Z}[1/p], G(\kappa)) = \operatorname{Hom}(\mathbb{Z}[1/p], G(\mathcal{O}_K)) = G(\mathcal{O}_K)^{p-\mathrm{div}}.$$

In particular, the p-divisible elements are insensitive to purely ramified extension. Note that the prime-to-p torsion, which is in bijection by reduction with prime-to-p torsion in $G(\kappa)$, is always p-divisible. From this it also follows that if $G(\kappa)/G[p^\infty](\kappa)$ admits a set of p-divisible representatives in $G(\kappa)$ then $\overline{I}(G[p^\infty](\mathcal{O}_K)) = \overline{I}(G(\mathcal{O}_K))$ — this occurs, e.g., when κ is algebraically closed or finite.

(2) The crystalline condition (b) is necessary — for example, when $G = \mu_{p^\infty}$, the non-crystalline Tate module of the semistable elliptic curve $\mathbb{G}_m^{\mathrm{an}}/p^{\mathbb{Z}}$ can be obtained from a y satisfying (a) and trivially (c).

(3) The condition (c) is automatically verified for G such that

$$\operatorname{Ext}_{\operatorname{Gal}(\overline{\kappa}/\kappa)}(\mathbb{Q}_p, V_pG(\overline{\kappa})) = 0.$$

For example, it suffices that κ be algebraically closed, or that $G[p^\infty]_\kappa$ be connected, or that G_κ be a semi-abelian scheme and κ be finite (see Theorem 1.3 below).

In the setting of a local field K and a p-adic formal semi-abelian scheme over $\mathcal{O}_K$, we prove that the integration map is actually injective on $G[p^\infty](\mathcal{O}_K)$.

Theorem 1.3. *Suppose* $[K : \mathbb{Q}_p] < \infty$, G *is a* p*-adic formal semi-abelian scheme over* $\mathcal{O}_K$, *and* $T_pG(\mathcal{O}_{K^{\mathrm{ur}}}) = 0$. *Then* $\overline{I}$ *factors as the projection*

$$G(\mathcal{O}_K) = G[p^\infty](\mathcal{O}_K) \times G(\mathcal{O}_K)^{\textit{prime-to-p torsion}} \to G[p^\infty](\mathcal{O}_K) \tag{1.3.1}$$

composed with the injection $\overline{I}\colon G[p^\infty](\mathcal{O}_K) \hookrightarrow (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$. *The image consists of the points* y *stabilized by* $\operatorname{Gal}(\overline{K}/K)$ *with* V_y *crystalline (notation as in the statement of Theorem 1.1).*

Remark 1.4. In the proof of Theorem 1.3, we see how to extend this result to certain p-divisible groups. Indeed, we will prove that for any p-divisible group $H/\mathcal{O}_K$ with $T_pH(\mathcal{O}_{K^{\mathrm{ur}}}) = 0$ and $T_pH(\kappa) = 0$, the $\mathcal{O}_K$-points of H can be identified with the set of crystalline points in $(\operatorname{Lie} H \otimes_{\mathcal{O}_K} C(1))/T_pH$.

We show in Section 6 that $\overline{I}$ can also be constructed using Fontaine integration [7], so that it agrees with the map studied in [10]. Theorem 1.3 thus generalizes the main result of [10], which treats the case where G is the

base change to $\mathcal{O}_K$ of a good reduction abelian variety over a finite unramified extension of $\mathbb{Q}_p$. In [10], the trivial decomposition of Eq. (1.3.1) into the points of the p-divisible group and the prime-to-p torsion is not clearly described, so that one factor is never identified as being the points of the p-divisible group. As a result, in [10], this uniformization is presented as a result about abelian varieties.

By contrast, we wish to emphasize that this uniformization is purely a result about p-divisible groups. Moreover, the main motivation for this work was to give a simpler and more general construction of the uniformization of loc. cit. rendering its key properties evident via the theory of p-divisible groups. Beyond extending the field of definition, our generality allows for, e.g.,

(1) the connected component of the Néron model of an arbitrary abelian variety (after finite extension to obtain semi-abelian reduction), and
(2) non-algebraizable good reduction abeloid varieties.

We refer to the characterization of the $\mathcal{O}_K$-points of certain p-divisible groups described in Theorem 1.3 and Remark 1.4 as the *conjugate uniformization* because the splitting of the Hodge–Tate filtration is an analog of the conjugate filtration in complex Hodge theory (and to distinguish it from the Scholze–Weinstein uniformization, recalled below in Section 7).

Remark 1.5. In [10] an emphasis is placed on the continuity properties of $\overline{I}$ on $G(\mathcal{O}_{\overline{K}}) := \bigcup_{K' \subseteq C, [K':K]<\infty} G(\mathcal{O}_{K'})$ — this aspect did not appear in Theorem 1.1 or Theorem 1.3 because it plays no role in the construction and because we have stated these results using points in a single p-adic field while the subtle topological issue in question apears only if we consider all $\mathcal{O}_{\overline{K}}$-points at once. See Remark 6.9 for further discussion.

Before describing our construction of $\overline{I}$ and explaining how it leads to a proof of Theorem 1.1, we take a brief detour to explain the analogous construction of the inverse uniformization map for complex abelian varieties.

1.6. **Uniformization of complex semi-abelian varieties.** Let G be a semi-abelian variety over $\mathbb{C}$, that is, an extension of an abelian variety by a torus. Via exponentiation, we obtain a uniformization

$$\exp\colon \operatorname{Lie} G / H_1(G(\mathbb{C}), \mathbb{Z}) \to G(\mathbb{C}).$$

The inverse can be constructed by integration: if we identify $\operatorname{Lie} G$ with the dual to the space of invariant differentials on G, then $x \in G(\mathbb{C})$ maps to

$$\overline{I}_{\mathbb{C}}(x)\colon \omega \mapsto \int_e^x \omega.$$

This integral factors through the category of 1-motives [5, §10] — that is, to the point x we can associate the 1-motive $G_x\colon \mathbb{Z} \to G$ where the map sends 1 to x. Taking homology gives an extension of mixed Hodge structures

$$0 \to H_1(G, \mathbb{Z}) \to H_1(G_x, \mathbb{Z}) \to \mathbb{Z} \to 0.$$

Concretely, $H_1(G_x, \mathbb{Z})$ is the homology of G relative to $\{e, x\}$, so that the class 1 in the quotient trivial mixed Hodge structure $\mathbb{Z}$ corresponds to any path from e to x. The integration map can then be obtained by quotienting by the Hodge filtration Fil^{-1} in the first two terms to obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(G,\mathbb{Z}) & \longrightarrow & H_1(G_x,\mathbb{Z}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Lie} G & = & \operatorname{Lie} G_x & \longrightarrow & \operatorname{Lie} G/H_1(G,\mathbb{Z}). & & \end{array}$$

Tracing through the construction, we find $\overline{I}_{\mathbb{C}}(x)$ is the image of $1 \in \mathbb{Z}$.

Example 1.7. When $G = \mathbb{G}_m$, if we trivialize $\operatorname{Lie} G$ via the vector field $x\partial_x$, then the uniformization can be identified with the complex exponential

$$\exp\colon \mathbb{C}/\mathbb{Z}(1) \to \mathbb{C}^\times$$

whose inverse is the logarithm map obtained by integrating the dual basis $\frac{dx}{x}$ for $\omega_{\mathbb{C}^\times}$ from 1 to x. It is a linear algebra exercise to see that there is a natural identification $\mathrm{Ext}_{\mathbb{Z}-\mathrm{MHS}}(\mathbb{Z}, \mathbb{Z}(1)) = \mathbb{C}/\mathbb{Z}(1)$, and to verify that the mixed Hodge structure on the first homology of the (twice punctured) nodal cubic obtained from $\mathbb{C}^\times$ by glueing 1 and x, which is canonically an extension of $\mathbb{Z}$ by $\mathbb{Z}(1)$, is matched in this identification with $\log(x)$.

1.8. **Construction of the map $\overline{I}$ and outline of proof.** Returning to the setup at the start of the introduction, we now construct the homomorphism

$$\overline{I}\colon G(\mathcal{O}_K) \to (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG.$$

The construction is functorial and compatible with extension $K \subset K' \subset C$. To construct it, to any point $x \in G(\mathcal{O}_K)$ we attach the Kummer extension

$$\mathcal{E}_x\colon 0 \to G[p^\infty] \to G_x[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

of p-divisible groups over $\mathcal{O}_K$ given by formally adjoining p-power roots of $-x$ to G. When G is a p-adic formal semi-abelian scheme, this can be identified with the p-divisible group of the p-adic formal 1-motive $\mathbb{Z} \xrightarrow{1\mapsto x} G$ with the extension structure coming from the weight filtration.

We then take Tate modules and apply Eq. (1.0.1) to obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_pG & \longrightarrow & T_pG_x & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1) & = & \operatorname{Lie} G_x \otimes_{\mathcal{O}_K} C(1) & \to & (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG. & & \end{array}$$

We define $\overline{I}(x)$ to be the image of $1 \in \mathbb{Z}_p$ in $(\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$.

To prove Theorem 1.1, we first establish that all *rigidified* extensions (Definition 4.3) of $\mathbb{Q}_p/\mathbb{Z}_p$ by $G[p^\infty]$ are of the form $\mathcal{E}_x$ for $x \in G[p^\infty](\mathcal{O}_K)$ (cf. Theorem 4.4). Using that $\mathcal{E}_x$ is split if and only if x is p-divisible and a simple argument again using that crystalline representations of Hodge–Tate weight zero are unramified, we can characterize the kernel on all of $G(\mathcal{O}_K)$

as the p-divisible elements. The crystalline characterization of the image is then immediate from the equivalence between lattices in crystalline representations with Hodge–Tate weights $\{0,1\}$ and p-divisible groups (see e.g., [12] or [14, Corollary 6.2.3]) — indeed, this gives an extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $G[p^\infty]$ attached to any y satisfying (a) and (b), and the analysis of extensions indicated already shows it comes from an $x \in G[p^\infty](\mathcal{O}_K)$ if and only if it also satisfies (c).

1.9. **Outline.** In Section 2, we recall some constructions regarding p-divisible groups, p-adic formal semi-abelian schemes, universal covers, and p-divisible elements. In Section 3, we give a construction of the p-divisible group attached to a 1-motive, and in Section 4, we discuss rigidified extensions of p-divisible groups. Because it is of independent interest and the proofs are not any more difficult, in Section 3 and Section 4, we work in more generality than will be needed for the application to Theorem 1.1 and also discuss some complements (e.g., the p-divisible group of a Raynaud-uniformized abeloid variety) — for the proof of Theorem 1.1, the key result is the identification of a p-divisible group G with the moduli of rigidified extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by G. In Section 5, we prove Theorem 1.1, and in Section 6, we show the equivalence with the definition of $\overline{I}$ via the Fontaine integral and give another construction using the crystalline incarnation of the universal cover. In Section 7, we discuss geometric aspects of the conjugate uniformization.

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2. Preliminaries

In this section, we recall the construction of the universal cover of a p-divisible group or p-adic formal semi-abelian scheme and its basic properties.

Let R be a p-adically complete ring, and let G be a p-divisible group or a p-adic formal semi-abelian scheme over R. The universal cover of G is the functor on p-adically complete R-algebras

$$\tilde{G} = \mathrm{Hom}(\mathbb{Z}[1/p], G) = \lim(G \xleftarrow{p} G \xleftarrow{p} \cdots),\ S \mapsto \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], G(S)).$$

When G is a p-divisible group, we can replace $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], \cdot)$ with $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p, \cdot)$.

Inside of $\tilde{G}$, we have the Tate module

$$T_pG = \mathrm{Hom}(\mathbb{Z}[1/p]/\mathbb{Z}, G) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, G) = \lim(1 \xleftarrow{p} G[p] \xleftarrow{p} G[p^2] \xleftarrow{p} \cdots).$$

Of course, T_pG only depends on $G[p^\infty]$, and it is the kernel of the projection $\tilde{G} \to G$ given by evaluation at 1 (an fpqc surjection when restricted to the category of p-nilpotent R-algebras endowed with the fpqc topology).

The key property of $\tilde{G}$ is that, by a construction due to Fontaine, it is invariant under topologically nilpotent thickenings:

Proposition 2.1. *Let S be a p-adically complete R-algebra and let I be a topologically nilpotent ideal in S. Then reduction induces an isomorphism*

$$\tilde{G}(S) = \tilde{G}(S/I).$$

Proof. By considering liftings from $S/(I, p^k)$ to S/I and passing to the limit, it suffices to suppose $p^k = 0$ on S and that I is nilpotent. In this case, let $(g_1, g_2, \ldots) \in \tilde{G}(S/I)$, and choose arbitrary element-wise lifts $\tilde{g}_1, \tilde{g}_2, \ldots$ to $G(S)$. Then, we claim that for N sufficiently large,

$$(p^N \tilde{g}_N, p^N \tilde{g}_{N+1}, \ldots)$$

is independent of choices and furnishes a lift to $\tilde{G}(S)$ that is independent of the choices thus unique. Indeed, two different lifts of g_i differ by an element of $\ker(G(S) \to G(S/I))$, and the Drinfeld construction [11, Lemma 1.1.2] shows a large power of p annihilates this subgroup. □

Remark 2.2. We note that Proposition 2.1 for G a p-divisible group appears as [14, Proposition 3.1.3(ii)], and the proofs are similar.

Definition 2.3. We say an element of $G(S)$ is (uniquely) p-divisible if it admits a (unique) compatible system of p-power roots in $G(S)$.

Note that an element of $G(S)$ is p-divisible if and only if it is in the image of $\tilde{G}(S) \to G(S)$, and that the p-divisible elements are uniquely p-divisible if and only if $T_pG(S) = 0$.

Example 2.4. If K is a p-adic field with residue field κ and $G/\mathcal{O}_K$, then Proposition 2.1 gives

$$\tilde{G}(\mathcal{O}_K) = \tilde{G}(\kappa).$$

In particular, $\tilde{G}(\mathcal{O}_K)$ and the p-divisible elements of $G(\mathcal{O}_K)$ are invariant under finite totally ramified extensions. Note that a p-divisible element lies in the formal neighborhood of the identity if and only if it comes from an element of $T_pG(\kappa)$, and when G is semi-abelian, it lies in the p-divisible group if and only if it comes from an element of $V_pG(\kappa) = T_pG(\kappa) \otimes \mathbb{Q}_p$.

Note that $T_pG(\kappa)$ is often much larger than $T_pG(\mathcal{O}_K)$ — for example, if κ is algebraically closed, then $G_\kappa[p^\infty] \cong G_\kappa^\wedge \times (\mathbb{Q}_p/\mathbb{Z}_p)^r$ where $G_\kappa^\wedge$ is the connected component of the identity[2] and r is the rank of $T_pG(\kappa)$, but for a generic lift of G_κ to $\mathcal{O}_K$ no copy of $\mathbb{Q}_p/\mathbb{Z}_p$ will lift to a subgroup so we will have $T_pG(\mathcal{O}_K) = 0$. In this case, $T_pG(\kappa)$ is identified with the subgroup of $G^\wedge(\mathcal{O}_K)$ consisting of points that are (uniquely) p-divisible in $G(\mathcal{O}_K)$, and in fact this subgroup determines the extension structure for the connected-étale sequence (see Example 4.9).

Remark 2.5. It also follows from Proposition 2.1 that, for H_1 and H_2 p-divisible groups over R and I an open topologically nilpotent ideal of R,

$$\mathrm{Hom}(\widetilde{H}_1, \widetilde{H}_2) = \mathrm{Hom}(H_{1,R/I}, H_{2,R/I}) \otimes \mathbb{Q}_p.$$

[2]Equivalently, we may consider this as a formal group.

In other words, to give a map of universal covers is the same as to give a map in the isogeny-category of p-divisible groups over R/I. This is a helpful way to encode the independence of the latter on the choice of I while also bringing to the forefront the Fontaine lifts as in the proof of Proposition 2.1. We use this only in the case when H_1 is étale, which immediately reduces to the case that $H_1 = \mathbb{Q}_p/\mathbb{Z}_p$ where it is an immediate consequence of Proposition 2.1 and the identity $T_pH_2(A) \otimes \mathbb{Q}_p = \tilde{H}_2(A)$ for any R-algebra A such that p is nilpotent in A.

3. The p-divisible group of a 1-motive

In this section, we construct the p-divisible group of a 1-motive. First, we recall the notion of a 1-motive (see [5, §10], [2]). Let R be a ring.

Definition 3.1. A 1-motive over $\operatorname{Spec} R$ is a homomorphism $\varphi\colon M \to G$ where G is a semi-abelian scheme over R and M is an étale $\mathbb{Z}$-local system on $\operatorname{Spec} R$.

Given a 1-motive φ and a prime p, we can construct an extension of group schemes over R

$$\mathcal{E}_\varphi\colon 0 \to G \to G_\varphi \to M \otimes (\mathbb{Z}[1/p]/\mathbb{Z}) \to 0$$

by formally adjoining p-power roots along φ. Precisely, G_φ is the push-out

$$\begin{array}{ccc} M & \xrightarrow[-\varphi]{} & G \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Z}[1/p] & \longrightarrow & G_\varphi \end{array} \tag{3.1.1}$$

This push-out can be realized concretely as a carrying law: if M is trivializeable then, fixing a trivialization $M \cong \mathbb{Z}^m$ and writing $I_p = \mathbb{Z}[1/p] \cap [0,1) \subset \mathbb{Q}$,

$$G_\varphi \cong \bigsqcup_{t \in I_p^m} G,\ g_s + h_t = (g + h - \varphi(\lfloor s + t \rfloor))_{\{s+t\}} \tag{3.1.2}$$

where here $\lfloor \cdot \rfloor$ denotes floor and $\{\cdot\}$ denotes the fractional part. The general case is reduced to the case of M trivializeable by descent from a finite étale cover.

Remark 3.2. If we take the short exact sequence

$$0 \to M \to M \otimes \mathbb{Z}[1/p] \to M \otimes (\mathbb{Z}[1/p]/\mathbb{Z}) \to 0$$

and apply the functor $\operatorname{Hom}(\cdot, G)$, the boundary map

$$\operatorname{Hom}(M, G) \to \operatorname{Ext}(M \otimes (\mathbb{Z}[1/p]/\mathbb{Z}), G).$$

sends φ to the extension $\mathcal{E}_\varphi$.

Again, after taking a trivialization of M, we have that

$$G_\varphi[p^n] = \bigsqcup_{t \in (1/p^n\mathbb{Z} \cap [0,1))^m} G_\varphi[p^n]_t \tag{3.2.1}$$

where $G_\varphi[p^n]_t$ is the fiber of multiplication by p^n on G above $\varphi(p^n t)$.

In particular, $G_\varphi[p^\infty]$ is a p-divisible group, and $\mathcal{E}_\varphi[p^\infty]$ is an extension by p-divisible groups

$$\mathcal{E}_\varphi[p^\infty]\colon 0 \to G[p^\infty] \to G_\varphi[p^\infty] \to M \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \to 0.$$

Definition 3.3. The p-divisible group attached to the 1-motive $\varphi\colon M \to G$ is $G_\varphi[p^\infty]$, and the extension structure $\mathcal{E}_\varphi[p^\infty]$ is the weight filtration.

Remark 3.4. Over $\mathbb{C}$, Eq. (3.2.1) agrees with the p^n torsion of a 1-motive as constructed in [5, (10.1.5)] (in particular, the minus sign in the push-out diagram Eq. (3.1.1) arises naturally from the Koszul sign rule). More generally, this construction agrees with the construction of the p-divisible group attached to a 1-motive in [2, §2.4].

Next, we define a p-adic formal 1-motive.

Definition 3.5. If R is p-adically complete, a p-adic formal 1-motive is a homomorphism $M \to G$ where M is an étale $\mathbb{Z}$-local system on $\operatorname{Spf} R$ (or just $\operatorname{Spec} R/p$) and G is a p-adic formal semi-abelian scheme over $\operatorname{Spf} R$.

If G is a p-adic formal semi-abelian scheme over R, then applying the previous construction over R/p^n for all n yields an extension of p-divisible groups $\mathcal{E}_\varphi[p^\infty]$ over R. Note that if φ factors through $G[p^\infty]$, then in the construction of $G_\varphi[p^\infty]$ we may dispense with G altogether and work from the beginning with $G[p^\infty]$ in its place. Note that the maps $\varphi\colon \mathbb{Z} \to G[p^\infty]$ correspond exactly to the points à la Tate,

$$G[p^\infty](R) := \lim_k \operatorname{colim}_n G[p^n](R/p^k),$$

which typically is much larger than $G(R)[p^\infty]$.

On the other hand, there are sometimes very interesting extensions of $G[p^\infty]$ that can only be seen by considering points in G, as the following example illustrates.

Example 3.6. Let $E/\mathbb{Z}(\!(q)\!)$ be the Tate elliptic curve, and let $\varphi\colon \mathbb{Z} \to \mathbb{G}_m(\mathbb{Z}(\!(q)\!))$ sending 1 to q. Then there is a canonical isomorphism $E[p^\infty] \cong (\mathbb{G}_m)_\varphi[p^\infty]$. Even if we p-adically complete, this extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by μ_{p^∞} still does not arise from a 1-motive factoring through μ_{p^∞}, which see only Serre–Tate extensions; this version was treated in detail in [8], where the construction of $\mathcal{E}_\varphi$ for $G = \mathbb{G}_m$ and $M = \mathbb{Z}$ was referred to as the theory of Kummer extensions and was used to unify computations for p-adic modular forms in Serre–Tate and cuspidal coordinates.

Example 3.7. There is a completely analogous construction for rigid analytic 1-motives, and using this one can construct the p-divisible group of a Raynaud uniformized abeloid variety in the same way using a rigid 1-motive. In this case one obtains a p-divisible group over K that does not extend to $\mathcal{O}_K$ (but can sometimes still be made sense of algebraically over a complete $\mathcal{O}_K$-algebra in the limit by adding a formal variable as in Example 3.6).

These kinds of examples will not play a serious role in the remainder of this work because of Theorem 4.6 below, which implies that for H a p-divisible group over the ring of integers in a p-adic field, up to a minor discrepancy all extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by H can be obtained already from points of H.

4. Rigidifed Extensions

In this section, we define rigidifed extensions and prove Theorem 4.6. Let R be a p-adically complete ring, and let H/R be a p-divisible group. Suppose M is a $\mathbb{Z}_p$-local system on $\operatorname{Spf} R$ (equivalently $\operatorname{Spec} R/p$) and $\varphi\colon M \to H$ is a $\mathbb{Z}_p$-linear homomorphism. We note also that the datum of M is equivalent to the étale p-divisible group $M \otimes \mathbb{Q}_p/\mathbb{Z}_p$ (from which M is recovered as the Tate module).

Remark 4.1. It is tempting to call φ a p-divisible 1-motive, but this would be a mistake (see Remark 4.10).

Example 4.2. Suppose $\varphi\colon M \to G$ is a p-adic formal 1-motive over $\operatorname{Spf} R$ such that φ factors through $G[p^\infty]$. Then φ extends uniquely to

$$\varphi \otimes \mathbb{Z}_p\colon M \otimes \mathbb{Z}_p \to G[p^\infty].$$

Given $\varphi\colon M \to H$, we form the pushout H_φ in the category of sheaves on p-nilpotent R-algebras with the fpqc topology. This is analogous to the earlier construction Eq. (3.1.1) with 1-motives but replacing $\mathbb{Z}[1/p]$ with $\mathbb{Q}_p$ and $\mathbb{Z}$ with $\mathbb{Z}_p$:

$$\begin{array}{ccc} M & \xrightarrow{-\varphi} & H \\ \downarrow & & \downarrow \\ M \otimes \mathbb{Q}_p & \longrightarrow & H_\varphi. \end{array}$$

Note that $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are equipped with their natural topologies and should be interpreted here as topological constant sheaves — that is, for any p-nilpotent R-algebra S, $\mathbb{Q}_p(S)$ (resp. $\mathbb{Z}_p(S)$) is the set of continuous maps from $\operatorname{Spec} S$ to $\mathbb{Q}_p$ (resp. $\mathbb{Z}_p$).

After a pro-finite étale cover to trivialize M, this admits an identical explicit description via a carrying law. In particular, we obtain a short exact sequence

$$\mathcal{E}_\varphi\colon 0 \to H \to H_\varphi \to M \otimes (\mathbb{Q}_p/\mathbb{Z}_p) \to 0$$

and at the level of universal covers we have the short exact sequence

$$\tilde{\mathcal{E}}_\varphi\colon 0 \to \tilde{H} \to \tilde{H}_\varphi \to M \otimes \mathbb{Q}_p \to 0$$

in the category of sheaves on p-nilpotent R-algebras with the fpqc topology. In fact, there is another important piece of data in the mix: there is a canonical section $s_\varphi\colon M \otimes \mathbb{Q}_p \to \tilde{H}_\varphi$ of the induced extension of universal covers $\tilde{\mathcal{E}}_\varphi$ coming from the canonical map $M \otimes \mathbb{Q}_p \to H_\varphi$ extending $-\varphi$.

Definition 4.3. With the notation as above, we refer to an extension $\mathcal{E}$ of $M \otimes (\mathbb{Q}_p/\mathbb{Z}_p)$ by H equipped with a section s of $\tilde{\mathcal{E}} \to M \otimes \mathbb{Q}_p$ as rigidified. We let $\mathrm{RigExt}(M \otimes (\mathbb{Q}_p/\mathbb{Z}_p), H)$ be the functor on p-adically complete R-algebras sending S to the set of isomorphism classes of rigidified extensions of $M_S \otimes (\mathbb{Q}_p/\mathbb{Z}_p)$ by H_S.

We now describe how one can interpret the functor $\mathrm{RigExt}(M \otimes (\mathbb{Q}_p/\mathbb{Z}_p), H)$ in terms of p-adic 1-motives.

Theorem 4.4. *The assignment $\varphi \mapsto (\mathcal{E}_\varphi, s_\varphi)$ is an isomorphism of functors from p-adically complete R-algebras to abelian groups*

$$\mathrm{Hom}(M, H) \to \mathrm{RigExt}(M \otimes (\mathbb{Q}_p/\mathbb{Z}_p), H)$$

where the right-hand side is equipped with the Baer sum.

Proof. If $(\mathcal{E}, s)$ is a rigidified extension, let s_0 denote the composition of s with projection to the first coordinate from $\tilde{\mathcal{E}} \to \mathcal{E}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{H} & \longrightarrow & \tilde{\mathcal{E}} & \underset{\longrightarrow}{\overset{s}{\longleftarrow}} & M \otimes \mathbb{Q}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & \overset{s_0}{\swarrow} & \downarrow & & \\ 0 & \longrightarrow & H & \longrightarrow & \mathcal{E} & \longrightarrow & M \otimes (\mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & 0 \end{array}$$

When we restrict s_0 to $M = M \otimes \mathbb{Z}_p \hookrightarrow M \otimes \mathbb{Q}_p$, this morphism factors through the kernel of the bottom right map, and so we recover a map $M \to H$. In other words, s_0 lies in $\mathrm{Hom}(M, H)$, and hence we obtain a canonical isomorphism from the push-out property

$$\mathcal{E}_{-s_0} \xrightarrow{\sim} \mathcal{E}$$

compatible with the sections. Note that $\mathcal{E}_{-s_0} \cong \mathcal{E}_{s_0}$, and so we have an isomorphism $\mathcal{E} \cong \mathcal{E}_{s_0}$.

The assignment $(\mathcal{E}, s) \mapsto s_0$ is well-defined and gives an inverse to the map in the statement of the theorem. That the map is compatible with the group structures is immediate by comparing the push-outs in the definition of $\mathcal{E}_\varphi$ and of the Baer sum of extensions. □

Lemma 4.5. *The kernel of the induced map*

$$\mathrm{Hom}(M, H)(R) \to \mathrm{Ext}(M \otimes (\mathbb{Q}_p/\mathbb{Z}_p), H)(R)$$

obtained by forgetting the rigidification is the image of $\mathrm{Hom}(M \otimes \mathbb{Q}_p, H)(R)$.

Proof. Indeed, by the push-out property any element of $\mathrm{Hom}(M \otimes \mathbb{Q}_p, H)(R)$ restricting to a given $\varphi \in \mathrm{Hom}(M, H)(R)$ gives rise to an isomorphism of $\mathcal{E}_\varphi$ with the trivial extension, and vice versa. □

In our main case of interest, we can also understand the image:

Theorem 4.6. *Suppose R/p is Artinian local with residue field κ. Then a rigidification of $\mathcal{E} \in \mathrm{Ext}(M \otimes (\mathbb{Q}_p/\mathbb{Z}_p), H)$ is equivalent to a splitting of $\mathcal{E}_\kappa$ in the isogeny category of p-divisible groups over κ. In particular,*

(1) *If the residue field κ is algebraically closed, then any extension of an étale p-divisible group by H can be rigidified.*
(2) *If the residue field κ is perfect, then to give a rigidification it is equivalent to give a splitting in the isogeny category of p-divisible groups over κ of the induced extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by $H^{\text{ét}}_\kappa$. In particular, if H_κ is connected, then any extension of an étale p-divisible group by H can be uniquely rigidified.*

Proof. The first part of the theorem follows from Remark 2.5. Suppose given an extension $\mathcal{E}$ as in (1) or (2). Then to split $\tilde{\mathcal{E}}$ it is equivalent to split $\mathcal{E}_\kappa$ up to isogeny. In the first case, the category of p-divisible groups up to isogeny over κ is semi-simple, so it is always split. In the second case, the category may not be semi-simple but the slope decomposition still descends to κ, so that a splitting occurs purely in the slope zero (étale) part. □

Remark 4.7. The p-divisible group of the Tate curve over $\mathbb{F}_p(\!(q)\!)$, an extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by μ_{p^∞} (see Example 3.6), shows that the assumption that the residue field is perfect in (2) cannot be removed.

Example 4.8. Consider the projection map

$$\tau\colon \operatorname{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \widehat{\mathbb{Z}} \to \mathbb{Z}_p.$$

The Galois representation $\begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}$ is the Tate module of a non-trivial extension of $\mathbb{Q}_p/\mathbb{Z}_p$ by itself over $\mathbb{Z}_p$ that cannot be rigidified even if we allow passage to arbitrary finite extensions.

Example 4.9. In this example, we describe how the canonical splitting of the connected-étale sequence over κ is related to Theorem 4.6-(1). Suppose (for simplicity) that κ is algebraically closed, and let G be a p-divisible group over $\mathcal{O}_K$. If $T_pG(\mathcal{O}_K) = 0$, then $T_pG(\kappa) \subset \tilde{G}(\kappa) = \tilde{G}(\mathcal{O}_K)$ is identified via projection to the first coordinate with the $\mathbb{Z}_p$-module M of elements in $G^\wedge(\mathcal{O}_K)$ that are p-divisible in $G(\mathcal{O}_K)$. We refer the reader to Example 2.4 for further discussion. The connected-étale sequence

$$0 \to G^\wedge \to G \to G^{\text{ét}} \to 0$$

induces an isomorphism of $M \otimes (\mathbb{Q}_p/\mathbb{Z}_p)$ with $G^{\text{ét}}$, and this is the extension of $G^\wedge$ determined by the inclusion map $\varphi\colon M \hookrightarrow G^\wedge$. Moreover, the rigidification s_φ from Theorem 4.6-(1) comes from the map φ.

Remark 4.10. The category of rigidified Breuil–Kisin–Fargues modules (see [1], [3, §4] provides a natural category of cohomological motives over $\mathbb{C}_p$ (for example, for a smooth proper formal scheme over $\mathcal{O}_{\mathbb{C}_p}$ with torsion-free crystalline cohomology, there is a rigidified Breuil–Kisin–Fargues module in each cohomological degree that recovers all other p-adic cohomology theories and their comparisons). The category of p-divisible groups over $\mathcal{O}_{\mathbb{C}_p}$ is equivalent to the full sub-category of the category of Breuil–Kisin–Fargues modules with slopes in $[0, 1]$, but if $H_{\overline{\mathbb{F}}_p}$ is not isoclinic then there is no

canonical choice of a rigidification for H — a rigidification here amounts to the choice of an isogeny $H_{\overline{\mathbb{F}}_p} \times_{\overline{\mathbb{F}}_p} \mathcal{O}_{\mathbb{C}_p}/p \to H_{\mathcal{O}_{\mathbb{C}_p}/p}$ inducing the identity modulo $\mathfrak{m}_{\mathbb{C}_p}$. If H is connected and equipped with a rigidification, then a rigidified extension as considered in this section is exactly an extension in the category of rigidified Breuil–Kisin–Fargues modules. The natural category analogous to 1-motives here is the category of rigidified Breuil–Kisin–Fargues modules with slopes in $[0,1]$, and Theorem 4.6-(2) expresses the fact that the choice of a rational structure over a discretely valued subfield induces a canonical rigidification.

5. Proof of main theorem

In this section we construct the map $\overline{I}$ and prove Theorem 1.1. Before doing so, we recall the setup and restate our main result.

Let K be a p-adic field and let G be a p-divisible group or p-adic formal semi-abelian scheme over $\mathcal{O}_K$. We will construct an integration homomorphism

$$\overline{I}\colon G(\mathcal{O}_K) \to (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG,$$

and prove the following results.

Theorem 5.1. *If $T_pG(K^{\mathrm{ur}}) = 0$, then*

(1) $\operatorname{Ker}(\overline{I}) = G(\mathcal{O}_K)^{p-\mathrm{div}}$, *the subgroup of p-divisible elements (see Definition 2.3).*
(2) $\overline{I}(G[p^\infty](\mathcal{O}_K))$ *is the set of $y \in (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$ such that*
   (a) *y is stabilized by $\operatorname{Gal}(\overline{K}/K)$,*
   and, for $V_y := T_y \otimes \mathbb{Q}_p$ where T_y is the extension of $\mathbb{Z}_p$ by T_pG obtained by pulling back along $\mathbb{Z}_p \to (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$, $1 \mapsto y$,
   (b) *V_y is a crystalline representation of $\operatorname{Gal}(\overline{K}/K)$, and*
   (c) *the maximal unramified quotient of V_y is a split extension of $\mathbb{Q}_p$ by $V_pG(\overline{\kappa})$ as a $\operatorname{Gal}(\overline{\kappa}/\kappa)$-representation.*

5.2. **Construction of the map $\overline{I}$.** We construct $\overline{I}$ as the composition of the following two homomorphisms:

(1) The homomorphism $\mathcal{E}\colon G(\mathcal{O}_K) \to \operatorname{Ext}(\mathbb{Q}_p/\mathbb{Z}_p, G[p^\infty])$, $x \mapsto \mathcal{E}_x[p^\infty]$, where
$$\mathcal{E}_x\colon 0 \to G \to G_x \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$
is the extension attached to $\varphi\colon \mathbb{Z} \to G$, $1 \mapsto x$ by the construction of Section 4.
(2) The homomorphism $\psi\colon \operatorname{Ext}(\mathbb{Q}_p/\mathbb{Z}_p, G[p^\infty]) \to (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$ sending
$$0 \to G[p^\infty] \to H \to \mathbb{Q}_p/\mathbb{Z}_p \to 0.$$
to the image of $1 \in \mathbb{Z}_p$ under the right vertical arrow of the diagram induced by applying the canonical splitting of the Hodge–Tate

filtration (Eq. (1.0.1)) in the left two terms,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_pG & \longrightarrow & T_pH & \longrightarrow & \mathbb{Z}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1) & = & \operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1) & \to & (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG. & & \end{array}$$

Remark 5.3. Note that the map $\mathbb{Z}[1/p] \to G_x$ extending $-\varphi$ in the pushout construction of G_x gives rise to a canonical system of p-power roots of $-x$ in $G_x(\mathcal{O}_K)$ via the images of $1/p^n$. We compile these as an element $\widetilde{-x}_{\mathrm{can}}$ of $\widetilde{G_x}(\mathcal{O}_K)$ lifting $-x \in G(\mathcal{O}_K) \subset G_x(\mathcal{O}_K)$ and projecting to $1 \in \widetilde{\mathbb{Q}_p/\mathbb{Z}_p} = \operatorname{Hom}(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p)$. In the explicit coordinates of Eq. (3.1.2),

$$\widetilde{-x}_{\mathrm{can}} := ((-x, 0), (0, 1/p), (0, 1/p^2), \dots).$$

When G is a p-divisible group, $\widetilde{-x}_{\mathrm{can}} = s_\varphi(1)$ for s_φ the canonical rigidification from Section 4. We will use the element $\widetilde{-x}_{\mathrm{can}}$ in our comparison with other constructions in Section 6.

Remark 5.4. We note that the homomorphism $\mathcal{E}$ arises naturally as a boundary map as follows. If we take the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[1/p] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

over $\mathcal{O}_K$ and apply the functor $\operatorname{Hom}(\cdot, G)$, we get

$$T_pG(\mathcal{O}_K) \to \tilde{G}(\mathcal{O}_K) \to G(\mathcal{O}_K) \to \operatorname{Ext}(\mathbb{Q}_p/\mathbb{Z}_p, G).$$

The image of a point $x \in G(\mathcal{O}_K)$ in $\operatorname{Ext}(\mathbb{Q}_p/\mathbb{Z}_p, G)$ gives us G_x, and after passage to the p-divisible group $G_x[p^\infty]$, we get an extension of $G[p^\infty]$.

5.5. **Proof of Theorem 1.1.** In this section, we prove Theorem 1.1. To begin, we establish two lemmas concerning the maps ψ and $\mathcal{E}$ defined in Section 5.2

Lemma 5.6. *If κ is algebraically closed, then the map ψ is injective.*

Proof. Suppose we have an extension such that $1 \mapsto 0$. That means there is a pre-image v of 1 in T_pH such that v maps to zero in $\operatorname{Lie} H \otimes C(1)$. The $\mathbb{Q}_p$-span M of the Galois orbit of v is thus contained in the kernel of this map, so $M \otimes C \subset \omega_{H^\vee} \otimes_{\mathcal{O}_K} C \subset T_pH \otimes C$. Thus M is of Hodge–Tate weight zero and crystalline (as a subrepresentation of V_pH), so the Galois action is trivial since κ is algebraically closed. Thus we obtain a splitting

$$v \in M \subseteq T_pH(K) = \operatorname{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, H[p^\infty]).$$

□

Lemma 5.7. *If κ is algebraically closed, then* $\operatorname{Ker}(\mathcal{E}) = G(\mathcal{O}_K)^{p-\mathrm{div}}$.

Proof. We first observe that if κ is algebraically closed, then by Theorem 4.4 and Theorem 4.6, $\mathcal{E}_{|_{G[p^\infty](\mathcal{O}_K)}}$ is surjective. Then by Lemma 4.5 it induces

$$G[p^\infty](\mathcal{O}_K)/G[p^\infty](\mathcal{O}_K)^{p-\mathrm{div}} = \operatorname{Ext}(\mathbb{Q}_p/\mathbb{Z}_p, G[p^\infty]). \tag{5.7.1}$$

Since $G[p^\infty](\mathcal{O}_K)^{p-\mathrm{div}}$ is a divisible $\mathbb{Z}$-module (it is a $\mathbb{Z}_p$-module and p-divisible), it is injective and thus a direct summand of the $\mathbb{Z}$-module $G[p^\infty](\mathcal{O}_K)$. Thus Eq. (5.7.1) implies $\mathrm{Ext}(\mathbb{Q}_p/\mathbb{Z}_p, G[p^\infty])$ has no non-zero p-divisible elements, so we conclude $G(\mathcal{O}_K)^{p-\mathrm{div}}$ is contained in the kernel of $\mathcal{E}$. On the other hand, since every element of $G(\kappa)$ is p-divisible thus admits a lift to a p-divisible element of $G(\mathcal{O}_K)$ by Example 2.4, we have the factorization as an amalgamated sum

$$G(\mathcal{O}_K) = G[p^\infty](\mathcal{O}_K) \sqcup_{G[p^\infty](\mathcal{O}_K)^{p-\mathrm{div}}} G(\mathcal{O}_K)^{p-\mathrm{div}}.$$

Combined with Eq. (5.7.1), we conclude[3] the kernel is identically $G(\mathcal{O}_K)^{p-\mathrm{div}}$. □

Thus, for a general K, we find the kernel of $\overline{I}$ is $G(\mathcal{O}_{(K^{\mathrm{ur}})^\wedge})^{p-\mathrm{div}} \cap G(\mathcal{O}_K)$. The claim about the kernel in Theorem 1.1 then follows from

Lemma 5.8. *If $T_pG(K^{\mathrm{ur}}) = 0$, then*

$$G(\mathcal{O}_{(K^{\mathrm{ur}})^\wedge})^{p-\mathrm{div}} \cap G(\mathcal{O}_K) = G(\mathcal{O}_K)^{p-\mathrm{div}}$$

Proof. Since $T_pG(K^{\mathrm{ur}}) = T_pG((K^{\mathrm{ur}})^\wedge) = 0$, any p-divisible element is uniquely p-divisible. Thus, by considering the Galois action, we find that if

$$x \in G(\mathcal{O}_K) \cap G(\mathcal{O}_{(K^{\mathrm{ur}})^\wedge})^{p-\mathrm{div}}$$

then x^{1/p^n} is also in $G(\mathcal{O}_K)$ for any n so $x \in G(\mathcal{O}_K)^{p-\mathrm{div}}$. □

It remains to show the crystalline characterization of the image in Theorem 1.1. On the one hand, any point in the image of $G[p^\infty](\mathcal{O}_K)$ satisfies $(a)-(c)$ because the Tate module of any p-divisible group is crystalline and from the construction the extensions are rigidified. Conversely, suppose given y satisfying $(a)-(c)$. Then the lattice T_y in the crystalline representation V_y has Hodge–Tate weights zero and one and thus comes from a p-divisible group H. By full-faithfulness of the Tate module, the extension

$$0 \to T_pG[p^\infty] \to T_pH \to \mathbb{Z}_p \to 0$$

comes from a diagram of p-divisible groups

$$0 \to G[p^\infty] \to H \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

which is an extension. It admits a rigidification by condition (c), Theorem 4.4, and Theorem 4.6, and so it comes from a point in $G[p^\infty](\mathcal{O}_K)$, thus we find y is in the image of $\overline{I}$. This concludes the proof of Theorem 1.1.

To conclude the section, we will prove Theorem 1.3, which we now recall.

Theorem 5.9. *Suppose $[K:\mathbb{Q}_p] < \infty$, G is a p-adic formal semi-abelian scheme over $\mathcal{O}_K$, and $T_pG(\mathcal{O}_{K^{\mathrm{ur}}}) = 0$. Then $\overline{I}$ factors as the projection*

$$G(\mathcal{O}_K) = G[p^\infty](\mathcal{O}_K) \times G(\mathcal{O}_K)^{\textit{prime-to-p torsion}} \to G[p^\infty](\mathcal{O}_K)$$

[3] Instead of using the amalgamated sum, one could alternatively use the fact that $G(\mathcal{O}_K)$ is generated by $G[p^\infty](\mathcal{O}_K)$ and $G(\mathcal{O}_K)^{p-\mathrm{div}}$ to deduce the desired claim.

composed with the injection $\overline{I}\colon G[p^\infty](\mathcal{O}_K) \hookrightarrow (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$. *The image consists of the points* y *stabilized by* $\operatorname{Gal}(\overline{K}/K)$ *with* V_y *crystalline (notation as in the statement of Theorem 1.1).*

Proof. Note the decomposition in Eq. (1.3.1) is immediate from the following properties of G and the reduction map $\operatorname{Red}\colon G(\mathcal{O}_K) \to G(\kappa)$:

(1) $G(\kappa)$ is a finite abelian group.
(2) The map Red induces an isomorphism
$$G(\mathcal{O}_K)^{\text{prime-to-}p \text{ torsion}} \cong G(\kappa)^{\text{prime-to-}p \text{ torsion}}$$
(3) $G[p^\infty](\mathcal{O}_K) = \operatorname{Red}^{-1}(G[p^\infty](\kappa)) = \operatorname{Red}^{-1}(G(\kappa)[p^\infty])$.

Now, as noted in Remark 1.2-(1), reduction induces
$$G(\mathcal{O}_K)^{p-\mathrm{div}} = \operatorname{Hom}(\mathbb{Z}[1/p], G(\mathcal{O}_K)) = \operatorname{Hom}(\mathbb{Z}[1/p], G(\kappa)).$$
Since $G(\kappa)$ is a finite abelian group, the right-hand-side is identified by evaluation at 1 with the prime-to-p torsion in $G(\kappa)$, thus the left-hand side is also prime-to-p torsion. Since every prime-to-p torsion point is p-divisible,
$$G(\mathcal{O}_K)^{p-\mathrm{div}} = G(\mathcal{O}_K)^{\text{prime-to-}p \text{ torsion}}.$$
It remains only to show that in this case condition (c) in the characterization of the image in Theorem 1.1 is superfluous. But this follows since here the Galois representation is determined by a single matrix corresponding to Frobenius, but the sub-representation $V_pG(\kappa)$ does not have 1 as an eigenvalue (since otherwise there would be a non-trivial fixed vector giving rise to infinitely many points in $G(\kappa)$), thus the extension is split. □

Remark 5.10. In Theorem 1.3, the identification of $G[p^\infty](\mathcal{O}_K)$ with the set of crystalline points in $(\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$ is formulated entirely in the world of p-divisible groups over $\mathcal{O}_K$ — the source, the map $\overline{I}|_{G[p^\infty](\mathcal{O}_K)}$, and the target all only depend on $G[p^\infty]$. Indeed, the only place we use the semi-abelian scheme in the proof of Theorem 1.3 is to conclude that $G[p^\infty](\kappa)$ has finitely many points. Thus this part of the theorem holds for any p-divisible group $H/\mathcal{O}_K$ with the same properties, which can be formulated by requiring that $T_pH(K^{\mathrm{ur}}) = 0$ and $T_pH(\kappa) = 0$.

6. Integration on the universal cover and other constructions

For G a p-divisible group or a p-adic formal semi-abelian scheme over $\mathcal{O}_K$, in this section we write $\overline{I}_G$ for the integration map defined in Section 5.2 to emphasize the dependence on G. The integration map $\overline{I}_G$ lifts to a homomorphism

$$I_G\colon \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K) \to \operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1) \tag{6.0.1}$$

which we define now: recall from Remark 5.3 that, given $x \in G(\mathcal{O}_K)$ we have a canonical compatible systems of p-power roots of $-x$, $\widetilde{-x}_{\mathrm{can}} \in \tilde{G}_x(\mathcal{O}_K)$. Given a compatible system of p-power roots of x, $\tilde{x} \in \tilde{G}(\mathcal{O}_C)$, the element $\tilde{x} + (\widetilde{-x}_{\mathrm{can}})$ lies in $T_pG_x(\mathcal{O}_C)$, and we define $I(\tilde{x})$ to be the image of $\tilde{x} +$

$(\widetilde{-x}_{\mathrm{can}})$ in $\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1) = \operatorname{Lie} G_x \otimes_{\mathcal{O}_K} C(1)$ under the canonical splitting of the Hodge–Tate filtration. It is immediate from the construction that this lifts $\overline{I}_G$.

In Section 6.2 and Section 6.4, we explain two other ways to construct the map I_G. The first is via Fontaine integration as in [10], while the second passes through the crystalline incarnation of the universal cover of a p-divisible group as in [14]. That all three constructions agree is a consequence of the following uniqueness result:

Theorem 6.1. *Let $\mathcal{C}$ be the full subcategory of group-valued functors on $\mathrm{Nilp}_{\mathcal{O}_K}$ consisting of functors represented by p-divisible groups over $\mathcal{O}_K$ or by p-adic formal abelian schemes over $\mathcal{O}_K$. The natural transformation of functors from $\mathcal{C}$ to abelian groups*

$$G \mapsto I_G \colon \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K) \to \operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1)$$

is the unique natural transformation that is Galois equivariant and agrees with the canonical splitting of the Hodge–Tate filtration on

$$T_p G(\mathcal{O}_C) \subset \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K).$$

Proof. We first verify that I satisfies these properties. The Galois equivariance is immediate; to show it agrees with the canonical splitting of the Hodge–Tate filtration on $T_p G(\mathcal{O}_C)$, note that if $x \in G[p^n](\mathcal{O}_K)$ then $\widetilde{-x}_{\mathrm{can}} \in V_p G_x(\mathcal{O}_K)$ thus its image under the canonical splitting of the Hodge–Tate filtration is zero (because zero is the only Galois invariant vector in $\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1)$ by [16, Theorem 2]).

Suppose now given another natural transformation I' satisfiying these properties and let $\tilde{x} = (x, x_1, \dots) \in \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K)$. Consider the extension

$$\mathcal{E}_x \colon 0 \to G[p^\infty] \to G_x[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

and the induced sequence of universal covers

$$0 \to \widetilde{G[p^\infty]} \to \widetilde{G_x[p^\infty]} \to \mathbb{Q}_p \to 0.$$

Again by Galois equivariance, $I'_{G_x}(\widetilde{-x}_{\mathrm{can}}) = 0$ and $I_{G_x}(\widetilde{-x}_{\mathrm{can}}) = 0$. Thus

$$I'_G(\tilde{x}) = I'_{G_x}(\tilde{x}) = I'_{G_x}(\tilde{x} + (\widetilde{-x}_{\mathrm{can}})) = I_{G_x}(\tilde{x} + (\widetilde{-x}_{\mathrm{can}})) = I_{G_x}(\tilde{x}) = I_G(\tilde{x})$$

where we have used functoriality and the inclusions $\tilde{G} \subset \tilde{G}_x$ and $\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1) = \operatorname{Lie} G_x \otimes_{\mathcal{O}_K} C(1)$ to make sense of the first and last equality, while the middle equality follows since both I and I' agree with the canonical splitting of the Hodge–Tate filtration on $T_p G_x(\mathcal{O}_C)$ and $\tilde{x} + (\widetilde{-x}_{\mathrm{can}}) \in T_p G_x(\mathcal{O}_C)$. □

6.2. **Construction via Fontaine integration.** For G an abelian scheme or p-divisible group over $\mathcal{O}_K$, in [7], Fontaine gave an explicit construction of the canonical splitting of the Hodge–Tate filtration via "integration" along elements of $T_p G$. Concretely, when G is an abelian scheme, given $(x_i) \in T_p G(\mathcal{O}_{\overline{K}})$ we may pullback any differential ω on G to obtain

$$(x_i^* \omega) \in V_p \Omega_{\mathcal{O}_{\overline{K}}/\mathcal{O}_K} = C(1),$$

where the last equality follows from [7, Théorème 1′]. For abelian schemes, it was observed in [10] that this definition extends naturally to a map

$$I\colon \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K) \to \operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1) \tag{6.2.1}$$

In fact, just as with the extension to p-divisible groups in [7, §5], the definition works also for any p-adic formal semi-abelian scheme $G/\operatorname{Spf}\mathcal{O}_K$: given an element $(x_i) \in \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K)$, each x_i gives a map $\operatorname{Spf}\mathcal{O}_{K_i} \to G$ where K_i/K is a finite extension, thus there is no problem with the pullback of differentials in the formation of the Fontaine integral (the a priori "issue" we need to circumvent is that one cannot run the argument with arbitrary $\mathcal{O}_C$-points, but for a p-adic formal semi-abelian scheme there is no such thing as an $\mathcal{O}_{\overline{K}}$-point since $\mathcal{O}_{\overline{K}}$ is not p-adically complete).

Proposition 6.3. *Let I_G (resp. I) denote the homomorphism from Eq. (6.0.1) (resp. Eq. (6.2.1)) We have that $I_G = I$.*

Proof. We will verify that the Fontaine integral I satisfies the conditions of Theorem 6.1. The Galois equivariance is immediate from the construction, and the agreement with the canonical splitting of the Hodge–Tate filtration is established in [7, §5] (for p-divisible groups, which suffices by functoriality). □

6.4. **Construction via crystalline incarnation of the universal cover.** We first restrict to the case that G is a p-divisible group over $\mathcal{O}_K$. Let $B^+_{\mathrm{crys}} \subseteq B^+_{\mathrm{dR}}$, $B_{\mathrm{crys}} \subseteq B_{\mathrm{dR}}$ denote the usual Fontaine period rings for K, $\theta\colon B^+_{\mathrm{dR}} \to C$ the usual map, and $\operatorname{Fil}^i B_{\mathrm{dR}} = t^i B^+_{\mathrm{dR}} \subseteq B_{\mathrm{dR}}$ for t any generator of $\ker\theta$. By Hensel's lemma, θ is a map of K-algebras.

We write D for the covariant isocrystal of G, which is the Dieudonné module of G_κ with p inverted and Frobenius divided by p (so, e.g., if $G = \mathbb{Q}_p/\mathbb{Z}_p$, then the Frobenius is 1, while if $G = \mathbb{G}_m$ it is $1/p$). We write $T = T_p G(\mathcal{O}_C)$ and $V = T[1/p]$. Evaluation of Dieudonné crystals on B_{crys} induces (by [14, Corollary 5.1.2]) an identification of $\widetilde{G}(\mathcal{O}_C)$ with $(D \otimes B^+_{\mathrm{crys}})^{\varphi=1}$ (note that this differs from the $\varphi = p$ in [14] because in our normalization we have divided the Frobenius on D by p), and the map

$$\mathrm{qlog}\colon \widetilde{G}(\mathcal{O}_C) = (D \otimes B^+_{\mathrm{crys}})^{\varphi=1} \to D \otimes B^+_{\mathrm{dR}} \xrightarrow{\theta} D_C$$

is the quasi-logarithm of [14, Definition 3.2.3]. We write $\widetilde{\mathrm{qlog}}$ for the first arrow

$$\widetilde{G}(\mathcal{O}_C) \to D \otimes B^+_{\mathrm{dR}}.$$

Remark 6.5. Alternatively, if we interpret the universal cover as the global sections of the corresponding vector bundle on the Fargues–Fontaine curve, then qlog is the restriction of global sections to the canonical point ∞_C while $\widetilde{\mathrm{qlog}}$ is the restriction to a formal neighborhood of ∞_C.

As in [14, Lemma 3.2.5], the logarithm

$$\log\colon \tilde{G}(\mathcal{O}_C) \to G(\mathcal{O}_C) \to (\operatorname{Lie} G)_C$$

can be realized as the composition of qlog with the projection

$$D_C \to \operatorname{Lie} G_C = \operatorname{Gr}^{-1}(D_K)_C = (D_K/\operatorname{Fil}^0 D_K)_C,$$

where $\operatorname{Fil}^0 D_K = \omega_{G^\vee}$ is the non-trivial step in the Hodge filtration on D_K with quotient $\operatorname{Lie} G = \operatorname{Gr}^{-1}(D_K) = D_K/\operatorname{Fil}^0 D_K$. We thus write

$$\widetilde{\log} : \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K) \to \operatorname{Lie} G_{B^+_{\mathrm{dR}}}$$

for the lift of log given by composing $\widetilde{\operatorname{qlog}}$ with the projection

$$D_{B^+_{\mathrm{dR}}} \to \operatorname{Lie} G_{B^+_{\mathrm{dR}}} = (D_K/\operatorname{Fil}^0 D_K)_{B^+_{\mathrm{dR}}}.$$

On the other hand, we also have the constant lift $\log \otimes_K B^+_{\mathrm{dR}}$ coming from the section $K \to B^+_{\mathrm{dR}}$ of θ. The difference

$$\widetilde{\log} - \log \otimes_K B^+_{\mathrm{dR}}$$

is a Galois equivariant homomorphism

$$f \colon \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K) \to \operatorname{Lie} G \otimes_K \operatorname{Fil}^1 B^+_{\mathrm{dR}}$$

and we write $\overline{f}$ for the map obtained by quotienting by $\operatorname{Lie} G \otimes_K \operatorname{Fil}^2 B^+_{\mathrm{dR}}$

$$\overline{f} \colon \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_K) \to \operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1). \tag{6.5.1}$$

Remark 6.6. To extend this definition to p-adic formal semi-abelian schemes $G/\operatorname{Spf} \mathcal{O}_K$, we may assume κ is algebraically closed then use the decomposition from Example 2.4

$$\tilde{G}(\mathcal{O}_C) = \widetilde{G^\wedge}(\mathcal{O}_C) \times \tilde{G}(\kappa)$$

and projection onto $\widetilde{G^\wedge}$ followed by $\overline{f}$.

Proposition 6.7. *Let I_G (resp. $\overline{f}$) denote the homomorphism from Eq. (6.0.1) (resp. Eq. (6.5.1)) We have that $I_G = \overline{f}$.*

Proof. We will verify the properties from Theorem 6.1 First, we have that $\overline{f}$ is Galois equivariant since $\widetilde{\log} - \log \otimes_K B^+_{\mathrm{dR}}$ is Galois equivariant.

We claim that $\overline{f}$ agrees on the Tate module T with the canonical splitting of the Hodge–Tate filtration. To see this, first note that log is identically zero on T, so that $\overline{f}_{|T}$ is just the reduction of $\widetilde{\log}$. We need to check that this is the canonical splitting of the Hodge–Tate filtration.

To that end, we note that the inclusion

$$T \hookrightarrow \tilde{G}(\mathcal{O}_C) = (D \otimes B^+_{\mathrm{crys}})^{\varphi=1}$$

is the restriction to T of the crystalline comparison isomorphism

$$V \otimes B_{\mathrm{crys}} = D \otimes B_{\mathrm{crys}}.$$

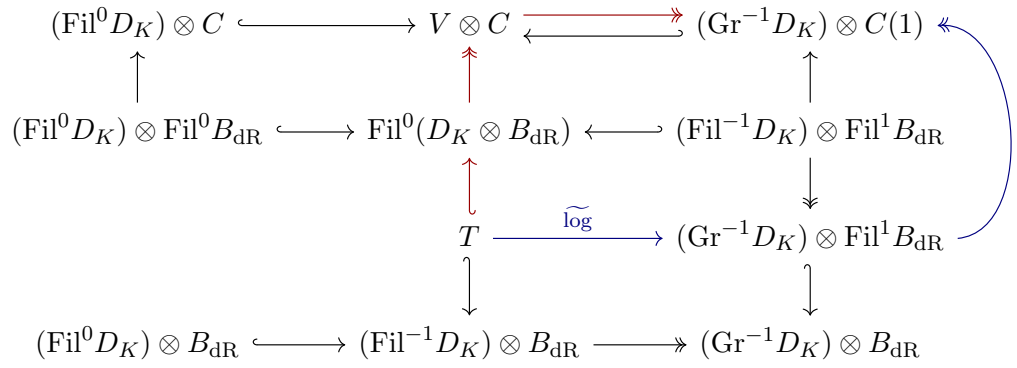
As with any de Rham representation, the ith component of the Hodge–Tate grading

$$V \otimes C = \bigoplus_i \operatorname{Gr}^{-i} D_K \otimes_K C(i)$$

is induced by first multiplying $\mathrm{Fil}^{-i} D_K$ with $\mathrm{Fil}^i B_{\mathrm{dR}}$ to land in $\mathrm{Fil}^0(D_{B_{\mathrm{dR}}}) = V_{B_{\mathrm{dR}}^+}$ and then projecting to V_C. The identification then follows from the following commutative diagram. To better navigate the diagram, recall the following identifications:

$$\mathrm{Fil}^0(D_K \otimes B_{\mathrm{dR}}) = V \otimes \mathrm{Fil}^0 B_{\mathrm{dR}} = V \otimes B_{\mathrm{dR}}^+, \ \mathrm{Fil}^{-1} D_K = D_K,$$
$$\mathrm{Gr}^0(D_K) = \mathrm{Fil}^0(D_K) = \omega_{G^\vee} \otimes K, \text{ and } \mathrm{Gr}^{-1} D_K = \mathrm{Lie}\, G \otimes K$$

Now the first two rows of the diagram illustrate the canonical splitting of the Hodge–Tate filtration and whose bottom row illustrates the definition of $\widetilde{\log}$. The above discussion shows that the image of T under $\widetilde{\log}$ will land in $(\mathrm{Gr}^{-1} D_K) \otimes \mathrm{Fil}^1 B_{\mathrm{dR}}$, which we have indicated in the third row.



□

Example 6.8. When $G = \mu_{p^\infty}$, one obtains the explicit formula

$$I(q, q^{1/p}, q^{1/p^2}, \ldots)\left(\frac{dt}{t}\right) = \log([q^\flat]) - \log([q] \equiv \frac{[q^\flat]}{[q]} - 1 \pmod{(\ker \theta)^2}$$

where $q \in 1 + \mathfrak{m}_{\overline{K}}$, $q^\flat = (q, q^{1/p}, q^{1/p^2}, \ldots) \in (1 + m_C)^\flat$ is a compatible family of p-power roots of q, and $[\cdot]$ denotes the multiplicative lift to $W(\mathcal{O}_{C^\flat}) \subset B_{\mathrm{dR}}^+$. We refer the reader to [10, Remark 3.1.2] for an additional discussion.

Remark 6.9. Although $\widetilde{\log}$ and log are both continuous on

$$\bigcup_{\substack{[K':K]<\infty \\ K' \subset \overline{K}}} \tilde{G}(\mathcal{O}_C) \times_{G(\mathcal{O}_C)} G(\mathcal{O}_{K'})$$

for the Banach-Colmez topology on $\tilde{G}(\mathcal{O}_C)$, $\log \otimes B_{\mathrm{dR}}^+/t^2$ is not continuous because the canonical section $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+/t^2$ provided by Hensel's lemma is not continuous for the p-adic topology induced by $\overline{K} \subset C$. Because of this, I is not continuous on this set (and this gives another way to explain why it does not extend to $\tilde{G}(\mathcal{O}_C)$). As in [10], this can be rectified at the level of

$$\overline{I}\colon \bigcup_{\substack{[K':K]<\infty \\ K' \subset \overline{K}}} G(\mathcal{O}_{K'}) \to (\mathrm{Lie}\, G \otimes_{\mathcal{O}_K} C(1))/T_p G,$$

by replacing the p-adic topology on the left with a stronger topology induced locally by the inclusion of $\overline{K} \subset B_{\mathrm{dR}}^+/t^2$ (cf. [10, Remark 2.4]). This issue is invisible if we only work with $\mathcal{O}_K$-points where the two topologies agree.

7. Geometric structure of the codomain of $\overline{I}$

In this short section, we discuss the geometric structure of the codomain of the integration map $\overline{I}$. Let G be a p-divisible group over $\mathcal{O}_K$. Below we work exclusively with the adic generic fibers as in [14] of G, $\tilde{G}$, and $\operatorname{Lie} G$.

The universal cover $\tilde{G}$ is an effective Banach–Colmez space and admits a natural geometric structure as a diamond [15, §15.2]. The projection to G realizes $G = \tilde{G}/T_pG$ (where the equality here is as diamonds), i.e. it gives a profinite étale Scholze–Weinstein uniformization of G by an effective Banach–Colmez space.

Although the conjugate uniformization is not geometric, its codomain does have a natural geometric structure: consider the maps

$$\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1) \twoheadrightarrow (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG \twoheadrightarrow (\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/V_pG.$$

The space $\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1)$ is the C-points of an effective Banach–Colmez space over K (a very non-trivially twisted affine space), while the space $(\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/V_pG$ is a negative Banach–Colmez space over K in the sense of [6]. Thus $(\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$ can be thought of as either a profinite-étale quotient of an effective Banach–Colmez space or as an étale cover of a negative Banach–Colmez space; in particular, all three spaces in the diagram are naturally diamonds over K.

We thus have an identification of the classical K-valued points of the diamond G with a subset of the classical K-valued points of the diamond $(\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$ cut out by a p-adic Hodge theoretic condition.

Question 7.1. Are there any higher dimensional rigid analytic subdiamonds of $(\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$? If so, can any be distinguished by conditions in relative p-adic Hodge theory and matched with rigid analytic subvarieties of G parameterizing rigidified extensions of $\mathbb{Q}_p/\mathbb{Z}_p$ by G?

We note that this question is not asking about identifying *morphisms* from any rigid analytic space over K to these diamonds. Rather, the question focuses on understanding the relationship between rigid analytic subdiamonds of $(\operatorname{Lie} G \otimes_{\mathcal{O}_K} C(1))/T_pG$ and rigid analytic subvarieties of G.

In some related contexts there are interesting answers to these kinds of questions. For example, the non-miniscule open Schubert cells in B_{dR}^+-affine Grassmannians are diamonds which are not rigid analytic, but their rigid analytic subvarieties still admit a nice description via p-adic Hodge theory — via the Bialynicki–Birula map of [4, Proposition 3.4.3], they are identified with maps to a flag variety satisfying Griffiths transversality (cf. [9, §5] and [13, §6]). Some related questions in the context of moduli spaces of rigidified Breuil–Kisin–Fargues modules (see Remark 4.10 for the connection to the present work) are treated in [9].

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