

SOLVING A DIRICHLET PROBLEM ON UNBOUNDED DOMAINS VIA A CONFORMAL TRANSFORMATION

RYAN GIBARA, RIIKKA KORTE, NAGESWARI SHANMUGALINGAM

ABSTRACT. In this paper, we solve the p -Dirichlet problem for Besov boundary data on unbounded uniform domains with bounded boundaries when the domain is equipped with a doubling measure satisfying a Poincaré inequality. This is accomplished by studying a class of transformations that have been recently shown to render the domain bounded while maintaining uniformity. These transformations conformally deform the metric and measure in a way that depends on the distance to the boundary of the domain and, for the measure, a parameter p . We show that the transformed measure is doubling and the transformed domain supports a Poincaré inequality. This allows us to transfer known results for bounded uniform domains to unbounded ones, including trace results and Adams-type inequalities, culminating in a solution to the Dirichlet problem for boundary data in a Besov class.

CONTENTS

1. Introduction	1
2. Construction of the transformation of metric and measure	4
3. Background related to metric measure spaces	7
4. Doubling property of μ_φ	13
5. Uniformity of $\Omega_\varphi \setminus \{\infty\}$	20
6. Poincaré inequalities	23
7. Transformation of potentials	25
8. Connecting potential theory on Ω to functions on $\partial\Omega$	30
9. Effect on the Dirichlet problem for (Ω, d, μ)	35
10. Some illustrative examples	37
Conflict of interest and data availability	38
References	38

1. INTRODUCTION

In studying Dirichlet and Neumann boundary-value problems on domains in metric measure spaces of bounded geometry, existence of the solution via the direct method of the calculus of variations requires that we are able to bound the L^p -norm of a Sobolev function on the domain (with zero

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boundary values) by the Sobolev energy norm of the function, thus ensuring the boundedness in the Sobolev norm of an energy minimizing sequence of Sobolev functions in the domain. When the domain is a bounded uniform domain, this is always possible thanks to the Poincaré inequality, for we can then envelop the domain in a sufficiently large ball. Bounded uniform domains play a central role in potential theory as many of the classical results about Dirichlet problems on smooth Euclidean domains hold for such domains in metric measure spaces. In particular, they are extension domains for several function spaces [7, 35] and traces, to the boundary, of Sobolev-class functions on the domain belong to certain Besov classes [32] of functions on the boundary. However, when the domain, albeit uniform, is not bounded, these properties might not hold. Therefore, it is beneficial to have a transformation of the domain into a bounded uniform domain.

One such transformation is sphericalization as defined in the work of [8], which transforms an unbounded metric space X into a bounded metric space whose completion is topologically its one-point compactification. A generalization of the familiar stereographic projection, sphericalization has been used in the study of quasiconformal geometry, including the study of quasi-Möbius maps and Gromov hyperbolic spaces, see for example [9, 10, 21, 29, 37]. Sphericalization is known to preserve many desirable properties of the metric space X , see for example [12, 13]. In particular, if X is a uniform domain (or a uniform space, in the language of [8]), then its sphericalization is also a uniform domain, see [10].

On the other hand, sphericalization distorts the metric of X everywhere, including near its boundary if X is not complete. This poses a problem if one is interested in gaining information about, or preserving the geometry of, the boundary of the original unbounded domain X when $\partial X = \overline{X} \setminus X$ itself is bounded, for example as in [11]. This issue was addressed in [15], where a class of transformations was identified such that unbounded uniform domains are transformed to bounded uniform domains in such a way that the inner length metric is not perturbed, locally, near the boundary. The purpose of the present paper is to explore potential theory on these transformed domains, with the view of applying this in ongoing work on boundary-value problems on unbounded domains.

The setting: We consider a locally compact, non-complete metric space (Ω, d) , equipped with a doubling measure μ , and supporting a Poincaré inequality, at least for balls with radius at most some fixed constant times the distance from its center to the boundary. We assume that Ω is unbounded and is uniform in its completion $\overline{\Omega}$ with bounded boundary $\partial\Omega := \overline{\Omega} \setminus \Omega$, and we fix a monotone decreasing continuous function $\varphi : (0, \infty) \rightarrow (0, \infty)$ that will act as a dampening function, see Definition 2.1 for the specific assumptions on φ . As Ω is a uniform domain, it is rectifiably connected, that is, pairs of points in Ω can be connected by curves in Ω of finite length. As such, we may use φ to construct a new metric d_φ on Ω by setting

$$d_\varphi(x, y) := \inf_{\gamma} \int_{\gamma} \varphi \circ d_{\Omega} ds,$$

with the infimum ranging over all rectifiable curves in Ω with end points $x, y \in \Omega$. Here, $\int_{\gamma} h ds := \int_{\gamma} h(\gamma(\cdot)) ds$ is the path integral with respect to the arc-length parametrization of the rectifiable curve γ , see for example [20, Chapter 5], and d_{Ω} is defined by $d_{\Omega}(x) = \text{dist}(x, \partial\Omega)$. We fix $1 \leq p < \infty$ and also construct a new measure μ_{φ} supported on Ω , absolutely continuous with μ , with Radon-Nikodym derivative $\varphi(d_{\Omega}(x))^p$.

In [15], it was shown that $\Omega_{\varphi} := \overline{\Omega \cup \partial\Omega^{\varphi}} \setminus \partial\Omega$, where $\overline{A^{\varphi}}$ is the completion of $A \subset \overline{\Omega}$ with respect to the metric d_{φ} , differs from Ω by one point, which we denote by ∞ . The transformed space $(\Omega_{\varphi}, d_{\varphi})$ was shown to be uniform in its completion and to have boundary $\partial\Omega_{\varphi} = \partial\Omega$. Moreover, d_{φ} and d are uniformly locally bi-Lipschitz near $\partial\Omega$. The present paper begins by showing the following (see Theorems 4.11 and 6.3).

Theorem 1.1. *The metric measure space $(\Omega_{\varphi}, d_{\varphi}, \mu_{\varphi})$ is doubling and supports a p -Poincaré inequality.*

In the process of proving the above theorem, we verify that $\Omega_{\varphi} \setminus \{\infty\}$ is also a uniform domain (see Theorem 5.2), supplementing the results from [15].

With these tools in hand, in Sections 7 and 8 we proceed to study potential theory on the domain Ω_{φ} . We show that a function is p -harmonic on (Ω, d, μ) if and only if it is p -harmonic on $(\Omega_{\varphi} \setminus \{\infty\}, d_{\varphi}, \mu_{\varphi})$. Furthermore, when p is sufficiently large and $\varphi(t) = \min\{1, t^{-\beta}\}$ for $t > 0$ and some fixed large enough $\beta > 1$, p -harmonic functions on $(\Omega_{\varphi} \setminus \{\infty\}, d_{\varphi}, \mu_{\varphi})$ can be extended to become p -harmonic on all of $(\Omega_{\varphi}, d_{\varphi}, \mu_{\varphi})$. We show in Proposition 7.7 that when the index p is small, the p -capacity of $\{\infty\}$ is zero; but if p is sufficiently large, then the p -capacity of $\{\infty\}$ is positive. We also show that when the measure on $\partial\Omega$ satisfies a codimensionality condition with respect to μ , the trace class of the Dirichlet-Sobolev space $D^{1,p}(\Omega, d, \mu)$ is a Besov space of functions on $\partial\Omega$, see Proposition 8.3, and an Adams-type inequality holds for the measure on $\partial\Omega$ and functions in $D^{1,p}(\Omega, d, \mu)$, see Theorem 8.7. Interestingly, it turns out that under this codimensionality condition for the boundary, each (relative) ball in $\partial\Omega$ has positive p -capacity in $\overline{\Omega_{\varphi}^{\varphi}}$, see Proposition 8.2.

Using the potential theory developed in Sections 7 and 8, we obtain the following culminating theorem regarding the Dirichlet problem in Section 9. In what follows, the boundary data f is taken to be in the Besov space $B_{p,p}^{1-\theta/p}$ on the boundary $\partial\Omega$ with respect to a codimensional measure ν , see Definition 3.12 for the definition of the Besov space and Proposition 3.13 regarding the trace operator T acting on the Dirichlet-Sobolev space $D^{1,p}(\Omega, \mu)$; for the description of $D^{1,p}$, we refer the reader to the paragraph before Definition 3.4 below.

Theorem 1.2. *Let $1 < p < \infty$ and ν be a measure on $\partial\Omega$ that is θ -codimensional with respect to the measure μ on Ω with $0 < \theta < p$, as described at the beginning of Section 8. Let $f \in B_{p,p}^{1-\theta/p}(\partial\Omega, \nu)$. Then there is a function $u \in D^{1,p}(\Omega, \mu)$ such that*

- u is p -harmonic in (Ω, d, μ) ,
- $Tu = f$ on $\partial\Omega$ ν -a.e..

If Ω is p -parabolic, then the solution u is unique. If Ω is p -hyperbolic, then for each solution u of the problem we have $\lim_{\Omega \ni y \rightarrow \infty} u(y)$ exists; this limit uniquely determines the solution.

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2. CONSTRUCTION OF THE TRANSFORMATION OF METRIC AND MEASURE

Let (Ω, d) be an unbounded, locally compact, non-complete metric space such that Ω is a *uniform domain* in its completion $\bar{\Omega}$. A uniform domain is one for which there exists a constant $C_U \geq 1$ satisfying the following property: for each $x, y \in \Omega$ with $x \neq y$, there is a *uniform curve* with end points x and y , that is, a curve γ such that

- the length (with respect to the metric d) of the curve γ satisfies $\ell_d(\gamma) \leq C_U d(x, y)$,
- for each z in the trajectory of γ , we have

$$\min\{\ell_d(\gamma[x, z]), \ell_d(\gamma[z, y])\} \leq C_U d_\Omega(z).$$

Here, for two points w_1, w_2 in the trajectory of γ , we represent each segment of γ with end points w_1, w_2 by $\gamma[w_1, w_2]$. Moreover, $d_\Omega(x) := \text{dist}(x, \partial\Omega)$ for $x \in \bar{\Omega}$, and $\partial\Omega = \bar{\Omega} \setminus \Omega$.

Throughout this paper, we set n_0 to be the smallest integer such that

$$2^{n_0-1} \leq C_U < 2^{n_0}.$$

We will also assume that $\partial\Omega$ is bounded, that μ is a doubling Radon measure supported on Ω with doubling constant $C_\mu \geq 1$, and that (Ω, d, μ) supports a sub-Whitney p -Poincaré inequality for some fixed $1 \leq p < \infty$, see Section 3 for the definitions.

The notation $A \lesssim B$ will be used to mean that there exists a constant $C > 0$, depending only on structural data, such that $A \leq CB$; furthermore, the notation $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

Definition 2.1. In this paper, we fix a monotone decreasing continuous function $\varphi : (0, \infty) \rightarrow (0, 1]$ such that the following hold:

- (1) $\varphi(t) = 1$ when $0 < t \leq 1$.
- (2) We have

$$\int_0^\infty \varphi(t) dt < \infty. \tag{2.2}$$

- (3) There is a constant $C_\varphi \geq 1$ for which we have $\varphi(t) \leq C_\varphi \varphi(2t)$ for all $t > 0$ (that is, φ satisfies a reverse doubling condition).
- (4) There is some $\tau > 2$ such that $\varphi(t) \geq \tau \varphi(2t)$ (thus requiring $C_\varphi > 2$, as well).
- (5) For all positive integers m , we have

$$2^m \varphi(2^m) \leq \sum_{n=m}^{\infty} 2^n \varphi(2^n) \lesssim 2^m \varphi(2^m) \quad (2.3)$$

(and, indeed, this condition follows from Condition (4) above, but we list it here for it is used extensively in this paper).

- (6) For all positive integer m ,

$$\varphi(2^m)^p \mu(\Omega_m) \leq \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n) \lesssim \varphi(2^m)^p \mu(\Omega_m), \quad (2.4)$$

where, for $n > 0$,

$$\Omega_n := \{x \in \Omega : 2^{n-1} < d_\Omega(x) \leq 2^n\}.$$

The last condition is needed in order to know that μ_φ is finite and doubling, see below for the definition of μ_φ . Examples of functions φ satisfying all the necessary conditions include $\varphi(t) = \min\{1, t^{-\beta}\}$ or $\varphi(t) = \min\{1, t^{-\beta} \log(e - 1 + t)\}$ for some sufficiently large fixed $\beta > 1$ depending on p and the doubling property of μ (see the condition on β in Lemma 7.6).

Construction: For $1 \leq p < \infty$, we wish to transform the geometry of Ω by weighting both the metric and the measure on Ω using φ in the following way.

We transform the metric d on Ω into d_φ by setting

$$d_\varphi(x, y) := \inf_{\gamma} \ell_\varphi(\gamma) := \inf_{\gamma} \int_{\gamma} \varphi(d_\Omega(\gamma(t))) dt$$

with the infimum ranging over all rectifiable curves γ in Ω with end points x and y in Ω . Note that Ω is rectifiably connected as it is uniform. The notation B_{d_φ} and B_d will be used for balls taken with respect to the metric d_φ and d , respectively. All balls will be assumed to come with a prescribed center and radius.

The measure μ on Ω is transformed into the measure μ_φ , absolutely continuous with respect to μ , with

$$d\mu_\varphi(x) = \varphi(d_\Omega(x))^p d\mu.$$

Note that μ_φ depends not only on φ but also on the choice of p ; however, $1 \leq p < \infty$ is fixed and we suppress the dependence on p in the notation. We credit [4] for the idea of considering transformations of measures that are allowed to depend on p .

Now we have two identities for Ω ; namely, (Ω, d, μ) and $(\Omega, d_\varphi, \mu_\varphi)$. Consider the set $\Omega_\varphi := \overline{\Omega \cup \partial\Omega^\varphi} \setminus \partial\Omega$, where the completion is taken with respect to d_φ and $\partial\Omega := \overline{\Omega} \setminus \Omega$. In [15], it was shown that there is only one point in $\Omega_\varphi \setminus \Omega$, which we denote by ∞ . Moreover, $\partial\Omega_\varphi = \partial\Omega$ and $(\Omega_\varphi, d_\varphi)$ is uniform in its completion. In [15], the uniform domain properties of the transformed domain Ω_φ were studied; many of the tools developed there will be used in

the present paper. The goal of this paper is to investigate properties related to μ_φ , and apply these properties to the study of potential theory and the Dirichlet problem on unbounded uniform domains with bounded boundaries.

Basic Lemmas: We now recall some preliminary lemmas from [15] as well as some simple consequences that will be useful throughout the paper. In what follows, we very often break up Ω into bands in the following way.

Definition 2.5. We set

$$\Omega_0 := \{x \in \Omega : d_\Omega(x) \leq 1\},$$

and for positive integers n we set

$$\Omega_n := \{x \in \Omega : 2^{n-1} < d_\Omega(x) \leq 2^n\}.$$

Note that $\Omega = \bigcup_{n \geq 0} \Omega_n$.

Since $\overline{\Omega_\varphi}$ is compact, see Proposition 4.12, we have that Ω_φ is locally compact, and hence a simple topological argument gives the following lemma.

Lemma 2.6. *Let $x \in \Omega_m$ for some positive integer m . Then there is a geodesic in Ω_φ , with respect to the metric d_φ , connecting x and ∞ .*

As a consequence of the above lemma, we can show that every pair of points in $\bigcup_{n \geq 1} \overline{\Omega_n}$ can be connected in Ω_φ by a d_φ -geodesic curve.

We next show that $\partial\Omega$ being bounded implies that consecutive bands have comparable measure. This follows from the fact that each band has bounded diameter. Indeed, setting $L = \text{diam}_d(\partial\Omega)$, it follows from the triangle inequality that $2^{n-1} \leq \text{diam}_d(\Omega_n) \leq 2^{n+1} + L$ for each non-negative integer n , and so we can find a constant $C_L > 0$ such that

$$C_L^{-1} 2^n \leq \text{diam}_d(\Omega_n) \leq C_L 2^n. \quad (2.7)$$

Lemma 2.8. *Let n be a non-negative integer. Then there exists a constant $C_0 > 0$ such that*

$$C_0^{-1} \mu(\Omega_n) \leq \mu(\Omega_{n+1}) \leq C_0 \mu(\Omega_n), \quad (2.9)$$

where C_0 depends solely on the doubling constant of μ and the constant C_L . Moreover, there exists a $y_n \in \Omega_n$ such that $\mu(\Omega_n) \approx \mu(B_d(y_n, 2^n))$.

Proof. Take $x \in \Omega_{n+1}$ for which $d_\Omega(x) = \frac{3}{2} 2^n$ and γ a uniform curve with respect to the metric d with one end point in $\partial\Omega$ and the other at x . The existence of such a curve is guaranteed by the uniformity of Ω with respect to d . We can then find y_n in the trajectory of γ so that $y_n \in \Omega_n$ with $d_\Omega(y_n) = 2^n$, and so, by (2.7) and the doubling property of μ ,

$$\mu(\Omega_n) \leq \mu(B_d(y_n, C_L 2^n)) \lesssim \mu(B_d(y_n, 2^n)) \lesssim \mu(B_d(x, 2^n/C_L)) \lesssim \mu(\Omega_{n+1}).$$

A similar argument gives us the opposite direction. $\square$

Lemma 2.10 (Lemma 2.10 of [15]). *Let $x \in \Omega_m$ for some integer $m \geq n_0 + 2$. Then*

$$\frac{5}{11} \sum_{n=m+1}^{\infty} 2^n \varphi(2^n) \leq d_\varphi(x, \infty) \leq C_U C_\varphi \sum_{n=m-n_0}^{\infty} 2^n \varphi(2^n).$$

Thanks to the above lemma, we know that $(\Omega_\varphi, d_\varphi)$ is bounded. Under the additional conditions imposed on φ in this paper (in particular, conditions (3) and (5) in Definition 2.1 above) as compared to [15], we obtain the following.

Lemma 2.11. *There exists a constant $\kappa > 1$ such that for all non-negative integers m and $x \in \Omega_m$, we have*

$$\kappa^{-1} 2^m \varphi(2^m) \leq d_\varphi(x, \infty) \leq \kappa 2^m \varphi(2^m).$$

Here κ depends only on C_U, C_φ , and the implied constant from (2.3).

Proof. We begin by assuming that $m \geq n_0 + 2$ and show that it follows from Lemma 2.10 that

$$d_\varphi(x, \infty) \approx \sum_{n=m}^{\infty} 2^n \varphi(2^n), \quad (2.12)$$

where the implicit constants depend only on C_U, C_φ , and n_0 . Indeed, since φ is decreasing and satisfies Condition (3) of Definition 2.1, it follows that

$$\sum_{n=m-n_0}^{m-1} 2^n \varphi(2^n) \leq n_0 2^{m-1} \varphi(2^{m-n_0}) \leq \frac{n_0}{2} C_\varphi^{n_0} 2^m \varphi(2^m),$$

and so

$$\begin{aligned} \sum_{n=m-n_0}^{\infty} 2^n \varphi(2^n) &\leq \frac{n_0}{2} C_\varphi^{n_0} 2^m \varphi(2^m) + \sum_{n=m}^{\infty} 2^n \varphi(2^n) \\ &\leq \left[\frac{n_0}{2} C_\varphi^{n_0} + 1 \right] \sum_{n=m}^{\infty} 2^n \varphi(2^n). \end{aligned}$$

Similarly,

$$\sum_{n=m}^{\infty} 2^n \varphi(2^n) \leq \frac{C_\varphi}{2} \sum_{n=m+1}^{\infty} 2^m \varphi(2^n),$$

and so (2.12) follows from Lemma 2.10. The desired result then follows from (2.3).

Finally, if $m \leq n_0 + 1$, then $1 \gtrsim d_\varphi(x, \infty) \gtrsim \varphi(2^{n_0+2}) 2^{n_0}$, and so again the above inequality holds even if $m \leq n_0 + 1$. $\square$

Lemma 2.13 (Lemma 2.8 of [15]). *Let $x \in \Omega_m$ for some non-negative integer m . There exist constants $C_A \geq 1$ and $0 < c < 1$, depending solely on C_φ and C_U , such that if $y \in \Omega$ satisfies $d_\varphi(x, y) < c \varphi(2^m) 2^m$, then*

$$C_A^{-1} \varphi(2^m) d(x, y) \leq d_\varphi(x, y) \leq C_A \varphi(2^m) d(x, y).$$

3. BACKGROUND RELATED TO METRIC MEASURE SPACES

In this section, we give the definitions of the notions associated with measures and first order calculus in metric measure spaces. Namely, we give the definition of doubling measures, first-order calculus on metric measure spaces using the approach of upper gradients, and then discuss associated Poincaré inequalities. We also discuss moduli of families of curves, and variational capacities related to the first-order calculus.

In this section we let U be an open and connected subset of a complete metric measure space (Z, d_Z, μ_Z) . In the rest of the paper, U will stand at various points for Ω , $\bar{\Omega}$, $\Omega \cup \{\infty\}$, or $\bar{\Omega} \cup \{\infty\}$, while d_Z stands in for either the original metric d or the transformed metric d_φ , and μ_Z stands in for either the original measure μ or the transformed measure μ_φ . Observe that as the completion $\bar{\Omega}$ of Ω is the metric space in which Ω is a subset, necessarily $\bar{\Omega}$ is open in the topology of $\bar{\Omega}$. Moreover, as Ω is locally compact, it follows that Ω is also open in the topology of $\bar{\Omega}$.

Recall that $1 \leq p < \infty$ is fixed throughout the paper.

Definition 3.1. We say that μ_Z is a *locally uniformly doubling* measure on U if μ_Z is a Radon measure and there is a constant $C_d \geq 1$ and for each $x \in U$ there exists $r_x > 0$ such that whenever $0 < r \leq r_x$ we have

$$0 < \mu_Z(B(x, 2r) \cap U) \leq C_d \mu_Z(B(x, r) \cap U) < \infty.$$

If there is some constant $A > 1$ such that we can choose $r_x = \frac{1}{A} \text{dist}(x, \partial U)$, then we say that μ_Z is *sub-Whitney doubling* on U . We say that μ_Z is *doubling on Z* if $U = Z$ and $r_x = \infty$ for each $x \in U$. Note that if μ_Z is doubling on Z , then whenever $U \subset Z$ is an open set with $\partial U \neq \emptyset$, we must have that μ_Z is sub-Whitney doubling on U . A ball $B(x, r)$ with $x \in U$ is said to be a *sub-Whitney ball* if $0 < r \leq \frac{1}{A} \text{dist}(x, \partial U)$.

A metric measure space (Z, d_Z, μ_Z) with a doubling measure is a *doubling metric space*, that is, there exists some positive integer N such that for each $r > 0$ and $x_0 \in Z$, and for each $A \subset B(x_0, r)$ such that for each $z, w \in A$ with $z \neq w$ we have $d_Z(z, w) \geq r/2$, then A has at most N number of elements. On the other hand, there are doubling metric spaces that do not support a doubling measure. If the doubling metric space is complete, however, then it does support a doubling measure, see for example [30, 36]. The completion of a doubling metric space is also doubling, and complete doubling metric spaces are proper (that is, closed and bounded subsets are compact).

Definition 3.2. The p -modulus of a collection Γ of non-constant, compact, and rectifiable curves in U is

$$\text{Mod}_p(\Gamma; U) := \inf_{\rho} \int_U \rho^p d\mu_Z,$$

where the infimum is taken over all admissible ρ , that is, all non-negative Borel functions ρ such that $\int_{\gamma} \rho ds \geq 1$ for each $\gamma \in \Gamma$. A useful property is that $\text{Mod}_p(\Gamma; U) = 0$ if and only if there is a non-negative Borel function $\rho \in L^p(U)$ such that $\int_{\gamma} \rho ds = \infty$ for every $\gamma \in \Gamma$, see [20, 28]. Note that a countable union of zero p -modulus collections of curves is also of p -modulus zero. When $U = Z$, we simply write $\text{Mod}_p(\Gamma; U) = \text{Mod}_p(\Gamma)$.

In subsequent sections of this paper, Mod_p will denote the p -modulus with respect to the metric d and measure μ , while Mod_p^φ will denote the p -modulus with respect to the metric d_φ and measure μ_φ .

Definition 3.3. Following [19, 20], we say that a Borel function $g : U \rightarrow [0, \infty]$ is an *upper gradient* of a function $u : U \rightarrow \mathbb{R}$ if

$$|u(y) - u(x)| \leq \int_{\gamma} g ds$$

whenever γ is a non-constant compact rectifiable curve in U , with x and y denoting the two end points of γ . For $1 \leq p < \infty$, we say that g is a p -weak upper gradient of u if the collection Γ of non-constant compact rectifiable curves for which the above inequality fails is of p -modulus zero.

It is not difficult to see that if g_1 and g_2 are both p -weak upper gradients of u , then so is $\lambda g_1 + (1 - \lambda)g_2$ whenever $0 \leq \lambda \leq 1$. Let $D_p(u)$ denote the collection of all p -weak upper gradients of u ; then $D_p(u) \cap L^p(U)$ is a closed convex subset of $L^p(U)$, and so if $D_p(u) \cap L^p(U)$ is non-empty, then it has a unique element g_u of smallest L^p -norm; this function g_u is called the *minimal p -weak upper gradient* of u in U . We invite the interested readers to see [20] for more details on p -weak upper gradients.

In subsequent sections of the paper, when the minimal p -weak upper gradient of u is taken with respect to the metric d , it will be denoted by $g_{u,d}$, while $g_{u,\varphi}$ will denote the minimal p -weak upper gradient when taken with respect to d_φ .

We say that u is in the *Dirichlet-Sobolev* class $D^{1,p}(U)$ if $D_p(u) \cap L^p(U)$ is non-empty. We say that u is in the *Newton-Sobolev* class $N^{1,p}(U)$ if $u \in D^{1,p}(U)$ with $\int_U |u|^p d\mu_Z$ finite.

Definition 3.4. We say that U supports a *uniformly local p -Poincaré inequality* if there are constants $C_P > 0$, $\lambda \geq 1$, and for each $x \in U$ there exists $r_x > 0$, such that whenever $0 < r \leq r_x$ and $u \in D^{1,p}(U)$, we have

$$\int_{B(x,r) \cap U} |u - u_{B(x,r) \cap U}| d\mu_Z \leq C_P r \left(\int_{B(x,\lambda r) \cap U} g_u^p d\mu_Z \right)^{1/p},$$

where

$$u_{B(x,r) \cap U} := \int_{B(x,r) \cap U} u d\mu_Z := \frac{1}{\mu_Z(B(x,r) \cap U)} \int_{B(x,r) \cap U} u d\mu_Z.$$

Moreover, U supports a *sub-Whitney p -Poincaré inequality* if there is a constant $A \geq 1$ such that for each $x \in U$ we can choose $r_x = \frac{1}{A} \text{dist}(x, \partial U)$. We say that Z supports a *p -Poincaré inequality* if $U = Z$ and we can choose $r_x = \infty$ for each $x \in U$.

Remark 3.5. If U is bounded and supports a uniformly local p -Poincaré inequality, then $D^{1,p}(U) = N^{1,p}(U)$ as vector spaces, but their norms are naturally different. The norm on $N^{1,p}(U)$ incorporates the L^p -norm of the function in addition to the energy seminorm inherited from $D^{1,p}(U)$; for a function $u \in D^{1,p}(U)$, its energy seminorm is $\|u\|_{D^{1,p}(U)} := \inf_g \left(\int_U g^p d\mu_Z \right)^{1/p}$, where the infimum is over all upper gradients g of u . To turn $D^{1,p}(U)$, with this energy seminorm, into a normed space, one would have to form a quotient space where two functions $u_1, u_2 \in D^{1,p}(U)$ are said to be equivalent if $\|u_1 - u_2\|_{D^{1,p}(U)} = 0$; in particular, two functions that differ by a constant would have to be considered to be equivalent. We do not wish to do so, and hence $D^{1,p}(U)$ is only a seminormed space.

Definition 3.6. Given two sets $E, F \subset U$, the variational p -capacity of the condenser $(E, F; U)$ is the number

$$\text{cap}_p(E, F; U) := \inf_u \int_U g_u^p d\mu_Z,$$

where the infimum is over all functions $u \in N^{1,p}(U)$ with $u \geq 1$ on E and $u \leq 0$ on F . When $U = Z$, we simply write $\text{cap}_p(E, F; U) = \text{cap}_p(E, F)$.

For a set $A \subset U$, by $\text{Cap}_p(A)$ we mean the Sobolev p -capacity

$$\text{Cap}_p(A) := \inf_u \int_U [|u|^p + g_u^p] d\mu_Z,$$

where the infimum is over all functions $u \in N^{1,p}(U)$ with $u \geq 1$ on A .

Should U support a local p -Poincaré inequality, then functions in $D^{1,p}(U)$ are necessarily p -quasicontinuous. Hence, in the above definitions of capacities, we can also insist on the admissible functions u satisfying $u \geq 1$ in a neighborhood of the sets E , A , respectively, and $u \leq 0$ in a neighborhood of the set F ; see for example [24] or [3, Theorem 6.11]. Moreover, by [3, Proposition 1.48], we have that $\text{Cap}_p(A) = 0$ if and only if $\mu_Z(A) = 0$ and $\text{Mod}_p(\Gamma_A) = 0$, where Γ_A consists of all non-constant compact rectifiable curves in U that intersect A .

Remark 3.7. By [20, Corollary 9.3.2], we know that

$$\text{cap}_p(E, F; U) = \text{Mod}_p(\Gamma(E, F; U); U) =: \text{Mod}_p(E, F; U),$$

where $\Gamma(E, F; U)$ is the collection of all rectifiable curves in U that intersect both E and F .

As with the modulus, the notation $\text{cap}_p, \text{Cap}_p$ will be used throughout the paper when taken with respect to the metric d and measure μ , and $\text{cap}_p^\varphi, \text{Cap}_p^\varphi$ when taken with respect to the metric d_φ and measure μ_φ .

Next we recall the definition of p -harmonic functions on a metric measure space.

Definition 3.8. A function u on U is said to be a p -minimizer if $u \in D^{1,p}(U)$ and whenever $v \in D^{1,p}(U)$ has compact support V contained in U , then

$$\int_V g_u^p d\mu_Z \leq \int_V g_v^p d\mu_Z.$$

If (U, d_Z, μ_Z) is locally doubling and locally supports a p -Poincaré inequality, then there is a locally Hölder continuous representative of u , see for example [26]. Continuous p -minimizers are called *p -harmonic functions*.

Definition 3.9. For $t \geq 0$, the t -codimensional Hausdorff measure of a set $A \subset U$ is defined as

$$\mathcal{H}^{-t}(A; U) = \lim_{\varepsilon \rightarrow 0^+} \mathcal{H}_\varepsilon^{-t}(A; U),$$

where for each $\varepsilon > 0$,

$$\mathcal{H}_\varepsilon^{-t}(A; U) = \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu_Z(B(x_i, r_i) \cap U)}{r_i^t} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < \varepsilon \right\}.$$

Lemma 3.10. Let $f \in L^p(U)$, $0 < t < p$, and $M > 0$. If μ_Z is locally uniformly doubling on U , then $\mathcal{H}^{-t}(E_M) = 0$, where

$$E_M = \left\{ x \in U : \limsup_{r \rightarrow 0^+} r^t \int_{B(x,r) \cap U} |f|^p d\mu_Z > M^p \right\}.$$

In the above lemma, the conclusion is valid even if $t \geq p$, but in our use of this lemma in the proof of Proposition 3.11 below we require that $t < p$. However, when t is larger than the lower mass bound exponent of μ_Z as discussed in Section 7, then it can be shown from the fact that $f \in L^p(U)$ that E_M is empty.

Proof. Fix $\varepsilon > 0$. From continuity of the integral there exists $\delta > 0$ such that for all measurable $V \subset U$,

$$\mu_Z(V) < \delta \implies \int_V |f|^p d\mu_Z < \varepsilon.$$

By the Lebesgue differentiation theorem and the fact that $t > 0$, we see that $\mu_Z(E_M) = 0$ and so there exists an open set $W \subset U$ with $E_M \subset W$ for which $\mu_Z(W) < \delta$. It follows that $\int_{W \cap U} |f|^p d\mu_Z < \varepsilon$.

We construct a cover of E_M by balls in the following way. For each $x \in E_M$, select $r_x > 0$ such that

- (1) $0 < r_x < \varepsilon/5$,
- (2) $B(x, 5r_x) \subset W$,
- (3) $r_x^t \int_{B(x, r_x) \cap U} |f|^p d\mu_Z > M^p$.

This follows from the definition of E_M and the fact that W is open. An application of the basic $5r$ -covering lemma (see for example [18, Theorem 1.2]) yields a countable pairwise disjoint subcollection $\{B(x_i, r_i)\}$ such that $E_M \subset \bigcup B(x_i, 5r_i)$. Hence,

$$\begin{aligned} \mathcal{H}_\varepsilon^{-t}(E_M) &\leq \sum_i \frac{\mu_Z(B(x_i, 5r_i))}{(5r_i)^t} \leq \frac{C_d^3}{5^t} \sum_i \frac{\mu_Z(B(x_i, r_i))}{r_i^t} \\ &< \frac{C_d^3}{5^t M^p} \int_{W \cap U} |f|^p d\mu_Z < \frac{C_d^3}{5^t M^p} \varepsilon. \end{aligned}$$

The result follows by sending $\varepsilon \rightarrow 0^+$. $\square$

The following proposition relates the p -capacity of a set to its codimensional Hausdorff measure. In the Euclidean setting, the following proposition can be found in [14, Section 4.7.2, Theorem 4].

Proposition 3.11. *Let U support a uniformly local p -Poincaré inequality, and let μ_Z be a locally uniformly doubling measure on U . If $\text{Cap}_p(A) = 0$ for $A \subset U$, then $\mathcal{H}^{-t}(A) = 0$ for all $0 < t < p$.*

Proof. If $\text{Cap}_p(A) = 0$, then for each $k \in \mathbb{N}$ there exists a function $u_k \in N^{1,p}(U)$ such that $u_k \geq 1$ on a neighborhood of A , $0 \leq u_k \leq 1$ on U , and

$$\int_U [u_k^p + g_k^p] d\mu_Z < \frac{1}{2^{kp}},$$

where $g_k := g_{u_k}$. Define $u = \sum_k u_k$. Then $g = \sum_k g_k$ is a p -weak upper gradient of u with

$$\left(\int_U [u^p + g^p] d\mu_Z \right)^{1/p} \leq \sum_k \left(\int_U [u_k^p + g_k^p] d\mu_Z \right)^{1/p} < \sum_k \frac{1}{2^k} < \infty.$$

It follows that $u \in N^{1,p}(U)$.

Since each u_k is at least 1 on a neighborhood of A , we have that for $M \geq 1$, $A \subset \{u \geq M\}^\circ$. Hence, for $x \in A$ there exists $r_x > 0$ such that $B(x, r_x) \subset \{u \geq M\}$, and so $u_{B(x,r) \cap U} \geq M$ for $0 < r \leq r_x$. Since this is true for all $M \geq 1$, it follows that $u_{B(x,r) \cap U} \rightarrow \infty$ as $r \rightarrow 0^+$ for each $x \in A$.

Fix $x \in A$ and assume that

$$\limsup_{r \rightarrow 0^+} r^t \int_{B(x, \lambda r) \cap U} g^p d\mu_Z < \infty.$$

Then for all $0 < r \leq 1$ there is some $M \geq 1$ for which

$$r^t \int_{B(x, \lambda r) \cap U} g^p d\mu_Z \leq M^p.$$

By the Poincaré inequality,

$$\int_{B(x, r) \cap U} |u - u_{B(x, r) \cap U}| d\mu_Z \leq C_P r \left(\int_{B(x, \lambda r) \cap U} g^p d\mu_Z \right)^{1/p} \leq C_P M r^{1-t/p}.$$

It follows then that

$$|u_{B(x, r/2) \cap U} - u_{B(x, r) \cap U}| \leq C_d C_P M r^{1-t/p}$$

and so, for $k > j$,

$$|u_{B(x, r/2^k) \cap U} - u_{B(x, r/2^j) \cap U}| \leq C_d C_P M r^{1-t/p} \sum_{i=j+1}^k 2^{(i-1)(1-t/p)}.$$

As $0 \leq t < p$, $1 - \frac{t}{p} > 0$ and so this is the tail of a convergent geometric series. From this we have that $\{u_{B(x, r/2^k) \cap U}\}$ is a Cauchy sequence in $\mathbb{R}$, contradicting the fact that $u_{B(x, r) \cap U} \rightarrow \infty$ as $r \rightarrow 0^+$. Therefore,

$$\limsup_{r \rightarrow 0^+} r^t \int_{B(x, r) \cap U} g^p d\mu_Z = \infty.$$

Since this is true for each $x \in A$, we have that $A \subset E_M$ for any $M \geq 1$, each of which has t -codimensional Hausdorff measure zero by Lemma 3.10. The result follows. $\square$

We end this section by defining two notions that are tools in the study of potential theory.

Definition 3.12. For $\alpha > 0$ and $1 \leq p < \infty$, we set the Besov space $B_{p,p}^\alpha(Z)$ to be the class of all functions $f \in L_{loc}^1(Z)$ such that

$$\|f\|_{B_{p,p}^\alpha(Z)}^p := \int_Z \int_Z \frac{|f(y) - f(x)|^p}{d_Z(x, y)^{\alpha p} \mu_Z(B(x, d_Z(x, y)))} d\mu_Z(y) d\mu_Z(x) < \infty.$$

It was shown in [11] that functions in the Besov class are in $L^p(Z, \mu_Z)$ if Z is bounded. Besov spaces arise naturally as the trace class of Sobolev spaces. While this is well-known in the setting of Euclidean spaces, see for example [23] or [33, Chapter 10], the following extension to the setting of metric spaces is found in [32, Theorem 1.1].

Proposition 3.13. *Suppose that (Z, d_Z, μ_Z) is doubling and supports a p -Poincaré inequality for some $1 \leq p < \infty$, and let $U \subset Z$ be a bounded uniform domain such that $\mu_Z|_U$ is also doubling. Suppose that there is a measure ν on ∂U and some positive $\theta < p$ such that for each $w \in \partial U$ and $0 < r \leq \text{diam } \partial U$, we have*

$$\nu(B(w, r) \cap \partial U) \approx \frac{\mu_Z(B(w, r) \cap U)}{r^\theta}.$$

Then there is a bounded linear surjective operator $T : N^{1,p}(U) \rightarrow B_{p,p}^{1-\theta/p}(\partial U)$ such that for ν -a.e. $x \in \partial U$,

$$\lim_{r \rightarrow 0^+} \int_{B(x,r) \cap U} |u - Tu(x)|^p d\mu_Z = 0.$$

Moreover, there is a bounded linear operator $E : B_{p,p}^{1-\theta/p}(\partial U) \rightarrow N^{1,p}(U)$ such that $T \circ E$ is the identity operator on $B_{p,p}^{1-\theta/p}(\partial U)$.

Definition 3.14. For a non-negative function u , we define the *Riesz potential* of u relative to U as

$$I_{1,U}u(x) = \int_U \frac{u(y) d_Z(x, y)}{\mu_Z(B(x, d_Z(x, y)))} d\mu_Z(y).$$

The following proposition is an application of [31, Corollary 4.2] to the setting where $X = \bar{U}$ and ν is the measure, as in Proposition 3.13 above, supported on ∂U . In this proposition, $Q_{\bar{Z}}$ plays the role of the lower mass bound exponent for the measure $\mu_Z|_U$:

$$\left(\frac{r}{R}\right)^{Q_{\bar{Z}}} \lesssim \frac{\mu_Z(B(x, r) \cap U)}{\mu_Z(B(x, R) \cap U)}$$

for all $x \in U$ and $0 < r < R < \infty$.

Proposition 3.15. *With U and ν as in Proposition 3.13, and let $1 < \bar{p} < \bar{q}$ such that*

$$Q_{\bar{Z}} + \bar{q} - \frac{Q_{\bar{Z}} \bar{q}}{\bar{p}} = \theta.$$

Then there is a constant $C \geq 1$ such that for all balls $B \subset Z$ centered at points in $\bar{U}$, setting $B_0 := B \cap \bar{U}$ and all $f \in L^{\bar{p}}(B_0, \mu_Z)$, we have

$$\left(\int_{B_0} (I_{1,B_0}|f|)^{\bar{q}} d\nu \right)^{1/\bar{q}} \leq C \mu_Z(B_0)^{1/\bar{q}-1/\bar{p}} \text{rad}(B)^{1-\theta/\bar{q}} \left(\int_{B_0} |f|^{\bar{p}} d\mu_Z \right)^{1/\bar{p}}.$$

4. DOUBLING PROPERTY OF μ_φ

In this section we establish the doubling property of the measure μ_φ . Recall the constant $\kappa > 1$ established in Lemma 2.11. Since it follows from (2.2) that $\sum_{n=1}^\infty 2^n \varphi(2^n)$ is finite, we can find $r_0 > 0$ such that whenever m is a positive integer with $2^m \varphi(2^m) \leq \kappa r_0$ we have that $m > n_0 + 2$. Here, as in Section 2, n_0 is the positive integer satisfying $2^{n_0-1} \leq C_U < 2^{n_0}$.

From Lemma 2.11 we know that if $x \in B_{d_\varphi}(\infty, r) \setminus \{\infty\}$ with $r \leq r_0$, then necessarily $x \in \Omega_m$ for some $m > n_0 + 2$.

Lemma 4.1. *For $0 < r < r_0$,*

$$\mu_\varphi(B_{d_\varphi}(\infty, r)) \leq \left[MC_\varphi^{Mp} C_0^{M+1} + 1 \right] \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n),$$

and

$$\mu_\varphi(B_{d_\varphi}(\infty, r)) \geq \left[MC_\varphi^{Mp} C_0^{M+1} + 1 \right]^{-1} \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n),$$

where $M = \frac{\log(\kappa^2)}{\log(\tau/2)}$ and m is any non-negative integer such that for some $x \in \Omega$ with $d_\varphi(x, \infty) = r$ we have $x \in \Omega_m$. Moreover, if k is also a non-negative integer such that Ω_k contains a point z with $d_\varphi(z, \infty) = r$, then $|k - m| \leq M$.

Proof. Let m_1 be the smallest non-negative integer such that $2^{m_1} \varphi(2^{m_1}) \leq \kappa r$ and m_2 be the largest non-negative integer such that $\kappa 2^{m_2} \varphi(2^{m_2}) \geq r$. As $r \leq r_0$, we have that $m_1 > n_0 + 2$ and $m_2 \geq n_0 + 2$. Since for each $m \geq 1$, every point in Ω_m can be connected to ∞ by a d_φ -geodesic by Lemma 2.6, there is some $x \in \Omega$ such that $d_\varphi(x, \infty) = r$. With m a positive integer such that $x \in \Omega_m$, by Lemma 2.11 we know that $m_1 \leq m \leq m_2$. From Lemma 2.11 again, we have that

$$B_{d_\varphi}(\infty, r) \subset \bigcup_{n=m_1}^{\infty} \Omega_n \quad \text{and} \quad \bigcup_{n=m_2}^{\infty} \Omega_n \subset B_{d_\varphi}(\infty, r),$$

from where it follows from the construction of μ_φ and Condition (3) that

$$\mu_\varphi(B_{d_\varphi}(\infty, r)) \leq \sum_{n=m_1}^{\infty} \mu_\varphi(\Omega_n) \leq C_\varphi^p \sum_{n=m_1}^{\infty} \varphi(2^n)^p \mu(\Omega_n) \quad (4.2)$$

and, this time from the fact that φ is decreasing, that

$$\mu_\varphi(B_{d_\varphi}(\infty, r)) \geq \sum_{n=m_2}^{\infty} \mu_\varphi(\Omega_n) \geq \sum_{n=m_2}^{\infty} \varphi(2^n)^p \mu(\Omega_n). \quad (4.3)$$

We now estimate $m - m_1$ and $m_2 - m$. Invoking Lemma 2.11, we have that

$$\kappa^2 2^m \varphi(2^m) \geq 2^{m_1} \varphi(2^{m_1}),$$

and so

$$\kappa^2 2^{m-m_1} \varphi(2^m) \geq \varphi(2^{m_1}) \geq \tau^{m-m_1} \varphi(2^m),$$

from where it follows that (recall that $\tau > 2$)

$$0 \leq m - m_1 \leq \frac{\log(\kappa^2)}{\log(\tau/2)} = M.$$

Similarly, we have $0 \leq m_2 - m \leq M$. Now by combining (2.9) with (4.2) we obtain

$$\begin{aligned} \mu_\varphi(B_{d_\varphi}(\infty, r)) &\leq \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n) + \sum_{n=m_1}^{m-1} \varphi(2^n)^p \mu(\Omega_n) \\ &\leq \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n) + M C_\varphi^{Mp} \varphi(2^m)^p C_0^{M+1} \mu(\Omega_m) \\ &\leq \left[1 + M C_\varphi^{Mp} C_0^{M+1}\right] \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n). \end{aligned}$$

By combining (2.9) with (4.3) instead, we obtain

$$\begin{aligned} \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n) &\leq \sum_{n=m_2}^{\infty} \varphi(2^n)^p \mu(\Omega_n) + \sum_{n=m}^{m_2-1} \varphi(2^n)^p \mu(\Omega_n) \\ &\leq \sum_{n=m_2}^{\infty} \varphi(2^n)^p \mu(\Omega_n) + M C_\varphi^{Mp} \varphi(2^{m_2})^p C_0^{M+1} \mu(\Omega_{m_2}) \\ &\leq \left[1 + M C_\varphi^{Mp} C_0^{M+1}\right] \sum_{n=m_2}^{\infty} \varphi(2^n)^p \mu(\Omega_n) \\ &\leq \left[1 + M C_\varphi^{Mp} C_0^{M+1}\right] \mu_\varphi(B_{d_\varphi}(\infty, r)), \end{aligned}$$

completing the proof. $\square$

We are now ready to prove the doubling property of μ_φ , and we do so via the following series of lemmata. The first lemma deals with balls centered at ∞ , the next two lemmata deal with balls that are far away from ∞ , and the final lemma deals with intermediate balls.

Lemma 4.4. *For $0 < r \leq r_0/2$, we have*

$$\mu_\varphi(B_{d_\varphi}(\infty, 2r)) \leq C_1 \mu_\varphi(B_{d_\varphi}(\infty, r)),$$

where

$$C_1 = \left[MC_\varphi^{Mp} C_0^{M+1} + 1\right]^2 \left[\widehat{M}C_\varphi^{\widehat{M}p} C_0^{\widehat{M}+1} + 1\right]$$

with

$$\widehat{M} = \frac{\log(2\kappa^2)}{\log(\tau/2)},$$

M is as in Lemma 4.1, and $\kappa > 1$ is as in Lemma 2.11.

Proof. Let $m, \widehat{m}$ be the largest positive integers such that there is some $x \in \Omega_m$ and $y \in \Omega_{\widehat{m}}$ with $d_\varphi(x, \infty) = r$ and $d_\varphi(y, \infty) = 2r$. Note then that $m \geq \widehat{m}$. Moreover, by the choice of r_0 , we know that $\widehat{m} \geq n_0 + 2$. By Lemma 4.1, we have

$$\left[MC_\varphi^{Mp} C_0^{M+1} + 1\right] \mu_\varphi(B_{d_\varphi}(\infty, r)) \geq \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n)$$

and

$$\mu_\varphi(B_{d_\varphi}(\infty, 2r)) \leq \left[MC_\varphi^{Mp} C_0^{M+1} + 1\right] \sum_{n=\widehat{m}}^{\infty} \varphi(2^n)^p \mu(\Omega_n).$$

We now use the fact that $m \geq n_0 + 2$ to estimate the size of $m - \widehat{m}$ in the same way as we estimated $m - m_1$ and $m_2 - m$ in the proof of Lemma 4.1. From Lemma 2.11 it follows that

$$\kappa 2^m \varphi(2^m) \geq r = \frac{1}{2}(2r) \geq \frac{1}{2\kappa} 2^{\widehat{m}} \varphi(2^{\widehat{m}}),$$

and so

$$2\kappa^2 2^{m-\widehat{m}} \varphi(2^m) \geq \varphi(2^{\widehat{m}}) \geq \tau^{m-\widehat{m}} \varphi(2^m),$$

from where it follows that (recall that $\tau > 2$)

$$m - \widehat{m} \leq \frac{\log(2\kappa^2)}{\log(\tau/2)} = \widehat{M}.$$

Thus,

$$\begin{aligned} \sum_{n=\widehat{m}}^{m-1} \varphi(2^n)^p \mu(\Omega_n) &\leq (m - \widehat{m}) \varphi(2^{\widehat{m}})^p C_0^{m-\widehat{m}+1} \mu(\Omega_m) \\ &\leq \widehat{M} C_\varphi^{\widehat{M}p} \varphi(2^m)^p C_0^{\widehat{M}+1} \mu(\Omega_m), \end{aligned}$$

and so

$$\begin{aligned} \sum_{n=\widehat{m}}^{\infty} \varphi(2^n)^p \mu(\Omega_n) &= \sum_{n=\widehat{m}}^{m-1} \varphi(2^n)^p \mu(\Omega_n) + \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n) \\ &\leq \left[\widehat{M} C_\varphi^{\widehat{M}p} C_0^{\widehat{M}+1} + 1 \right] \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n). \end{aligned}$$

Combining this with Lemma 4.1, we obtain the desired inequality. $\square$

Lemma 4.5. *Let $x \in \Omega_m$ for some positive integer m . For $0 < r < r_0/2$, if $\infty \in B_{d_\varphi}(x, r/2)$, then*

$$\mu_\varphi(B_{d_\varphi}(x, 2r)) \leq C_2 \mu_\varphi(B_{d_\varphi}(x, r)),$$

where C_2 depends on the constant C_1 from Lemma 4.4 above.

Proof. It follows from $\infty \in B_{d_\varphi}(x, r/2)$ that $B_{d_\varphi}(\infty, r/2) \subset B_{d_\varphi}(x, r)$ and $B_{d_\varphi}(x, 2r) \subset B_{d_\varphi}(\infty, 4r)$. Combining this with Lemma 4.4 yields

$$\mu_\varphi(B_{d_\varphi}(x, 2r)) \leq \mu_\varphi(B_{d_\varphi}(\infty, 4r)) \lesssim \mu_\varphi(B_{d_\varphi}(\infty, r/2)) \leq \mu_\varphi(B_{d_\varphi}(x, r)).$$

$\square$

Now we consider balls that are far away from ∞ .

Lemma 4.6. *Let $x \in \Omega$ and $0 < r \leq r_0/2$ such that $\infty \notin B_{d_\varphi}(x, C_*r)$, where $C_* = 4\kappa/c$ with κ from Lemma 2.11 and c from Lemma 2.13. Then*

$$\mu_\varphi(B_{d_\varphi}(x, 2r)) \leq C_3 \mu_\varphi(B_{d_\varphi}(x, r)),$$

where C_3 depends only on the structural constants C_μ , $\widehat{M}$ from Lemma 4.4, and C_A from Lemma 2.13.

Moreover, with m a non-negative integer such that $x \in \Omega_m$, we have that

$$B_{d_\varphi}(x, 2r) \subset B_d(x, 2C_A \varphi(2^m)^{-1}r) \quad \text{and} \quad B_d(x, C_A^{-1} \varphi(2^m)^{-1}r) \subset B_{d_\varphi}(x, r).$$

Furthermore, for all $y \in B_{d_\varphi}(x, 2r)$ we have that $\varphi(2^m) \approx \varphi(d_\Omega(y))$, with comparison constant independent of x, y, r, m .

It follows from the above lemma that $B_{d_\varphi}(x, r)$ is a quasiball with respect to the metric d , which is to say that there exists a constant $C > 0$, independent of x and r , such that $B_d(x, C^{-1}r) \subset B_{d_\varphi}(x, r) \subset B_d(x, Cr)$.

Proof. Since $\infty \notin B_{d_\varphi}(x, C_*r)$, from Lemma 2.11 we have that $x \in \Omega_m$ with

$$r \leq \frac{\kappa}{C_*} 2^m \varphi(2^m).$$

Note that as $r \leq r_0$, by the choice of r_0 , the above inequality holds if $m \leq n_0 + 1$. So we needed Lemma 2.11 only for the case that $m \geq n_0 + 2$. By our choice of C_* , which is greater than $2\kappa/c$, for all $y \in B_{d_\varphi}(x, 2r)$,

$$d_\varphi(x, y) < 2r \leq \frac{2\kappa}{C_*} 2^m \varphi(2^m) < c 2^m \varphi(2^m),$$

and so it follows from Lemma 2.13 that

$$\frac{1}{C_A} \varphi(2^m) d(x, y) \leq d_\varphi(x, y) \leq C_A \varphi(2^m) d(x, y). \quad (4.7)$$

Therefore,

$$B_{d_\varphi}(x, 2r) \subset B_d(x, 2C_A \varphi(2^m)^{-1}r) \quad \text{and} \quad B_d(x, C_A^{-1} \varphi(2^m)^{-1}r) \subset B_{d_\varphi}(x, r).$$

The latter inclusion is seen by noting that, thanks to Lemma 2.13 and (4.7), we must have $B_d(x, C_A^{-1} \varphi(2^m)^{-1}r) \cap B_{d_\varphi}(x, 2r) \subset B_{d_\varphi}(x, r)$, and as balls with respect to the metric d are connected with respect to both metrics d and d_φ , we must have that $B_d(x, C_A^{-1} \varphi(2^m)^{-1}r) \subset B_{d_\varphi}(x, 2r)$ for otherwise, $B_d(x, C_A^{-1} \varphi(2^m)^{-1}r) \cap B_{d_\varphi}(x, 2r) \setminus B_{d_\varphi}(x, r)$ would be non-empty.

Suppose first that $m \geq n_0 + 2$. As in the proof of the previous two lemmata, we have that if $y \in B_{d_\varphi}(x, r)$ and $\tilde{m} \in \mathbb{N}$ such that $y \in \Omega_{\tilde{m}}$, then $|m - \tilde{m}| \leq \tilde{M}$. Indeed, in Lemma 2.13 we can fix the choice of C_A and then always make c smaller and thus C_* larger, and so without loss of generality we have that $C_A \kappa / C_* < 1/2$. We also have from the limitation on r and from (4.7) that $d(x, y) < C_A \frac{\kappa}{C_*} 2^m$. It follows from the fact that $x \in \Omega_m$ and $y \in \Omega_{\tilde{m}}$ that $d_\Omega(x) \geq 2^m$ and $d_\Omega(y) \leq 2^{\tilde{m}+1}$; so by the triangle inequality,

$$2^m - 2^{\tilde{m}+1} \leq C_A \frac{\kappa}{C_*} 2^m < 2^{m-1},$$

whence we obtain $2^{m-1} \leq 2^{\tilde{m}+1}$, that is, $m - \tilde{m} \leq 2$. Similarly we have that $2^{\tilde{m}} - 2^{m+1} < 2^{m-1}$, and so $2^{\tilde{m}} < 2^{m+2}$, that is, $\tilde{m} - m < 2$. So the choice of $\tilde{M} = 2$ satisfies the above statement about $|m - \tilde{m}|$. It follows now from the assumptions on φ that $\varphi(2^m) \approx \varphi(2^{\tilde{m}})$ for all $y \in B_{d_\varphi}(x, 2r)$, and so by the doubling property of μ ,

$$\begin{aligned} \mu_\varphi(B_{d_\varphi}(x, 2r)) &\lesssim \varphi(2^m)^p \mu(B_{d_\varphi}(x, 2r)) \\ &\lesssim C_\mu^{k_0} \varphi(2^m)^p \mu(B_d(x, C_A^{-1} \varphi(2^m)^{-1}r)) \\ &\lesssim C_\mu^{k_0} \varphi(2^m)^p \mu(B_{d_\varphi}(x, r)) \\ &\lesssim C_\mu^{k_0} \mu_\varphi(B_{d_\varphi}(x, r)). \end{aligned}$$

Here, $k_0 = \log(2C_A^2)$.

Finally, we take care of the case that $m \leq n_0 + 1$. In this case, by Condition (3) of Definition 2.1, we have $1 \geq \varphi(2^m) \geq \varphi(2^{n_0+1}) \geq C_\varphi^{n_0+1} \varphi(1) =$

$C_\varphi^{m_0+1}$, that is, for $m \leq n_0 + 1$ we have that $\varphi(2^m) \approx 1$. By Lemma 2.13 again, we have that whenever $y \in B_{d_\varphi}(x, 2r)$, necessarily

$$d(x, y) \leq \frac{2C_A}{\varphi(2^m)} r \leq \frac{2C_A}{\varphi(2^m)} r_0,$$

and so by choosing a positive integer k_1 such that $2^{k_1} > \frac{2C_A^2}{\varphi(2^m)} r_0$, we have that each $y \in B_{d_\varphi}(x, 2C_A r)$ is in some Ω_n with $n < k_1$, and so $\varphi(d_\Omega(y)) \approx 1$ as well. Therefore,

$$\begin{aligned} \mu_\varphi(B_{d_\varphi}(x, 2r)) &\approx \mu(B_{d_\varphi}(x, 2r)) \leq \mu(B_d(x, 2C_A \varphi(2^m)^{-1} r)) \\ &\lesssim \mu(B_d(x, C_A^{-1} \varphi(2^m)^{-1} r)) \\ &\lesssim \mu(B_{d_\varphi}(x, r)) \\ &\lesssim \mu_\varphi(B_{d_\varphi}(x, r)), \end{aligned}$$

finishing the proof. $\square$

Finally, we take care of the intermediate balls. In what follows, set

$$T = \frac{2C_A}{c}. \quad (4.8)$$

Lemma 4.9. *Let $x \in \Omega$ and $0 < r \leq \frac{r_0}{8(C_*+1)}$ such that $\infty \in B_{d_\varphi}(x, C_* r) \setminus B_{d_\varphi}(x, r/2)$, where C_* is as in Lemma 4.6, then*

$$\mu_\varphi(B_{d_\varphi}(x, 2r)) \leq C_4 \mu_\varphi(B_{d_\varphi}(x, r)).$$

Moreover, with m a non-negative integer such that $x \in \Omega_m$ and fixing $C_\Lambda \geq 1$, independent of x , m , and r , so that

$$\frac{2^m \varphi(2^m)}{C_\Lambda} \leq r \leq C_\Lambda 2^m \varphi(2^m),$$

for each $y \in B_{d_\varphi}(x, r/(8TC_\Lambda))$ we have $\varphi(d_\Omega(y)) \approx \varphi(2^m)$ and

$$\mu(\Omega_m) \approx \mu(B_d(x, 2^m/(8TC_\Lambda^2 C_A))),$$

and

$$B_d(x, 2^m/(8TC_\Lambda^2 C_A)) \subset B_{d_\varphi}(x, r/(8TC_\Lambda)).$$

The constant C_4 depends only on the structural constants C_μ , C_φ , and the constants C_0 from Lemma 2.8, κ from Lemma 2.11, C_A from Lemma 2.13, and M from Lemma 4.1.

Proof. By our assumptions, $B_{d_\varphi}(x, 2r) \subset B_{d_\varphi}(\infty, (C_* + 2)r)$. Therefore

$$\mu_\varphi(B_{d_\varphi}(x, 2r)) \leq \mu_\varphi(B_{d_\varphi}(\infty, 2(C_* + 1)r)).$$

Since $2(C_* + 1)r \leq r_0/4$, it follows from Lemma 4.1 that with the choice of m so that $x \in \Omega_m$,

$$\mu_\varphi(B_{d_\varphi}(x, 2r)) \lesssim \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n) \lesssim \varphi(2^m)^p \mu(\Omega_m), \quad (4.10)$$

and as $d_\varphi(x, \infty) \geq r/2$, we have that $d_\varphi(x, \infty) \approx r$. Now by Lemma 2.11 we have that

$$2^m \varphi(2^m) \approx d_\varphi(x, \infty) \approx r.$$

Thus there is a constant $C_\Lambda > 1$, which is independent of x , m , and r , so that

$$\frac{2^m \varphi(2^m)}{C_\Lambda} \leq r \leq C_\Lambda 2^m \varphi(2^m).$$

Thus with our choice of T from (4.8), from Lemma 2.13 we have that if $y \in \Omega$ such that $d_\varphi(x, y) \leq \frac{r}{8TC_\Lambda}$, then

$$\frac{d(x, y)}{C_A} \varphi(2^m) \leq d_\varphi(x, y) \leq C_A d(x, y) \varphi(2^m).$$

It follows that $d(x, y) \leq \frac{C_A 2^m}{8T}$. By our choice of T , we know that $T > C_A$, and so we have that $d(x, y) < 2^{m-3}$. Therefore

$$d_\Omega(y) \geq d_\Omega(x) - 2^{m-3} \geq 2^{m-2} \quad \text{and} \quad d_\Omega(y) \leq d_\Omega(x) + 2^{m-3} \leq 2^{m+1}.$$

It follows that for each $y \in B_{d_\varphi}(x, r/(8TC_\Lambda))$ we have $\varphi(d_\Omega(y)) \approx \varphi(2^m)$. Hence

$$\mu_\varphi(B_{d_\varphi}(x, r/(8TC_\Lambda))) \approx \varphi(2^m)^p \mu(B_{d_\varphi}(x, r/(8TC_\Lambda))).$$

Moreover, if $y \in \Omega$ such that $d(x, y) < 2^m/(8TC_\Lambda^2 C_A)$, then $d_\varphi(x, y) < r/(8TC_\Lambda)$, and so $y \in B_{d_\varphi}(x, r/(8TC_\Lambda))$. By the doubling property of μ , we know that with $\zeta \in \partial\Omega$ (and, without loss of generality, assuming that $\text{diam}_d(\partial\Omega) \leq 1$),

$$\mu(B_d(x, 2^m/(8TC_\Lambda^2 C_A))) \approx \mu(B_d(\zeta, 2^{m+1})) \approx \mu\left(\bigcup_{n=0}^{m+1} \Omega_n\right),$$

and so we have that

$$\mu(\Omega_m) \lesssim \mu(B_d(x, 2^m/(8TC_\Lambda^2 C_A))).$$

From the above discussion, we now have that

$$\begin{aligned} \varphi(2^m)^p \mu(\Omega_m) &\lesssim \varphi(2^m)^p \mu(B_d(x, 2^m/(8TC_\Lambda^2 C_A))) \\ &\lesssim \varphi(2^m)^p \mu(B_{d_\varphi}(x, r/(8TC_\Lambda))) \\ &\approx \mu_\varphi(B_{d_\varphi}(x, r/(8TC_\Lambda))). \end{aligned}$$

Combining the above estimate with (4.10) we obtain

$$\mu_\varphi(B_{d_\varphi}(x, 2r)) \lesssim \varphi(2^m)^p \mu(\Omega_m) \lesssim \mu_\varphi(B_{d_\varphi}(x, r/(8TC_\Lambda))) \leq \mu_\varphi(B_{d_\varphi}(x, r)),$$

as desired. $\square$

The above lemmata together prove that μ_φ is a uniformly locally doubling measure on $(\Omega_\varphi, d_\varphi)$. In fact, μ_φ is globally doubling as the assumption that $\partial\Omega$ is bounded implies the compactness of $\overline{\Omega_\varphi}^\varphi = \overline{\Omega} \cup \{\infty\}$ with respect to the metric d_φ . We summarize this in the following theorem.

Theorem 4.11. *The metric measure space $(\Omega_\varphi, d_\varphi, \mu_\varphi)$ is doubling.*

Conversely, $\overline{\Omega_\varphi}^\varphi$ cannot be compact with respect to d_φ (hence μ_φ cannot be doubling on Ω_φ) without $\partial\Omega$ being bounded, as we will see now.

Proposition 4.12. *$\overline{\Omega_\varphi}^\varphi = \overline{\Omega} \cup \{\infty\}$ is compact with respect to the metric d_φ if and only if $\partial\Omega$ is bounded with respect to the metric d .*

Proof. Assume that $\partial\Omega$ is bounded with respect to d . Consider a sequence $(x_k) \subset \overline{\Omega}_\varphi$. If infinitely-many of the terms in the sequence equal ∞ or if $\liminf d_\varphi(x_k, \infty) = 0$, then there is a subsequence of (x_k) converging to infinity with respect to d_φ . As such, we assume without loss of generality that $(x_k) \subset \overline{\Omega}$ and that $d_\varphi(x_k, \infty) \geq \frac{\tau}{2}$ for all k , where $\tau := \liminf d_\varphi(x_k, \infty) > 0$. It follows from Lemma 2.11 that there is some N_0 for which

$$x_k \in \bigcup_{n=0}^{\overline{N_0}} \Omega_n = \bigcup_{n=0}^{N_0} \overline{\Omega}_n$$

for all k . Each set $\overline{\Omega}_n$ is closed in $\overline{\Omega}$ with respect to d and bounded since $\text{diam}_d(\Omega_n) \leq 2^{n+2} + \text{diam}_d(\partial\Omega) < \infty$. Hence, $\overline{\Omega}_n$ is compact with respect to d as $\overline{\Omega}$ is proper. By Lemma 2.13, d and d_φ are locally bi-Lipschitz equivalent and so the two metrics are homeomorphic on $\overline{\Omega}_n$. Hence, $\bigcup_{n=0}^{N_0} \overline{\Omega}_n$ is compact with respect to d_φ . Therefore, there exists a d_φ -convergent subsequence of (x_k) .

Assume now that $\partial\Omega$ is unbounded with respect to d . Then we may find a sequence $(\zeta_k) \subset \partial\Omega$ such that $d(\zeta_k, \zeta_j) \geq \tau \in (0, \frac{1}{10}]$ for all $k \neq j$. It follows from the local isometry of d and d_φ on $\partial\Omega$ (see Lemma 2.6 of [15]) that $d_\varphi(\zeta_k, \zeta_j) \geq \tau$ for all $k \neq j$, and so (ζ_k) cannot converge in $\overline{\Omega}$ with respect to d_φ . Moreover, $d_\varphi(\zeta_k, \infty) \geq 1$ for all k and so cannot converge to ∞ with respect to d_φ . $\square$

5. UNIFORMITY OF $\Omega_\varphi \setminus \{\infty\}$

In [15] it was shown that $\Omega_\varphi = \Omega \cup \{\infty\}$ is a uniform domain with respect to the metric d_φ and that $\partial\Omega_\varphi = \partial\Omega$. It is not always the case that the removal of a point from a uniform domain results in a uniform domain, as seen by the example domain $(-1, 0] \times [0, 1] \cup [0, 1] \times (-1, 0] \subset X$ where the metric space $X = \mathbb{R}^2 \setminus ((-1, 0)^2 \cup (0, 1)^2)$. However, in our setting, $\infty \in \Omega_\varphi$ has a special role, given the fact that Ω itself is a uniform domain with respect to the metric d .

In the rest of this section, by increasing the uniformity constant K if needed, we can find uniform curves for which every subcurve is also a uniform curve (all with respect to the metric d_φ), see [8, Theorem 2.10]. In the remainder of this section, when we choose a K -uniform curve with respect to the metric d_φ , we will also implicitly assume that every subcurve is also K -uniform with respect to d_φ .

Lemma 5.1. *Let $0 < r \leq r_0/C$ and $x, y \in B_{d_\varphi}(\infty, r) \setminus B_{d_\varphi}(\infty, r/2)$, where $C = 2\kappa^2 C_U C_\varphi$. Then there is a curve $\beta \subset B_{d_\varphi}(\infty, Cr) \setminus B_{d_\varphi}(\infty, r/C)$ with end points x, y such that $\ell_\varphi(\beta) \approx d_\varphi(x, y)$.*

Proof. Let m be a positive integer such that $x \in \Omega_m$. Then by (2.3) and by Lemma 2.11 we know that $d_\varphi(x, \infty) \approx 2^m \varphi(2^m)$. Thus, with $y \in \Omega_k$ we also have $2^k \varphi(2^k) \approx 2^m \varphi(2^m)$. It follows that there is some $N_0 \in \mathbb{N}$ such that $|k - m| \leq N_0$, see the proof of Lemma 4.1 above. Moreover, by the choice of r_0 we also have that $k \geq 2n_0$ and $m \geq 2n_0$.

Let β be a C_U -uniform curve in Ω , with respect to the original metric d , connecting x to y . Then, from (2.7) we see that $d(x, y) \lesssim 2^m$, and so

$\ell_d(\beta) \lesssim 2^{m+n_0}$. Hence

$$\beta \subset \bigcup_{j=m-n_0-N_0}^{m+n_0+N_0} \Omega_j,$$

whence we obtain

$$\ell_\varphi(\beta) \approx \varphi(2^m) \ell_d(\beta) \approx \varphi(2^m) d(x, y) \lesssim \varphi(2^m) 2^m \approx r.$$

By Lemma 2.11 above and our choice of C , we have

$$\beta \subset B_{d_\varphi}(\infty, Cr) \setminus B_{d_\varphi}(\infty, r/C).$$

It also follows that $d_\varphi(x, y) \approx \varphi(2^m) d(x, y) \approx \ell_\varphi(\beta)$, thus completing the proof. $\square$

In what follows, let $K \geq 1$ denote the uniformity constant of Ω_φ . If γ_1 is a curve with end points x and y , and γ_2 is a curve with end points y and z , then we denote by $\gamma_1 + \gamma_2$ the concatenation of two curves γ_1 and γ_2 , having end points x and z .

Theorem 5.2. *The set $\Omega_\varphi \setminus \{\infty\}$ is a uniform domain with respect to the metric d_φ .*

Proof. From [15] we know that $(\Omega_\varphi, d_\varphi)$ is a uniform domain. Let $x, y \in \Omega_\varphi \setminus \{\infty\}$ with $x \neq y$, and let γ be a K -uniform curve in Ω_φ , with the uniformity with respect to the metric d_φ , with end points x, y . Without loss of generality let $d_\varphi(x, \infty) \leq d_\varphi(y, \infty)$. We now consider two cases.

Case 1: We suppose first that

$$\gamma \subset \Omega_\varphi \setminus B_{d_\varphi}(\infty, d_\varphi(x, \infty)/4CK),$$

where $C = 2\kappa^2 C_U C_\varphi$ is from Lemma 5.1 above. In this case, for $z \in B_{d_\varphi}(\infty, 4d_\varphi(x, \infty)) \cap \gamma$, we have that

$$\frac{d_\varphi(x, \infty)}{4CK} \leq d_\varphi(z, \infty) < 4d_\varphi(x, \infty)$$

and so

$$\ell_\varphi(\gamma[x, z]) \leq K d_\varphi(x, z) < 5K d_\varphi(x, \infty) \leq 20CK^2 d_\varphi(z, \infty).$$

For $z \in \gamma \setminus B_{d_\varphi}(\infty, 4d_\varphi(x, \infty))$, we have that

$$\ell_\varphi(\gamma[x, z]) \leq K d_\varphi(x, z) \leq K[d_\varphi(x, \infty) + d_\varphi(z, \infty)] \leq 2K d_\varphi(z, \infty).$$

Combining the above two subcases, we have that for each $z \in \gamma$,

$$d_\varphi(z, \infty) \geq \frac{1}{20CK^2} \ell_\varphi(\gamma[x, z]) \geq \frac{1}{20CK^2} \min\{\ell_\varphi(\gamma[x, z]), \ell_\varphi(\gamma[z, y])\}.$$

Case 2: If, instead, we have

$$\gamma \cap B_{d_\varphi}(\infty, d_\varphi(x, \infty)/4CK) \neq \emptyset,$$

then let $w_1, w_2 \in \gamma$ such that

$$\gamma[x, w_1] \cup \gamma[w_2, y] \subset \Omega_\varphi \setminus B_{d_\varphi}(\infty, d_\varphi(x, \infty)/2CK)$$

and

$$d_\varphi(\infty, w_1) = d_\varphi(\infty, w_2) = \frac{d_\varphi(x, \infty)}{2CK}.$$

Let β be a curve from Lemma 5.1 with end points w_1, w_2 . For $z \in \gamma[x, w_1]$, a repetition of the argument of Case 1 above yields $d_\varphi(z, \infty) \geq \frac{1}{20CK^2} \ell_\varphi(\gamma[x, z])$. For $z \in \beta$, we have that

$$d_\varphi(z, \infty) \geq \frac{d_\varphi(x, \infty)}{2C^2K},$$

and

$$\begin{aligned} \ell_\varphi(\gamma[x, w_1]) + \ell_\varphi(\beta[w_1, z]) &\leq K d_\varphi(x, w_1) + \ell_\varphi(\beta) \\ &\lesssim d_\varphi(x, w_1) + d_\varphi(w_1, w_2) \\ &\lesssim d_\varphi(x, \infty) + 2d_\varphi(w_1, \infty) + d_\varphi(w_2, \infty) \\ &\lesssim d_\varphi(x, \infty). \end{aligned}$$

It follows that $d_\varphi(z, \infty) \gtrsim \ell_\varphi(\gamma[x, w_1] + \beta[w_1, z])$. For $z \in \gamma[w_2, y] \cap B_{d_\varphi}(\infty, 4d_\varphi(x, \infty))$, we have that

$$d_\varphi(z, \infty) \geq \frac{d_\varphi(x, \infty)}{2CK},$$

and

$$\ell_\varphi(\gamma[x, w_1] + \beta + \gamma[w_2, z]) \lesssim d_\varphi(x, \infty) + 4d_\varphi(x, \infty) \lesssim d_\varphi(x, \infty),$$

and therefore

$$d_\varphi(z, \infty) \gtrsim \ell_\varphi(\gamma[x, w_1] + \beta + \gamma[w_2, z]).$$

For $z \in \gamma[w_2, y] \setminus B_{d_\varphi}(\infty, 4d_\varphi(x, \infty))$, we have

$$\begin{aligned} \ell_\varphi(\gamma[w_2, z]) &\leq K d_\varphi(w_2, z) \leq K[d_\varphi(w_2, \infty) + d_\varphi(z, \infty)] \\ &= K[d_\varphi(x, \infty)/(2CK) + d_\varphi(z, \infty)] \\ &\lesssim d_\varphi(z, \infty). \end{aligned}$$

Observe also from the above discussion that

$$\ell_\varphi(\gamma[x, w_1] + \beta) \lesssim d_\varphi(x, \infty) \lesssim d_\varphi(z, \infty).$$

It then again follows that $d_\varphi(z, \infty) \gtrsim \ell_\varphi(\gamma[x, w_1] + \beta + \gamma[w_2, z])$. Combining the above four possibilities in this case, we obtain for each $z \in \hat{\gamma} := \gamma[x, w_1] + \beta + \gamma[w_2, y]$ that

$$d_\varphi(z, \infty) \gtrsim \ell_\varphi(\hat{\gamma}[x, z]).$$

From Cases 1 and 2 above we see that there is a curve $\hat{\gamma}$ with end points x, y such that for each $z \in \hat{\gamma}$ we have that

$$d_\varphi(z, \infty) \gtrsim \ell_\varphi(\hat{\gamma}[x, z]),$$

and moreover, $\ell_\varphi(\hat{\gamma}) \lesssim d_\varphi(x, y)$. Here, in Case 1 above, we merely set $\hat{\gamma} = \gamma$, the original uniform curve with respect to d_φ , connecting x to y . So in Case 1 we have from the K -uniformity of γ with respect to d_φ that $\ell_\varphi(\hat{\gamma}) \leq K d_\varphi(x, y)$. In Case 2 we have that $\ell_\varphi(\hat{\gamma}) \leq \ell_\varphi(\gamma) + \ell_\varphi(\beta) \leq K d_\varphi(x, y) + \ell_\varphi(\beta)$. Moreover, by the choice of β , we have that $\ell_\varphi(\beta) \lesssim d_\varphi(w_1, w_2) \leq \ell_\varphi(\gamma)$ because w_1, w_2 belong to γ , and hence again we have that $\ell_\varphi(\hat{\gamma}) \lesssim d_\varphi(x, y)$, thus justifying the inequality given above. Thus to show that $\hat{\gamma}$ is a uniform curve in $\Omega_\varphi \setminus \{\infty\}$, it now only remains to check $\text{dist}_\varphi(z, \partial\Omega)$ for each $z \in \hat{\gamma}$.

In the situation considered in Case 1 above, the curve $\widehat{\gamma}$ is also a K -uniform curve in Ω_φ with respect to d_φ , and so we immediately have

$$\text{dist}_\varphi(z, \partial\Omega) \geq \frac{1}{K} \min\{\ell_\varphi(\widehat{\gamma}[x, z]), \ell_\varphi(\widehat{\gamma}[z, y])\}$$

as desired. In the situation considered in Case 2, the above inequality holds also when $z \in \gamma[x, w_1] \cup \gamma[w_2, y]$. We set $r = d_\varphi(x, \infty)/2CK$. Note that then $\ell_\varphi(\gamma[x, w_1]) \geq (2CK - 1)r$, and hence by the uniformity of the curve γ , we have that

$$\text{dist}_\varphi(w_1, \partial\Omega) \geq \frac{2CK - 1}{K} r.$$

Hence, for each $z \in \beta$, we obtain

$$\begin{aligned} \text{dist}_\varphi(z, \partial\Omega) &\geq \text{dist}_\varphi(w_1, \partial\Omega) - d_\varphi(w_1, \infty) - d_\varphi(z, \infty) \\ &\geq \frac{2CK - 1}{K} r - r - Cr \\ &= (C - 1 - K^{-1})r. \end{aligned}$$

Here we used the fact that $\beta \subset B_{d_\varphi}(\infty, Cr) \setminus B_{d_\varphi}(\infty, r/C)$. Note that $\ell_\varphi(\gamma[x, w_1] + \beta) \lesssim r$. As $C > 2$, we now have the desired inequality $\text{dist}_\varphi(z, \partial\Omega) \gtrsim \ell_\varphi(\gamma[x, w_1] + \beta)$. The claim follows. $\square$

Remark 5.3. The proof of the above theorem can be modified to show that if Ω is a uniform domain and $z_0 \in \Omega$ is such that annular quasiconvexity as in Lemma 5.1 holds with z_0 playing the role of ∞ , then $\Omega \setminus \{z_0\}$ is also a uniform domain.

Remark 5.4. The utility of uniformity of $\Omega_\varphi \setminus \{\infty\}$ stems from the fact that when we transform Ω under the metric d_φ and the measure μ_φ , all the functions u in the Dirichlet-Sobolev class $D^{1,p}(\Omega)$ belong to the Dirichlet-Sobolev class of the transformed space $D^{1,p}(\Omega_\varphi \setminus \{\infty\})$; note that as sets, $\Omega = \Omega_\varphi \setminus \{\infty\}$. In order to gain control over the behavior of transformed functions in the uniform domain Ω_φ , we need to know that functions in $D^{1,p}(\Omega_\varphi \setminus \{\infty\})$ also have an extension to ∞ that belongs to $D^{1,p}(\Omega_\varphi)$; see the discussion in Section 7 and Proposition 7.5 below. Knowledge of uniformity of $\Omega_\varphi \setminus \{\infty\}$, together with the information that $\Omega_\varphi \setminus \{\infty\}$ satisfies a p -Poincaré inequality when Ω itself does (see Section 6 below) aids us in this extension.

6. POINCARÉ INEQUALITIES

The goal of this section is to demonstrate that $(\Omega_\varphi, d_\varphi, \mu_\varphi)$ supports a p -Poincaré inequality when (Ω, d, μ) supports a sub-Whitney p -Poincaré inequality as in Definition 3.4. This result can be proved using a variant of the Boman chain condition method that Hajlasz and Koskela [17] used to prove that if all balls satisfy a Poincaré inequality, then all sets satisfying a chain condition also satisfy a Poincaré inequality. We do not know *a priori* whether all balls in $(\Omega, d_\varphi, \mu_\varphi)$ satisfy a Poincaré inequality, but all balls in Ω_φ can be covered with chains of smaller (i.e., sub-Whitney) balls on which φ is approximately constant and which therefore inherit a Poincaré inequality from (Ω, d, μ) . We will also see that these small balls (with respect to the metric d) that make up the chain are quasiballs with respect to the

metric d_φ , as for example in the proof of Lemma 4.6. For the readers' convenience, we provide a complete proof here. Our proof uses an application of [7, Theorem 4.4]. We start with the following chain condition.

Definition 6.1. Let $\mathcal{B}$ be a family of balls in a metric measure space X and $\lambda, M \geq 1$ and $a > 1$. We say that $A \subset X$ satisfies the chain condition $C(\mathcal{B}, \lambda, M, a)$ if there exists a distinguished ball $B_0 \subset A$ that belongs to $\mathcal{B}$ such that for every $x \in A$ there exists an infinite sequence of balls $\{B_i\}_{i=0}^\infty \subset \mathcal{B}$ (called a “chain”) with the following properties:

- (i) $\lambda B_i \subset A$ for $i = 0, 1, 2, \dots$ and B_i is centered at x for all sufficiently large i ;
- (ii) for $i \geq 0$, the radius r_i of B_i satisfies $M^{-1}(\text{diam } A) a^{-i} \leq r_i \leq M(\text{diam } A) a^{-i}$; and,
- (iii) the intersection $B_i \cap B_{i+1}$ contains a ball B'_i such that $B_i \cup B_{i+1} \subset MB'_i$ for all $i \geq 0$.

In this section, we consider sub-Whitney balls corresponding to the constant $2\lambda M$, see Definition 3.1.

Remark 6.2. Given a uniform domain Ω , we set the collection $\mathcal{B}$ to consist of balls centered at points in Ω and with radii such that the ball is also contained in Ω . Then [7, Lemma 4.3] tells us that when Ω is a uniform domain, there are constants a, λ and M such that for each $x \in \Omega$ and $r > 0$, the set $B(x, r) \cap \Omega$ satisfies the chain condition $C(\mathcal{B}, \lambda, M, a)$ with x the center of the distinguished ball B_0 . The above chain condition is equivalent to the chain condition given in [7, Lemma 4.3]. While the chain of balls in [7] are not of strictly dyadically decreasing radii, there are at most L balls of the same radius, with L depending solely on the uniformity constant of Ω and the choice of λ ; hence, for sufficiently large M in our definition above, the chains constructed in [7, Lemma 4.3] satisfy the conditions in Definition 6.1. Therefore we can exploit [7, Theorem 4.4].

Theorem 6.3. *The uniform domains $\Omega_\varphi \setminus \{\infty\}$ and Ω_φ , as well as $\overline{\Omega_\varphi}^\varphi$, equipped with the metric d_φ and the measure μ_φ , all satisfy p -Poincaré inequality if (Ω, d, μ) satisfies a sub-Whitney p -Poincaré inequality.*

Proof. We wish to choose $\sigma > 1$ in applying [7, Lemma 4.3] such that when $x \in \Omega_m$ and $r > 0$ such that $\infty \notin B_{d_\varphi}(x, 4\sigma r)$, then $\infty \notin B_{d_\varphi}(x, 5C_A^2\lambda r)$ and $4C_A^2\lambda r < c2^m\varphi(2^m)$, where c, C_A are the constants from Lemma 2.13. We do this as follows.

Clearly the first condition is satisfied if $B_{d_\varphi}(x, 5C_A^2\lambda r) \subset B_{d_\varphi}(x, 4\sigma r)$, that is, whenever $\sigma \geq 5C_A^2\lambda/4$. To make sure that the second condition is also satisfied, we only consider the radii r for which $\infty \notin B_{d_\varphi}(x, 4\sigma r)$, that is, $d_\varphi(x, \infty) \geq 4\sigma r$. Combining this with Lemma 2.11, we obtain

$$4\sigma r \leq d_\varphi(x, \infty) \leq \kappa 2^m \varphi(2^m).$$

If $\sigma \geq C_A^2\lambda\kappa/c$, the above inequality implies that the second condition $4C_A^2\lambda r < c2^m\varphi(2^m)$ is satisfied. Henceforth, we fix a choice of

$$\sigma > \max \left\{ \frac{5C_A^2\lambda}{4}, \frac{C_A^2\lambda\kappa}{c} \right\}.$$

For each ball B_{d_φ} in $\Omega_\varphi \setminus \{\infty\}$ (that is, it is the intersection of a ball in $\overline{\Omega_\varphi}$ with $\Omega_\varphi \setminus \{\infty\}$), we can appeal to [7, Lemma 4.3] to construct chains of balls $B_i = B_{d_\varphi}(x_i, r_i)$, $i \in \mathbb{N}$, corresponding to the above choice of σ . From the above discussion and by Lemma 2.13, we have that

$$B_i \subset B_d(x_i, \frac{C_A}{\varphi(2^m)} r_i) \subset C_A^2 B_i \subset \Omega_\varphi \setminus \{\infty\}.$$

As $C_A^2 B_i \subset B_{d_\varphi}$ and $\infty \notin B_{d_\varphi}$, the last inclusion above holds. Moreover, the weight $\varphi(d_\Omega(y))^p$ is approximately constant on $C_A^2 B_i$, with the comparison constant independent of the ball. Therefore,

$$u_{B_d(x, \frac{C_A}{\varphi(2^m)} r)} := \int_{B_d(x, \frac{C_A}{\varphi(2^m)} r)} u \, d\mu_\varphi \approx \int_{B_d(x, \frac{C_A}{\varphi(2^m)} r)} u \, d\mu =: c_u.$$

In what follows, by $g_{u,d}$ we mean the minimal p -weak upper gradient of u with respect to the metric d , while $g_{u,\varphi}$ denotes the minimal p -weak upper gradient with respect to d_φ . Hence, by the sub-Whitney Poincaré inequality for (Ω, d, μ) , we have

$$\begin{aligned} \int_{B_i} |u - c_u| d\mu_\varphi &\lesssim \int_{B_d(x, \frac{C_A}{\varphi(2^m)} r)} |u - c_u| d\mu \\ &\lesssim \text{diam}_d(B_d(x, \frac{C_A}{\varphi(2^m)} r)) \left(\int_{B_d(x, \frac{C_A}{\varphi(2^m)} r)} g_{u,d}^p d\mu \right)^{1/p} \\ &\lesssim r_i \left(\int_{C_A^2 B_i} g_{u,\varphi}^p d\mu_\varphi \right)^{1/p}. \end{aligned}$$

Here we use the estimate

$$\text{diam}_d(B_d(x, \frac{C_A}{\varphi(2^m)} r)) \approx \varphi(2^m)^{-1} \text{diam}_\varphi(B_d(x, \frac{C_A}{\varphi(2^m)} r)) \approx \varphi(2^m)^{-1} r_i$$

from Lemma 2.13, together with

$$g_{u,d} \approx \varphi(2^m) g_{u,\varphi}$$

to justify the last step. It follows that $(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$ satisfies a sub-Whitney p -Poincaré inequality, i.e., with respect to the balls B_i . See Definition 3.4 above for these concepts. Recall that $\Omega_\varphi \setminus \{\infty\}$ is a uniform domain, see Theorem 5.2 above. Now we invoke [7, Theorem 4.4] to conclude that $(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$ satisfies a p -Poincaré inequality with respect to all balls. While the statement of [7, Theorem 4.4] requires that p -Poincaré inequality be valid with respect to all balls in an ambient space containing the uniform domain, *the proof there only needed the validity of p -Poincaré inequality with respect to the balls in the chain*.

Now the remaining claims follow from [1, Proposition 7.1], for we have that $\Omega_\varphi \setminus \{\infty\} \subset \Omega_\varphi \subset \overline{\Omega_\varphi}^\varphi = \overline{\Omega_\varphi} \setminus \{\infty\}^\varphi$. $\square$

7. TRANSFORMATION OF POTENTIALS

In this section, we return to the original motivation for the problems studied in the prior sections of this paper. We assume that Ω is a unbounded locally compact, non-complete uniform domain with bounded boundary,

equipped with a doubling measure μ that supports a sub-Whitney p -Poincaré inequality for some fixed $1 \leq p < \infty$. Here, of course, we extend the measure μ_φ to $\partial(\Omega_\varphi \setminus \{\infty\})$ by zero.

Recall that as a set, $\Omega = \Omega_\varphi \setminus \{\infty\}$. From the results in the prior sections, we know that $(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$ is doubling and supports a p -Poincaré inequality; hence by [1] we know that

$$N^{1,p}(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi) = N^{1,p}(\Omega_\varphi, d_\varphi, \mu_\varphi) = N^{1,p}(\overline{\Omega_\varphi}^\varphi, d_\varphi, \mu_\varphi). \quad (7.1)$$

With $g_{u,d}$ the minimal p -weak upper gradient of a function $u \in N^{1,p}(\Omega, d, \mu)$ with respect to the original metric d , and $g_{u,\varphi}$ the minimal p -weak upper gradient of u with respect to the metric d_φ , we have the relationship

$$g_{u,\varphi} = \frac{1}{\varphi \circ d_\Omega} g_{u,d}. \quad (7.2)$$

It follows that

$$\int_{\Omega_\varphi} g_{u,\varphi}^p d\mu_\varphi = \int_{\Omega} g_{u,d}^p d\mu. \quad (7.3)$$

As a consequence of (7.2), we have the following proposition (see Definition 3.8 for the definition of p -harmonicity).

Proposition 7.4. *A function u is p -harmonic in the metric measure space (Ω, d, μ) if and only if it is p -harmonic in $(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$.*

Also from (7.2) we obtain the following proposition.

Proposition 7.5. *Let $u \in D^{1,p}(\Omega, d, \mu)$. Then $u \in L^p(\Omega, \mu_\varphi)$ with*

$$\int_{\Omega} |u - c_u|^p (\varphi \circ d_\Omega)^p d\mu \leq C \int_{\Omega} g_{u,d}^p d\mu.$$

Here $c_u = \int_{\Omega} u d\mu_\varphi$. In particular, $u \in N^{1,p}(\Omega_\varphi, d_\varphi, \mu_\varphi)$.

As Ω is unbounded with respect to the metric d , we cannot conclude that $u \in L^p(\Omega, \mu)$. Note that with the choice of $\varphi(t) = \min\{1, t^{-\beta}\}$ for sufficiently large fixed $\beta > 1$, the above lemma is an analog of a Hardy-Sobolev inequality with distance to $\partial_\varphi \overline{\Omega}^d = \{\infty\}$ playing the role of distance to the boundary. Readers interested in the topic of Hardy-Sobolev spaces are referred to [33, Section 1.3.3], [27] (for the Euclidean setting), and [6, Corollary 6.1] (for a metric setting) and the references therein.

Proof. Since $u \in D^{1,p}(\Omega, d, \mu)$, we know that the minimal p -weak upper gradient $g_{u,d}$ of u in Ω , with respect to the metric d , is in $L^p(\Omega, \mu)$. Then by (7.2) above, we know that $u \in D^{1,p}(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$, recalling that $\Omega = \Omega_\varphi \setminus \{\infty\}$. Note that then $u \in N^{1,p}(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$ by Remark 3.5. From Theorem 6.3 and Theorem 4.11, we know that $\Omega_\varphi \setminus \{\infty\}$ is bounded with respect to the metric d_φ , and supports a p -Poincaré inequality with respect to d_φ and μ_φ . Thus, for any $x \in \Omega_\varphi \setminus \{\infty\}$ and for sufficiently large $R > 0$, we have $\Omega_\varphi \setminus \{\infty\} = B_{d_\varphi}(x, R)$, and moreover, $(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$ also supports the following (p, p) -Poincaré inequality, see [17, Theorem 5.1], [20, Theorem 9.1.2]: for $u \in N^{1,p}(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$, we have by (7.3) that

$$\int_{\Omega_\varphi \setminus \{\infty\}} |u - c_u|^p d\mu_\varphi \leq C \int_{\Omega_\varphi \setminus \{\infty\}} g_{u,\varphi}^p d\mu_\varphi = C \int_{\Omega} g_{u,d}^p d\mu.$$

Note that C also depends on R . Now the desired conclusion follows from noting that, by definition, $d\mu_\varphi = (\varphi \circ d_\Omega)^p d\mu$ and by using (7.1). $\square$

As μ is doubling, there exists some $Q_\mu^- > 0$, called the *lower mass bound exponent* of μ , such that

$$\frac{\mu(B_d(x, r))}{\mu(B_d(y, R))} \gtrsim \left(\frac{r}{R}\right)^{Q_\mu^-}$$

for all $x, y \in \Omega$ with $x \in B_d(y, R)$ and $0 < r \leq R < \infty$, where the implied constant depends only on C_μ , the doubling constant of μ , see for instance [18, (4.16)] or [20, Lemma 8.1.13]. Moreover, as Ω is connected, there exists $Q_\mu^+ > 0$, called the *upper mass bound exponent* of μ , such that

$$\frac{\mu(B_d(x, r))}{\mu(B_d(y, R))} \lesssim \left(\frac{r}{R}\right)^{Q_\mu^+}$$

for all $x, y \in \Omega$ with $x \in B_d(y, R)$ and $0 < r \leq R < \infty$, see [3, Corollary 3.8]. Note that $Q_\mu^+ \leq Q_\mu^-$.

Lemma 7.6. *Let $1 \leq p < \infty$, and $\varphi(t) = \min\{1, t^{-\beta}\}$ for $t > 0$ and $\beta > 1$ such that $\beta p > Q_\mu^-$. For all $0 < r \leq R < r_0$, where r_0 is as in Section 4,*

$$\left(\frac{r}{R}\right)^{Q_\beta^+} \gtrsim \frac{\mu_\varphi(B_{d_\varphi}(\infty, r))}{\mu_\varphi(B_{d_\varphi}(\infty, R))} \gtrsim \left(\frac{r}{R}\right)^{Q_\beta^-},$$

where $Q_\beta^- = \frac{\beta p - Q_\mu^+}{\beta - 1}$ and $Q_\beta^+ = \frac{\beta p - Q_\mu^-}{\beta - 1}$. Moreover, the function φ satisfies the conditions of Definition 2.1.

Proof. Fix $0 < r < R < r_0$. Take non-negative integers m_r and m_R such that Ω_{m_r} contains a point x_r satisfying $d_\varphi(x_r, \infty) = r$ and Ω_{m_R} contains a point x_R satisfying $d_\varphi(x_R, \infty) = R$. By Lemma 4.1 and the assumption (2.4),

$$\begin{aligned} \frac{\mu_\varphi(B_{d_\varphi}(\infty, r))}{\mu_\varphi(B_{d_\varphi}(\infty, R))} &\approx \frac{\sum_{n=m_r}^{\infty} \varphi(2^n)^p \mu(\Omega_n)}{\sum_{n=m_R}^{\infty} \varphi(2^n)^p \mu(\Omega_n)} \approx \left(\frac{\varphi(2^{m_r})}{\varphi(2^{m_R})}\right)^p \frac{\mu(\Omega_{m_r})}{\mu(\Omega_{m_R})} \\ &= \left(\frac{2^{m_R \beta}}{2^{m_r \beta}}\right)^p \frac{\mu(\Omega_{m_r})}{\mu(\Omega_{m_R})}. \end{aligned}$$

From Lemma 2.8, there exist $y_r \in \Omega_{m_r}$ with $d_\Omega(y_r) = 2^{m_r}$ and $y_R \in \Omega_{m_R}$ with $d_\Omega(y_R) = 2^{m_R}$, so that

$$\frac{\mu(\Omega_{m_r})}{\mu(\Omega_{m_R})} \approx \frac{\mu(B_d(y_r, 2^{m_r}))}{\mu(B_d(y_R, 2^{m_R}))}.$$

As $\partial\Omega$ is bounded with respect to d , we can consider an integer upper bound K for $\text{diam}_d(\partial\Omega)$. The ball $B_d(y_r, 2K2^{m_r})$ engulfs $\partial\Omega$ and therefore also $B_d(y_R, 2^{m_R})$ since from the above estimates we can conclude that $2^{m_R} \lesssim 2^{m_r}$. Hence, by the doubling property of μ ,

$$\frac{\mu(B_d(y_r, 2^{m_r}))}{\mu(B_d(y_R, 2^{m_R}))} \approx \frac{\mu(B_d(y_r, 2K2^{m_r}))}{\mu(B_d(y_R, 2^{m_R}))}.$$

Applying the upper and lower mass bound estimates for μ to this right-hand quantity, we arrive at

$$\left(\frac{2^{m_r}}{2^{m_R}}\right)^{Q_\mu^+} \lesssim \frac{\mu(B_d(y_r, 2^{m_r}))}{\mu(B_d(y_R, 2^{m_R}))} \lesssim \left(\frac{2^{m_r}}{2^{m_R}}\right)^{Q_\mu^-}.$$

By Lemma 2.11, we have that $r = d_\varphi(x_r, \infty) \approx 2^{m_r} \varphi(2^{m_r}) = 2^{m_r(1-\beta)}$ and similarly for R . Therefore,

$$\frac{\mu_\varphi(B_{d_\varphi}(\infty, r))}{\mu_\varphi(B_{d_\varphi}(\infty, R))} \gtrsim \left(\frac{2^{m_R}}{2^{m_r}}\right)^{\beta p - Q_\mu^+} \approx \left(\frac{R^{\frac{1}{1-\beta}}}{r^{\frac{1}{1-\beta}}}\right)^{\beta p - Q_\mu^+} = \left(\frac{r}{R}\right)^{\frac{\beta p - Q_\mu^+}{\beta - 1}},$$

with the opposite relationships holding with Q_μ^+ replaced by Q_μ^- .

The first five conditions of Definition 2.1 are clear for this choice of φ . Condition (6) of that definition follows from Lemma 2.8. Indeed, in this case, for $n, m \in \mathbb{N}$ with $n > m > n_0$, we have the existence of points $y_n \in \Omega_n$ and $y_m \in \Omega_m$ such that $\mu(\Omega_m) \approx \mu(B_d(y_m, 2^m))$ and $\mu(\Omega_n) \approx \mu(B_d(y_n, 2^n))$. Moreover, $B_d(y_m, 2^m)$ intersects $B_d(y_n, 2^n)$, and so we have that

$$\frac{\mu(\Omega_n)}{\mu(\Omega_m)} \lesssim \frac{\mu(B_d(y_n, 2^n))}{\mu(B_d(y_m, 2^m))} \lesssim \left(\frac{2^n}{2^m}\right)^{Q_\mu^-},$$

and so

$$\begin{aligned} \sum_{n=m}^{\infty} \varphi(2^n)^p \mu(\Omega_n) &\lesssim \sum_{n=m}^{\infty} 2^{-\beta p n} 2^{(n-m)Q_\mu^-} \mu(\Omega_m) \\ &= 2^{-mQ_\mu^-} \mu(\Omega_m) \sum_{n=m}^{\infty} 2^{n(Q_\mu^- - \beta p)} \\ &= 2^{-\beta p m} \mu(\Omega_m) \sum_{j=0}^{\infty} 2^{j(Q_\mu^- - \beta p)}. \end{aligned}$$

Since $\beta p > Q_\mu^-$, the latter series converges, and hence Condition (6) follows. $\square$

Proposition 7.7. *Let $1 \leq p < \infty$, $\varphi(t) = \min\{1, t^{-\beta}\}$ for $t > 0$ and $\beta > 1$ such that $\beta p > Q_\mu^-$, and let Q_β^+, Q_β^- be as in Lemma 7.6. Let Γ be the collection of all curves in Ω_φ that are non-constant, compact, and rectifiable with respect to the metric d_φ , and ending at ∞ .*

- (1) *If $p > Q_\beta^-$, then $\text{Mod}_p^\varphi(\Gamma) > 0$.*
- (2) *If $p < Q_\beta^+$ or $1 < p = Q_\beta^+$, then $\text{Mod}_p^\varphi(\Gamma) = 0$.*

Note that $p > Q_\beta^-$ if and only if $p < Q_\mu^+$, and $p < Q_\beta^+$ if and only if $p > Q_\mu^-$.

Proof. We first prove (1). Fix $0 < r < R < r_0$, and choose a positive integer k_0 such that $\mu_\varphi(B_{d_\varphi}(\infty, R))/\mu_\varphi(B_{d_\varphi}(\infty, 2^{k_0-1}R)) \leq 1/2$. We begin by showing that $\text{Mod}_p(\overline{B}_{d_\varphi}(\infty, r); \Omega_\varphi \setminus B_{d_\varphi}(x_0, R)) \geq CR^{Q_\beta^- - p}$ for some constant $C > 0$ that is independent of r . Since the p -modulus of the condenser is equal to the variational p -capacity of the condenser with $U = \Omega_\varphi$ (see Remark 3.7), we will work with the latter and utilize the Poincaré inequality.

To this end, consider a function $u \in N^{1,p}(\Omega_\varphi)$ satisfying $u = 1$ on $\overline{B_{d_\varphi}(\infty, r)}$ and $u = 0$ in $\Omega_\varphi \setminus B_{d_\varphi}(\infty, R)$. Then, ∞ is a Lebesgue point of u , and so for each positive integer k setting $B_k = B_{d_\varphi}(\infty, 2^{k_0-k}R)$, we have that

$$\begin{aligned} 1 - u_{B_1} &= |u(\infty) - u_{B_1}| \leq \sum_{k=1}^{\infty} |u_{B_k} - u_{B_{k+1}}| \\ &\lesssim \sum_{k=1}^{\infty} 2^{k_0-k} R \left(\int_{B_k} g_{u,\varphi}^p d\mu_\varphi \right)^{1/p} \\ &\lesssim \mu_\varphi(B_1)^{-1/p} \sum_{k=1}^{\infty} (2^{k_0-k} R)^{1-\frac{Q_\beta^-}{p}} \left(\int_{B_k} g_{u,\varphi}^p d\mu_\varphi \right)^{1/p}, \end{aligned}$$

where we used Lemma 7.6 in the last step. From the choice of k_0 ,

$$u_{B_1} = \frac{1}{\mu_\varphi(B_{d_\varphi}(\infty, 2^{k_0-1}R))} \int_{B_{k_0}} u d\mu_\varphi \leq \frac{\mu_\varphi(B_{d_\varphi}(\infty, R))}{\mu_\varphi(B_{d_\varphi}(\infty, 2^{k_0-1}R))} \leq \frac{1}{2}.$$

Since $p > Q_\beta^-$, from the above we obtain

$$\frac{1}{2} \leq 1 - u_{B_1} \lesssim \mu_\varphi(B_1)^{-1/p} R^{1-\frac{Q_\beta^-}{p}} \left(\int_{\Omega_\varphi} g_{u,\varphi}^p d\mu_\varphi \right)^{1/p},$$

from where it follows that

$$2^{-p} \mu_\varphi(B_1) R^{Q_\beta^- - p} \lesssim \int_{\Omega_\varphi} g_{u,\varphi}^p d\mu_\varphi,$$

and, taking the infimum over all such u ,

$$\text{cap}_p(\overline{B_{d_\varphi}(\infty, r)}, \Omega_\varphi \setminus B_{d_\varphi}(x_0, R)) \gtrsim \mu_\varphi(B_1) R^{Q_\beta^- - p} > 0.$$

Letting $r \rightarrow 0^+$, from the Choquet property of variational p -capacity (see [3, Theorem 6.7(viii)]) we obtain

$$\text{cap}_p(\{\infty\}, \Omega_\varphi \setminus B_{d_\varphi}(x_0, R)) \gtrsim \mu_\varphi(B_1) R^{Q_\beta^- - p} > 0.$$

The result then follows from Remark 3.7.

We now move to proving (2). Fix $0 < r < \frac{R}{4} < R < r_0$, and let n_r be the unique positive integer such that $2^{-n_r}R \leq r < 2^{1-n_r}R$. Let ρ be the function on Ω_φ given by $\rho(x) = 2[n_r d_\varphi(x, \infty)]^{-1} \chi_{B_{d_\varphi}(\infty, R) \setminus B_{d_\varphi}(\infty, r)}$. For each $\gamma \in \Gamma(\overline{B_{d_\varphi}(\infty, r)}, \Omega_\varphi \setminus B_{d_\varphi}(\infty, R), \Omega_\varphi)$, we have $\int_\gamma \rho ds_\varphi \geq 1$.

Setting $B_k = B_{d_\varphi}(\infty, 2^{1-k}R)$ for $k = 1, \dots, n_r$, we see that

$$\int_{\Omega_\varphi} \rho^p d\mu_\varphi \leq \sum_{k=1}^{n_r} \int_{B_k \setminus B_{k+1}} \frac{2^p}{n_r^p d_\varphi(x, \infty)^p} d\mu_\varphi(x) \leq \sum_{k=1}^{n_r} \frac{2^{(1+k)p}}{n_r^p R^p} \mu_\varphi(B_k \setminus B_{k+1})$$

since $x \notin B_{k+1}$ implies that $d_\varphi(x, \infty) \geq 2^{-k}R$. We now estimate $\mu_\varphi(B_k \setminus B_{k+1})$. From Lemma 2.11, it follows that $B_k \setminus B_{k+1} \subset \Omega_{m_k}$ for some non-negative integer m_k ; moreover, $2^{-k}R \approx d_\varphi(x, \infty) \approx 2^{m_k(1-\beta)}$ for $x \in B_k \setminus$

B_{k+1} . Thus, for some $y_k \in \Omega_{m_k}$ (see Lemma 2.8),

$$\begin{aligned} \mu_\varphi(B_k \setminus B_{k+1}) &= \int_{B_k \setminus B_{k+1}} d_\Omega(x)^{-p\beta} d\mu \approx 2^{-m_k p\beta} \mu(B_k \setminus B_{k+1}) \\ &\leq 2^{-m_k p\beta} \mu(\Omega_{m_k}) \leq 2^{-m_k p\beta} \mu(B_d(y_k, 2^{m_k})) \\ &\lesssim 2^{-m_k p\beta} (2^{m_k})^{Q_\mu^-} \end{aligned}$$

where the lower-mass bound on μ was used in the last step.

From here, we have

$$\int_{\Omega_\varphi} \rho^p d\mu_\varphi \lesssim \sum_{k=1}^{n_r} \frac{2^{kp}}{n_r^p R^p} 2^{m_k(Q_\mu^- - p\beta)} \approx \frac{R^{Q_\beta^+ - p}}{n_r^p} \sum_{k=1}^{n_r} (2^k)^{p - Q_\beta^+}. \quad (7.8)$$

Recall that $Q_\beta^+ = \frac{p\beta - Q_\mu^-}{\beta - 1}$ and $2^{m_k} \approx (2^{-k} R)^{\frac{1}{1-\beta}}$.

If $1 < p = Q_\beta^+$, then the sum on the right-hand side of (7.8) equals n_r , and so

$$\text{Mod}_p^\varphi(\overline{B}_{d_\varphi}(\infty, r); \Omega_\varphi \setminus B_{d_\varphi}(x_0, R)) \lesssim n_r^{1-p} \approx \left[\log \left(\frac{R}{r} \right) \right]^{1-p},$$

the right side of which tends to zero as $r \rightarrow 0^+$. If $p < Q_\beta^+$, then the sum on the right-hand side of (7.8) is dominated by the convergent series obtained by summing over all positive integers k , and so

$$\text{Mod}_p^\varphi(\overline{B}_{d_\varphi}(\infty, r); \Omega_\varphi \setminus B_{d_\varphi}(x_0, R)) \lesssim n_r^{-p} \approx \left[\log \left(\frac{R}{r} \right) \right]^{-p},$$

the right side of which also tends to zero as $r \rightarrow 0^+$.

In either case, it then follows that $\text{Mod}_p^\varphi(\{\infty\}; \Omega_\varphi \setminus B_{d_\varphi}(\infty, R)) = 0$. From this it follows that the p -modulus (with respect to d_φ and μ_φ) of the collection of all non-constant compact rectifiable curves in Ω_φ that intersect ∞ is zero as well. $\square$

Remark 7.9. As a consequence of the above results and by [2], we have that if u is p -harmonic in (Ω, d, μ) and $p > Q_\mu^-$ or $p = Q_\mu^- > 1$, then u is p -harmonic in $(\Omega_\varphi, d_\varphi, \mu_\varphi)$. From [2, Proposition 7.2] it also follows that when $p < Q_\mu^+$ and the Sobolev p -capacity $\text{Cap}_p(\partial\Omega)$ is positive, then there is a bounded p -harmonic function in $(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$ which has no p -harmonic extension to Ω_φ .

8. CONNECTING POTENTIAL THEORY ON Ω TO FUNCTIONS ON $\partial\Omega$

Let ν be a Borel regular measure on $\partial\Omega = \partial\Omega_\varphi$ and $\theta \geq 0$. We say that ν is θ -codimensional to μ_φ if

$$\nu(B_{d_\varphi}(\zeta, r) \cap \partial\Omega) \approx \frac{\mu_\varphi(B_{d_\varphi}(\zeta, r) \cap \Omega_\varphi)}{r^\theta}$$

for all $\zeta \in \partial\Omega$ and $0 < r < 2 \text{diam}_{d_\varphi}(\partial\Omega)$. Note that as μ_φ is doubling on $\Omega_\varphi \setminus \{\infty\}$, we must necessarily have that ν is doubling on $\partial\Omega$ with respect to the metric d_φ (and equivalently, with respect to the metric d).

Lemma 8.1. *Assume there exists a Borel regular measure ν on $\partial\Omega$ that is θ -codimensional to μ for some $\theta \geq 0$. Then, $\mathcal{H}^{-\theta}(A; \overline{\Omega}) \approx \nu(A)$ for all $A \subset \partial\Omega$.*

If $\theta > 0$, then $\widehat{\mu}_\varphi(\partial\Omega) = 0$ for any doubling measure $\widehat{\mu}_\varphi$ on $\overline{\Omega}_\varphi^\varphi$ such that $\widehat{\mu}_\varphi = \mu_\varphi$ on Ω_φ and $\nu(B_{d_\varphi}(\zeta, r) \cap \partial\Omega) \approx \frac{\widehat{\mu}_\varphi(B_{d_\varphi}(\zeta, r))}{r^\theta}$ for all $\zeta \in \partial\Omega$ and $0 < r < 2 \text{diam}_{d_\varphi}(\partial\Omega)$.

As both d_φ and d are bi-Lipschitz equivalent on $\partial\Omega$ and $\mu = \mu_\varphi$ on Ω_0 , computing the codimensional Hausdorff measure $\mathcal{H}^{-\theta}$ of a subset of $\partial\Omega$ (as in Definition 3.9 with $U = \overline{\Omega}$) with respect to the metric d and the measure μ is equivalent to computing it with respect to d_φ and μ_φ . In doing the computation of $\mathcal{H}^{-\theta}(A; \overline{\Omega})$, we extend the measures μ and μ_φ by zero to $\partial\Omega$ so that $\mu(\partial\Omega) = 0$.

Proof. We fix $A \subset \partial\Omega$.

For each $\varepsilon > 0$ let $\{B_{d_\varphi}(x_i, r_i)\}$ be a cover of A with $r_i \leq \varepsilon$ and $x_i \in A$. Then

$$\sum_i \frac{\mu_\varphi(B_{d_\varphi}(x_i, r_i))}{r_i^\theta} \gtrsim \sum_i \nu(B_{d_\varphi}(x_i, r_i) \cap \partial\Omega) \geq \nu(A).$$

Taking the infimum over all such covers, and then letting $\varepsilon \rightarrow 0^+$, this implies that $\mathcal{H}^{-\theta}(A; \overline{\Omega}_\varphi) \gtrsim \nu(A)$.

Since ν is Borel regular, it follows that for each $\eta > 0$ there is a set $U \supset A$ such that U is open in $\overline{\Omega}$ and $\nu(U) \leq \nu(A) + \eta$. Fix $\varepsilon > 0$, and consider a cover $\{B_{d_\varphi}(x_i, r_i)\}$ of A with $r_i \leq \varepsilon/5$, $x_i \in A$, and $B_{d_\varphi}(x_i, 5r_i) \subset U$. By the basic $5r$ -covering lemma as in [18, Theorem 1.2], there is a countable pairwise disjoint subcollection $\{B_{d_\varphi}(x_j, r_j)\}$ such that $\{B_{d_\varphi}(x_j, 5r_j)\}$ covers A . Then, by the doubling property of ν ,

$$\begin{aligned} \mathcal{H}_\varepsilon^{-\theta}(A; \overline{\Omega}_\varphi) &\leq \sum_j \frac{\mu_\varphi(B_{d_\varphi}(x_j, 5r_j))}{(5r_j)^\theta} \lesssim \sum_j \frac{\mu_\varphi(B_{d_\varphi}(x_j, r_j))}{r_j^\theta} \\ &\lesssim \sum_j \nu(B_{d_\varphi}(x_j, r_j) \cap \partial\Omega) \\ &= \nu\left(\bigcup_j B_{d_\varphi}(x_j, r_j) \cap \partial\Omega\right) \\ &\leq \nu(U \cap \partial\Omega) \leq \nu(A) + \eta. \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$, we obtain $\mathcal{H}^{-\theta}(A; \overline{\Omega}_\varphi) \lesssim \nu(A) + \eta$. Letting $\eta \rightarrow 0^+$ now yields the desired conclusion of the first part of the lemma.

To prove the second claim of the lemma, we suppose that $\theta > 0$ and argue as in the above proof to obtain that for each $\varepsilon > 0$ and any cover $\{B_{d_\varphi}(x_i, r_i)\}$ of $\partial\Omega$ with $x_i \in \partial\Omega$ and $r_i \leq \varepsilon$, to see that

$$\frac{\widehat{\mu}_\varphi(\partial\Omega)}{\varepsilon^\theta} \leq \frac{\sum_i \widehat{\mu}_\varphi(B_{d_\varphi}(x_i, r_i))}{\varepsilon^\theta} \leq \sum_i \frac{\mu_\varphi(B_{d_\varphi}(x_i, r_i))}{r_i^\theta} \lesssim \nu(\partial\Omega) + \eta,$$

from which we obtain that $\widehat{\mu}_\varphi(\partial\Omega) \lesssim \varepsilon^\theta [\nu(\partial\Omega) + \eta]$. Letting $\varepsilon \rightarrow 0^+$ yields the conclusion as $\nu(\partial\Omega)$ is finite (note that $\partial\Omega$ is bounded). $\square$

Proposition 8.2. *Let $1 \leq p < \infty$, and assume there exists a Borel regular measure ν on $\partial\Omega$ that is θ -codimensional to μ_φ for $0 < \theta < p$. Then, for all $\zeta \in \partial\Omega$ and $r > 0$, we have $\text{Cap}_p^\varphi(B_{d_\varphi}(\zeta, r) \cap \partial\Omega) > 0$.*

Proof. From Theorem 4.11 and Theorem 6.3 we have that $\overline{\Omega}^\varphi$, equipped with the metric d_φ and the measure μ_φ , is doubling and supports a p -Poincaré inequality. Note that $\mu = \mu_\varphi$ on Ω_0 and d_φ is bi-Lipschitz equivalent to d on $\Omega_0 \cup \partial\Omega$.

Due to Proposition 3.11, it suffices to show that $\mathcal{H}^{-\theta}(B_{d_\varphi}(\zeta, r) \cap \partial\Omega) > 0$. From Lemma 8.1, $\mathcal{H}^{-\theta}(B_{d_\varphi}(\zeta, r) \cap \partial\Omega) \approx \nu(B_{d_\varphi}(\zeta, r) \cap \partial\Omega)$, which must be positive because ν is doubling. $\square$

Recall that the space $D^{1,p}(\Omega, d, \mu)$ consists of measurable functions on Ω which have an upper gradient in $L^p(\Omega, \mu)$. This space is naturally equipped with the seminorm $\|u\|_{D^{1,p}(\Omega, d, \mu)} := \|g_{u,d}\|_{L^p(\Omega, \mu)}$; note from (7.2) that $\|g_{u,d}\|_{L^p(\Omega, \mu)} = \|g_{u,\varphi}\|_{L^p(\Omega_\varphi, \mu_\varphi)}$.

Proposition 8.3. *Let $1 \leq p < \infty$, and assume there exists a Borel regular measure ν on $\partial\Omega$ that is θ -codimensional to μ for $0 < \theta < p$. For every $u \in D^{1,p}(\Omega, d, \mu)$ there exists $Tu \in B_{p,p}^{1-\theta/p}(\partial\Omega, \nu)$ such that*

$$\lim_{r \rightarrow 0^+} \int_{B_d(\zeta, r) \cap \Omega} |u(x) - Tu(\zeta)| d\mu(x) = 0 \quad (8.4)$$

for ν -a.e. $\zeta \in \partial\Omega$. Moreover, the operator $u \mapsto Tu$ is bounded from $D^{1,p}(\Omega, d, \mu)$ to $B_{p,p}^{1-\theta/p}(\partial\Omega, \nu)$.

Recall that since $\partial\Omega$ is bounded, $B_{p,p}^{1-\theta/p}(\partial\Omega, \nu) \subset L^p(\partial\Omega, \nu)$.

Proof. By Lemma 7.5 and (7.1), it suffices to look at $u \in N^{1,p}(\Omega_\varphi, d_\varphi, \mu_\varphi)$. Since Ω_φ is a uniform domain with compact closure (see Proposition 4.12), it follows from Proposition 3.13 that there exists $Tu \in B_{p,p}^{1-\theta/p}(\partial\Omega, \nu)$ satisfying

$$\lim_{r \rightarrow 0^+} \int_{B_{d_\varphi}(\zeta, r) \cap \Omega_\varphi} |u(x) - Tu(\zeta)| d\mu_\varphi(x) = 0 \quad (8.5)$$

for ν -a.e. $\zeta \in \partial\Omega$ and $\|Tu\|_{B_{p,p}^{1-\theta/p}(\partial\Omega, \nu)} \lesssim \|u\|_{D^{1,p}(\Omega_\varphi, d_\varphi, \mu_\varphi)} = \|u\|_{D^{1,p}(\Omega, d, \mu)}$.

For $\zeta \in \partial\Omega$ and for sufficiently small $r > 0$, we have that $\mu_\varphi = \mu$ on $B_d(\zeta, r)$ which is bi-Lipschitz equivalent to $B_{d_\varphi}(\zeta, r)$. Therefore, (8.5) is equivalent to (8.4). $\square$

We now turn our attention to proving an Adams-type inequality on $\overline{\Omega}_\varphi^\varphi$ with respect to the measure ν supported on $\partial\Omega$.

Remark 8.6. From Theorems 4.11 and 6.3, it follows that $(\overline{\Omega}_\varphi^\varphi, d_\varphi, \mu_\varphi)$ is doubling and supports a p -Poincaré inequality. Hence by [25] (see also [20, Theorem 12.3.9]), $(\overline{\Omega}_\varphi^\varphi, d_\varphi, \mu_\varphi)$ also supports a $\tilde{p}$ -Poincaré inequality for some $1 \leq \tilde{p} < p$.

In the following, $Q_{\mu_\varphi}^-$ is the lower mass bound for the measure μ_φ .

Theorem 8.7. *Let $1 < p < \infty$, and assume there exists a Borel regular measure ν on $\partial\Omega$ that is θ -codimensional to μ_φ with $0 < \theta < p < Q_{\mu_\varphi}^-$, and let $q > p$ be a real number given by*

$$\theta = -\frac{Q_{\mu_\varphi}^- q}{p} + Q_{\mu_\varphi}^- + \frac{q}{\tilde{p}},$$

where $1 \leq \tilde{p} < p$ is from Remark 8.6. Then for every $u \in N^{1,p}(\overline{\Omega_\varphi}^\varphi, d_\varphi, \mu_\varphi)$ and ball $B_{d_\varphi} \subset \overline{\Omega_\varphi}^\varphi$,

$$\inf_{c \in \mathbb{R}} \left(\int_{B_{d_\varphi}} |u - c|^q d\nu \right)^{1/q} \lesssim \frac{\text{rad}_{d_\varphi}(B_{d_\varphi})^{1-\theta/q}}{\mu_\varphi(B_{d_\varphi})^{1/p-1/q}} \left(\int_{2B_{d_\varphi}} g_{u,\varphi}^p d\mu_\varphi \right)^{1/p}.$$

In particular,

$$\inf_{c \in \mathbb{R}} \left(\int_{\partial\Omega} |u - c|^q d\nu \right)^{1/q} \lesssim \frac{\text{diam}_{d_\varphi}(\Omega_\varphi)^{1-\theta/q}}{\mu_\varphi(\Omega_\varphi)^{1/p-1/q}} \left(\int_{\Omega_\varphi} g_{u,\varphi}^p d\mu_\varphi \right)^{1/p}.$$

Proof. Fix a ball $B_{d_\varphi} := B_{d_\varphi}(y, r) \subset \overline{\Omega_\varphi}^\varphi$ and $u \in \text{Lip}(\overline{\Omega_\varphi}^\varphi)$. Since $\overline{\Omega_\varphi}^\varphi$ is a compact length space, it follows that it is a geodesic space, and compactness implies that u is a bounded continuous function. Moreover, it satisfies a $\tilde{p}$ -Poincaré inequality by Remark 8.6. Hence, setting

$$u_{2B_{d_\varphi}} := \int_{2B_{d_\varphi}} u d\mu_\varphi,$$

it follows from [18, Theorem 9.5] that

$$|u(x) - u_{2B_{d_\varphi}}|^{\tilde{p}} \lesssim r^{\tilde{p}-1} I_{2B_{d_\varphi}}(g_{u,\varphi}^{\tilde{p}})(x)$$

for $x \in B_{d_\varphi}$, where $I_{2B_{d_\varphi}}$ is the Riesz potential, see Definition 3.14.

Integrating and using Proposition 3.15 with $\tilde{p} = \frac{p}{\tilde{p}}$ and $\tilde{q} = \frac{q}{\tilde{p}}$ yields

$$\begin{aligned} \left(\int_{B_{d_\varphi}} |u - u_{2B_{d_\varphi}}|^q d\nu \right)^{1/q} &= \left(\int_{B_{d_\varphi}} (|u - u_{2B_{d_\varphi}}|^{\tilde{p}})^{q/\tilde{p}} d\nu \right)^{1/q} \\ &\lesssim r^{\frac{\tilde{p}-1}{\tilde{p}}} \left(\left(\int_{2B_{d_\varphi}} I_{2B_{d_\varphi}}(g_{u,\varphi}^{\tilde{p}})^{q/\tilde{p}} d\nu \right)^{\tilde{p}/q} \right)^{1/\tilde{p}} \\ &\lesssim r^{\frac{\tilde{p}-1}{\tilde{p}} + \frac{Q_{\mu_\varphi}^-}{p} - \frac{Q_{\mu_\varphi}^-}{q}} \mu_\varphi(2B_{d_\varphi})^{1/q-1/p} \left(\left(\int_{2B_{d_\varphi}} (g_{u,\varphi}^{\tilde{p}})^{p/\tilde{p}} d\mu_\varphi \right)^{\tilde{p}/p} \right)^{1/\tilde{p}} \\ &\lesssim r^{\frac{\tilde{p}-1}{\tilde{p}} + \frac{Q_{\mu_\varphi}^-}{p} - \frac{Q_{\mu_\varphi}^-}{q}} \mu_\varphi(B_{d_\varphi})^{1/q-1/p} \left(\int_{2B_{d_\varphi}} g_{u,\varphi}^p d\mu_\varphi \right)^{1/p}. \end{aligned}$$

The result follows from the density of the Lipschitz functions in $N^{1,p}(\overline{\Omega_\varphi}^\varphi)$, see, for example, [20, Theorem 8.2.1]. $\square$

The above Adams-type inequality also gives us a way to link traces, to $\partial\Omega$, of Dirichlet-Sobolev functions on Ω , and through this, we will see next that the relative capacities of compact subsets of $\partial\Omega$ are governed by the ν -measure of those subsets. As a consequence of the above Adams-type

inequality, we have the following corollary giving us a lower bound on relative capacities of subsets of $\partial\Omega$.

Given a domain Ω and a point $x \in \partial\Omega$, the point x is called a *regular point* for the domain if for each continuous function $f : \partial\Omega \rightarrow \mathbb{R}$ we have

$$\lim_{\Omega \ni y \rightarrow x} H_\Omega f(y) = f(x),$$

where $H_\Omega f$ is the Perron solution to the Dirichlet problem of finding p -harmonic functions on Ω with trace f . We refer the interested reader to [3, Chapter 10–11] or [5] for more on classification of boundary points and Perron solutions.

Corollary 8.8. *Let $1 < p < \infty$. For each $0 < r \leq \min\{1, \text{diam}_{d_\varphi}(\partial\Omega)\}/2$ and $\zeta \in \partial\Omega$, we have*

$$\nu(B_{d_\varphi}(\zeta, r) \cap \partial\Omega) \lesssim r^{p-\theta} \text{cap}_p^\varphi(B_{d_\varphi}(\zeta, r) \cap \partial\Omega, B_{d_\varphi}(\zeta, 2r)).$$

Consequently, as $0 < \theta < p$, then each point of $\partial\Omega$ is a regular point for the domain Ω_φ .

Recall that μ_φ is doubling, and $\overline{\Omega_\varphi}^\varphi$ is connected, and so we have a reverse doubling property: there is a positive constant $c < 1$ such that for all $x \in \overline{\Omega_\varphi}^\varphi$ and $0 < r < \frac{1}{2} \text{diam}_{d_\varphi}(\Omega_\varphi)$, we have $\mu_\varphi(B_{d_\varphi}(x, r)) \leq c \mu_\varphi(B_{d_\varphi}(x, 2r))$.

Proof. Fix ζ and r as in the statement of the corollary. Let $u \in N^{1,p}(\Omega_\varphi, d_\varphi, \mu_\varphi)$ such that $u = 1$ on $B_{d_\varphi}(\zeta, r) \cap \partial\Omega$, $0 \leq u \leq 1$ on Ω_φ , and $u = 0$ on $\Omega_\varphi \setminus B_{d_\varphi}(\zeta, 2r)$. Then,

$$\begin{aligned} \nu(B_{d_\varphi}(\zeta, r) \cap \partial\Omega)^{1/q} &\leq \left(\int_{B_{d_\varphi}(\zeta, r)} u^q d\nu \right)^{1/q} \\ &\leq \left(\int_{B_{d_\varphi}(\zeta, 4r)} u^q d\nu \right)^{1/q} \\ &\lesssim \left(\int_{B_{d_\varphi}(\zeta, 4r)} |u - u_{B_{d_\varphi}(\zeta, 4r)}|^q d\mu \right)^{1/q} \\ &\lesssim r^{1-\theta/q} \mu_\varphi(B_{d_\varphi}(\zeta, 4r))^{1/q-1/p} \left(\int_{B_{d_\varphi}(\zeta, 2r)} g_{u,\varphi}^p d\mu_\varphi \right)^{1/p}. \end{aligned}$$

In obtaining the penultimate inequality, we used Hölder's inequality and the reverse doubling property of μ_φ , while in the ultimate inequality we used Theorem 8.7 applied to $B_{d_\varphi}(\zeta, 4r)$ and the doubling property of μ_φ . Now an application of the codimensionality relationship between ν and μ together with the fact that for $r < 1/2$ the measure $\mu_\varphi = \mu$, and then taking the infimum over all such u on the right-hand side, yields the desired inequality.

To verify the second claim of the corollary, we use the results of [6]. Note that from [3], for each $\zeta \in \partial\Omega$ and $r > 0$, we have

$$\text{cap}_p^\varphi(B_{d_\varphi}(\zeta, r), B_{d_\varphi}(\zeta, 2r)) \approx \frac{\mu_\varphi(B_{d_\varphi}(\zeta, r))}{r^p}.$$

So if $0 < r < 1/2$, then using the codimensionality of ν with respect to μ (and hence μ_φ), we see that

$$\begin{aligned} \text{cap}_p^\varphi(B_{d_\varphi}(\zeta, r), B_{d_\varphi}(\zeta, 2r)) &\approx \frac{\nu(B_{d_\varphi}(\zeta, r) \cap \partial\Omega)}{r^{p-\theta}} \\ &\lesssim \text{cap}_p^\varphi(B_{d_\varphi}(\zeta, r) \cap \partial\Omega, B_{d_\varphi}(\zeta, 2r)). \end{aligned}$$

Thus, $\partial\Omega$ is uniformly p -fat with respect to the domain Ω_φ , that is,

$$\frac{\text{cap}_p^\varphi(B_{d_\varphi}(\zeta, r) \cap \partial\Omega, B_{d_\varphi}(\zeta, 2r))}{\text{cap}_p^\varphi(B_{d_\varphi}(\zeta, r), B_{d_\varphi}(\zeta, 2r))} \gtrsim 1,$$

and so by the results in [6, Theorem 5.1] the conclusion follows. $\square$

9. EFFECT ON THE DIRICHLET PROBLEM FOR (Ω, d, μ)

Now we have the tools necessary to verify Theorem 1.2. We split the proof into two theorems below.

To show existence of solutions to a Dirichlet problem for bounded domains can be done via the direct method of calculus of variations: any sequence of functions, with the same boundary value, and with p -energy converging to the infimum of the p -energies of the class of all Dirichlet-Sobolev functions satisfying the fixed boundary conditions is done by first showing that this sequence is bounded in the Sobolev class (which is reflexive), and this in turn is accomplished by using the Poincaré inequality, as the domain is bounded and hence sits inside a large ball to which the Poincaré inequality can be applied. See [34] for details regarding this method in the setting of metric spaces. When the domain is unbounded however, this method cannot work.

In this final section of the paper, we use the tools developed in the previous two sections to study existence and uniqueness issues related to solutions to the Dirichlet problem regarding p -harmonic functions on an unbounded domain whose boundary is bounded. In [16] a Perron method from [5] was adapted to solve the Dirichlet problem corresponding to continuous boundary data on unbounded domains that were p -parabolic. As in [22], an unbounded domain Ω is p -parabolic if $\text{Mod}_p(\Gamma_\infty) = 0$, where Γ_∞ consists of all locally rectifiable curves in Ω that leave every compact subset of $\overline{\Omega}^d$. Note that in our setting, curves in Γ_∞ are the restriction to Ω of the curves in Γ studied in Proposition 7.7, and so by (7.2), the domain Ω is p -parabolic in the sense of [16] if and only if $\text{Mod}_p^\varphi(\Gamma) = 0$. The domain is said to be p -hyperbolic if it is not p -parabolic.

The following theorem extends the result of [16] to boundary data in Besov classes.

Theorem 9.1. *Let $1 < p < \infty$, (Ω, d, μ) be a doubling metric measure space satisfying a p -Poincaré inequality such that (Ω, d) is a locally compact, non-complete, unbounded uniform domain with bounded boundary, and ν a Borel regular measure that is θ -codimensional with respect to μ for some $0 < \theta < p$.*

Suppose that Ω is p -parabolic. Then, for every $f \in B_{p,p}^{1-\theta/p}(\partial\Omega, \nu)$, there is a unique function $u \in D^{1,p}(\Omega, \mu)$ such that

- u is p -harmonic in (Ω, d, μ) ,
- $Tu = f$ on $\partial\Omega$ ν -a.e..

Proof. Recall from Proposition 7.5 that

$$D^{1,p}(\Omega, d, \mu) = N^{1,p}(\Omega_\varphi, d_\varphi, \mu_\varphi).$$

Hence it suffices to find a function $u \in N^{1,p}(\Omega_\varphi, d_\varphi, \mu_\varphi)$ that is p -harmonic in $(\Omega_\varphi, d_\varphi, \mu_\varphi)$, for p -harmonicity in this metric measure space implies p -harmonicity in the original metric measure space (Ω, d, μ) thanks to Proposition 7.4.

Since $f \in B_{p,p}^{1-\theta/p}(\partial\Omega, \nu)$, and this Besov space is the trace space of $N^{1,p}(\Omega_\varphi)$ by the proof of Proposition 8.3, and as $\text{Cap}_p^\varphi(\partial\Omega) > 0$ by Proposition 8.2, the direct method of calculus of variations as described in the comment before the statement of the theorem applies; note that we also have $(\Omega_\varphi, d_\varphi, \mu_\varphi)$ is doubling and supports a p -Poincaré inequality. Thus there is a function $u \in N^{1,p}(\Omega_\varphi, d_\varphi, \mu_\varphi)$ that is p -harmonic in Ω_φ with respect to the metric d_φ and the measure μ_φ , such that $Tu = f$.

Now suppose that v is also a p -harmonic function in (Ω, d, μ) with $Tv = f$. We also have that v is p -harmonic in $\Omega_\varphi \setminus \{\infty\}$ by Proposition 7.4. As $\text{Cap}_p^\varphi(\{\infty\}) = 0$ by the assumption of p -parabolicity, it follows from [2] (see Remark 7.9 above) that v is p -harmonic in Ω_φ . By the uniqueness of solutions to Dirichlet problem in $(\Omega_\varphi, d_\varphi, \mu_\varphi)$ (see [34]), we see that necessarily $v = u$ in Ω_φ and hence in Ω . This demonstrates the uniqueness of the solution. $\square$

Theorem 9.2. *Let $1 < p < \infty$, (Ω, d, μ) be a doubling metric measure space satisfying a p -Poincaré inequality such that (Ω, d) is a locally compact, non-complete, unbounded uniform domain with bounded boundary, and ν a Borel regular measure that is θ -codimensional with respect to μ for some $0 < \theta < p$.*

Suppose that Ω is p -hyperbolic. Then, for every $f \in B_{p,p}^{1-\theta/p}(\partial\Omega, \nu)$, there is a function $u \in D^{1,p}(\Omega, \mu)$ such that

- u is p -harmonic in (Ω, d, μ) ,
- $Tu = f$ on $\partial\Omega$ ν -a.e.,
- $\lim_{\Omega \ni y \rightarrow \infty} u(y)$ exists as a real value.

In the above theorem, by $\lim_{\Omega \ni y \rightarrow \infty} u(y)$ we mean a real number τ such that for some (and hence each) $x_0 \in \Omega$, for each $\varepsilon > 0$ we can find $R > 0$ such that $|u(y) - \tau| < \varepsilon$ whenever $y \in \Omega$ with $d(x_0, y) > R$.

Proof. As in the proof of Theorem 9.1, we see that solutions do exist, but unlike in that theorem, we do not have uniqueness. However, as Ω is p -hyperbolic, we have that $\text{Cap}_p^\varphi(\{\infty\}) > 0$. Let v be any such p -harmonic solution; then $v \in N^{1,p}(\Omega_\varphi, d_\varphi, \mu_\varphi)$ by Proposition 7.5. Thus, extending $f \in B_{p,p}^{1-\theta/p}(\partial\Omega, \nu)$ to ∞ by setting

$$f(\infty) := \lim_{r \rightarrow 0^+} \int_{B_{d_\varphi}(\infty, r)} v \, d\mu_\varphi,$$

and noting that p -capacity almost every point is a Lebesgue point of v (see [20, Theorem 9.2.8]), the above extension is well-defined. Thus v solves the Dirichlet problem on $(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$ with boundary data as the extended function f , with f continuous at (and hence, as ∞ is an isolated point of $\partial[\Omega_\varphi \setminus \{\infty\}]$, in a neighborhood of) ∞ . By Corollary 8.8, we

know that ∞ is a regular point for the domain $(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$. Hence $\lim_{\Omega \ni y \rightarrow \infty} v(y) = f(\infty)$ exists. $\square$

We in fact obtain more; if v and u are both solutions to the original problem, and if $\lim_{\Omega \ni y \rightarrow \infty} v(y) = \lim_{\Omega \ni y \rightarrow \infty} u(y)$, then necessarily $v = u$ in Ω by the uniqueness of solutions to Dirichlet problem in $(\Omega_\varphi \setminus \{\infty\}, d_\varphi, \mu_\varphi)$ with boundary data on $\partial[\Omega_\varphi \setminus \{\infty\}] = \partial\Omega \cup \{\infty\}$. Moreover, of all the solutions to the Dirichlet problem on Ω with boundary data f , there is only one solution that is also a solution in the domain Ω_φ with respect to the metric d_φ and the measure μ_φ .

10. SOME ILLUSTRATIVE EXAMPLES

In this section, we consider some examples.

Example 10.1. Let $Z = [-1, 1] \times [0, \infty)$, equipped with the restriction of the Euclidean metric and the 2-dimensional Lebesgue measure from $\mathbb{R}^2$, and $\Omega = [-1, 1] \times (0, \infty)$. Then $\partial\Omega = [-1, 1] \times \{0\}$ is bounded. We have that the measure on Z is doubling and supports the strongest of all Poincaré inequality, the 1-Poincaré inequality. Moreover, Ω is a uniform domain that is also unbounded. For $\beta > 1$, with the choice of $\varphi(t) = \min\{1, t^{-\beta}\}$ for $t > 0$, the domain Ω is transformed into $\Omega_\varphi = \Omega \cup \{\infty\}$. Note that in the proof of Proposition 7.7, we need only consider the indices Q_μ^- and Q_μ^+ corresponding to large values of r and R ; and in this case, we can set $Q_\mu^- = Q_\mu^+ = 1$, though when considering all scales of r , we have $Q_\mu^- = 2$ and $Q_\mu^+ = 1$. So reading the hypotheses of Proposition 7.7 in this setting, when $1 = Q_\mu^\pm < p$, we are in the case (2) of the proposition, and then $\text{Mod}_p^\varphi(\Gamma) = 0$. It follows that Ω is p -parabolic, and the solution to the Dirichlet problem on Ω is unique. Note that $\nu = \mathcal{H}^1|_{[-1, 1] \times \{0\}}$ has co-dimensional relation with respect to the 2-dimensional Lebesgue measure at scales $0 < r \leq R_0$ with $\theta = 1$. Solutions to the Dirichlet problem satisfy a homogeneous Neumann condition on the half-lines $\{\pm 1\} \times (0, \infty)$ when seen as a function on $(-1, 1) \times (0, \infty)$. For $p > 1 = \theta$, the Dirichlet problem is always solvable, and the solution is unique.

Example 10.2. $Z := \{(x, y) \in \mathbb{R}^2 : y \geq \max\{0, |x| - 1\}\}$ again be equipped the restriction of the Euclidean metric and Lebesgue measure from $\mathbb{R}^2$, and $\Omega = Z \setminus [-1, 1] \times \{0\}$. In this case, at all scales of r and R , we have $Q_\mu^- = Q_\mu^+ = 2$, and Ω is p -parabolic when $p \geq 2$ and is p -hyperbolic when $1 \leq p < 2$. Moreover, we again have $\theta = 1$. Here solutions to the Dirichlet problem on Ω satisfy a homogeneous Neumann condition on the rays $\{(x, x-1) : x > 1\}$ and $\{(x, 1-x) : x < -1\}$. Here, for $p > 1$ the Dirichlet problem is always solvable, but the solution is unique only when $p \geq 2$.

On the other hand, if, with Z as above, we have $\Omega = Z \setminus K \times \{0\}$ with K the standard 1/3-rd Cantor set, then again Ω is a uniform domain that is p -parabolic when $p \geq 2$ and p -hyperbolic when $1 \leq p < 2$, but now with ν the $\frac{\log 2}{\log 3}$ -dimensional Hausdorff measure supported on $K \times \{0\} = \partial\Omega$, we have that $\theta = 2 - \frac{\log 2}{\log 3} > 1$. In this case, the Dirichlet problem is always solvable when $p > 2 - \frac{\log 2}{\log 3}$, but the solution is unique only when $p \geq 2$.

Solutions to the Dirichlet problem on Ω satisfy the homogeneous Neumann condition on the two above-mentioned rays, but in addition, they also satisfy that Neumann condition on $([-1, 1] \setminus K) \times \{0\}$.

In contrast to the above examples, the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : |y| > \max\{0, |x| - 1\}\} \cup (-1, 1) \times \{0\}$, considered as a domain in $Z = \mathbb{R}^2$, is not a uniform domain and hence the mechanisms developed in this paper do not apply to such Ω . However, from the fact that this domain is obtained by gluing two copies of the second example domain along their common boundary, we see that Ω is p -hyperbolic precisely when $1 \leq p < 2$, and this example indicates that Ω should have a two-point compactification rather than the one-point compactification considered here. We will not address this issue further in the present paper.

Example 10.3. With $Z = \mathbb{R}^2$ equipped with the 2-dimensional Lebesgue measure and the Euclidean metric, let $\Omega = Z \setminus (K \times K)$, where K is the standard 1/3-rd Cantor set. Then Ω is a uniform domain, and it is p -hyperbolic precisely when $1 \leq p < 2$. In this case, ν is the restriction, to $\partial\Omega = K \times K$, of the $\frac{\log 2}{\log 3}$ -dimensional Hausdorff measure; thus $\theta = 2[1 - \frac{\log 2}{\log 3}] < 1$. In this case the Dirichlet problem is solvable for each $p > 1$, but the solution is unique only for $p \geq 2$.

CONFLICT OF INTEREST AND DATA AVAILABILITY

On behalf of all authors, the corresponding author states that there is no conflict of interest. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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Address:

R.G.: Department of Mathematical Sciences, P.O. Box 210025, University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.

E-mail: ryan.gibara@gmail.com

R.K.: Aalto University Department of Mathematics and Systems Analysis, P.O. Box 11100, FI-00076 Aalto, Finland.

E-mail: riikka.korte@aalto.fi

N.S.: Department of Mathematical Sciences, P.O. Box 210025, University of Cincinnati, Cincinnati, OH 45221-0025, U.S.A.

E-mail: shanmun@uc.edu