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ARTICLE

Supersaturation of even linear cycles in linear hypergraphs

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Abstract

A classical result of Erdős and, independently, of Bondy and Simonovits [3] says that the maximum number of edges in an n-vertex graph not containing C_{2k}, the cycle of length $2k$, is $O(n^{1+1/k})$. Simonovits established a corresponding supersaturation result for C_{2k}'s, showing that there exist positive constants C, c depending only on k such that every n-vertex graph G with $\mathrm{e}(G) \geqslant Cn^{1+1/k}$ contains at least $c(\mathrm{e}(G)/\mathrm{v}(G))^{2k}$ copies of C_{2k}, this number of copies tightly achieved by the random graph (up to a multiplicative constant).

In this paper we extend Simonovits' result to a supersaturation result of r-uniform linear cycles of even length in r-uniform linear hypergraphs. Our proof is self-contained and includes the $r = 2$ case. As an auxiliary tool, we develop a reduction lemma from general host graphs to almost-regular host graphs that can be used for other supersaturation problems, and may therefore be of independent interest.

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1. Introduction

One of the central problems in extremal graph theory is the Turán problem, where, for a fixed graph H and fixed n, we wish to determine the maximum number of edges an n-vertex graph can have without creating a copy of H as a subgraph. This number is called the *Turán number* of H and is denoted by $\mathrm{ex}(n, H)$. The celebrated Erdős–Stone–Simonovits [6] theorem says that

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)n^2 + o(n^2),$$

where $\chi(H)$ is the chromatic number of the graph H. This solves the Turán problem asymptotically for all non-bipartite graphs H. However, asymptotic results or exact results are known for only a handful of bipartite graphs. While the Turán problem asks for the threshold on the number of edges on n vertices that guarantees at least one copy of H, it is natural to ask what is the minimum number of copies of H guaranteed in a host graph once its number of edges exceeds $\mathrm{ex}(n, H)$. Such problems are referred to as *supersaturation problems*. When H is non-bipartite, we

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know the correct order of magnitude of the answer. Let H be a graph with $\chi(H) = p \geqslant 3$ and $\mathrm{v}(H)$ vertices. A simple averaging argument (see *e.g.* Lemma 2.1 in [17]) can be used to show that for any $\varepsilon > 0$ there exist $\delta, n_0 > 0$ such that if G is a graph on $n \geqslant n_0$ vertices with

$$\mathrm{e}(G) \geqslant \left(1 - \frac{1}{p-1} + \varepsilon\right)\binom{n}{2},$$

then G contains at least $\delta\binom{n}{\mathrm{v}(H)}$ copies of H. This count is tight up to a multiplicative constant, as shown by the random graph of the same edge density as G. The threshold on the number of edges on G for which the count is valid is also asymptotically best possible, as shown by the Turán graph $T_{n,p-1}$, which is defined as the balanced blowup of the complete graph on $(p-1)$ vertices. For the supersaturation problem for bipartite graphs, Erdős and Simonovits [9] and also separately Simonovits [26] made the following conjecture in the 1980s.

Conjecture 1 ([9, 26]). *Let H be a bipartite graph with v vertices and e edges. Suppose that $\mathrm{ex}(n, H) = O(n^{2-\alpha})$ for some real $0 < \alpha < 1$. Then there exist $\alpha' \leqslant \alpha$ and constants $C, c > 0$ such that if G is an n-vertex graph with*

$$\mathrm{e}(G) \geqslant Cn^{2-\alpha'} \tag{1.1}$$

edges, then G contains at least $c(\mathrm{e}(G))^e/n^{2e-v}$ copies of H.

The Erdős–Rényi random graph $G(n, p)$ with $p = \mathrm{e}(G)/\binom{n}{2}$ shows that if Conjecture 1 is true then it is best possible up to a multiplicative constant. Simonovits [26] made a weaker conjecture that for every bipartite graph H with v vertices and e edges, there exist reals $\alpha, C, c > 0$ such that every n-vertex graph G with $e(G) > Cn^{2-\alpha}$ contains at least $c(\mathrm{e}(G))^e/n^{2e-v}$ copies of H. A slight reformulation of this conjecture (with c replaced by 1 and the number of copies of H replaced by the number of homomorphic copies of H) was made by Sidorenko [24, 25]. Note that this weakened version and Sidorenko's conjecture focus on the counting statement eventually being true when the host graph becomes very dense. When we study these conjectures, we are therefore concerned with host graphs with $\Omega(n^{2-c})$ edges where c is as small as we want. On the contrary, works on Conjecture 1 typically aim at finding the best possible threshold beyond which the counting statement holds and hence the host graphs we work with are (much) sparser. In fact, Erdős and Simonovits [9] made two conjectures even stronger than Conjecture 1 by replacing condition (1.1) with $\mathrm{e}(G) \geqslant C \cdot \mathrm{ex}(n, H)$ and $\mathrm{e}(G) \geqslant (1 + \varepsilon)\mathrm{ex}(n, H)$, respectively. At this stage, resolving these stronger versions seems hopeless since the exact value or even just the order of magnitude of $\mathrm{ex}(n, H)$ is only known for very few bipartite graphs H.

Now we turn our attention to the main focus of this paper, that is, the supersaturation of linear cycles of even length in linear r-uniform hypergraphs, or in short, r-graphs. First let us give the background for $r = 2$. A classical result of Erdős (unpublished) and of Bondy and Simonovits [3] establishes that $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$. The explicit upper bound that Bondy and Simonovits gave was $\mathrm{ex}(n, C_{2k}) \leqslant 100kn^{1+1/k}$. This upper bound was later improved by Verstraëte to $8(k-1)n^{1+1/k}$ for sufficiently large n, by Pikhurko [21] to $(k-1)n^{1+1/k} + O(n)$, and by Bukh and Jiang [4] to $80\sqrt{k}\log kn^{1+1/k} + O(n)$. Erdős and Simonovits conjectured that $\mathrm{ex}(n, C_{2k}) = \Omega(n^{1+1/k})$ also holds. This is known to be true for $k = 2, 3, 5$.

For supersaturation of even cycles, it was mentioned in [9] that Simonovits proved Conjecture 1 with $\alpha = \alpha' = 1 - 1/k$. The proof was not published at the time but is expected to appear in an upcoming paper of Faudree and Simonovits [13]. Very recently, Morris and Saxton [20] developed a balanced version of the supersaturation result for even cycles, which they use to obtain a sharp result on the number of C_{2k}-free graphs via the container method. Since Morris and Saxton require a balanced version of supersaturation where the collection of C_{2k}'s they obtain are, informally speaking, uniformly distributed, their proof is quite involved. In this paper we extend the result by Simonovits to a supersaturation result of even linear cycles in linear r-graphs, for all $r \geqslant 2$. Our proof is self-contained and includes the $r = 2$ case.

Before stating our main result, we need a few definitions. An r-graph G is called *linear* if any two edges share at most one vertex. For instance, all 2-graphs are linear. The *linear Turán number* of an r-graph H, denoted by $\mathrm{ex}_\ell(n, H)$, is defined to be the maximum number of edges an n-vertex linear r-graph can have without having a copy of H. The study of linear Turán numbers of linear r-graphs is motivated in part by their similarity to the Turán numbers of 2-graphs. Also, such studies were implicit in some classical extremal hypergraph problems, such as the famous $(6, 3)$-problem (see [2] and [23]), which is asymptotically equivalent to determining the linear Turán number of a linear 3-cycle. The $(6, 3)$-problem asks for the maximum size of an n-vertex 3-graph in which no six vertices span three or more edges. Note that the usual Turán number $\mathrm{ex}(n, H)$ of a linear r-graph H and the linear Turán number $\mathrm{ex}_\ell(n, H)$ of H are typically very different. The former is already at least $\binom{n-1}{r-1}$ as long as H contains two disjoint edges, while the latter is $O(n^2)$.

An r-uniform *linear cycle* $C_m^{(r)}$ of length m is obtained from a 2-uniform m-cycle $v_1 v_2 \ldots v_m v_1$ by extending each $v_i v_{i+1}$ (indices taken modulo m) with an $(r-2)$-tuple I_i such that the tuples I_i are pairwise disjoint for distinct indices. Collier-Cartaino, Graber and Jiang [5] extended the aforementioned result of Bondy and Simonovits on even cycles in 2-graphs to linear cycles in linear r-graphs. They showed that for all $r \geqslant 3$ and $m \geqslant 4$,

$$\mathrm{ex}_\ell(n, C_m^{(r)}) = O(n^{1+1/\lfloor m/2 \rfloor}).$$

For even linear cycles, their result also works for uniformity $r = 2$. In the case $r = 3$, it is known that the upper bound above is sharp when $m = 5, 7, 9, 13$ (see [11]). It is interesting to note that when $r \geqslant 3$, the linear Turán number of an odd linear cycle resembles that of an even linear cycle, which is very different from the situation for $r = 2$.

Our main result is the supersaturation version of the linear Turán result for linear cycles, but only for even linear cycles.

Theorem 1.1. *Given $k, r \geqslant 2$, there exist constants $C = C(k, r)$ and $c = c(k, r)$ such that if G is an n-vertex linear r-graph with $\mathrm{e}(G) \geqslant Cn^{1+1/k}$, then G contains at least $c(\mathrm{e}(G)/n)^{2k}$ copies of $C_{2k}^{(r)}$.*

It is not hard to see that this lower bound on the number of copies of linear cycles is tight, up to a multiplicative constant. Indeed, for $r = 2$, in the Erdős–Rényi graph $G(n, p)$, in expectation there are $\Theta(p^{2k} n^{2k})$ $2k$-cycles. For $r \geqslant 3$, one may consider random subgraphs of almost complete partial Steiner systems. An (n, λ, r, q)-*Steiner system* is defined to be an r-graph on n vertices such that every q-tuple is in exactly λ r-edges. A *partial* (n, λ, r, q)-*Steiner system* is defined to be an r-graph on n vertices such that every q-tuple is in at most λ r-edges. Rödl [22] proved that for all n there are partial $(n, 1, r, q)$-Steiner systems with

$$(1 - o(1))\binom{n}{q} \Big/ \binom{r}{q}$$

edges. (Note that this is also implied by a recent solution of the existence conjecture by Keevash [18], while the $q = 2$ case was proved much earlier by Wilson [28, 29, 30].) By taking random subgraphs of such partial Steiner systems with $\lambda = 1$ and $q = 2$, one can show that for every n and $0 \leqslant e \leqslant \binom{n}{2}$ there is a linear r-graph G on n vertices and $\mathrm{e}(G) = e$ in which the number of copies of the linear cycles of length $2k$ is $O((e/n)^{2k})$ (see Proposition 2.1).

Our Theorem 1.1 includes $r = 2$ as a special case and has a much simpler proof than that of Morris and Saxton of their stronger version of supersaturation. We use an approach developed by Faudree and Simonovits [12] in the study of the Turán numbers of theta graphs, which in its original form is not well suited for effective counting of C_{2k}'s. So we adapt their approach to facilitate counting. The proofs are greatly simplified via a reduction tool which allows us to reduce the supersaturation problem in a general host r-graph to one which has some regularity property. This regularization tool is an analogue of a regularization theorem of Erdős and Simonovits for the Turán problem and can be used for supersaturation problems in general. We refer the reader to Section 3 for the precise statement (see Theorem 3.3).

We organize the rest of the paper roughly as follows. In Section 2 we present notation and definitions. In Section 3 we develop our reduction results. In Section 4 we give an overview of the proof for $r=2$. In Sections 5 and 6 we give proofs of the $r=2$ and $r\geqslant 3$ cases of Theorem 1.1, respectively. Even though we could have proved Theorem 1.1 for general r directly, we feel that proving the $r=2$ case first helps to illustrate the main ideas. However, we will give two slightly different proofs for $r=2$ and $r\geqslant 3$, modulo the reduction mentioned earlier. Both of these proofs could be written for all $r\geqslant 2$. The proof we present for $r=2$ is more constructive and gives a better bound on constants. The proof we present for $r\geqslant 3$ follows the approach of Faudree and Simonovits more closely and is perhaps more intuitive and easier to follow for some readers. In Section 7 we give some concluding remarks.

2. Notation and definitions

For an r-graph G, we let $\mathrm{v}(G)$ and $\mathrm{e}(G)$ denote its number of vertices and edges, respectively. Let $\Delta(G)$, $\delta(G)$ denote the maximum and minimum degree of G, respectively. If S is a subset of $V(G)$, then we use $G-S$ to denote the graph obtained from G by deleting the vertices in S as well as edges of G that contain at least one vertex in S. Given a vertex $v\in V(G)$, the *link* of v in G, denoted by $L_G(v)$, is defined to be

$$L_G(v)=\{I\in[V(G)]^{(r-1)}\mid I\cup\{v\}\in E(G)\},$$

recalling that $[V(G)]^{(r-1)}$ refers to the family of all $(r-1)$-subsets of $V(G)$.

Given a real $q\geqslant 1$, we say that an r-graph G is *q-almost-regular* if $\Delta(G)\leqslant q\delta(G)$ holds. Given positive constants C,γ, we say that an r-graph G is (C,γ)-dense if $\mathrm{e}(G)\geqslant C(\mathrm{v}(G))^{\gamma}$.

We let $t_H(G)$ denote the number of copies of an r-graph H in an r-graph G. Given a 2-graph F, the *r-expansion* of F is the r-graph obtained by replacing each edge e of F with $e\cup I_e$, where I_e is an $(r-2)$-tuple of new vertices, such that the I_e's are pairwise disjoint for distinct edges e. Note that $F^{(2)}=F$. If G is an r-expansion of a 2-graph F, then we call F a *skeleton* of G. So, for example, the linear r-cycle, $C_m^{(r)}$ is the r-expansion of the 2-uniform m-cycle. Now we define the notion of supersaturation of expansions in linear r-graphs.

Definition 2. Given a 2-graph F with v vertices and e edges, $r\geqslant 2$ and c a positive real, we say that a linear r-graph G *c-supersaturates $F^{(r)}$* if

$$t_{F^{(r)}}(G)\geqslant c\frac{(\mathrm{e}(G))^e}{(\mathrm{v}(G))^{2e-v}}.$$

Note that for $r=2$ this definition is the usual supersaturation for 2-graphs (as in Conjecture 1), that is, a graph G c-supersaturates another graph F with e edges and v vertices if

$$t_F(G)\geqslant c\frac{(\mathrm{e}(G))^e}{(\mathrm{v}(G))^{2e-v}}.$$

As we have discussed earlier, for 2-graphs the bound on the number of copies of $F^{(r)}=F$ in Definition 2 is achieved up to a multiplicative constant by the random graph of the same edge density. For general $r\geqslant 3$, the bound is tight as well, obtained by random subgraphs of appropriate edge density in almost complete partial Steiner systems. The proof of this is quite standard but we include it here for brevity.

Proposition 2.1. *Let F be a 2-uniform graph with v vertices, e edges and no isolated vertices. Let $r \geqslant 2$. For all n sufficiently large and all $0 < E \leqslant n^2/4r^2$ there exists an n-vertex linear r-graph G with $\mathrm{e}(G) \geqslant E$ and $O(E^e/n^{2e-v})$ copies of $F^{(r)}$.*

Proof. Since F has no isolated vertex we must have $v \leqslant 2e$. By our earlier discussion we know there exist almost complete r-uniform Steiner systems, that is, we can find an n-vertex linear r-graph G with $e(G) \geqslant n^2/2r^2$. If $v = 2e$ then the bound we are trying to prove on the number of $F^{(r)}$ is $O(E^e)$, which holds trivially. So we may assume $v < 2e$. Note that the number of copies of $F^{(r)}$ in G is at most n^v. This is because there are at most n^v injections σ from $V(F)$ to $V(G)$, and since G is linear for each $uv \in E(F)$ there is at most one edge of G containing $\{\sigma(u), \sigma(v)\}$.

Now, let $p = 2E/e(G)$. Let H be a random subgraph of G obtained by picking each edge of G independently with probability p. Let X denote the number of edges in H and let Y denote the number of copies of $F^{(r)}$ in H. Then $\mathbb{E}[X] = 2E$ and $\mathbb{E}[Y] \leqslant n^v p^e < E^e/n^{2e-v}$. If $E^e/n^{2e-v} < E$, then by linearity of expectation, we have $\mathbb{E}[X - Y] = \mathbb{E}[X] - \mathbb{E}[Y] \geqslant E$. So there is a subgraph H of G with $X - Y \geqslant E$. By deleting one edge from each copy of $F^{(r)}$ in H, we get a subgraph G' of H with at least E edges and no copy of $F^{(r)}$. Hence we may assume that $E^e/n^{2e-v} > E$, from which we get

$$\mathbb{E}[X] = 2E = \Omega(n^{2-(v-2)/(e-1)}) = \Omega(n^{\alpha}) \quad \text{for some } \alpha > 0$$

(here we use $v < 2e$). Using the Chernoff bound (Lemma 3.4), one can show that $\mathbb{P}[X < E] < 1/2$ for sufficiently large n. Also, by Markov's inequality, $\mathbb{P}[Y > 2\mathbb{E}[Y]] < 1/2$. So with positive probability, H has at least E edges and contains at most $O(E^e/n^{2e-v})$ copies of $F^{(r)}$. □

Let us remark that the bound given in Definition 2 is specific to the setting where the host graph is linear and embedded graphs is an expansion. In a different setting, the supersaturation problem typically becomes very different and the expected optimal count is expected to be different. In fact, as mentioned in the Introduction, the thresholds for forcing even just one copy of $F^{(r)}$ can be very different depending on whether or not we require the host graph to be linear.

An r-graph G is *r-partite* if there exists a partition of $V(G)$ into r subsets $A_1, \ldots, A_r$ such that each edge of G contains exactly one vertex from each A_i. We call $(A_1, \ldots, A_r)$ an *r-partition* of G. Given an r-partite r-graph G with an r-partition $(A_1, \ldots, A_r)$ and $1 \leqslant i < j \leqslant r$, we define the *$(i, j)$-projection* of G, denoted by $P_{i,j}(G)$, to be a 2-graph with edge set

$$E(P_{i,j}(G)) = \{e \cap (A_i \cup A_j) \mid e \in E(G)\}.$$

We call a 2-graph H a *2-projection* of G if $H = P_{i,j}(G)$ for some $1 \leqslant i < j \leqslant r$. The following proposition follows immediately from the definition of $P_{i,j}(G)$ and the linearity of G.

Proposition 2.2. *Let G be a linear r-partite r-graph with an r-partition $(A_1, \ldots, A_r)$. Let $1 \leqslant i < j \leqslant r$. Then the mapping $f\colon E(G) \to E(P_{i,j}(G))$ defined by $f(e) = e \cap (A_i \cup A_j)$ is bijective. In particular, $\mathrm{e}(G) = e(P_{i,j}(G))$.*

Next we give the following slightly technical definition of what we call *projection-restricted supersaturation* in linear r-partite r-graphs. In the next section we show that we can obtain the supersaturation of an expansion $F^{(r)}$ in linear r-graphs from projection-restricted supersaturation in linear r-graphs which have an almost-regular 2-projection.

Definition 3. Given a 2-graph F with v vertices and e edges, $r \geqslant 2$ and c a positive real. For a linear, r-partite r-graph G and any 2-projection of it, say P, we say that (G, P) c-supersaturates $F^{(r)}$ if

$$t_{F^{(r)}}(G) \geqslant c \frac{(\mathrm{e}(P))^e}{(\mathrm{v}(P))^{2e-v}}.$$

Note that for 2-graphs, Definition 3 coincides with Definition 2.

3. Reduction results

Erdős and Simonovits [7] proved the following 'regularization' theorem for 2-graphs.

Theorem 3.1 ([7]). *Let $0 < \alpha < 1$ be a real and $q = 20 \cdot 2^{(1/\alpha)^2}$. There exists $n_0 = n_0(\alpha)$ such that if G is a $(1, 1+\alpha)$-dense graph on $n \geqslant n_0$ vertices then there exists a q-almost-regular subgraph of G, say G', which is $(2/5, 1+\alpha)$-dense such that $v(G') > n^{\alpha(1-\alpha)/(1+\alpha)}$.*

Theorem 3.1 is a useful tool for the Turán problem for H. Indeed, given a dense enough G, we may first find an almost-regular subgraph G' that has similar density to G and look for a copy of H in G'. This theorem itself is not sufficient for establishing supersaturation results for 2-graphs since we aim to force many copies of H in G. By going into G', we might lose many copies of H. What we probably need is the existence of a collection of dense enough almost-regular subgraphs of G which together supply the number of copies of H that we need. Indeed, this is the rough idea behind the following lemma, which may be viewed as some kind of extension of Theorem 3.1. For any s, t integers, $t \geqslant s \geqslant 1$, and a graph H, we define

$$f(H, s, t) = \frac{(\mathrm{e}(H))^s}{(\mathrm{v}(H))^t}.$$

Lemma 3.2. *Let α be a real and s, t integers, where $0 < \alpha < 1$ and $t \geqslant s \geqslant 1$, then there exist positive reals $C_0 = C_0(\alpha, s, t)$ and $q = q_{3.2}(\alpha, s, t)$ such that the following holds. For every $C \geqslant C_0$, if G is a $(C, 1+\alpha)$-dense graph then it contains a collection of edge-disjoint subgraphs $G_1, \ldots, G_m$ satisfying*

(1) *for all $i \in [m]$, G_i is q-almost-regular and $(\frac{1}{4}C, 1+\alpha)$-dense,*
(2) *$\sum_{i=1}^{m} f(G_i, s, t) \geqslant \frac{1}{4^s} f(G, s, t)$.*

Proof. While we specify the choice of q explicitly, we do not do so for C_0. We assume C_0 is sufficiently large as a function of α, s and t. Let

$$p = \left\lceil 2^{\max\{4/\alpha, (2s+t)/(t-s+1)\}} \right\rceil$$

and $q = 8p$. By the definition of p, we have

$$p^{\alpha} \geqslant 16 \quad \text{and} \quad p^{t-s+1} \geqslant 2^{2s+t}. \tag{3.1}$$

Suppose G has n vertices. Let us partition $V(G)$ into p sets $A_1, \ldots, A_p$ of almost equal sizes, *i.e.* each of size $\lceil n/p \rceil$ or $\lfloor n/p \rfloor$, such that A_1 contains vertices of the highest degrees in G. For convenience, we will drop the ceilings and floors in our arguments as doing so does not affect the arguments except for the slight changes to constants.

We now prove our statement by induction on n. When $n < q$ the claim holds trivially, since either G itself is q-almost-regular or no $(C, 1+\alpha)$-dense graph on $n < q$ vertices exists. For the induction step, we consider two cases.

Case 1. The number of edges in G with at least one endpoint in A_1 is at most $\mathrm{e}(G)/2$.

Let $d = d(G)$ be the average degree of G. By our definition of A_1, for each vertex $v \in V(G) \setminus A_1$, we have $d_G(v) \leqslant pd$; otherwise $\sum_{u \in A_1} d_G(u) > pd(n/p) = nd$, a contradiction. Let $G' = G - A_1$. Then

$$\Delta(G') \leqslant pd,$$

and $\mathrm{e}(G') \geqslant \mathrm{e}(G)/2$, by initial assumptions. By iteratively deleting vertices whose degree becomes less than $d/8$, we obtain a subgraph $G'' \subseteq G'$ with

$$e(G'') \geqslant \mathrm{e}(G') - \frac{nd}{8} \geqslant \frac{\mathrm{e}(G)}{4} \quad \text{and} \quad \delta(G'') \geqslant \frac{d}{8}.$$

Since $\Delta(G'') \leqslant \Delta(G') \leqslant pd$ and $\delta(G'') \geqslant d/8$, G'' is $8p$-almost-regular, that is, G'' is q-almost-regular. Also,

$$e(G'') \geqslant \frac{1}{4}\,\mathrm{e}(G) \geqslant \frac{1}{4}C[\,\mathrm{v}(G)]^{1+\alpha} \geqslant \frac{1}{4}C[\,\mathrm{v}(G'')]^{1+\alpha}.$$

Thus, G'' is $(\frac{1}{4}C, 1+\alpha)$-dense. Now

$$f(G'', s, t) = \frac{[e(G'')]^s}{[\,\mathrm{v}(G'')]^t} \geqslant \frac{[\,\mathrm{e}(G)/4]^s}{[\,\mathrm{v}(G)]^t} \geqslant \frac{1}{4^s}\frac{[\,\mathrm{e}(G)]^s}{[\,\mathrm{v}(G)]^t} = \frac{1}{4^s}f(G, s, t).$$

So the claim holds by letting our collection of subgraphs be $\{G''\}$.

Case 2. The number of edges in G with at least one endpoint in A_1 is more than $\frac{1}{2}\,\mathrm{e}(G)$.

For each $i = 2, \ldots, p$, let $G_i = G[A_1 \cup A_i]$, $n_i = \mathrm{v}(G_i)$ and $e_i = \mathrm{e}(G_i)$. Then, for each $i \in \{2, \ldots, p\}$, $n_i = 2n/p$. Also,

$$\sum_{i=2}^{p} e_i \geqslant \frac{\mathrm{e}(G)}{2}.$$

Let $\mathscr{I} = \{2, \ldots, p\}$. Define

$$\mathscr{I}_1 = \{i \in \mathscr{I} : e_i \geqslant Cn_i^{1+\alpha}\} \quad \text{and} \quad \mathscr{I}_2 = \mathscr{I} \setminus \mathscr{I}_1.$$

Recall that $p^\alpha \geqslant 16$. By the definition of $\mathscr{I}_2$ and the fact that $n_i = 2n/p$ for each $i \in [p]$, we have

$$\sum_{i\in\mathscr{I}_2} e_i \leqslant \frac{C}{p^{1+\alpha}} \sum_{i\in\mathscr{I}_2} (2n)^{1+\alpha} \leqslant \frac{C|\mathscr{I}_2|2^{1+\alpha}n^{1+\alpha}}{p^{1+\alpha}} \leqslant \frac{4Cn^{1+\alpha}}{p^\alpha} \leqslant \frac{C}{4}n^{1+\alpha} \leqslant \frac{\mathrm{e}(G)}{4}.$$

Hence

$$\sum_{i\in\mathscr{I}_1} e_i \geqslant \frac{\mathrm{e}(G)}{4}. \tag{3.2}$$

For each $i \in \mathscr{I}_1$ since $e_i \geqslant Cn_i^{1+\alpha}$ and $n_i < n$, by the induction hypothesis, G_i contains a collection of edge-disjoint subgraphs $G_i^1, \ldots, G_i^{m_i}$ each of which is q-almost-regular and $(\frac{1}{4}C, 1+\alpha)$-dense such that

$$\sum_{j=1}^{m_i} f(G_i^j, s, t) \geqslant \frac{1}{4^s}f(G_i, s, t).$$

Hence

$$\sum_{i\in\mathscr{I}_1}\sum_{j=1}^{m_i} f(G_i^j, s, t) \geqslant \frac{1}{4^s}\sum_{i\in\mathscr{I}_1} f(G_i, s, t) = \frac{1}{4^s}\sum_{i\in\mathscr{I}_1}\frac{e_i^s}{n_i^t} = \frac{p^t}{4^s(2n)^t}\sum_{i\in\mathscr{I}_1} e_i^s. \tag{3.3}$$

Hence, by (3.2), (3.3), $p^{t-s+1} \geqslant 2^{2s+t}$, and the convexity of the function x^s, we have

$$\sum_{i\in\mathscr{I}_1}\sum_{j=1}^{m_i} f(G_i^j, s, t) \geqslant \frac{p^t}{4^s(2n)^t}\frac{(\sum_{i\in\mathscr{I}_1} e_i)^s}{|\mathscr{I}_1|^{s-1}} \geqslant \frac{p^{t-s+1}}{4^{2s}2^t}\frac{e^s}{n^t} \geqslant \frac{1}{4^s}f(G, s, t).$$

Hence the claims holds by letting $\{G_i^j : i \in \mathscr{I}_1, 1 \leqslant j \leqslant m_i\}$ be our collection of subgraphs of G. This completes Case 2 and the proof. □

Now we are ready to state our main result of the section. Informally, it says that we can obtain the supersaturation of an expansion $F^{(r)}$ in linear r-graphs from projection-restricted supersaturation in linear r-graphs which have an almost-regular 2-projection.

Theorem 3.3. *Let $\alpha \in (0, 1)$ be a real and $r \geqslant 2$. Let F be a graph with v vertices and e edges, where $e \geqslant v$. There exists a real $q = q(\alpha, F) \geqslant 1$ such that the following holds. Suppose $C, c > 0$ are constants such that, for every linear r-partite r-graph G that has a $(C, 1 + \alpha)$-dense and q-almost-regular 2-projection P, (G, P) c-supersaturates $F^{(r)}$. Then there exist C', c' such that every linear r-partite r-graph that is $(C', 1 + \alpha)$-dense c'-supersaturates $F^{(r)}$.*

Proof. Let $s = e$, $t = 2e - v$. By our assumption, $t \geqslant s \geqslant 1$. We show that the theorem holds for q to be chosen as $q_{3.2}(\alpha, s, t)$, derived from Lemma 3.2 applied with constant α, s and t. Finally, let

$$C' = 4C, \quad c' = \frac{cr^{2e-v}}{2^{4e-v}}.$$

Let G be an r-partite r-graph on n vertices such that $\mathrm{e}(G) \geqslant 4Cn^{1+\alpha}$. Suppose G has an r-partition $(A_1, A_2, \ldots, A_r)$, such that $|A_1| \geqslant |A_2| \geqslant \ldots |A_r|$. It follows that $|A_1 \cup A_2| \geqslant 2n/r$, and $H = P_{1,2}(G)$ is a $(4C, 1 + \alpha)$-dense graph. By Lemma 3.2, there exists a collection of edge-disjoint subgraphs $H_1, \ldots, H_m$ of H, each of which is $(C, 1 + \alpha)$-dense and q-almost-regular, such that

$$\sum_{i=1}^{m} f(H_i, s, t) \geqslant \frac{1}{4^s} f(H, s, t).$$

For each $i \in [m]$, let G_i be the subgraph of G such that $P_{1,2}(G_i) = H_i$. For each $i \in [m]$, since H_i is $(C, 1 + \alpha)$-dense and q-almost-regular, by the hypothesis of the theorem, (G_i, H_i) c-supersaturates $F^{(r)}$. That is,

$$t_{F^{(r)}}(G_i) \geqslant c \frac{(e(H_i))^s}{(v(H_i))^t} = cf(H_i, s, t).$$

Since the H_i's are edge-disjoint, the G_i's are also edge-disjoint (*i.e.* there is no edge contained in two different G_i's). Thus we have

$$\begin{aligned} t_{F^{(r)}}(G) &\geqslant \sum_{i=1}^{m} t_{F^{(r)}}(G_i) \\ &\geqslant c \sum_{i=1}^{m} f(H_i, s, t) \\ &\geqslant \frac{c}{4^s} f(H, s, t) \\ &= \frac{c}{4^e} \cdot \frac{(\mathrm{e}(H))^e}{(\mathrm{v}(H))^{2e-v}} \\ &\geqslant \frac{cr^{2e-v}}{2^{4e-v}} \cdot \frac{(\mathrm{e}(G))^e}{n^{2e-v}}. \end{aligned}$$ □

Theorem 3.3 says that to establish supersaturation of $F^{(r)}$ in an n-vertex linear r-partite r-graph G, we may assume G has a dense enough almost-regular 2-projection P. Our next lemma can be used to show that we may further assume P to have edge density exactly $\Theta(\mathrm{v}(P)^{1+\alpha})$, where α is any fixed real for which $\mathrm{ex}(n, F^{(r)}) = O(n^{1+\alpha})$. The proof uses random sampling and the classical Chernoff bound, which we state here for completeness.

Lemma 3.4 (Corollary 2.3 of [16]). *Given a binomially distributed variable $X \in \mathrm{BIN}(n, p)$ we have $\mathbb{P}(|X - \mathbb{E}[X]| \geqslant a\mathbb{E}[X]) \leqslant 2e^{-(a^2/3)\mathbb{E}[X]}$, as long as $0 < a \leqslant 3/2$.*

Lemma 3.5. *Let $r \geqslant 2$ be an integer. Let $\alpha \in (0, 1)$ be a real. Let F be a graph with v vertices and e edges, where $e \geqslant v$. There is a constant $m_0 = m_0(\alpha)$ such that the following holds for all $M \geqslant m_0$.*

Suppose $D, q, c > 0$ are reals where $q \geqslant 1$ such that, for every linear r-partite r-graph G' that has a 2-projection P' on $m \geqslant M$ vertices satisfying $Dm^\alpha \leqslant \delta(P') \leqslant \Delta(P') \leqslant 3qDm^\alpha$, we have that (G', P') c-supersaturates $F^{(r)}$. Then, for every linear r-partite r-graph G that has a $(qD, 1+\alpha)$-dense and q-almost-regular 2-projection P on at least M vertices, (G, P) $c/2^{e+1}$-supersaturates $F^{(r)}$.

Proof. The choice of m_0 will be given implicitly in the proof. Let G be a linear r-partite graph with an r-partition $(A_1, A_2, \dots, A_r)$, where without loss of generality $P = P_{1,2}(G)$ is $(qD, 1+\alpha)$-dense, q-almost-regular, and has $m = \mathrm{v}(P) \geqslant M$ vertices. The idea is to sample randomly a subgraph G' of G with an appropriate edge probability, count $F^{(r)}$ in G', and then use it to bound the count of $F^{(r)}$ in G. Now let $\delta(P)$, $\Delta(P)$ and d denote the minimum, maximum and average degrees in P, respectively. Then

$$d = \frac{2\,\mathrm{e}(P)}{\mathrm{v}(P)} \geqslant \frac{2qDm^{1+\alpha}}{m} = 2qDm^\alpha.$$

Since P is q-almost-regular, we have $\delta(P) \geqslant d/q \geqslant 2Dm^\alpha$.

Let $p = 2Dm^\alpha/\delta(P)$ and let G' be a random subgraph of G, obtained by including each edge of G in G' independently with probability p. Then

$$\mathbb{E}[\,\mathrm{e}(G')] = p\,\mathrm{e}(G) \quad \text{and} \quad \mathbb{E}[t_{F^{(r)}}(G')] = t_{F^{(r)}}(G)\cdot p^e,$$
$$\mathbb{E}[d_{G'}(v)] = p d_G(v) \quad \text{for all } v \in V(G')v \in V(G').$$

Since G is linear and P is a 2-projection, for each $v \in V(P) = A_1 \cup A_2$ we have $d_G(v) = d_P(v)$. Since P is q-almost-regular, $\Delta(P)/\delta(P) \leqslant q$. So, for each $v \in A_1 \cup A_2$, we have

$$\mathbb{E}[d_{G'}(v)] = p d_G(v) = p d_P(v) \leqslant \frac{2Dm^\alpha}{\delta(P)}\Delta(P) \leqslant 2qDm^\alpha,$$

and similarly $\mathbb{E}[d_{G'}(v)] \geqslant 2Dm^\alpha$.

Now random variables $d_{G'}(v)$ and $\mathrm{e}(G')$ have binomial distributions. Hence, using Markov's inequality and Chernoff's inequality, one can show that

$$\mathbb{P}[t_{F^{(r)}}(G') > 2t_{F^{(r)}}(G)\cdot p^e] < \frac{1}{2},$$
$$\mathbb{P}[\,\mathrm{e}(G') < \frac{p}{2}\,\mathrm{e}(G)] < \frac{1}{4}$$

and

$$\mathbb{P}[\exists v \in A_1 \cup A_2 \text{ such that } d_{G'}(v) < Dm^\alpha \text{ or } d_{G'}(v) > 3qDm^\alpha] < 1/4.$$

In some of the inequalities above, we used Chernoff (and the union bound for the last one). The desired inequalities hold when m is large enough as a function of α, which is guaranteed by choosing m_0 to be large enough. So there exists a subgraph G' of G satisfying

$$\mathrm{e}(G') \geqslant \frac{p}{2}\,\mathrm{e}(G),\ t_{F^{(r)}}(G') \leqslant 2t_{F^{(r)}}(G)\cdot p^e, \tag{3.4}$$

and that for each $v \in A_1 \cup A_2$

$$Dm^\alpha \leqslant d_{G'}(v) \leqslant 3qDm^\alpha. \tag{3.5}$$

Now let $P' = P_{1,2}(G')$. Since there is a bijection between $E(G')$ and $E(P')$, for each $v \in V(P)$, we have $d_{P'}(v) = d_{G'}(v)$. By (3.5), for each $v \in V(P') = A_1 \cup A_2$, we have

$$Dm^\alpha \leqslant d_{P'}(v) \leqslant 3qDm^\alpha.$$

Thus, by the hypothesis of our theorem, (G', P') c-supersaturates $F^{(r)}$. By (3.4), we have

$$t_{F^{(r)}}(G) \geqslant \frac{1}{2p^e} t_{F^{(r)}}(G') \geqslant \frac{c}{2p^e}\cdot\frac{(\mathrm{e}(G'))^e}{m^{2e-v}} \geqslant \frac{c}{2p^e}\cdot\frac{p^e(\mathrm{e}(G))^e}{2^e m^{2e-v}} = \frac{c}{2^{e+1}}\cdot\frac{(\mathrm{e}(G))^e}{m^{2e-v}}. \qquad \square$$

Applying Theorem 3.3 and Lemma 3.5 we obtain the following reduction tool for supersaturation of expansions. In this paper we use it on $C_{2k}^{(r)}$.

Corollary 3.6. *Let $r \geqslant 2$ be an integer and $\alpha \in (0, 1)$ a real. Let F be a graph with v vertices and e edges, where $e \geqslant v$. Let $q = q_{3.3}(\alpha, F)$ be given as in Theorem 3.3 and $m_0 = m_0(\alpha)$ be given as in Lemma 3.5. Suppose there exist reals $D, \lambda, M, c > 0$, where $\lambda \geqslant 3q$, $M \geqslant m_0$, such that, for every linear r-partite r-graph G that has a 2-projection P on $m \geqslant M$ vertices satisfying $Dm^\alpha \leqslant \delta(P) \leqslant \Delta(P) \leqslant \lambda Dm^\alpha$, we have that (G, P) c-supersaturates $F^{(r)}$. Then there exist C', c' such that every $(C', 1+\alpha)$-dense linear r-graph G c'-supersaturates $F^{(r)}$.*

Proof. Suppose there exist reals $D, \lambda > 0$, where $\lambda \geqslant 3q$, such that, for every linear r-partite r-graph G that has a 2-projection P on $m \geqslant M$ vertices satisfying $Dm^\alpha \leqslant \delta(P) \leqslant \Delta(P) \leqslant \lambda Dm^\alpha$, we have that (G, P) c-supersaturates $F^{(r)}$. By Lemma 3.5, there exists a constant $c_{3.5}$ such that, for every linear r-partite r-graph G that has a $(qD, 1+\alpha)$-dense q-almost-regular 2-projection P on at least M vertices, (G, P) $c_{3.5}$-supersaturates $F^{(r)}$. Set $C = \max\{qD, M\}$. Then, for every linear r-partite r-graph G that has a $(C, 1+\alpha)$-dense q-almost-regular 2-projection P, we have that (G, P) $c_{3.5}$-supersaturates $F^{(r)}$ (as P must have at least $C \geqslant M$ vertices). Applying Theorem 3.3 with the C above and $c = c_{3.5}$, there exist constant $C'_{3.3}$ and $c'_{3.3}$ such that every linear r-partite r-graph that is $(C'_{3.3}, 1+\alpha)$-dense $c'_{3.3}$-supersaturates $F^{(r)}$.

Let $C' = (r^r/r!)C'_{3.3}$. Let G be a linear $(C', 1+\alpha)$-dense r-graph. By a well-known fact, G contains an r-partite subgraph G' with $\mathrm{e}(G') \geqslant (r!/r^r)\,\mathrm{e}(G)$. Clearly, G' is $(C'_{3.3}, 1+\alpha)$-dense. By our discussion above, G' $c'_{3.3}$-supersaturates $F^{(r)}$. Hence G $c'_{3.3}$-supersaturates $F^{(r)}$. □

Corollary 3.7. *Let $r, k \geqslant 2$ be integers. There exist constants m_k, q_k depending only on k such that the following holds. Suppose there are reals $D, \lambda, M, c > 0$, where $\lambda \geqslant q_k D$ and $M \geqslant m_k$, such that, for every n-vertex linear r-partite r-graph G that has a 2-projection P on $m \geqslant M$ vertices satisfying $Dm^{1/k} \leqslant \delta(P) \leqslant \Delta(P) \leqslant \lambda Dm^{1/k}$, we have $t_{C_{2k}^{(r)}}(G) \geqslant cm^2$. Then there exist constants C', c' such that every n-vertex linear r-graph G with $\mathrm{e}(G) \geqslant C'n^{1+1/k}$ satisfies $t_{C_{2k}^{(r)}}(G) \geqslant c'(\mathrm{e}(G)/n)^{2k}$.* □

4. Overview of the proof of Theorem 1.1

In this short section, we give a brief outline of the proof strategy for $r = 2$. The strategy for $r \geqslant 3$ is similar and relies on establishing some coloured version of the $r = 2$ case. See Section 6 for details. By Corollary 3.7, to prove Theorem 1.1 for $r = 2$, it suffices to prove that every almost-regular n-vertex bipartite graph with average degree $\Theta(n^{1/k})$ contains at least $\Omega(n^2)$ linear cycles of length $2k$. Suppose we are given an n-vertex almost-regular bipartite graph G with average degree $\Theta(n^{1/k})$. The key ingredient is to show that for each vertex x, there exists a $j = j(x) \in [k-1]$ such that the number of C_{2k}'s containing some vertex that lies in the jth distance classes from x is at least $\Omega(n^{1+j/k})$. Suppose we have established that. Then, for some $t \in [k-1]$, there are $\Omega(n)$ different vertices x in G with $j(x) = t$. Since $\Delta(G) = O(n^{1/k})$ by almost-regularity, any vertex in G lies in the jth distance class of at most $O(n^{t/k})$ different vertices x. Hence G contains at least $\Omega(n \cdot n^{1+t/k}/n^{t/k}) = \Omega(n^2)$ distinct C_{2k}'s and we are done.

To establish the above-mentioned key ingredient, we adapt the Faudree–Simonovits method as follows. From each vertex x we use breadth-first search (BFS) to obtain the distance classes $L_0 = \{x\}, L_1, L_2, \ldots$. Using the minimum degree assumption, we can find the smallest $i \in [k-1]$ such that the subgraph $G[L_i \cup L_{i+1}]$ of G induced by $L_i \cup L_{i+1}$ has average degree at least some given constant, depending on k and i. The choice of i together with the condition $\delta(G) = \Omega(n^{1/k})$

also ensures that $|L_i| = \Omega(n^{i/k})$ and $e(G[L_i \cup L_{i+1}]) = \Omega(n^{(i+1)/k})$. Let T be a BFS tree rooted at x with vertex set $L_0 \cup L_1 \cup \cdots \cup L_i$. We divide vertices in L_{i+1} into L^+_{i+1} and L^-_{i+1}, where the former consists of those in L_{i+1} that send edges in G to many components of $T - x$ (*i.e.* many different subtrees of T under x) and the latter consists of the rest of L_{i+1}. If most of the edges of $G[L_i \cup L_{i+1}]$ go into L^+_{i+1}, we build many paths of length $2k - 2i$ in $G[L_i \cup L_{i+1}]$ whose two ends lie in L_i and under different children of x in T. These paths then extend to different C_{2k}'s going through x. If most edges of $G[L_i \cup L_{i+1}]$ go into L^-_{i+1}, then we do some cleaning to find vertex-disjoint subgraphs of $G[L_i \cup L_{i+1}]$ which together capture at least a constant proportion of the edges of $G[L_i \cup L_{i+1}]$ such that each is attached to a different component of $T - x$. Since the height of these components of $T - x$, viewed as subtrees rooted at children of x, is one shorter than that of T, we can apply some form of induction to show that these subgraphs together with the different subtrees of T that they are attached to collectively supply the needed collection of C_{2k}'s.

Let us now describe a variant of this method, which avoids induction (on the height of a BFS tree) and appears more versatile. The key to the Faudree–Simonovits method is that any path of length p in $G[L_i \cup L^+_{i+1}]$ that starts in some vertex a in L_i and ends in some vertex b in L^+_{i+1} can be extended at b to end at a vertex c in L_i so that a and c lie under different children of x in T. This allows us to round out the path into a cycle through x of length $p + 2i + 1$, as the paths in T from a and c to x are internally disjoint. The useful feature of a vertex v in L^+_{i+1} is that it is well-linked to x by some measure. A variant to the Faudree–Simonovits method is to locate for each vertex v in L_{i+1} some vertex $r(v)$ in T (which may not be x itself) that v is well-linked to, which we may call the 'local root' for v (see Lemma 5.1 for details). Then we partition L_{i+1} based on the level $r(v)$ lies in over all $v \in L_{i+1}$. This new approach has the advantage that even if T is replaced with some non-tree structure, we will still be able to extend a path in $G[L_i \cup L_{i+1}]$ to a cycle through a local root in a well-controlled way. Indeed, in [5], this variant is key to establishing that $\mathrm{ex}_\ell(n, C^{(r)}_{2k+1}) = O(n^{1+1/k})$.

In our proof of Theorem 1.1 we will use the Faudree–Simonovits inductive approach for $r \geqslant 3$ and the 'local root' approach for $r = 2$. As we mentioned before, one could prove Theorem 1.1 in its generality using one of the two approaches.

5. Supersaturation of even cycles in graphs

5.1 Preliminary lemmas

Lemma 5.1. *Let T be a tree of height h with a root x. For each $v \in V(T)$, let T_v be the subtree of T rooted at v. Let b be a positive integer. Let S be a set of at least $bh + 1$ vertices in T. Then there exists a vertex y at distance i from x, for some $0 \leqslant i \leqslant h - 1$, such that $|V(T_y) \cap S| \geqslant |S| - ib$ and that for any child z of y in T, $|V(T_z) \cap S| \leqslant |V(T_y) \cap S| - b$.*

Proof. We define a sequence of vertices as follows. Let $x_0 = x$. Among all the children of x_0, let x_1 be one such that T_{x_1} contains the maximum number of vertices in S. Among all the children of x_1, let x_2 be one that contains the maximum number of vertices in S, and so on. Suppose the sequence we define this way is $x_0, x_1, \ldots, x_p$, where $p \leqslant h$ and $|V(T_{x_p}) \cap S| = 1$. Since $|V(T_{x_0}) \cap S| \geqslant bh + 1$ and $|V(T_{x_p}) \cap S| = 1$, there must exist a smallest index $0 \leqslant i < p$ such that $|V(T_{x_{i+1}}) \cap S| \leqslant |V(T_{x_i}) \cap S| - b$. Let $y = x_i$. Then y satisfies the claim. $\square$

Given a bipartite graph G with a bipartition (A, B), we will use $d_A(G)$ and $d_B(G)$ to denote the average degree in G of vertices in A and in B, respectively. Also, we will use $\delta_A(G)$ and $\delta_B(G)$ to denote the minimum degree in G of vertices in A and in B, respectively. When the graph G is clear, we will drop G from the notation above and write $d_A, d_B, \delta_A, \delta_B$, respectively.

Lemma 5.2. *Given a bipartite graph G with a bipartition (A, B), there exists a subgraph G' of G with a bipartition (A', B') where $A' \subseteq A$, $B' \subseteq B$ such that $\mathrm{e}(G') \geqslant \frac{1}{2}\,\mathrm{e}(G)$ and that $\delta_{A'}(G') \geqslant \frac{1}{4} d_A(G)$ and $\delta_{B'}(G') \geqslant \frac{1}{4} d_B(G)$.*

Proof. Let $d_A = d_A(G)$ and $d_B = d_B(G)$. Let us iteratively delete any vertex in A whose degree becomes less than $\frac{1}{4} d_A$ and any vertex in B whose degree becomes less than $\frac{1}{4} d_B$. We continue until we no longer have such vertices or run out of vertices. Let G' denote the remaining graph. Let A' denote the set of remaining vertices in A and B' the set of remaining vertices in B. The number of edges removed in the process is at most

$$\frac{1}{4} d_A |A| + \frac{1}{4} d_B |B| \leqslant \frac{1}{4}\,\mathrm{e}(G) + \frac{1}{4}\,\mathrm{e}(G) = \frac{\mathrm{e}(G)}{2}.$$

Hence G' is non-empty. By the procedure, each vertex in A' has degree at least $\frac{1}{4} d_A$ and each vertex in B' has degree at least $\frac{1}{4} d_B$ in G'. □

We also need the following crude bound on the number of paths of a given length in an asymmetric bipartite graph. Even though sharper estimates exist in the literature, the lemma suffices for our purposes and is self-contained.

Lemma 5.3. *Let p be a positive integer. Let G be a bipartite graph with a bipartition (A, B). Let d_A, d_B denote the average degrees of vertices in A and in B, respectively. Suppose $d_A, d_B \geqslant 8p$. Then the number of paths of length $2p + 1$ in G is at least*

$$\frac{1}{2^{6p+1}}\,\mathrm{e}(G)(d_A d_B)^p.$$

Proof. By Lemma 5.2, G contains a subgraph G' with a bipartition (A', B') such that $\mathrm{e}(G') \geqslant \frac{1}{2}\,\mathrm{e}(G)$ and that $\delta_{A'}(G') \geqslant \frac{1}{4} d_A$, $\delta_{B'}(G') \geqslant \frac{1}{4} d_B$. Consider growing a $(2p - 1)$-path $v_1 v_2 \ldots v_{2p}$ where $v_1 \in A$, $v_{2p} \in B$. There are $\mathrm{e}(G')$ ways to pick $v_1 v_2$. Then there are at least $\prod_{i=1}^{p} (\delta_{B'} - i)(\delta_{A'} - i)$ ways to pick the remaining vertices one by one. Since $\delta_{A'} \geqslant \frac{1}{4} d_A \geqslant 2p$ and $\delta_{B'} \geqslant \frac{1}{4} d_B \geqslant 2p$, we have $\delta_{A'} - i \geqslant \frac{1}{2} d_{A'} \geqslant \frac{1}{8} d_A$ and $\delta_{B'} - i \geqslant \frac{1}{2} d_{B'} \geqslant \frac{1}{8} d_B$ for all $i \in [p]$. Hence the number of $(2p + 1)$-paths in G is at least

$$\frac{1}{2}\,\mathrm{e}(G) \cdot \frac{1}{2^{6p}} (d_A d_B)^p = \frac{1}{2^{6p+1}}\,\mathrm{e}(G)(d_A d_B)^p. \qquad \square$$

5.2 Proof of Theorem 1.1 for r = 2

Lemma 5.4. *Let h, k be positive integers where $h \leqslant k$. Let G be a graph. Let T be a tree of height h in G with a root x. For each $i \in [h]$, let L_i be the set of vertices at distance i from x in T. Let W be a set of vertices in $V(G) \setminus V(T)$. Let F denote the bipartite subgraph of G containing all the edges of G between L_h and W. Let d_L, d_W be the average degree in F of vertices in L_h and in W, respectively. Suppose $d_L, d_W \geqslant 16k^2$. Then there exists $j \in [h]$ such that the number of C_{2k}'s in G that contain some vertex in L_j is at least $\alpha_k |L_h| d_L^{k-h+j} d_W^{k-h+j-1}$, where*

$$\alpha_k = \frac{1}{2^{6k}} \left(\frac{1}{2k}\right)^{2k-2}.$$

Proof. For each vertex y in T, let T_y denote the subtree of T rooted at y. We clean up F to get a subgraph F' of F as follows. First we delete vertices w in W with $d_F(w) \leqslant kh$. Let W' denote the set of remaining vertices in W. Let $w \in W'$. Applying Lemma 5.1 to T and $S = N_F(w)$, we conclude that there exists some vertex $r(w) \in L_j$ for some $j \in [h-1]$ such that there are at least $|N_F(w)| - kj$ members of $N_F(w)$ that lie in $T_{r(w)}$. Furthermore, for any child z of $r(w)$ in T, there are at least k members of $N_F(w) \cap V(T_{r(w)})$ that lie outside T_z. To form F', we include edges between w and $N_F(w) \cap V(T_{r(w)})$ for each $w \in W'$. By our assumptions, in forming F' from F we have deleted at most kh edges incident to each $w \in W$. Hence

$$e(F') \geqslant e(F) - kh|W|.$$

For each $j \in [h-1]$, let $W_j = \{w \in W' : r(w) \in L_j\}$ and let F_j be the subgraph of F' induced by $L_h \cup W_j$. Let us choose an $j \in [h-1]$ such that $e(F_j)$ is maximum. Then (as $h \leqslant k$)

$$e(F_j) \geqslant \frac{1}{h} e(F') \geqslant \frac{1}{k} e(F') \geqslant \frac{1}{k}(e(F) - kh|W|) \geqslant \frac{1}{2k} e(F), \tag{5.1}$$

where the last inequality follows from the fact that $e(F) = d_W|W| \geqslant 16k^2|W| \geqslant 2kh|W|$.

Suppose $L_j = \{z_1, z_2, \ldots, z_t\}$. For each $e \in [t]$, let $S_e = L_h \cap V(T_{z_e})$. By our definition of F' and F_j, in F_j each $w \in W_j$ has edges to precisely one S_e. Let $N_e = N_{F_j}(S_e)$. Then $N_1, \ldots, N_t$ partition W_j and F_j is the vertex-disjoint union of $F_j[S_e \cup N_e]$, for $e \in [t]$. Let $m = k - (h - j)$.

Claim 1. *Every $(2m-1)$-path P in F_j extends to a C_{2k} in G that contains a vertex in L_j.*

Proof of claim. Consider any $(2m-1)$-path P in F_j. By our discussion, $P \subseteq F_j[S_e \cup N_e]$ for some $e \in [t]$. Suppose $Q = v_1 v_2 \ldots v_{2m}$, where $v_1 \in S_e$ and $v_{2m} \in N_e$. Then $r(v_2) = r(v_{2m}) = z_e$. Let a denote the child of z_e in T_{z_e} such that v_1 lies under a. Since $r(v_{2m}) = z_e$, by definition, v_{2m} has at least k neighbours in T_{z_e} that lies outside T_a. Among them, at least one, say u, lies outside $V(P)$. Let Q, Q' denote the unique (v_1, z_e)-path and the unique (u, z_e)-path in T_{z_e}, respectively. Since v_1 and u lie under different children of z_e, $V(Q) \cap V(Q') = \{z_e\}$. Now $P \cup v_{2m}u \cup Q \cup Q'$ is a cycle of length $2m - 1 + 1 + 2(k - m) = 2k$ in G that contains z_e. □

By Claim 1 the number of C_{2k}'s in G that contain a vertex in L_j is at least the number of $(2m-1)$-paths in F_j. To complete our proof, it suffices to find a corresponding lower bound on the number of $(2m-1)$-paths in F_j. For convenience, let $A = L_h$ and $B = W_j$. Let d_A, d_B denote the average degrees in F_j of vertices in A and B, respectively. By (5.1),

$$d_A \geqslant \frac{1}{2k} d_L \geqslant 8k \geqslant 8m \quad \text{and} \quad d_B \geqslant \frac{1}{2k} d_W \geqslant 8k \geqslant 8m.$$

By Lemma 5.3 with $p = m - 1$, the number of $(2m-1)$-paths in F_j is at least

$$\begin{aligned} \frac{1}{2^{6(m-1)+1}} e(F_j)[d_A d_B]^{m-1} &\geqslant \frac{1}{2^{6m}} \left(\frac{1}{2k}\right)^{2m-2} e(F_j) d_L^{m-1} d_W^{m-1} \\ &\geqslant \alpha_k |L_h| d_L^m d_W^{m-1} \\ &= \alpha_k |L_h| d_L^{k-h+j} d_W^{k-h+j-1}, \end{aligned}$$

where

$$\alpha = \frac{1}{2^{6k}} \left(\frac{1}{2k}\right)^{2k-2}.$$

This completes our proof. □

In the next theorem, we use Lemma 5.4 to quickly obtain the desired lower bound on the number of C_{2k}'s in almost-regular n-vertex graphs whose number of edges is $\Theta(n^{1+1/k})$.

Theorem 5.5. *Let $k \geqslant 2$ be an integer. Let $D, \lambda > 0$ be constants where $D \geqslant 64k^2$ and $\lambda \geqslant 1$. Let $n_0 = (8\lambda)^k$. Let G be an n-vertex bipartite graph, $n \geqslant n_0$ such that, for each $v \in V(G)$, $Dn^{1/k} \leqslant d(v) \leqslant \lambda Dn^{1/k}$. Then there exists a positive constant $\beta = \beta(D, \lambda, k)$ such that $t_{C_{2k}}(G) \geqslant \beta n^2$.*

Proof. For each $x \in V(G)$, let $L_i(x)$ denote the set of vertices at distance i from x. Let $h(x)$ be the minimum $i \leqslant k-1$ such that $|L_{i+1}(x)|/|L_i(x)| < n^{1/k}$. Clearly $h(x)$ exists or else we run out of vertices. Let $h = h(x)$. Let T be a BFS tree rooted at x that includes $L_0(x), L_1(x), \ldots, L_h(x)$. By our assumption,

$$|L_h(x)| \geqslant n^{h/k} \quad \text{and} \quad |L_{h+1}(x)| < n^{1/k}|L_h(x)|.$$

Recall that $|L_{i+1}(x)|/|L_i(x)| \geqslant n^{1/k}$ for all $i = 0, 1, \ldots, h-1$. Since $n \geqslant n_0 \geqslant (8\lambda)^k$, $n^{1/k} \geqslant 8\lambda$. By our assumption,

$$|V(T) \setminus L_h(x)| \leqslant |L_h| \sum_{i=1}^{h-1} \left(\frac{1}{8\lambda}\right)^i \leqslant \frac{1}{4\lambda}|L_h(x)|.$$

Let F be the bipartite subgraph of G consisting of all the edges of G between $L_h(x)$ and $L_{h+1}(x)$. The total number of edges of G incident to $L_h(x)$ is at least $Dn^{1/k}|L_h(x)|/2$. Among them, the number of edges that are incident to $V(T) \setminus L_h(x)$ is at most

$$(1/4\lambda)\lambda Dn^{1/k}|L_h(x)| = (1/4)Dn^{1/k}|L_h(x)|.$$

Hence

$$e(F) \geqslant (1/4)Dn^{1/k}|L_h(x)|.$$

Let d_A, d_B denote the average degrees in F of vertices in $L_h(x)$ and $L_{h+1}(x)$, respectively. Then

$$d_A \geqslant (1/4)Dn^{1/k} \geqslant 16k^2 n^{1/k} \geqslant 16k^2,$$

and

$$d_B \geqslant (1/4)Dn^{1/k}|L_h(x)|/|L_{h+1}(x)| \geqslant (1/4)D \geqslant 16k^2.$$

By Lemma 5.4, there exists a $j \in [h-1]$ such that the number of C_{2k}'s in G that contain a vertex in $L_j(x)$ is at least

$$\alpha_k |L_h| d_A^{k-h+j} \geqslant \alpha_k n^{h/k} (n^{1/k})^{k-h+j} = \alpha_k n^{1+j/k}.$$

Let us denote this j value by $j(x)$. For each $t \in [h-1]$, let $S_t = \{x \in V(G) \colon j(x) = t\}$. By the pigeonhole principle, for some $t \in [h-1]$, we have $|S_t| \geqslant n/(h-1)$. Let us fix such a t. By our discussion, for each $x \in S_t$, the number of C_{2k}'s that contain a vertex in $L_t(T_x)$ is at least $\alpha_k n^{1+t/k}$. On the other hand, a vertex y lies in $L_t(T_x)$ for at most $[\lambda Dn^{1/k}]^t$ different x. Hence the number of distinct C_{2k}'s in G is at least

$$|S_t| \alpha_k n^{1+t/k} / \lambda^t D^t n^{t/k} \geqslant (\alpha_k / k\lambda^k D^k) n^2.$$

The claim holds by setting $\beta = \alpha_k / k\lambda^k D^k$. □

Proof of the $r = 2$ case of Theorem 1.1. Theorem 5.5 applies as long as $D \geqslant 64k^2$, $\lambda \geqslant 1$, and $n_0 \geqslant (8\lambda)^k$. To apply Corollary 3.7, we set $D = \max\{64k^2, m_k\}$, $\lambda = q_k D$ and $M_k = (8\lambda)^k$, where m_k, q_k are as given in Corollary 3.7. The claim follows readily from Corollary 3.7. □

6. Supersaturation of even linear cycles in linear hypergraphs

6.1 Notation and preliminary results

Let H be a graph and S be a set of vertices, where possibly $S \cap V(H) \neq \emptyset$. Let φ be any colouring of the edges of H using non-empty subsets of S. Given any subgraph F of H, we let

$$\mathscr{C}(F) = \bigcup_{e \in E(F)} \varphi(e),$$

and call it the *colour set of F under* φ. We say that φ is *strongly proper* on H if, for any $e, e' \in E(H)$ that share a vertex, we have $\varphi(e) \cap \varphi(e') = \emptyset$. We say that φ is *rainbow* on F (or that F is *rainbow under* φ) if, for every two edges e, e' in F, we have $\varphi(e) \cap \varphi(e') = \emptyset$ and that $\mathscr{C}(F)$ is disjoint from $V(F)$. Note that if φ uses $(r-2)$-subsets of S and F is rainbow under φ, then $F \cup \mathscr{C}(F)$ forms an r-expansion of F. Observe that if G is an r-partite linear r-graph with an r-partition $(A_1, \ldots, A_r)$, then the *natural* colouring φ of $P_{i,j}(G)$, where, for all $f \in E(P_{i,j}(G))$, $\varphi(f)$ is the unique $(r-2)$-tuple I_f for which $f \cup I_f \in E(G)$, is strongly proper on $P_{i,j}(G)$ by the linearity of G.

Let G be an r-graph and $v \in V(G)$. Recall the definition of $L_G(v)$ from the Introduction. For any subset $S \subseteq V(G)$, we let $L_G(v)|_S$ denote the restriction of the link of v to S, that is,

$$L_G(v)|_S = \{I \subseteq S \mid I \in L_G(v)\}.$$

We give a very crude analogue of Lemma 5.3, this time counting rainbow paths of a given length in an asymmetric bipartite graph.

Lemma 6.1. *Let p, m be positive integers and H be a bipartite graph with a bipartition (A, B). Let φ be a strongly proper edge-colouring of H using m-sets. If $\delta_A, \delta_B \geqslant 4p(m+1)$ then the number of rainbow paths of length $2p+1$ in H is at least*

$$\frac{1}{2^{2p}} \mathrm{e}(H)(\delta_A \delta_B)^p.$$

Proof. Consider growing a rainbow path $P = v_1 v_2 \ldots v_{2p+2}$ where $v_1 \in A$ and $v_{2p+2} \in B$. There are $\mathrm{e}(H)$ choices for $v_1 v_2$. In general, suppose the subpath $P_t = v_1 v_2 \ldots v_t$ has been grown, where $2 \leqslant t \leqslant 2p+1$. If $v_t \in A$ then we let v_{t+1} be a neighbour of v_t in B such that $\{v_{t+1}\} \cup \varphi(v_t v_{t+1})$ is disjoint from $(V(P_t) \setminus \{v_t\}) \cup \mathscr{C}(P_t)$. If $v_t \in B$, v_{t+1} is defined symmetrically. Assume first that $v_t \in A$. Note that

$$|(V(P_t) \setminus \{v_t\}) \cup \mathscr{C}(P_t))| \leqslant t - 1 + (t-1)m \leqslant 2p(m+1).$$

Since φ is strongly proper, the set $\{u \cup \varphi(u) \colon u \in N_H(v_t)\}$ is an $(m+1)$-uniform matching of size $d_H(v_t)$. At most $2p(m+1)$ of these members contain a vertex in $(V(P_t) \setminus \{v_t\}) \cup \mathscr{C}(P_t)$. So there are at least $d_H(v_t) - 2p(m+1) \geqslant \delta_A - 2p(m+1) \geqslant \frac{1}{2}\delta_A$ choices for v_{t+1}. Similarly, if $v_t \in B$, there are at least $\frac{1}{2}\delta_B$ choices for v_{t+1}. Hence the number of ways to grow P is at least

$$\mathrm{e}(H)\left(\frac{1}{2}\delta_A\right)^p \left(\frac{1}{2}\delta_B\right)^p = \frac{1}{2^{2p}} \mathrm{e}(H)(\delta_A \delta_B)^p. \qquad \square$$

Lemma 6.2 (splitting lemma)**.** *Suppose we are given $D \in \mathbb{R}^+$, $\gamma \in (0,1)$ and integers $k, r \geqslant 2$. There exists $n_0 = n_{6.2}(D, k, r, \gamma)$ such that, for all $n \geqslant n_0$, if G is a linear r-partite r-graph such that two of its r-partition classes, say A and B, satisfy that $|A \cup B| = n$ and that $|L_G(v)| \geqslant Dn^\gamma$ for each $v \in A \cup B$, then there exists a partition of $V(G)$ into $S_1, S_2, \ldots, S_k$ such that, for every $v \in A \cup B$ and every $i \in [k]$, we have*

$$|L_G(v)|_{S_i}| \geqslant \frac{Dn^\gamma}{2k^{r-1}}.$$

Proof. Let us independently assign each vertex x in $V(G)$ a colour from $[k]$ chosen uniformly at random. Let S_i be the vertices of assigned colour i. For a vertex $v \in A \cup B$, we let $X_i(v)$ denote the number of edges (which are $(r-1)$-sets) in $L_G(v)$ that are completely contained in S_i. For each $I \in L_G(v)$,

$$\mathbb{P}[I \subseteq S_i] = \frac{1}{k^{r-1}}.$$

Since G is linear, edges in $L_G(v)$ are pairwise disjoint. Hence the events $\{I \subseteq S_i\}$ are independent for different $I \in L_G(v)$'s. Therefore $X_i(v)$ has the binomial distribution BIN $(d_G(v), 1/k^{r-1})$. Writing d for $d_G(v)$, we have $\mathbb{E}(X_i(v)) = d/k^{r-1}$. By the Chernoff bound,

$$\mathbb{P}\left[X_i(v) < \frac{d}{2k^{r-1}}\right] \leqslant P\left[\left|X_i(v) - \frac{d}{k^{r-1}}\right| > \frac{d}{2k^{r-1}}\right] < 2e^{-d/(12k^{r-1})} < 2e^{-Dn^{\gamma}/(12k^{r-1})}.$$

Therefore the probability that the event $\{X_i(v) < d/2k^{r-1}\}$ occurs, for some vertex $v \in A \cup B$ and some $i \in [k]$, is less than

$$kn \cdot 2e^{-Dn^{\gamma}/(12k^{r-1})} < 1,$$

when n_2 is large enough and $n \geqslant n_2$. Thus there exists some colouring which guarantees that every vertex $v \in A \cup B$ satisfies

$$|L_G(v)|_{S_i}| \geqslant \frac{d}{2k^{r-1}} \geqslant \frac{Dn^{\gamma}}{2k^{r-1}}. \qquad \square$$

Before we establish supersaturation of $C_{2k}^{(r)}$'s in linear r-partite r-graphs that have an almost-regular 2-projection with the right density, we need another lemma. Given an r-graph G, where $r \geqslant 2$, and $S \subseteq V(G)$, S is a *vertex cover* of G if S contains at least one vertex of each edge of G.

Lemma 6.3. *Let $r \geqslant 2$. Let G be an r-graph and S a vertex cover of G. There exist a subset $S' \subseteq S$ and a subgraph $G' \subseteq G$ such that* $\mathrm{e}(G') \geqslant (r/2^r)\,\mathrm{e}(G)$ *and that, for all $e \in E(G')$, $|e \cap S'| = 1$.*

Proof. Let S' be a random subset of S obtained by including each vertex of S randomly and independently with probability $1/2$. Let e be any edge of G. Suppose $|e \cap S| = m$. Then $1 \leqslant m \leqslant r$. The probability that exactly one of these m vertices of $e \cap S$ is chosen for S' is $m/2^m \geqslant r/2^r$. So the expected number of edges of G that meet S' in exactly one vertex is at least $(r/2^r)\,\mathrm{e}(G)$. So there exists $S' \subseteq S$ such that at least $(r/2^r)\,\mathrm{e}(G)$ of the edges meet S' in exactly one vertex. Let G' be the subgraph of G consisting of these edges. $\square$

6.2 Rainbow rooted trees

We now introduce the following adaptation of the BFS tree to linear hypergraphs.

Definition 4 (maximal rooted rainbow tree)**.** Given $r \geqslant 3$, let G be a linear r-partite r-graph with two of its partition classes being A and B, and let $t \geqslant 0$ be an integer. Suppose there exists a partition of $V(G)$ into $S_1, S_2, \ldots, S_t$ such that, for every $v \in A \cup B$ and for every $i \in [t]$,

$$L_G(v)|_{S_i} \neq \emptyset \tag{6.1}$$

For every $x \in A \cup B$, we define a tree T_x, rooted at x and of height t, together with a colouring φ of its edges by $(r-2)$-sets as follows. We define the tree by defining its levels L_i iteratively. The L_i's will alternate between being completely inside A and being completely inside B. Without loss of generality, suppose $x \in A$. The tree T_x is defined symmetrically if $x \in B$.

(1) Let $L_0 = \{x\}$.
(2) Having defined L_i, we define L_{i+1} as follows. Without loss of generality, suppose $L_i \subseteq A$. Let $F = \bigcup_{v \in L_i} L_G(v)|_{S_{i+1}}$. Since G is r-partite with A, B being two partite sets and $L_i \subseteq A$, $V(F)$ is disjoint from A and each edge of F contains exactly one vertex in B. Let M_{i+1} be a maximum matching in F. By condition (6.1), M_{i+1} is non-empty. Define

$$L_{i+1} = M_{i+1}|_B = \{b \in B \mid \exists I \in M_{i+1} \text{ such that } b \in I\}.$$

It remains to define how the vertices of L_i are connected to L_{i+1}. For each $b \in L_{i+1}$, there exists a unique $I_b \in M_{i+1}$ which contains b, and due to linearity of G there is a unique $v \in L_i$ such that $I_b \cup \{v\} \in E(G)$. We add the edge vb to T_x and let $\varphi(vb) = I_b \setminus \{b\}$.
(3) Repeat step (2) until all vertices of G are exhausted or $i > t$.

Proposition 6.4. *Under the assumptions of Definition 4, T_x is a tree of height t rooted at x that is rainbow under the assigned colouring φ. In particular, if P is a path in T_x then $P \cup \mathscr{C}(P)$ is a linear path of the same length in G with P being its skeleton.*

Proof. It is clear from the definition that T_x is a height t tree rooted at x. We now show that T_x is rainbow under c. By the way we define T_x and c, $\mathscr{C}(T_x) \cap V(T_x) = \emptyset$. Let e, e' be any two edges in T_x. Suppose e joins a vertex in L_i to L_{i+1} and e' joins a vertex in $L_{i'}$ to $L_{i'+1}$. If $i \neq i'$, then $\varphi(e) \cap \varphi(e') = \emptyset$, since $\varphi(e) \subseteq S_{i+1}$ and $\varphi(e') \subseteq S_{i'+1}$ and $S_{i+1} \cap S_{i'+1} = \emptyset$. If $i = i'$, then $e \subseteq I$ and $e' \subseteq I'$ for two different members $I, I' \in M_{i+1}$. Since M_{i+1} is a matching, $\varphi(e) \cap \varphi(e') = \emptyset$. So T_x is rainbow under c.

The second statement follows immediately from our discussion in Section 6.1 that a rainbow subgraph F together with its colours forms an expansion of F. □

We are now ready to prove the following analogue of Lemma 5.4. As mentioned in the Introduction, we will give a proof slightly different from that of Lemma 5.4. Instead of using Lemma 5.1, we will use the strong/weak level notion used by Faudree and Simonovits [12] in the study of theta graphs. Note that we could also prove Lemma 6.5 using Lemma 5.1 and Lemma 6.1, but we think that the Faudree–Simonovits approach is more intuitive and less technical for the hypergraph case.

Lemma 6.5. *Let i, k, m be integers where $k \geqslant i + 1 \geqslant 1$, $m \geqslant 1$. Let b, d be reals satisfying $b, d \geqslant 16^i(2m + 2)k$. Let T_x be a tree of height i rooted at x. For each $j = 0, \ldots, i$, let L_j be the set of vertices in T_x at distance j from x. Let W be some set of vertices disjoint from $V(T)$ and let H be a bipartite graph with bipartition (L_i, W) such that*

$$\mathrm{e}(H) \geqslant \max\{d|L_i|, b|W|\}.$$

Suppose c is an edge-colouring of $G = T_x \cup H$ such that c is rainbow on T_x and strongly proper on H and that $\mathscr{C}(G) \cap V(G) = \emptyset$ and $\mathscr{C}(T_x) \cap \mathscr{C}(H) = \emptyset$. Then there exist $0 \leqslant q \leqslant i$ and some positive real $a_i = a_i(i, k)$ such that there are at least $a_i(bd)^{k-i-1+q}\,\mathrm{e}(H)$ rainbow C_{2k}'s in G that contain a vertex in L_q.

Proof. We proceed by induction on the height i of the tree. It holds vacuously for $i = 0$. For all $i \geqslant 1$ we prove the result by splitting the argument into two cases, and we use induction in only one of the cases. It is important to point out that when $i = 1$ we are in Case 1 and thus need not use the vacuous case of $i = 0$ as our induction hypothesis.

Let $x_1, x_2, \ldots, x_p$ denote the children of x in T_x. For each $j \in [p]$, let $T(x_j)$ be the subtree of T_x rooted at x_j. For each $j \in [p]$, we define the jth *sector* to be $S_j = L_i \cap V(T(x_j))$. Note that since T_x is a tree, the S_j's are pairwise disjoint. For a vertex $v \in L_i$, we let $S(v)$ denote the sector that v lies in.

We say that a sector S_j is *dominant* for a vertex $w \in W$ if

$$|N_H(w) \cap S_j| > \max\left\{|N_H(w)| - 2km, \frac{|N_H(w)|}{2}\right\}.$$

We say that $w \in W$ is *strong* if it has no dominant sector and *weak* otherwise. Note that, by our definition, if $w \in W$ has a dominant sector then there is only one such dominant sector for w.

Let W_s be the set of strong vertices and let W_w be the set of weak vertices, respectively. Let H_s denote the subgraph of H induced by L_i and W_s, and let H_w denote the subgraph of H induced by L_i and W_w. The argument splits into two cases, depending on whether the majority of the edges of H lie in H_s or in H_w. In the first case we build the necessary number of rainbow $2k$-cycles going through the vertex x (so in the outcome of the theorem we have $j = 0$ as $x \in L_0$). In the second case we use induction to find rainbow $2k$-cycles in $T(x_j)$'s for many j.

Case 1.

$$e(H_s) \geqslant e(H)/2. \tag{6.2}$$

Let $d_{\text{avg}}(L_i)$ and $d_{\text{avg}}(W_s)$ denote the average degrees in H for vertices in L_i and W_s respectively. Then by (6.2), we have $d_{\text{avg}}(L_i) \geqslant d/2$, $d_{\text{avg}}(W_s) \geqslant b/2$. By Lemma 5.2, there is a subgraph H' of H_s with bipartition (A, B), $A \subseteq L_i$, $B \subseteq W_s$, such that

$$e(H') \geqslant \frac{e(H_s)}{2} \geqslant \frac{e(H)}{4}, \quad \delta_A(H') \geqslant \frac{d_{\text{avg}}(L_i)}{4} \geqslant \frac{d}{8}, \quad \delta_B(H') \geqslant \frac{d_{\text{avg}}(W_s)}{4} \geqslant \frac{b}{8}. \tag{6.3}$$

Since $b, d \geqslant 16^i(2m+2)k$, clearly $b/8, d/8 \geqslant (4m+4)k \geqslant (4m+4)(k-i-1)$. Since c is strongly proper on H, by Lemma 6.1 with $p = k-i-1$, the number of rainbow paths of length $2(k-i)-1$ in H' is at least

$$\frac{1}{2^{2(k-i-1)}} e(H')(\delta_A(H')\delta_B(H'))^{k-i-1} \geqslant \frac{1}{2^{5(k-i-1)+2}}\, e(H)(bd)^{k-i-1}.$$

Claim 2. *Every rainbow path $P = v_1v_2 \ldots v_{2(k-i)}$ of length $2(k-i)-1$ extends to a rainbow C_{2k} in G that contains x.*

Proof of claim. By symmetry, we may assume that $v_1 \in A$, $v_{2(k-i)} \in B$. For convenience, let $t = 2(k-i)$. It suffices to show that there exists $u \in N_H(v_t)$ (note that u lies in L_i but does not necessarily lie in A) such that $P \cup v_tu$ is a rainbow path in H and that $S(v_1) \neq S(u)$. Indeed, suppose such a u exists. Then, since $S(v_1) \neq S(u)$, the unique path Q_1 in T_x from v_1 to x and the unique path Q_2 from u to x intersect only at x. Since T_x is rainbow, $Q_1 \cup Q_2$ is rainbow. By our assumption, $\mathscr{C}(T_x) \cap \mathscr{C}(H) = \emptyset$. Thus P, Q_1, Q_2 together form a rainbow C_{2k} in G.

Now we show that such a u exists. Since $v_t \in W_s$, by definition, $|N_H(v_t) \setminus S(v_1)| \geqslant 2km$. Since φ is a strongly proper edge-colouring using m-sets, $\{w \cup \varphi(w) \colon w \in N_H(v_t) \setminus S(v_1)\}$ is an $(m+1)$-uniform matching of size $|N_H(v_t) \setminus S(v_1)| \geqslant 2km$. Since clearly $|V(P) \cup \mathscr{C}(P)| < 2km$, there exists $u \in N_H(v_t) \setminus S(v_1)$ such that $(w \cup \varphi(w)) \cap (V(P) \cup \mathscr{C}(P)) = \emptyset$. It is easy to see that $P \cup v_tu$ is a rainbow path in H. Also, $u \notin S(v_1)$ by choice. $\square$

Case 2. $e(H_w) \geqslant e(H)/2$.

In this case we have

$$e(H_w) \geqslant \frac{d}{2}|L_i|, \quad e(H_w) \geqslant \frac{b}{2}|W|. \tag{6.4}$$

Recall that $x_1, \ldots, x_p$ are the children of the root x and for each $j \in [p]$, $S_j = V(T(x_j)) \cap L_i$. For each $j \in [p]$, let W_j be the set of vertices in W_w whose dominant sector is S_j. Now we run the following 'cleaning' procedure. For every vertex $y \in W_w$ we only keep those edges in H_w joining y

to vertices in its dominant sector. Let H'' denote the resulting subgraph of H_w. By the definition of W_w, every vertex $y \in W_w$ satisfies

$$d_{H''}(y) \geqslant |N_H(y)| - 2km.$$

Hence

$$e(H'') \geqslant e(H_w) - 2km|W_w|.$$

Since $b \geqslant 8km$, by (6.4) $e(H_w) \geqslant 4km|W|$. Therefore

$$e(H'') \geqslant \frac{1}{2} e(H_w) \geqslant \frac{1}{4}\, e(H). \tag{6.5}$$

For each $j \in [p]$, let H_j denote the subgraph of H'' induced by $S_j \cup W_j$. Note that the H_j's are pairwise vertex-disjoint. We want to apply induction to those $T(x_j) \cup H_j$ where H_j is relatively dense from both partite sets. For that purpose we partition the index set $[p]$ as follows. Let

$$\mathscr{I}_1 = \left\{ j \in [p] \colon e(H_j) \leqslant \frac{d}{16}|S_j| \right\}, \quad \mathscr{I}_2 = \left\{ i \in [p] \colon e(H_j) \leqslant \frac{b}{16}|W_j| \right\}, \quad \mathscr{I}_3 = [p] \setminus (\mathscr{I}_1 \cup \mathscr{I}_2).$$

By the definition and disjointness of the H_j's, we have

$$\sum_{j \in \mathscr{I}_1 \cup \mathscr{I}_2} e(H_j) \leqslant \frac{d}{16}|L_i| + \frac{b}{16}|W| \leqslant \frac{1}{8}\, e(H).$$

Hence

$$\sum_{j \in \mathscr{I}_3} e(H_j) \geqslant \frac{1}{8}\, e(H). \tag{6.6}$$

For each $j \in \mathscr{I}_3$, by definition, we have

$$e(H_j) \geqslant \frac{d}{16}|S_j| \quad \text{and} \quad e(H_j) \geqslant \frac{b}{16}|W_j|.$$

Since $T(x_j)$ has height $i-1$ and $d/16, b/16 > (16)^{i-1}(2m+2)k$, by the induction hypothesis with d, b replaced with $d/16$ and $b/16$ respectively, there exists $q = q(j)$ such that the number of rainbow $2k$-cycles in $T(x_j) \cup H_j$ that contain a vertex in level $q(j)$ of $T(x_j)$ is at least

$$a_{i-1} \left(\frac{bd}{16^2} \right)^{k-(i-1)-1+q(j)} e(H_j) = a_{i-1} \left(\frac{bd}{16^2} \right)^{k-i+q(j)} e(H_j).$$

For each $t = 0, \ldots, i-2$, let $\mathscr{I}_{3,t} = \{ j \in \mathscr{I}_3 \colon q(j) = t \}$. By the pigeonhole principle, there exists $t \in \{0, \ldots, i-2\}$ such that

$$\sum_{j \in \mathscr{I}_{3,t}} e(H_j) \geqslant \frac{1}{i-1} \sum_{j \in \mathscr{I}_3} e(H_j) \geqslant \frac{1}{8k}\, e(H).$$

Let us fix such a t. By our earlier discussion and the fact that vertices in level t of each $T(x_j)$ for $j \in \mathscr{I}_{3,t}$ lie in level $t+1$ of T_x, the number of rainbow $2k$-cycles in G that contain a vertex from L_{t+1} is at least

$$\sum_{j \in \mathscr{I}_{3,t}} a_{i-1} \left(\frac{bd}{16^2} \right)^{k-i+t} e(H_j) \geqslant a_i (bd)^{k-i-1+(t+1)}\, e(H),$$

with the choice of

$$a_i = \frac{a_{i-1}}{2^{8(k-i+l)+3} k}.$$

Hence in this case the lemma holds for $q = t+1$. $\square$

6.3 Proof of the $r \geqslant 3$ case of Theorem 1.1

We are finally ready to prove the supersaturation statement of $C_{2k}^{(r)}$ for linear r-partite r-graphs G that have a 2-projection on two parts A, B that is almost-regular and have a number of edges exactly $\Theta(|A \cup B|^{1+1/k})$. By Corollary 3.7 this would imply Theorem 1.1 for all $r \geqslant 3$. For this we first define an adequate partition $V(G)$ into $S_1, \ldots, S_k$. From each vertex x we define the maximal rainbow tree T_x rooted at x relative to the partition $(S_1, \ldots, S_k)$. Then we apply Lemma 6.5 to find many rainbow $2k$-cycles containing a vertex from some fixed level of T_x, which corresponds to linear $2k$-cycles in G. Summing over all x and eliminating overcount, we get a lower bound on the number of $2k$-cycles in G.

Theorem 6.6. *Let $k, r \geqslant 2$ be integers. Let D be a constant such that $D \geqslant 2^{r+1} rk^r (16)^k$. There exist n_0 such that if G is a linear r-partite r-graph with an r-partition $A_1, \ldots, A_r$ such that $|A_1 \cup A_2| = n \geqslant n_0$ and for every $v \in A_1 \cup A_2$,*

$$Dn^{1/k} \leqslant |L_G(v)| \leqslant \lambda D n^{1/k},$$

where $\lambda \geqslant 1$ is a real, then there exists $\alpha = \alpha(k, r, \lambda)$ such that $t_{C_{2k}^{(r)}}(G) \geqslant \alpha n^2$.

Proof. The choice of α will be specified at the end of the proof. We will choose n_0 to be large enough that $n_0 \geqslant n_{6.2}(D, k, r, 1/k)$, where $n_{6.2}$ is specified in Lemma 6.2. Let $S_1, S_2, \ldots, S_k$ be a partition obtained by applying Lemma 6.2 to G. In particular, for each $x \in A_1 \cup A_2$ and $j \in [k]$, we have

$$|L_G(x)|_{S_j}| \geqslant \frac{Dn^{1/k}}{2k^{r-1}}. \tag{6.7}$$

For each $x \in A_1 \cup A_2$, let T_x be a maximal rainbow tree of height k rooted at x relative to the partition $S_1, \ldots, S_k$, as described in Definition 4. The proof is similar to that of Theorem 5.5. For each x, we find an $i \in [k]$ such that

(i) there exists some set W' and a bipartite subgraph H_x induced by L_i and W' which has high average degree from both partite sets;
(ii) the colouring c on T_x is extended to also include a strongly proper edge-colouring of H_x such that $\mathscr{C}(H_x) \cap \mathscr{C}(T_x) = \emptyset$.

We then use Lemma 6.5 to find many rainbow $2k$-cycles that contain some vertex in some fixed level of T_x. Below are the details.

Fix x and write T for T_x. For $j = 0, \ldots, k$, let L_j be defined as in Definition 4 and let φ be the assigned edge-colouring of T given in Definition 4. Since $|L_1| \geqslant Dn^{1/k} > n^{1/k}$ and $|L_k| \leqslant n$, there exists a smallest $i \in [k-1]$ such that, for all $1 \leqslant j \leqslant i$, $|L_j| > n^{j/k}$ but

$$|L_{i+1}| \leqslant n^{(i+1)/k}.$$

Let T' be the subtree of T induced by $\bigcup_{j=0}^{i} L_j$. Let $F = \bigcup_{v \in L_i} L_G(v)|_{S_{i+1}}$. Since S_{i+1} is disjoint from $S_1 \cup \cdots \cup S_i$ and since $L_i \subseteq A_1$ where A_1 is a partite set in an r-partition of G, we have $V(F) \cap V(T') = \emptyset$. By the construction of T, $|L_{i+1}|$ is equal to the size of a maximum matching in F. Since F is an $(r-1)$-graph, we have $\tau(F) \leqslant (r-1)\alpha'(F)$, where $\tau(F)$ and $\alpha'(F)$ denote the vertex cover number and matching number of F, respectively. Let W be a minimum vertex cover of F. Then

$$|W| \leqslant (r-1)|L_{i+1}| \leqslant (r-1)n^{(i+1)/k}.$$

By Lemma 6.3, there exist $W' \subseteq W$ and $F' \subseteq F$ such that

$$e(F') \geqslant \frac{r-1}{2^{r-1}} e(F) \quad \text{and} \quad |e \cap W'| = 1 \text{ for all } e \in e(F').$$

We define a bipartite graph H_x between L_i and W' and extend the edge-colouring φ restricted on T' to an edge-colouring of $T' \cup H_x$ as follows. We go through the edges of F' one by one. For each $e \in E(F')$, since G is linear, there is a unique $v \in L_i$ such that $v \cup e \in E(G)$. Also by our definition of F', $e \cap W'$ has exactly one vertex w. We include vw in H_x and let $\varphi(vw) = e \setminus \{w\}$. By the linearity of G and our discussion so far, each edge of F' yields a different edge of H_x. There is a bijection between $E(F')$ and $E(H_x)$. Moreover, $\mathscr{C}(H_x) \cap \mathscr{C}(T') = \emptyset$, since colours used on H_x are $(r-2)$-sets in S_{i+1} while $\mathscr{C}(T') \subseteq S_1 \cup \cdots \cup S_i$.

Since G is linear and $r \geqslant 3$, for all $v, v' \in L_i$ we have $L_G(v)|_{S_{i+1}} \cap L_G(v')|_{S_{i+1}} = \emptyset$. By (6.7),

$$e(F) = \sum_{v \in L_i} |L_G(v) \cap S_{i+1}| \geqslant \frac{Dn^{1/k}}{2k^{r-1}} |L_i|.$$

Hence we have

$$e(H_x) = e(F') \geqslant \frac{r-1}{2^{r-1}} e(F) \geqslant \frac{D(r-1)}{2^r k^{r-1}} n^{1/k} |L_i|. \tag{6.8}$$

Also, by our choice of i, $|L_{i+1}| \leqslant n^{1/k}|L_i|$. Recall also that $|W'| \leqslant |W| \leqslant (r-1)|L_{i+1}|$. Hence

$$e(H_x) \geqslant \frac{D(r-1)}{2^r k^{r-1}} |L_{i+1}| \geqslant \frac{D}{2^r k^{r-1}} |W| \geqslant \frac{D}{2^r k^{r-1}} |W'|. \tag{6.9}$$

Let

$$b = \frac{D}{2^r k^{r-1}} \quad \text{and} \quad d = \frac{D(r-1)}{2^r k^{r-1}} n^{1/k}.$$

Since $D \geqslant 2^{r+1} r k^r (16)^k$, we have $d > b \geqslant (2(r-2)+2)k(16)^k$. So T' and H_x satisfy the conditions of Lemma 6.5 with constants b, d and $m = r-2$. By Lemma 6.5, there exists some $q = q(x)$ with $0 \leqslant q \leqslant i$ and some $a_i = a_i(i,k) > 0$ such that there are at least

$$a_i (bd)^{k-i-1+q} e(H_x)$$

rainbow C_{2k}'s in $T' \cup H_x$ that contain some vertex in level L_q of T'. Now $|L_i| \geqslant n^{i/k}$ by definition, $e(H_x) \geqslant \Omega(n^{(i+1)/k})$ by (6.8). Also, $d = \Omega(n^{1/k})$. Hence the number of rainbow C_{2k}'s in $T' \cup H_x$ that contain some vertex in L_q is at least

$$\beta n^{(k-i-1+q)/k} \cdot n^{(i+1)/k} = \beta n^{(k+q)/k} \quad \text{for some } \beta = \beta(k,r) > 0.$$

So in G there are at least $\beta n^{(k+q)/k}$ different linear $2k$-cycles each of whose skeletons contains some vertex in L_q.

For each $t \in [k-1]$, let $S_t = \{x \in V(G) \mid q(x) = t\}$. By the pigeonhole principle, for some $t \in [k-1]$, $|S_t| \geqslant n/(k-1)$. Let us fix such a t. Let M denote the number of triples (C, x, y), where $x \in S_t$, C is a linear $2k$-cycle in G whose skeleton contains a vertex in $L_t(T_x)$ and y is a vertex on the skeleton of C that lies in $L_t(T_x)$. Let μ denote the number of different linear $2k$-cycles C in G that are involved in these triples. By our discussion above, for each $x \in S_t$, there are at least $\beta n^{(k+q)/k}$ different C. For each such C there is at least one y. So

$$M \geqslant |S_t| \beta n^{(k+t)/k} > (\beta/k) n^{2+t/k}. \tag{6.10}$$

On the other hand, for each of the μ linear $2k$-cycles C involved, there are at most $2k$ different choices of y. For fixed y, there are at most $(\lambda D n^{1/k})^t$ choices of x since such an x is at distance at most t from y in the $(1,2)$-projection $P_{1,2}(G)$ of G, which has maximum degree at most $\lambda D n^{1/k}$. So

$$M \leqslant \mu (2k)(\lambda D n^{1/k})^t. \tag{6.11}$$

Combining (6.10) and (6.11) and solving for μ, we get

$$\mu \geqslant \frac{\beta}{2k^2(\lambda D)^t} n^2.$$

Let

$$\alpha = \frac{\beta}{2k^2(\lambda D)^k}.$$

Then α is a function of k, r, λ and we have $t_{C_{2k}^{(r)}}(G) \geqslant \mu \geqslant \alpha n^2$. □

We are now ready to prove the $r \geqslant 3$ case of Theorem 1.1.

Proof of the $r \geqslant 3$ case of Theorem 1.1. First note that Theorem 6.6 can be rephrased by saying that if G is linear r-partite r-graph that has a 2-projection P on at least $m \geqslant n_0$ vertices such that $Dm^{1/k} \leqslant \delta(P) \leqslant \Delta(P) \leqslant \lambda D m^{1/k}$, then $t_{C_{2k}^{(r)}}(G) \geqslant \alpha m^2$. The statement holds as long as $D \geqslant 2^{r+1} rk^r (16)^k$, $\lambda \geqslant 1$ and $m \geqslant n_0$. To apply Corollary 3.7, we set

$$D = \max\{2^{r+1} rk^r (16)^k, m_k\}, \quad \lambda = q_k D \quad \text{and} \quad M = \max\{n_0, m_k\},$$

where m_k, q_k are as given in Corollary 3.7. The claim follows readily from Corollary 3.7. □

7. Concluding remarks

First we would like to mention that the reduction to proving the supersaturation of C_{2k} for n-vertex host graphs G with density exactly at $\Theta(n^{1+1/k})$ is crucial to establishing our general theorem. For graphs G with density much higher than $n^{1+1/k}$, the bound resulting directly from BFS is worse than the optimal $c(\mathrm{e}(G)/\mathrm{v}(G))^{2k}$. One reason is that the subgraph of G induced by the early levels of a BFS tree is potentially much denser than a tree. In this case, if we use the BFS structure we may lose many C_{2k}'s in counting.

For all integers $k, p \geqslant 2$, the theta graph $\Theta_{p,k}$ is the graph consisting of p internally disjoint paths of length k sharing the same endpoints. Faudree and Simonovits [12] showed that $\mathrm{ex}(n, \Theta_{p,k}) = O(n^{1+1/k})$. The method of our paper can be used to establish the supersaturation of the r-expansion $\Theta_{p,k}^{(r)}$ (where $r \geqslant 2$) of $\Theta_{p,k}$ in linear r-graphs. When $r = 2$ this establishes the truth of Conjecture 1 for $H = \Theta_{p,k}$ with $\alpha = \alpha' = 1 - 1/k$. Again, the lower bound is tight up to a multiplicative constant, obtained by taking a random graph of an almost complete Steiner system. Since the arguments are essentially the same, we have not included the results involving theta graphs in this paper.

It would be very interesting to establish the supersaturation of odd linear cycles in linear r-graphs, for $r \geqslant 3$. Toward this end, in [5] it is shown that when $r \geqslant 3$ we have $\mathrm{ex}_\ell(n, C_{2k+1}^{(r)}) = O(n^{1+1/k})$, which is very different from the 2-uniform case, where for all sufficiently large n it is known that $\mathrm{ex}(n, C_{2k+1}) = \lceil n/2 \rceil \lfloor n/2 \rfloor$. The proof of this theorem is much more involved than its counterpart for even linear cycles. It is unclear if a supersaturation statement similar to Theorem 1.1 holds for $C_{2k+1}^{(r)}$. At least, our methods do not readily give this. We raise it as an open question.

Question 5. Let k, r be integers where $k \geqslant 2, r \geqslant 3$. Do there exist positive constants C and c depending only on k and r such that every n-vertex linear r-graph G with $\mathrm{e}(G) \geqslant Cn^{1+1/k}$ contains at least $c(\mathrm{e}(G)/\mathrm{v}(G))^{2k+1}$ copies of $C_{2k+1}^{(r)}$?

Very recently, Balogh, Narayanan and Skokan [1] obtained a balanced supersaturation result for linear cycles of all lengths in general r-graphs. Note that this is a different setting from ours, as in our case host graphs are linear, and hence are sparse, while they are working with dense ones. As Morris and Saxton did for even cycles in graphs, Balogh, Narayanan and Skokan used their supersaturation result to obtain a bound on the number of of n-vertex $C_m^{(r)}$-free r-graphs. It would be interesting to obtain such a balanced version of supersaturation for even linear cycles in linear r-graphs.

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