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Abstract

We categorify the inclusion–exclusion principle for partially ordered topological spaces and schemes to a filtration on the derived category of sheaves. As a consequence, we obtain functorial spectral sequences that generalize the two spectral sequences of a stratified space and certain Vassiliev-type spectral sequences; we also obtain Euler characteristic analogs in the Grothendieck ring of varieties. As an application, we give an algebro-geometric proof of Vakil and Wood's homological stability conjecture for the space of smooth hypersurface sections of a smooth projective variety. In characteristic zero this conjecture was previously established by Aumonier via topological methods.

1. Introduction

In this work, we explore consequences of topological poset theory when applied to partially ordered topological spaces (pospaces) and partially ordered schemes (poschemes). In particular, we investigate some ramifications of a simplicial proof of the inclusion–exclusion formula (described in § 1.1) in these enriched contexts.

We reinterpret this simplicial proof as a categorified inclusion–exclusion principle in topology and algebraic geometry via rank filtrations on the derived category of sheaves (see § 1.2). We give a simple criterion for the nerve of a pospace or poscheme to satisfy cohomological descent (Theorem B), and obtain in these cases functorial cohomological spectral sequences attached to a rank function (Theorem C). These spectral sequences give a common generalization of the two spectral sequences of a stratified space (cf. [Pet17]), Vassiliev-type[1] spectral sequences (as in, e.g., [Vas95, Tom14, Vas99]), and other related spectral sequences that have appeared in the literature (e.g. [Ban19]). We also give a combinatorial analog in the Grothendieck ring of varieties, Theorem D, which generalizes the motivic inclusion–exclusion principle of [BH21, VW15].

For our main application, consider a smooth projective variety X over an algebraically closed field κ. Let $\mathcal{L}$ be an ample line bundle on X and let V_d be the vector space of global sections of $\mathcal{L}^d$. Let $U_d \subset V_d$ be the open subscheme of sections with non-singular vanishing locus. We refine and prove Vakil and Wood's [VW15] homological stabilization conjecture for U_d as $d \to \infty$. To state the result, let $H_\bullet(-)$ denote either rational ℓ-adic étale homology for $\mathrm{char}\,\kappa \neq \ell$ or rational singular homology if $\kappa = \mathbb{C}$. Note that $H_\bullet(-)$ has a canonical weight filtration [BBD82, Del80], which we denote by W and normalize by treating $H_i(Y)$ for Y smooth projective as having degree

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[1] Indeed, we have essentially adopted Vassiliev's method of topological order complexes and run it directly in the simplicial category instead of passing to a geometric realization.

$-i$ and weight zero. For $\kappa = \mathbb{C}$, weights can be detected by mixed Hodge theory; for arbitrary κ (including also $\kappa = \mathbb{C}$) they are detected by spreading out and using Frobenius eigenvalues over finite fields.

For κ of characteristic zero, the following is a consequence of stronger topological results of Aumonier [Aum21]; we give an algebro-geometric proof valid for arbitrary κ using a categorified inclusion–exclusion principle.

THEOREM A (See Theorem 8.0.1). *Let X be a smooth projective variety over an algebraically closed field κ equipped with an ample line bundle $\mathcal{L}$, and let U_d/κ be the variety of nonsingular sections of $\mathcal{L}^{\otimes d}$. For $d \gg 0$ depending on i, there are natural isomorphisms*

$$\mathrm{Gr}_W^{-k} H_i(U_d) \xrightarrow{\sim} H_{i-k}(X^k)[\mathrm{sgn}](k), \tag{1.0.0.1}$$

where $H_{i-k}(X^k)[\mathrm{sgn}](k)$ is the isotypic part for the sign character of the symmetric group permuting the coordinates, shifted and twisted into degree $-i$ and weight $-k$.

Remark 1.0.1. In the case of étale cohomology, if X and $\mathcal{L}$ are defined over a subfield $\kappa_0 \subseteq \kappa$ (not necessarily algebraically closed), then both sides of (1.0.0.1) carry actions of $\mathrm{Aut}(\kappa/\kappa_0)$ and the isomorphism is equivariant for these actions. Indeed, the isomorphism is deduced from a spectral sequence which is defined already over κ_0 (replacing compactly supported cohomology H^i_c with $R^i f_!$ for f the structure map to κ_0; cf. Remark 1.2.4). In the case of singular homology for $\kappa = \mathbb{C}$, Aumonier [Aum21, Proposition 8.6] has shown the weight filtration is split and that one obtains an isomorphism of mixed Hodge structures. The argument given in [Aum21] uses a compatibility with cup products that is not apparent from the spectral sequence we use.

Vakil and Wood [VW15, VW20] proved an analogous stabilization in the Grothendieck ring of varieties, where the limit is a special value of the inverse Kapranov zeta function. The latter is given by a convergent infinite sum in a completed Grothendieck ring, and in the introduction to § 8 we explain how the weight $-k$ part in Theorem A naturally corresponds to the kth term in this sum. More generally, we define motivic and sheaf-theoretic incidence algebras for the poscheme of effective zero-cycles on X, then explain the inversion formula and 'cohomological special values' of the inverse Kapranov zeta function as natural outputs of Möbius inversion. To prove Theorem A, we use a cohomological approximate inclusion–exclusion formula derived from the relative poscheme of effective zero-cycles of a natural resolution of the discriminant locus $V_d \backslash U_d$. This is a cohomological analog of the motivic inclusion–exclusion that arises in Vakil and Wood's proof of stabilization in the Grothendieck ring, and here it yields an E_1-spectral sequence. We show the sequence degenerates on E_1 by comparing with a simpler complex introduced by Banerjee; vanishing of the E_1 differential is then a direct computation and vanishing of the higher differentials follows from a weight argument.

Thus, beyond just proving Theorem A, this work draws a straight line from the classical inclusion–exclusion principle through to a modern homological stability result while rendering transparent the relation with special values of the zeta function and highlighting a common thread in various geometric and topological incarnations of inclusion–exclusion that have previously appeared in the literature.

Remark 1.0.2. Our proof of homological stabilization is distinct from Aumonier's [Aum21]; however, the description of the stable cohomology in [Aum21] and the definitive knowledge that certain spectral sequences must degenerate at E_1 played an important role in the development of the proof presented here. We are indebted also to Banerjee for sharing her insights on the sign cohomology of configuration spaces as well as the existence of the complex referred to above: the former plays a key role in our proof, while the latter led to a considerable simplification

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through the replacement of the rank spectral sequence with the skeletal spectral sequence. We note that Tommasi has announced closely related results (see [Aum21, § 1.1] for a statement), and Banerjee has announced a closely related spectral sequence.

We now outline the remainder of the introduction: in § 1.1 we take a detour to explain a categorification of the inclusion–exclusion formula for finite sets as a toy model for our computations with pospaces and poschemes. In § 1.2 we explain the spectral sequences obtained from a suitable pospace or poscheme. In § 1.3 we explain the decategorification to the Grothendieck ring of varieties, and in § 1.4 we give an outline of the body of the article. Further discussion of Theorem A is deferred to § 8; the reader interested primarily in this application may wish to skip immediately to that section and return to the rest of the paper as needed.

1.1 Categorifying the inclusion–exclusion formula

The inclusion–exclusion formula for finite sets states:

$$\text{If } X = \bigcup_{i=1}^{n} X_i \text{ is a finite set, then } |X| = \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} (-1)^{|J|-1} \Big| \bigcap_{j \in J} X_j \Big|. \tag{1.1.0.1}$$

This statement admits a topological proof: consider the poset

$$\mathcal{P} := \bigsqcup_i X_i = \{(x,i) \,|\, x \in X_i\} \subset X \times \{1,\ldots,n\}, \quad (x,i) \leq (y,j) \iff x = y \text{ and } i \leq j.$$

We form the *nerve* $N\mathcal{P}$, a simplicial set whose m-simplices are the ordered chains $a_0 \leq a_1 \leq \cdots \leq a_m$ of length $m+1$ in $\mathcal{P}$. The sum on the right-hand side of (1.1.0.1) computes the Euler characteristic of $N\mathcal{P}$ by counting the non-degenerate simplices (i.e. the strict chains $a_0 < a_1 < \cdots < a_m$). However, $N\mathcal{P}$ is homotopic to the constant simplicial set X: because the fiber $\mathcal{P}_x$ is totally ordered for any point $x \in X$, we can contract all chains above x to the constant chain on any choice of element in $\mathcal{P}_x$. Because homotopy preserves Euler characteristic, we recover (1.1.0.1):

$$|X| = \chi(X) = \chi(N\mathcal{P}) = \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} (-1)^{|J|-1} \Big| \bigcap_{j \in J} X_j \Big|.$$

The argument is point by point, so this lifts to an identity in $K_0(\mathrm{FinSet}/X) \cong \mathbb{Z}^X$:

$$\mathbf{1}_X = \sum_{\emptyset \neq J \subset \{1,\ldots,n\}} (-1)^{|J|-1} \mathbf{1}_{\bigcap_{j\in J} X_j}. \tag{1.1.0.2}$$

Note that we can rewrite the right-hand side as $(1 - \prod_{j=1}^{n}(1 - \mathbf{1}_{X_j} t))|_{t=1}$ and then (1.1.0.2) can be established immediately by evaluating at elements $x \in X = \bigcup_{i=1}^{n} X_i$. Summing over X then gives a proof of (1.1.0.1) that avoids any topology.

The advantage of the topological approach is that it leads to an interesting *categorification* of the inclusion–exclusion formula: we can think of the indicator function $\mathbf{1}_X$ as being the function on X assigning to a point $x \in X$ the rank at x of the constant local system $\mathbb{Q}$ on X. More generally, writing $D_{\mathrm{c}}(X, \mathbb{Q})$ for the bounded derived category of complexes of $\mathbb{Q}$-sheaves on X (a $\mathbb{Q}$-sheaf on X is just the choice of a $\mathbb{Q}$-vector space $\mathcal{V}_x$ for each $x \in X$) with finite-dimensional cohomology sheaves, we have

$$K_0(D_{\mathrm{c}}(X,\mathbb{Q})) \xrightarrow{\sim} \mathbb{Z}^X, \quad K \mapsto \Big(x \mapsto \sum_i (-1)^i \dim_{\mathbb{Q}} H^i(K)_x \Big)$$

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identifying $\mathbb{Q} = \mathbb{Q}[0]$ with $\mathbf{1}_X$. The simplicial homotopy described above gives

$$\mathbb{Q}[0] \cong C^\bullet(N\mathcal{P}, \mathbb{Q}) \quad \text{in } D_{\mathrm{c}}(X, \mathbb{Q}),$$

where $C^\bullet(N\mathcal{P}, \mathbb{Q})$ denotes the complex whose fiber at x is the simplicial cochain complex $C^\bullet(N\mathcal{P}_x, \mathbb{Q})$. This is a categorification of the inclusion–exclusion formula because $C^\bullet(N\mathcal{P}, \mathbb{Q})$ can be equipped with a filtration $F^\bullet C^\bullet(N\mathcal{P}, \mathbb{Q})$ such that

$$\mathbf{1}_X = \chi_X(\mathbb{Q}[0]) = \chi_X(C^\bullet(N\mathcal{P}, \mathbb{Q})) = \sum_i \chi_X(\mathrm{Gr}^i\, C^\bullet(N\mathcal{P}, \mathbb{Q}))$$

realizes (1.1.0.2). Indeed, any increasing filtration of the simplicial set $N\{1, \ldots, n\} \cong \Delta^{n-1}$ induces a filtration $F_\bullet N\mathcal{P}$, and we can then define

$$F^i C^\bullet(N\mathcal{P}, \mathbb{Q}) = \ker C^\bullet(N\mathcal{P}, \mathbb{Q}) \xrightarrow{\text{restriction}} C^\bullet(F_i N\mathcal{P}, \mathbb{Q}).$$

Noting that the non-empty subsets $J \subset \{1, \ldots, n\}$ correspond exactly to the non-degenerate simplices of $N\{1, \ldots, n\}$, we find that

$$\chi_X(\mathrm{Gr}^i\, C^\bullet(N(\mathcal{P}), \mathbb{Q})) = \sum_{J \in F_{i+1} N\mathcal{P} \setminus F_i N\mathcal{P}} (-1)^{|J|-1} \mathbf{1}_{\bigcap_{j \in J} X_J}.$$

In particular, if we filter $N\{1, \ldots, n\} \cong \Delta^{n-1}$ by first adding in all of the zero-simplices one at a time, then all of the one-simplices, etc., each term in (1.1.0.2) will correspond to exactly one graded piece of the complex. There are also other interesting ways of grouping the terms corresponding to filtrations: for example, the skeletal filtration will group the terms by $|J|$, while the filtration $\Delta^0 \subset \Delta^1 \subset \Delta^2 \subset \cdots \subset \Delta^{n-1}$ (corresponding via nerves to the poset filtration $\{1\} \subset \{1, 2\} \subset \cdots \subset \{1, \ldots, n\}$) will group terms according to the largest element contained in J.

None of these sheaf-theoretic constructions depend on starting with the complex $\mathbb{Q}[0]$ and, in fact, what we have really categorified is the decomposition of *multiplication* by $\mathbf{1}_X$: in other words, we have factored the identity functor on $D_{\mathrm{c}}(X, \mathbb{Q})$ through the filtered derived category $DF^+(X, \mathbb{Q})$ in a way compatible with multiplication by the two sides of (1.1.0.2) after taking Euler characteristics.

1.2 Inclusion–exclusion for sheaves

In the previous section, we obtained the poset $\mathcal{P}$ fibered over the finite set X from a representation of X as a union of subsets. However, these origins were immaterial to the reasoning, and only arose in the interpretation of the final formulas. We now generalize: A *pospace* (respectively, *poscheme*) $\mathcal{P}$ over a topological space (respectively, scheme) X is a topological space (respectively, scheme) over X equipped with a closed partial ordering relation $\leq_{\mathcal{P}} \subset \mathcal{P} \times_X \mathcal{P}$. In the introduction, we assume all pospaces/poschemes $\mathcal{P}/X$ are *split*, i.e. that the diagonal $\Delta_{\mathcal{P}}$ is clopen in $\leq_{\mathcal{P}}$. We say $\mathcal{P}/X$ is *proper* if $\mathcal{P} \to X$ is proper. We say $\mathcal{P}/X$ is *ranked* if it is equipped with a strictly increasing map of pospaces/poschemes $\mathrm{rk} : \mathcal{P} \to \mathbb{Z}$; here $\mathbb{Z}$ is treated as a discrete pospace or poscheme, and a map of pospaces or poschemes is a map of topological spaces or schemes respecting the ordering (see also Definition 3.1.4(i)).

Given a pospace (respectively, poscheme) $\mathcal{P}/X$, we form its nerve $N\mathcal{P}$ (or order complex), the simplicial topological space (respectively, simplicial scheme) whose m-simplices are the space (respectively, scheme) of ordered chains of length $m+1$ in $\mathcal{P}$. We wish to compute the cohomology of sheaves on X by pullback to $N\mathcal{P}$, so that we can exploit filtrations on the nerve in order to obtain filtrations on this cohomology. This is possible when $N\mathcal{P}/X$ is of cohomological descent,

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and we now state a criterion for this in terms of the partial ordering. A center (or weak center in the scheme-theoretic case) is a section p such that $\mathcal{P} = \{\leq p\} \cup \{\geq p\}$ (see Definition 3.1.6).

Theorem B (Fiberwise center descent criterion; see also Theorem 4.0.1).

(i) *Let X be a topological space. If $\mathcal{P}/X$ is a proper pospace and $\mathcal{P}_x$ admits a center for all $x \in X$, then $N\mathcal{P}/X$ is of cohomological descent for abelian sheaves on X.*
(ii) *Let X be a scheme. If $\mathcal{P}/X$ is a proper poscheme and $\mathcal{P}_x$ admits a weak center for every geometric point $x : \operatorname{Spec} \kappa \to X$ (κ algebraically closed), then:*
   (a) *$N\mathcal{P}/X$ is of cohomological descent for étale torsion sheaves on X;*
   (b) *if X is furthermore Noetherian and L is an algebraic extension of $\mathbb{Q}_\ell$ for some ℓ invertible on X, then $\mathcal{P}/X$ is of cohomological descent for constructible $\mathcal{O}_L$ or L-sheaves on X.*

Remark 1.2.1. More generally, one has cohomological descent on suitable complexes; see Theorem 4.0.1 for a precise statement. We note that in part (ii)(b) we use the formalism of [BS15] for ℓ-adic sheaves; this is useful in order to construct filtrations by working in the derived category of an abelian category of sheaves.

Remark 1.2.2. Theorem B is a version of Quillen's Theorem A (see [Qui73]) and of the Vietoris mapping theorem (see, e.g., [Bre97, Theorem 11.7]) in that it converts fiberwise contractibility to a global equivalence. The topological case can be deduced from a suitable version of the Vietoris mapping theorem.

Remark 1.2.3. In the toy model (§ 1.1), we used the total ordering on each fiber to deduce that the nerve was contractible: for a totally ordered set, any point is a center. More generally, a maximum or minimum element in a poset is a simple example of a center. In the topological/scheme theoretic setting, one can expect to obtain a useful fiberwise criterion only when $\mathcal{P}/X$ is proper, so that we can invoke proper base change. With properness imposed, any condition on the fibers that gives (weak) contractibilty of the nerve would suffice, and in § 3.2 we develop some more general tools for verifying contractibility using the partial ordering.

A finite filtration of $N\mathcal{P}$ by closed simplicial subspaces/subschemes induces a support filtration on sheaves on $N\mathcal{P}$. When $N\mathcal{P}/X$ is of cohomological descent, this, in turn, induces a derived filtration on sheaves on X (just as in the toy model of § 1.1, the filtration can only be seen after replacing $\mathcal{F}$ with a resolution obtained by working on $N\mathcal{P}$). We are most interested in the case where the filtration on $N\mathcal{P}$ is inherited from some structure on $\mathcal{P}$ itself: in particular, if $\mathcal{P}/X$ is finitely ranked, then we can consider the rank filtration of $N\mathcal{P}$ by $N(\mathrm{rk} \leq i)$ and the induced derived filtration on sheaves on X. In this case, the cohomology of the graded parts can be interpreted as the reduced cohomology complexes of nerves of certain auxiliary pospaces/poschemes constructed from $\mathcal{P}$; in practice, these reduced cohomology complexes are computable. Concretely, they are defined as follows: let $\mathcal{P}^+$ be the pospace or poscheme obtained by adjoining a disjoint minimum section $-\infty$ over X. We choose an extension of the rank function to $\mathcal{P}^+$, and let $\mathcal{P}^+_r := \mathrm{rk}^{-1}(r)$. In this setting, we define (up to quasi-isomorphism) a complex of sheaves $\tilde{C}(-\infty, \mathcal{P}^+_r, \mathcal{F})$ over $\mathcal{P}^+_r$ that at each geometric point $p \in \mathcal{P}^+_r$ computes the reduced cohomology of $\mathcal{F}$ on $N(-\infty, p)$. By convention[2] as in [Pet17], if $r = \mathrm{rk}(-\infty)$ so $\mathcal{P}^+_r \cong X$, then $\tilde{C}(-\infty, \mathcal{P}^+_r, \mathcal{F}) := \mathcal{F}[2]$. We write $\mathcal{P}_r = \mathcal{P}^+_r \cap \mathcal{P}$, which is equal to $\mathcal{P}^+_r$ unless $r = \mathrm{rk}(-\infty)$ in which case $\mathcal{P}_r = \emptyset$.

[2] In our context, this can be explained as follows: for the other terms, it is not the reduced cohomology $\tilde{C}(-\infty, \mathcal{P}^+_r, \mathcal{F})$ that shows up most naturally, but rather a shift by 1 (geometrically this is because it is actually the suspension of this nerve that appears). This accounts for a shift by 1 here; the second shift by 1 comes because the reduced cohomology of the empty set, viewed as the cone of $\mathcal{F}[0] \to 0$, is equal to $\mathcal{F}[1]$.

THEOREM C (See also Theorems 4.0.2 and 4.0.3). *Suppose X is a topological space and $\mathcal{F}$ is a sheaf on X, or X is a separated finite-type scheme over an algebraically closed field and $\mathcal{F}$ is a pro-étale sheaf on X. Suppose $\pi : \mathcal{P} \to X$ is a proper finitely ranked pospace or poscheme and let $Z = \pi(\mathcal{P})$ with complement $U := X \backslash Z$. Suppose also that $\mathcal{P}|_Z$ is of cohomological descent for $\mathcal{F}|_Z$ (e.g. it meets the criterion of Theorem B). Then there are spectral sequences, functorial in $\mathcal{F}$:*

(i) *for compactly supported cohomology on Z,*

$$E_1^{p,q} = H_c^{p+q-1}(\mathcal{P}_{p+1}, \tilde{C}(-\infty, \mathcal{P}_{p+1}^+, \mathcal{F})) \Rightarrow H_c^{p+q}(Z, \mathcal{F});$$

(ii) *for compactly supported cohomology on U,*

$$E_1^{p,q} = H_c^{p+q-2}(\mathcal{P}_p^+, \tilde{C}(-\infty, \mathcal{P}_p^+, \mathcal{F})) \Rightarrow H_c^{p+q}(U, \mathcal{F}).$$

The spectral sequences in parts (i) and (ii) are related by the long exact sequence in compactly supported cohomology and are essentially equivalent. We state them individually for convenience in applications.

Remark 1.2.4. The more general setup of Theorem 4.0.3 allows us, in particular, to treat varieties X over non-algebraically closed fields κ_0. In this case, compactly supported cohomology is replaced with $R^\bullet f_!$ for f the structure map to $\operatorname{Spec} \kappa_0$. This is equivalent to working with compactly supported cohomology over an algebraic closure $\kappa = \overline{\kappa}_0$ and remembering the Galois action of $\operatorname{Aut}(\kappa/\kappa_0)$, so that in this setting one deduces the spectral sequences of Theorem C are $\operatorname{Aut}(\kappa/\kappa_0)$-equivariant.

Remark 1.2.5. As in the toy model, these spectral sequences arise from a functorial filtration on a suitable category of complexes and this is made precise in Theorem 4.0.2. Thus, we can replace $\mathcal{F}$ with a complex of sheaves and $H_c^\bullet$ with Rf_* or $Rf_!$ for a general morphism $f : X \to S$; see Theorem 4.0.3.

Remark 1.2.6. In the topological case, if we take geometric realizations, then the terms are computing the compactly supported cohomology of the complement of one stratum of $|N\mathcal{P}|$ in the next, and the spectral sequence above arises from interpreting these complements using the geometric realizations of the nerves of auxiliary pospaces. In particular, for constant coefficients A, if we assume for part (ii) that $\operatorname{rk}(-\infty) = 0$, we can rewrite these spectral sequences as:

(i) $E_1^{p,q} = H_c^{p+q}(|N(-\infty, \mathcal{P}_{p+1}]| - |N(-\infty, \mathcal{P}_{p+1})|; A) \Rightarrow H_c^{p+q}(Z; A)$;
(ii) $E_1^{p,q} \Rightarrow H_c^{p+q}(U; A)$, with

$$E_1^{p,q} = \begin{cases} 0 & \text{if } p < 0, \\ H_c^q(X; A) & \text{if } p = 0, \\ H_c^{p+q-1}(|N(-\infty, \mathcal{P}_p]| - |N(-\infty, \mathcal{P}_p)|; A) & \text{if } p > 0. \end{cases}$$

Here $(-\infty, \mathcal{P}_r] = (\mathcal{P} \times \mathcal{P}_r) \cap \leq_{\mathcal{P}}$, a pospace over $\mathcal{P}_r$. For $p \in \mathcal{P}_r$ the nerve of the fiber $(-\infty, p]$ is a cone with vertex p and base $N(\infty, p)$, so the fiber of $|N(-\infty, \mathcal{P}_r]| - |N(-\infty, \mathcal{P}_r)|$ over $p \in \mathcal{P}_r$ is the *open cone* on $|N(-\infty, p)|$. In applications, the space $|N(-\infty, \mathcal{P}_r]| - |N(-\infty, \mathcal{P}_r)|$ is often a bundle over $\mathcal{P}_r$. For instance, for the configuration poscheme we study in § 6, it is a disk (open simplex) bundle over $\mathcal{P}_r$ with sign monodromy, which gives a topological explanation for the ubiquity of the sign representation in § 6 and in our application to Vakil and Wood's conjecture (see § 8).

We conclude this subsection with some applications of Theorem C. The first two applications recover the two spectral sequences of a stratified space described in the introduction to [Pet17].

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1.2.7 *The first spectral sequence of a stratified space.* Suppose a topological space or scheme X is a finite union of disjoint locally closed sets S_α, such that the closure $Z_\alpha := \overline{S_\alpha}$ is a union of strata S_β. The index set is a poset under the order $\alpha \geq \beta$ if $Z_\alpha \supseteq Z_\beta$.

We consider the pospace or poscheme $\mathcal{P} = \bigsqcup_\alpha Z_\alpha \to X$ where the relation is given by $z_\alpha \geq z_\beta$ if $z_\alpha = z_\beta$ as elements of X and $Z_\alpha \supseteq Z_\beta$. It is a finite disjoint union of closed immersions into X, thus proper. The fiber over a geometric point z has an isolated minimum, given by z_β for $z \in S_\beta$. There is a map from $\mathcal{P}$ to the index poset, so a rank on the latter induces a rank on $\mathcal{P}$. Thus, Theorem C(i) applies with $Z = X$ and yields (see § 4.4.1) the spectral sequence of a stratified space

$$E_1^{p,q} = \bigoplus_{\mathrm{rk}(\alpha)=p+1} H_c^{p+q}(S_\alpha, \mathcal{F}) \Rightarrow H_c^{p+q}(X, \mathcal{F}).$$

1.2.8 *The Petersen spectral sequence of a stratified space.* Here we adopt the same setup as above, but *reverse the ordering*: now $\alpha \leq \beta$ if $Z_\alpha \supseteq Z_\beta$. We also assume there is a unique stratum $Z_\eta = X$, which we remove from $\mathcal{P}$. Now after fixing a rank function on the indices for this reversed order, we have

$$\mathcal{P}_p = \bigsqcup_{\mathrm{rk}(\alpha)=p+1} Z_\alpha \quad \text{and} \quad (-\infty, \mathcal{P}_{p+1}) = \bigsqcup_{\mathrm{rk}(\alpha)=p+1} Z_\alpha \times (\eta, \alpha)$$

Thus, the pospaces/poschemes appearing in $E_1^{p,q}$ in Theorem C are locally constant. In particular, if X is a topological space or a variety over an algebraically closed field, then applying Theorem C(ii), we obtain Petersen's [Pet17] spectral sequence:

$$E_1^{p,q} = \bigoplus_{\mathrm{rk}(\alpha)=p} H_c^{p+q-2}(Z_\alpha, \tilde{C}^\bullet(\eta, \alpha, \mathbb{Z}) \otimes \mathcal{F}) \Rightarrow H_c^{p+q}(S_\eta, \mathcal{F}).$$

Taking $\star = *$ in Theorem 4.0.3(ii) gives the variant with supports of [Pet17].

1.2.9 *Approximate inclusion–exclusion and Vassiliev-type sequences.* Given a proper map of varieties $f : Z \to X$, there are several natural poschemes one can construct that resolve the image $f(Z)$. We investigate some of these in § 6, with an emphasis on the poscheme of relative effective zero-cycles; the latter gives a resolution of $f(Z)$ when f is finite, but is useful more generally because it can be used to give an approximate inclusion–exclusion principle when the finite locus of f has complement of high codimension (see Theorem 6.4.1). The terms of the rank spectral sequence of Theorem C in this case are given by the compactly supported sign cohomology of relative configuration spaces, and indeed the spectral sequence is closely related to the stable part of the Vassiliev spectral sequences for discriminant loci appearing in [Tom14]. In the case where the map is not finite or well-approximated by a finite map, one can use the full Hilbert poscheme to get a full Vassiliev spectral sequence closely related to the sequences appearing in, e.g., [Vas99]. We refer the reader to the introduction of § 6 for more on these points.

1.2.10 *Other applications.* Theorem C also recovers Banerjee's [Ban19, Theorem 1] spectral sequence of a symmetric semisimplicial filtration; see § 4.4.2.

1.3 Decategorifications and Grothendieck rings

Recall that, in the toy model of § 1.1, one could also interpret the classical inclusion–exclusion formula as an identity in the (combinatorial) Grothendieck ring of the category of finite sets, which was moreover equal to the Grothendieck ring of constructible sheaves.

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If X is a Noetherian scheme, then this story is enriched: for ℓ invertible on X, we can form the Grothendieck ring of sheaves $K_0(D_{\mathrm{Cons}}(X, \mathbb{Q}_\ell))$. In the setting of Theorem B, the filtration then induces corresponding Euler characteristic identities in $K_0(D_{\mathrm{Cons}}(X, \mathbb{Q}_\ell))$. We can also form the (modified; see § 5.1) Grothendieck ring of varieties $K_0(\mathrm{Var}/X)$, and there is a compactly supported cohomology map

$$K_0(\mathrm{Var}/X) \to K_0(D_{\mathrm{Cons}}(X, \mathbb{Q}_\ell)).$$

Unlike the case of finite sets, however, it is not typically an isomorphism, so we have two different decategorifications (one combinatorial and one abelian). There is also an inclusion–exclusion principle in the combinatorial decategorification $K_0(\mathrm{Var}/X)$ lifting the inclusion–exclusion principle in the abelian decategorification $K_0(D_{\mathrm{Cons}}(X, \mathbb{Q}_\ell))$ along this map: for a simplicial scheme $S_\bullet/X$, let

$$\chi(S_\bullet) := \sum_{k=0}^{\infty} (-1)^k [\mathcal{S}_k^\circ] \in K_0(\mathrm{Var}/X) \quad \text{and} \quad \tilde{\chi}(S_\bullet) := \chi(S_\bullet) - [X/X] \in K_0(\mathrm{Var}/X), \quad (1.3.0.1)$$

where $\mathcal{S}_k^\circ$ denotes the subscheme of non-degenerate k-simplices and we only consider $S_\bullet$ such that these spaces are of finite type over X and this sum is finite. When $S_\bullet = N\mathcal{P}$ for a poscheme $\mathcal{P}/X$, the non-degenerate simplices are the strictly ordered chains, and the sum is finite exactly when $\mathcal{P}$ is of bounded length, i.e. there is some bound on the length of strict chains. We show the following.

Theorem D (Motivic inclusion–exclusion; proof in § 5.2). *Let X be a Noetherian scheme and let $\mathcal{P}/X$ be a poscheme over X of finite type that admits a weak center in each geometric fiber. Then, in $K_0(\mathrm{Var}/X)$, $[X/X] = \chi(N\mathcal{P})$. If $\mathcal{P}$ is equipped with a rank, then this is furthermore equal to*

$$-\sum_p \tilde{\chi}(N(-\infty, \mathcal{P}_p)) = -\sum_p \left(\left(\sum_{k=0}^{\infty} (-1)^k [N(-\infty, \mathcal{P}_p)_k^\circ] \right) - [\mathcal{P}_p] \right) \in K_0(\mathrm{Var}/X). \quad (1.3.0.2)$$

Note that the reduced Euler characteristic $\tilde{\chi}(N(-\infty, \mathcal{P}_p))$ appearing here is first formed in $K_0(\mathrm{Var}/\mathcal{P}_p)$, then viewed as a class in $K_0(\mathrm{Var}/X)$ by the forgetful map $K_0(\mathrm{Var}/\mathcal{P}_p) \to K_0(\mathrm{Var}/X)$.

Remark 1.3.1. There is no properness assumption in Theorem D.

Remark 1.3.2. As in the toy model § 1.1, the expression (1.3.0.2) coming from a rank function is just a rearrangement of the terms in the Euler characteristic. This is sometimes quite useful.

This generalizes the motivic inclusion–exclusion principle of [BH21, VW15], which is recovered by applying Theorem D to the configuration poscheme $\mathbf{C}_X^\bullet(Z)$ of a morphism $Z \to X$ (see Theorem 6.4.1 for a closely related approximate motivic inclusion–exclusion principle along with a cohomological analog).

1.4 Outline

In § 2 we give a brief overview of simplicial spaces and simplicial schemes and filtered derived categories, recalling precisely the points that we need later on. In particular, we explain how to translate a geometric filtration into a support filtration on the derived category of sheaves on a simplicial space or simplicial scheme and establish some useful lemmas related to this procedure. In § 3 we define and study elementary properties of pospaces and poschemes and the nerve construction.

With these preliminaries established, in § 4 we prove our cohomological inclusion–exclusion results, Theorems B and C. The key ideas as described in § 1.2 come from elementary poset topology, so that the main work is just to be careful with the technical details in this enriched setting. In § 5 we prove our motivic inclusion–exclusion result, Theorem D; this follows the same general pattern as the cohomological case and is in some sense the simpler of the two, but some extra work is needed to translate the contracting homotopies used in § 4 into identifications of enriched Euler characteristics.

In § 6, we study the relative poscheme of effective zero-cycles for a projective morphism of varieties $f : Z \to X$; in particular, we compute the graded components for the rank spectral sequence and the E_1 page of the skeletal spectral sequence, and prove an approximate inclusion–exclusion theorem (Theorem 6.4.1). These results play an important role in our application to homological stability, Theorem A. Note that we systematically use divided powers in place of symmetric powers, which leads to a more technically satisfying theory in positive and mixed characteristics.

In § 7 we explain how to construct motivic and sheaf-theoretic incidence algebras attached to poschemes. In particular, we explain how to realize the Kapranov zeta function as an element of the reduced incidence algebra of the poscheme of effective zero-cycles, then use this to give a new perspective on the inversion formula for the Kapranov zeta function as an instance of Möbius inversion (that Hasse–Weil zeta functions can be realized inside incidence algebras of posets of zero-cycles is well-known and classical; see [Kob20] for a recent survey, which also raised the question of whether such an approach could exist for the Kapranov zeta function). Motivated by this construction, we also define cohomological special values of the inverse Kapranov zeta function; these describe the stable homology in Theorem A.

Finally, in § 8 we prove Theorem A. As indicated earlier, § 8 begins with a self-contained introduction that provides a much more detailed discussion of Vakil and Wood's conjecture and related work leading to Theorem A.

2. Preliminaries

2.1 Simplicial objects

2.1.1 *Definitions.* We write Δ for the simplex category, which we take as the category of non-empty finite subsets of $\mathbb{Z}_{\geq 0}$ with morphisms given by non-decreasing maps (this is the minimal version of Δ which allows the clean formulation of the join construction in Example 2.1.2(iii) below). We write $[n] = \{0, \dots, n\} \in \Delta$. A simplicial object in a category $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$. We write $s\mathcal{C}$ for the category of simplicial objects in $\mathcal{C}$. We will often specify a simplicial object by specifying its restriction to the full subcategory consisting of the objects $[n]$ for $n \geq 0$, which is canonically equivalent to Δ, and given $A \in s\mathcal{C}$ we write $A_k := A([k])$. We write Δ^k for the simplicial set $\mathrm{Hom}(\bullet, [k])$.

Example 2.1.2.

(i) Given an object $X \in \mathcal{C}$, we can form the constant simplicial object $X_\bullet \in s\mathcal{C}$ such that $X_k = X$ for all k and all maps are Id_X.

(ii) If $\mathcal{C}$ admits coproducts, then for any $A \in s\mathcal{C}$ and simplicial set $T \in s\mathrm{Set}$, we can form $A \times T \in s\mathcal{C}$ as in [Sta19, Tag 017C] by the formula

$$(A \times T)_k = \bigsqcup_{t \in T_k} A_k.$$

If $\mathcal{C}$ only admits finite coproducts then this still makes sense for T a finite simplicial set, in particular for $T = \Delta^k$.

(iii) If $\mathcal{C}$ admits finite coproducts and products and a final object $*$, we define the join $A \star B$ of $A, B \in s\mathcal{C}$ by

$$(A \star B)_k = \bigsqcup_{j=-1}^{k} A([j]) \times B([k]\backslash[j])$$

where we interpret $[-1] = \emptyset$ and $A(\emptyset) = B(\emptyset) = *$. With the same hypotheses, for $A \in s\mathcal{C}$, we define

$$\text{Cone}(A) = (*_\bullet) \star A \quad \text{and} \quad \text{Cocone}(A) = A \star (*_\bullet).$$

2.1.3 *Homotopy.* Suppose $\mathcal{C}$ admits finite coproducts. Then, for any simplicial object $A \in s\mathcal{C}$, we can form $A \times \Delta^1$ as in Example 2.1.2(ii). If $f, g : A \to B$ are maps in $s\mathcal{C}$, a homotopy from f to g is a map $h : A \times \Delta^1 \to B$ such that $\{0 \mapsto 0\}^* h = f$ and $\{0 \mapsto 1\}^* h = g$. We say f and g are homotopic if there is a chain of maps $f = f_1, f_2, f_3, \ldots, f_n = g$ such that for each i, there is a homotopy from f_i to f_{i+1} or from f_{i+1} to f_i.

A map $f : A \to B$ in $s\mathcal{C}$ is a homotopy equivalence if there exists a map $g : B \to A$ such that $f \circ g$ is homotopic to Id_A and $g \circ f$ is homotopic to Id_B. A functor $\mathcal{C} \to \mathcal{D}$ induces a functor $s\mathcal{C} \to s\mathcal{D}$ which preserves homotopies (by applying the functor to the maps realizing the homotopies) and, thus, also homotopic maps and homotopy equivalences. If $\mathcal{C}$ has a final object $*$, then we say $A \in s\mathcal{C}$ is contractible if the unique map $A \to *_\bullet$ is a homotopy equivalence.

We will mostly encounter the following special type of homotopy equivalence: we say $\iota : A \hookrightarrow B$ is a deformation retract if there is a map $r : B \to A$ such that $r \circ \iota = \text{Id}_A$ and $\iota \circ r$ is homotopic to Id_B. If A is contractible, then any $\iota : *_\bullet \hookrightarrow A$ is a homotopy inverse for $A \to *_\bullet$ and, therefore, a deformation retract.

Example 2.1.4. If $\mathcal{C}$ admits finite coproducts and has a final object $*$, then, for any $A \in s\mathcal{C}$, $\text{Cone}(A)$ and $\text{Cocone}(A)$ (as defined in Example 2.1.2(iii)) are contractible. Indeed, the homotopy inverse ι as above is given by inclusion of the vertex/tip.

2.1.5 *Augmentations and contractible objects.* For $X \in \mathcal{C}$ and $A \in s\mathcal{C}$, an augmentation from A to X is a map $\epsilon : A \to X_\bullet$. The category $s\mathcal{C}_{/X_\bullet}$ of simplicial objects equipped with an augmentation is canonically equivalent to $s\mathcal{C}_{/X} := s(\mathcal{C}_{/X})$, the simplicial objects in the category $\mathcal{C}_{/X}$ of objects in $\mathcal{C}$ with a map to X. We say $A/X_\bullet$ is contractible if it is contractible as an object of $s\mathcal{C}_{/X}$, i.e. if the map $\epsilon : A \to X_\bullet$ is a homotopy equivalence with homotopy inverse given by a section of ϵ.

2.1.6 *Cosimplicial objects in abelian categories.* A cosimplicial object in $\mathcal{C}$ is a simplicial object in $\mathcal{C}^{\text{op}}$. If $\mathcal{C}$ is abelian, there are two natural functors from $s\mathcal{C}^{\text{op}}$ to the category of cochain complexes in $\mathcal{C}$ concentrated in positive degree (see, e.g., [Sta19, Tag 0194]).

(i) The Moore complex (of objects in $\mathcal{C}$) attached to $A \in s\mathcal{C}^{\text{op}}$ is

$$A_0 \xrightarrow{d_1} A_1 \xrightarrow{d_2} A_2 \xrightarrow{d_3} \cdots,$$

where $d_k = \sum_{i \in [k]} (-1)^i \delta_i$ for δ_i the morphism obtained by applying A to the standard ith face map $[k-1] \to [k]$ that skips the vertex $i \in [k]$.

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(ii) The normalized cochain complex attached to $A \in s\mathcal{C}^{\mathrm{op}}$ is the subcomplex obtained by replacing each term A_k with the kernel of the total degeneracy map.

The inclusion of the normalized complex in the Moore complex is a chain homotopy equivalence,[3] and both functors send simplicial homotopies to chain homotopies [Sta19, Tag 019S].

2.2 Sheaves on simplicial spaces and simplicial schemes

For X a topological space, we write $\mathrm{Sh}(X)$ for the category of abelian sheaves on X. For X a scheme, we write $\mathrm{Sh}(X)$ for either the category of abelian étale sheaves or the category of pro-étale $\mathcal{O}_L$ or L modules for $L/\mathbb{Q}_\ell$ an algebraic extension (as defined in [BS15]).

If A is a simplicial space (which for us means simplicial topological space) or a simplicial scheme, we extend these definitions in the standard way to define $\mathrm{Sh}(A)$: concretely, we define a sheaf on A to be a family of sheaves $\mathcal{F}_n$ on the simplex spaces A_n equipped with a compatible family of maps $A(f)^*\mathcal{F}_n \to \mathcal{F}_m$ for $f : [n] \to [m]$ in Δ. It can be shown that these form an abelian category that is naturally identified with the category of sheaves on a site built from A, so that, in particular, there are enough injectives and the standard formalism of derived categories of sheaves applies. The case of simplicial spaces is treated concretely in [Sta19, Tag 09VK], or can be considered in parallel with the case of schemes by first passing to the corresponding simplicial site and then applying the formalism of [Sta19, Tag 09WB], Case (A).

Given a simplicial space or simplicial scheme A with an augmentation ϵ towards X, we obtain adjoint pushforward and pullback functors

$$\epsilon_* : \mathrm{Sh}(A) \to \mathrm{Sh}(X), \epsilon^* : \mathrm{Sh}(X) \to \mathrm{Sh}(A)$$

such that $\epsilon^*\mathcal{G}$ is given by the obvious system of $\epsilon_n^*\mathcal{G}$ on A_n and

$$\epsilon_*(\mathcal{F}) = \mathrm{Eq}(\epsilon_{0*}\mathcal{F}_0 \rightrightarrows \epsilon_{1*}\mathcal{F}_1).$$

For $K \in D^+(A)$, the bounded below derived category of $\mathrm{Sh}(A)$, there is a functorial skeletal spectral sequence [Sta19, Tag 0D7A]

$$E_1^{p,q} = R^q\epsilon_{p_*}K \Rightarrow R^{p+q}\epsilon_*K. \tag{2.2.0.1}$$

such that for each q the cochain complex $E_1^{\bullet,q}$ is the Moore complex of the cosimplicial sheaf $[p] \mapsto R^q\epsilon_{p_*}K$ on X (see § 2.1.6).

Definition 2.2.1. We write $\tilde{C}(A/X, \bullet)$ for the relative reduced cohomology complex functor $D^+(X) \to D^+(X)$, i.e. the cone of the adjunction unit $u_{A/X} : \mathrm{Id}_{D^+(X)} \to R\epsilon_*\epsilon^*$, so that there is a functorial exact triangle for $K \in D^+(X)$

$$K \to R\epsilon_*\epsilon^*K \to \tilde{C}(A/X, K) \to K[1].$$

We write its cohomology sheaves as

$$\tilde{H}^q(A/X, \bullet) := H^q(\tilde{C}(A/X, \bullet)).$$

We say A/X satisfies cohomological descent on a full subcategory $D \subset D^+(X)$ if $u_{A/X}|_D$ is an isomorphism of functors or, equivalently, if $\tilde{C}(A/X, K) \cong 0$ for each $K \in D$, or equivalently if $\tilde{H}^q(A/X, K) = 0$ for each $K \in D$ and each $q \in \mathbb{Z}$.

[3] For a simplicial object in an abelian category, the Moore complex is quasi-isomorphic to its quotient that in each degree is the cokernel of the total degeneracy map; see, e.g., [Sta19, Tag 019C]. Here we are applying this to a cosimplicial object by viewing it as a simplicial object in the opposite abelian category, which turns the cokernel of the total degeneracy map into the kernel of the total degeneracy map.

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LEMMA 2.2.2. *If $f : A/X \to B/X$ is a homotopy equivalence of simplicial spaces or simplicial schemes over X, then f induces an isomorphism of functors on $D^+(X)$*

$$R\epsilon_{A*}\epsilon_A^* \cong R\epsilon_{B*}\epsilon_B^*.$$

In particular, if A/X is contractible, then A/X satisfies cohomological descent.

Proof. We obtain a map f^* between the skeletal spectral sequences (2.2.0.1) for A/X and B/X. On the qth column $E_1^{\bullet,q}$ this is the map on Moore complexes coming from the map of cosimplicial sheaves

$$f^* : ([p] \mapsto R^q\epsilon_{B_p *}\epsilon_{B_p}^* K) \to ([p] \mapsto R^q\epsilon_{A_p *}\epsilon_{A_p}^* K).$$

Because f is a homotopy equivalence, so is the induced map on cosimplicial sheaves: it comes from applying the functor on simplicial spaces or simplicial schemes over X induced by the functor on topological spaces or schemes over X sending $\pi : Y \to X$ to $R^q\pi_*\pi^*K$. A functor constructed in this way preserves homotopy equivalences (see the second paragraph of § 2.1.3 above). The induced map on complexes is then a homotopy equivalence, thus an isomorphism on cohomology, so that we obtain an isomorphism of spectral sequences starting at the E_2 page. □

2.3 Filtered derived categories and simplicial filtrations

2.3.1 *Filtered derived categories.* We follow [Sta19, Tag 05RX, Tag 015O]. To summarize: for an abelian category $\mathcal{A}$, we write $\mathcal{A}^f$ for the exact category of finitely filtered objects in $\mathcal{A}$. It admits exact filtered piece, graded part, and forgetful functors F^p, Gr^p, Forget : $\mathcal{A}^f \to \mathcal{A}$. For the category of sheaves on a topological space, scheme, simplicial space, or simplicial scheme X, we write $DF^+(X)$ for the bounded below filtered derived category, i.e. the bounded below derived category of $\mathrm{Sh}(X)^f$. The filtered piece, graded and forgetful functors induced triangulated functors $DF^+(X) \to D^+(X)$. Moreover, if A/X is a simplicial space or simplicial scheme augmented towards X, then we have a filtered derived functor $R\epsilon^f{}_* : DF^+(A) \to DF^+(X)$ and a canonical isomorphism

$$\mathrm{Forget} \circ R\epsilon^f{}_* = R\epsilon_* \circ \mathrm{Forget}\,.$$

2.3.2 *Simplicial filtrations.* Let A be a simplicial space or simplicial scheme. We say a map of simplicial spaces or simplicial schemes $B \to A$ is a closed (respectively, open; respectively, clopen) immersion if for all $k \geq 0$, $B_k \to A_k$ is a closed (respectively, open; respectively, clopen) immersion.

A filtration of A is an increasing sequence of closed sub-simplicial spaces/schemes $\iota_i : F_iA \hookrightarrow A, i \in \mathbb{Z}$. It is finite if $F_iA = \emptyset$ for $i \ll 0$ and $F_iA = A$ for $i \gg 0$. It is *split* if ι_i is a clopen immersion for all i.

Example 2.3.3. For A a simplicial space or simplicial scheme over X, Cone(A/X) (see Example 2.1.2) has a natural finite split filtration with

$$F_0 = X_\bullet \quad \text{and} \quad F_1\mathrm{Cone}(A/X) = \mathrm{Cone}(A/X),$$

and similarly for Cocone(A/X).

Attached to a filtration $F_\bullet$ of A we have filtered and graded piece functors

$$F^i : \mathrm{Sh}(A) \to \mathrm{Sh}(A), \mathcal{F} \mapsto F^i\mathcal{F} = \ker(\mathcal{F} \to \iota_{i*}\iota_i{}^*\mathcal{F}),$$

$$\mathrm{Gr}^i : \mathrm{Sh}(A) \to \mathrm{Sh}(A), \mathcal{F} \mapsto \mathrm{Gr}^i\,\mathcal{F} = F^i\mathcal{F}/F^{i+1}\mathcal{F}.$$

These functors are exact: this can be checked on m-simplices, where it immediately reduces to the corresponding assertion for filtrations of spaces/schemes by closed subspaces/subschemes;

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indeed, if we write j for the locally closed immersion $F_{i+1}A_m \backslash F_i A_m \hookrightarrow A_m$, then there is a canonical identification of sheaves on A_m

$$j_! j^* \mathcal{F}_m = (\mathrm{Gr}^i \mathcal{F})_m.$$

We can reinterpret this description of the graded pieces geometrically as the following useful lemma which also describes the simplicial structure.

Lemma 2.3.4. *For $F_\bullet A$ a finite filtration, let $\iota_p : F_p A \to A$. The unit $F^p\mathcal{F} \to {\iota_{p+1}}_* \iota_{p+1}^* F^p \mathcal{F}$ induces an isomorphism $\mathrm{Gr}^p \mathcal{F} = {\iota_{p+1}}_* \iota_{p+1}^* F^p \mathcal{F}$.*

As a consequence of exactness, applying F^i or Gr^i termwise to complexes induces a functor $D^+(A) \to D^+(A)$. When the filtration $F_\bullet A$ is finite, the functors F^i assemble to a functor

$$\mathrm{Fil}_{F_\bullet} : \mathrm{Sh}(A) \to \mathrm{Sh}(A)^f$$

that is exact (i.e. maps quasi-isomorphisms to filtered quasi-isomorphisms; this is equivalent to the assertion above that each filtered or graded piece functor is exact), so that termwise application to complexes induces a functor $\mathrm{Fil}_{F_\bullet} : D^+(A) \to DF^+(A)$ whose associated filtered and graded piece functors are canonically identified with F^i and Gr^i as above and such that there is a canonical identification $\mathrm{Forget} \circ \mathrm{Fil} = \mathrm{Id}_{D^+(A)}$.

3. Pospaces and poschemes

In this section we treat some elementary properties of pospaces and poschemes and their nerves. In § 3.1 we make the basic definitions, and in § 3.2 we develop some useful tools for establishing contractibility of the nerve. In particular, we show that the nerve of a pospace or poscheme with a maximum/minimum/center is contractible.

3.1 Definitions and first properties

Definition 3.1.1 (Pospaces/poschemes).

- (i) For X a topological space, a *pospace over* X is a continuous map $\mathcal{P} \to X$ equipped with a closed poset relation $\leq_{\mathcal{P}} \subset \mathcal{P} \times_X \mathcal{P}$.
- (ii) For X a scheme, a *poscheme over* X is a morphism $\mathcal{P} \to X$ equipped with a closed poset relation $\leq_{\mathcal{P}} \subset \mathcal{P} \times_X \mathcal{P}$ (in this case, by a poset relation we mean that it should induce a poset structure on $\mathcal{P}(T)$ for any T/X).

A pospace (respectively, poscheme) $\mathcal{P}/X$ is *proper* if $\mathcal{P} \to X$ is proper as a map of topological spaces (respectively, schemes). A map $f : \mathcal{P}_1 \to \mathcal{P}_2$ of pospaces (respectively, poschemes) over X is a map of topological spaces (respectively, schemes) over X respecting the order relation, i.e. such that $f \times f|_{\leq_{\mathcal{P}_1}}$ factors through $\leq_{\mathcal{P}_2}$.

For $\mathcal{P}/X$ a pospace or poscheme, we write $\geq_{\mathcal{P}}$ for the closed relation obtained by swapping the coordinates; we have $\Delta_{\mathcal{P}} = \geq_{\mathcal{P}} \cap \leq_{\mathcal{P}}$, thus $\Delta_{\mathcal{P}}$ is closed, so $\mathcal{P}/X$ is separated as a map of topological spaces/schemes. We will also write $<_{\mathcal{P}} = \leq_{\mathcal{P}} \backslash \Delta_{\mathcal{P}}$, the complement of the diagonal in $\leq_{\mathcal{P}}$, and similarly for $>_{\mathcal{P}}$.

Example 3.1.2. Any poset $\mathcal{P}$ induces a constant pospace (respectively, poscheme) over any topological space (respectively, scheme) X, $\mathcal{P} \times X$.

It will be convenient to use the following relative interval notation.

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DEFINITION 3.1.3 (Intervals). Suppose $\mathcal{P}/X$ is a pospace or poscheme, and suppose given $T_1 \to \mathcal{P}$ and $T_2 \to \mathcal{P}$. The *open interval* (T_1, T_2) is the fiber product

$$\begin{array}{ccc} (T_1, T_2) & \longrightarrow & T_1 \times_X T_2 \\ \downarrow & & \downarrow \\ <_{\mathcal{P}} \times_{\mathcal{P}} <_{\mathcal{P}} & \xrightarrow{(a<b<c)\mapsto(a,c)} & \mathcal{P} \times_X \mathcal{P} \end{array}$$

viewed as a pospace or poscheme over $T_1 \times T_2$ with ordering pulled back from the middle coordinate. The closed and mixed intervals $[T_1, T_2]$, $(T_1, T_2]$, $[T_1, T_2)$ over $T_1 \times T_2$ are defined by changing between $\leq$ and $<$ appropriately in the bottom left.

Given $T \to \mathcal{P}$, we similarly define the intervals $(-\infty, T)$, $(-\infty, T]$, (T, ∞) and $[T, \infty)$ as poschemes or pospaces over T. These latter can be interpreted literally in the above notation as intervals in the pospace or poscheme obtained by adding disjoint maximum and minimum sections ∞ and $-\infty$ to $\mathcal{P}/X$.

DEFINITION 3.1.4 (Rankings and split pospaces and poschemes).

(i) A *ranking* on a pospace (respectively, poscheme) $\mathcal{P}/X$ is a strictly increasing map of pospaces (respectively, poschemes) $\mathrm{rk} : \mathcal{P} \to \mathbb{Z}$, where $\mathbb{Z}$ with its usual ordering is viewed as a constant pospace (respectively, poscheme) as in Example 3.1.2 and strictly increasing means that for any $k \in \mathbb{Z}$, the restriction of $\leq_{\mathcal{P}}$ to $\mathrm{rk}^{-1}(\{k\})$ is the diagonal. The ranking is *finite* if it factors through $\{-n, \ldots, n\} \subset \mathbb{Z}$ for some n. A pospace or poscheme $\mathcal{P}/X$ is (*finitely*) *ranked* if it is equipped with a (finite) ranking, in which case we write $\mathcal{P}_k := \mathrm{rk}^{-1}(\{k\})$, $\mathcal{P}_{\leq k} := \mathrm{rk}^{-1}((-\infty, k])$, etc.

(ii) A pospace or poscheme $\mathcal{P}/X$ is *split* if $\Delta_{\mathcal{P}}$ is open in $\leq_{\mathcal{P}}$ (or, equivalently, in $\geq_{\mathcal{P}}$), in which case $\leq_{\mathcal{P}} = \Delta_{\mathcal{P}} \sqcup <_{\mathcal{P}}$ (and $\geq_{\mathcal{P}} = \Delta_{\mathcal{P}} \sqcup >_{\mathcal{P}}$).

Any pospace or poscheme that admits a ranking is split, and our motivating applications all fall into this category. It is possible to construct examples that are not split, but imposing this condition will simplify some hypotheses later (because then maxima and minima are centers; see Remark 3.1.8).

DEFINITION 3.1.5. If $\mathcal{P}$ is a pospace (respectively, poscheme) over a topological space (respectively, scheme) X:

(i) a *subpospace* (respectively, *subposcheme*) $\mathcal{P}' \subset \mathcal{P}$ is a subspace (respectively, subscheme) equipped with the induced relation

$$\leq_{\mathcal{P}'} = \leq_{\mathcal{P}} \times_{\mathcal{P} \times_X \mathcal{P}} (\mathcal{P}' \times_X \mathcal{P}');$$

(ii) if $\mathcal{P}' \subset \mathcal{P}$ is a subpospace (respectively, subposcheme), a *retraction of* $\mathcal{P}$ *onto* $\mathcal{P}'$ is a map $r : \mathcal{P} \to \mathcal{P}'$ of pospaces (respectively, poschemes) over X such that $r|_{\mathcal{P}'} = \mathrm{Id}_{\mathcal{P}'}$.

DEFINITION 3.1.6. If $\mathcal{P}$ is a pospace (respectively, poscheme) over a topological space (respectively, scheme) X:

(i) a *maximum* of $\mathcal{P}/X$ is a section $m : X \to \mathcal{P}$ such that $\mathcal{P} = (-\infty, m]$;
(ii) a *minimum* of $\mathcal{P}/X$ is a section $m : X \to \mathcal{P}$ such that $\mathcal{P} = [m, \infty)$;
(iii) a *center* of $\mathcal{P}/X$ is a section $c : X \to \mathcal{P}$ such that

$$\mathcal{P} = (-\infty, c) \sqcup c(X) \sqcup (c, \infty)$$

(in particular, c is isolated, i.e. $c(X)$ is clopen).

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If X is a scheme and $\mathcal{P}/X$ is a poscheme, a *weak* maximum/minimum/center is a section such that the corresponding identity holds topologically but not necessarily scheme-theoretically.

Example 3.1.7. If κ is a field, the poscheme $\mathcal{P} = \operatorname{Spec} \kappa[\epsilon]/(\epsilon^2)$ over $\operatorname{Spec} \kappa$ with trivial ordering relation $\leq_{\mathcal{P}} = \Delta_{\mathcal{P}}$ admits a weak center given by the reduced subscheme $\operatorname{Spec} \kappa = \mathcal{P}^{\mathrm{red}} \hookrightarrow \mathcal{P}$.

Remark 3.1.8. For a split pospace or poscheme, if $\mathcal{P} = (-\infty, c] \cup [c, \infty)$ set theoretically, then it follows immediately that c is a center in the topological case and a weak center in the scheme theoretic case. In particular, maxima and minima are centers for split $\mathcal{P}$. For nonsplit $\mathcal{P}$ and c satisfying only $\mathcal{P} = (-\infty, c] \cup [c, \infty)$, there can be connected T/X such that c is not a center of $\mathcal{P}(T)$ in the naive sense, which is why we require that c be isolated. This issue does not occur with maxima and minima even when they are not isolated.

Definition 3.1.9. The *nerve* (or order complex) of a pospace (respectively, poscheme) $\mathcal{P}/X$ is the simplicial space (respectively, simplicial scheme) $N\mathcal{P}$, canonically augmented to X, whose topological space (respectively, scheme) of m-simplices is the topological space (respectively, scheme) of chains of length $m+1$ in $\mathcal{P}$,

$$N\mathcal{P}_m := \underbrace{\leq_{\mathcal{P}} \times_{\mathcal{P}} \leq_{\mathcal{P}} \times_{\mathcal{P}} \cdots \times_{\mathcal{P}} \leq_{\mathcal{P}}}_{m-1 \text{ terms}}$$

with the obvious transition maps. In other words, it is the topological space (respectively, scheme) over X of pospace (respectively, poscheme) maps from $[m] \times X$ to $\mathcal{P}$. We also write

$$N\mathcal{P}_m^{\circ} := \underbrace{<_{\mathcal{P}} \times_{\mathcal{P}} <_{\mathcal{P}} \times_{\mathcal{P}} \cdots \times_{\mathcal{P}} <_{\mathcal{P}}}_{m-1 \text{ terms}},$$

for the subspace (respectively, subscheme) of $N\mathcal{P}_m$ consisting of non-degenerate m-simplices (i.e. strictly ordered chains). We say $\mathcal{P}/X$ is *of bounded length* if $N\mathcal{P}_m^{\circ} = \emptyset$ for m sufficiently large, i.e. if there is a bound on the length of chains of proper inequalities in $\mathcal{P}/X$.

Note that a finitely ranked pospace or poscheme is bounded, and that any ranked topologically Noetherian poscheme is finitely ranked.

Example 3.1.10. We have the following examples:

(i) $N[n] = \Delta^n$;
(ii) $N\mathcal{P}_2$ is naturally identified with the closed interval $[\mathcal{P}, \mathcal{P}]$ with the map to $\mathcal{P} \times \mathcal{P}$ induced by the edge $0 < 2$ of Δ^2 and the ordering pulled back from the map to $\mathcal{P}$ induced by the vertex 1 of Δ^2; the non-degenerate 2-simplices $N\mathcal{P}_2^{\circ}$ inside are identified with the open interval $(\mathcal{P}, \mathcal{P}) \subset [\mathcal{P}, \mathcal{P}]$;
(iii) $N(N\mathcal{P})$ is the barycentric subdivision of $N\mathcal{P}$, where $N(N\mathcal{P})$ makes sense by the abuse of notation in which we treat $N\mathcal{P}$ as the pospace or poscheme $\bigsqcup_{k=0}^{\infty} N\mathcal{P}_k$ with ordering by inclusion of chains;
(iv) if $\mathcal{P}/X$ is the disjoint union of $\mathcal{A} < \mathcal{B}$, then $N\mathcal{P}$ is the join $N\mathcal{A} \star N\mathcal{B}$ (see Example 2.1.2); in particular, if $\mathcal{P}$ has a center c, then

$$N\mathcal{P} = N(-\infty, c] \star N(c, \infty) = N(-\infty, c) \star N[c, \infty);$$

if c is, in fact, an isolated minimum m (respectively, isolated maximum M), then $N\mathcal{P}$ is naturally identified with $\operatorname{Cone}(N(m, \infty)/X)$ (respectively, $\operatorname{Cocone}(N(-\infty, M)/X)$) as defined in Example 2.1.2(iii).

The following is immediate from the definitions.

Lemma 3.1.11. *Let X be a topological space (respectively, scheme). If $\mathcal{P}/X$ is a split pospace (respectively, poscheme), in particular, if it is ranked, then $N\mathcal{P}$ is split as a simplicial space (respectively, simplicial scheme), i.e. each degeneracy map is an isomorphism onto a clopen set. In particular, in this case $N\mathcal{P}_m^\circ$ is clopen in $N\mathcal{P}_m$ for all m.*

3.2 A contractibility criterion

We now show that if a pospace or poscheme $\mathcal{P}/X$ admits a maximum/minimum/center then $N\mathcal{P}/X$ is contractible. This is essentially an immediate consequence of Example 3.1.10(iv): the case of isolated maxima and minima follows from the contractibility of cones and cocones, while the case of a center follows from contracting separately the two cones forming the join. We make this precise as a consequence of the following more general result, which will have further applications later on.

Lemma 3.2.1. *Let X be a topological space (respectively, scheme), let $\mathcal{P}/X$ be a pospace (respectively, poscheme), and let $\mathcal{P}' \subset \mathcal{P}$ be a subpospace (respectively, subposcheme). If either:*

(i) $(-\infty, \mathcal{P}] \cap \mathcal{P}' \times \mathcal{P}$ *has a maximum (as a pospace or poscheme over $\mathcal{P}$) ; or*
(ii) $[\mathcal{P}, \infty) \cap \mathcal{P} \times \mathcal{P}'$ *has a minimum (as a pospace or poscheme over $\mathcal{P}$);*

then the inclusion of $N\mathcal{P}'$ in $N\mathcal{P}$ is a deformation retract over X.

Proof. We treat the first case: let r denote the composition of the maximum, a map $\mathcal{P} \to (-\infty, \mathcal{P}] \cap \mathcal{P}' \times \mathcal{P}$, with the projection to the first factor in $\mathcal{P}'$. It is evidently a retraction of $\mathcal{P}$ onto $\mathcal{P}'$, so we only need to show that Nr, viewed as a map from $N\mathcal{P}$ to itself, is homotopic to the identity. We define a homotopy

$$N\mathcal{P} \times \Delta^1 \to N\mathcal{P}$$

on simplices as follows: given a k-simplex $(p_0, \ldots, p_k) \times \alpha$ where $\alpha : [k] \to [1]$ is non-decreasing, send it to $(r^{1-\alpha(0)}(p_0), r^{1-\alpha(1)}(p_1), \ldots)$ where $r^0 = \mathrm{Id}$ and $r^1 = r$. The identity $r(p_i) \le p_i$ ensures this is a chain because α is non-decreasing, and evidently it is a homotopy from r to the identity.

For the second case, we define r using the minimum, then obtain a homotopy from r to $\mathrm{Id}_{N\mathcal{P}}$ similarly by replacing $r^{1-\alpha(i)}$ with $r^{\alpha(i)}$ above. □

Remark 3.2.2. By Yoneda, the first statement is equivalent to the statement that, for any $t : T \to \mathcal{P}$, the intersection of the poset $(-\infty, t) \in \mathcal{P}(T)$ with $\mathcal{P}'(T)$ has a maximum, and similarly for the second statement. Thus, one can read the statement as a version of Quillen's Theorem A [Qui73] in a particularly simple case. It would be interesting to see if this interpretation can be pushed any further.

Remark 3.2.3. To give a map $r : \mathcal{P} \to \mathcal{P}'$ as in the proof in case (i) of Lemma 3.2.1 is equivalent to giving a retraction $r : \mathcal{P} \to \mathcal{P}'$ that, when viewed as a map $\mathcal{P} \to \mathcal{P}$, satisfies $r \le \mathrm{Id}_{\mathcal{P}}$ in the poset $\mathcal{P}(\mathcal{P})$. We call such a map a *falling retraction*, since it can be equivalently characterized as a retraction such that, for all t, $r(t) \le t$. On the other hand, one easily sees that, given a falling retraction r, $r \times \mathrm{Id}$ is a maximum of $(-\infty, \mathcal{P}] \cap \mathcal{P}' \times \mathcal{P}$, so to verify part (i) in the statement of Lemma 3.2.1 is the same as to give a falling retraction. Similarly, to verify part (ii) is the same as to give a rising retraction, i.e. a retraction $r : \mathcal{P} \to \mathcal{P}'$ such that $r(t) \ge t$ for all t.

As an application, we find the following.

Lemma 3.2.4. *Let X be a topological space (respectively, scheme). If $\mathcal{P}/X$ is a pospace (respectively, poscheme) with a maximum, minimum, or center, then $N\mathcal{P}/X$ is contractible.*

Proof. Given a maximum (respectively, minimum) m, we can apply Lemma 3.2.1 to the subposcheme $\mathcal{P}' = m(X)$. Given a center c, we can first apply Lemma 3.2.1(i) to $(-\infty, c] \subset \mathcal{P}$; here, the maximum on $(-\infty, \mathcal{P}] \cap (-\infty, c] \times \mathcal{P}$ is given by $\mathrm{Id} \times \mathrm{Id}$ on the clopen set $(\infty, c]$ and by $c \times \mathrm{Id}$ on the clopen $(c, \infty]$. We then use that c is a maximum of $(-\infty, c]$. □

4. Cohomological inclusion–exclusion

In this section we elaborate on the setup from § 1.2 to prove Theorems B and C. We recall that Theorem B gives a criterion for cohomological descent, while Theorem C gives the spectral sequence of a rank function in the presence of cohomological descent. In the introduction we restricted to a single sheaf and cohomology with compact support, but here we will obtain the more general statements alluded to in Remarks 1.2.1 and 1.2.5.

We first state a version of Theorem B that applies to complexes. The conditions imposed are those necessary to ensure a suitable proper base change theorem holds.

Theorem 4.0.1 (Refinement of Theorem B).

(i) *If X is a topological space and $\mathcal{P}/X$ is a proper pospace such that $\mathcal{P}_x$ admits a center for all $x \in X$, then $\mathcal{P}/X$ is of cohomological descent on $D^+(X)$.*
(ii) *If X is a scheme and $\mathcal{P}/X$ is a proper poscheme such that $\mathcal{P}_x$ admits a weak center for every geometric point $x : \operatorname{Spec} \kappa \to X$, then:*
  (a) *$\mathcal{P}/X$ is of cohomological descent on the subcategory of $D^+(X_{\text{ét}})$ consisting of complexes with torsion cohomology sheaves;*
  (b) *if X is, furthermore, Noetherian and L is an algebraic extension of $\mathbb{Q}_\ell$ for some ℓ invertible on X, then $\mathcal{P}/X$ is of cohomological descent on the subcategories of $D^+(X_{\text{proét}}, \mathcal{O}_L)$ or $D^+(X_{\text{proét}}, L)$ consisting of complexes with constructible cohomology sheaves.*

In § 4.1 we explain the reduction of cohomological descent to pointwise contractibility via proper base change (and topological invariance of the étale site in the schematic case), then in § 4.2 we put this together with the contractibility criteria of § 3.2 to finish the proof of Theorem 4.0.1.

For any ranked $\pi : \mathcal{P} \to X$, we obtain a rank filtration $F^{\mathrm{rk}}_\bullet$ on $N\mathcal{P}$. As in § 2.3, this induces a filtration on sheaves, and if the ranking is finite we obtain a functor

$$\mathrm{Fil}_{\mathcal{P},\mathrm{rk}} = R\epsilon^f_* \circ \mathrm{Fil}_{F^{\mathrm{rk}}_\bullet} \circ \epsilon^* : D^+(\mathrm{Sh}(X)) \to DF^+(\mathrm{Sh}(X))$$

along with graded parts functors $\mathrm{Gr}^p_{\mathcal{P},\mathrm{rk}}$ (the latter are defined even if the filtration is not finite). Up to a shift, we will identify the graded parts with the reduced cohomology complex functors that were described in the introduction. Precisely, we show:

Theorem 4.0.2. *Let X be a topological space (respectively, scheme) and suppose $\pi : \mathcal{P} \to X$ is a ranked pospace (respectively, poscheme). For $\tilde{C}$ as defined in Definition 2.2.1, there are canonical isomorphisms of functors $D^+(X) \to D^+(X)$*

$$\mathrm{Gr}^p_{\mathcal{P},\mathrm{rk}}(\bullet) \xrightarrow{\sim} \tilde{C}(N(-\infty, \mathcal{P}_{p+1})/\mathcal{P}_{p+1}, \pi^*(\bullet))[-1].$$

If $\mathcal{P}/X$ is finitely ranked and of cohomological descent on a subcategory $\mathcal{C} \hookrightarrow D^+(X)$, then there is a canonical isomorphism from the inclusion $\mathcal{C} \hookrightarrow D^+(X)$ to $\mathrm{Forget} \circ \mathrm{Fil}_{\mathcal{P},\mathrm{rk}}$.

This immediately implies a functorial spectral sequence for any derived functor restricted to $\mathcal{C}$. We state a useful general case refining Theorem C as Theorem 4.0.3 below: in § 4.4 we will provide the justification, as well as details for the examples given in § 1.2. To state the result, as in the introduction (paragraph preceding Theorem C), we extend the rank function to $\mathcal{P}^+/X$ (obtained by adding a disjoint minimum section $-\infty$), then define (cf. Definition 2.2.1)

$$\tilde{C}(-\infty, \mathcal{P}_r^+, \bullet) : D^+(X) \to D^+(\mathcal{P}_r^+)$$
$$K \mapsto \begin{cases} \tilde{C}(N(-\infty, \mathcal{P}_r)/\mathcal{P}_r, \pi^* K) & r \neq \mathrm{rk}(-\infty) \\ K[2] & r = \mathrm{rk}(-\infty). \end{cases}$$

THEOREM 4.0.3 (Refinement of Theorem C). *Suppose X is a topological space (respectively, scheme), $K \in D^+(X)$, and $\pi : \mathcal{P} \to X$ is a proper finitely ranked pospace (respectively, poscheme). Let $Z = \pi(\mathcal{P})$ with complement $U = X \backslash Z$, $j : U \hookrightarrow X$. Suppose also that $\mathcal{P}|_Z$ is of cohomological descent for $K|_Z$. Let $\star = *$ or $!$ and let $f : X \to S$ be a morphism; in the scheme case, if $\star =!$, suppose further that S is qcqs and f is separated and of finite type (so $Rf_!$ is defined). Then there are spectral sequences, functorial in K:*

(i) *for $Rf_\star$ on Z:*
$$E_1^{p,q} = R^{p+q-1}(f \circ \pi|_{\mathcal{P}_{p+1}})_\star \tilde{C}(-\infty, \mathcal{P}_{p+1}^+, K) \Rightarrow R^{p+q} f|_{Z\star}(K|_Z);$$
(ii) *for $Rf_\star$ on U:*
$$E_1^{p,q} = R^{p+q-2}(f \circ \pi|_{\mathcal{P}_p^+})_\star \tilde{C}(-\infty, \mathcal{P}_p^+, K) \Rightarrow R^{p+q} f_\star j_! j^* K.$$

4.1 Technical criteria for cohomological descent

The following summarizes the application of proper base change that allows us to spread out cohomological descent from (geometric) points when it holds; in the schematic case we must put some restrictions in place to obtain a suitable proper base change theorem.

LEMMA 4.1.1 (Spreading out cohomological descent by proper base change).

(i) (Spaces). *Let X be a topological space, let $\mathcal{P}/X$ be a proper pospace, and let $K \in D^+(X)$. If $N\mathcal{P}_x/\{*\}$ is of cohomological descent for K_x for all $x \in X$, then $N\mathcal{P}/X$ is of cohomological descent for K.*

(ii) (Schemes étale). *Let X be a scheme, let $\mathcal{P}/X$ be a proper poscheme and let $K \in D^+(X_{\text{ét}})$ have torsion cohomology sheaves. If $N\mathcal{P}_x$ is of cohomological descent for K_x for all geometric points $x : \operatorname{Spec} \kappa \to X$, then $N\mathcal{P}/X$ is of cohomological descent for K.*

(iii) (Schemes pro-étale). *Let X be a Noetherian scheme, let $\mathcal{P}/X$ be a proper poscheme, and let $K \in D^+(X_{\text{proét}}, \mathcal{O}_L)$ or $D^+(X_{\text{proét}}, L)$ for L an algebraic extension of $\mathbb{Q}_\ell$, ℓ invertible on X, have constructible cohomology sheaves. If $N\mathcal{P}_x/\operatorname{Spec} \kappa$ is of cohomological descent for K_x for all geometric points $x : \operatorname{Spec} \kappa \to X$, then $N\mathcal{P}/X$ is of cohomological descent for K.*

Proof. We first treat case (i): to establish cohomological descent for K, it suffices to show that $C := \tilde{C}(N\mathcal{P}/X, K)$ is quasi-isomorphic to zero. This is equivalent to checking that each cohomology sheaf $\mathcal{H}^i(C)$ of this complex is zero, which can be checked by showing the stalks $\mathcal{H}^i(C)_x$ are zero, and this is equivalent to showing C_x is quasi-isomorphic to zero. To conclude, we invoke proper base change [Sta19, Tag 09V6] to identify C_x with

$$\tilde{C}(N\mathcal{P}_x/\{x\}, K_x),$$

which is quasi-isomorphic to zero by the assumption that cohomological descent holds on fibers.

The argument in case (ii) is identical after replacing points with geometric points and applying proper base change in the form of [Sta19, Tag 0DDE]. For case (iii), it is not true that a general pro-étale sheaf is zero if it is zero after evaluation at all geometric points, but this is

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true for a constructible pro-étale sheaf (where the statement reduces to the étale case). Thus, the same argument also goes through in case (iii), applying proper base change in the form of [BS15, Lemma 6.7.5 and Proposition 6.8.14]. □

To show a pospace or poscheme over a point satisfies cohomological descent, we will typically appeal to contractibility of the nerve; in § 3.2 above we already established some useful tools that can be used to deduce this contractibility. For schemes we will often need one more reduction before we can apply these contractibility criteria; the result is encoded in the following lemma. We say $f : \mathcal{P}_1/X \to \mathcal{P}_2/X$ is a universal homeomorphism of poschemes over X if it is a map of poschemes over X that is universal homeomorphism as a map of schemes and induces a universal homeomorphism $\leq_{\mathcal{P}_1} \to \leq_{\mathcal{P}_2}$.

Lemma 4.1.2 (Contractibility and cohomological descent).

(i) *Let X be a topological space. If $\mathcal{P}/X$ is a pospace and $N\mathcal{P}/X$ is contractible, then $N\mathcal{P}/X$ is of cohomological descent on $D^+(X)$.*
(ii) *Let X be a scheme. If $f : \mathcal{P}/X \to \mathcal{Q}/X$ is a universal homeomorphism of poschemes over X and $N\mathcal{P}/X$ is contractible, then $N\mathcal{Q}/X$ satisfies cohomological descent on $D^+(X_{\text{ét}})$, $D^+(X_{\text{proét}}, \mathcal{O}_L)$, and $D^+(X_{\text{proét}}, L)$.*

Proof. The statement for topological spaces is a reformulation of Lemma 2.2.2. The schemes statement combines Lemma 2.2.2 with the topological invariance of the étale site [Sta19, Tag 03SI]; see [BS15, Lemma 5.4.2] for an explanation of how this leads also to an equivalence of the corresponding pro-étale topoi. The key point is then that a universal homeomorphism of poschemes induces a universal homeomorphism on the simplex spaces of the nerve. □

4.2 Weak centers and the proof of Theorem 4.0.1

Proof of Theorem 4.0.1. The topological case of Theorem 4.0.1 can now be established: we apply Lemma 4.1.1, where the punctual condition is satisfied by combining Lemmas 3.2.4 and 4.1.2. Applying Lemma 4.2.1 below, the argument for the scheme-theoretic case is the same. □

The last lemma used in the proof is an alternative characterization of weak centers: recall from § 3.1.6 that, for $\mathcal{P}/X$ a poscheme, $c \in \mathcal{P}(X)$ is a *weak* center/minimum/maximum if the defining decomposition of $\mathcal{P}$ holds topologically but not necessarily scheme-theoretically.

Lemma 4.2.1. *A poscheme $\mathcal{P}/X$ admits a weak center/maximum/minimum if and only if there is a poscheme $\mathcal{P}'/X$ with a center/maximum/minimum and a universal homeomorphism of poschemes $\mathcal{P}'/X \to \mathcal{P}/X$.*

Proof. We treat the case of a center, the others being similar. If such a $\mathcal{P}'$ exists, then, for $c \in \mathcal{P}'(X)$ a center, the induced point in $\mathcal{P}(X)$ is clearly a weak center. Conversely, if $c \in \mathcal{P}(X)$ is a weak center of $\mathcal{P}$, we can take $\mathcal{P}'$ to be the closed subposcheme $\{< c\} \bigsqcup c(X) \bigsqcup \{> c\}$ of $\mathcal{P}$. □

4.3 Construction of filtration and computation of graded components

Proof of Theorem 4.0.2. Let $\mathcal{P}/X$ be a finitely ranked pospace or poscheme (with no assumptions of cohomological descent or contractibility!). We consider a split simplicial filtration $F_i N\mathcal{P} = \mathrm{rk}^{-1}(-\infty, i]$ of $N\mathcal{P}$. By the formalism of § 2.3, we obtain

$$\mathrm{Fil}_{F_\bullet N\mathcal{P}} : D^+(N\mathcal{P}) \to DF^+(N\mathcal{P})$$

and a canonical isomorphism

$$\mathrm{Id}_{D^+(N\mathcal{P})} = \mathrm{Forget} \circ \mathrm{Fil}_{F_\bullet N\mathcal{P}}\,.$$

We then consider the functor $\mathrm{Fil}_{\mathcal{P},\mathrm{rk}} : D^+(X) \to DF^+(X)$ given by composition as

$$\mathrm{Fil}_{\mathcal{P},\mathrm{rk}} := R\epsilon^f_* \circ \mathrm{Fil}_{F_\bullet N(\mathcal{P})} \circ \epsilon^*,$$

where $\epsilon : N\mathcal{P} \to X$ is the augmentation. We have a canonical natural transformation

$$\mathrm{Id}_{D^+(X)} \to \mathrm{Forget} \circ \mathrm{Fil}_{\mathcal{P},\mathrm{rk}} \tag{4.3.0.1}$$

given as the composition of

$$\mathrm{Id}_{D^+(X)} \xrightarrow{u_{\mathcal{P}}} R\epsilon_* \circ \epsilon^* \cong R\epsilon_* \circ \mathrm{Forget} \circ \mathrm{Fil}_{F_\bullet N\mathcal{P}} \circ \epsilon^* \cong \mathrm{Forget} \circ R\epsilon^f_* \circ \mathrm{Fil}_{F_\bullet N\mathcal{P}} \circ \epsilon^*.$$

In particular, (4.3.0.1) restricts to an isomorphism of functors $D \to D^+(X)$ on any subcategory D on which $\mathcal{P}/X$ is of cohomological descent. We thus obtain the desired statement, up to the computation of the graded pieces (which no longer requires the finiteness hypothesis for the rank). This computation is given in Lemma 4.3.3 below. □

Before giving the general computation of graded pieces in Lemma 4.3.3 to complete the above proof, we consider a useful example that computes the graded pieces for a simplicial cocone.

Example 4.3.1. If $\epsilon : A \to X$, then $\epsilon_{\mathrm{Cocone}(A)} : \mathrm{Cocone}(A) \to X$ is contractible, and thus satisfies cohomological descent. If we take the filtration as in Example 2.3.3, then

$$R\epsilon_{\mathrm{Cocone}(A)_*} \mathrm{Gr}^p_{F_\bullet \mathrm{Cocone}(A)/X} K \cong \begin{cases} R\epsilon_{A*}\epsilon^*_A K & \text{if } p = -1 \\ \tilde{C}(A/X, K)[-1] & \text{if } p = 0. \end{cases} \tag{4.3.1.1}$$

Indeed, because $F_1\mathrm{Cocone}(A) = \mathrm{Cocone}(A)$, $F^1\epsilon^*_{\mathrm{Cocone}(A)} K = 0$, thus

$$F^0\epsilon^*_{\mathrm{Cocone}(A)} K = \mathrm{Gr}^0\,\epsilon^*_{\mathrm{Cocone}(A)} K.$$

In particular, we have an exact triangle

$$\mathrm{Gr}^0\,\epsilon^*_{\mathrm{Cocone}(A)} K \to F^{-1}\epsilon^*_{\mathrm{Cocone}(A)} K \to \mathrm{Gr}^{-1}\,\epsilon^*_{\mathrm{Cocone}(A)} K \to \cdots.$$

Since $F_{-1}\mathrm{Cocone}(A) = 0$, the middle term is $\epsilon^*_{\mathrm{Cocone}(A)} K$. Expanding the definition, we find the right term is $\epsilon^*_A K$, where we have identified $A = F_0\mathrm{Cocone}(A)$. Since $\mathrm{Cocone}(A)$ is contractible, applying $R\epsilon_{\mathrm{Cocone}(A)_*}$ gives an identification of the exact triangle

$$\begin{aligned} R\epsilon_{\mathrm{Cocone}(A)_*} \mathrm{Gr}^0\,\epsilon^*_{\mathrm{Cocone}(A)} K &\to R\epsilon_{\mathrm{Cocone}(A)_*} F^{-1}\epsilon^*_{\mathrm{Cocone}(A)} K \\ &\to R\epsilon_{\mathrm{Cocone}(A)_*} \mathrm{Gr}^{-1}\,\epsilon^*_{\mathrm{Cocone}(A)} K \to \cdots. \end{aligned}$$

with

$$R\epsilon_{\mathrm{Cocone}(A)_*}(\mathrm{Gr}^0\,\epsilon^*_{\mathrm{Cocone}(A)} K) \to K \to R\epsilon_{A*}\epsilon^*_A K \to \cdots$$

and (4.3.1.1) follows from the definition of $\tilde{C}(A/X, K)$ (Definition 2.2.1).

The argument for computing the graded pieces then is a reduction to this example via excision. Consider for each p the natural map (cf. Example 3.1.10(iv))

$$r_p : \mathrm{Cocone}(N(-\infty, \mathcal{P}_p)/\mathcal{P}_p) \cong N(-\infty, \mathcal{P}_p] \to N\mathcal{P}.$$

Lemma 4.3.2. *Let X be a topological space (respectively, scheme) and let $\mathcal{P}/X$ be a finitely ranked pospace (respectively, poscheme). For any $K \in D^+(N\mathcal{P})$, $\mathrm{Gr}^{p-1}_{\mathrm{Fil}_\bullet N\mathcal{P}} K = Rr_{p_*}\mathrm{Gr}^0\, r^*_p K$ where the graded on the right is for the cocone filtration as in Example 4.3.1.*

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Proof. The point is that $(r_p)_m$ maps $F_0\mathrm{Cocone}(N(-\infty,\mathcal{P}_p))$ into $F_{p-1}N\mathcal{P}$ and on m-simplices restricts to an isomorphism

$$\mathrm{Cocone}(N(-\infty,\mathcal{P}_p)/\mathcal{P}_p)_m \backslash F_0\mathrm{Cocone}(N(-\infty,\mathcal{P}_p)/\mathcal{P}_p)_m \xrightarrow{\sim} (F_pN\mathcal{P})_m\backslash(F_{p-1}N\mathcal{P})_m.$$

Suppose we represent K by a complex of injectives. Then, K_m is a complex of injectives for each m, and similarly for $F^{p-1}K_m$. Then $(r_p^*F^{p-1}K)_m = (F^0 r_p^*K)_m = (\mathrm{Gr}^0\, r_p^*K)_m$ is a complex of injectives. Thus, pushforward of this complex computes Rr_{p*}, but that is just identified with the pushpull from $F_pN\mathcal{P}$ and by Lemma 2.3.4 we conclude this is $\mathrm{Gr}^{p-1} K$. □

Lemma 4.3.3. *Let X be a topological space (respectively, scheme) and let $\mathcal{P}/X$ be a finitely ranked pospace (respectively, poscheme). For $K \in D^+(X)$, there is a functorial (in K) identification*

$$\mathrm{Gr}^p_{\mathcal{P},\mathrm{rk}} K = R\pi_*\tilde{C}(N(-\infty,\mathcal{P}_{p+1})/\mathcal{P}_{p+1},\pi^*K)[-1].$$

Proof. Combining Lemma 4.3.2 with Example 4.3.1, we obtain

$$\begin{aligned}\mathrm{Gr}^p_{\mathcal{P},\mathrm{rk}} K &= R\epsilon_*\,\mathrm{Gr}^p\,\epsilon^*K\\ &= R(\epsilon\circ r_{p+1})_*\,\mathrm{Gr}^0(\epsilon\circ r_{p+1})^*K\\ &= R(\pi\circ\epsilon_{\mathrm{Cocone}(N(-\infty,\mathcal{P}_{p+1}))})_*\,\mathrm{Gr}^0(\pi\circ\epsilon_{\mathrm{Cocone}(N(-\infty,\mathcal{P}_{p+1}))})^*K\\ &= R\pi_*\tilde{C}(N(-\infty,\mathcal{P}_{p+1})/\mathcal{P}_{p+1},\pi^*K)[-1]. \end{aligned}\tag{4.3.3.1}$$

□

4.4 Spectral sequences

Proof of Theorem 4.0.3. Write $\tilde{K} = \mathrm{Fil}_{\mathcal{P},\mathrm{rk}}(K|_Z)$. Then, since we assume $\mathcal{P}/Z$ is of cohomological descent for $K|_Z$, we have $\mathrm{Forget}(\tilde{K}) = K|_Z$. We then apply [Sta19, Tag 015W] to obtain the spectral sequence (i), whose terms are described by (4.3.3.1); it remains only to observe that π is proper by assumption so that when $\star =!$, $R(f\circ\pi)_! = Rf_!\circ R\pi_*$. For part (ii) we extend to a filtration of $j_!j^*K$ by

$$i_*i^*K[-1] \to j_!j^*K \to K \quad \text{for } j: U \hookrightarrow X \text{ and } i: Z\hookrightarrow X$$

before applying the spectral sequence (and make a similar observation when $\star =!$). □

4.4.1 *The first spectral sequence of a stratified space.* Recall the setup from § 1.2.7: $X = \bigcup_\alpha S_\alpha$ for disjoint locally closed sets S_α and $Z_\alpha := \overline{S_\alpha} = \bigcup_{\beta\le\alpha} S_\beta$. We consider the pospace or poscheme given by $\mathcal{P} = \bigsqcup_\alpha Z_\alpha \to X$ with $z_\alpha \ge z_\beta$ if $z_\alpha = z_\beta$ as elements of X and $\alpha\ge\beta$ and with a ranking given by a ranking on the indexing set.

We have, $\mathcal{P}_{p+1} = \bigsqcup_{\mathrm{rk}(\alpha)=p+1} Z_\alpha$, and we will also write $j: U_{p+1}\to\mathcal{P}_{p+1}$ and $i:\partial\mathcal{P}_{p+1}\to\mathcal{P}_{p+1}$ where

$$U_{p+1} := \bigsqcup_{\mathrm{rk}(\alpha)=p+1} S_\alpha, \quad \partial\mathcal{P}_{p+1} := \mathcal{P}_{p+1}\backslash U_{p+1} = \bigsqcup_{\mathrm{rk}(\alpha)=p+1} Z_\alpha\backslash S_\alpha.$$

We then claim that $\tilde{C}(N(-\infty,\mathcal{P}_{p+1})/\mathcal{P}_{p+1},K) = j_!j^*K[1]$. Indeed, $(-\infty,\mathcal{P}_{p+1})/\mathcal{P}_{p+1}$ is proper with image $\partial\mathcal{P}_{p+1}$, and the fiber over any geometric point in $\partial\mathcal{P}_{p+1}$ has a minimum. Thus, writing $\epsilon: N(-\infty,\mathcal{P}_{p+1})\to\mathcal{P}_{p+1}$, $R\epsilon_*\epsilon^*K = i_*i^*K$ and, thus, the cone is $j_!j^*K[1]$, as desired. In particular, if X is a topological space or a variety over an algebraically closed field, then taking f to be the structure map to a point and $\star =!$, we obtain the standard spectral sequence for a

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stratified space

$$E_1^{p,q} = \bigoplus_{\mathrm{rk}(\alpha)=p+1} H_{\mathrm{c}}^{p+q}(S_\alpha, K) \Rightarrow H_{\mathrm{c}}^{p+q}(X, K).$$

4.4.2 *The Banerjee spectral sequence of a symmetric semisimplicial filtration.* Here we recover the spectral sequence of Banerjee [Ban19, Theorem 1]. In that setting, we are given 'face maps' $f_i : M^p \times X_{n+e} \to M^{p-1} \times X_n$ for $0 \leq i < p$, fixed e and $n, p > 0$, satisfying certain conditions that define a *symmetric semisimplicial filtration of* $\{X_n\}$ *by powers of* M. Take

$$\mathcal{P} = \bigsqcup_{p>0} \mathbf{S}^p M \times X_{n-ep}$$

over $Z_n := f_0(M \times X_{n-p}) \subset X_n$, where $\mathbf{S}^p M = M^p/\mathfrak{S}_p$ and $\leq_{\mathcal{P}}$ is given by all possible compositions of the face maps (this is well-defined on $\mathbf{S}^\bullet M$ by the assumptions on the face maps [Ban19, Definition 2.10]). Given $x \in Z_n$, the 'equalizer' and 'embedding' assumptions imply that there is a maximal p and a unique $(S, x) \in \mathbf{S}^p(M) \times X_{n-ep}$ that maps to x. To us, this means that the pospace $\mathcal{P}$ admits a fiberwise maximum. Then Theorem C gives the desired

$$E_1^{p,q} = H_{\mathrm{c}}^q(M^p \times X_{n-ep}; \mathbb{Q}) \otimes_{\mathfrak{S}_p} \mathrm{sgn} \Rightarrow H_{\mathrm{c}}^{p+q}(X_n - Z_n; \mathbb{Q}).$$

The sign representation sgn appears here for the same reason as in Corollary 7.2.6.

5. Motivic inclusion–exclusion

In this section we prove Theorem D. To set the stage, in § 5.1 we first discuss an abelian decategorification of the results of the previous section for constructible sheaves and explain how this relates to the finer combinatorial decategorification in the Grothendieck ring of varieties given by Theorem D. In § 5.2 we prove some combinatorial contractibility criteria and deduce Theorem D.

5.1 Decategorifications

Suppose X is a noetherian scheme; fix $L/\mathbb{Q}_\ell$ an algebraic extension for ℓ invertible on X. Then we can consider the abelian category $\mathrm{Cons}(X, L)$ of constructible L-sheaves on (the pro-étale site of) X and its Grothendieck ring $K_0(\mathrm{Cons}(X, L))$. We consider the constructible derived category $D_{\mathrm{Cons}}(X, L)$, the subcategory of the bounded derived category $D^b(X, L)$ consisting of complexes with constructible cohomology sheaves. There is an Euler characteristic $\mathrm{Ob}(D_{\mathrm{Cons}}(X, L)) \to K_0(\mathrm{Cons}(X, L))$,

$$K \mapsto [K] = \sum_k (-1)^k [H^i(K)].$$

By [BS15, Remark 6.8.15], the constructible derived category is preserved by proper pushforward. Thus, we can decategorify the work of the previous section: for $\pi : \mathcal{P} \to X$ a proper finitely ranked poscheme of cohomological descent for K, $\mathrm{Fil}_{\mathcal{P},\mathrm{rk}}(K)$ induces

$$[K] = \sum_p [\mathrm{Gr}^p_{\mathcal{P},\mathrm{rk}}(K)] = -\sum_p [R\pi_* \tilde{C}(-\infty, \mathcal{P}_{p+1}, K)] \tag{5.1.0.1}$$

where the shift in (4.3.3.1) manifests as a minus sign. In fact, there is an inclusion–exclusion formula independent of any rank function: under the same hypotheses, we have

$$[K] = \sum_{k \geq 0} (-1)^k [R\epsilon^\circ_{k*} \epsilon^{\circ *}_k K)], \tag{5.1.0.2}$$

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where ϵ_k° denotes the restriction of ϵ_k to the non-degenerate k-simplices $N\mathcal{P}_k^\circ$ (i.e. the scheme of strict $(k+1)$ chains). Indeed, since $\mathcal{P}$ admits a rank function, Lemma 3.1.11 shows that $N\mathcal{P}$ is split, i.e. that each degeneracy map is an isomorphism onto connected components. Then, we apply the spectral sequence (2.2.0.1) and use that each column on the E_1 page is quasi-isomorphic to the normalized complex given by the kernel of the total degeneracy map (see § 2.1.6), which by the above consideration is exactly the restriction to the space of non-degenerate simplices. The choice of a rank function gives a way to break up each simplex space into connected components, thus breaks up each term of (5.1.0.2); rearranging and reassembling, one can recover (5.1.0.1); indeed, we saw exactly this rearrangement phenomenon already in our toy model for classical inclusion–exclusion in the introduction (§ 1.1).

It turns out we can also give a combinatorial decategorification of the inclusion–exclusion formula that lifts (5.1.0.1) and (5.1.0.2): we work in the modified Grothendieck ring of varieties $K_0(\mathrm{Var}/X)$; recall that in characteristic zero this is the standard Grothendieck ring defined by cut and paste relations, but in non-zero characteristic one must also mod out by radicial surjective maps more general than constructible decompositions (e.g. purely inseparable field extensions). We refer to [BH21] for a detailed discussion and other perspectives. Here let us just highlight that there is a natural compactly supported cohomology homomorphism

$$K_0(\mathrm{Var}/X) \to K_0(D_{\mathrm{Cons}}(X,L)), [f : Y \to X] \mapsto [Rf_! L],$$

so that it makes sense to ask for a combinatorial lift of (5.1.0.2) to $K_0(\mathrm{Var}/X)$. This lift is exactly what is provided by Theorem D, which says that if $\mathcal{P}/X$ is a bounded poscheme with weak geometric centers, then

$$[X/X] = \chi(N\mathcal{P}/X) = \sum_{k\geq 0}(-1)^k[N\mathcal{P}_k^\circ/X] \text{ in } K_0(\mathrm{Var}/X).$$

Remark 5.1.1. We discuss some interesting points of comparison between the abelian and combinatorial decategorifications.

(i) In $K_0(\mathrm{Var}/X)$ we only give a statement for $[X/X]$. However, the analogous statement for $[Y/X]$ or any other class is obtained simply by multiplying by $[Y/X]$. In fact, the statement in $K_0(\mathrm{Cons}(X,L))$ also reduces to just the statement for the constant sheaf L by the projection formula [BS15, Lemma 6.7.4].

(ii) We do not know whether Theorem D holds if we only require that each fiber is weakly contractible: we are only able to prove the identity in the Grothendieck ring by using an Euler characteristic analog of Lemma 3.2.1 which directly cancels out isomorphic components in different simplex spaces.

(iii) The statement of Theorem D does not require any properness hypothesis. Combined with the projection formula (or by directly running the same proof for sheaves), we obtain under the same hypotheses as Theorem D

$$[K] = \sum_k(-1)^k[R\epsilon_{k!}^\circ\epsilon_k^{\circ *}K] \text{ in } K_0(D_{\mathrm{Cons}}(X,L)).$$

5.2 Euler characteristic contractibility criterion and proof of Theorem D

To prove Theorem D, we will spread out our geometric weak centers to reduce to the following elementary Euler characteristic version of Lemma 3.2.4.

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Lemma 5.2.1. *Suppose X is a Noetherian scheme and $\mathcal{P}/X$ is a finite type bounded poscheme with a weak center/maximum/minimum. Then*

$$\chi(N\mathcal{P}/X) = [X/X] \text{ in } K_0(\mathrm{Var}/X).$$

Proof. Since we are working in the Grothendieck ring, we can apply Lemma 4.2.1 to assume there is a genuine center/maximum/minimum. Then, in the case of a maximum or minimum, by passing to a constructible decomposition we can assume it is isolated so that it is a center.

Thus, we may assume we have a center c. We write $A_k \subset N\mathcal{P}_k^\circ$ for the clopen subscheme of strict chains passing through c, and B_k for its complement, the strict chains that do not pass through c. Then for $k \geq 1$, $B_k \cong A_{k+1}$ by insertion of c in the unique possible spot. Thus, the sum

$$\sum_{k\geq 0}(-1)^k[N\mathcal{P}_k^\circ] = \sum_k (-1)^k([A_k] + [B_k])$$

telescopes and is equal to $A_0 = [X]$. □

Proof of Theorem D. By Noetherian induction it suffices to show that if X is irreducible and reduced, then the identity holds after restriction to a non-empty open $U \subset X$. We write η for the generic point of X and fix an algebraic closure $\overline{K(\eta)}$ of $K(\eta)$ and a weak center $c \in \mathcal{P}(\overline{K(\eta)})$. We first observe that c is defined over a finite purely inseparable extension $L/K(\eta)$: first, since $\mathcal{P}/X$ is of finite type, c can be defined over some finite normal extension $M/K(\eta)$. Writing $G = \mathrm{Aut}(M/K(\eta))$, we have $L = M^G/K(\eta)$ is purely inseparable, thus it suffices to show c is fixed by G. Thus, suppose $\sigma \in G$. Then, by the definition of a weak center, since c is a field-valued point, either $c \leq \sigma(c)$ or $c \geq \sigma(c)$; by replacing σ with σ^{-1}, we can assume $c \leq \sigma(c)$. Then, since σ preserves the order relation (because it is defined over $K(\eta)$), we find $\sigma(c) \leq \sigma^2(c)$, $\sigma^2(c) \leq \sigma^3(c)$, etc., so that for $k > 1$ a multiple of the order of G,

$$c \leq \sigma(c) \leq \cdots \leq \sigma^k(c) = c.$$

Thus, $c \leq \sigma(c) \leq c$ so $\sigma(c) = c$, as desired.

Now, we can spread out $c : \operatorname{Spec} L \to \operatorname{Spec} K(\eta)$ to a finite radicial surjective map $\tilde{U} \to U$ over a non-empty open $U \subset X$. By shrinking U further, we can assume this spreading out is a weak center of $\mathcal{P} \times_X \tilde{U}$. Lemma 5.2.1 then gives

$$\sum_{k\geq 0}(-1)^k[N(\mathcal{P}\times_X \tilde{U})_k^\circ/\tilde{U}] = [\tilde{U}/\tilde{U}] \text{ in } K_0(\mathrm{Var}/\tilde{U}).$$

By composing with the map $\tilde{U} \to U$, we find

$$\begin{aligned}\sum_{k\geq 0}(-1)^k[N\mathcal{P}_k^\circ|_U][\tilde{U}/U] &= \sum_{k\geq 0}(-1)^k[N\mathcal{P}_k^\circ \times_X \tilde{U}/U] \\ &= \sum_{k\geq 0}(-1)^k[N(\mathcal{P}\times_X \tilde{U})_k^\circ/U] = [\tilde{U}/U] \text{ in } K_0(\mathrm{Var}/U).\end{aligned}$$

Recalling that in the modified Grothendieck ring $[\tilde{U}/U] = [U/U] = 1$, we conclude. □

Because it is will be useful in later sections, we also give an analog of Lemma 3.2.1 using a similar argument. Both this and the previous criterion should be special cases of a more general poscheme Euler characteristic version of Quillen's Theorem A [Qui73].

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Theorem 5.2.2. *Let X be a Noetherian scheme, let $\mathcal{P}/X$ be a poscheme that is bounded and of finite type. Let $\mathcal{P}' \subset \mathcal{P}$ be a sub-poscheme and suppose that, for every geometric point $t : \operatorname{Spec}\kappa \to X$, one of the following holds:*

(i) $(-\infty, \mathcal{P}_t] \cap \mathcal{P}'_t \times \mathcal{P}_t/\mathcal{P}_t$ *has a weak maximum; or*
(ii) $[\mathcal{P}_t, \infty) \cap \mathcal{P}_t \times \mathcal{P}'_t/\mathcal{P}_t$ *has a weak minimum.*

Then

$$\chi(N\mathcal{P}/X) = \chi(N\mathcal{P}'/X) \text{ in } K_0(\mathrm{Var}/X). \tag{5.2.2.1}$$

Proof. By Noetherian induction, it suffices to assume X is irreducible and to show that there exists a non-empty open $U \subset X$ where (5.2.2.1) holds. We write η for the generic point and $\overline{\eta} : \operatorname{Spec}\overline{K(\eta)} \to X$ for a geometric point above η. We assume case (i) holds at $\overline{\eta}$, case (ii) being similar. We write

$$m_{\overline{\eta}} : \mathcal{P}_{\overline{\eta}} \to (-\infty, \mathcal{P}_{\overline{\eta}}] \cap (\mathcal{P}' \times_X \mathcal{P})_{\overline{\eta}}$$

for the weak maximum. First note that $m_{\overline{\eta}}$ can be defined over a finite subextension $M/K(\eta)$, which we may take to be normal. Then, if we let $G = \operatorname{Aut}(M/K(\eta))$, we must have that $m_{\overline{\eta}}$ is fixed by the action of G because it is a maximum and the order relation is defined over $K(\eta)$. It follows that $m_{\overline{\eta}}$ can be defined over $L = M^G$, a finite purely inseparable extension of $K(\eta)$: that is, writing $\eta_L : \operatorname{Spec} L \to X$, we obtain $m_{\overline{\eta}}$ as the base change to $\overline{K(\eta)}$ of

$$m_{\eta_L} : \mathcal{P}_{\eta_L} \to (-\infty, \mathcal{P}_{\eta_L}] \cap (\mathcal{P}' \times_X \mathcal{P})_{\eta_L}.$$

It is a weak maximum still because this can be checked on geometric points (for a section to be a weak maximum it is necessary and sufficient that it be a maximum on any set of geometric points). Now, we can spread out the purely inseparable map $\operatorname{Spec} L \to \operatorname{Spec} K(\eta)$ to a radicial surjective $\tilde{U} \to U$ where U is open in X such that m_{η_L} spreads out to

$$m_{\tilde{U}} : \mathcal{P}_{\tilde{U}} \to (-\infty, \mathcal{P}_{\tilde{U}}] \cap (\mathcal{P}' \times_X \mathcal{P})_{\tilde{U}}.$$

Now $(\infty, m_{\tilde{U}}]^c$ is open, so its image in $\tilde{U}$ is constructible by Chevalley's theorem [Sta19, Tag 054K]. Since this image does not contain η_L, we deduce that its complement, the locus where $m_{\tilde{U}}$ is a weak maximum, contains a non-empty open set. Thus, replacing $\tilde{U}$ with this non-empty open set, we can assume, furthermore, that $m_{\tilde{U}}$ is a weak maximum.

Now, for any geometric point $p_0 < p_1 < \cdots < p_k$ of $(N\mathcal{P}_{\tilde{U}})^\circ_k$, either the chain stays entirely in $\mathcal{P}'_{\tilde{U}}$ or first leaves at an index i. Of those that leave, we can break them up into the subset A_k such that for this first i, $p_{i-1} = m_{\tilde{U}}(p_i)$, and the subset B_k where this is not satisfied; this gives a constructible decomposition

$$(N\mathcal{P}_{\tilde{U}})^\circ_k = (N\mathcal{P}'_{\tilde{U}})^\circ_k \sqcup A_k \sqcup B_k. \tag{5.2.2.2}$$

Moreover, we claim that $[A_{k+1}/\tilde{U}] = [B_k/\tilde{U}]$. Indeed, we can decompose B_k as $\bigsqcup_{i=0}^k B_{k,i}$ where $B_{k,i}$ is the constructible set that first leaves at i. Then we have a map $B_{k,i} \to A_{k+1}$ such that

$$p_0 < \cdots < p_k \mapsto p_0 < \cdots < p_{i-1} < m_{\tilde{U}}(p_i) < p_i \cdots < p_k$$

and the induced map $\bigsqcup_{i=0}^k B_{k,i} \to A_k$ is a bijection on geometric points, so gives the desired equality in the Grothendieck ring. Because we have $A_0 = \emptyset$, the $A_\bullet$ and $B_\bullet$ terms cancel when we use (5.2.2.2) to compute the Euler characteristic, giving the result. □

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6. The configuration, effective zero-cycle, and Hilbert poschemes

Let $Z \to X$ be a map of schemes. We write $\mathbf{C}^k_X(Z)$ for the kth unordered configuration space of Z, relative to X,

$$\mathbf{C}^k_X(Z) = \Big(\underbrace{Z \times_X Z \times_X \cdots \times_X Z}_{k} \setminus \Delta\Big) \Big/ \mathfrak{S}_k, \tag{6.0.0.1}$$

where Δ is the big diagonal where two coordinates agree and $\mathfrak{S}_k$ is the symmetric group on k elements acting by permutation. We define the configuration poscheme of Z over X

$$\mathbf{C}^\bullet_X(Z) := \coprod_{k=1}^{\infty} \mathbf{C}^k_X(Z) \text{ and its augmented variant } \mathbf{C}^{+,\bullet}_X(Z) := \coprod_{k=0}^{\infty} \mathbf{C}^k_X(Z).$$

The order relation is by inclusion: to make this precise and verify that this indeed defines a poscheme one may, for example, identify $\mathbf{C}^k_X(Z)$ with the reduced locus in the relative Hilbert scheme of length k subschemes. The formation of the configuration poscheme commutes with arbitrary change of base.

When Z/X is finite étale surjective of degree d, then $\mathbf{C}^\bullet_X(Z)$ has a maximum $X \cong \mathbf{C}^d_X(Z)$ (whose fiber over any geometric point $\overline{x}$ is the configuration of all d-points in $Z_{\overline{x}}$). As a consequence, if $Z \to X$ is quasi-finite surjective and X is Noetherian, then the geometric fibers of $\mathbf{C}^\bullet_X(Z)$ have weak maxima, and thus Theorem D applies. This can be applied fruitfully to an arbitrary finite type Z/X by considering the truncations $\mathbf{C}^{\leq k}_X(Z)$ which satisfy the hypotheses of Theorem D after restriction to an open locus where $Z \to X$ is quasi-finite of degree $\leq k$. This yields an approximate motivic inclusion–exclusion formula, Theorem 6.4.1(i), which captures one of the main combinatorial methods used in the motivic stabilization arguments of [VW15, BH21].

When Z/X is projective, we will give a matching cohomological approximate inclusion–exclusion formula in Theorem 6.4.1(ii). To obtain it, we need to compactify $\mathbf{C}^\bullet_X(Z)$. There are (at least) two obvious candidates.

(i) The Chow poscheme of effective zero-cycles.
(ii) The Hilbert poscheme of finite length subschemes (or its good component).

Here by *the* Chow poscheme of effective zero-cycles, we actually mean the divided powers scheme $\Gamma^\bullet_X(Z)$, ordered by inclusion of zero-cycles. This scheme was introduced by Rydh [Ryd08], and provides a canonical scheme structure on the Chow variety with respect to any sufficiently ample projective embedding.

The divided power $\Gamma^k_X(Z)$ is closely related to the symmetric power

$$\mathbf{S}^k_X(Z) := \Big(\underbrace{Z \times_X Z \times_X \cdots \times_X Z}_{k}\Big) \Big/ \mathfrak{S}_k. \tag{6.0.0.2}$$

Indeed, there is a natural universal homeomorphism

$$\mathrm{SG} : \mathbf{S}^k_X(Z) \to \Gamma^k_X(Z)$$

induced by the addition of points $Z \times_X \ldots \times_X Z \to \Gamma^k_X(Z)$. The map SG is an isomorphism when Z/X is flat (in particular, when $X = \operatorname{Spec} \kappa$ for κ any field), or when X is of characteristic zero. It is the divided powers schemes, however, that provide a natural interpolation of symmetric powers from fields to arbitrary bases; in particular, the formation of $\Gamma^\bullet_X(Z)$ is stable under arbitrary base change (symmetric powers are not). This is more than just an aesthetic choice: we are actually not certain whether the monoid structure on the symmetric powers induces a

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poscheme structure in full generality (i.e. whether it is cancelative in a scheme-theoretic sense), whereas we can prove this for $\Gamma^{\bullet}_X(Z)$.

For the purposes of proving an approximate cohomological inclusion–exclusion formula, it is possible to work with either the poscheme of effective zero-cycles or the Hilbert poscheme. In either case, the cohomology of the graded pieces for the rank filtration will be identified with the compactly supported sign cohomology of configuration spaces, so that the specific choice of compactification is irrelevant; more precisely, there is a natural map $\mathrm{HG} : \mathbf{H}^{\bullet}_X(Z) \to \Gamma^{\bullet}_X(Z)$, and it induces an isomorphism of the rank spectral sequences for cohomology.

In the majority of this section, we will thus focus on the approach via the poscheme of effective zero-cycles because it best highlights the role of symmetric powers and the relation with the Kapranov zeta function. However, in § 6.6, we will briefly summarize the argument using punctual Hilbert schemes and also study a larger Vassiliev-style Hilbert poscheme that gives an exact cohomological inclusion–exclusion formula (at the price of introducing difficult-to-compute terms).

We now outline the contents of this section: in § 6.1 we study the configuration and symmetric posets of a finite set Z. These are simple and classical objects: the configuration poset of Z is the lattice of subsets of Z, and the symmetric poset is the lattice of multisets of Z. In the latter case, it is often useful to interpret the lattice of multisets as the free commutative monoid on Z, with the poset ordering induced by the monoid multiplication. We prove the (surely well-known) result that the nerve of the symmetric poset deformation retracts to the nerve of the configuration poset, and observe that, for $Z = [n] = \{0, \dots, n\}$, the nerve of the configuration poset is the barycentric subdivision of Δ^n. Our computations in the scheme-theoretic case are accomplished by reduction to these elementary results.

In § 6.2, we define the poscheme of effective zero-cycles using the divided powers scheme of [Ryd08]. The reader interested only in the characteristic zero (or topological) case may replace these with symmetric powers; the key point in any case is to show that the natural monoid structure induces a poscheme structure (see Proposition 6.2.2, Remark 6.2.3, and the paragraph following them).

In § 6.3 we show the cohomology of the graded pieces for the rank filtration on the poscheme of effective zero-cycles are naturally identified (up to a shift) with the extension by zero of the sign local system on configuration spaces.

In § 6.4 we prove approximate motivic and cohomological inclusion–exclusion, Theorem 6.4.1. In both the motivic and cohomological settings, the result can be thought of as describing how closely the k-truncated poscheme of relative effective zero-cycles approximates the image of the morphism.

In § 6.5 we study the skeletal spectral sequence for the poscheme of effective zero-cycles. In the case of rational coefficients, there is a natural quasi-isomorphism from the E_1-page to a complex consisting of the sign part of the cohomology of powers of the cartesian products; we first learned of this latter complex from Banerjee, who has studied it from a different perspective and has announced a spectral sequence with this complex on its E_1 page (that is surely closely related to the one studied here). The terms of this complex can be identified with the terms of the E_1-page for the rank spectral sequence, but the advantage of the skeletal sequence is that the differential is completely explicit. There seems to be an intimate relation between the skeletal filtration and the rank filtration: in particular, the skeletal spectral sequence is compatible with the rank filtration, and we can use it to also study also the differential on the E_1 page of the rank spectral sequence. In the rational case we find that it has the same kernel as the differential on the Banerjee complex, and it seems likely that the E_1 page for the rank spectral sequence is,

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in fact, quasi-isomorphic to the Banerjee complex, though we do not know how to prove this, or whether any deeper comparisons hold; see Remark 6.5.5.

6.1 Configuration and symmetric posets

For Z a non-empty set, the configuration poset $\mathbf{C}^\bullet(Z)$ is the lattice of finite non-empty subsets of Z. The symmetric semigroup $\mathbf{S}^\bullet(Z)$ is the semigroup of finite non-empty multisets of Z under disjoint union, i.e. the free commutative semigroup on Z. It is contained in the symmetric monoid $\mathbf{S}^{\bullet,+}(Z)$, the free commutative monoid on Z, where we allow also the empty set (which gives an identity element for the monoid operation). We may view $\mathbf{S}^{\bullet,+}(Z)$ (and, thus, also the subset $\mathbf{S}^\bullet(Z)$) as a poset, where $a \le c$ if and only if there is a (necessarily unique) b in $\mathbf{S}^{\bullet,+}(Z)$ such that $ab = c$. In other words, if we think of this poset as a category, then each morphism is labeled by an element of $\mathbf{S}^{\bullet,+}(Z)$ (i.e. the underlying graph is the directed Cayley graph of the monoid with the identity vertex removed). In particular, we can think of elements of the nerve as labeled by multisets in two different ways. The poset interpretation gives that a k-simplex in $N(\mathbf{S}^{\bullet,+}(Z))$ is a chain of multisets $I_0 \le \cdots \le I_k$, while the monoid interpretation gives that a k-simplex is an ordered list of finite multisets in Z, $(J_0, J_1, \ldots, J_k)$, with the bijection given by $I_s = J_0 + \cdots + J_s$, $J_s = I_s - I_{s-1}$ (here set $I_{-1} = \emptyset$), and where the subtraction exists by definition of the order relation and is unique because the monoid is cancelative. The simplicial subset $N\mathbf{S}^\bullet \subseteq N\mathbf{S}^{\bullet,+}$ consists of the simplices where $I_0 = J_0 \neq \emptyset$, and $N\mathbf{C}^{\bullet,+} \subset N\mathbf{S}^{\bullet,+}$ consists of the simplices where each I_s is a set (i.e. each element has multiplicity zero or one) or where the J_s are pairwise disjoint.

There is a natural rank function $\mathbf{S}^{\bullet,+}(Z) \to \mathbb{Z}_{\ge 0}$, given by cardinality, and the rank k component can be described as

$$\mathbf{S}^k(Z) = Z^k/\mathfrak{S}_k.$$

In the monoid interpretation of the nerve, we can write

$$N\mathbf{S}^{\bullet,+}(Z)_p = \bigsqcup_{\underline{k}=(k_0,\ldots,k_p)\in\mathbb{Z}^p_{\ge 0}} \mathbf{S}^{\underline{k}}(Z),$$

where $\mathbf{S}^{\underline{k}}(Z) = \prod_{i=0}^p \mathbf{S}^{k_i}(Z)$. The nerve of $\mathbf{S}^\bullet(Z)$ consists of those simplices where $k_0 \neq 0$. In this description, the face map

$$\delta_i : N\mathbf{S}^{\bullet,+}(Z)_p \to N\mathbf{S}^{\bullet,+}(Z)_{p-1}$$

is described as follows: for $0 \le i < p$, it merges the ith and $(i+1)$th set, while for $i = p$ it forgets the pth set. The nerve of the configuration poset $\mathbf{C}^\bullet(Z)$ consists of the simplex spaces

$$\mathbf{C}^{\underline{k}}(Z) = \big(Z^{\sum \underline{k}} \backslash \Delta\big)/\mathfrak{S}_{\underline{k}},$$

where $\sum \underline{k} = k_0 + \cdots + k_p$ and $\mathfrak{S}_{\underline{k}} = \prod_{i=0}^p \mathfrak{S}_{k_i}$ denotes the subgroup of $\mathfrak{S}_{\sum \underline{k}}$ preserving subsequent blocks of size k_i.

Example 6.1.1. For $[n] = \{0, \ldots, n\}$, $N\mathbf{C}^\bullet([n])$ can be identified with the barycentric subdivision of the standard n-simplex Δ^n. Indeed, the vertices correspond to non-empty subsets of $[n]$, i.e. to collections of vertices of Δ^n, and a k-simplex corresponds to a chain of such under inclusion. If we consider the sub-poset $\mathbf{C}^{\le n}([n])$ of configurations of rank at most n, obtained by removing the maximum from $C^\bullet([n])$, then the subcomplex $N\mathbf{C}^{\le n}([n])$ is the barycentric subdivision of $\partial\Delta^n$.

An element of $\mathbf{S}^\bullet(Z)$ can be written as a formal sum $\sum_{z\in Z} n_z z$ where $n_z \in \mathbb{Z}_{\ge 0}$, $n_z = 0$ for all but finitely many $z \in Z$ and $n_z > 0$ for at least one z; the rank is given by $\sum_{z\in Z} n_z$.

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There is a natural support map

$$\text{Support} : \mathbf{S}^\bullet(Z) \to \mathbf{C}^\bullet(Z), \sum n_x z \mapsto \{z \in Z | n_z > 0\}.$$

We equip both $N\mathbf{S}^\bullet(Z)$ and $N\mathbf{C}^\bullet(Z)$ with the rank filtration. We then obtain immediately the following result.

LEMMA 6.1.2. *Let Z be a set. The map* Support : $\mathbf{S}^\bullet(Z) \to \mathbf{C}^\bullet(Z)$ *is a falling retraction of ranked posets (see Remark 3.2.3). Thus, for any sub-poset $\mathcal{P}$ of $\mathbf{S}^\bullet(Z)$ containing $\mathbf{C}^\bullet(Z)$, $N\mathbf{C}^\bullet(Z)$ is a filtered deformation retract of $N\mathcal{P}$. In particular, if Z is a finite set, then $N\mathcal{P}$ is contractible (since $N\mathbf{C}^\bullet(Z)$ is by Example 6.1.1).*

6.2 The poscheme of effective zero-cycles

In the following we assume that Z/X satisfies the condition (AF) that any finite set in a single fiber is contained in a quasi-affine open (see [Ryd08, Paper III, Appendix A.1]); this holds, in particular, if Z/X is quasi-projective. The symmetric powers $\mathbf{S}^k_X(Z)$ defined by the quotient (6.0.0.2) then exist as schemes, and we consider the symmetric power monoid

$$\mathbf{S}^{\bullet,+}_X(Z) = \bigsqcup_{k=0}^{\infty} \mathbf{S}^k_X(Z),$$

where the monoid multiplication

$$\mathbf{S}^{k_1}_X(Z) \times \mathbf{S}^{k_2}_X(Z) \to \mathbf{S}^{k_1+k_2}_X(Z)$$

is induced by the quotient property from the natural map

$$Z^{\times_X k_1} \times_X Z^{\times_X k_2} \to Z^{\times_X k_1+k_2} \to \mathbf{S}^{k_1+k_2}_X(Z)$$

and the identification

$$\left(Z^{\times_X k_1} \times_X Z^{\times_X k_2}\right)/(\mathfrak{S}_{k_1} \times \mathfrak{S}_{k_2}) = \mathbf{S}^{k_1}_X(Z) \times \mathbf{S}^{k_2}_X(Z).$$

In characteristic zero, this provides a good notion of moduli of effective zero-cycles, compatible with arbitrary base change, and the monoid map can be used to define a poscheme structure compatible with the obvious poset structure on geometric points (see below). In positive and mixed characteristic there are some well-known perversities of symmetric powers (see, e.g., [Lun08]) and, in particular, it is not clear that the monoid structure defines a poscheme structure for a general Z/X.

We can address this by using divided powers schemes: in [Ryd08, Paper III], there is defined, for any scheme Z/X and $r \geq 0$, a divided powers schemes $\Gamma^r(Z/X)$, which we write here as $\Gamma^r_X(Z)$. Because we have assumed that Z/X satisfies the condition (AF), each $\Gamma^r_X(Z)$ is represented by a scheme. In [Ryd08, Paper III] it is shown that:

(i) the formation of $\Gamma^r_X(Z)$ is stable under arbitrary change of base $X' \to X$;
(ii) if $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} B$, then $\Gamma^r_X(Z) = \operatorname{Spec} \Gamma^r_A(B)$, where $\Gamma^r_A(B)$ is the rth divided power of B as an A-module;
(iii) there is a commutative monoid structure on

$$\Gamma^{\bullet,+}_X(Z) = \bigsqcup_{k=0}^{\infty} \Gamma^k_X(Z)$$

compatible with the degree, i.e. restricting to maps

$$\Gamma^r_X(Z) \times_X \Gamma^s_X(Z) \to \Gamma^{r+s}_X(Z);$$

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(iv) there is a canonical universal homeomorphism that induces isomorphisms on residue fields

$$\mathrm{SG} : \mathbf{S}^{\bullet;+}_X(Z) \to \Gamma^{\bullet;+}_X(Z);$$

the map SG is compatible with the monoid structures, and if Z/X is flat (e.g. if $X = \operatorname{Spec}\kappa$ for κ a field) or if $X/\mathbb{Q}$, then it is an isomorphism;

(v) there is a dense open non-degenerate locus $\Gamma^r_X(Z)^{\mathrm{nd}}$ such that SG restricts to an isomorphism $\mathbf{C}^r_X(Z) \to \Gamma^r_X(Z)^{\mathrm{nd}}$;

(vi) if Z/X is projective, then, for any sufficiently high power $\mathcal{L}^n$ of a relatively ample bundle $\mathcal{L}$, $\Gamma^r_X(Z)$ is naturally identified with the Chow scheme of effective zero-cycles of degree r on Z for $\mathcal{L}^n$ (or, more accurately, its reduced subscheme is identified with the Chow variety, and this identification equips the latter with a natural scheme structure).

By part (iv), $\Gamma^\bullet_X(Z)$ agrees with $\mathbf{S}^\bullet_X(Z)$ for X of characteristic zero or when X is a geometric point, but $\Gamma^\bullet_X(Z)$ is better behaved for X of positive or mixed characteristic. Over a geometric point, we will sometimes write the more familiar $\mathbf{S}^\bullet$ instead of $\Gamma^\bullet$, with the implicit understanding that these are canonically isomorphic in this case.

Proposition 6.2.2 below says that the monoid scheme $\Gamma^{\bullet;+}_X(Z)$ is cancelative in a scheme-theoretic sense. As a consequence, we will see that the monoid multiplication induces a natural poscheme structure. In the proof, we use some standard properties of divided powers modules recalled in the following lemma.

Lemma 6.2.1. *Let A be a commutative ring.*

(i) *If $M \twoheadrightarrow N$ is a surjection of A-modules, then $\Gamma^r_A(M) \to \Gamma^r_A(N)$ is a surjection.*

(ii) *If M is a flat A module, then the natural map $\Gamma^r_A(M) \to (M^{\otimes_A r})^{\mathfrak{S}_r}$ is an isomorphism.*

Proof. Statement (i) is explained, e.g., in [Ryd08, Paper I, (1.2.9)]. Statement (ii) is explained, e.g., in [Ryd08, Paper I, (1.2.13), second paragraph] □

Proposition 6.2.2. *For any map of schemes $Z \to X$ satisfying (AF), the map*

$$\Gamma^r_X(Z) \times \Gamma^s_X(Z) \to \Gamma^r_X(Z) \times \Gamma^{r+s}_X(Z)$$
$$(x, y) \mapsto (x, x + y)$$

is a closed immersion.

Proof. We may assume $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} B$. Let m^* be the map of rings $\Gamma^{r+s}_A(B) \to \Gamma^r_A(B) \otimes \Gamma^s_A(B)$ inducing the addition of cycles map $(x, y) \mapsto (x + y)$. We must show the map

$$\Gamma^r_A(B) \otimes \Gamma^{r+s}_A(B) \to \Gamma^r_A(B) \otimes \Gamma^s_A(B)$$
$$f \otimes 1 \mapsto f \otimes 1$$
$$1 \otimes g \mapsto m^*(g)$$

is surjective. This diagram is functorial in the A-algebra B, so if we choose a surjection from a free A-algebra $F \twoheadrightarrow B$, we obtain a commutative diagram

$$\begin{array}{ccc} \Gamma^r_A(B) \otimes \Gamma^{r+s}_A(B) & \longrightarrow & \Gamma^r_A(B) \otimes \Gamma^s_A(B) \\ \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ \Gamma^r_A(F) \otimes \Gamma^s_A(F) & \longrightarrow & \Gamma^r_A(F) \otimes \Gamma^s_A(F) \end{array}$$

where the two vertical arrows are surjections by Lemma 6.2.1(i). To verify the top horizontal arrow is surjective, it thus suffices to show the lower horizontal arrow is surjective.

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Now, since F is free as an A-module, the divided powers are identified with the symmetric tensors by Lemma 6.2.1(ii), and under this identification the map m^* is restriction

$$(F^{\otimes r+s})^{\mathfrak{S}_{r+s}} \hookrightarrow (F^{\otimes r+s})^{\mathfrak{S}_r \times \mathfrak{S}_s} = (F^{\otimes r})^{\mathfrak{S}_r} \otimes (F^{\otimes s})^{\mathfrak{S}_s}.$$

We thus need to show $(F^{\otimes r})^{\mathfrak{S}_r} \otimes (F^{\otimes s})^{\mathfrak{S}_s}$ is generated as an A-algebra by

$$(F^{\otimes r})^{\mathfrak{S}_r} \otimes 1 \text{ and } (F^{\otimes r+s})^{\mathfrak{S}_{r+s}}.$$

Clearly, it suffices to show the subalgebra T generated by these contains

$$1 \otimes (F^{\otimes s})^{\mathfrak{S}_s}.$$

To that end, we consider the following construction: fix a basis $\mathcal{B}$ of the free A-module F such that $1 \in \mathcal{B}$. Given $n \geq 0$ and a multiset S of elements in $\mathcal{B}$ with $|S| \leq n$, we define $t_n(S) \in (F^{\otimes n})^{\mathfrak{S}_n}$ by choosing any ordering $S = \{f_1\} + \cdots + \{f_k\}$, $f_i \in \mathcal{B}$, then summing up all elements in the $\mathfrak{S}_n$-orbit of $f_1 \otimes \cdots \otimes f_k \otimes 1 \otimes \cdots \otimes 1$ to obtain $t_n(S)$. As we vary over all multisets S with $|S| = n$, the elements $t_n(S)$ span $(F^{\otimes n})^{\mathfrak{S}_n}$.

With this notation in place, we see that it suffices to show that T contains $1 \otimes t_s(S)$ for any multiset S with $|S| \leq s$. We argue this by induction on $|S| \leq s$: the base case $|S| = 0$ is $1 \otimes \cdots \otimes 1 \in T$. Suppose it holds for $j < k$ and let S be a multiset with $|S| = k$. We have $t_{r+s}(S) \in T$, but on the other hand we also have

$$t_{r+s}(S) = 1 \otimes t_s(S) + \sum_{\emptyset \neq S' \leq S} t_r(S') \otimes t_s(S - S').$$

By the inductive hypothesis it is clear that all terms in the sum on the right are also contained in T, so we conclude. □

Remark 6.2.3. If we use symmetric powers instead of divided powers, then arguing with geometric points we easily deduce that the corresponding map

$$\mathbf{S}^r_X(Z) \times \mathbf{S}^s_X(Z) \to \mathbf{S}^r_X(Z) \times \mathbf{S}^{r+s}_X(Z)$$
$$(x, y) \mapsto (x, x + y)$$

is finite and a universal homeomorphism onto its image, but we do not know if it is a closed immersion outside of the cases covered by the proposition (i.e. $X/\mathbb{Q}$ or Z/X flat, when the divided powers and symmetric powers agree). In the proof above we have crucially used the surjectivity of Lemma 6.2.1(i) to reduce to the case of a free module, but this surjectivity does not hold, in general, for symmetric powers [Lun08].

In the setting of Proposition 6.2.2, we find that $\Gamma^{\bullet;+}_X(Z)$ is a ranked poscheme with $\leq_{\Gamma^{\bullet;+}_X(Z)}$ the closed subscheme defined by the closed immersion

$$\Gamma^{\bullet;+}_X(Z) \times \Gamma^{\bullet;+}_X(Z) \to \Gamma^{\bullet;+}_X(Z) \times \Gamma^{\bullet;+}_X(Z)$$
$$(x, y) \mapsto (x, x + y).$$

Indeed, the only non-trivial poscheme axiom to verify is the transitivity axiom (on T-valued points for an arbitrary test scheme T) that $a \leq b$ and $b \leq c$ implies $a \leq c$, and this is shown by representing $b = a + a'$ and $c = b + b'$ to obtain $c = a + (a' + b')$.

6.2.4 *Notation for symmetric powers, divided powers, and configurations.* It will be helpful to introduce some notation for labeled symmetric powers, divided powers, and configuration spaces, where the labelings are prescribed by a multiset. Explicitly, a finite multiset on a set S

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can be identified with a function $\underline{a} : S \to \mathbb{Z}_{\geq 0}$ with finite support, i.e. $\underline{a} \in \mathbb{Z}_{\geq 0}$ such that $\underline{a}(s) = 0$ for all but finitely many $s \in S$. Given such a multiset $\underline{a}$, we write

$$\Gamma_X^{\underline{a}}(Z) := \prod_{s\in S} \Gamma_X^{\underline{a}(s)}(Z) \quad \text{and} \quad \mathbf{S}_X^{\underline{a}}(Z) = \prod_{s\in S} \mathbf{S}_X^{\underline{a}(s)}(Z).$$

We can also make the canonical identification

$$\mathbf{S}_X^{\underline{a}}(Z) = \left(\prod_{s\in S} Z^{\times_X \underline{a}(s)}\right) \Big/ \prod_{s\in S} \mathfrak{S}_{\underline{a}(s)}.$$

There is a natural universal homeomorphism

$$\mathrm{SG} : \mathbf{S}_X^{\underline{a}}(Z) \to \Gamma_X^{\underline{a}}(Z)$$

and inside the former there is an open configuration locus $\mathbf{C}_X^{\underline{a}}(Z)$ where *all* points are distinct, not just those in each group (i.e. formed by taking the quotient after removing the big diagonal). The homeomorphism SG restricts to an isomorphism of this locus with its open image, so that we may also consider $\mathbf{C}_X^{\underline{a}}(Z)$ as an open subscheme of $\Gamma_X^{\underline{a}}(Z)$.

We note, in particular, that if we write $\Gamma_X^p(Z)$, this is the pth relative divided powers scheme as above, but if we write $\Gamma_X^{[p]}(Z)$, then this means to interpret $[p] = \{0, 1, \ldots, p\}$ as a finite (multi)set so that by the above

$$\Gamma_X^{[p]}(Z) = \Gamma_X^1(Z) \times \cdots \times \Gamma_X^1(Z) = Z \times_X Z \times_X \cdots \times_X Z = Z^{\times_X |[p]|},$$

where there are $p + 1 = |[p]|$ terms in the fiber product.

Example 6.2.5. As in the case of finite sets, it is useful to observe that we obtain a monoidal description of $N\Gamma_X^{\bullet;+}(Z)$. In this description, the p-simplices are

$$\coprod_{\underline{a}\in\mathbb{Z}_{\geq 0}^{[p]}} \Gamma_X^{\underline{a}}(Z), \tag{6.2.5.1}$$

and the map to the poscheme nerve is given by

$$(c_0, \ldots, c_p) \mapsto (c_0, c_0 + c_1, \ldots, c_0 + c_1 + \cdots + c_p).$$

The face and degeneracy maps are described as in the case of finite sets: in particular, the face map δ_i for $0 \leq i < k$ is given by summing the ith and $(i+1)$th coordinates, so sends the term corresponding $\underline{a} = (a_0, \ldots, a_p)$ to the term $\underline{a} = (a_0, \ldots, a_i + a_{i+1}, a_{i+2}, \ldots, a_p)$ and δ_k forgets the last coordinate, so sends the term corresponding to $\underline{a} = (a_0, \ldots, a_p)$ to $\underline{a} = (a_0, \ldots, a_{p-1})$.

Inside of (6.2.5.1), we can identify the p-simplices of $N\mathbf{C}_X^{\bullet;+}(Z)$ as $\coprod_{\underline{a}\in\mathbb{Z}_{\geq 0}^{[p]}} \mathbf{C}_X^{\underline{a}}(Z)$. In these interpretations $N\Gamma_X^{\bullet}(Z)$ and $N\mathbf{C}_X^{\bullet}(Z)$ correspond to $\underline{a} \neq 0$, the rank is given by $\sum \underline{a}$, and the non-degenerate simplices are those corresponding to $\underline{a} \in \mathbb{Z}_{>0}^{[p]}$ as well as $\underline{a} = 0$ when $p = 0$.

6.3 Graded pieces for the rank filtration

Suppose Z/X is projective. In this section, we compute the cohomology of the graded pieces for the rank filtration on $N\Gamma_X^{\bullet}(Z)$.

Lemma 6.3.1. *Suppose κ is algebraically closed and $Z/\operatorname{Spec}\kappa$ is finite and reduced. If $\mathcal{P}$ is a subposcheme of $\mathbf{S}_{\operatorname{Spec}\kappa}^{\bullet}(Z)$ containing $\mathbf{C}_{\operatorname{Spec}\kappa}^{\bullet}(Z)$, then $N\mathcal{P}$ is contractible.*

Proof. The category of finite reduced schemes over $\operatorname{Spec}\kappa$ is equivalent to the category of finite sets, and this equivalence is compatible with the formation of configuration spaces and symmetric powers. The result is then immediate from Lemma 6.1.2. □

PROPOSITION 6.3.2. *Let X be a locally Noetherian scheme, let Z/X be projective, and let $A = \mathcal{O}_L$ or L for L an algebraic extension of $\mathbb{Q}_\ell$, ℓ invertible on X. Then, for $K \in D_{\mathrm{Cons}}(X, A)$,*

$$\begin{aligned}\tilde{C}(N(-\infty, \Gamma_X^k(Z))/\Gamma_X^k(Z), K) &= j_! \tilde{C}(N(-\infty, \mathbf{C}_X^k(Z))/\mathbf{C}_X^k(Z), K)\\ &= j_!(\underline{\mathrm{sgn}}[2-k] \otimes K),\end{aligned}$$

where $j : \mathbf{C}_X^\bullet(Z) \to \Gamma_X^\bullet(Z)$ and $\underline{\mathrm{sgn}}$ denotes the sign local system.

Proof. By the projection formula, it will suffice to treat $K = A$. The poscheme $(-\infty, \Gamma_X^k(Z))/\Gamma_X^k(Z)$ is proper, so, for $t : \mathrm{Spec}\, \kappa \to \Gamma_X^k(Z)$ a geometric point,

$$\tilde{C}(N(-\infty, \Gamma_X^k(Z))/\Gamma_X^k(Z), A)_t = \tilde{C}(N(-\infty, t)/\mathrm{Spec}\, \kappa, A).$$

We first show that this is 0 if t does not factor through $\mathbf{C}_X^k(Z)$: in this case, the geometric support of t is a closed reduced finite subscheme of degree strictly less than k, $F \subset Z_t$. There is a natural closed immersion of poschemes

$$\mathbf{S}^\bullet_{\mathrm{Spec}\, \kappa}(F) = \Gamma^\bullet_{\mathrm{Spec}\, \kappa}(F) \hookrightarrow \Gamma_X^\bullet(Z)$$

and the map t factors through $t' : \mathrm{Spec}\, \kappa \to \mathbf{S}^\bullet_{\mathrm{Spec}\, \kappa} F$ and induces an isomorphism of poschemes $(-\infty, t') \to (-\infty, t)$. By assumption, $C^\bullet_{\mathrm{Spec}\, \kappa}(F) \subset (-\infty, t')$, so Lemma 6.3.1 shows $N(-\infty, t')$ is contractible and Lemma 4.1.2(ii) concludes.

It remains to establish the identity over $\mathbf{C}_X^k(Z)$. In this case, if we pullback the entire situation to $\mathbf{C}_X^{(1,\ldots,1)}(Z)$, then we are considering the reduced cohomology complex of the constant poscheme $(-\infty, \{0, \ldots, k-1\}) \subseteq \mathbf{C}^\bullet(\{0, \ldots, k-1\})$ over $\mathbf{C}_X^{(1,\ldots,1)}(Z)$. By Example 6.1.1, the nerve of this poset is identified with the barycentric subdivision of $\partial \Delta^{k-1}$. The reduced cohomology complex is naturally isomorphic to A supported in degree $k-2$, and the action of $\mathfrak{S}_{[k]}$ by reordering the vertices of the $(k-1)$-simplex yields the sign representation on this copy of A, thus we conclude. □

We also have a version in the Grothendieck ring.

PROPOSITION 6.3.3. *Let X be a Noetherian scheme and let Z/X be quasi-projective. Then, for $\tilde{\chi}$ as defined in* (1.3.0.1),

$$\tilde{\chi}(N(-\infty, \Gamma_X^k(Z))) = \tilde{\chi}(N(-\infty, \mathbf{C}_X^k(Z))) \in K_0(\mathrm{Var}/\Gamma_X^k(Z))$$

Proof. Argue as in the previous proof to invoke Theorem 5.2.2 over the complement of the configuration locus. □

6.4 Approximate-inclusion exclusion

Let X be a Noetherian scheme and let $f : Z \to X$ be a quasi-projective variety over X. We write $\dim_{/X} Z$ for the maximum over the dimensions of all irreducible components of geometric fibers of f. We write $X_{\geq k}$ for the closure of the image of $\mathbf{C}_X^k(Z)$ in X, i.e. the closure of the locus of geometric points $\overline{x}$ of X such that $Z_{\overline{x}}$ contains at least k geometric points. If f is proper, then note that $X_{\geq 1} = f(Z)$. We define X_I for I an interval in the obvious way, e.g., $X_{[1,k]} = X_{\geq 1} \backslash X_{\geq k+1}$ and $X_{>k} = X_{\geq k+1}$. We write $X_\infty = \bigcap_k X_{\geq k}$.

The main idea is that over $X_{[1,k]}$, the k-truncated poscheme of effective zero-cycles satisfies cohomological descent, so that under further constraints on the dimensions of the fibers its nerve provides a good approximation of $f(Z)$. Here a good approximation means matching 'up to high codimension', and this is made precise in compactly support cohomology by obtaining isomorphisms in sufficiently high degrees and in $K_0(\mathrm{Var}/\kappa)$ by matching classes modulo a suitable index of the the dimension filtration on the Grothendieck ring $K_0(\mathrm{Var}/\kappa)$. For the latter, recall that $\mathrm{Fil}^{-n} K_0(\mathrm{Var}/\kappa)$ is the sub-$\mathbb{Z}$-module of $K_0(\mathrm{Var}/\kappa)$ spanned by classes $[X]$ where $\dim X \leq n$.

Theorem 6.4.1. *Let κ be an algebraically closed field, let X/κ be a variety, and let $f : Z \to X$ be a surjective map of varieties. Suppose $k > 0$ is such that*

$$\dim(X_{>k}) \geq \dim X_\infty + k \dim_{/X} Z. \tag{6.4.1.1}$$

(i) Motivic approximate inclusion–exclusion:

$$\begin{aligned}[X] &\equiv \chi(N\mathbf{C}_X^{\leq k}(Z)) \\ &\equiv \sum_{p\geq 0}(-1)^p \sum_{\underline{a}\in\mathbb{Z}_{>0}^{[p]}, \sum \underline{a}\leq k} [\mathbf{C}_X^{\underline{a}}(Z)] \\ &\quad \mod \mathrm{Fil}^{-\dim X_{>k}} K_0(\mathrm{Var}/\kappa).\end{aligned}$$

(ii) Cohomological approximate inclusion–exclusion:
Suppose, furthermore, that f is projective, and let $\mathcal{F} \in \mathrm{Cons}(X, A)$ for $A = L$ or $\mathcal{O}_L$, L an algebraic extension of $\mathbb{Q}_\ell$ with ℓ invertible in κ. The adjunction unit $\mathcal{F} \to R\epsilon_*\epsilon^*\mathcal{F}$ for the augmentation $\epsilon : N\Gamma_X^{\leq k}(Z) \to X$ induces isomorphisms for all $q \geq k + 2\dim X_{>k} + 1$

$$H^q_c(X, \mathcal{F}) \xrightarrow{\sim} H^q_c(X, R\epsilon_*\epsilon^*\mathcal{F}),$$

where we note that if X/κ is proper, then the right-hand side is equal to

$$H^q(N\Gamma_X^{\leq k}(Z), \epsilon^*\mathcal{F}).$$

Remark 6.4.2. At the price of complicating the proof by working with a finer stratification, the inequality (6.4.1.1) can be replaced with

$$\dim X_{>k} \geq \max_{x\in X}(k \dim Z_x + \dim \overline{\{x\}}).$$

Remark 6.4.3. Cohomological approximate inclusion–exclusion as in Theorem 6.4.1(ii) can be completed to a cohomological inclusion–exclusion *formula* by combining with the rank spectral sequence for $\Gamma_X^{\leq k}(Z)$,

$$E_1^{p,q} = \begin{cases} H_c^{q-p+1}(\mathbf{C}_X^{p+1}(Z), \underline{\mathrm{sgn}} \otimes \mathcal{F}) & 0 \leq p \leq k-1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow H_c^{p+q}(X, R\epsilon_*\epsilon^*\mathcal{F}),$$

where the computation of the terms $E_1^{p,q}$ follows from Proposition 6.3.2. This rank spectral sequence will also be used fiberwise over X in the proof of Theorem 6.4.1.

In the case of rational coefficients, it is more useful to use the closely related skeletal spectral sequence in place of the rank spectral sequence; see § 6.5.

Proof of Theorem 6.4.1. Motivic case. We will use motivic inclusion–exclusion. To do so, we consider the k-truncated configuration poscheme $\mathbf{C}_X^{\leq k}(Z)$. For any geometric point $\overline{x}$ in $X_{[1,k]}$, $\mathbf{C}_{X_{[1,k]}}^{\leq k}(Z)_{\overline{x}} = \mathbf{C}^{\leq k}(Z_{\underline{x}})$ has a weak maximum because, by definition of $X_{[1,k]}$, $Z_{\underline{x}}^{\mathrm{red}}$ is finite reduced of degree $\leq k$. By Theorem D, we then find

$$[X] = [X_{[1,k]}] + [X_{>k}] = \chi(N\mathbf{C}_{X_{[1,k]}}^{\leq k}(Z)) + [X_{>k}].$$

Clearly we have $[X] \equiv [X] + [X_{>k}] \mod \mathrm{Fil}^{-\dim_{/S} X_{>k}}$, so that it remains to compute the terms appearing in the Euler characteristic. If we compute the latter using Example 6.2.5, then from

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the identity

$$[\mathbf{C}^{\underline{a}}_X(Z)] = [\mathbf{C}^{\underline{a}}_{X_{[1,k]}}(Z)] + [\mathbf{C}^{\underline{a}}_{X_{>k}}(Z)],$$

we find that it suffices to show for any $\underline{a} \in \mathbb{Z}^{[p]}_{>0}$ with $\sum \underline{a} \leq k$, that

$$\dim_{/S} \mathbf{C}^{\underline{a}}_{X_{>k}}(Z) \leq \dim_{/S} X_{>k}.$$

To see this, clearly it suffices to treat the case of the unordered configuration space $\mathbf{C}^j$ for $j \leq k$. Then

$$\dim_{/S} \mathbf{C}^j_{X_{>k}}(Z) \leq \max(\dim \mathbf{C}^j_{X_{(k,\infty)}}(Z), \dim \mathbf{C}^j_{X_\infty}(Z)).$$

Since Z is quasifinite over $X_{(k,\infty)}$, the first term in the maximum is $\leq \dim X_{>k}$, while the second term is bounded by $j \dim_{/X} Z + \dim X_\infty$, so that the hypothesis (6.4.1.1) yields the result.

Cohomological case. Let K denote the cone of $\mathcal{F} \to R\epsilon_*\epsilon^*\mathcal{F}$ so that there is an exact triangle $\mathcal{F} \to R\epsilon_*\epsilon^*\mathcal{F} \to K$. By the corresponding long exact sequence, it suffices to show $H^q_c(X, K)$ vanishes in degree $q \geq k + 2\dim_{X_{>k}}$.

We first note that the hypotheses imply that $\mathcal{P}$ satisfies cohomological descent over $X_{[1,k]}$, so K is supported in the closed set $X_{>k}$. Indeed: for $\overline{x}$ a geometric point of $X_{[1,k]}$, $\mathbf{S}^{\leq k}(Z_{\overline{x}}) = \Gamma^{\leq k}_X(Z)_{\overline{x}}$ is universally homeomorphic to $\mathbf{S}^{\leq k}((Z_{\overline{x}})^{\mathrm{red}})$. The latter is contractible by Lemma 6.3.1, so that we obtain cohomological descent by Lemma 4.1.2.

We now decompose $X_{>k}$ into the open $X_{(k,\infty)}$ and closed X_∞, so that we have a long exact sequence

$$\cdots \to H^q_c(X_{(k,\infty)}, K) \to H^q_c(X_{>k}, K) = H^q_c(X, K) \to H^q_c(X_\infty, K) \to \cdots$$

where the equality in the middle term comes from the support condition on K that we have obtained above. Thus, it suffices to prove the vanishing on $X_{(k,\infty)}$ and X_∞ separately.

Over $X_{(k,\infty)}$, f is finite. As a consequence, we claim $K|_{X_{(k,\infty)}}$ is supported in degrees $\leq k-1$: it suffices to check this for $R\epsilon_*\epsilon^*\mathcal{F}$ in place of K, and, by proper base change it suffices to check at a geometric point of $X_{(k,\infty)}$. Applying the rank spectral sequence as in Remark 6.4.3 to such a fiber, we find the E_1 terms are zero for $q > 0$ and for $p > k-1$, thus we find that $R\epsilon_*\epsilon^*\mathcal{F}$ is supported in degrees $\leq k-1$, verifying the claim. Then, as $K|_{X_{(k,\infty)}}$ is supported in degrees $\leq k-1$, its compactly supported cohomology vanishes above degree $k - 1 + 2\dim X_{(k,\infty)}$, so

$$H^q_c(X_{(k,\infty)}, K) = 0 \text{ for } q \geq k + 2\dim X_{>k} > k - 1 + 2\dim X_{(k,\infty)}.$$

Similarly, over X_∞, if we apply the rank spectral sequence to geometric fibers, we find that K is supported in degrees $\leq 2k\dim_{/X}(Z) + k - 1$. Thus, $H^q_c(X_{>k}, K) = 0$ for

$$q \geq 2\dim X_{>k} + k > 2\dim X_\infty + 2k\dim_{/X}(Z) + k - 1,$$

where here we have used (6.4.1.1). □

Remark 6.4.4. The instance of Theorem D used in the proof of Theorem 6.4.1(i) can be replaced with the earlier version of [BH21, Theorem 7.2.4] that is proved using motivic Euler products (by a method analogous to the proof of the classical inclusion–exclusion formula described after (1.1.0.2)).

6.5 The skeletal spectral sequence

We now study the skeletal spectral sequence for the effective zero-cycles poscheme. We assume Z/X is projective and X/S is proper. Below we will consider cohomology relatively over S; to that end, we write ϵ for the augmentation $N\Gamma^\bullet_X(Z) \to S$.

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6.5.1 *The skeletal spectral sequence.* Suppose $K \in D^+(N\Gamma^\bullet_X(Z))$ (see § 2.2). Using the monoidal description of the nerve (see Example 6.2.5), the skeletal sequence is

$$E_1^{s,t}(K) = \bigoplus_{\underline{a}\in\mathbb{Z}^{[s]}_{>0}} R^t(\Gamma^{\underline{a}}_X(Z) \to S)_* K|_{\Gamma^{\underline{a}}_X(Z)} \Rightarrow R^{s+t}\epsilon_* K.$$

Here the differential from $E^{s-1,t}$ to $E^{s,t}$ is given by $\sum_{i=0}^s (-1)^i \delta_i^*$ where δ_i is as described in Example 6.2.5.

There is a natural rank filtration on the complex $E_1(K)$, defined by taking $\mathrm{Fil}^p(E_1(K))$ to be the subcomplex consisting of summands with $\sum \underline{a} > p$. This filtration is induced by the rank filtration on K: the natural map of spectral sequences $E(\mathrm{Fil}^p(K)) \to E(K)$ is an injection on the first page and identifies $E(\mathrm{Fil}^p(K))_1$ with $\mathrm{Fil}^p(E(K))_1$. More generally, we obtain a canonical identification $E(\mathrm{Gr}^{[a,b]} K)_1 = \mathrm{Gr}^{[a,b]} E(K)_1$, where $\mathrm{Gr}^{[a,b]} = \mathrm{Fil}^a / \mathrm{Fil}^{b+1}$.

In all of these, we may pass to the quasi-isomorphic complex $E(\bullet)_1^\circ$ of cochains in the kernel of the total degeneracy map (see § 2.1.6). Term by term, this is given by the summands corresponding to $\underline{a} \in \mathbb{Z}^{[s]}_{>0}$ (this follows from the description of the degenerate simplices in Example 6.2.5).

6.5.2 *The Banerjee complex.* We now consider $K \in D^+(X)$, and consider the complex $E(K)_1^\circ := E(K|_{N\Gamma^\bullet_X(Z)})_1^\circ$. We will compare $E(K)_1^\circ$, together with its filtration, to a simpler complex, which we define now: we first consider the non-degenerate complex whose degree p term is

$$R^\bullet(Z^{\times_X [p]} \to S)_* K|_{Z^{\times_X [p]}},$$

where the differential from degree $p-1$ to p is $\sum_{i=0}^p (-1)^i \alpha_i^*$, with α_i the map forgetting the ith coordinate. This is the non-degenerate subcomplex of the E_1 page of a spectral sequence for the cohomology over S of the pullback of K to the simplicial scheme $[p] \mapsto Z^{\times_X [p]}$, but we will not use this here. As we learned from Banerjee, the isotypic components for the sign character sgn form a subcomplex

$$B(K)^p = R^\bullet(Z^{\times_X [p]} \to S)_* K|_{Z^{\times_X [p]}}[\mathrm{sgn}].$$

That this is a subcomplex follows from the following elementary observation.

Lemma 6.5.3. *For $\sigma \in \mathfrak{S}_{[p]}$ with $\sigma(i) = j$ and α_i, α_j as above,*

$$\alpha_j \circ \sigma = (p\ \ p-1\ \ \cdots\ \ j) \circ \sigma \circ (i\ \ i+1\ \ \cdots\ \ p) \circ \alpha_i,$$

where the two other permutations on the right are given in cycle notation.

Proof. Left to the reader. □

Indeed, given this lemma, we find that for c in degree p,

$$\begin{aligned}
\sigma \cdot d(c) &= (\sigma^{-1})^* \sum_{j=0}^p (-1)^j \alpha_j^* c \\
&= \sum_{j=0}^p (-1)^j \alpha_{\sigma(j)}^* \big((p\, p-1\ \cdots\ j) \circ \sigma^{-1} \circ (\sigma(j)\ \sigma(j)+1\ \ldots\ p)\big)^*(c) \\
&= \sum_{j=0}^p (-1)^j \alpha_{\sigma(j)}^* \big((p\ \cdots\ \sigma(j))\sigma(j\ \cdots\ p)\big) \cdot c
\end{aligned}$$

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$$= \sum_{j=0}^{p} (-1)^j \alpha^*_{\sigma(j)} \mathrm{sgn}(\sigma)(-1)^{\sigma(j)-p}(-1)^{p-j} c$$

$$= \mathrm{sgn}(\sigma) \sum_{j=0}^{p} (-1)^{\sigma(j)} \alpha^*_{\sigma(j)} c$$

$$= \mathrm{sgn}(\sigma) d(c).$$

We define an antisymmetrization map

$$\mathrm{asym} : E(K)^\circ_1 \to B(K)$$

summand by summand as follows:

(i) for c in the degree p summand corresponding to $\underline{a} = [p] = (1, \ldots, 1)$, for which $\Gamma^{[p]}_X(Z) = Z^{\times_X [p]}$, we define

$$\mathrm{asym}(c) = \sum_{\sigma \in \mathfrak{S}_{[p]}} \mathrm{sgn}(\sigma)\sigma \cdot c;$$

(ii) for c in any other summand, $\mathrm{asym}(c) = 0$.

Theorem 6.5.4. *As in our running assumption for this section, suppose X/S is proper and Z/X is projective. For any $K \in D^+(X)$ (see § 2.2), the map*

$$\mathrm{asym} : E(K)^\circ_1 \to B(K)$$

is a map of filtered complexes. If K is a complex of L-modules, then it is a filtered quasi-isomorphism.

Proof. It is clear from the definition that the map preserves the filtration, but we must verify that it is a map of complexes, i.e. that $d \circ \mathrm{asym} = \mathrm{asym} \circ d$. We check this summand by summand.

Suppose c is a local section of the degree $p-1$ summand corresponding to $\underline{a}$. We consider three different cases:

(i) $\underline{a} = [p]$, i.e. $\underline{a}(i) = 1$ for all $i \in [p-1]$;
(ii) $\underline{a}(i) = 2$ for exactly one $i \in [p-1]$ and is 1 for all other i;
(iii) the remaining $\underline{a}$.

In the third case, the $[p]$ component of $\delta^*_i c$ is zero for all $0 \leq i \leq p$, so that $\mathrm{asym}(d(c)) = 0 = d(0) = d(\mathrm{asym}(c))$, as desired.

In the second case, let $0 \leq i \leq p-1$ be the unique index such that $\underline{a}(i) = 2$. Then $\delta^*_j c$ has non-trivial $[p]$-component if and only if $i = j$. However, $\delta^*_j c$ is invariant under action of the transposition $(i\ i+1)$, thus its antisymmetrization is zero, so $\mathrm{asym}(d(c)) = 0 = d(0) = d(\mathrm{asym}(c))$.

In the first case, $\delta^*_i c$ is non-zero only for $i = p$, in which case it is concentrated in the $[p]$-component. Thus, $d(c) = (-1)^p \delta^*_p c = (-1)^p \alpha^*_p c$. Using Lemma 6.5.3, its image under antisymmetrization is

$$(-1)^p \sum_{\sigma \in \mathfrak{S}_{[p]}} \mathrm{sgn}(\sigma)\sigma \cdot \alpha^*_p c = (-1)^p \sum_{\sigma \in \mathfrak{S}_{[p]}} \mathrm{sgn}(\sigma)(\alpha_p \circ \sigma^{-1})^* c$$

$$= (-1)^p \sum_{\sigma \in \mathfrak{S}_{[p]}} \mathrm{sgn}(\sigma)\big(\sigma^{-1}(\sigma(p)\ \sigma(p)+1\ \cdots\ p) \circ \alpha_{\sigma(p)}\big)^* c$$

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$$= (-1)^p \sum_{\sigma \in \mathfrak{S}_{[p]}} \operatorname{sgn}(\sigma)\alpha^*_{\sigma(p)}(p\ \cdots\ \sigma(p))\sigma \cdot c$$
$$= \sum_{\sigma \in \mathfrak{S}_{[p]}} (-1)^{\sigma(p)} \alpha^*_{\sigma(p)}\big(\operatorname{sgn}((p\ \cdots\ \sigma(p))\sigma)(p\ \cdots\ \sigma(p))\sigma \cdot c\big).$$

If we group terms according to the coset for $\mathfrak{S}_{[p-1]}$, i.e. according to $\sigma(p)$, then we obtain the differential in $B(K)$ applied to the antisymmetrization of c, as desired.

It remains to see that when K is a complex of L-modules, then this is a filtered quasi-isomorphism. We then consider the map

$$\operatorname{Gr}^p(E(K)^\circ_1) \to \operatorname{Gr}^p(B(K)).$$

The complex on the right is concentrated in degree p where it equals

$$R^\bullet(Z^{\times_X [p]} \to S)_* K[\operatorname{sgn}].$$

Multiplication by $1/(p+1)!$ and inclusion into the $[p]$-component of $\operatorname{Gr}^p(E(K)^\circ_1)$ give a section. This section is a quasi-isomorphism: one can check that

$$\operatorname{Gr}^p(E(K)^\circ_1) \cong \big(\tilde{C} \otimes_E R^\bullet(Z^{\times_X [p]} \to S)_* K\big)^{\mathfrak{S}_{[p]}},$$

where $\tilde{C}$ is the non-degenerate relative cohomology complex for $N\mathbf{C}^{\leq p}([p]) \subset N\mathbf{C}^\bullet([p])$. By Example 6.1.1, this inclusion is naturally identified with the barycentric subdivision of the inclusion of $\partial\Delta^p$ in Δ^p, thus $\tilde{C}$ is quasi-isomorphic as a complex of $L[\mathfrak{S}_{[p]}]$-modules to $\underline{\operatorname{sgn}}[-p]$, and we conclude. □

Suppose now that $K \in D_{\mathrm{Cons}}(X, L)$. Then Theorem 6.5.4 gives an alternate computation of $R\epsilon_* \operatorname{Gr}^p \epsilon^* K$ (computed earlier by Proposition 6.3.2). Indeed, since the map asym is a filtered quasi-isomorphism, we have

$$E_1(\operatorname{Gr}^p \epsilon^* K) = \operatorname{Gr}^p E_1(K) \simeq \operatorname{Gr}^p B(K) = R^\bullet(Z^{\times_X [p]} \to S)_* K[\operatorname{sgn}],$$

where the right hand-side is supported in the column $s = p$. Thus the sequence degenerates at E_1 to give isomorphisms

$$R\epsilon^{i+p}_*(\operatorname{Gr}^p \epsilon^* K) \cong R^i(Z^{\times_X [p]} \to S)_* K[\operatorname{sgn}].$$

As we learned from Banerjee, there is a simple direct argument that shows these match with the expression in terms of $\underline{\operatorname{sgn}}$ cohomology of relative configuration spaces as computed by Proposition 6.3.2. This will be explained and compared with Grothendieck ring computations of Vakil and Wood in § 7.2.

Remark 6.5.5. Using the skeletal spectral sequence and Theorem 6.5.4, we can also compute information about the differential of the rank spectral sequence. Indeed, this differential is induced by the connecting homomorphism in cohomology from the short exact sequence

$$0 \to \operatorname{Gr}^p \epsilon^* K \to \operatorname{Gr}^{[p-1,p]} \epsilon^* K \to \operatorname{Gr}^{p-1} \epsilon^* K \to 0. \tag{6.5.5.1}$$

In particular, the kernel of the connecting homomorphism in $R^i\epsilon_* \operatorname{Gr}^{p-1} \epsilon^* K$ is equal to the image of $R^i\epsilon_* \operatorname{Gr}^{[p-1,p]} \epsilon^* K$ under the natural map. Using the filtered quasi-isomorphism of Theorem 6.5.4 and compatibility of the rank filtration with the E_1 page of the skeletal sequence, we can deduce that the skeletal sequence for $\operatorname{Gr}^{[p-1,p]} \epsilon^* K$ degenerates on E_2 (because it is supported in the columns $s = p$ and $s = p - 1$), and then compute this connecting map on the E_2 page. We deduce that the kernel of the differential on the E_1 page of the rank spectral sequence is the same as the kernel of the differential in the Banerjee complex (assuming the

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coefficients are rational). As indicated in the introduction to this section, it would be interesting to understand if there is a deeper connection between the skeletal and rank spectral sequences.

6.6 Hilbert schemes and Vassiliev sequences

As indicated in the introduction to this section, analogs of Proposition 6.3.2 and Theorem 6.4.1 also exist using the Hilbert poscheme of points

$$\mathbf{H}_X^{\bullet}(Z) := \bigsqcup_{k=1}^{\infty} \mathbf{H}_X^k(Z).$$

Here the poscheme structure is induced by inclusion of closed subschemes (note there is no monoid structure here!), and there is a natural open immersion $\mathbf{C}_X^{\bullet}(Z) \subseteq \mathbf{H}_X^{\bullet}(Z)$ induced by the universal configuration over $\mathbf{C}_X^{\bullet}(Z)$. The key step in carrying out the argument in this setting is to prove the correct analog of the contractibility Lemma 6.3.1; in this case the right statement is that, for $Z/\operatorname{Spec}\kappa$ finite (but not necessarily reduced), geometric support induces a deformation retraction from $\mathbf{H}_X^{\bullet}(Z)^{\mathrm{red}}$ to $\mathbf{C}_X^{\bullet}(Z)^{\mathrm{red}} = \mathbf{C}_X^{\bullet}(Z^{\mathrm{red}})$. There is a natural Hilbert–Chow morphism $\mathrm{HG} : \mathbf{H}_X^{\bullet}(Z) \to \Gamma_X^{\bullet}(Z)$ that restricts to an isomorphism on $\mathbf{C}_X^{\bullet}(Z)$ and induces the identity map on the E_1 page of the corresponding rank spectral sequences, so that in a strong sense one does not obtain anything new from this approach.

If we instead take a larger subset of the relative Hilbert poscheme allowing fibers of arbitrary dimension, then we obtain an exact inclusion–exclusion sequence and formula. This is closely related to certain Vassiliev spectral sequences, as we explain briefly now. These results will not be used in the rest of the paper.

Let X be a locally Noetherian scheme and let Z/X be quasi-projective. Then we can form the relative Hilbert scheme, $\mathrm{Hilb}_X(Z)$: recall [Gro62, FGI+05] that this is the moduli scheme over X of proper flat families of closed subschemes of Z; and it is equipped with a natural poscheme structure by the inclusion relation. By definition, its formation is compatible with base change.

If we fix a relatively ample $\mathcal{L}/Y$, then for any numerical polynomial f we obtain a clopen subscheme $\mathrm{Hilb}_{Y/X}^{\mathcal{L},f}$ parameterizing families with Hilbert polynomial f. Given any set of numerical polynomials S, we write $\mathrm{Hilb}_{Y/X}^{\mathcal{L},S}$ for the union of these clopen subschemes, whose formation is again compatible with base change.

Proposition 6.6.1. *Suppose X is a locally Noetherian scheme, Z/X is projective, and S is a finite set of numerical polynomials. Then $\mathrm{Hilb}_{Z/X}^{\mathcal{L},S}$ is a projective poscheme over X admitting a rank function. Moreover, if X is Noetherian, then there is a finite set of numerical polynomials that appear as the Hilbert polynomial of a geometric fiber $Z_{\overline{x}}$, and if S contains this finite set, then $\mathrm{Hilb}_{Z/X}^{\mathcal{L},S}$ admits maxima over geometric points in the image of Z.*

Proof. Projectivity is a standard result for Hilbert schemes (see, e.g., [FGI+05, Theorem 5.14 of Part II on p.127]), and it admits a rank because it admits a strictly increasing map to the finite poset S (we say $f \leq g$ if $f(n) \leq g(n)$ for $n \gg 0$), and any finite poset admits a rank function.

We now restrict to the Noetherian case to show the last statement of the proposition. By Noetherian induction and generic flatness, there is a finite set of polynomials that appear as the Hilbert polynomial of $(Z_x, \mathcal{L}_y)$ for x a geometric point of X. The maximum over a geometric point x is the point of the Hilbert scheme corresponding to the fiber Z_x (by the same argument one obtains a maximum over any $T \to X$ such that Z_T/T is flat). □

As a consequence, Theorem C and the choice of a rank function give an inclusion–exclusion spectral sequence for the cohomology of the closed subscheme $f(Z)$ as well as the compactly

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supported cohomology of its complement, and Theorem D allows us to compute the class $[f(Z)]$ (and, thus, trivially also the class of its complement $[X - f(Z)] = [X] - [f(Z)]$). For a suitable choice of S, the Hilbert poscheme $\mathrm{Hilb}^{\mathcal{L},S}_{Z/X}$ includes a truncated punctual Hilbert poscheme $\mathbf{H}^{\leq k}_X(Z)$, so that, for a suitable choice of rank function, the resulting spectral sequence will admit a natural map to the approximate inclusion–exclusion sequence studied above that is a quotient map on E_1; the other terms thus give precise control over the error in approximate inclusion–exclusion.

Example 6.6.2. Suppose we have a family of varieties $f : V \to X$ (e.g. the universal degree d hypersurface in $\mathbb{P}^n$ as in § 8), and $Z \subseteq V$ is the relative singular locus. Then we can write the discriminant locus D (consisting of the points $x \in X$ such that the fiber V_x is singular) as $D = f(Z)$. The spectral sequence from the Hilbert poscheme of $f|_Z$ is a Vassiliev-type spectral sequence for the compactly supported cohomology of D, and motivic inclusion–exclusion for $f|_Z$ gives a combinatorial analog in the Grothendieck ring.

6.6.3 *Geometrically reduced variant.*

Lemma 6.6.4. *Suppose X is a Noetherian scheme and Z/X is projective. There is a finite set of numerical polynomials S_{gr} that can occur as the Hilbert polynomial of $Z^{\mathrm{red}}_{\overline{x}}$ for $\overline{x}$ a geometric point.*

Proof. We argue by Noetherian induction: assume X is irreducible and let $\overline{\eta}$ be a geometric point lying above the generic point η of X. Suppose $Z_{\overline{\eta}}$ is reduced; then, by [Sta19, Tag 0C0E], the geometric fibers are reduced in an open locus, and, by generic flatness, if we restrict to a potentially smaller open then they all have the same Hilbert polynomial. Otherwise, by [Sta19, Tag 04KT], we can find a finite purely inseparable extension $k(\eta)'/k(\eta)$ such that $(Z_{k(\eta)'})^{\mathrm{red}}$ is geometrically reduced (in other words, if the geometric fiber $Z_{\overline{\eta}}$ is not reduced, we can see its nilpotents already after base change to some finite purely inseparable subextension). We may then spread out $k(\eta)'/k(\eta)$ to a dominant map $X' \to X$ for X' irreducible of the same dimension such that $((Z_{X'})^{\mathrm{red}})_{\overline{\eta}}$ is reduced, then we conclude as above. □

Thus, we obtain a variant by taking $S \supseteq S_{\mathrm{gr}}$ and then considering the open (by [Sta19, Tag 0C0E]) geometrically reduced locus $\mathrm{Hilb}^{\mathrm{gr},\mathcal{L},S}_{X/Y} \subset \mathrm{Hilb}^{\mathcal{L},S}_X(Z)$. This is a (no longer proper) poscheme over X with geometric maxima over points in the image of f: the maximum at a geometric point $\overline{x}$ is given by the point corresponding to $Z^{\mathrm{red}}_{\overline{x}}$. The corresponding motivic inclusion–exclusion formula is more useful than the non-reduced variant and mirrors cohomological calculations as in [Vas99, Das21]. For example, in the setting of Example 6.6.2, this allows us to consider the geometric support of the singular locus rather than its scheme structure.

Remark 6.6.5. Since this poscheme is not proper, we do not obtain a cohomological spectral sequence. In the special case of discriminant loci for polynomials over $\mathbb{C}$ of [Vas99], however, one obtains a spectral sequence in cohomology by equipping the geometric realization of the $\mathbb{C}$-points of $\mathrm{Hilb}^{\mathrm{gr},\mathcal{L},S_{\mathrm{gr}}}_X(Z)$ with a different topology, obtained as a quotient topology from the geometric realization of its closure by contracting non-reduced schemes at the boundary to their geometric support. We do not see a way to understand such a construction using purely simplicial methods without passing to a geometric realization, so we cannot apply it in the general scheme-theoretic case. However, in the part of the Hilbert poscheme corresponding to finite subschemes, this reduction is equivalent to our earlier computation that the terms on the E_1 page only depend on the configuration spaces.

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7. Incidence algebras and Möbius inversion

In this section we construct motivic and sheaf-theoretic incidence algebras of poschemes and study Möbius inversion therein. The most important case for us, treated in § 7.2 below, is the reduced incidence algebra of the poscheme of effective zero-cycles, where the Möbius element gives a lift of the inverse Kapranov zeta function; from this we will recover Vakil and Wood's inversion formula for the Kapranov zeta function and also motivate a definition of cohomological special values of the Kapranov zeta function (see § 7.3). Kobin [Kob20] has also recently raised the question of finding an incidence algebra interpretation of the Kapranov zeta function.

Because reduced incidence algebras are a slightly *ad hoc* construction (they require a notion of when two intervals in a poset are the same), in § 7.1 we first give the construction of the non-reduced incidence algebra for a general poscheme. In both cases the discussion for poschemes is preceded by some recollections on the classical constructions for posets.

7.1 Incidence algebras

We briefly recall incidence algebras of posets and then give a categorification.

7.1.1 *The incidence algebra of a locally finite poset.* A poset $\mathcal{P}$ is locally finite if for any $a, b \in \mathcal{P}$, the interval $[a, b]$ is a finite poset. In this case, one defines the incidence algebra $I(\mathcal{P})$ as the space of functions (with values in a commutative ring) on 1-simplices in $N(\mathcal{P})$ (i.e. length 2 chains $a \leq b$) with the convolution product

$$f \star g(a \leq b) = \sum_{a \leq x \leq b} f(a \leq x) g(x \leq b).$$

The multiplicative identity is the function $f(a \leq b) = 1$ if $a = b$, 0 if $a \neq b$. The zeta function $\zeta_{\mathcal{P}} \in I(P)$ is the constant function $\zeta_{\mathcal{P}}(a \leq b) = 1$, and the Möbius function $\mu_{\mathcal{P}} \in I(\mathcal{P})$ is the inverse of ζ_P for the convolution product; this is one formulation of Möbius inversion. We recall the topological expression for $\mu_{\mathcal{P}}$.

Lemma 7.1.2. *For $\tilde{\chi} = \chi - 1$ the reduced Euler characteristic,*

$$\mu_{\mathcal{P}}(a \leq b) = \begin{cases} \tilde{\chi}(N(a,b)) & \text{if } a < b \\ 1 & \text{if } a = b, \end{cases}$$

where (a, b) denotes the open interval of $p \in \mathcal{P}$ such that $a < p < b$.

Proof. Taking μ to be the element defined by the equation in the statement, we have to show that $\mu \star \zeta_{\mathcal{P}}(a \leq b)$ is 1 if $a = b$ and 0 otherwise. When $a = b$ this is trivial. For $a < b$, this 0 will be interpreted as 1 minus the Euler characteristic of the contractible poset $(a, b]$. Indeed, we have

$$\mu \star \zeta_{\mathcal{P}}(a \leq b) = 1 + \sum_{a < x \leq b} \tilde{\chi}(N(a, x))$$

and we can group simplices contributing to the sum as follows: a non-degenerate simplex of $N(a, x)$ can be identified with a non-degenerate simplex of one degree higher in $N(a, b]$ by adding x to the end. This establishes a bijection between the non-degenerate k-simplices of $N(a, x)$ and the non-degenerate $k + 1$ simplices ending at x of $N(a, b]$. Thus, we miss only the 0-simplices in $N(a, b]$, however, the reduced Euler characteristics appearing in the sum, which subtract off 1

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for each $a < x < b$, accounts for these, so that we have established

$$\sum_{a<x\leq b} \tilde{\chi}(N(a,x)) = -\chi(N(a,b]) = -1,$$

(where the minus sign comes in because our bijection went from k-simplices to $k+1$-simplices). Adding 1 gives zero, as desired. □

7.1.3 *Categorification.* Let X be a Noetherian scheme and let L be an algebraic extension of $\mathbb{Q}_\ell$ with ℓ invertible in X. Let $\mathcal{P}/X$ be of finite type over X, so that $\leq_\mathcal{P}$ is also a Noetherian scheme. We consider the sheaf theoretic and motivic incidence algebras $I_{\mathrm{sh}}(\mathcal{P})$ and $I_{\mathrm{mot}}(\mathcal{P})$: as abelian groups, we have

$$I_{\mathrm{sh}}(\mathcal{P}) = K_0(\mathrm{Cons}(\leq_\mathcal{P}, L)) \quad \text{and} \quad I_{\mathrm{mot}}(\mathcal{P}) = K_0(\mathrm{Var}/\leq_\mathcal{P}).$$

The multiplication, however, is given by the convolution product: on $I_{\mathrm{sh}}(\mathcal{P})$, this is induced by the convolution functor

$$D_{\mathrm{Cons}}(\leq_\mathcal{P}, L) \times D_{\mathrm{Cons}}(\leq_\mathcal{P}, L) \xrightarrow{\star} D_{\mathrm{Cons}}(\leq_\mathcal{P}, L)$$
$$(K_1, K_2) \mapsto K_1 \star K_2 := R\delta^1_!\left(\delta^{2^*} K_1 \otimes_L \delta^{0^*} K_2\right),$$

where δ^i are the standard face maps $N(\mathcal{P})_2 \to N(\mathcal{P})_1 = \leq_\mathcal{P}$. On $I_{\mathrm{mot}}(\mathcal{P})$ it is induced by

$$\mathrm{Var}/\leq_\mathcal{P} \times \mathrm{Var}/\leq_\mathcal{P} \to \mathrm{Var}/\leq_\mathcal{P}$$
$$(Y_1/\leq_\mathcal{P}) \times (Y_2/\leq_\mathcal{P}) \mapsto Y_1 \star Y_2 = \delta^{2^*} Y_1 \times_{N(\mathcal{P})_2} \delta^{0^*} Y_2$$
$$\text{with structure map to } \leq_\mathcal{P} \text{ induced by } N(\mathcal{P})_2 \xrightarrow{\delta^1} \leq_\mathcal{P}.$$

By standard arguments $I_{\mathrm{sh}}(\mathcal{P})$ (respectively, $I_{\mathrm{mot}}(\mathcal{P})$) is an algebra over $K_0(\mathrm{Cons}(X,L))$ (respectively, $K_0(\mathrm{Var}/X)$) and the compactly supported cohomology map is a map of $K_0(\mathrm{Var}/X)$-algebras $I_{\mathrm{mot}}(\mathcal{P}) \to I_{\mathrm{sh}}(\mathcal{P})$ (where the latter is a $K_0(\mathrm{Var}/X)$-algebra through the map $K_0(\mathrm{Var}/X) \to K_0(\mathrm{Cons}(X,L))$).

Remark 7.1.4. When describing the incidence algebra for posets above we allowed locally finite posets instead of finite posets; this is quite useful in practice (e.g. for realizing Hasse–Weil zeta functions). In the geometric setting, we can generalize similarly: we say a poscheme $\mathcal{P}/X$ is locally finite if it is locally of finite type as a map of schemes and, for any finite type open subscheme $U/X \subset \mathcal{P}/X$,

$$[U,U] = U \times_X \mathcal{P} \times_X U \cap N(\mathcal{P})_2$$

is of finite type over X. In this case, for each such U one can define the convolution products over $U \times U \cap \leq_\mathcal{P}$, then take the inverse limit of Grothendieck rings over all such U. We treat just the simpler finite-type case here, as it illustrates the main points without the technical burden of making the previous sentence precise. In § 7.2, however, we will study a version of this more general construction for the reduced incidence algebra of the poscheme of effective zero-cycles (which is only locally finite), using the monoid structure to make it completely explicit.

The identity for the convolution product in the incidence algebra $I_\mathcal{M}(\mathcal{P})$ is $[\Delta_\mathcal{P}/\leq_\mathcal{P}]$. We define $\zeta_\mathcal{P} = [\leq_\mathcal{P}/\leq_\mathcal{P}]$. We then have the following Euler characteristic formula for Möbius inversion formula generalizing Lemma 7.1.2:

Theorem 7.1.5. *Let X be a Noetherian scheme and let $\mathcal{P}/X$ be a poscheme of finite type. Let $(\mathcal{P},\mathcal{P})$ be the open interval poscheme, viewed as a poscheme over $<_\mathcal{P} \subset \mathcal{P} \times_X \mathcal{P}$; i.e. $N(\mathcal{P})^\circ_2$*

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viewed as a scheme over $<_{\mathcal{P}} = N(\mathcal{P})_1^\circ$ *via* δ^1 $(a < b < c \mapsto a < c)$ *and ordered by pullback from the projection to middle of the chain* $(a < b < c \mapsto b)$*. For*

$$\mu_{\mathcal{P}} := 1 + \tilde{\chi}(N(\mathcal{P}, \mathcal{P})) \in I_{\mathrm{mot}}(\mathcal{P}),$$

we have

$$\mu_{\mathcal{P}} \star \zeta_{\mathcal{P}} = 1 \in I_{\mathrm{mot}}(\mathcal{P}), \tag{7.1.5.1}$$

where we recall that in the definition of $\mu_{\mathcal{P}}$ *and* (7.1.5.1), $1 = [\Delta_{\mathcal{P}}/\leq_{\mathcal{P}}] \in I_{\mathrm{mot}}(\mathcal{P})$.

Proof. The proof is the same as the proof for posets: we simply need to upgrade our bijection between simplices to a radicial surjective map (in fact, an isomorphism). To that end, suppose

$$t_0 < t_1 < \cdots < t_k \in \delta_2^* N(\mathcal{P}, \mathcal{P})_k^\circ(T)$$

with image $a < b < c$ in $N(\mathcal{P})_2^\circ(T)$; note that this means

$$a < t_0 < t_1 < \cdots < t_k < b.$$

Then we can map this to the point

$$t_0 < t_1 < \cdots < t_k < b \in N(\mathcal{P}, \mathcal{P}]_{k+1}^\circ(T)$$

above the point $a < c \in \leq_{\mathcal{P}}(T)$. By Yoneda, this gives an isomorphism of schemes over $\leq_{\mathcal{P}}$ between $\delta_2^* N(\mathcal{P}, \mathcal{P})_k^\circ$, viewed as a scheme over $\leq_{\mathcal{P}}$ by δ^1, with $N(\mathcal{P}, \mathcal{P}]_{k+1}^\circ$. Thus, we find

$$\chi(N(\mathcal{P}, \mathcal{P})) \star \zeta_{\mathcal{P}} = -\chi(N(\mathcal{P}, \mathcal{P}]) + [N(\mathcal{P}, \mathcal{P}]_0^\circ] = -\chi(N(\mathcal{P}, \mathcal{P}]) + [(\mathcal{P}, \mathcal{P}]].$$

Now, $\tilde{\chi}(N(\mathcal{P}, \mathcal{P})) = \chi(N(\mathcal{P}, \mathcal{P})) - [<_{\mathcal{P}}]$, and $[<_{\mathcal{P}}] \star \zeta_{\mathcal{P}} = [(\mathcal{P}, \mathcal{P}]]$. Thus,

$$\tilde{\chi}(N(\mathcal{P}, \mathcal{P})) \star \zeta_{\mathcal{P}} = -\chi(N(\mathcal{P}, \mathcal{P}]) = -[<_{\mathcal{P}}/\leq_{\mathcal{P}}],$$

where the final equality is by Theorem D because $(\mathcal{P}, \mathcal{P}]/<_{\mathcal{P}}$ has a maximum. We conclude that

$$\mu_{\mathcal{P}} \star \zeta_{\mathcal{P}} = (1 + \tilde{\chi}(N(\mathcal{P}, \mathcal{P}))) \star \zeta_{\mathcal{P}} = [\leq_{\mathcal{P}}/\leq_{\mathcal{P}}] - [<_{\mathcal{P}}/\leq_{\mathcal{P}}] = [\Delta_{\mathcal{P}}/\leq_{\mathcal{P}}] = 1. \qquad \square$$

Remark 7.1.6. Specializing via $I_{\mathrm{mot}}(\mathcal{P}) \to I_{\mathrm{sh}}(\mathcal{P})$, we obtain a Möbius inversion formula also in $I_{\mathrm{sh}}(\mathcal{P})$.

7.2 The reduced incidence algebra for the poscheme of effective zero-cycles

In § 7.1, we considered incidence algebras for pospaces and poschemes. There is another closely related notion in classical poset theory: for any locally finite poset $\mathcal{P}$ such that there is a good notion of two intervals $[a, b]$ and $[a', b']$ being 'the same', one considers the reduced incidence algebra consisting of functions on intervals such that $f(a \leq b) = f(a' \leq b')$ whenever $[a, b]$ is the same as $[a', b']$; it is a subalgebra of the full incidence algebra. This applies to the poset attached to a free commutative monoid such as the divisor poset in $\mathbb{Z}_{>0}$ (which is the free commutative monoid generated by primes under multiplication, and where $[1, m]$ is 'the same' as $[n, mn]$), whose reduced incidence algebra is a natural combinatorial home for the Riemann zeta function and other formal Dirichlet series.

More generally, the poset of effective zero-cycles on a finite-type scheme over $\mathbb{Z}$ gives a reduced incidence algebra containing the zeta function of the variety as a natural element. We now lift this to the Grothendieck ring by using the poscheme of effective zero-cycles; this gives a new interpretation of Bilu's [Bil23] lift of the Kapranov zeta function used in the definition of motivic Euler products, and, through the Möbius inversion formula, Vakil and Wood's [VW15] formula for its inverse.

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For Z/X, we consider reduced motivic (respectively, sheaf theoretic) incidence algebras defined via the divided powers schemes of § 6:

$$\tilde{I}_{\mathrm{mot}}(\Gamma_X^{\bullet,+}(Z)) := \prod_{k=0}^{\infty} K_0(\mathrm{Var}/\Gamma_X^k(Z)),$$

$$\tilde{I}_{\mathrm{sh}}(\Gamma_X^{\bullet,+}(Z)) := \prod_{k=0}^{\infty} K_0(\mathrm{Cons}(\Gamma_X^k(Z), L)),$$

equipped with the convolution products $x \star y = m_!(\pi_1^*(x)\pi_2^*(y))$ (respectively, $Rm_! \dots$) where π_1, π_2, and m are the first projection, second projection, and multiplication maps, respectively,

$$\Gamma^{\bullet,+} \times \Gamma^{\bullet,+} \to \Gamma^{\bullet,+},$$

and we recall that for $f : U \to V$, $f_!$ denotes the map on the relative Grothendieck ring of varieties which sends $[g : T \to U]$ to $[f \circ g : T \to V]$.

We note that $\pi : \Gamma_X^k(Z) \to X$ induces natural algebra homomorphisms

$$\pi_! : \tilde{I}_{\mathrm{mot}}(\Gamma_X^{\bullet,+}(Z)) \to \tilde{K}_0(\mathrm{Var}/X)[[t]],\ ([A_k/\Gamma_X^k(Z)])_k \mapsto \sum_{k \geq 0} [A_k/X]t^k$$

$$R\pi_! : \tilde{I}_{\mathrm{sh}}(\Gamma_X^{\bullet,+}(Z)) \to K_0(\mathrm{Cons}(X, L))[[t]],\ ([K_k])_k \mapsto \sum_{k \geq 0} [R\pi_! K_k]t^k.$$

More generally, a map $Z \to Z'$ over X induces maps of the associated reduced incidence algebras, and the above are the maps obtained from the structure map $Z/X \to X/X$ and the identification $\Gamma_X^{\bullet,+}(X) = (\mathbb{Z}_{\geq 0} \times X)/X$.

The (relative to X) Kapranov zeta function $Z_{Z/X}^{\mathrm{Kap}}(t) \in K_0(\mathrm{Var}/X)[[t]]$ naturally lifts along $\pi_!$ to the motivic incidence algebra as

$$\zeta = (1, 1, 1, \dots), \quad \text{where we note } 1 = [\Gamma_X^k(Z)/\Gamma_X^k(Z)] \in K_0(\mathrm{Var}/\Gamma_X^k(Z)).$$

Here we recall from § 6 that there is a natural universal homeomorphism $\mathbf{S}_X^k(Z) \to \Gamma_X^k(Z)$ so that, in particular, they have the same class in the modified Grothendieck ring $K_0(\mathrm{Var}/X)$ and one can define the Kapranov zeta function using either $\mathbf{S}^\bullet$ or $\Gamma^\bullet$.

Remark 7.2.1. The convolution structure on $\tilde{I}_{\mathrm{mot}}(\Gamma_X^{\bullet,+}(Z))$ along with this lift of the Kapranov zeta function was essentially considered by Bilu [Bil17, Bil23] (with $\mathbf{S}^\bullet$ in place of $\Gamma^\bullet$) in her definition of motivic Euler products, but the interpretation as a reduced incidence algebra appears to have gone unnoticed.

7.2.2 *Möbius inversion formula for the Kapranov zeta function.*

Theorem 7.2.3 (Möbius inversion and Kapranov zeta). *Let X be a Noetherian scheme and let Z/X be quasi-projective. In $\tilde{I}_{\mathrm{mot}}(\Gamma_X^{\bullet,+}(Z))$, writing the Möbius elements as $\mu = (\mu_0, \mu_1, \dots)$, for $k \geq 1$,*

$$\mu_k = \sum_{p=0}^{\infty} (-1)^{p-1} \sum_{\underline{a} \in \mathbb{Z}_{>0}^{[p]}, \sum \underline{a} = k} [\Gamma_X^{\underline{a}}(Z)/\Gamma_X^k(Z)]$$

$$= \sum_{p=0}^{\infty} (-1)^{p-1} \sum_{\underline{a} \in \mathbb{Z}_{>0}^{[p]}, \sum \underline{a} = k} [\mathbf{C}_X^{\underline{a}}(Z)/\Gamma_X^k(Z)].$$

In $\tilde{I}_{\mathrm{sh}}(\Gamma_X^{\bullet;+}(Z))$, *writing the Möbius elements as* $\mu=(\mu_0,\mu_1,\ldots)$, *for* $k\geq 1$,

$$\begin{aligned}\mu_k &= (-1)^k[(s_*L)[\mathrm{sgn}]] \quad \text{for } s: Z\times_X\cdots\times_X Z\to \Gamma_X^k(Z)\\ &= (-1)^k[(s^\circ_! L)[\mathrm{sgn}]] \quad \text{for } s^\circ := s|_{\mathbf{C}_X^{(1,\ldots,1)}(Z)}\\ &= (-1)^k[j_!\underline{\mathrm{sgn}}] \quad \text{for } j:\mathbf{C}_X^k(Z)\hookrightarrow \Gamma_X^k(Z),\end{aligned}$$

where sgn *denotes the sign character of* $\mathfrak{S}_k$ *on* L, $\underline{\mathrm{sgn}}$ *denotes the corresponding local system on* $\mathbf{C}_X^k(Z)$, *and* $V[\mathrm{sgn}]$ *denotes the isotypic part for a* $\mathfrak{S}_k$*-action on* V.

Remark 7.2.4. Passing the second equality in $\tilde{I}_{\mathrm{mot}}$ to $K_0(\mathrm{Var}/\kappa)[[t]]$ via $\pi_!$ recovers Vakil and Wood's formula

$$Z_{Z/X,\mathrm{Kap}}(t)^{-1} = 1+\sum_{p=0}^{\infty}(-1)^{p+1}\sum_{\underline{a}\in\mathbb{Z}_{>0}^{[p]}}[\mathbf{C}_X^{\underline{a}}(Z)/X]t^{\sum\underline{a}}.$$

Passing the first equality in $\tilde{I}_{\mathrm{sh}}$ to a cohomological Grothendieck ring recovers (8.0.0.2).

Proof. The first two equalities in $\tilde{I}_{\mathrm{sh}}$ can be deduced immediately from the two equalities in $\tilde{I}_{\mathrm{mot}}$ by using the formula for the character of the sign representation in terms of permutation representation given by the reduced Euler characteristic of the simplicial cohomology complex computing $H^\bullet(\partial\Delta^{k-1})$ for the barycentric subdivision of $\partial\Delta^{k-1}$ (see § 6.1). The third equality in $\tilde{I}_{\mathrm{sh}}$ is a reformulation of the second. See the paragraph following this proof for an alternative deduction of these sheaf identities without passing through the motivic identities.

It remains to treat the motivic case: by essentially the same argument as Theorem 7.1.5,

$$\mu = 1+\tilde{\chi}(N(-\infty,\Gamma_X^\bullet(Z))). \tag{7.2.4.1}$$

Thus, to compute a formula for μ we need only to compute this Euler characteristic. From the definitions and the monoidal description of the nerve in Example 6.2.5, one then obtains

$$\mu|_{\Gamma_X^k(Z)} = \sum_{p=0}^{\infty}(-1)^{p-1}\sum_{\underline{a}\in\mathbb{Z}_{>0}^{[p]},\sum\underline{a}=k}[\Gamma_X^{\underline{a}}(Z)/\Gamma_X^k(Z)],$$

where $(-1)^{p-1}$ is because a point $(t_0,\ldots,t_p)\in\Gamma_X^{\underline{a}}(Z)(T)$ corresponds to the $p-1$ simplex $t_0< t_0+t_1<\cdots<t_0+\cdots+t_{p-1}$ in $(-\infty,t_0+t_1+\cdots+t_p)$.

On the other hand, by Proposition 6.3.3,

$$\tilde{\chi}(N(-\infty,\Gamma_X^\bullet(Z)))=\tilde{\chi}(N(-\infty,\mathbf{C}_X^\bullet(Z)))$$

and we also obtain the formula in terms of configuration spaces similarly. □

In the proof we gave a direct argument in the motivic case and deduced the sheaf-theoretic version by a character identity. Arguing instead directly in the sheaf-theoretic case and invoking Proposition 6.3.2, one obtains naturally the third expression, which is trivially equivalent to the second already at the level of the sheaves on $\Gamma_X^k(Z)$. Arguing with the skeletal spectral sequence as in § 6.5, one would instead obtain the first expression. Alternatively, an argument we learned from Banerjee using the spectral sequence of a stratified space shows directly the equality between these two expressions. As it will be useful in § 8, we record this result here in the case of a constructible sheaf using a slightly different proof.

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PROPOSITION 7.2.5. *Let X be a locally Noetherian scheme and let Z/X be quasi-projective. Let S be a finite set, and $\mathcal{F}$ a constructible sheaf of L-modules on X. Let $\pi : (Z/X)^S = \Gamma_X^S(Z) \to \Gamma_X^{|S|}(Z)$ denote the addition of cycles map, and let $j : \mathbf{C}_X^{|S|}(Z) \hookrightarrow \Gamma_X^{|S|}(S)$. Then*

$$\pi_*\mathcal{F}|_{\Gamma_X^S(Z)}[\mathrm{sgn}] = \pi_!\mathcal{F}|_{\mathbf{C}_X^S(Z)}[\mathrm{sgn}] = j_!\Big(\mathcal{F}|_{\mathbf{C}_X^{|S|}(Z)} \otimes \underline{\mathrm{sgn}}\Big), \tag{7.2.5.1}$$

where $[\mathrm{sgn}]$ denotes the isotypic component for the sign representation of $\mathfrak{S}_S$, i.e. the image of the idempotent

$$\frac{1}{|S|!}\sum_{\sigma\in\mathfrak{S}_S} \mathrm{sgn}(\sigma)\sigma. \tag{7.2.5.2}$$

In particular, if $X \to Y$, then

$$\begin{aligned}\Big(R^\bullet(\Gamma_X^S(Z)\to Y)_!\mathcal{F}|_{\Gamma_X^S(Z)}\Big)[\mathrm{sgn}] &= \Big(R^\bullet(\mathbf{C}_X^S(Z)\to Y)_!\mathcal{F}|_{\mathbf{C}_X^S(Z)}\Big)[\mathrm{sgn}]\\ &= R^\bullet\big(\mathbf{C}_X^{|S|}(Z)\to Y\big)_!\Big(\mathcal{F}|_{\mathbf{C}_X^{|S|}(Z)}\otimes\underline{\mathrm{sgn}}\Big).\end{aligned} \tag{7.2.5.3}$$

Proof. The identities (7.2.5.3) are immediate from the sheaf-theoretic equalities (7.2.5.1) since π is a finite map so $\pi_! = R\pi_!$. In (7.2.5.1), the second equality is almost tautological, so it remains to show the first. Since $\pi_*\mathcal{F}|_{\Gamma_X^S(Z)}$ is constructible, so is $\pi_*\mathcal{F}|_{\Gamma_X^S(Z)}[\mathrm{sgn}]$, a direct summand, and it suffices to show that its stalk vanishes at every geometric point in the closed set $\Gamma_X^{|S|}(Z)\backslash\mathbf{C}_X^{|S|}(Z)$. Thus, let $\overline{c} : \operatorname{Spec}\kappa \to \Gamma_X^{|S|}(Z)\backslash\mathbf{C}_X^{|S|}(Z)$ for κ algebraically closed, and write $\overline{x}$ for its image in X. Then $\overline{c}$ corresponds to an effective zero-cycle

$$\sum_{z\in Z_{\overline{x}}(\kappa)} a_z Z$$

with $\sum a_Z = |S|$ and $a_z \geq 2$ for at least one z. Since π is finite,

$$\Big(\pi_*\mathcal{F}|_{\Gamma_X^S(Z)}\Big)_{\overline{c}} = \bigoplus_{\pi^{-1}(\overline{c})} \mathcal{F}_{\overline{x}}.$$

The preimage $\pi^{-1}(\overline{c}) \subset \Gamma_X^S(Z)(\kappa)$ indexing the direct sum consists of the maps

$$\tilde{c} : S \to Z_{\overline{x}}(\kappa)$$

such that $\sum_{s\in S}\tilde{c}(s) = \overline{c}$ (as effective zero-cycles) with the obvious action of $\mathfrak{S}_S$.

Passing to the sign component commutes with taking stalks, so it suffices to show that the idempotent (7.2.5.2) acts by zero on this stalk. However, since $a_z \geq 2$ for some z, any $\tilde{c}$ as above is preserved by a transposition in $\mathfrak{S}_S$. Breaking the sum in the definition of the idempotent (7.2.5.2) into cosets of this transposition shows that anything in the image of the idempotent is zero in the $\tilde{c}$ component, and since this holds for each $\tilde{c}$, we conclude that the idempotent is identically zero. □

In particular, applying the Kunneth formula, one obtains the following.

COROLLARY 7.2.6 (Banerjee). *For any variety Y over κ algebraically closed (and with Λ_{gr} denoting the graded exterior power):*

$$H_c^\bullet(\mathbf{C}^p(Y), \underline{\mathrm{sgn}}) = H_c^\bullet(Y^p)[\mathrm{sgn}] \tag{7.2.6.1}$$

$$= \Lambda_{\mathrm{gr}}^p H_c^\bullet(Y, \mathbb{Q}_\ell) \tag{7.2.6.2}$$

$$= \bigoplus_{i=0}^{p} \Lambda^i H_c^{\mathrm{even}}(Y,\mathbb{Q}_\ell) \otimes \operatorname{Sym}^{p-i} H_c^{\mathrm{odd}}(Y,\mathbb{Q}_\ell). \tag{7.2.6.3}$$

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7.3 Cohomological special values of Kapranov zeta

Suppose now that X/κ is a smooth projective algebraic variety over an algebraically closed field. Building on the constructions above, we define cohomological special values of the inverse Kapranov zeta function as the bigraded (by weight and degree) vector spaces

$$\begin{aligned} \zeta_X^{-1,\mathrm{Coh}}(n) &:= \bigoplus_{k=0}^{\infty} (\zeta_X^{-1,\mathrm{Coh}})_k \otimes \mathbb{L}_{\mathrm{Coh}}^{\otimes -nk},\ \mathbb{L}_{\mathrm{Coh}} := H_c^{\bullet}(\mathbb{A}^1), \\ (\zeta_X^{-1,\mathrm{Coh}})_k &:= H_c^{\bullet-k}(\mathbf{C}^k(X), \underline{\mathrm{sgn}}) = H^{\bullet-k}(X^k)[\mathrm{sgn}], \end{aligned} \tag{7.3.0.1}$$

where the last equality is by Corollary 7.2.6. Note that by our conventions, the kth summand in $\zeta_X^{-1,\mathrm{Coh}}(n)$ sits in weight $-k$.

This definition is motivated as follows: we have seen above that $Z_X^{\mathrm{Kap}}(t)^{-1}$ lifts naturally to the Möbius element μ in $\widetilde{I}_{\mathrm{Mot}}$. On the other hand, (7.2.4.1) gives a formula for the kth component, $k \geq 1$, as the Euler characteristic

$$\mu_k = \tilde{\chi}(N(-\infty, \Gamma^k(X))).$$

Thus, to obtain a cohomological analog, we should replace the Euler characteristic with the corresponding reduced cohomology sheaf $j_!\underline{\mathrm{sgn}}[2-k]$ as computed in Proposition 6.3.2. The analog of the forgetful map from $K_0(\mathrm{Var}/\Gamma^k(X))$ to $K_0(\mathrm{Var}/\kappa)$ is compactly supported cohomology, and, after a shift by 2, this yields the formula (7.3.0.1) for special values (using the obvious interpretation of $\mathbb{L}$). The shift by two here is natural for various reasons (in particular, in applications it is canceled out by the same shift by two that occurs in Theorem C(ii)), so that we incorporate it into the definition.

In § 8, a special role is played by the special value for $n = \dim X + 1$. By the above definition and Poincaré duality,

$$\zeta_X^{-1,\mathrm{Coh}}(\dim X + 1) = \bigoplus_{k=0}^{\infty} H_{\bullet-k}(X^k)[\mathrm{sgn}] \otimes \mathbb{L}_{\mathrm{Coh}}^{\otimes -k}. \tag{7.3.0.2}$$

This is naturally identified with the graded symmetric algebra (for the Koszul sign rule with commutativity constraint given by degree) of $H_{\bullet-1}(X)(1)$, where the degree shift and Tate twist place $H_i(X)$ in degree $-i-1$ and weight -1; this will be explained again in the introduction to § 8 where it connects our stabilization results to those of Aumonier [Aum21]; see also (7.2.6.1).

8. Stability for the space of smooth hypersurface sections

In this section we prove Theorem A. We first discuss the context of this result and related work, expanding on the discussion in the introduction and starting with the Grothendieck ring stabilization of Vakil and Wood [VW15, VW20] (which we will reprove below in parallel with Theorem A to illustrate the close relation between the methods). To ensure this discussion is accessible to readers who have skipped here directly from the introduction, we will recall some notation along the way.

Let X be a smooth projective variety over an algebraically closed field κ. Let $\mathcal{L}$ be an ample line bundle on X and let V_d be the affine space of global sections of $\mathcal{L}^d$. Let $U_d \subset V_d$ be the open subscheme of sections with non-singular vanishing locus: its complement D_d is the image of the incidence variety of I_d parameterizing $(x, f) \in X \times V_d$ such that the degree-one Taylor expansion of f at x is zero.

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We write $\mathbb{L} = [\mathbb{A}^1]$ in any Grothendieck ring of varieties. For κ any field, Vakil and Wood [VW15, VW20] showed that, in the completion of $K_0(\mathrm{Var}/\kappa)[\mathbb{L}^{-1}]$ for the dimension filtration (where elements of the form $[Z]/\mathbb{L}^n$ are small if $n \gg \dim Z$),

$$\lim_{d\to\infty} \frac{[U_d]}{\mathbb{L}^{\dim U_d}} = \zeta_X^{\mathrm{Kap}}(\dim X + 1)^{-1}, \qquad (8.0.0.1)$$

an inverse special value of the Kapranov zeta function

$$\zeta_X^{\mathrm{Kap}}(n) := Z_X^{\mathrm{Kap}}(\mathbb{L}^{-n}), \quad Z_X^{\mathrm{Kap}}(t) := \sum_{k=0}^{\infty} [\mathbf{S}^k X] t^k \in K_0(\mathrm{Var}/\kappa)[[t]], \quad \mathbf{S}^k X := X^k/\mathfrak{S}_k.$$

If X is defined over a finite field $\mathbb{F}_q$, the dth term in the limit on the left of (8.0.0.1) specializes by point-counting to the probability that a random smooth degree d hypersurface section of X defined over $\mathbb{F}_q$ is smooth. The Kapranov zeta function specializes by point-counting to the Hasse–Weil zeta function, and the result of Vakil and Wood is a Grothendieck ring analog of a point-counting result of Poonen [Poo04] for varieties over finite fields that shows these probabilities converge to the same special value of the Hasse–Weil zeta function (recall, however, that the Grothendieck ring stabilization does not imply the point-counting stabilization because point-counting is not continuous for the dimension topology; see [BDH22] for a recent discussion). This point-counting result is itself an extension from curves to arbitrary varieties of the function field analog of the classical statement that the asymptotic probability that an integer is squarefree is $\zeta_{\mathbb{Z}}(2)^{-1}$ for $\zeta_{\mathbb{Z}}(s) = \sum 1/n^s$ the Riemann zeta function (see [Poo04] or [BH21, § 1.1] for more details on this point).

We now assume κ is algebraically closed, and denote by $H^\bullet(-)$ (respectively, $H^\bullet_{\mathrm{c}}(-)$; respectively, $H_\bullet(-)$) either ℓ-adic étale cohomology for ℓ invertible in κ or rational singular cohomology if $\kappa = \mathbb{C}$ (respectively, compactly supported cohomology; respectively, homology). There is a natural compactly supported Euler characteristic map from $K_0(\mathrm{Var}/\kappa)[\mathbb{L}^{-1}]$ to a weight-graded cohomological Grothendieck ring K_0^{Coh} (e.g. of Hodge structures or germs of Galois representations),

$$[Y] \mapsto \sum_i (-1)^i [H^i_{\mathrm{c}}(Y)],$$

and by Poincaré duality the class of $[U_d]/\mathbb{L}^{\dim V_d}$ is sent to

$$\sum_i (-1)^i [H_i(U_d)].$$

The result of Vakil and Wood implies that this generalized homological Euler characteristic stabilizes as $d \to \infty$ in the completion of K_0^{Coh} for the weight grading to the image in the same ring of $\zeta_X^{\mathrm{Kap}}(\dim X + 1)^{-1}$. Based on this observation, Vakil and Wood conjectured that the rational homology of U_d also stabilizes, but without specifying a natural stable value except in cases where the special value is of a particularly simple form. In those cases, they conjectured an Occam's razor principle that the cohomology should be in a sense the simplest possible.

Tommasi [Tom14] established homological stability in the case $X = \mathbb{P}^n$ by combining a Vassiliev-type spectral sequence with an E_1-degeneration argument specific to the case of $\mathbb{P}^n$. Interestingly, Tommasi's computation showed that the most naive Occam's razor does not hold in this case, but for good reasons; in this case, the orbit map for the natural action of $\mathrm{PGL}_{n+1}(\mathbb{C})$ describes the cohomology completely, so that the stable cohomology is equal to the cohomology of $\mathrm{PGL}_{n+1}(\mathbb{C})$. The Vassiliev spectral sequence of [Tom14] applies to general $X/\mathbb{C}$, and indeed our approximate inclusion–exclusion formula is an algebro-geometric version of this sequence.

The degeneration argument and the simple description of the stable cohomology as that of $\mathrm{PGL}_{n+1}(\mathbb{C})$ are very specific to the case $X = \mathbb{P}^n$.

Recently Aumonier [Aum21] has obtained stabilization for general smooth projective $X/\mathbb{C}$ via an h-principle comparing continuous and holomorphic sections of a jet bundle. The end result includes a beautiful and simple description of the stable cohomology of U_d in terms of the cohomology of X itself. In hindsight, there is a simple heuristic that leads directly from Vakil and Wood's stabilization to Aumonier's description and can be viewed as a refined Occam's razor: the Kapranov zeta function defines a pre-λ ring structure on the Grothendieck ring and, using the notation of the corresponding power structure [GLM04] (see also [How20, BH21]), we can write $Z_{X,\mathrm{Kap}}(t)^{-1} = (1-t)^{[X]}$. Specializing to the cohomological Grothendieck ring, we may then expand as

$$\begin{aligned}(1-t)^{\sum_i (-1)^i [H^i(X)]} &= (1-t)^{[H^{\mathrm{even}}(X)]}(1-t)^{-[H^{\mathrm{odd}}(X)]}\\ &= \Big(\sum_j (-1)^j \Big[\bigwedge^j H^{\mathrm{even}}(X)\Big] t^j\Big)\Big(\sum_j \big[\mathrm{Sym}^j H^{\mathrm{odd}}(X)\big] t^j\Big)\\ &= \sum_j (-1)^j \Big[\bigwedge_{\mathrm{gr}}^j H^\bullet(X)\Big] t^j. \end{aligned} \tag{8.0.0.2}$$

Here the subscript gr on the last line denotes exterior power is of graded vector spaces with the Koszul sign rule. If we substitute $t = \mathbb{L}^{-(\dim X+1)}$, then this is the natural class attached to the graded symmetric algebra of $H_{\bullet-1}(X)(1)$, and in [Aum21] it is shown that the cohomology ring of U_d stabilizes to the dual graded symmetric algebra of $H^{\bullet-1}(X)(-1)$ (our result here does not describe the cup product).

On the other hand, the kth graded exterior power of a graded vector space V is isomorphic to the sign-isotypic summand for the $\mathfrak{S}_k$-action on the graded tensor product $V^{\otimes k}$ (with the Koszul sign rule), so by Kunneth (8.0.0.2) equals

$$\sum_{k=0}^{\infty} (-1)^k [H^\bullet(X^k)[\mathrm{sgn}]] t^k. \tag{8.0.0.3}$$

This incarnation is how the special value appears in our homological stabilization.

To explain this, recall that there are two key ingredients in Vakil and Wood's proof of the motivic stabilization (8.0.0.1) (see also [BH21]): an approximate motivic inclusion–exclusion formula describing D_d using the resolution $I_d \to D_d$, and an inversion formula for $Z_X^{\mathrm{Kap}}(t)$. In our language, the approximate inclusion–exclusion is provided by Theorem 6.4.1(i), which is of a particularly simple form because for d sufficiently large the relative configurations of I_d are vector bundles over configurations of X. The inversion formula was treated in § 7, where we explained how (8.0.0.3) and other closely related formulas can be obtained from Möbius inversion on the poscheme of effective zero-cycles on X. Since Möbius inversion is an incarnation of inclusion–exclusion, it is no surprise that the two should be related.

For homological stabilization, the approximate inclusion–exclusion formula is provided by Theorem 6.4.1(ii). For the inversion formula, in section § 7.3, motivated by incidence algebra constructions, we defined bigraded (by degree and weight) cohomological special values of the inverse Kapranov zeta function

$$\zeta_X^{-1,\mathrm{Coh}}(n) = \bigoplus_{k=0}^{\infty} H^{\bullet-k}(X^k)[\mathrm{sgn}] \otimes \mathbb{L}_{\mathrm{Coh}}^{\otimes -nk}, \quad \mathbb{L}_{\mathrm{Coh}} := H_c^\bullet(\mathbb{A}^1).$$

In particular, the weight $-k$ summand of $\zeta_X^{-1,\mathrm{Coh}}(-\dim X+1)$ is

$$H^{\bullet-k}(X^k)[\mathrm{sgn}]\otimes \mathbb{L}_{\mathrm{Coh}}^{\otimes-(\dim X+1)k}=H_{\bullet+k}(X^k)[\mathrm{sgn}]\otimes \mathbb{L}_{\mathrm{Coh}}^{\otimes -k}.$$

These terms will be matched with the E_1 page of the skeletal spectral sequence for the truncated poscheme of effective zero-cycles that arises in approximate inclusion–exclusion, so that the main point is to prove this spectral sequence degenerates at E_1. We show this with a direct analysis on the E_1 page and a weight argument to treat the later differentials in order to prove a slight refinement of Theorem A.

Theorem 8.0.1. *Let X be a smooth projective variety over an algebraically closed field κ, equipped with an ample line bundle $\mathcal{L}$. As above, let V_d/κ be the variety of global sections of $\mathcal{L}^{\otimes d}$, let $I_d\subseteq X\times V_d$ be the incidence variety parameterizing singular points of global sections of $\mathcal{L}^{\otimes d}$, let D_d denote the image of I_d in V_d (the discriminant locus), and let $U_d=V_d\backslash D_d$.*

For $d\gg 0$ depending on k, the skeletal spectral sequence for the k-truncated poscheme of effective zero-cycles for (a compactification of) I_d/D_d induces a canonical isomorphism of bigraded vector spaces

$$\mathrm{Gr}_W\, H_{\leq k}(U_d)\cong \zeta_X^{-1,\mathrm{Coh}}(\dim X+1)_{\deg\geq -k}$$

where we recall W denotes the weight filtration and $H_i(U_d)$ sits in degree $-i$.

The rest of this section is organized as follows: in § 8.1 we recall the argument for Vakil and Wood's motivic stabilization (8.0.0.1) via motivic approximate inclusion–exclusion as in Theorem 6.4.1(i); this is the argument of [VW15, VW20] (see also [BH21]), but the results of the previous sections render completely transparent the relation between the approximate motivic inclusion–exclusion formula and the inversion formula for the Kapranov zeta function. In § 8.2 we carry out the first steps of the proof of Theorem A by cohomological approximate inclusion–exclusion as in Theorem 6.4.1(ii). These first steps mirror the motivic argument, but the final, critical step, which has no analog in the Grothendieck ring, is to show the degeneration at E_1 of the skeletal spectral sequence for the truncated symmetric power poscheme. We carry this out in § 8.3: on E_1 we can analyze the differentials explicitly, while on higher pages we obtain vanishing by a weight argument using purity of E_1.

8.1 Motivic stabilization

We now explain how to obtain (8.0.0.1). The main geometric input is as follows.

Lemma 8.1.1 [VW15, Lemma 3.2]. *For any $M>0$, there is a B sufficiently large such that for all $d\geq B$ and $a_1+a_2+\cdots+a_m=j\leq M$, $\mathbf{C}_{V_d}^{(a_1,\ldots,a_m)}(I)$ is a vector bundle of rank $r(j):=\dim V_d-j(\dim X+1)$ over $\mathbf{C}^{(a_1,\ldots,a_m)}(X)$ and, in particular, $\dim \mathbf{C}_{V_d}^{(a_1,\ldots,a_m)}(I)=\dim V_d-j$.*

As a first consequence of this lemma, we show that, if we fix a k, then for $d\gg 0$ the hypothesis of Theorem 6.4.1 is met for k. Recall that, in the notation of Theorem 6.4.1, for any j, $V_{d,>j}$ is the closure of the image of $\mathbf{C}_{V_d}^{j+1}I_d$ in V_d, and $V_{d,\infty}=\cap_j V_{d,>j}$.

Lemma 8.1.2. *For a fixed k and for $d\gg 0$,*

$$\dim(V_{d,>k})=\dim V_d-(k+1)\geq \dim V_{d,\infty}+k\dim_{/V_d} I_d.$$

Proof. We first show that, for any j and for $d\gg 0$, $\mathrm{codim} V_{d,>j}=j+1$. To that end, let $d\geq B$ where B is as in Lemma 8.1.1 for $M=j+2$. It follows that $\dim \mathbf{C}_{V_d}^{j+1}I_d=\dim V_d-(j+1)$. Thus, its image in V_d has codimension at least $j+1$, and so does the closure $V_{>j}$. To see the codimension is exactly $j+1$, note that we also have $\dim \mathbf{C}^{j+2}_{V_d} I_d = \dim V_d - (j+2)$. Thus,

$$\dim \left(\mathbf{C}^{j+1}_{V_d} I_d - \mathbf{C}^{j+2}_{V_d} I_d\right) = \dim V_d - (j+1),$$

but over this locus the projection to V_d is quasi-finite, so the image of this locus in V_d also has dimension $\dim V_d - (j+1)$. Since this image is contained in $V_{>j}$, we conclude $\mathrm{codim} V_{>j} = j+1$.

We now prove the inequality on dimensions. In terms of codimension, it is equivalent to

$$\mathrm{codim}(V_{d,>k}) \leq \mathrm{codim} V_{d,\infty} - k \dim_{/V_d} I_d.$$

Note that $k \dim_{V_d} I_d = k \dim X$ (consider the fiber over $0 \in V_d$), which does not change with d. Thus, it suffices to show that, by choosing $d \gg 0$, we can make the difference between $\mathrm{codim}(V_{d,>k})$ and $\mathrm{codim}(V_{d,\infty})$ arbitrarily large. However, by what we have shown above, for any $a > 0$, taking $d \gg 0$ gives $\mathrm{codim} V_{d,>k} = k+1$ and $\mathrm{codim} V_{d,\infty} \geq \mathrm{codim} V_{d,>k+a} = k+1+a$, so we conclude. □

Now fix a k and assume d is larger than the bound D of Lemma 8.1.1 for $M = k+1$ and also large enough for Lemma 8.1.2 to hold. Then, Theorem 6.4.1(i) applies, giving

$$[U_d] = [V_d] - [D_d] = [V_d] + \sum_{\substack{(a_1,a_2,\ldots,a_m) \\ a_i>0, \sum a_i \leq k}} (-1)^m \left[\mathbf{C}^{(a_1,a_2,\ldots,a_m)}_{V_d}(I_d)\right]$$
$$\mod \mathrm{Fil}^{k+1-\dim V_d} K_0(\mathrm{Var}/\kappa)[\mathbb{L}^{-1}].$$

Lemma 8.1.1 allows us to rewrite the terms as

$$\left[\mathbf{C}^{(a_1,\ldots,a_m)}_{V_d}(I_d)\right] = \left[\mathbf{C}^{(a_1,\ldots,a_m)}(X)\right]\mathbb{L}^{\dim V_d - k(\dim X+1)}.$$

Dividing everything by $\mathbb{L}^{\dim V_d}$, we obtain

$$\frac{[U_d]}{\mathbb{L}^{\dim V_d}} \equiv 1 + \sum_{\substack{(a_1,a_2,\ldots,a_m) \\ a_i>0, \sum a_i \leq k}} (-1)^m \left[\mathbf{C}^{(a_1,a_2,\ldots,a_m)}_{X}(Z)\right]\mathbb{L}^{-k(\dim X+1)}$$
$$\equiv Z^{\mathrm{Kap}}_X(\mathbb{L}^{-(\dim X+1)}) \mod \mathrm{Fil}^{k+1} K_0(\mathrm{Var}/\kappa)[\mathbb{L}^{-1}],$$

where the final line follows from the inversion formula in Remark 7.2.4. Taking k larger and larger gives the desired result

$$\lim_{d\to\infty} \frac{[U_d]}{\mathbb{L}^{\dim V_d}} = Z^{\mathrm{Kap}}_X(\mathbb{L}^{-(\dim X+1)}) =: \zeta^{\mathrm{Kap}}_X(\dim X+1).$$

8.2 Homological stabilization: first steps

To prove Theorem 8.0.1, we must compute the weight graded of $H_i(U_d, \mathbb{Q}_\ell)$. By Poincaré duality, this is equivalent to computing the weight graded of $H_c^{2\dim V_d - i}(U_d, \mathbb{Q}_\ell)$. By the long exact sequence of compactly supported cohomology for the decomposition of V_d into the open U_d and its closed complement D_d, we have

$$H^i_c(U_d) = \begin{cases} H^{i-1}_c(D_d) & \text{for } 1 \leq i < 2\dim V_d, \\ H^{2\dim V_d}_c(V_d) = (\mathbb{L}^{\mathrm{Coh}})^{\dim V_d} & \text{for } i = 2\dim V_d, \\ 0 & \text{otherwise.} \end{cases} \tag{8.2.0.1}$$

Thus, it will suffice to compute the weight graded of $H^i_c(D_d)$.

It will be useful later on in our argument to compute this using an explicit compactification, so we introduce it into our setup from the beginning: let $\overline{V}_d = \mathbb{P}(V_d(\kappa) \oplus \kappa)$ be the compactification

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of V_d to a projective space over $\operatorname{Spec}\kappa$. The incidence variety I_d evidently extends to

$$\overline{I}_d \subseteq X \times \overline{V}_d$$

(indeed, the points we have added correspond to $\mathbb{P}(V_d)$, and evidently the property of being singular at a point depends only on the equation up to multiplication by a scalar), and $\overline{I}_d$ is the closure of I_d. Let $\overline{D}_d$ denote the image of $\overline{I}_d$ in $\overline{V}_d$, which is also the closure of D_d in $\overline{V}_d$. Writing $j : D_d \hookrightarrow \overline{D}_d$ for the open immersion, we have $H^i_c(D_d, \mathbb{Q}_\ell) = H^i(\overline{D}_d, j_!\mathbb{Q}_\ell)$.

We will compute these cohomology groups using the k-truncated poscheme of effective zero-cycles $\Gamma^{\leq k}_{\overline{V}_d}(\overline{I}_d)$. Thus, we need to invoke Theorem 6.4.1(ii) to show that for a fixed k this is a good approximation if $d \gg 0$. If we fix a k, then, arguing as in the motivic case above, we may assume $d \gg 0$ is large enough so that:

(i) $\mathbf{C}^p_{D_d}(I_d)$ is a vector bundle of rank $r(p) = \dim V_d - p(\dim X + 1)$ over $\mathbf{C}^p(X)$ for all $1 \leq p \leq k$, and $\mathbf{C}^p_{\overline{D}_d}(\overline{I}_d)$ is the compactifying projective bundle obtained by taking the closure of $\mathbf{C}^p_{D_d}(I_d)$ inside of $\mathbf{C}^p(X) \times \overline{V}_d$; note here that, since $I_d \to V_d$ factors through D_d, $\mathbf{C}^p_{D_d} I_d = \mathbf{C}^p_{V_d} I_d$; similarly, $\mathbf{C}^p_{\overline{D}_d} \overline{I}_d = \mathbf{C}^p_{\overline{V}_d} \overline{I}_d$;

(ii) $\dim \overline{V}_{d,>k} = \dim \overline{V}_d - (k+1)$;

(iii) the dimension hypothesis of Theorem 6.4.1 holds.

We apply Theorem 6.4.1 to $j_!\mathbb{Q}_\ell$. Taking the skeletal spectral sequence and applying Theorem 6.5.4 to simplify the E_1 page, we thus obtain

$$E_1^{p,q} = \begin{cases} H^q\big((\overline{I}_d/\overline{D}_d)^{[p]}, j_!\mathbb{Q}_\ell\big)[\mathrm{sgn}] & \text{if } 0 \leq p \leq k-1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} E_1^{p,q} &\Rightarrow H^{p+q}\big(\Gamma^{\leq k}_{\overline{D}_d}(\overline{I}_d), \epsilon^* j_!\mathbb{Q}_\ell\big) \\ &= H^{p+q}(\overline{D}_d, j_!\mathbb{Q}_\ell) \qquad \text{if } p+q \geq 2\dim V_d - k + 1 \\ &= H^{p+q}_c(D_d, \mathbb{Q}_\ell). \end{aligned}$$

By (8.2.0.1), this is sufficient to compute $H^i_c(U_d, \mathbb{Q}_\ell)$ for $i \geq 2\dim V_d - k$.

We observe that we can rewrite, for $0 \leq p \leq k-1$,

$$\begin{aligned} E_1^{p,\bullet} &= H^\bullet((\overline{I}_d/\overline{D}_d)^{[p]}, j_!\mathbb{Q}_\ell)[\mathrm{sgn}] \\ &= H^\bullet_c((I_d/D_d)^{[p]}, \mathbb{Q}_\ell)[\mathrm{sgn}] \\ &= H^\bullet_c(\mathbf{C}^{[p]}_{D_d}(I_d), \mathbb{Q}_\ell)[\mathrm{sgn}] \\ &= H^\bullet_c(\mathbf{C}^{[p]}(X), \mathbb{Q}_\ell)[\mathrm{sgn}] \otimes \mathbb{L}_{\mathrm{Coh}}^{\dim V_d - (\dim X+1)(p+1)} \\ &= H^\bullet(X^{[p]}, \mathbb{Q}_\ell)[\mathrm{sgn}] \otimes \mathbb{L}_{\mathrm{Coh}}^{\dim V_d - (\dim X+1)(p+1)}. \end{aligned}$$

The third and fifth equalities follow from Proposition 7.2.5, while the fourth follows from the projection formula. Renormalizing,

$$E_1^{p,\bullet} \otimes \mathbb{L}_{\mathrm{Coh}}^{-\dim V_d} = H^\bullet(X^{[p]}, \mathbb{Q}_\ell)[\mathrm{sgn}] \otimes \mathbb{L}_{\mathrm{Coh}}^{-(\dim X+1)(p+1)}.$$

The main point is then to establish the following.

PROPOSITION 8.2.1. *The spectral sequence $E_\bullet$ degenerates at E_1.*

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Proof of Theorem 8.0.1 assuming Proposition 8.2.1. Assume Proposition 8.2.1 holds. Then, the graded for the filtration on $H^\bullet_{\mathrm{c}}(D_d, \mathbb{Q}_\ell) \otimes \mathbb{L}_{\mathrm{Coh}}^{-\dim V_d}$ induced by the spectral sequence in the stable range $\bullet \geq 1-k$ satisfies, for $0 \leq p \leq k-1$,

$$\mathrm{Gr}^p\, H^\bullet_{\mathrm{c}}(D_d, \mathbb{Q}_\ell) \otimes \mathbb{L}_{\mathrm{Coh}}^{-\dim V_d} = H^{\bullet-p}(X^{p+1}, \mathbb{Q}_\ell)[\mathrm{sgn}] \otimes \mathbb{L}_{\mathrm{Coh}}^{-(\dim X+1)(p+1)}$$

and is zero for $p \geq k$. Then, using (8.2.0.1), the induced filtration on $H^\bullet_{\mathrm{c}}(U_d, \mathbb{Q}_\ell) \otimes \mathbb{L}_{\mathrm{Coh}}^{-\dim V_d}$ in the stable range $\bullet \geq -k$ satisfies for $0 \leq p \leq k$

$$\mathrm{Gr}^p\, H^\bullet_{\mathrm{c}}(U_d, \mathbb{Q}_\ell) \otimes \mathbb{L}_{\mathrm{Coh}}^{-\dim V_d} = H^{\bullet-p}(X^{p}, \mathbb{Q}_\ell)[\mathrm{sgn}] \otimes \mathbb{L}_{\mathrm{Coh}}^{-(\dim X+1)(p)}$$

and is zero for $p \geq k+1$. Then, we obtain Theorem 8.0.1 by comparison with (7.3.0.2): the filtration induced by the spectral sequence agrees with the weight filtration by a spreading out argument similar to Lemma 8.3.1 below. □

It remains to prove Proposition 8.2.1. We do so in the next subsection.

Remark 8.2.2. The use of Theorem 6.5.4 to simplify the terms of the spectral sequence, as well as the weight arguments below, requires that we work with rational coefficients. In fact, the analog of Theorem 8.0.1 in singular homology with $\mathbb{Z}/2$ coefficients fails for $X = \mathbb{CP}^2$, as shown in [Aum21, Proposition 8.12].

8.3 Homological stabilization: degeneration of the spectral sequence

We first make a weight argument to show degeneration at E_2.

Lemma 8.3.1. *The spectral sequence $E_\bullet$ degenerates at E_2.*

Proof. We first observe that we may spread X out to a smooth projective $X_0/\mathrm{Spec}\, A$ for a finitely generated $\mathbb{Z}$-algebra $A \subseteq \kappa$ (i.e. $X_0 \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, \kappa = X$) in which ℓ is invertible. We then may spread out V_d to $V_{d,0}$, and I_d to $I_{d,0}$ by the obvious definitions, and similarly to obtain $\overline{V}_{d,0}$ and $\overline{I}_{d,0}$. Then $\Gamma^{\leq k}_{\overline{V}_d}(\overline{I}_d)$ spreads out to $\Gamma^{\leq k}_{\overline{V}_{d,0}}(\overline{I}_{d,0})$. The E_1 page of the skeletal spectral sequence for cohomology relative to $\mathrm{Spec}\, A$ is again quasi-isomorphic to the relative Banerjee complex; in particular, the terms are locally constant sheaves on $\mathrm{Spec}\, A$. Thus, so are the terms on the higher pages, so it suffices to show that the differential $d_r^{p,q}$ for $r \geq 2$ vanishes after specialization to geometric points over closed points of $\mathrm{Spec}\, A$. Since A is of finite type over $\mathbb{Z}$ and ℓ is invertible in A, any maximal ideal has residue field a finite field $\mathbb{F}_a$, $\ell \nmid a$, thus by specializing above such a point we may assume that $\kappa = \overline{\mathbb{F}}_a$ and $X_0/\mathbb{F}_a$.

In this case, from Deligne's [Del74] purity theorem and the expression of the E_1 terms above, we find that geometric Frobenius acts on $E_1^{p,q}$ with eigenvalues a-Weil integers of weight q. The spectral sequence is Galois equivariant, and since $E_r^{p,q}$ is a subquotient of $E_1^{p,q}$, the eigenvalues of geometric Frobenius on $E_r^{p,q}$ are also of weight q. Because d_r is Galois-equivariant it preserves generalized eigenspaces for geometric Frobenius, thus because d_r is of degree $(r, 1-r)$ it must be identically zero for $r \geq 2$, as claimed. □

It remains to show that the differentials vanish also on E_1. To that end, recall that the E_1 differential is obtained by restricting the map

$$H^\bullet_{\mathrm{c}}((I_d/D_d)^{[p-1]}, \mathbb{Q}_\ell) \xrightarrow{\sum_{i\in[p]}(-1)^i \alpha_i^*} H^\bullet_{\mathrm{c}}((I_d/D_d)^{[p]}, \mathbb{Q}_\ell)$$

to the sign component, where α_i forgets the ith coordinate. We would like to show this map is zero; we will do by computing on a simpler space. Note that we have natural maps of schemes over $X^{[p]}$

$$(I_d/D_d)^{[p]} \hookrightarrow (\overline{I}_d/\overline{D}_d)^{[p]} \hookrightarrow X^{[p]} \times \overline{V}_d,$$

and a corresponding set of maps on the configuration loci. If we consider the induced maps on cohomology, we obtain the following commutative $\mathfrak{S}_{[p]}$-equivariant diagram.

$$\begin{array}{ccc} H_c^\bullet((I_d/D_d)^{[p]}, \mathbb{Q}_\ell) & \longleftarrow & H_c^\bullet(\mathbf{C}^{[p]}_{D_d}(I_d), \mathbb{Q}_\ell) \\ \downarrow & & \downarrow \\ H^\bullet((\overline{I}_d/\overline{D}_d)^{[p]}, \mathbb{Q}_\ell) & \longleftarrow & H_c^\bullet(\mathbf{C}^{[p]}_{\overline{D}_d}(\overline{I}_d), \mathbb{Q}_\ell) \\ \uparrow & & \uparrow \\ H^\bullet(X^{[p]}, \mathbb{Q}_\ell) \otimes H^\bullet(\overline{V}_d, \mathbb{Q}_\ell) & \longleftarrow & H_c^\bullet(\mathbf{C}^{[p]}(X), \mathbb{Q}_\ell) \otimes H^\bullet(\overline{V}_d, \mathbb{Q}_\ell) \end{array} \tag{8.3.1.1}$$

The top right vertical map in (8.3.1.1) is induced by the open embedding of a vector bundle inside its compactifying projective bundle, and the bottom right vertical map is induced by the closed embedding of a projective bundle in an ambient trivial projective bundle. If we expand $H^\bullet(\overline{V}_d, \mathbb{Q}_\ell) = \bigoplus_{k=0}^{2\dim V_d} \mathbb{L}^k_{\mathrm{Coh}}$, we deduce that the right column of (8.3.1.1) is identified with the following subquotient diagram.

$$\begin{array}{c} H_c^\bullet(\mathbf{C}^{[p]}(X), \mathbb{Q}_\ell) \otimes \mathbb{L}_{\mathrm{Coh}}^{2\dim V_d-(p+1)(\dim X+1)} \\ \big\downarrow \\ H_c^\bullet(\mathbf{C}^{[p]}(X), \mathbb{Q}_\ell) \otimes \bigoplus_{k=0}^{2\dim V_d-(p+1)(\dim X+1)} \mathbb{L}^k_{\mathrm{Coh}} \\ \big\uparrow \\ H_c^\bullet(\mathbf{C}^{[p]}(X), \mathbb{Q}_\ell) \otimes \bigoplus_{k=0}^{2\dim V_d} \mathbb{L}^k_{\mathrm{Coh}} \end{array}$$

By Proposition 7.2.5, the horizontal arrows in (8.3.1.1) become isomorphisms after passing to the sign component for the action of $\mathfrak{S}_{[p]}$, so, the sign component of the left column of (8.3.1.1) is then identified with the following subquotient diagram.

$$\begin{array}{c} H^\bullet(X^{[p]}, \mathbb{Q}_\ell)[\mathrm{sgn}] \otimes \mathbb{L}_{\mathrm{Coh}}^{2\dim V_d-(p+1)(\dim X+1)} \\ \big\downarrow \\ H^\bullet(X^{[p]}, \mathbb{Q}_\ell)[\mathrm{sgn}] \otimes \bigoplus_{k=0}^{2\dim V_d-(p+1)(\dim X+1)} \mathbb{L}^k_{\mathrm{Coh}} \\ \big\uparrow \\ H^\bullet(X^{[p]}, \mathbb{Q}_\ell)[\mathrm{sgn}] \otimes \bigoplus_{k=0}^{2\dim V_d} \mathbb{L}^k_{\mathrm{Coh}} \end{array}$$

Moreover, the analogous identifications for $p-1$ are compatible with the differential $\sum_{i\in[p]}(-1)^i \alpha_i^*$. The vanishing of the differential is then immediate, since in the bottom the differential preserves each summand corresponding to a power $\mathbb{L}^k_{\mathrm{Coh}}$, but the summands contributing in degree p and $p-1$ in the top are distinct.

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Conflicts of interest

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